Final Project Report

Linear Algebra

Fractals

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Abstract

The following report is a brief introduction to topology, fractals and computation of fractals. Fractals are objects that have a Hausdorff dimension that does not match with its Lebsque covering dimension. Self similar fractals can be generated using iterations of affine transformation, which is a composition of a linear transformation and a translation. The application of fractals is discussed briefly and a fascinating fractal- the Mandelbrot Set is introduced at the end as an extended exploration section.

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1. Sets

1.1. Bounded sets

Definition 1.1: *Bounded sets are sets that have defined upper and lower limits that are finite, i.e, contained in a finite interval.*

1.2. Open sets

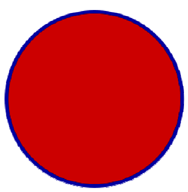
An open set is a sort of generalization of the idea of an interval. When we take an interval between x and y on the real number line, we say the interval is

i. **Open** if the interval does not include x and y, i.e.,$(x, y) =]x, y[= \{t \in \mathbb{R}\, | x < t < y\}$.

ii. **Closed** if the interval includes x and y, i.e., $[x, y] = \{t \in \mathbb{R}\, | x \leq t \leq y\}$.

iii. **Half-open** if the interval includes either x or y, i.e., $[x, y) = [x, y[= \{t \in \mathbb{R}\, | x \leq t < y\}$ and $(x, y] =]x, y] = \{t \in \mathbb{R}\, | x < t \leq y\}$

Definition 1.2: *An open set is one where the boundary conditions/ points are a part of the set itself.*



Example: The blue circle represents the set of points (x, y) satisfying x2 + y2 = r2. The red disk represents the set of points (x, y) satisfying x2 + y2 < r2. The red set is an open set, the blue set is its boundary set, and the union of the red and blue sets is a closed set.[1]

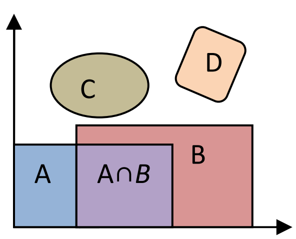
1.3. Overlapping and Non-overlapping sets

1.3.1. Overlapping sets

Definition 1.3.1: *Two sets 'A' and 'B' are said to be overlapping if A∩B has at least one element.*

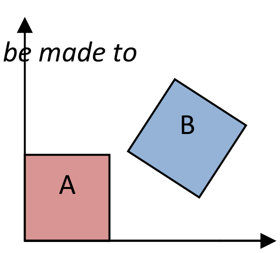
1.3.2. Non-overlapping sets

Definition 1.3.1: *Two sets 'C' and 'D' are said to be overlapping if C∩D is a null set.*



1.4. Congruent sets

Definition 1.4: *Two sets 'A' and 'B' are congruent if they are be made to coincide/ exactly overlap each other by appropriate translations and/ or rotations, in the domain.*



[1] By Richard Giuly at English Wikipedia - Transferred from en.wikipedia to Commons. Converted to SVG by Oleg Alexandrov 02:27, 2 September 2007 (UTC), Public Domain, https://commons.wikimedia.org/w/index.php?curid=2667413

1.5. Self-similar sets

A set ‘S’ that can be written as the union of non-overlapping sets that are congruent to S and are scaled by a particular factor is called a self-similar set.

<u>Definition 1.5:</u> *A closed and bounded subset ‘S’ of* $\mathbb{R}^2$ *is said to be self-similar if it can be expressed as the union of non-overlapping sets that are congruent to S scaled by the same factor ‘s’ (0<s<1).*

$$S = S_1 \cup S_2 \cup \ldots \cup S_k$$

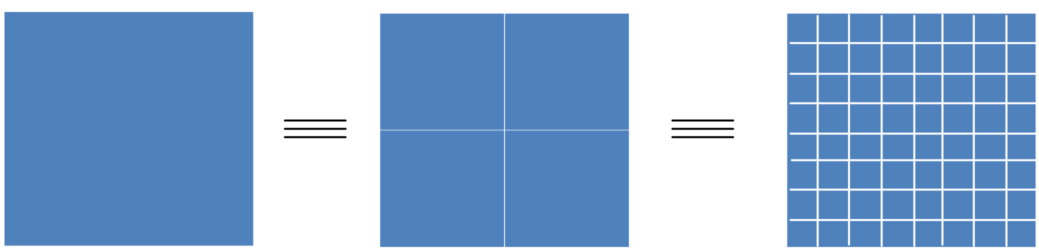
Line segment

Take a line segment of length ‘l’. This can be made of 2 line segments of length ‘1/2 l’ or 3 line segments of length ‘1/3l’ or N line segments of length ‘1/N’.

