JOURNAL OF COMBINATORIAL THEORY, Series B **63**, 65–110 (1995)

Graph Minors.
XIII. The Disjoint Paths Problem

NEIL ROBERTSON*

Department of Mathematics, Ohio State University,
231 West 18th Avenue, Columbus, Ohio 43210

AND

P. D. SEYMOUR

Bellcore, 445 South Street,
Morristown, New Jersey 07960

Received September 10, 1990

We describe an algorithm, which for fixed $k \geqslant 0$ has running time $O(|V(G)|^3)$, to solve the following problem: given a graph G and k pairs of vertices of G, decide if there are k mutually vertex-disjoint paths of G joining the pairs.

1. INTRODUCTION

Consider the following algorithmic problem:

Disjoint Paths (DP).
Instance: A graph G, and pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices of G.
Question: Do there exist paths $P_1, \ldots, P_k$ of G, mutually vertex-disjoint, such that P_i joins s_i and t_i $(1 \leqslant i \leqslant k)$?

This problem was shown to be NP-complete by Karp [8], if k is a variable part of the input. For fixed k, however, the problem is more tractable. For instance, in [26, 27, 29] there are polynomial algorithms to solve DP with $k=2$. (In contrast, the corresponding question for directed graphs G, where we seek directed paths $P_1, \ldots, P_k$, is NP-complete even with $k=2$ [5].) The main result of this paper is that for any fixed k there is a polynomial algorithm to solve DP. It has running time $O(v^3)$, where

* Research partially supported by NSF Grant DMS 8504054 and partially performed under a consulting agreement with Bellcore.

0095-8956/95 $6.00

$v = v(G)$ is defined to be $|V(G)| + 1$. The algorithm is not practically feasible, since it involves the manipulation of enormous constants.

Although the algorithm is presented in this paper in full, the proof of its correctness is not, for that requires a lemma, (10.2), that is not proved in this paper. We postpone proving (10.2) to [24], because its proof is an application of the main results of this series, which will also appear in later papers. A draft version of the full proof of (10.2) has been made available to the referees, but due to its great length (>300 pages) it is unreasonable to expect them to guarantee its correctness.

As a consequence of our DP algorithm we have a polynomial algorithm for one of Garey and Johnson's [6] open problems, which they call Subgraph Homeomorphism (for Fixed Graph H), the following. A *subdivision* of a graph H is a graph which may be obtained from a graph isomorphic to H by repeatedly replacing an edge by two edges in series and a new vertex of valency 2. Let H be a fixed graph. The problem is:

H-Subdivision Subgraph.
Instance: A graph G.
Question: Is there a subgraph of G which is a subdivision of H?

Clearly one can use the algorithm for DP to obtain an algorithm for H-Subdivision Subgraph with running time $O(v^{|V(H)|+3})$.

The following problem is more general than DP, but for fixed k there is an $O(v^3)$ algorithm for its solution as well:

Disjoint Connected Subgraphs.
Instance: A graph G and non-null subsets $Z_1, ..., Z_t \subseteq V(G)$ with $\Sigma\, |Z_i| \leqslant k$.
Question: Do there exist connected subgraphs $G_1, ..., G_t$ of G, mutually vertex-disjoint, with $Z_i \subseteq V(G_i)$ $(1 \leqslant i \leqslant t)$?

(This does generalize DP; for given the (s_i, t_i)'s of DP, we set $Z_i = \{s_i, t_i\}$.)

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. A further result of this paper is that for every fixed graph H there is an $O(v^3)$ algorithm to solve the following:

H-Minor Containment.
Instance: A graph G.
Question: Is some minor of G isomorphic to H?

The fact that there is a polynomial algorithm for this for all H can be deduced from the solvability for all H of H-Subdivision Subgraph; because for any H there is a finite list $H_1, ..., H_k$ of graphs such that a general graph G has a minor isomorphic to H if and only if it contains a subdivision of one of $H_1, ..., H_k$. To make the degree of the polynomial independent of H needs a different method, however.

As an application, let $\mathscr{F}$ be a class of graphs such that if $G \in \mathscr{F}$ and H is isomorphic to a minor of G then $H \in \mathscr{F}$. (Such a class is *minor-closed.*) We shall show in [23] that there is a finite list $H_1, ..., H_k$ of graphs, such that a general graph G is in $\mathscr{F}$ if and only if it has no minor isomorphic to any of $H_1, ..., H_k$. We deduce that there is an $O(v^3)$ algorithm for

$\mathscr{F}$-Membership.
Instance: A graph G.
Question: Is $G \in \mathscr{F}$?

For we use the algorithm for H-Minor Containment to test for the presence of $H_1, ..., H_k$ as minors. (This proves the *existence* of the algorithm; however, we do not know how to *find* the algorithm, even for concrete classes of $\mathscr{F}$, because we do not know how to find $H_1, ..., H_k$.) We show also that if $\mathscr{F}$ does not contain every planar graph, there is an $O(v^2)$ algorithm to determine membership.

For example,

(i) Let Σ be a surface; there is an $O(v^3)$ algorithm to test if a graph G may be drawn in Σ. This improves a result of Filotti, Miller, and Reif [4].

(ii) Let $\omega \geqslant 0$ be an integer; there is an $O(v^2)$ algorithm to test if a graph G has tree-width $\leqslant \omega$ (or branch-width $\leqslant \omega$, similarly). (For definitions, see [15] and Section 3.) This improves a result of Arnborg, Corneil, and Proskurowski [1].

We remark that for both the above examples, the finiteness of the list $H_1, ..., H_k$ of excluded minors does not depend on the result of [23]; it is shown in [17, 14], respectively.

A clarification: how can the above algorithms have running time just dependent on v and not on the number of edges? Certainly, for simple graphs, if the number of edges is large enough in terms of v then the problems (i) and (ii) are both easily answered (the answer is "no"). But what about graphs with multiple edges? We would like to be able to construct the underlying simple graph H of a graph G in time which depends only on $|E(H)|$, not on $|E(G)|$. This can of course be done if our graphs are presented with an appropriate data structure, and we assume this is so. Similarly, we shall always assume that we can decide, for a graph G and vertices u, v of G, whether there are at least k edges of G joining u and v, in time only dependent on k.

2. Folios

To solve the problems of the last section, we shall solve a common generalization of Disjoint Connected Subgraphs and H-Minor Containment, which we describe in this section. For maximum generality we work with directed graphs, although that is not necessary for the applications. Thus each edge has a head and a tail (the same, for a loop). The *underlying graph* of a digraph is obtained by forgetting the directions of its edges. We shall work almost completely in terms of the underlying graph, and shall often fail to distinguish it from the digraph, for notational and other convenience. In particular, a digraph is *connected* if the underlying graph is connected.

Let G, H be both graphs or both digraphs. A *model* ϕ of H in G assigns to each edge e of H an edge $\phi(e)$ of G, and to each vertex v of H a non-null connected subgraph $\phi(v)$ of G, such that

(i) the graphs $\phi(v)$ $(v \in V(H))$ are mutually vertex-disjoint, the edges $\phi(e)$ $(e \in E(H))$ are all distinct, and for $v \in V(H)$ and $e \in E(H)$, $\phi(e) \notin E(\phi(v))$

(ii) for $e \in E(H)$, if e has head u and tail v (or, in the undirected case, has ends u, v) then $\phi(e)$ has a head in $V(\phi(u))$ and a tail in $V(\phi(v))$ (respectively, $\phi(e)$ has one end in $V(\phi(u))$ and the other in $V(\phi(v))$.

Thus, H is isomorphic to a minor of G if and only if there is a model of H in G; indeed, this could be taken as the definition of when H is isomorphic to a minor of G.

We would like to extend the idea of a minor to "rooted" digraphs. Intuitively, a rooted digraph is a digraph G, of which some vertices, $v_1, ..., v_k$ say, are selected and called roots. If we take minors in an unrestricted sense, however, some of these roots may be deleted and some pairs of them may be identified, and this is notationally too complicated to be satisfactory. The best compromise seems to be to permit roots to become identified, but not to permit them to be deleted.

Thus, let us say that $(G, v_1, ..., v_k)$ is a *rooted digraph* if G is a digraph and $v_1, ..., v_k \in V(G)$ (not necessarily distinct). For $1 \leqslant i \leqslant k$, v_i is the ith *root*. Isomorphism of rooted digraphs is defined in the natural way; thus, isomorphic rooted digraphs have the same number of roots, and the isomorphism maps the ith root of one to the ith root of the other, for all i. Let $(G, v_1, ..., v_k)$, $(H, u_1, ..., u_k)$ be rooted digraphs. A *model* of the second in the first is a model ϕ of H in G such that $v_i \in V(\phi(u_i))$ $(1 \leqslant i \leqslant k)$, and if there is such a model, we say that $(H, u_1, ..., u_k)$ is a *minor* of $(G, v_1, ..., v_k)$. The set of all minors of $(G, v_1, ..., v_k)$ is closed under isomorphism, and we call it the *folio* of $(G, v_1, ..., v_k)$. Clearly the folio is finite up to isomorphism. If $\delta \geqslant 0$ is an integer, we say that $(H, u_1, ..., u_k)$ has *detail* $\leqslant \delta$ if $|E(H)| \leqslant \delta$ and $|V(H) - \{u_1, ..., u_k\}| \leqslant \delta$. The *$\delta$-folio* of $(G, v_1, ..., v_k)$ is the set of all minors of $(G, v_1, ..., v_k)$ with detail $\leqslant \delta$. Its size (up to isomorphism) is bounded above by a function of k and δ, as is easily seen.

For example, let G be the digraph with four vertices u, v_1, v_2, v_3 and three edges uv_1, uv_2, uv_3; we shall examine the 0-folio of (G, v_1, v_2, v_3). For $i = 1, 2, 3$ let N_i be the digraph with vertex set $\{1, ..., i\}$ and no edges. Then the 0-folio of (G, v_1, v_2, v_3) contains $(N_1, 1, 1, 1)$, $(N_2, 1, 1, 2)$, $(N_2, 1, 2, 1)$, $(N_2, 2, 1, 1)$, $(N_3, 1, 2, 3)$, and all rooted digraphs isomorphic to one of these.

Now the δ-folio of $(G, v_1, ..., v_k)$ can be determined from the δ-folio of $(G, u_1, ..., u_l)$ if $\{u_1, ..., u_l\} = \{v_1, ..., v_k\}$, as is easily seen. Thus, if $Z \subseteq V(G)$, by the *δ-folio of G relative to Z* we mean the δ-folio of $(G, v_1, ..., v_k)$ for any choice of $v_1, ..., v_k$ with $k = |Z|$ and $Z = \{v_1, ..., v_k\}$.

For fixed δ, $\xi \geq 0$ we shall give an algorithm with running time $O(v^3)$ to solve the following problem.

Folio.
Instance: A digraph G, and a subset $Z \subseteq V(G)$ with $|Z| \leq \xi$.
Question: What is the δ-folio of G relative to Z?

Let us see how Folio can be used to solve the two problems of the previous section. For Disjoint Connected Subgraphs, with input $G = (V, E)$ and $Z_1, ..., Z_t \subseteq V$, we may assume that $Z_1, ..., Z_t$ are mutually disjoint; we direct the edges of G arbitrarily; we compute the 0-folio of G relative to $Z_1 \cup \cdots \cup Z_t$ and from this we read off whether the desired connected subgraphs exist. (Compare the example above.) For H-Minor Containment, we direct the edges of G arbitrarily and compute the δ-folio of G relative to $\emptyset$, where $\delta = \max(|V(H)|, |E(H)|)$. These both have running time $O(v^3)$.

We shall need some easy observations on folio management, the proofs of which we leave to the reader:

(2.1) *If $Z' \subseteq Z \subseteq V(G)$ and $\delta \geq 0$, the δ-folio of G relative to Z' is determined by a knowledge of the δ-folio of G relative to Z.*

A *separation* of a graph or digraph G is a pair (A, B) of subgraphs such that $A \cup B = G$ and $E(A \cap B) = \emptyset$, where for subgraphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ of G, $G_1 \cup G_2$ and $G_1 \cap G_2$ are the subgraphs $(V_1 \cup V_2, E_1 \cup E_2)$ and $(V_1 \cap V_2, E_1 \cap E_2)$, respectively. The *order* of the separation (A, B) is $|V(A \cap B)|$.

(2.2) *Let (A, B) be a separation of a digraph G, and let $V(A \cap B) \subseteq Z \subseteq V(G)$. For $\delta \geq 0$, the δ-folio of G relative to Z is determined by a knowledge of the δ-folios of A relative to $Z \cap V(A)$ and of B relative to $Z \cap V(B)$.*

For $Z \subseteq V(G)$, a vertex v of G is *irrelevant* (*to the δ-folio of G relative to Z*) if $v \notin Z$ and the δ-folios of G and $G \backslash v$ relative to Z are equal. (We shall use $\backslash$ for deletion.) From (2.1) and (2.2), the following follows easily.

(2.3) *Let (A, B) be a separation of a digraph G, and let $Z \subseteq V(A)$. If $v \in V(B)$ is irrelevant to the δ-folio of B relative to $V(A \cap B)$ then it is irrelevant to the δ-folio of G relative to Z.*

The idea of our algorithm for Folio is as follows: If G can be divided into trivial pieces by cutsets of bounded size we can solve Folio directly, using a method of Arnborg and Proskurowski [2]. If not there is an irrelevant vertex, and we give a procedure to find such a vertex. We delete it, and repeat the process as often as necessary, until the graph is reduced so much that the Arnborg–Proskurowski method can be applied. The first part of the paper (Sections 3 and 4) cover detecting and utilizing that G has bounded "branch-width," that is, that G can be divided into trivial pieces by cutsets of bounded size. The remainder of the paper concerns finding an irrelevant vertex. This turns out to be easy if we can find a large clique minor (Sections 5 and 6); the main problem is what should be done if we have not found such a minor. In that case it follows from an earlier result of this series that G has a large grid minor; and the remainder of the paper concerns how to exploit a large grid minor to obtain either a large clique minor or an irrelevant vertex. For this we shall make use of a lemma which is a consequence of the main results of this series and which will be proved in a future paper. (The lemma is an assertion that under certain circumstances, a given vertex is irrelevant.)

3. Branch-Width

A *branch-decomposition* of a graph or digraph G is a pair (T, τ), where

(i) T is a tree and every vertex of T has valency 1 or 3

(ii) τ is a bijection from $E(G)$ to the set of leaves of T.

(The *leaves* of T are the vertices with valency 1.) For each $f \in E(T)$ the edges e of G are divided into two sets, depending on which component of $T \backslash f$ contains $\tau(e)$. The *order* of f is the number of vertices of G incident with an edge in each set. The *width* of (T, τ) is the maximum order of edges of T, and the *branch-width* of G is the minimum width of all branch-decompositions of G. (If there are no branch-decompositions of G, then $|E(G)| \leqslant 1$ and G has branch-width 0 by convention.) Branch-width is closely related to "tree-width"—see [19].

Our first objective is an algorithm which decides either that G has branch-width $\leqslant 3\omega$, or that G has branch-width $\geqslant \omega$. For that we need the following lemma.

(3.1) LEMMA. *Let G be a graph with branch-width $\leqslant \omega$, where $\omega \geqslant 2$, and let $Z \subseteq V(G)$. Then there is a separation (A, B) of G of order $\leqslant \omega$ with*

$$|Z - V(A)|, |Z - V(B)| \leqslant \tfrac{2}{3}|Z| - \tfrac{1}{2}|Z \cap V(A \cap B)|.$$

Proof. Let $\mathscr{T}$ be the set of all separations (A, B) of G of order $\leqslant \omega$ with

$$|Z - V(A)| + \tfrac{1}{2}|Z \cap V(A \cap B)| > \tfrac{2}{3}|Z|.$$

We observe that

(1)(i) *If $(A, B) \in \mathscr{T}$ then $V(A) \neq V(G)$,*

(ii) *if (A_1, B_1), $(A_2, B_2) \in \mathscr{T}$ then B_1 is not a subgraph of A_2*

(iii) *if A_1, A_2, A_3 are mutually edge-disjoint subgraphs of G with union G, then not all of $(A_1, A_2 \cup A_3)$, $(A_2, A_3 \cup A_1)$, $(A_3, A_1 \cup A_2)$ belong to $\mathscr{T}$.*

Subproof. For (i), let (A, B) be a separation of G with $V(A) = V(G)$. Then

$$|Z - V(A)| + \tfrac{1}{2}|Z \cap V(A \cap B)| \leqslant \tfrac{1}{2}|Z|,$$

and so $(A, B) \notin \mathscr{T}$. This proves (i).

For (ii), suppose that (A_1, B_1), $(A_2, B_2) \in \mathscr{T}$. Then

$$|Z \cap V(B_1)| = |Z - V(A_1)| + |Z \cap V(A_1 \cap B_1)| > \tfrac{2}{3}|Z|,$$

but

$$\begin{aligned}|Z| - \tfrac{1}{2}|Z \cap V(A_2)| &= |Z - V(A_2)| + \tfrac{1}{2}|Z \cap V(A_2)| \\ &\geqslant |Z - V(A_2)| + \tfrac{1}{2}|Z \cap V(A_2 \cap B_2)| > \tfrac{2}{3}|Z|,\end{aligned}$$

that is, $|Z \cap V(A_2)| < \frac{2}{3}|Z|$. Consequently (ii) holds.

