A first regularization method

Consider the integral operator $A : L^2(0,1) \to L^2(0,1)$,

$$Af(x) = \int_0^x f(t)\,\mathrm{d}t\,.$$

In order to discretize A, define a grid $x_i = \left(i - \frac{1}{2}\right) h$, $i = 1, \dots, N$ with $h = \frac{1}{N}$ and $N \in \mathbb{N}$. The function f is approximated by piece-wise constant step function $\bar{f}$ with values $\bar{f}_i = f(x_i)$ on the interval $[(i-1)h, ih)$ and the function Af by the vector $\mathbf{g}$ with

$$g_i = \left(A\bar{f}\right)(x_i) = \int_0^{x_i} \bar{f}(t)\ \mathrm{d}t.$$

This results in a linear system $\mathbf{g} = \mathbf{A}\mathbf{f}$ with

$$\mathbf{A} = h \begin{pmatrix} \frac{1}{2} & 0 & \cdots & 0 \\ 1 & \frac{1}{2} & & \vdots \\ \vdots & & \ddots & 0 \\ 1 & \cdots & 1 & \frac{1}{2} \end{pmatrix} \qquad \text{and } \mathbf{f} = \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \vdots \\ \bar{f}_N \end{pmatrix}.$$

- Compute data $\mathbf{g}$ and noisy data $\mathbf{g}^\delta$ with $\left\|\mathbf{g} - \mathbf{g}^\delta\right\| \leq 0.19$ for the function $f(x) = \exp(-2x)\cos(5x)$ and $N = 100$.
- Implement the truncated singular value decomposition (TSVD) which is given by the the finite series

$$\mathbf{A}^+\mathbf{g} = \sum_{j=1}^{K} \sigma_j^{-1} \langle \mathbf{g}, \mathbf{u}_j \rangle\, \mathbf{v}_j$$

  with singular system $\{(\sigma_j, \mathbf{v}_j, \mathbf{u}_j), j \in \mathbb{N}\}$ of A.
- Test the TSVD with $\mathbf{g}^\delta$. Which K gives the best reconstruction of the original function f? Compare your reconstruction with the minimal-norm-solution.