$$k = N, s = \frac{1}{N}$$

Square

Take a square of side ‘a’. This square could be made of four smaller squares of side ‘1/2 a’ or 16 squares of side ‘1/4 a’ or a grid of N squares of side ‘$1/N^2$.



$$k = N$$

$$s = \frac{1}{N^2}$$

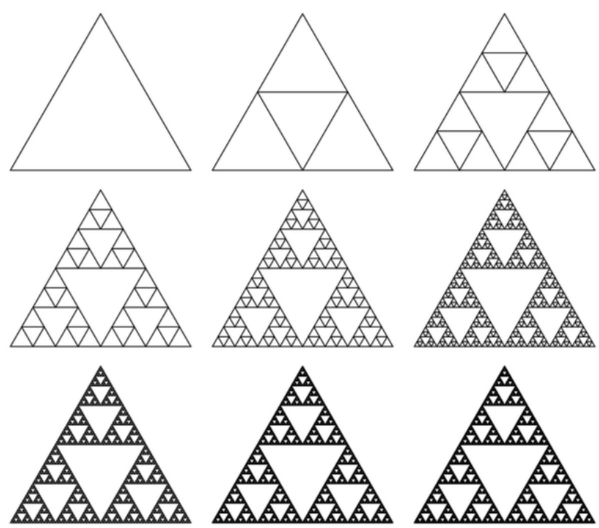
Sierpinski Triangle

The Sierpinski Triangle is a fractal that is taking a triangle and drawing a triangle inside it, and then drawing a triangle in the three outer triangles and doing the same for the next scales of outer triangles (as seen in the diagram on the right).

$$k = 3N$$

$$s = \frac{1}{2N}$$

Figure(right) Construction of Sierpinski Triangle[2]



[2] http://www.oftenpaper.net/sierpinski.htm

2. Topology

Topology is often called rubber sheet geometry and is seen to be about geometrical structures that can be reshaped by bending and stretching but retain the same properties. In essence, topology is the study of the limit-point concept.[3] It is an abstract field that arises from the how dependent many fields of mathematics are on limit points, connectedness and other associated properties. In this report, we shall be covered certain topics under topology, such as topological spaces and dimensions.

2.1. A "Topology"

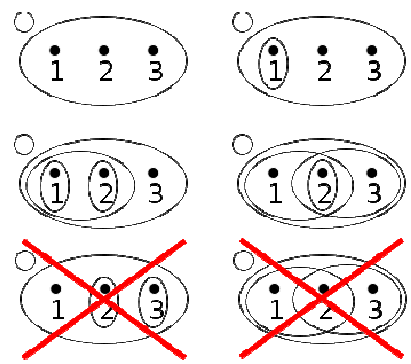
Definition 2.1.1: *A topology on a set X is a collection T of subsets of X that obey the following rules:*

i. $\{\emptyset \cap X\} \in T$ *[The null set and X belong to T]*

ii. $\forall\, U_1, U_2, \ldots, U_n \in T$, $\left(\bigcup_{i=1}^{k\leq n} U_i\right) \in T$ *[Any union of elements of T belongs to T.]*

iii. $\forall\, U_1, U_2, \ldots, U_n \in T$, $\left(\bigcap_{i=1}^{k\leq n} U_i\right) \in T$ *[Any intersections of elements of T belongs to T.]*

Definition 2.1.2: *The elements of T are called open sets.*



Example: Take a set $X = \{1,2,3\}$

Along with some collections: $A = \{\emptyset, \{1,2,3\}\}$

$B = \{\emptyset, \{1,2,3\}, \{1\}\}$

$C = \{\emptyset, \{1,2,3\}, \{1,2\}, \{1\}, \{2\}\}$

$D = \{\emptyset, \{1,2,3\}, \{2\}, \{1,2\}, \{2,3\}\}$

$E = \{\emptyset, \{1,2,3\}, \{2\}, \{3\}\}$

$F = \{\emptyset, \{1,2,3\}, \{1,2\}, \{2,3\}\}$

Looking at Definition 2.1.1, condition *i* is satisfied for all the above, condition *ii* is satisfied for A, B, C, D and F, and condition *iii* is satisfied by for A, B, C, D and E. Hence, only A, B, C and D are considered topologies for *X* as shown on right[4]

2.2. Topological space

Definition 2.1.1: *A topological space is an ordered pair of the set and its topology: (X, T).*

Example: Sierpinksi Space is a topological space S = (X_S, T_S) where X_S= {0, 1} and T_S = {∅,{0},{0,1}}.