For (iii), suppose that all three of the separations belong to $\mathscr{T}$. For $i = 1, 2, 3$ let $Z_i = Z \cap V(A_i)$. Since $(A_1, A_2 \cup A_3) \in \mathscr{T}$ we deduce that

$$|Z - Z_1| + \tfrac{1}{2}|Z_1 \cap (Z_2 \cup Z_3)| > \tfrac{2}{3}|Z|,$$

that is,

$$\tfrac{1}{3}|Z| > |Z_1| - \tfrac{1}{2}|Z_1 \cap Z_2| - \tfrac{1}{2}|Z_1 \cap Z_3| + \tfrac{1}{2}|Z_1 \cap Z_2 \cap Z_3|.$$

By adding this and the two similar inequalities, we deduce that

$$|Z| > \sum_{1 \leqslant i \leqslant 3} |Z_i| - \sum_{1 \leqslant i < j \leqslant 3} |Z_i \cap Z_j| + \tfrac{3}{2}|Z_1 \cap Z_2 \cap Z_3|,$$

contrary to the inclusion–exclusion formula, since $Z = Z_1 \cup Z_2 \cup Z_3$. This proves (iii) and so completes the proof of (1).

Suppose that for every separation (A, B) of G of order $\leqslant \omega$, $\mathscr{T}$ contains one of (A, B), (B, A). By (1) and [19, Theorem (4.5)], $\mathscr{T}$ is a "tangle of order $\omega+1$" in G, in the terminology of that paper. But this contradicts [19, Theorem (4.3)], since G has branch-width $\leqslant \omega$ and every edge of G has $\leqslant 2 \leqslant \omega$ ends. It follows that there is a separation (A, B) of G of order $\leqslant \omega$, such that $(A, B), (B, A) \notin \mathscr{T}$; but then (A, B) satisfies the theorem. ∎

To apply (3.1) we use the following lemma.

(3.2) LEMMA. *If $\omega \geqslant 2$ is an integer and G is a simple graph with branch-width $\leqslant \omega$, then $|E(G)| \leqslant (\frac{3}{2}\omega - 1)\,|V(G)|$.*

Proof. Put $\frac{3}{2}\omega - 1 = \alpha$. We proceed by induction on $|V(G)|$. If $|V(G)| = 0$ the result is trivial, and we assume that $|V(G)| > 0$. By [19, Theorem (5.1)], G has tree-width $\leqslant \alpha$ (see that paper for definitions) and hence has a vertex v of valency $\leqslant \alpha$. From the inductive hypothesis, $G \backslash v$ has $\leqslant \alpha(|V(G)| - 1)$ edges, and so

$$|E(G)| \leqslant \alpha(|V(G) - 1) + \alpha = \alpha\,|V(G)|,$$

as required. ∎

We use (3.1) to show that if we "greedily" attempt to find a branch-decomposition of G of width $\leqslant 3\omega$, and fail, then G has branch-width $\geqslant \omega$. Actually, we can obtain slightly more than this, as follows. For a graph or digraph G, we define $\varepsilon = \varepsilon(G) = |E(G)| + |V(G)| + 1$. Let $\omega \geqslant 2$ be some integer.

(3.3) ALGORITHM.
Input: A graph G with $|E(G)| \geqslant 2$.
Output: A subgraph G' of G and a branch-decomposition of G' of width $\leqslant 3\omega$, such that either $G' = G$ or G' has branch-width $\geqslant \omega$.
Running time: $O(v^2 + \varepsilon)$.
Description: We construct the simple graph H_1 underlying G, by deleting all loops and multiple edges. We choose a maximal subgraph H_2 of H_1 with $|E(H_2)| \leqslant (\frac{3}{2}\omega - 1)\,|V(H_1)| + 1$. Thus, either $H_2 = H_1$ or H_2 has branch-width $\geqslant \omega$, by (3.2). We assume that $|E(H_2)| \geqslant 2$, for otherwise the problem is trivial. We shall construct a subgraph H_3 of H_2 and a branch-decomposition of H_3 of width $\leqslant 3\omega$, such that either $H_3 = H_2$ or H_3 has branch-width $\geqslant \omega$. Given such an H_3, the algorithm is completed as follows. If $H_3 \neq H_2$, we set $G' = H_3$ and stop. If $H_3 = H_2$, we have a branch-decomposition of H_2 of width $\leqslant 3\omega$. If $H_2 \neq H_1$ we set $G' = H_2$ and stop. If $H_2 = H_1$, we convert our branch-decomposition to one for G, set $G' = G$, and stop.

Thus it remains to show how to construct H_3. We initialize by setting $G_1 = H_2$, we let T_1 be a tree with two vertices t, t', and we define $\tau_1: E(G_1) \to V(T_1)$ by setting $\tau_1(e) = t$ for some edge e of G_1, and $\tau_1(f) = t'$ for all other edges f of G_1. We perform a recursion. At the beginning of the ith iteration we have

(i) a subgraph G_i of H_2 such that either $G_i = H_2$ or G_i has branch-width $\geqslant \omega$

(ii) a tree T_i each vertex of which has valency 1 or 3

(iii) a surjection τ_i from $E(G_i)$ to the set of leaves of T_i, such that each edge of T_i has order $\leqslant 3\omega$.

(The *order* of $f \in E(T_i)$ is defined to be the number of vertices of G_i incident with an edge e_1 and an edge e_2 such that $\tau_i(e_1)$ and $\tau_i(e_2)$ are in different components of $T_i \backslash f$.)

The iteration proceeds as follows:

(1) If τ_i is a bijection, we see $H_3 = G_i$ and stop. (T_i, τ_i) is a branch-decomposition of H_3 of width $\leqslant 3\omega$, and either $H_3 = H_2$ or H_3 has branch-width $\geqslant \omega$, as required.

(2) If τ_i is not a bijection we choose a leaf v of T_i such that $v = \tau_i(e)$ for at least two edges e of G_i. Let S, S' be the components of $T_i \backslash f$, where $f \in E(T_i)$ is incident with v, and $V(S) = \{v\}$. Let R be the subgraph of G_i formed by all edges e with $\tau_i(e) \in V(S)$ and their ends, and define R' similarly. Then $E(R \cap R') = \emptyset$, $E(R \cup R') = E(G_i)$, and $|V(R \cap R')| \leqslant 3\omega$. Let $Z = V(R \cap R')$. Let T' be a tree obtained by adding two new vertices v_1, v_2 to T_i, both adjacent to v.

(3) If $Z = \emptyset$ we choose an edge g of R, and if $0 < |Z| < 3\omega$ we choose $g \in E(R)$ with an end in Z. (This is possible since $E(R) \neq \emptyset$ and every vertex in Z is incident with an edge of R.) We set $T_{i+1} = T'$, $G_{i+1} = G_i$, and define τ_{i+1} by

$$\begin{aligned} &\tau_{i+1}(e) = \tau_i(e) && (e \in E(G_i), \tau_i(e) \neq v) \\ &\tau_{i+1}(e) = v_1 && (e \in E(G_i) - \{g\}, \tau_i(e) = v) \\ &\tau_{i+1}(g) = v_2. \end{aligned}$$

We return to (1) for the next iteration. It is easy to see that G_{i+1}, T_{i+1}, τ_{i+1} satisfy (i), (ii), and (iii).

(4) If $|Z| = 3\omega$ we test if there is a separation (A, B) of R of order $\leqslant \omega$ with

$$|Z - V(A)|, |Z - V(B)| \leqslant \tfrac{2}{3}|Z| - \tfrac{1}{2}|Z \cap V(A \cap B)|.$$

This can be done by enumerating all partitions (Z_1, Z_2, Z_3) of Z with $|Z_1|$, $|Z_2| \leqslant 2\omega - \frac{1}{2}|Z_3|$ and then testing for separations (A, B) of order $\leqslant \omega$ with $V(A) \cap Z = Z_1 \cup Z_3$ and $V(B) \cap Z = Z_2 \cup Z_3$. (Such a separation does not exist if and only if in $R \backslash Z_3$ there are $> \omega - |Z_3|$ paths from Z_1 to Z_2, mutually vertex-disjoint.)

(5) If (A, B) is a separation as in (4), we set $T_{i+1} = T'$, $G_{i+1} = G_i$, and define τ_{i+1} by

$$\begin{aligned} \tau_{i+1}(e) &= \tau_i(e) && (e \in E(G_i), \tau_i(e) \neq v) \\ \tau_{i+1}(e) &= v_1 && (e \in E(A)) \\ \tau_{i+1}(e) &= v_2 && (e \in E(B)). \end{aligned}$$

We return to (1) for the next iteration.

Let us verify that $G_{i+1}, T_{i+1}, \tau_{i+1}$ satisfy (i), (ii), and (iii). Now (i) is clear, since $G_{i+1} = G_i$, and (ii) is clear from construction. To verify (iii) we observe that $E(A) \neq \emptyset \neq E(B)$; for if $E(A) = \emptyset$, say, then $B = R$ since R has no isolated vertices, and so

$$|Z - V(A)| \leqslant \tfrac{2}{3}|Z| - \tfrac{1}{2}|Z \cap V(A)|,$$

that is, $|Z| \leqslant \frac{3}{2}|Z \cap V(A)|$, which is impossible since $|Z| = 3\omega$ and

$$|Z \cap V(A)| \leqslant |V(A)| = |V(A \cap B)| \leqslant \omega.$$

Thus $E(A)$, $E(B) \neq \emptyset$ and so τ_{i+1} is a surjection. Every edge of T_{i+1} has the same order as it does in T_i except for the two new edges. The new edges have orders $|(Z \cup V(A)) \cap V(B)|$ and $|(Z \cup V(B)) \cap V(A)|$; and

$$\begin{aligned} |(Z \cup V(A)) \cap V(B)| &= |Z - V(A)| + |V(A) \cap V(B)| \\ &\leqslant \tfrac{2}{3}|Z| + |V(A \cap B)| \leqslant 2\omega + \omega = 3\omega. \end{aligned}$$

Similarly, $|(Z \cup V(B)) \cap V(A)| \leqslant 3\omega$. This verifies (iii).

(6) If there is no separation (A, B) as in (4), we choose $g \in E(R)$, set $G_{i+1} = G_i \backslash g$, $T_{i+1} = T_i$, and let τ_{i+1} be the restriction of τ_i to $E(G_{i+1})$. We return to (1) for the next iteration.

Again, let us verify that $G_{i+1}, T_{i+1}, \tau_{i+1}$ satisfy (i), (ii), (iii). From (3.1), G_i has branch-width $> \omega$, and so G_{i+1} has branch-width $\geqslant \omega$ (we leave as an exercise the fact that deleting an edge reduces branch-width by at most one, except when branch-width $\leqslant 2$ which does not occur here since $\omega \geqslant 2$). This verifies (i), and (ii) and (iii) are clear. This concludes the description of the algorithm.

To find its running time, we observe that at each iteration but the last the quantity $|E(G_i)| - |E(T_i)|$ is reduced by 1; it starts at $O(|V(G)|)$ and finishes at 1, and so there are $O(v)$ iterations. Each one takes a time $O(v)$. The time is spent principally in step 4; there, each partition (Z_1, Z_2, Z_3) can be checked in time $O(|E(G_i)|)$ and, hence, $O(v)$, since $|E(G_i)| \leqslant O(v)$ and the number of partitions to be checked is bounded.

Thus constructing H_3 and its branch-decomposition takes time $O(v^2)$. Converting it to a branch-decomposition of G takes time $O(\varepsilon)$, and so the whole algorithm takes time $O(v^2 + \varepsilon)$. ∎

Actually, Hans Bodlaender [3] (and see also [12]) has recently found an algorithm that could be used in place of (3.3), with running time $O(v)$ for simple graphs G, using tree-width instead of branch-width. Its use could speed up the applications of (3.3); for instance, (7.2) could be made to run in time $O(\varepsilon)$. This would not affect the total running time of our Folio algorithm, and so we omit further details.

4. Folio Determination with Bounded Branch-Width

If G is a digraph, $Z \subseteq V(G)$, and $\delta \geqslant 0$, a *modelling* of the δ-folio of G relative to Z means some choice of model, for each member of the δ-folio of G relative to Z (up to isomorphism). A method of Arnborg and Proskurowski [2] can be used to solve Folio for graphs of bounded branch-width, as follows. Let δ, $\xi \geqslant 0$, $\omega \geqslant 1$ be integers.

(4.1) Algorithm.

Input: A digraph G, and a subset $Z \subseteq V(G)$ with $|Z| \leqslant \xi$, and a branch-decomposition (T, τ) of G of width $\leqslant \omega$.

Output: The δ-folio of G relative to Z, and a modelling of it.

Running time: $O(\varepsilon)$.

Description: Since the effect of isolated vertices of G on the folio is clear, we may arrange (by deleting them) that G has no isolated vertices. Let z be a leaf of T, and number the edges of T as $f_1, ..., f_m$, where $m = |E(T)|$, so that the numbers on every path leaving z are decreasing. (Our branch-decomposition in applications usually comes already equipped with such a numbering; but if not, one can be found in time $O(\varepsilon)$.) For $1 \leqslant i \leqslant m$, let S_i, T_i be the components of $T \backslash f_i$, with $z \in V(S_i)$. Let A_i be the subgraph of G formed by the edges e with $\tau(e) \in V(S_i)$, and their ends; and define B_i similarly. Since G has no isolated vertices, (A_i, B_i) is a separation of G of order $\leqslant \omega$. Let $Z_i = (V(A_i) \cup Z) \cap V(B_i)$. We shall compute the δ-folio of B_i relative to Z_i for each i, by a recursion as follows. At the start of the ith iteration, the δ-folio of B_j relative to Z_j, has been determined for $1 \leqslant j \leqslant i$.

(1) If $i \leqslant m$ and $|V(T_i)| = 1$ then $|E(B_i)| = 1$ and $|V(B_i)| \leqslant 2$. We determine the folio of Z_i in B_i and return to (1) for the next iteration.

(2) If $i \leqslant m$ and $|V(T_i)| > 1$, let f_j, f_k be the two edges of T_i with a common end with f_i. Then $j, k < i$, and (B_j, B_k) is a separation of B_i, and the δ-folios of B_j relative to Z_j and B_k relative to Z_k have already been determined. Since $V(B_j \cap B_k) \subseteq Z_j \cap Z_k$ we can calculate the δ-folio of $B_j \cup B_k$ relative to $Z_j \cup Z_k$ from this information, by (2.2). Since $Z_i \subseteq Z_j \cup Z_k$ and $B_j \cup B_k = B_i$, the δ-folio of B_i relative to Z_i can be calculated by (2.1). (Note that, since δ-folios have size bounded by a constant, these calculations take constant time.) We return to (1) for the next iteration.

(3) If $i = m+1$, the folio of B_m relative to Z_m has been determined. We determine the δ-folio of A_m relative to $(Z \cup V(B_m)) \cap V(A_m)$ (this is easy, since $|V(A_m)| \leqslant 2$), and we use (2.1) and (2.2) to determine the δ-folio of Z in $A_m \cup B_m = G$.

(4) For each member of the δ-folio of Z in G (up to isomorphism) we backtrack through this computation to compute a corresponding model, in the natural way.

Each iteration takes constant time, and there are $|E(T)| + 1$ iterations. Thus, steps (1)–(3) take time $O(\varepsilon)$. In step (4), the backtracking to determine how each edge of G should be used to produce the desired member of the δ-folio takes time $O(\varepsilon)$ and converting this information into a model again takes time $O(\varepsilon)$. Thus the total running time is $O(\varepsilon)$. ∎

To apply (4.1), we use the following theorem, a consequence of Theorems (1.5) of [15] and (5.1) of [19]. See also [25] for a numerical improvement.

(4.2) THEOREM. *For every planar graph H there is a number ω such that every graph with branch-width $\geqslant \omega$ has a minor isomorphic to H.*

For a first application of (4.1), we have the following, improving a result of [13].

(4.3) *Let $\mathscr{F}$ be a minor-closed class of graphs such that some planar graph is not in $\mathscr{F}$. Then there is an algorithm to determine whether $G \in \mathscr{F}$ with running time $O(v^2)$.*

Proof. By Theorem (1.4) of [14], there is a finite list of graphs $H_1, \ldots, H_k$ such that a graph G is in $\mathscr{F}$ if and only if it has no minor isomorphic to any of $H_1, \ldots, H_k$. Since $\mathscr{F}$ does not contain every planar graph, some H_i (say H_1) is planar. By (4.2) there is a number $\omega \geqslant 2$ such that every graph with branch-width $\geqslant \omega$ has a minor isomorphic to H_1. Let $\delta = \max(\varepsilon(H_1), \ldots, \varepsilon(H_k))$. We now describe an algorithm to test membership of $\mathscr{F}$. Let G be the input graph. We may assume that $|E(G)| \geqslant 2$.

(1) Choose a maximal subgraph G_1 of G with at most δ loops at each vertex and with no $\delta+1$ edges mutually parallel. Then $G_1 \in \mathscr{F}$ if and only if $G \in \mathscr{F}$, as is easily seen.

(2) We apply (3.3) to G_1. We obtain a subgraph G' of G_1, and a branch-decomposition of G' of width $\leqslant 3\omega$, such that either $G' = G_1$ or G' has branch-width $\geqslant \omega$.

(3) If $G' \neq G_1$ then G' has branch-width $\geqslant \omega$, and so G' has a minor isomorphic to H_1 and $G' \notin \mathscr{F}$; thus, $G \notin \mathscr{F}$ and we stop.

(4) If $G' = G_1$, we have a branch-decomposition of G_1 of width $\leqslant 3\omega$. We direct the edges of G_1 arbitrarily, we apply (4.1) to G_1 with $Z = \varnothing$ (with 3ω replacing ω) and compute the δ-folio of G_1 relative to $\varnothing$. From this we deduce whether G_1 has a minor isomorphic to any of $H_1, ..., H_k$ and hence determine whether $G_1 \in \mathscr{F}$. This completes the algorithm.

Since at step 1 we arrange that $\varepsilon(G_1) \leqslant O(v^2)$, the algorithm has running time $O(v^2)$. ∎

A second, similar, application of (4.1) is the following. Let H be a planar graph. By (4.3) (with $\mathscr{F}$ the class of graphs with no minor isomorphic to H) we can test if an input graph has a minor isomorphic to H in time $O(v^2)$. But what if we want to *find* such a minor? The following improves a result of [13].

