Exercise 3. Differentiation

Let $g^\delta(x)$, $x \in \mathbb{R}$ be point-wise measurements of the function $g \in C^3(\mathbb{R})$ with

$$\|g^\delta - g\| \leq \delta\,.$$

The values of the derivative $g'(x)$, $x \in \mathbb{R}$, are approximated with help of the centered difference quotient of the measurements, i.e.

$$d_h g^\delta(x) = \frac{1}{2h}\left(g^\delta(x+h) - g^\delta(x-h)\right).$$

Determine the optimal step size $h^* > 0$ such that the total error

$$\left|g^{'}(x) - d_h\left(g^\delta\right)(x)\right|$$

of the approximation is minimal. Which behavior in terms of δ do you observe for the total error?

Exercise 4. Numerical differentiation in practice

Implement the differentiation method from exercise 3: Use for g the exponential function over the interval $[-1, 1]$ as test function and generate noisy data g^δ by adding normal distributed noise with mean 0 and standard deviation 0.01 to g.

1. Find the optimal choice of h^+ by visual comparison with the true derivative g'.
2. Plot $g'(x)$ and $D_{h^+}\left(g^\delta(x)\right)$.