2.3. Cover

Definition 2.3: *A cover C of a topological space Y is a collection of the subsets U_i of Y, where the union of these subsets is the whole space Y.*

2.3.1. Open Cover

Definition 2.3.1: *A cover C of a topological space Y is said to be an open cover if each of its members is contained in the topology of Y.*

2.3.2. Refinement

Definition 2.3.2: *A refinement of a cover C of a topological space Y is a cover of Y such that every member of D is subset of some member of C.*

[3] (Hocking and Young)

[4] By Dcoetzee - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=4114048

3. Transformations

If T is a transformation such that $T: Q \rightarrow T(Q)$, then $T(Q) = M_T Q$ where M_T is the transformation matrix of transformation T.

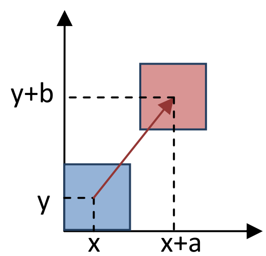
3.1. Symmetry

<u>Definition 3.1:</u> *If a transformation 'T' leaves a pattern invariant, T is a symmetry.*

Three basic symmetries:

1. Translation 'T'

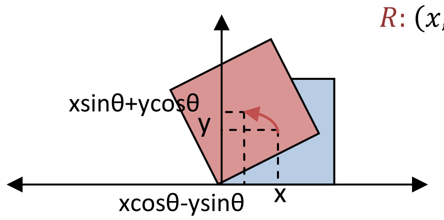
$$T: (x, y) \rightarrow (x + a, y + b)$$



$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{y} \\ \frac{x}{b} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

2. Rotation 'R'

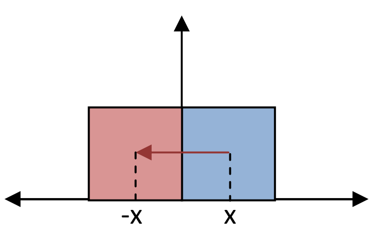
$$R: (x, y) \rightarrow (xcos\theta - ysin\theta, xsin\theta + ycosd\theta)$$



$$R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

3. Reflection 'S'

$$S: (x, y) \rightarrow (-x, y)$$



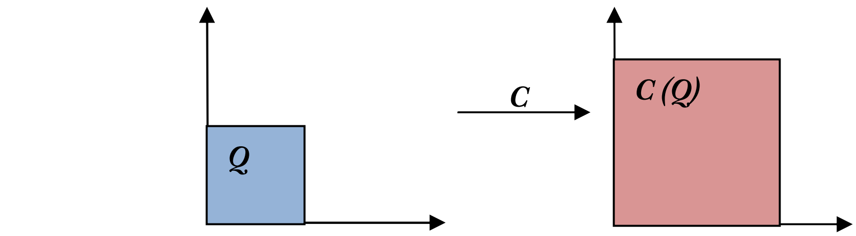
$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

3.1. Scaling (Contraction)

Scaling '$C(Q)$' is a linear transformation, where a set 'Q' is scaled by a factor 's'.

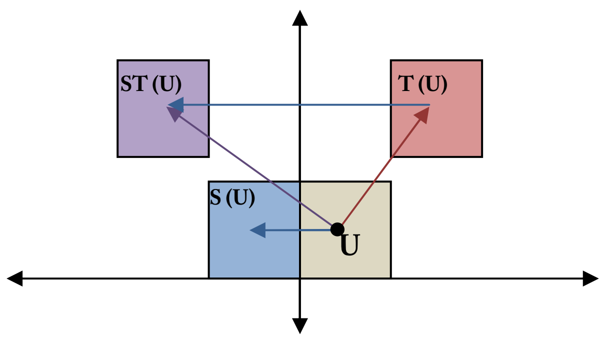
$$C: (x, y) \rightarrow (sx, sy)$$

$$C\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$



3.2. Composition

Definition 3.2:*A composition of transformations '**T**' and '**S**' is '**ST**', when '**T**' is performed first and then '**S**' is performed. '**TS**' is a composition when S is performed first and then '**T**' is performed.*