(4.4) ALGORITHM.
Input: A graph G.
Output: Decides if there is a model of H in G, and outputs such a model if one exists.
Running time: $O(v^2)$.
Description: Let ω be as in (4.2), with $\omega \geqslant 2$.

(1) We choose a maximal subgraph G_1 of G with at most $|E(H)|$ loops at every vertex and at most $|E(H)|+1$ edges mutually parallel. There is a model of H in G if and only if there is one in G_1.

(2) We apply (3.3) to G_1. We obtain a subgraph G_2 of G_1, and a branch-decomposition of G_2 of width $\leqslant 3\omega$, such that either $G_2 = G_1$ or G_2 has branch-width $\geqslant \omega$. From the choice of ω, it follows that either $G_2 = G_1$ or there is a model of H in G_2. Consequently there is a model of H in G_1 if and only if there is one in G_2.

(3) We test if G_2 has an H-minor, and if so find a model of it in G_2, by using (4.1) to compute the $\varepsilon(H)$-folio of G_2 relative to $\varnothing$ and a modelling of it.

From step (1), $|E(G_1)| \leqslant O(v^2)$. Step (2) therefore takes time $O(v^2)$, and so does step (3). Thus the algorithm has running time $O(v^2)$ in total. ∎

5. Generic Folios

So far we have discussed how to detect small branch-width and how to solve Folio in digraphs of bounded branch-width. Now we come to the more difficult problem of how to solve Folio in digraphs of large branch-width, and here our strategy is to locate an irrelevant vertex and delete it (and repeat, until we attain small branch-width). The problem of finding such a vertex splits into two subproblems: how to find an irrelevant vertex given a large clique minor; and how to find either an irrelevant vertex or a large clique minor. We treat the former problem in this section and the next.

The δ-folio of a digraph G relative to $Z \subseteq V(G)$ is *generic* if the δ-folio of the rooted digraph $(G, v_1, ..., v_\xi)$ contains every rooted digraph with ξ roots and with detail $\leqslant \delta$, where $|Z| = \xi$ and $Z = \{v_1, ..., v_\xi\}$.

(5.1) *Let G be a digraph with $|V(G)| \geqslant 2\xi + 3\delta$, and let $Z \subseteq V(G)$ with $|Z| = \xi$. Suppose that $u, v \in Z$, for all distinct $u, v \in V$ that are non-adjacent in G. Then the δ-folio of G relative to Z is generic.*

Proof. Let $Z = \{v_1, ..., v_\xi\}$, and let $(H, u_1, ..., u_\xi)$ be a rooted digraph with detail $\leqslant \delta$. We must show that $(H, u_1, ..., u_\xi)$ is a minor of $(G, v_1, ..., v_\xi)$. For each $v \in V(H)$, choose $X_v \subseteq V(G)$ such that $|X_v - Z| = 1$, $X_v \cap Z = \{v_i : 1 \leqslant i \leqslant \xi,\ u_i = v\}$, and for distinct $v, v' \in V(H)$, $X_v \cap X_{v'} = \emptyset$. For each edge $e \in E(H)$, choose a non-loop edge $\phi(e)$ of G with head s_e and tail t_e, such that $s_e, t_e \notin Z \cup \bigcup (X_v : v \in V(H))$, and for distinct $e, e' \in E(H)$, $s_e, t_e, s_{e'}, t_{e'}$ are all distinct. Such choices are possible since

$$|V(G) - Z| \geqslant (\xi + \delta) + 2\delta \geqslant |V(H)| + 2\,|E(H)|.$$

For each $v \in V(H)$, let $\phi(v)$ be the restriction of $G \backslash \{\phi(e) : e \in E(H)\}$ to

$$X_v \cup \{s_e : e \in E(H) \text{ has head } v\} \cup \{t_e : e \in E(H) \text{ has tail } v\}.$$

Then G_v is connected (since $X_v \nsubseteq Z$) and it is easy to see that ϕ is a model of $(H, u_1, ..., u_\xi)$ in $(G, v_1, ..., v_\xi)$, as required. ∎

If G is a digraph and $X, Y \subseteq V(G)$ or X, Y are subdigraphs of G, an edge of G is said to be *between* X and Y if it has one end in X and the other in Y.

(5.2) *Let G be a digraph, let $Z \subseteq V(G)$ with $|Z| = \xi$, and let $k = 2\xi + 3\delta$. Suppose that there are k non-null connected subgraphs $G_1, ..., G_k$ of G, mutually vertex-disjoint, such that*

(i) $|Z \cap V(G_i)| = 1$ $(1 \leqslant i \leqslant \xi)$, *and (hence)* $Z \cap V(G_i) = \emptyset$ $(\xi + 1 \leqslant i \leqslant k)$

(ii) *for* $1 \leqslant i < j \leqslant k$, *if there is no edge of* G *between* G_i *and* G_j *then* $i, j \leqslant \xi$.

Then the δ*-folio of* G *relative to* Z *is generic.*

Proof. We delete every vertex of G not in $V(G_1) \cup \cdots \cup V(G_k)$, and then contract every edge of each G_i. We obtain a digraph G' with $|V(G')| = k$. Let $Z' \subseteq V(G')$ correspond to Z. By (5.1), the δ-folio of G' relative to Z' is generic, and so the δ-folio of G relative to Z is generic, as required. ∎

We shall use (5.2) to prove the following lemma.

(5.3) *Let* G *be a graph and let* $Z \subseteq V(G)$ *with* $|Z| = \xi$. *Let* $k \geqslant \lfloor \frac{3}{2}\xi \rfloor$ *and let* $G_1, \ldots, G_k$ *be subgraphs of* G, *mutually vertex-disjoint, such that*

(i) *for* $1 \leqslant i \leqslant k$, *either* G_i *is connected or every component of* G_i *has a vertex in* Z,

(ii) *for* $1 \leqslant i < j \leqslant k$, *either* $V(G_i)$ *and* $V(G_j)$ *both meet* Z *or there is an edge of* G *between* G_i *and* G_j, *and*

(iii) *for* $1 \leqslant i \leqslant k$, *there is no separation* (A, B) *of* G *of order* $< \xi$ *with* $Z \subseteq V(A)$ *and with* $A \cap G_i$ *null.*

Then for some μ *with* $0 \leqslant \mu \leqslant \xi$ *there are* $\kappa = k - \lfloor \frac{1}{2}(\xi - \mu) \rfloor$ *non-null connected subgraphs* $H_1, \ldots, H_\kappa$ *of* G, *mutually vertex-disjoint, such that*

(a) $|V(H_i) \cap Z| = 1$ $(1 \leqslant i \leqslant \xi)$ *and* $V(H_i) \cap Z = \emptyset$ $(\xi + 1 \leqslant i \leqslant \kappa)$, *and*

(b) *for* $1 \leqslant i < j \leqslant \kappa$, *if there is no edge of* G *between* H_i *and* H_j *then* $i, j \leqslant \mu$.

In particular, for any $\delta \geqslant 0$, *if* $k \geqslant \lfloor \frac{5}{2}\xi \rfloor + 3\delta$ *and* G *is a digraph then the* δ*-folio of* G *relative to* Z *is generic.*

Proof. The second assertion follows from the first and (5.2). For if $\mu, \kappa, H_1, \ldots, H_\kappa$ satisfy the first assertion of the theorem, and $k \geqslant \lfloor \frac{5}{2}\xi \rfloor + 3\delta$, then

$$\kappa \geqslant k - \lfloor \tfrac{1}{2}\xi \rfloor \geqslant 2\xi + 3\delta$$

and so the second assertion follows from (5.2).

We prove the first assertion by contradiction. Suppose it is false and choose a counterexample G, Z with $|V(G)| + |E(G)|$ minimum. Clearly (using hypothesis (iii) for vertices in Z), G has no loops or multiple edges, or isolated vertices. We shall show that

(1) *There is no separation* (A, B) *of* G *of order* $\leqslant \xi$, *such that* $Z \subseteq V(A)$, $A \cap G_i$ *is null for some* i $(1 \leqslant i \leqslant k)$, *and* $B \neq G$.

Subproof. Suppose that (A, B) is such a separation, with $A \cap G_k$ null, say. By hypothesis (iii), (A, B) has order ξ. Let $Z' = V(A \cap B)$, and for $1 \leqslant i \leqslant k$ let $G_i' = G_i \cap B$. Then $G_1', ..., G_k'$ are mutually vertex-disjoint subgraphs of B. Moreover,

(i)′ *For* $1 \leqslant i \leqslant k$, *either* G_i' *is connected or every component of* G_i' *has a vertex in* Z'. For if G_i' has more than one component and one of them does not meet Z', then it is also a component of G_i and does not meet Z and does not equal G_i, contrary to hypothesis.

(ii)′ *For* $1 \leqslant i < j \leqslant k$, *either* $V(G_i')$ *and* $V(G_j')$ *both meet* Z' *or there is an edge of* B *between* $V(G_i')$ *and* $V(G_j')$. For if $V(G_i') \cap Z' = \emptyset$ then $G_i' = G_i$ and $V(G_i) \cap Z = \emptyset$. Thus there is an edge of G between $V(G_i)$ and $V(G_j)$. Since $V(G_i') \cap Z' = \emptyset$, this edge is an edge of B, and hence its other end is in $V(G_j')$, as required.

(iii)′ *For* $1 \leqslant i \leqslant k$,, *there is no separation* (A', B') *of* B *of order* $< \xi$ *such that* $Z' \subseteq V(A')$ *and* $G_i' \cap A'$ *is null.* For if (A', B') is such a separation, then $(A \cup A', B')$ is a separation of G of order $< \xi$, with $Z \subseteq V(A \cup A')$ and with $G_i' \cap (A \cup A')$ null, a contradiction since $G_i' = G_i$.

From (i)′, (ii)′, (iii)′ and the minimality of G, we deduce, since B is smaller than G, that for some μ with $0 \leqslant \mu \leqslant \xi$ there are $\kappa = k - \lfloor \frac{1}{2}(\xi - \mu) \rfloor$ non-null connected subgraphs $H_1', ..., H_\kappa'$ of B, mutually vertex-disjoint, such that

(a) $|V(H_i') \cap Z'| = 1$ $(1 \leqslant i \leqslant \xi)$, $V(H_i') \cap Z' = \emptyset$ $(\xi + 1 \leqslant i \leqslant \kappa)$, and

(b) for $1 \leqslant i < j \leqslant \kappa$, if there is no edge of B between $V(H_i')$ and $V(H_j')$ then $i, j \leqslant \mu$.

By Menger's theorem and hypothesis (iii) there are ξ paths $P_1, ..., P_\xi$ of A from Z to Z', mutually vertex-disjoint. Let us choose the numbering so that for $1 \leqslant i \leqslant \xi$ the end of P_i in Z' is in $V(H_i')$. Let

$$H_i = P_i \cup H_i' \qquad (1 \leqslant i \leqslant \xi)$$
$$H_i = H_i' \qquad (\xi + 1 \leqslant i \leqslant \kappa).$$

Then it is easy to see that $H_1, ..., H_\kappa$ satisfy the theorem, a contradiction. This proves (1).

(2) *Every edge of* G *is between* $V(G_i)$ *and* $V(G_j)$ *for some choice of* i, j *with* $1 \leqslant i < j \leqslant k$.

Subproof. If $e \in E(G)$ is not between any two distinct $V(G_i)$'s, contracting e produces a smaller counterexample; for by (1) e does not join two members of Z, and by (1) again, condition (iii) holds after contraction of e. This proves (2).

Let $X = V(G) - Z$, and $Y = \bigcup (V(G_i) : 1 \leqslant i \leqslant k,\ |V(G_i)| \geqslant 2)$.

(3) *$Y \subseteq Z$; every vertex $x \in X$ is adjacent to every member of $(X - \{x\}) \cup (Z - Y)$; and $|X| - |Y| \geqslant k - \lfloor \frac{1}{2} |Y| \rfloor - \xi \geqslant 0$.*

Subproof. From (2) and hypothesis (i), it follows that for $1 \leqslant i \leqslant k$, either $|V(G_i)| = 1$ or $V(G_i) \subseteq Z$; and so $Y \subseteq Z$. Since no vertex is isolated, it follows from (2) that

$$V(G) = \bigcup (V(G_i) : 1 \leqslant i \leqslant k).$$

Hence the second claim holds and, moreover,

$$X \cup (Z - Y) = \bigcup (V(G_i) : 1 \leqslant i \leqslant k,\ |V(G_i)| = 1).$$

Consequently,

$$|X| + |Z - Y| = |\{1 \leqslant i \leqslant k,\ |V(G_i)| = 1\}|;$$

but

$$\lfloor \tfrac{1}{2} |Y| \rfloor \leqslant |\{i : 1 \leqslant i \leqslant k,\ |V(G_i)| \geqslant 2\}|$$

and, adding, we deduce the first inequality. The second holds since $k \geqslant \lfloor \frac{3}{2} \xi \rfloor$ and $|Y| \leqslant \xi$. This proves (3).

(4) *There is a matching $M \subseteq E(G)$ with $|M| = |Y|$, such that every member of M has one end in Y and the other in X.*

Subproof. Let $Y' \subseteq Y$ and let $X' \subseteq X$ be the set of vertices of X with a neighbour in Y'. Suppose that $|X'| < |Y'|$. Then $X' \neq X$, since $|X| \geqslant |Y| \geqslant |Y'|$ by (3); choose $x \in X - X'$. Choose i with $1 \leqslant i \leqslant k$ and $V(G_i) = \{x\}$. Let (A, B) be a separation of G with $V(A) = Z \cup X'$, $V(B) = V(G) - Y'$. Then $A \cap G_i$ is null, and yet (A, B) has order $|Z - Y'| + |X'| < \xi$, contrary to (iii). Thus $|X'| \geqslant |Y'|$, and (4) follows from Hall's theorem.

Let $\mu = |Z - Y|$, let $\kappa = k - \lfloor \frac{1}{2} |Y| \rfloor$, and let M be as in (4). For $1 \leqslant i \leqslant \kappa$ we define H_i as follows, mutually vertex-disjoint:

(i) for $1 \leqslant i \leqslant \mu$, H_i consists of a vertex of $Z - Y$

(ii) for $\mu + 1 \leqslant i \leqslant \xi$, H_i consists of an edge of M and its ends

(iii) for $\xi + 1 \leqslant i \leqslant \kappa$, H_i consists of a vertex of X incident with no member of M.

These choices are possible by (3). Since every vertex of X and every (different) vertex of $X \cup (Z - Y)$ are adjacent, $H_1, \ldots, H_\kappa$ satisfy the theorem, a contradiction, as required. ∎

As a consequence of (5.3) we have the following, which we shall need in later papers.

(5.4) *Let G be a graph and let $Z \subseteq V(G)$ with $|Z| = \xi$. Let $k \geqslant \lfloor \frac{3}{2}\xi \rfloor$ and let $G_1, \ldots, G_k$ be connected subgraphs of G, mutually vertex-disjoint, such that for $1 \leqslant i < j \leqslant k$ there is an edge of G between G_i and G_j. Suppose that there is no separation (A, B) of G of order $< |Z|$ with $Z \subseteq V(A)$ and $A \cap G_i$ null for some i $(1 \leqslant i \leqslant k)$. Then for every partition $(Z_1, \ldots, Z_n)$ of Z into non-empty subsets, there are n connected subgraphs $T_1, \ldots, T_n$ of G, mutually disjoint and with $V(T_t) \cap Z = Z_t$ $(1 \leqslant t \leqslant n)$.*

Proof. By (5.3) we may choose μ with $0 \leqslant \mu \leqslant \xi$ and $\kappa = k - \lfloor \frac{1}{2}(\xi - \mu) \rfloor$, and $H_1, \ldots, H_\kappa$ as in (5.3). Let $Z = \{z_1, \ldots, z_\xi\}$, where $z_i \in V(H_i)$ $(1 \leqslant i \leqslant \xi)$. For $1 \leqslant t \leqslant n$, let S_t be the subgraph of G induced on the set

$$\bigcup \{V(H_i) : 1 \leqslant i \leqslant \xi, z_i \in Z_t\}.$$

Then $S_1, \ldots, S_n$ are mutually vertex-disjoint subgraphs of G, and $S_t \cap Z = Z_t$ for each t, but they may not be connected. If S_t is connected, define $T_t = S_t$. Let $N \subseteq \{1, \ldots, n\}$ be

$$N = \{t : 1 \leqslant t \leqslant n, S_t \text{ is not connected}\}.$$

(1) *If $t \in N$ then $|Z_t| \geqslant 2$ and $Z_t \subseteq \{z_1, \ldots, z_\mu\}$.*

For if $|Z_t| = 1$ then S_t is connected since each H_i is connected. If $z_i \in Z_t$ for some i with $\mu + 1 \leqslant i \leqslant \xi$, then since there is an edge between H_i and every other H_j again it follows that S_t is connected.

From (1) it follows that $|N| \leqslant \lfloor \frac{1}{2}\mu \rfloor$. Since $k \geqslant \lfloor \frac{3}{2}\xi \rfloor$ it follows that

$$\kappa - \xi = k - \lfloor \tfrac{1}{2}(\xi - \mu) \rfloor - \xi \geqslant \lfloor \tfrac{3}{2}\xi \rfloor - \lfloor \tfrac{1}{2}(\xi - \mu) \rfloor - \xi \geqslant \lfloor \tfrac{1}{2}\mu \rfloor \geqslant |N|.$$

Let f be an injection from N into $\{1, \ldots, \kappa\}$ such that $V(H_{f(t)}) \cap Z = \varnothing$ for all $t \in N$. (Such an injection f exists since there are $\kappa - \xi \geqslant |N|$ values of $i \in \{1, \ldots, \kappa\}$ with $V(H_i) \cap Z = \varnothing$). For each $t \in N$ let T_t be the subgraph of G induced on $V(S_t) \cup V(H_{f(t)})$. Then T_t is connected, and $V(T_t) \cap Z = V(S_t) \cap Z = Z_t$ for each $t \in N$; and so $T_1, \ldots, T_n$ satisfy the theorem. ∎

(5.4) implies a theorem of Larman and Mani [10]. The latter asserts that if $s_1, t_1, \ldots, s_n, t_n$ are distinct vertices of a $2n$-connected graph G which has a subgraph that is a subdivision of K_{3n}, then there are n mutually disjoint paths joining s_i, t_i $(1 \leqslant i \leqslant n)$. It can be derived from (5.4) by setting

$\xi = 2n$, $k = 3n$, $Z = \{s_1, ..., s_n, t_1, ..., t_n\}$, $Z_1 = \{s_1, t_1\}, ..., Z_n = \{s_n, t_n\}$ and by choosing the G_i's suitably from the subdivided K_{3n}.

Let us digress for a moment to get a corollary of (5.4) of interest in "pure" graph theory. The following is due to Thomason [28].

(5.5) *Let $p \geqslant 1$ be an integer, and let G be a non-null simple graph with*

$$|E(G)| \geqslant 2.68\, p(\log_2(p))^{1/2}\, |V(G)|.$$

Then G has a K_p-minor.

A graph G is said to be *p-linked*, where $p \geqslant 0$ is an integer, if for all $s_1, ..., s_p,\ t_1, ..., t_p \in V(G)$ with $s_i, t_i \neq s_j, t_j$ for $1 \leqslant i < j \leqslant p$, there are p mutually vertex-disjoint paths of G linking s_i and t_i for $1 \leqslant i \leqslant p$. It is known that for any p, every sufficiently connected graph is p-linked, but it is an open problem to determine the minimum $f(p)$ such that every $f(p)$-connected graph is p-linked. It seems that, up to now, the only known bounds on $f(p)$ were exponential in p. But from (5.4) and (5.5) we obtain that $f(p) \leqslant 16.08p\,(\log_2(3p))^{1/2}$. More precisely,

(5.6) *For every integer $p \geqslant 0$, every $2p$-connected simple graph G with $|E(G)| \geqslant 8.04p(\log_2 3p)^{1/2}\, |V(G)|$ is p-linked.*

Proof. By (5.5) (since we may assume that $p \geqslant 1$ and $V(G) \neq \emptyset$) G has a K_{3p}-minor. Hence there are $3p$ connected subgraphs $G_1, ..., G_{3p}$ of G, mutually vertex-disjoint, such that for $1 \leqslant i < j \leqslant 3p$ there is an edge of G between G_i and G_j. Let $Z \subseteq V(G)$ with $|Z| \leqslant 2p$. There is no separation (A, B) of G of order $< |Z|$ with $Z \subseteq V(A)$ and $A \cap G_i$ null for some i, since G is $2p$-connected. From (5.4), setting $Z = \{s_1, ..., s_p, t_1, ..., t_p\}$, it follows that G is p-linked, as required. ∎

6. What To Do with a Clique

In this section we use (5.3) to show that, in solving Folio, if we find a large clique minor we can find an irrelevant vertex. We need the following.

(6.1) *Let G be a digraph and let $Z \subseteq V(G)$ with $|Z| \leqslant \xi$. Let $k = \lfloor \frac{5}{2}\xi \rfloor + 3\delta + 1$ and let $G_1, ..., G_k$ be mutually vertex-disjoint connected subgraphs of G such that for $1 \leqslant i < j \leqslant k$ there is an edge of G between G_i and G_j. Let (A, B) be a separation of G such that*

(i) *$A \cap G_i$ is null for some i $(1 \leqslant i \leqslant k)$*

(ii) *$Z \subseteq V(A)$*

(iii) *subject to* (i) *and* (ii), (A, B) *has minimum order*

(iv) *subject to* (i), (ii), *and* (iii), A *is maximal.*

Let $v \in V(B) - V(A)$. *Then* v *is irrelevant to the* δ*-folio of* G *relative to* Z.

Proof. By (2.3), it suffices to show that v is irrelevant to the δ-folio of B relative to $V(A \cap B)$. To prove this, it suffices to show that the δ-folio of $B \backslash v$ relative to $V(A \cap B)$ is generic. Let $V(A \cap B) = Z'$, and $|Z'| = \xi' \leqslant \xi$. Let $G_1, ..., G_k$ be numbered so that $v \notin V(G_1) \cup \cdots \cup V(G_{k-1})$, and let $G_i' = G_i \cap B$ $(1 \leqslant i \leqslant k-1)$. Let $G' = B \backslash v$. We shall apply (5.3) to $G', Z', G_1', ..., G_{k-1}'$. Let us verify its hypotheses. We observe that $k - 1 \geqslant \lfloor \frac{5}{2}\xi' \rfloor + 3\delta$, and $G_1', ..., G_{k-1}'$ are subgraphs of G', mutually vertex-disjoint. Moreover, for $1 \leqslant i \leqslant k-1$, either G_i' is connected or every component of G_i' has a vertex in Z'; for if some component of G_i' has no vertex in Z', then it is a component of G_i, and so equals G_i since G_i is connected; and so G_i' is connected. Furthermore, for $1 \leqslant i < j \leqslant k-1$, either $V(G_i')$ and $V(G_j')$ both meet Z or there is an edge of G' between $V(G_i')$ and $V(G_j')$; for if say $V(G_i') \cap Z = \varnothing$ then $G_i' = G_i$, and since there is an edge G between $V(G_i)$ and $V(G_j)$, this edge is therefore an edge of G' between $V(G_i')$ and $V(G_j')$. Finally, we claim

(1) *For* $1 \leqslant i \leqslant k-1$, *there is no separation* (A', B') *of* G' *of order* $< \xi'$ *with* $Z' \subseteq V(A')$ *and with* $A' \cap G_i'$ *null.*

Subproof. If (A', B') is such a separation, there is a separation (A'', B'') of B of order $\leqslant \xi'$ with $v \in V(A'' \cap B'')$ and $A'' \backslash v = A'$, $B'' \backslash v = B'$. Then $(A \cup A'', B'')$ is a separation of G of order $\leqslant \xi'$. Now $Z \subseteq V(A \cup A'')$, and $(A \cup A'') \cap G_i$ is null, and $A \cup A'' \neq A$ since $v \in V(A'')$. This contradicts hypothesis (iv), and proves (1).

From (1) and (5.3) we deduce that the δ-folio of G' relative to Z' is generic, as required. ∎

We use (6.1) for the following algorithm. Let $\xi, \delta \geqslant 0$ be integers, and let $p = \lfloor \frac{5}{2}\xi \rfloor + 3\delta + 1$.

(6.2) ALGORITHM.

Input: A digraph G, a subset $Z \subseteq V(G)$ with $|Z| \leqslant \xi$, and p mutually vertex-disjoint connected subgraphs $G_1, ..., G_p$ of G such that for $1 \leqslant i < j \leqslant p$ there is an edge of G between G_i and G_j.

Output: A vertex of G irrelevant to the δ-folio of G relative to Z.

Running time: $O(\varepsilon)$.

Description: For each i with $1 \leqslant i \leqslant p$ such that $Z \cap V(G_i) = \varnothing$ we find a separation (A_i, B_i) of G with $Z \subseteq V(A_i)$ and $A_i \cap G_i$ null, of minimum order and subject to that with A_i maximal. (The usual max-flow min-cut algorithm provides a "min-cut" which yields a separation (A, B) with A

automatically maximal, if we run the augmenting paths starting from $V(G_i)$. Thus this is easy.) We choose ξ' minimum such that some (A_i, B_i) has order ξ', and we choose i with A_i maximal such that (A_i, B_i) has order ξ'. We choose $v \in V(G_i)$. Then v is the desired irrelevant vertex, by (6.1), since (A_i, B_i) satisfies the conditions for (A, B) in (6.1).

The algorithm involves computing (A_i, B_i) for $\leqslant p$ values of i; and each computation can be performed with a standard network flow algorithm in time $O(\varepsilon)$. Thus the algorithm has running time $O(\varepsilon)$. ∎

(6.2) will form one of the key steps of our algorithm for Folio, but there is also a peripheral application for it, to reduce our general problem to the case when $|E(G)| \leqslant O(v)$ and, thus, to improve the overall running time of the main algorithm. In the remainder of this section we explain this.

If G is a graph and $X \subseteq V(G)$, we denote by $N(X)$ the set of vertices in $V(G) - X$ with a neighbour in X. First, we need the following.

(6.3) ALGORITHM.
Input: An integer $k > 0$ and a non-null simple graph G with $|E(G)| \geqslant k\,|V(G)|$.
Output: A subset $X \subseteq V(G)$ such that

(i) the restriction of G to X is non-null and connected

(ii) $N(X)$ is non-null, and

(iii) every vertex of $N(X)$ has $\geqslant k$ neighbours in $N(X)$.

Running time: $O(v\varepsilon)$.
Description: (1) Choose a component H of G with $|E(H)| \geqslant k\,|V(H)|$, and choose $v_1 \in V(H)$.

(2) We perform a recursion. Set $X_1 = \{v_1\}$. At the start of the ith step of the recursion we have a subset $X_i \subseteq V(H)$ with $|X_i| = i$ such that X_i meets at most $|N(X_i)| + k(i-1)$ edges, and such that the restriction of G to X_i is connected.

(3) If there exists $v \in N(X_i)$ with $<k$ neighbours in $N(X_i)$, we set $X_{i+1} = X_i \cup \{v\}$ and return for the next iteration. To check the inequality, let v have d_1 neighbours in $N(X_i)$ and d_2 in $V(H) - (X_i \cup N(X_i))$. Then

$$|N(X_{i+1})| = |N(X_i)| - 1 + d_2$$

and X_{i+1} meets $d_1 + d_2 \leqslant k - 1 + d_2$ edges not met by X_i. It follows that X_{i+1} meets at most

$$|N(X_i)| + k(i-1) + k - 1 + d_2 = |N(X_{i+1})| + ki$$

edges in total, and so X_{i+1} has the properties claimed.

(4) If there is no such vertex v, we set $X = X_i$ and stop. We claim that X has the desired properties for the output. For certainly (i) and (iii) hold; suppose that $N(X)$ is null. Then $X_i = V(H)$ and so $i = |V(H)|$. But X_i meets at most $|N(X_i)| + k(i-1)$ edges, and $N(X_i) = \varnothing$, and so

$$|E(H)| \leqslant k(|V(H)| - 1),$$

contrary to our choice of H. Thus $N(X)$ is non-null, as required.

There are at most $v(G)$ iterations of the recursion, and each takes time $O(\varepsilon)$. Thus the total running time is $O(v\varepsilon)$. ∎

We use (6.3) for the following, which is related to (5.5) and to a theorem of Mader [11]. Let $p \geqslant 1$ be an integer.

(6.4) ALGORITHM.
Input: A non-null simple graph G with $|E(G)| \geqslant 2^{p-3}|V(G)|$.
Output: A model of K_p in G.
Running time: $O(v^2)$.
Description: We may assume that $p \geqslant 3$, for otherwise the problem is trivial.

(1) Choose a subgraph $H \subseteq G$ with $V(H) = V(G)$ and $|E(H)| = 2^{p-3}|V(H)|$.

(2) We perform a recursion. Let $Y_1 = V(H)$. At the start of the ith iteration we have $i-1$ disjoint connected subgraphs $G_1, ..., G_{i-1} \subseteq H$ and a non-null subset $Y_i \subseteq V(H) - V(G_1 \cup \cdots \cup G_{i-1})$ such that

(i) there are at least $2^{p-2-i}|Y_i|$ edges of H with both ends in Y_i,

(ii) every vertex in Y_i has a neighbour in each of $V(G_1), ..., V(G_{i-1})$,

and

(iii) for $1 \leqslant j < j' \leqslant i-1$, each vertex of $V(G_{j'})$ has a neighbour in $V(G_j)$.

The ith iteration proceeds as follows.

(3) We apply (6.3) to the restriction of H to Y_i, taking $k = 2^{p-2-i}$. We obtain $X \subseteq Y_i$ such that the restriction of H to X is non-null and connected, $N(X) \cap Y_i$ is non-null, and every vertex of $N(X) \cap Y_i$ has at least 2^{p-2-i} neighbours in $N(X) \cap Y_i$. We set G_i to be the restriction of H to X, and $Y_{i+1} = N(X) \cap Y_i$. If $i = p-2$ the recursion terminates, and otherwise we return for the next iteration.

(4) Let G_{p-1}, G_p be one-vertex graphs with $V(G_{p-1}) = \{u\}$, $V(G_p) = \{v\}$, for some adjacent $u, v \in Y_{p-1}$. (These exist since $Y_{p-1} \neq \varnothing$ and every

$u \in Y_{p-1}$ has a neighbour in Y_{p-1}.) Then $G_1, \ldots, G_p$ are disjoint, non-null, connected subgraphs of G, and for $1 \leqslant i < j \leqslant p$ every vertex of G_j has a neighbour in G_i. From this we obtain a model of K_p.

There are $p-2$ iterations of the recursion, and each takes time $O(v(H)\,\varepsilon(H))$. Since $|E(H)| = 2^{p-3}\,|V(G)|$, the running time is $O(v(G)^2)$. ∎

Next, let $\xi, \delta \geqslant 0$ be integers.

(6.5) Algorithm.

Input: A digraph G and a subset $Z \subseteq V(G)$ with $|Z| \leqslant \xi$.

Output: A subdigraph G' of G with $Z \subseteq V(G')$, such that G and G' have the same δ-folio relative to Z, and such that $|E(G')| \leqslant C\,|V(G')|$, where $C = \delta + 2(\delta+1)2^{p-3}$ and where $p = \lfloor \frac{5}{2}\xi \rfloor + 3\delta + 1$.

Running time: $O(v^3)$.

Description:

(1) Let H be a maximal subdigraph of G with at most δ loops at each vertex and at most $\delta+1$ edges with head u and tail v, for all distinct $u, v \in V(G)$. Certainly G and H have the same δ-folios relative to Z.

(2) We set $G_1 = H$ and perform a recursion. At the start of the ith iteration we have a subdigraph G_i of H with $Z \subseteq V(G_i)$ and $|V(G_i)| = |V(H)| - i + 1$ such that G_i and H (and hence G) have the same δ-folios relative to Z. If $|E(G_i)| \leqslant C\,|V(G_i)|$ we set $G' = G_i$ and stop. Otherwise, the ith iteration proceeds as follows.

(3) Let J_i be the simple (undirected) graph underlying G_i. Then

$$\begin{aligned}(\delta + 2(\delta+1)2^{p-3})\,|V(J_i)| = C\,|V(G_i)| &\leqslant |E(G_i)| \\ &\leqslant \delta\,|V(J_i)| + 2(\delta+1)\,|E(J_i)|\end{aligned}$$

and so $2^{p-3}\,|V(J_i)| \leqslant |E(J_i)|$. We apply (6.4) to J_i and obtain a model of K_p in J_i (and hence in G_i).

(4) We apply (6.2) to G_i and the model of K_p and find a vertex v of G_i irrelevant to the δ-folio of G_i relative to Z. We set $G_{i+1} = G_i \backslash v$ and return for the next iteration.

There are at most $v(G)$ iterations, and each iteration involves an application of (6.4) to J_i, which takes time $O(v^2)$, and an application of (6.2) to G_i, which takes time $O(\varepsilon(G_i)) \leqslant O(v^2)$. Thus the total running time for constant δ, ξ is $O(v^3)$.

(6.3)–(6.5) can all be implemented more quickly than stated, but for us it makes no difference.

7. Walls

Now we come to the third part of the paper, which concerns what to do to solve Folio when the branch-width of our graph is too high to apply the methods of Section 4 and yet we have not found a large clique minor. In such a graph there is a large grid minor, by (4.2), and, roughly, our method is to find a section of such a minor that does not stand out among the other sections of the grid close to it in any significant way (with regard to how the rest of G is attached to it.) From a theorem which we postpone to a later paper, any vertex in the middle of such an "ordinary" section is irrelevant. Here we give an algorithm which, given a grid minor, either finds a large clique minor, or finds an "ordinary" section of the grid, or moves to a better grid minor, when we start again.

Although grids are natural objects to use from some viewpoints, grid minors introduce a complication (due to the fact that their vertices are mostly 4-valent) which is unnecessary and can be avoided by using *elementary walls* instead (which are parts of the hexagonal lattice instead of the square lattice). Figure 1 shows an elementary wall of height 8. It has four distinguished vertices, called its *corners*, shaded in the figure. The *elementary wall of height* h (where $h \geqslant 2$ is even) is similar; there are h rows of "bricks," and h "bricks" in each row, and four "corners." Thus there are $h+1$ "horizontal" paths and its "vertical" edges divide into $2h+2$ "collinear" sets of $\frac{1}{2}h$ edges each, and the corners are the ends of the first and the last horizontal paths. The *perimeter* is the circuit which bounds the infinite region when drawn as in Fig. 1.

A *wall of height* h (where $h \geqslant 2$ is even) is a subdivision of an elementary wall of height h. It too therefore has four distinguished vertices, called its *corners*. A wall is thus a pair (H, P), where H is a graph and $P \subseteq V(H)$ is

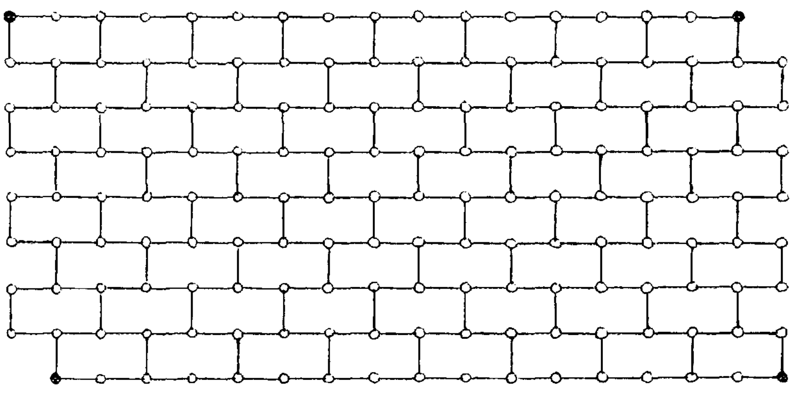

FIG. 1. Elementary 8-wall.

the set of corners, but for convenience we speak of a wall as a graph and leave the corners implicit. The *perimeter* of a wall is defined as before.

Now since every vertex of an elementary wall H of height h has valency $\leqslant 3$, if a graph has a minor isomorphic to H then it has a wall of height h as a subgraph (and algorithmically, it is easy to see how to construct the wall in time $O(\varepsilon)$, given a model of H).

(7.1) *For $h \geqslant 2$, even, if every graph with branch-width $\geqslant \omega/3$ has a minor isomorphic to the elementary wall of height h, then there is an algorithm as in* (7.2), *with running time $O(v^2 + \varepsilon)$.*

(7.2) ALGORITHM.
Input: A graph G.
Output: Either a branch-decomposition of G of width $\leqslant \omega$, or a wall of height h which is a subgraph of G.
Description: Given input G, we apply (4.4) to G to determine if there is a model of the elementary wall of height H in G and to find one if one exists. If we find one, we construct from it a wall of height h and stop. If there is none, then G has branch-width $< \omega/3$. We apply (3.3) to G and obtain a subgraph G' of G and a branch-decomposition of G' of width $\leqslant \omega$, such that either $G' = G$ or G' has branch-width $\geqslant \omega/3$. Thus $G' = G$, and we have a branch-decomposition of G of width $\leqslant \omega$. We stop.

This algorithm involves one application of (4.4) and one of (3.3); thus, it has running time $O(v^2 + \varepsilon)$. ∎

If H is a wall, drawn in the plane as in Fig. 1, its *distance function* d is defined as follows. For s, t in the plane, $d(s, t) = 0$ if $s = t$, and otherwise

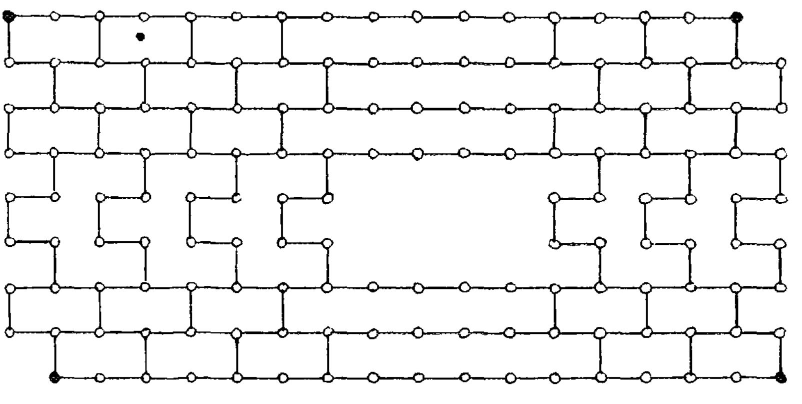

FIG. 2. A hole in a wall.

$d(s, t)$ is the minimum of the number of points of F in the drawing, taken over all subsets F in the plane homeomorphic to $[0, 1]$ with ends s and t. We shall speak of the distance function of H without reference to any drawing, but we assume implicitly that there is given some fixed drawing of H. If $u, v \in V(H)$, we define $d(u, v) = d(s, t)$, where s, t are the points of the drawing representing u, v. If H and H' are both walls and H' is a subgraph of H, let the corresponding distance functions (using the drawing of H' obtained from the given drawing of H) be d, d'. If $d'(s, t) \geqslant d(s, t) - k$ for all points s, t of the plane, we write $d' \geqslant d - k$.

Given a wall H of height h, we can obtain another wall of height $h-2$ as indicated in Fig. 2, omitting half the horizontal edges of two consecutive rows and all the vertical edges in four consecutive columns and the four single-edge components that result. By generalizing this construction, we deduce

(7.3) *Let H be a wall of height h with distance function d and let $v \in V(H)$. Let $r \geqslant 0$ be an integer with $h \geqslant 2r + 4$. Then there is a subgraph H' of H, which is a wall of height $h - 2r - 2$, such that*

(i) *$d' \geqslant d - 4r - 4$, where d' is the distance function in H'*

(ii) *there is no vertex u of H' with $d(u, v) \leqslant r$.*

The proof is left to the reader. (The most difficult case is when v is close to the infinite region of H, when it is necessary to delete a band of H around the outside.)

Let G be a graph. A *wall in G* is a subgraph of G which is a wall. Let H be a subgraph of G. An *H-path* in G is a path with distinct ends, both in $V(H)$, and with no edge or internal vertex in H. We shall need the following lemma.

(7.4) *For any $p \geqslant 2$ there exists $r \geqslant 0$ such that the following holds. Let H be a wall in a graph G. Let $P_1, ..., P_{\binom{p}{2}}$ be mutually vertex-disjoint H-paths in G, with their $p(p-1)$ ends mutually at distance $\geqslant r$. Then G has a minor isomorphic to K_p. Moreover, there is an algorithm, with running time $\leqslant C\varepsilon(G)$, where C depends only on p, which, given G, H, and $P_1, ..., P_{\binom{p}{2}}$ as above, finds a model of K_p in G.*

This follows from the main theorem of [16], or more directly from Theorem (4.4) of [20], taking $\mathscr{T}$ to be the set of all separations (A, B) of H of order $< r$ such that A contains no horizontal path of H, taking Σ to be a sphere, and taking $\Gamma = H$.

Let H be a wall in G. For $l \geqslant 3$, $m \geqslant 0$ an *(l, m)-star over H in G* is a subgraph $P_1 \cup \cdots \cup P_l$ of G such that for some vertex v,

(i) $P_1, ..., P_l$ are paths of G and all have one end v, and are otherwise mutually vertex-disjoint

(ii) for $1 \leqslant i \leqslant l$, let the ends of P_i be v, u_i; then $u_i \neq v$, and $u_i \in V(H)$, and no edge or internal vertex of P_i is in H

(iii) for $1 \leqslant i < j \leqslant l$, u_i and u_j are at distance $\geqslant m$ in H.

We say that v is the *center* of the star. (It is unique.) The star is *external* if $v \notin V(H)$.

(7.5) *For any $p \geqslant 2$ there are integers $l \geqslant 3$, $m \geqslant 0$ with the following property. Let G be a graph and let H be a wall in G. Suppose that there are $\binom{p}{2}$ external (l, m)-stars over H with distinct centers (but not necessarily disjoint). Then G has a K_p-minor.*

Proof. We shall prove the following for $k, m \geqslant 0$ by induction on k.

Let $l = 2k^2 - k + 3$. For any k external (l, m)-stars $S_1, ..., S_k$ over H with distinct centers there are k mutually vertex-disjoint H-paths of G, each a subgraph of $S_1 \cup \cdots \cup S_k$, with their $2k$ ends mutually at distance $\geqslant \frac{1}{2}m$.

The claim is true for $k = 0$, and we assume that $k > 0$. For $1 \leqslant i \leqslant k-1$, let S'_i be an $(l-1, m)$-star with the same center as S_i, with $S'_i \subseteq S_i$ and with $v_k \notin V(S'_i)$, where v_k is the center of S_k. Since $l - 1 \geqslant 2(k-1)^2 - (k-1) + 3$, there are (from our inductive hypothesis) $k - 1$ mutually vertex-disjoint H-paths $Q_1, ..., Q_{k-1}$ of G, with

$$Q_1 \cup \cdots \cup Q_{k-1} \subseteq S'_1 \cup \cdots \cup S'_{k-1}$$

and with their $2(k-1)$ ends mutually at distance $\geqslant \frac{1}{2}m$.

Let $S_k = P_1 \cup \cdots \cup P_l$, with the P_i's as in the definition of a star. Suppose that for some j with $1 \leqslant j \leqslant k-1$ there are $\geqslant 2k$ values of i with $1 \leqslant i \leqslant l$ such that $P_i \cap Q_j$ is non-null. For each such i, let u_i be the first vertex of P_i which belongs to $V(Q_j)$ (first as P_i is traversed starting from its end in $V(H)$). Choose distinct $i_1, i_2, ..., i_{2k}$ such that $P_{i_t} \cap Q_j$ is non-null for each t, numbered so that $u_{i_1}, ..., u_{i_{2k}}$ occur in this order on Q_j. Then we may construct k H-paths $R_1, ..., R_k$ with their ends mutually at distance $\geqslant m \geqslant \frac{1}{2}m$, where each R_t is formed from subpaths of $P_{i_{2t-1}}$ and $P_{i_{2t}}$ and the path of Q_j between $u_{i_{2t-1}}, u_{i_{2t}}$. We may assume therefore that there is no such j.

Hence the total number of pairs i, j with $P_i \cap Q_j$ non-null is at most $(k-1)(2k-1)$; and so there are at least

$$l - (k-1)(2k-1) \geqslant 2k$$

values of i such that $P_i \cap Q_j$ is null for all j with $1 \leqslant j \leqslant k-1$. For $1 \leqslant i \leqslant l$, let u_i, v_k be the ends of P_i, and for $1 \leqslant j \leqslant k-1$ let s_j, t_j be the ends of Q_j. Now for any $x \in \{s_1, t_1, ..., s_{k-1}, t_{k-1}\}$, $d(u_i, x) < \frac{1}{2}m$ for at most one value of i $(1 \leqslant i \leqslant l)$, since $d(u_i, u_{i'}) \geqslant m$ for $i \neq i'$. Thus there are at least two values of i such that

(i) $P_i \cap Q_j$ is null for $1 \leqslant j \leqslant k-1$, and

(ii) $d(u_i, x) \geqslant \frac{1}{2}m$ for all $x \in \{s_1, t_1, ..., s_{k-1}, t_{k-1}\}$.

The union of the corresponding two P_i's yields an H-path Q_k, and $Q_1, ..., Q_k$ are the desired H-paths. This completes the induction.

In particular, the statement holds if $k = \binom{p}{2}$. Let $l = 2\binom{p}{2}^2 - \binom{p}{2} + 3$, choose r as in (7.4) and let $m = 2r$. Then l, m satisfy the theorem, by (7.4). ∎

(7.6) *Let H be a wall in a graph G. Suppose there are k $(l, m+4k)$-stars in G over H with distinct centers $v_1, ..., v_k$, where $m \geqslant 3$. Then there is a wall $H' \subseteq H$ with k external (l, m)-stars over H', with centers $v_1, ..., v_k$.*

Proof. Let $v_1, ..., v_k \in V(G)$ be distinct, and each the center of an $(l, m+4k)$-star over H. Let $H_0 = H$. By k applications of (7.3), with $r = 0$ and setting $v = v_1, ..., v_k$ in turn, we deduce that there are walls $H_1, ..., H_k$ such that for $1 \leqslant i \leqslant k$

(i) H_i is a subgraph of H_{i-1}, and $v_i \notin V(H_i)$

(ii) $d_i \geqslant d_{i-1} - 4$, where $d_0, ..., d_k$ are the corresponding distance functions.

(1) *If $v \in V(G)$ is the center of an $(l, m+4(k-i+1))$-star over H_{i-1}, then v is the center of an $(l, m+4(k-i))$-star over H_i, for $1 \leqslant i \leqslant k$.*

Subproof. Let $P_1 \cup \cdots \cup P_l$ be an $(l, m+4(k-i+1))$-star over H_{i-1}, with center v. For $1 \leqslant j \leqslant l$, let P_j have ends v, u_j; and let Q_j be a minimal path of H_{i-1} between u_j and $V(H_i)$, with ends u_j, w_j say. Now for distinct j, j',

$$d_i(u_j, u_{j'}) \geqslant d_{i-1}(u_j, u_{j'}) - 4 \geqslant m + 4(k-i+1) - 4 = m + 4(k-i) \geqslant 3.$$

Consequently, u_j and $u_{j'}$ do not belong to the closure of the same region of H_i, and hence Q_j and $Q_{j'}$ are vertex-disjoint. It follows that

$$(P_1 \cup Q_1) \cup (P_2 \cup Q_2) \cup \cdots \cup (P_l \cup Q_l)$$

is an $(l, m+4(k-i))$-star over H_i, as required.

From (1), we deduce that $v_1, ..., v_k$ are all centers of (l, m)-stars over H_k; and these stars are all external since $v_1, ..., v_k \notin V(H_k)$. The result follows. ∎

From (7.5) and (7.6) we deduce

(7.7) *For any $p \geqslant 2$ there are integers $l \geqslant 3$, $m \geqslant 0$ with the following property. Let G be a graph and let H be a wall in G, so that there are $\binom{p}{2}$ (l, m)-stars over H, with distinct centers; then G has a K_p-minor. Moreover, there is an algorithm, with running time $\leqslant C\varepsilon(G)$, where C depends only on p, which, given $\binom{p}{2}$ (l, m)-stars with distinct centers over a wall H in a graph G, finds a model of K_p in G.*

The first assertion is clear from (7.5) and (7.6). The second follows since the proofs of (7.5) and of (7.6) may easily be converted to algorithms.

8. The 2-Paths Problem

Let G be a graph and let Ω be a cyclic permutation of a subset (denoted by $\bar{\Omega}$) of $V(G)$. We call (G, Ω) a *society*. Actually, for our application in this paper we could assume that $|\bar{\Omega}| = 4$ throughout, but we have chosen to give the results in general. A *cross* in a society (G, Ω) is a subgraph $P \cup Q$ of G, where P, Q are vertex-disjoint paths of G with ends p_1, p_2 and q_1, q_2, respectively, and $p_1, q_1, p_2, q_2 \in \bar{\Omega}$ and occur in this order or its reverse under Ω. We need the following.

(8.1) Algorithm.
Input: A society (G, Ω).
Output: A cross in (G, Ω) if one exists.
Running time: $O(v\varepsilon)$.
Description: We assume that $|\bar{\Omega}| \geqslant 4$, for otherwise no cross exists. Let H be the cubic graph formed by two vertex-disjoint circuits C_1, C_2, both of length $|\bar{\Omega}|$, and for $1 \leqslant i \leqslant |\bar{\Omega}|$ an edge joining the ith vertices of C_1 and C_2. Identify $V(C_2)$ with $\bar{\Omega}$, in order, and let s_1, s_2, t_1, t_2 be distinct vertices of C_1, in order. We apply an algorithm of Shiloach [27] to find two paths of $G \cup H$, vertex-disjoint, joining s_1, t_1 and s_2, t_2, respectively, if they exist. If we find two such paths it is easy to derive a cross in (G, Ω); and if there are not two such paths it is easy to see that (G, Ω) has no cross. (Since Shiloach's algorithm has running time $O(v\varepsilon)$, so does (8.1).) ▮

Let (G, Ω) be a society. A subset $Z \subseteq V(G)$ is *trisecting* if $\bar{\Omega} \subseteq Z$, and for every $v \in V(G) - Z$ there are $\leqslant 3$ vertices $z \in Z$ such that there is a path of G between v and z with no other vertex in Z. Thus, $V(G)$ is always trisecting.

A *core* C in (G, Ω) is a simple graph with vertex set a trisecting subset $Z \subseteq V(G)$, in which distinct $u, v \in Z$ are adjacent if and only if there is a path in G between u and v with no internal vertex in Z. (Note that $E(C) \nsubseteq E(G)$ in general.) If C is a core, we say $v \in V(C)$ *is 4-linked in* C if there is no separation (A, B) of C of order $\leqslant 3$ with $v \in V(A) - V(B)$ and $\bar{\Omega} \subseteq V(B)$. A core C is *4-linked* if every $v \in V(C)$ is 4-linked in C.

(8.2) *Let C be a core of (G, Ω), and let $v \in V(C)$ not be 4-linked. Let (A, B) be a separation of C with $v \in V(A) - V(B)$ and $\bar{\Omega} \subseteq V(B)$, of minimum order, and let C' be the simple graph obtained from B by adding edges joining all pairs of vertices in $V(A \cap B)$ if they are not already adjacent. Then C' is a core of (G, Ω), and each $v' \in V(C')$ which is 4-linked in C is also 4-linked in C'.*

Proof. (1) *$V(B)$ is trisecting.*

Subproof. Let $u \in V(G) - V(B)$ and suppose that $z_1, ..., z_4 \in V(B)$ are distinct, and for $1 \leqslant i \leqslant 4$ P_i is a path of G between u and z_i with no vertex in $V(B)$ except z_i. Since $|V(A \cap B)| \leqslant 3$, we may assume that $z_4 \notin V(A)$. Since $V(C)$ is trisecting, some vertex of $P_1 \cup \cdots \cup P_4$ belongs to $V(A) - V(B)$. Hence there is a path of $P_1 \cup \cdots \cup P_4$ between z_4 and $V(A) - V(B)$ with no internal vertex in $V(C)$. But then the ends of this path are adjacent in C, contradicting that (A, B) is a separation of C. This proves (1).

(2) *If $a, b \in V(A \cap B)$ are distinct, there is a path of G with ends a, b and with no internal vertex in $V(B)$.*

Subproof. By the minimality of $|V(A \cap B)|$ and Menger's theorem, there are $|V(A \cap B)|$ paths of A between v and $V(A \cap B)$, mutually disjoint except for v, and the union of two of these yields a path P of A with ends a and b and with no internal vertex in $V(B)$. For each $e \in E(P)$, let P_e be a path of G with the same ends as e with no internal vertex in $V(C)$. Then $\bigcup (P_e : e \in E(P))$ is connected, and meets $V(B)$ in precisely $\{a, b\}$, and so it includes a path of G between a and b with no internal vertex in $V(B)$, as required.

(3) *For distinct $a, b \in V(B)$, there is a path of G with ends a, b and with no internal vertex in $V(B)$ if and only if a, b are adjacent in C'.*

Subproof. Since every two vertices in $V(A \cap B)$ are adjacent in C', we may assume by (2) that $b \notin V(A \cap B)$. If a, b are adjacent in C', they are adjacent in C and are therefore joined by a path of G as required. Conversely, suppose that P is a path of G with ends a, b and with no internal vertex in $V(B)$. Since b is not adjacent in C to any vertex in $V(A) - V(B)$, it follows that $V(P) \cap V(A) \subseteq \{a\}$, and so a, b are adjacent in C and hence in C', as required.

From (1) and (3) we see that C' is a core.

(4) *If $v' \in V(C')$ is 4-linked in C then it is 4-linked in C'.*

Subproof. Suppose that (A', B') is a separation of order $\leqslant 3$ of C' with $v' \in V(A') - V(B')$ and $\bar{\Omega} \subseteq V(B')$. Since $V(A \cap B)$ is the vertex set of a complete subgraph of C', either $V(A \cap B) \subseteq V(A')$ or $V(A \cap B) \subseteq V(B')$. In the first case $((A \cup A') \cap C, B' \cap C)$ and in the second case $(A' \cap C, (A \cup B') \cap C)$ is a separation of C of order $\leqslant 3$ showing that v' is not 4-linked in C, as required.

From (4), the result follows. ∎

From (8.2) we obtain

(8.3) ALGORITHM.
Input: A society (G, Ω).
Output: A 4-linked core of (G, Ω).
Running time: $O(v\varepsilon)$.
Description: Initially, set $C = G$; then C is a core, but it need not be 4-linked. For each vertex v of G in turn, if $v \in V(C)$ test if v is 4-linked in C; and if v is not 4-linked in C, choose (A, B) as in (8.2), define C' as in (8.2), and replace C by C'. When this terminates, C is a 4-linked core, by the last assertion of (8.2). (For each $v \in V(G)$ this takes time $O(\varepsilon)$; and so the total running time is $O(v\varepsilon)$.) ∎

Next, we need the following.

(8.4) *If C is a core of (G, Ω) then (G, Ω) has a cross if and only if (C, Ω) has a cross.*

The proof is easy and we omit it (compare [18, Theorem (11.10)]).

We say that (G, Ω) is *rural* if it can be drawn in a disc with the vertices of $\bar{\Omega}$ drawn in the boundary of the disc, in order according to Ω. The following is related to results of [9, 18, 26, 27, 29].

(8.5) *Let C be a 4-linked core of (G, Ω). Then (G, Ω) has no cross if and only if (C, Ω) is rural.*

Proof. By (8.4), (G, Ω) has no cross if and only if (C, Ω) has no cross. But by [18, Theorems (2.3) and (2.4)], (C, Ω) has no cross if and only if (C, Ω) is rural, since C is 4-linked. ∎

Let (G, Ω) be a society. For a subgraph A of G we define ∂A to be the set of all $v \in V(A)$ such that either $v \in \bar{\Omega}$ or v is incident with an edge of G not in A. A *division* of (G, Ω) is a set $\mathscr{A}$ of subgraphs of G such that

(i) $\bigcup (A : A \in \mathscr{A}) = G$

(ii) for all distinct $A, A' \in \mathscr{A}$, $E(A \cap A') = \emptyset$ and $V(A \cap A') = \partial A \cap \partial A'$

(iii) for each $A \in \mathscr{A}$, $|\partial A| \leqslant 3$ and there are $|\partial A|$ mutually vertex-disjoint paths of G between ∂A and $\bar{\Omega}$

(iv) for each $A \in \mathscr{A}$ and all $u, v \in \partial A$, there is a path of A between u and v with no internal vertex in ∂A

(v) if $A, A' \in \mathscr{A}$ are distinct then $\partial A \neq \partial A'$

(vi) for each $A \in \mathscr{A}$, some vertex or edge of A belongs to no other member of $\mathscr{A}$.

If $\mathscr{A}$ is a division, we write $\partial \mathscr{A} = \bigcup (\partial A : A \in \mathscr{A})$. Thus, $\bar{\Omega} \subseteq \partial \mathscr{A}$. We say a division $\mathscr{A}$ of (G, Ω) is *rural* if (G', Ω) is rural, where G' is the bipartite graph with vertex set $\mathscr{A} \cup \partial \mathscr{A}$ in which $A \in \mathscr{A}$ is adjacent to $v \in \partial \mathscr{A}$ if $v \in \partial A$.

(8.6) ALGORITHM.
Input: A society (G, Ω).
Output: Either a cross in (G, Ω), or a rural division of (G, Ω).
Running time : $O(v\varepsilon)$.
Description: First we use (8.1) to find a cross if there is one. If not, we use (8.3) to find a 4-linked core C. Let $\mathscr{A}'$ be the set of all subgraphs A of G such that either

(i) A consists of an edge of G with both ends in $V(C)$, and its ends, or

(ii) A consists of a component of $G \setminus V(C)$, together with all edges of G between this component and $V(C)$, and their ends, or

(iii) A consists of a vertex $v \in \bar{\Omega}$ incident with no edges of G.

We compute $\mathscr{A}'$. Finally, say $A_1, A_2 \in \mathscr{A}'$ are *equivalent* if $\partial A_1 = \partial A_2$, let $\mathscr{A}_1, \ldots, \mathscr{A}_k$ be the equivalence classes into which $\mathscr{A}'$ is partitioned by this relation, and let

$$\mathscr{A} = \left\{ \bigcup (A : A \in \mathscr{A}_i) : 1 \leqslant i \leqslant k \right\}.$$

Output $\mathscr{A}$, and stop.

We must prove that $\mathscr{A}$ is a rural division of (G, Ω). First, we observe:

(1) *For each $A \in \mathscr{A}'$, $|V(A) \cap V(C)| \leqslant 3$ and there are $|V(A) \cap V(C)|$ mutually vertex-disjoint paths of G between $V(A) \cap V(C)$ and $\bar{\Omega}$.*

Subproof. Now $|V(A) \cap V(C)| \leqslant 3$ since $V(C)$ is trisecting. Since C is 4-linked, there are $|V(A) \cap V(C)|$ mutually vertex-disjoint paths of C between $V(A) \cap V(C)$ and $\bar{\Omega}$; and these can be converted easily to paths in G.

We deduce

(2) *For each* $A \in \mathscr{A}'$, $V(A) \cap V(C) = \partial A$.

Subproof. Now $V(A) \cap V(C) \subseteq \partial A$, by (1). For the converse, let $v \in V(A)$. Then $v \in V(A)$, and we claim that $v \in V(C)$. Since $\bar{\Omega} \subseteq V(C)$ we may assume that $v \notin \bar{\Omega}$, and so there is an edge e of G incident with v, with $e \notin E(A)$. By definition of $\mathscr{A}'$, $v \in V(C)$ as required.

From (1) and (2) it follows that $\mathscr{A}'$ satisfies conditions (i), (ii), (iii), (iv) in the definition of a division.

(3) *Let* $X \subseteq V(G)$, *let* $\mathscr{B} = \{A \in \mathscr{A}' : \partial A = X\}$, *and let* $B = \bigcup (A : A \in \mathscr{B})$. *Then* $\partial B = X$, *if* $\mathscr{B} \neq \emptyset$.

Subproof. Certainly $\partial B \subseteq X$. For the converse, since $\mathscr{B} \neq \emptyset$, there exists $A \in \mathscr{B}$, and by (1) there are $|X|$ mutually vertex-disjoint paths of G between X and $\bar{\Omega}$. Each of these paths has no vertex in any $A' \in \mathscr{B}$ except its end in X, because $\partial A' = X$. It follows that each of these paths has no vertex in B except its end in X, and so $\partial B = X$, as required.

Since $\mathscr{A}'$ satisfies (i)–(iv) in the definition of a division, it follows from (3) that $\mathscr{A}$ satisfies (i)–(v) and hence is a division. To see that $\mathscr{A}$ is rural, we observe first that (C, Ω) is rural, by (8.5). For each $A \in \mathscr{A}$, the vertices of $V(A) \cap V(C)$ form a complete subgraph of C, and since C is 4-linked, this is either empty, a single vertex, an edge of C, or a triangle of C bounding a region of the drawing of C. Since no two members of $\mathscr{A}$ correspond to the same complete subgraph of C (and in particular, no two correspond to the same triangular region) it follows that $\mathscr{A}$ is rural, as required.

Thus the algorithm gives the desired output. Moreover, its running time is $O(v\varepsilon)$, since (8.1) and (8.3) both take time $O(v\varepsilon)$ and computing $\mathscr{A}'$ and $\mathscr{A}$ both take time $O(\varepsilon)$. ∎

9. Finding a Flat Section of a Wall

We begin by defining a *subwall* of a wall. First, let J be an elementary wall of height h. Draw J in the plane as in Fig. 1, and let us name the vertices of J by x, y-coordinates, taking the origin in the bottom left of the figure. Thus, J has vertex set

$$\{(x, y) : 0 \leqslant x \leqslant 2h+1, 0 \leqslant y \leqslant h\} - \{(0, 0), (2h+1, h)\}$$

and corners $(1, 0)$, $(2h+1, 0)$, $(0, h)$, $(2h, h)$. Let h' be even with $2 \leqslant h' \leqslant h$, choose i, j, even, with $0 \leqslant i \leqslant 2h - 2h'$ and $0 \leqslant j \leqslant h - h'$, and let J' be the subgraph of J induced on the vertex set

$$\{(x, y) : i \leqslant x \leqslant i + 2h' + 1, j \leqslant y \leqslant j + h'\} - \{(i, j), (i + 2h' + 1, j + h')\}$$

with corners $(i+1, j)$, $(i+2h'+1, j)$, $(i, j+h')$, $(i+2h, j+h')$. Then J' is an elementary wall of height h'. Now let H be a subdivision of J, and let H' be the corresponding subdivision of J'; we say that the wall H' is a *subwall* of H. It follows that if H' is a subwall of H, then H' consists of its perimeter C together with the portion of H inside C. (Thus the wall H' of (7.3) is in general *not* a subwall of H.) If H' is a subwall of H, we shorten "inside the perimeter of H'" to "inside H'", etc. If H is a wall, and H_1, H_2 are non-null subgraphs of H, we define $d(H_1, H_2)$ to be the minimum of $d(v_1, v_2)$ over all $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$, where d is the distance function of H.

Let H be a wall in a graph G, and let p_1, p_2, p_3, p_4 be the corners of H, in cyclic order. A *cross over H in G* is a subgraph $P \cup Q$, where P, Q are mutually vertex-disjoint paths of G with ends p_1, p_3 and p_2, p_4, respectively.

(9.1) *For any integer $p \geqslant 0$ there exist $k, r \geqslant 0$ such that the following holds. Let H be a wall in G, and let $H_1, ..., H_k$ be subwalls of H with $d(H_i, H_j) \geqslant r$ $(1 \leqslant i < j \leqslant k)$, where d is the distance function of H. For $1 \leqslant i \leqslant k$, let $P_i \cup Q_i$ be a cross over H_i, such that*

(i) $(P_i \cup Q_i) \cap H \subseteq H_i$, *for* $1 \leqslant i \leqslant k$

(ii) $(P_i \cup Q_i) \cap (P_j \cup Q_j)$ *is null for* $1 \leqslant i < j \leqslant k$.

Then G has a K_p-minor. Moreover, a model of K_p in G can be found in time $\leqslant C\varepsilon(G)$, where C depends only on p and the height of H.

Proof. Let Σ be a sphere, and for p as given, let θ, t be as in [19, Theorem (4.5)]. We may assume that $t \geqslant 2$. Let $k = t+1$ and let $r = 4\theta + 4$. Now let $H, G, H_1, ..., H_k$ and P_i, Q_i $(1 \leqslant i \leqslant k)$ be as in the theorem, and let H be drawn in Σ. All but one of the regions of H are "bricks;" let s be a point of Σ, from the region that is not a brick. For $1 \leqslant i \leqslant k$, let d_i be the minimum of $d(s, x)$ taken over all $x \in V(H_i)$. We may assume that $d_1, ..., d_t \geqslant d_k$. Since for $1 \leqslant i \leqslant t$ we have $r \leqslant d(H_i, H_k) \leqslant d_i + d_k$, it follows that $d_1, ..., d_t \geqslant \frac{1}{2}r$.

Let H have n horizontal paths, and for each of them (say the jth), let P_j be a subpath of length $\frac{1}{2}r - 2 = 2\theta$ with one end in the perimeter of H. Since $d_i \geqslant \frac{1}{2}r = 2\theta + 2$ for each i, it follows that P_j is disjoint from H_i, for all i $(1 \leqslant i \leqslant t)$ and j $(1 \leqslant j \leqslant n)$. Let Γ be obtained from H by deleting all of H_i except its perimeter, for $1 \leqslant i \leqslant t$. Then for $1 \leqslant j \leqslant j' \leqslant n$ there are θ paths of Γ between $V(P_j)$ and $V(P_{j'})$, mutually vertex-disjoint. Let $\mathscr{T}$ be the set of all separations (A, B) of Γ of order $< \theta$ such that none of $P_1, ..., P_n$ is a subgraph of A. The result follows easily from [19, Theorem (4.5)] applied to $\Sigma, \Gamma, \mathscr{T}$. ∎

Let H' be a subwall of a wall H in a graph G. We say H' is *dividing* if there is no H-path in G with one end inside H' and the other end outside, and *non-dividing* otherwise. The following result is closely related to Theorem (6.3) of [21] and its proof is along similar lines (although much easier).

(9.2) *For any $l \geqslant 3$, and $m, p \geqslant 0$, there exist $k, r \geqslant 0$ such that the following holds. Let H be a wall with distance function d in a graph G, and let $H_1, \ldots, H_k$ be non-dividing subwalls of H, with*

$$d(H_i, H_j) \geqslant r \qquad (1 \leqslant i < j \leqslant k).$$

Then either there is an (l, m)-star over H or G has a K_p-minor. Moreover, there is an algorithm with running time $\leqslant C\varepsilon(G)$, where C depends only on l, m, p and the height of H, which given such subwalls $H_1, \ldots, H_k$, finds either an (l, m)-star over H or a model of K_p in G.

Proof. Let k, r be sufficiently large, in terms of l, m, p. (We shall specify later how to choose k and r.) Then $H_1, \ldots, H_k$ are mutually disjoint.

Let F be a minimal subgraph of G such that for $1 \leqslant i \leqslant k$, F contains an H-path from inside H_i to outside H_i. Thus $E(F \cap H) \neq \varnothing$. We can find F in time $O(\varepsilon)$, for fixed l, m, p, k, r. Since any circuit of F has $\geqslant 2$ vertices in $V(H)$ (by the minimality of F) it follows that we may express F as a union of trees S_j $(1 \leqslant j \leqslant t)$, where $S_j \cap S_{j'} \subseteq H$ for all $j \neq j'$, and every vertex of each S_j in $V(H)$ is a leaf of S_j, and each leaf of S_j is in $V(H)$.

A vertex $v \in V(F \cap H)$ is *essential* if v is inside H_i for some i, and v has valency 1 in F, and no other vertex of F is inside H_i.

(1) *For $1 \leqslant j \leqslant t$, if $e \in E(S_j)$ there exists i with $1 \leqslant i \leqslant k$ such that some leaf v of S_j inside H_i is essential, and every path of S_j between v and another leaf uses e.*

Subproof. From the minimality of F, there exists i $(1 \leqslant i \leqslant k)$ such that every H-path in F from inside H_i to outside H_i uses e. In particular, every such H-path is a path of S_j, and so for $j' \neq j$, either no leaf of $S_{j'}$ is inside H_i or no leaf of $S_{j'}$ is outside H_i. The latter contradicts the minimality of F, for then any edge of $S_{j'}$ could be deleted from F. Hence for $j' \neq j$, no leaf of $S_{j'}$ is inside H_i. Let v be a leaf of S_j inside H_i, and suppose that some leaf $u \neq v$ of S_j is either inside H_i or in the perimeter of H_i. Let f be the edge of S_j incident with u. There is a path of S_j from v to outside H_i, and it does not use f. Also, for $i' \neq i$, there is a path of F from inside $H_{i'}$ to outside $H_{i'}$, and it can be chosen not to use f, by rerouting it to v instead of to u if necessary. But this contradicts the minimality of F. Consequently, v

is the only leaf of S_j inside H_i or in its perimeter. Let $u \neq v$ be a leaf of S_j. We have already seen that u is outside H_i, and hence the path of S_j between u and v uses e. This proves (1).

We deduce

(2) *For $1 \leqslant j \leqslant t$, at most one vertex of S_j has valency $\geqslant 3$ in S_j; and if S_j has $\geqslant 3$ leaves they are all essential, and if S_j has two leaves then at least one is essential.*

Subproof. The first claim follows from (1) applied to an edge e of the path of S_j between two vertices of valency $\geqslant 3$. The second and third claims also follow from (1).

(3) *If $r \geqslant m$ then any vertex of F of valency $\geqslant l$ (in F) is the center of an (l, m)-star in G over H.*

Subproof. If v is such a vertex and $v \notin V(H)$ then v is the unique vertex of valency $\geqslant 3$ of some S_j by (2), and we may choose the desired (l, m)-star from S_j. If $v \in V(H)$, then v is a leaf of at least l different S_j's, each of which is an H-path from v to some essential vertex, by (2); thus the union of l of these S_j's yields the desired (l, m)-star.

By (3) we may assume that every vertex of F has valency $< l$. It follows that

(4) *For any $n \geqslant 0$, if $k \geqslant n(l-1)^2$, then there exist n mutually vertex-disjoint H-paths $Q_1, ..., Q_n \subseteq F$ with ends s_i, t_i $(1 \leqslant i \leqslant n)$, and we may renumber $H_1, ..., H_k$ so that s_i is inside H_i, t_i is outside H_i, and no other s_j or t_j is inside H_i $(1 \leqslant i \leqslant n)$.*

Subproof. By (2), for $1 \leqslant j \leqslant k$ at most one vertex of S_j belongs to $S_{j'}$ for some $j' \neq j$, and that vertex only belongs to $\leqslant l-2$ $S_{j'}$'s for $j' \neq j$. Thus $V(S_j \cap S_{j'}) \neq \emptyset$ for $\leqslant l-2$ values of $j' \neq j$. Choose $J \subseteq \{1, ..., t\}$ maximal such that $S_j \cap S_{j'}$ is null for all distinct $j, j' \in J$. Since each $S_{j'}$ $(j' \notin J)$ meets S_j for some $j \in J$, and any S_j $(j \in J)$ meets $\leqslant l-2$ $S_{j'}$'s $(j' \notin J)$, it follows that $t - |J| \leqslant (l-2)\,|J|$, that is, $|J| \geqslant t(l-1)^{-1}$. Now each S_j $(1 \leqslant j \leqslant t)$ has $\leqslant l-1$ leaves, and each H_i $(1 \leqslant i \leqslant k)$ contains a leaf of some S_j, and so $(l-1)t \geqslant k$. Thus $|J| \geqslant k(l-1)^{-2} \geqslant n$. For each $j \in J$ let $Q_j \subseteq S_j$ be an H-path with one end an essential vertex v, and the other end outside H_i, where v is inside H_i; then the claim follows.

Later we shall specify five numbers r_1, n_1, n_2, n_3, r depending on l, m, p but not on G. Then n will be defined in terms of r_1, n_1, n_2, n_3, r and k will be determined by (4).

(5) *For all sufficiently large r_1 and n_1, if there exists $I \subseteq \{1, ..., n\}$ with $|I| = n_1$ such that $d(s_i, t_i) \geqslant r_1$ for $i \in I$ and $d(t_i, t_{i'}) \geqslant r_1$ for distinct $i, i' \in I$, then G has a K_p-minor.*

Subproof. If $i \in I$, there are at most two $i' \in I - \{i\}$ such that either $d(s_i, t_{i'}) \leqslant \frac{1}{2}r_1$ or $d(t_i, s_{i'}) \leqslant \frac{1}{2}r_1$. Thus we may choose $I' \subseteq I$ with $|I'| \geqslant \frac{1}{3}n_1$ such that for distinct $i, i' \in I'$,

$$d(s_i, s_{i'}),\ d(s_i, t_{i'}),\ d(t_i, s_{i'}),\ d(t_i, t_{i'}) > \tfrac{1}{2}r_1.$$

By (7.4), if r_1 and n_1 are sufficiently large, G has a K_p-minor. This proves (5).

Choose r_1, n_1 to satisfy (5).

(6) *If n_2 and r are sufficiently large, and there exists $I \subseteq \{1, ..., n\}$ with $|I| = n_2$ such that $d(s_i, t_i) \leqslant r_1$ for all $i \in I$, then G has a K_p-minor.*

Subproof. Now $d(s_i, C) \geqslant \frac{1}{2}r - 1$ for all $i \in I$ except possibly one, where C is the perimeter of H. Hence we may enlarge $n_2 - 1$ of the H_i's to subwalls H_i', say of H, still mutually far apart but now with t_i inside H_i'. These H_i''s have crosses as in (9.1), and the claim follows from (9.1).

(7) *If n_3 and r are sufficiently large, and there exist i_0 with $1 \leqslant i_0 \leqslant n$ and $I \subseteq \{1, ..., n\} - \{i_0\}$ such that $|I| = n_3$ and $d(t_i, t_{i_0}) \leqslant r_1$ for $i \in I$, then G has a K_p-minor.*

Subproof. By (7.3) we may choose a wall $H' \subseteq H$ with $t_i \notin V(H')$ $(i \in I)$ and with all the t_i's $(i \in I)$ in the same component of $H \backslash V(H')$. Since $H \backslash V(H')$ has maximum valency $\leqslant 3$, we may pair many of the t_i's by vertex-disjoint paths of H disjoint from H'. This yields many H'-paths in G with their ends mutually far apart, and G has a K_p-minor by (7.4), with suitable choice of n_3 and r. This proves (7).

Choose n_2, n_3, and r to satisfy (6) and (7). (All these choices are independent of G.) Finally, we observe that

(8) *If n is sufficiently large, then there exists $I \subseteq \{1, ..., n\}$ as in* (5), (6), *or* (7).

We choose n to satisfy (8) and k as in (4). The result follows.

With k and r chosen as in the theorem, the algorithm to find an (l, m)-star over H or a model of K_p is trivial. First, we determine F. This takes time $O(\varepsilon)$. Since the size of F is bounded (exept for divalent vertices) by a function of l, m, and p, we can find an (l, m)-star or a model of K_p in $H \cup F$ in time $\leqslant C\varepsilon(G)$, where C depends only on l, m, p, and the height of H. ∎

From (9.2) we obtain

(9.3) *For any $p \geqslant 0$ there exists k, $r \geqslant 0$ such that there is an algorithm as in* (9.4), *with running time $\leqslant C\varepsilon(G)$, where C depends on p, t, and the height of H.*

(9.4) ALGORITHM.
Input: A graph G, a wall $H \subseteq G$ with distance function d, and subwalls $H_1, \ldots, H_t$ of H with $d(H_i, H_j) \geq r$ $(1 \leq i < j \leq t)$.
Output: Either

(i) a model of K_p in G, or

(ii) a subset $X \subseteq V(G)$ with $|X| < \binom{p}{2}$ and a subset $I \subseteq \{1, \ldots, t\}$ with $|I| \geq t-k$ such that for all $i \in I$, H_i is dividing in $(G \backslash X) \cup H$.

Proof of (9.3). Let $l \geq 4$, $m \geq 0$ satisfy (7.7), and let $k, r \geq 0$ satisfy (9.2). Now we describe the algorithm for (9.4). We set $Y_1 = \emptyset$, and perform a recursion. At the beginning of the ith iteration we have a set Y_i of $i-1$ (l, m)-stars over H in G, with distinct centers. We proceed as follows:

(1) If $i = \binom{p}{2}$ we apply (7.7) to obtain a model of K_p in G, and stop.

(2) Otherwise $i < \binom{p}{2}$. Let X be the set of centers of the stars in Y_i. Let $G' = (G \backslash X) \cup H$. Then H is a wall in G', and no (l, m)-star in G' has center in X (since $l \geq 4$ and every vertex of G' in X has valency ≤ 3).

(3) We test which H_i's are dividing in G'. If at least $t-k$ of them are dividing in G' we stop.

(4) Otherwise, at least k of the H_i's are non-dividing. We choose k of them and apply (9.2). We obtain either a model of K_p in G' and hence in G, when we stop, or an (l, m)-star S over H in G', when we set $Y_{i+1} = Y_i \cup \{S\}$ and return for the next iteration.

This algorithm has running time $\leq C\varepsilon(G)$, where C depends only on p, t, and the height of H, because there are at most $\binom{p}{2}$ iterations, each involving t tests of "is H_i dividing?" which takes time $O(\varepsilon)$, together possibly with an application of (7.7) or (9.2). ∎

Let H be a wall in a graph G, with perimeter C. Let $W \subseteq V(G)$ be the union of $V(C)$ and the vertex set of the unique component of $G \backslash V(C)$ that contains a vertex of H. We define the *compass of H in G* to be the restriction of G to W. It follows that H is a subgraph of its compass, and its compass is connected.

(9.5) *Let H be a wall in a graph G. If H_1 is a dividing subwall of H with compass K_1, then $K_1 \cap H \subseteq H_1$; and if H_2 is another dividing subwall of H with compass K_2, and $H_1 \cap H_2$ is null, then $K_1 \cap K_2$ is null.*

Proof. Since H_1 is an induced subgraph of H, to prove the first claim it suffices to show that $V(K_1 \cap H) \subseteq V(H_1)$. Suppose that $t \in V(K_1 \cap H) - V(H_1)$. Since t belongs to the compass of H_1 and not to the perimeter C_1 of H_1, there is a path of $G \backslash V(C_1)$ between t and the inside of H_1, and hence there is an H-path with one end inside H_1 and the other end outside. But this contradicts that H_1 is dividing and proves the first claim.

For the second claim, certainly $K_1 \cap H_2$ is null, since $K_1 \cap H_2 \subseteq K_1 \cap H \subseteq H_1$, and similarly $K_2 \cap H_1$ is null. Suppose that $t \in V(K_1 \cap K_2)$. Since $t \in V(K_1)$ and $t \notin V(H)$, there is a path of $G \backslash V(C_1)$ between t and the inside of H_1. By combining this and a similar path for H_2, we obtain an H-path from the inside of H_1 to the inside of H_2, a contradiction. Hence $K_1 \cap K_2$ is null, as required. ▮

A subwall H of G is *flat* in G if there is no cross $P \cup Q$ over H such that $P \cup Q$ is a subgraph of the compass of H in G.

(9.6) *For any $p, s \geq 0$ and $h \geq 2$, even, there exists $h' \geq p^2 + 2$, even, and an algorithm as in* (9.7) *with running time $O(v\varepsilon)$ for fixed p, s, h.*

(9.7) ALGORITHM.
Input: A graph G, and a wall H in G of height h'.
Output: Either

(i) a model of K_p in G, or

(ii) a subset $X \subseteq V(G)$ with $|X| < \binom{p}{2}$ and s subwalls $H_1, \ldots, H_s$ of H of height h, mutually vertex-disjoint, and each flat and dividing in $(G \backslash X) \cup H$, and each disjoint from X.

Proof of (9.6). We may assume that $p \geq 2$. Let $k = k_1$, $r = r_1$ satisfy (9.3), and let $k = k_2$, $r = r_2$ satisfy (9.1) for the given p. Let h' be some even number satisfying

$$h' \geq \left(s + k_1 + k_2 + \binom{p}{2} + 1\right) \cdot (h + \max(r_1, r_2) + 1).$$

Our algorithm is as follows. Given G, H as input, we choose $s + k_1 + k_2 + \binom{p}{2}$ subwalls of H, each of height h, and mutually at distance $\geq \max(r_1, r_2)$ (this is possible from the choice of h'). We apply (9.4) and obtain either a model of K_p in G when we stop, or a subset $X \subseteq V(G)$ with $|X| < \binom{p}{2}$ and some $s + k_2 + \binom{p}{2}$ of our subwalls, each dividing in $(G \backslash X) \cup H$. Now we test which of these subwalls are flat in $(G \backslash X) \cup H$ and find crosses over them in their compasses, if possible, using (8.1). Suppose that we find crosses for at least k_2 of them. By (9.5), these crosses satisfy the hypotheses of (9.1). By (9.1) there is a model of K_p in the union of H and these crosses, and we can find it in time $O(\varepsilon)$. On the other hand, if fewer than k_2 of our subwalls yield crosses, then at least $s + \binom{p}{2}$ of them are flat in $(G \backslash X) \cup H$, and since $|X| < \binom{p}{2}$ we may choose s of them disjoint from X. Again we stop. ▮

Let H be a wall in G. The *society of H in G* is the society (K, Ω), where K is the compass of H in G and Ω is the cyclic permutation of the corners of H corresponding to their order in the perimeter of H. Let $\mathscr{A}$ be a rural division of the society of H in G. We say that $A \in \mathscr{A}$ is *interior* if $A \cap C$ is null, where C is the perimeter of H.

(9.8) *For all $p \geqslant 0$ and $h \geqslant 2$, even, there exists $\omega \geqslant 0$ such that there is an algorithm as in* (9.9), *with running time $O(v\varepsilon)$ for fixed p, h.*

(9.9) ALGORITHM.
Input: A graph G.
Output: Either

(i) a branch-decomposition of G of width $\leqslant \omega$, or

(ii) a model of K_p in G, or

(iii) a subset $X \subseteq V$ with $|X| < \binom{p}{2}$ and a flat wall H in $G \backslash X$ of height h, and a rural division $\mathscr{A}$ of the society of H in $G \backslash X$, and a branch-decomposition of width $\leqslant \omega$ of each interior member of $\mathscr{A}$.

Proof of (9.8). We may assume that $p \geqslant 2$. Let $s = \binom{p}{2} + 5$. Choose $h' \geqslant 2$, even, as in (9.6). Choose ω, by (4.2), so that every graph with branch-width $\geqslant \omega/3$ contains a wall of height h'. Now we describe the algorithm. We are given G as input.

(1) We apply (7.2) to G. (This is possible by (7.1).) If we obtain a branch-decomposition of G of width $\leqslant \omega$ we stop. Otherwise we obtain a wall H in G of height h'.

(2) Now we perform a recursion. We set $G_1 = G$, $Z_1 = \emptyset$, $H_1 = H$. At the beginning of the ith iteration we have

(i) a subgraph G_i of G with $|\varepsilon(G_i)| \leqslant (\frac{1}{2})^{i-1} |\varepsilon(G)|$

(ii) a subset $Z_i \subseteq V(G_i)$ with $|Z_i| \leqslant s-3$ such that no vertex of $G_i \backslash Z_i$ is incident with an edge of G not in G_i,

(iii) a wall H_i in G_i of height h'.

The iteration proceeds as follows.

(3) We apply (9.7) to G_i, H_i. If we obtain a model of K_p in G_i and hence in G we stop. Otherwise, we obtain a subset $X \subseteq V(G_i)$ with $|X| < \binom{p}{2}$ and s subwalls $H^1, ..., H^s$ of H_i of height h, mutually vertex-disjoint, and each flat and dividing in $(G_i \backslash X) \cup H_i$, and each disjoint from X. Let K^j be the compass of H^j in $(G_i \backslash X) \cup H_i$ $(1 \leqslant j \leqslant s)$.

(4) Since the H^j's are all dividing in $(G_i \backslash X) \cup H_i$, it follows that $K^1, ..., K^s$ are mutually vertex-disjoint and that each contains no vertex in X, by (9.5). Since $|Z_i| \leqslant s-3$, we may choose three of the H^j's, say H^1, H^2, and H^3, such that $V(K^j) \cap Z_i = \emptyset$ $(1 \leqslant j \leqslant 3)$.

Since H^3 has height $h' \geqslant p^2+2$ it follows that $\varepsilon(K^3) \geqslant \varepsilon(H^3) \geqslant p^2+2 \geqslant |X|+2$. Let there be d_j edges in G_i with one end in X and the other in $V(K^j)$, for $1 \leqslant j \leqslant 3$. Then (we recall that by definition, $\varepsilon(G) = |E(G)| + |V(G)| + 1$) we have

$$\varepsilon(K^1)+\varepsilon(K^2)+\varepsilon(K^3)-2+d_1+d_2+d_3+|X| \leqslant \varepsilon(G_i).$$

But $\varepsilon(K^3) \geqslant |X|+2$, and so

$$\varepsilon(K^1)+\varepsilon(K^2)+d_1+d_2+2\,|X| \leqslant \varepsilon(G_i).$$

Choose one of them, K^1 say, so that $\varepsilon(K^1)+d_1+|X| \leqslant \frac{1}{2}\varepsilon(G_i)$.

(5) We apply (8.6) to obtain a rural division $\mathscr{A}$ of the society of H^1 in $(G_i \backslash X) \cup H_i$.

(6) For each interior $A \in \mathscr{A}$, we apply (7.2) to A and obtain either a branch-decomposition of A of width $\leqslant \omega$, or a wall in A of height h'.

(7) If for each interior $A \in \mathscr{A}$ we obtain a branch-decomposition of A of width $\leqslant \omega$, we stop.

(8) If for some interior $A \in \mathscr{A}$ we obtain a wall H_{i+1} in A of height h', we let G_{i+1} be the subgraph of G_i with vertex set $V(A) \cup X$ and edges those edges of G_i with an end in $V(A)$ and the other end in $V(A) \cup X$. Since A is a subgraph of K^1, it follows that

$$\varepsilon(G_{i+1}) \leqslant \varepsilon(A)+d_1+|X| \leqslant \varepsilon(K^1)+d_1+|X| \leqslant \tfrac{1}{2}\varepsilon(G_i) \leqslant (\tfrac{1}{2})^i \varepsilon(G).$$

Let ∂A be the set of all vertices v of A such that either v is a corner of H^1 or v is incident with an edge of $(G_i \backslash X) \cup H_i$ not in $E(A)$; then $|\partial A| \leqslant 3$ since $\mathscr{A}$ is a division. Set $Z_{i+1} = X \cup \partial A$; then $|Z_{i+1}| \leqslant \binom{p}{2}+2 = s-3$, since $|X| \leqslant \binom{p}{2}-1$. We return for the next iteration.

We must verify that the conditions (i)–(iii) of (2) hold with G_i, Z_i, H_i replaced by $G_{i+1}, Z_{i+1}, H_{i+1}$. Certainly (i) holds, as we saw above, and so does (iii). Let us verify (ii). Let $v \in V(G_{i+1} \backslash Z_{i+1})$, and let $e \in E(G)$ be incident with v, and suppose for a contradiction that $e \notin E(G_{i+1})$. Since $v \in V(G_{i+1} \backslash Z_{i+1}) = V(A) - \partial A$ and $V(A) \subseteq V(K^1)$ and $V(K^1) \cap Z_i = \varnothing$, it follows that $v \notin Z_i$, and so $e \in E(G_i)$. Since e has one end in $V(A)$ and $e \notin E(G_{i+1})$, its other end is not in $V(A) \cup X$, and in particular, $e \in E((G_i \backslash X) \cup H_i)$. Since A is internal and hence v does not belong to the perimeter of H^1, it follows that $e \in E(K^1)$, since $v \in V(A) \subseteq V(K^1)$. But every edge of K^1 incident with v belongs to $E(A)$, since $\mathscr{A}$ is a division of the society of H^1 in $(G_i \backslash X) \cup H_i$ and $v \in V(A) - \partial A$. This contradicts that $e \notin E(G_{i+1})$ as required, and completes the verification of (ii).

This completes the description of the algorithm. To estimate running time we observe that step (1) takes time $O(v^2+\varepsilon)$. Steps (3) and (5) both take time $O(v\varepsilon)$. Step (6) takes time $\Sigma(Cv(A')\varepsilon(A') : A' \in \mathscr{A})$ for some constant C. But for $A' \in \mathscr{A}$, $\varepsilon(A') \leqslant 4(\varepsilon(A') - |\partial A'| - 1)$, since some vertex or edge of A' belongs to no other member of $\mathscr{A}$, and hence, summing over $A' \in \mathscr{A}$,

$$\begin{aligned}\Sigma v(A')\,\varepsilon(A') &\leqslant \Sigma 4v(A')(\varepsilon(A') - |\partial A'| - 1)\\ &\leqslant 4v(G_i)\,\Sigma(\varepsilon(A') - |\partial A| - 1) \leqslant 4v(G_i)\,\varepsilon(G_i).\end{aligned}$$

Consequently, step (6) takes time $O(v(G_i)\,\varepsilon(G_i))$. Thus the ith iteration takes time $O(v(G_i)\,\varepsilon(G_i))$. Since $\varepsilon(G_i) \leqslant (\frac{1}{2})^{i-1}\varepsilon(G)$ and $v(G_i) \leqslant v(G)$, the whole recursion takes time $O(v\varepsilon)$. Hence the algorithm takes time $O(v\varepsilon)$, as claimed. ∎

10. Homogeneously Labelled Walls

Our last task is to show how to make use of the third possible outcome of (9.8) when solving Folio. The following condition is a common hypothesis of several results to follow and so we give it a name.

Hypothesis C. *G is a digraph, $X \subseteq V(G)$ with $|X| = q$, H is a wall in $G \backslash X$, and $\mathscr{A}$ is a rural division of the society of H in $G \backslash X$.*

Under Hypothesis C, let (K, Ω) be the society of H in $G \backslash X$. Let $A \in \mathscr{A}$ be interior, and let $|\partial A| = k$; then $k \leqslant 3$. Choose a numbering $\{s_1, ..., s_k\}$ of ∂A, where if $k = 3$ we number "clockwise;" that is, we number s_1, s_2, s_3 so that there are three paths P_1, P_2, P_3 of K from ∂A to $\bar{\Omega}$, mutually vertex-disjoint, where P_i has ends s_i, t_i $(i = 1, 2, 3)$, and t_1, t_2, t_3 occur in this order under Ω. Let $X = \{x_1, ..., x_r\}$; the sequence $(s_1, ..., s_k, x_1, ..., x_r)$ is called an *attachment sequence* of A, and we denote it by $\pi(A)$. For each $A \in \mathscr{A}$, choose a subgraph $\tilde{A}$ of G with $X \subseteq V(\tilde{A})$ and with $\tilde{A} \backslash X = A$, such that every edge of G with both ends in $X \cup V(K)$ is in $E(\tilde{A})$ for exactly one $A \in \mathscr{A}$. Let T_A be the δ-folio of $(\tilde{A}, \pi(A))$.

Now for each interior $A \in \mathscr{A}$, let $v(A) \in V(H)$ such that there is a path of K from ∂A to $v(A)$ with no vertex in $V(H)$ except $v(A)$. (Such a path always exists since K is connected and A is non-null.) This information (that is, for each interior $A \in \mathscr{A}$, a choice of $\tilde{A}$, an attachment sequence $\pi(A)$ of A, the δ-folio of $(\tilde{A}, \pi(A))$, and a vertex $v(A)$) is called a *vision*. For fixed ω and q, we need the following.

(10.1) Algorithm.
Input: $G, X, H, \mathscr{A}$ satisfying Hypothesis C, and an integer $\delta \geqslant 0$, and for each interior $A \in \mathscr{A}$ a branch-decomposition of A of width $\leqslant \omega$.

Output: A vision.
Running time: $O(\varepsilon^2)$.
Description: First we compute an attachment sequence for each interior $A \in \mathscr{A}$. For each A this takes time $O(\varepsilon(G))$, and so we can compute all the attachment sequences in time $O(\varepsilon^2)$, since $|\mathscr{A}| \leqslant |E(G)|$.

For each interior $A \in \mathscr{A}$ we choose $\tilde{A}$, and find a branch-decomposition of $\tilde{A}$ of width $\leqslant \omega + q$ (this exists, since $\tilde{A}$ is obtained from A by adding q extra vertices), by adapting the given branch-decomposition of A in the natural way. This takes time $O(\varepsilon)$.

Third, we compute the δ-folio of $(\tilde{A}, \pi(A))$ for each interior $A \in \mathscr{A}$, by (4.1). This takes time $O(v\varepsilon)$, since $\omega + q$ is a constant.

Fourth, for each interior $A \in \mathscr{A}$ we compute some $v(A)$, for instance, by extending a spanning tree of H to a spanning tree of its compass. This takes time $O(v\varepsilon)$ again, and completes the algorithm.

Under Hypothesis C and given some vision, a subwall H' of H is said to be *h-homogeneous* if for every internal $A \in \mathscr{A}$ with $v(A)$ inside H' and for every subwall H'' of H' of height h, there exists some internal $A' \in \mathscr{A}$ with $v(A')$ inside H'' such that $(A', \pi(A'))$ has the same δ-folio as $(A, \pi(A))$. The following lemma is proved in [24]; it is an application of the main results of [21, 22].

(10.2) *For any q, $\delta \geqslant 0$ and $h \geqslant 2$, even, there exists $f(h) \geqslant h$, even, such that under Hypothesis* C, *if $H' \subseteq H$ is an h-homogeneous subwall of height $f(h)$ then both middle vertices of H' are irrelevant to the δ-folio of G relative to X.*

(The *middle* vertices of a wall are the two vertices corresponding to the two middle vertices of the elementary wall of which the wall is a subdivision.) We use (10.2) to show

(10.3) *For all $q, \delta \geqslant 0$ there exists $h \geqslant 0$, even, such that for all $\omega \geqslant 0$ there is an algorithm as in* (10.4) *with running time $O(\varepsilon^2)$.*

(10.4) ALGORITHM.
Input: G, X, H, $\mathscr{A}$ satisfying Hypothesis C, where H has height h, and for each interior $A \in \mathscr{A}$ a branch-decomposition of A of width $\leqslant \omega$.
Output: A vertex irrelevant to the δ-folio of G relative to X.

Proof of (10.3). Let f be the function of (10.2) (for given q, δ). Let $\mathscr{L}$ be the class of all rooted digraphs with $\leqslant q+3$ roots and with detail $\leqslant \delta$. Then $\mathscr{L}$ is the union of finitely many isomorphism classes $W_1, \ldots, W_t$, say.

For any rooted graph with $\leqslant q+3$ roots, its δ-folio is a union of some of $W_1, ..., W_l$. Let

$$\mathscr{T}_1 = \left\{ \bigcup (W_i : i \in I) : I \subseteq \{1, ..., l\} \right\}.$$

Thus, $\mathscr{T}_1$ contains the δ-folio of every rooted graph with $\leqslant q+3$ roots, and $\mathscr{T}_1$ is finite. Let $|\mathscr{T}_1| = n$, let $h_{n+1} = 2$, for $1 \leqslant i \leqslant n$ let $h_i = f(h_{i+1})$, and let $h = h_1$. Then h satisfies (10.3), for an algorithm proceeds as follows. Given input as in (10.4), we first use (10.1) to find a vision. Second we set $H_1 = H$ and perform a recursion. At the beginning of the ith iteration we have H_i and $\mathscr{T}_i$ such that

(i) H_i is a subwall of H of height h_i

(ii) $\mathscr{T}_i \subseteq \mathscr{T}_1$ and $|\mathscr{T}_i| = n+1-i$

(iii) for each internal $A \in \mathscr{A}$, if $v(A)$ is inside H_i, then $\mathscr{T}_i$ contains the δ-folio of $(\tilde{A}, \pi(A))$.

The iteration proceeds as follows. We check if H_i is h_{i+1}-homogeneous. If it is, we may output one of its middle vertices and stop, by (10.2). Otherwise we choose a subwall H_{i+1} of H_i of height h_{i+1} such that some $T \in \mathscr{T}_1$ is the δ-folio of $(\tilde{A}, \pi(A))$ for some internal $A \in \mathscr{A}$ with $v(A)$ inside H_i, and T is not the δ-folio of $(\tilde{A}, \pi(A))$ for any internal $A \in \mathscr{A}$ with $v(A)$ inside H_{i+1}. Then $T \in \mathscr{T}_i$; we set $\mathscr{T}_{i+1} = \mathscr{T}_i - \{T\}$, and return for the next iteration.

Each iteration takes time $O(v)$ (since there are only a bounded number of subwalls of H_i to check) and there are at most $l+1$ iterations. Finding a δ-vision initially takes time $O(\varepsilon^2)$, and so the algorithm also has running time $O(\varepsilon^2)$. ∎

Now we give the complete algorithm for Folio. Let ξ and $\delta \geqslant 0$ be fixed integers.

(10.5) ALGORITHM.

Input: A digraph G and a subset $Z \subseteq V(G)$ with $|Z| \leqslant \xi$.

Output: The δ-folio of G relative to Z.

Running time: $O(v^3)$.

Description: (1) Let p, C be as in (6.5). We apply (6.5) to G, Z, and obtain a subdigraph $G' \subseteq G$ with $Z \subseteq V(G')$ such that G and G' have the same δ-folio relative to Z and such that $|E(G')| \leqslant C\,|V(G')|$.

Let $q = \binom{p}{2} + \xi - 1$; let h be as in (10.3); let ω be as in (9.8). We set $G_1 = G'$ and perform a recursion. At the beginning of the ith iteration, we have a subdigraph G_i of G' with $|V(G_i)| = |V(G')| - i + 1$ and with $Z \subseteq V(G_i)$, such that the δ-folios of G_i and G' relative to Z are equal. The iteration proceeds as follows. We apply (9.9) to $G_i \backslash Z$.

(2) If we obtain a branch-decomposition of $G_i \backslash Z$ of width $\leqslant \omega$, we convert it to a branch-decomposition of G_i of width $\leqslant \omega + \xi$, compute the δ-folio of G_i relative to Z by (4.1), and stop.

(3) If we obtain a model of K_p in $G_i \backslash Z$ and hence in G_i, we apply (6.2) to find a vertex v of G_i irrelevant to the δ-folio of G_i relative to Z; we set $G_{i+1} = G_i \backslash v$ and return for the next iteration.

(4) Otherwise, we obtain X, H, $\mathscr{A}$ as in Hypothesis C (with G replaced by $G_i \backslash Z$), where $|X| \leqslant \binom{p}{2} - 1$ and H has height h. Put $X' = X \cup Z$; then G_i, X', H, $\mathscr{A}$ also satisfy Hypothesis C. We apply (10.4) to G_i, X', H, $\mathscr{A}$, to compute an irrelevant vertex v for the δ-folio of G_i relative to X'. Since $Z \subseteq X'$, v is also irrelevant for the δ-folio of G_i relative to Z, by (2.1). We set $G_{i+1} = G_i \backslash v$, and return for the next iteration.

The application of (6.5) in step (1) takes time $O(\nu^3)$. In the recursion, each iteration takes time $O(\varepsilon(G')^2)$. Since there are $\leqslant \nu(G)$ iterations, and $\varepsilon(G') \leqslant O(\nu(G))$, the algorithm takes time $O(\nu^3)$. ∎

We would like to say that not only does there exist an algorithm as in (10.5), but we can *construct* it, and that depends on being able to construct the various constants p, h, ω used in the course of the proof, for given numerical values of ξ and δ. Now p is easy, but h and ω are defined by (10.3) and (9.8), and these are formulated as purely existential statements, containing no method for calculating h and ω. ("For all q, $\delta \geqslant 0$ there exists $h \geqslant 0$....") Nevertheless the proofs of (10.3) and (9.8) do yield procedures for calculating h and ω, except that (10.3) uses the function f of (10.2), and (9.8) uses h' of (9.5) and another ω, from (4.2). The sequences of definitions can be traced further and further back, through other papers in this series, until they terminate; and all the proofs used to define these constants are constructive. Even, say, Theorem (3.5) of [16], which asserts that "For every matching G in Σ and every homoplasty class $\mathscr{D}$ there exists $H \in \mathscr{D}$ with $U(G) \cap U(H)$ finite" can be proved constructively, in the sense that it is easy to give an algorithm which, given explicitly some G and $\mathscr{D}$, will compute an appropriate H. Thus, we do indeed have an algorithm which, with input numbers ξ and $\delta \geqslant 0$, finds the algorithm of (10.5); and also the constant C in the bound on its running time can be constructed.

References

1. S. Arnborg, D. G. Corneil, and A. Proskurowski, Complexity of finding embeddings in k-trees, *SIAM J. Algebraic Discrete Methods* **8** (1987), 277–284.
2. S. Arnborg and A. Proskurowski, Linear time algorithms for NP-hard problems restricted to partial k-trees, *Discrete Applied Math.* **23** (1989), 11–24.

3. H. Bodlaender, A linear time algorithm for finding tree-decompositions of small tree-width, manuscript, 1993.
4. I. S. Filotti, G. L. Miller, and J. Reif, On determining the genus of a graph in $O(v^{O(g)})$ steps, *in* "Proceedings, 11th Annual ACM Symp. on the Theory of Computing," pp. 27–37, Assoc. Comput. Mach., New York, 1979.
5. S. Fortune, J. E. Hopcroft, and J. Wyllie, The directed subgraph homeomorphism problem, *J. Theoret. Comput. Sci.* **10** (1980), 111–121.
6. M. R. Garey and D. S. Johnson, "Computers and Intractability," Freeman, San Francisco, 1979.
7. H. A. Jung, Eine Verallgemeinerung des n-fachen Zusammenhangs für Graphen, *Math. Ann.* **187** (1970), 95–103.
8. R. M. Karp, On the complexity of combinatorial problems, *Networks* **5** (1975), 45–68.
9. A. K. Kelmans, Finding special subdivisions of K_4 in a graph, *in* "Finite and Infinite Sets," Colloq. Math. Soc. János Bolyai, Vol. 37, pp. 487–508, North Holland, Amsterdam/New York, 1981.
10. D. G. Larman and P. Mani, On the existence of certain configurations within graphs and the 1-skeletons of polytopes, *Proc. London Math. Soc. (3)* **20** (1970), 144–160.
11. W. Mader, Hinreichende Bedingungen für die Existenz von Teilgraphen, die zu einem vollständigen Graphen homöomorph sind, *Math. Nachr.* **53** (1972), 145–150.
12. B. Reed, Finding approximate separators and computing tree-width quickly, manuscript, 1992.
13. N. Robertson and P. D. Seymour, Graph minors. II. Algorithmic aspects of tree-width, *J. Algorithms* **7** (1986), 309–322.
14. N. Robertson and P. D. Seymour, Graph minors. IV. Tree-width and well-quasi-ordering, *J. Combin. Theory Ser. B* **48** (1990), 227–254.
15. N. Robertson and P. D. Seymour, Graph minors. V. Excluding a planar graph, *J. Comb. Theory Ser. B* **41** (1986), 92–114.
16. N. Robertson and P. D. Seymour, Graph minors. VII. Disjoint paths on a surface, *J. Combin. Theory Ser. B* **45** (1988), 212–254.
17. N. Robertson and P. D. Seymour, Graph minors. VIII. A Kuratowski theorem for general surfaces, *J. Combin. Theory Ser. B* **48** (1990), 255–288.
18. N. Robertson and P. D. Seymour, Graph minors. IX. Disjoint crossed paths, *J. Combin. Theory Ser. B* **49** (1990), 40–77.
19. N. Robertson and P. D. Seymour, Graph minors. X. Obstructions to tree-decomposition, *J. Combin. Theory Ser. B* **52** (1991), 153–190.
20. N. Robertson and P. D. Seymour, Graph minors. XII. Distance on a surface, *J. Combin. Theory Ser. B* **62** (1994).
21. N. Robertson and P. D. Seymour, Graph minors. XVI. Excluding a non-planar graph, submitted.
22. N. Robertson and P. D. Seymour, Graph minors. XVII. Taming a vortex, submitted.
23. N. Robertson and P. D. Seymour, Graph minors. XX. Wagner's conjecture, in preparation.
24. N. Robertson and P. D. Seymour, Graph minors. XXII. Irrelevant vertices in linkage problems, in preparation.
25. N. Robertson, P. D. Seymour, and R. Thomas, Quickly excluding a planar graph, *J. Combin. Theory Ser. B* **62** (1994), 323–348.
26. P. D. Seymour, Disjoint paths in graphs, *Discrete Math.* **29** (1980), 293–309.
27. Y. Shiloach, A polynomial solution to the undirected two paths problem, *J. Assoc. Comput. Mach.* **27** (1980), 445–456.
28. A. Thomason, An extremal function for contractions of graphs, *Math. Proc. Cambridge Philos. Soc.* **95** (1984), 261–265.
29. C. Thomassen, 2-linked graphs, *Europ. J. Combin.* **1** (1980), 371–378.