$$ST: U \to S\big(T(U)\big) = ST(U)$$

$$ST(U) = M_S M_T U$$

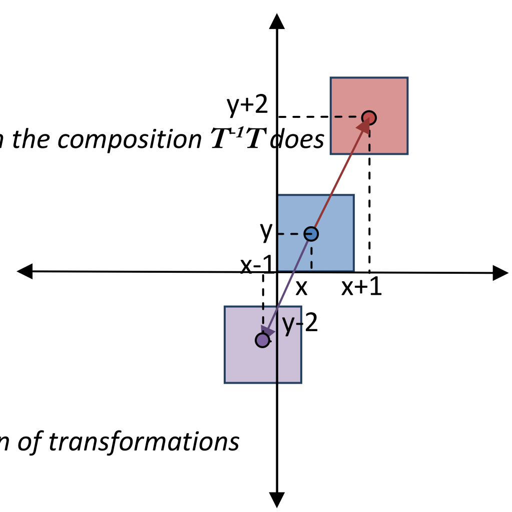
$$M_{ST} = M_S M_T$$

Inverse of a transformation

Definition 3.2.1:***T*$^{-1}$** *is the inverse of a transformation when the composition **T^{-1}T** does nothing, i.e, **T^{-1}T=I***

Example: If $T: \mathbb{R}_2 \to \mathbb{R}_2 := \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x+1 \\ y+2 \end{pmatrix}$,

then $T^{-1}: \mathbb{R}_2 \to \mathbb{R}_2 := \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}$



3.3. Groups of transformations

Definition 3.3:*A group of transformations '**G**' is a collection of transformations such that:*

i) *If **S**, **T** ∈ **G**, then **ST** ∈ **G**.*

ii) *If **S** ∈ **G**, then **S^{-1}** ∈ **G**.*

4. Dimensions

4.1. Topological Dimension

A common fairly intuitive idea of a dimension is as follows:

Definition 4.1: *A set is of zero dimension if for any point $p \in X$ if there are arbitrary small neighbourhoods of p whose boundary is empty. A set is of dimension n if there are arbitrarily small neighbourhoods of any point p whose boundary is of dimension $\leq n - 1$.* [5]

Set of rational numbers in real numbers

The real number line is continuous: made of irrational and rational numbers. There are neighbourhoods for every rational number where irrational numbers exist. This creates gaps in the rational number set. Hence, the dimension of the rational number set is zero. Similarly, the set of irrational numbers is of dimension zero.

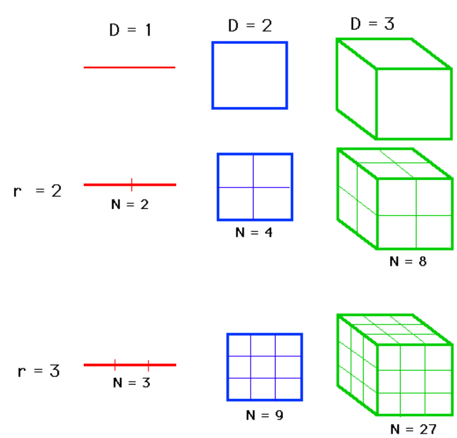
Talking about dimensions in this way implies that sets with a dimension 'n' is made of proper sub-sets of dimension 'n-1'. The real number line is of dimension one and is made of points of dimension zero. This is a little odd to consider and think about. Therefore, we could define a topological dimension in a different way.

4.1.1. Lebsque covering dimension

Definition 4.1.1: *Lebsque covering dimension or topological dimension of a topological space is defined to be the minimum value of n, such that any open cover has a refinement in which no point is included in more than n+ 1 elements.* (Kohavi and Davdovich)

The topological dimension of some object can be taken as the sum of the dimensions of its underlying boundaries. For example, a pyramid has surface of dimension '2', the surface has line boundaries of dimension '1' and these lines have a '0' dimensional point boundaries; hence, a pyramid is (2+1+0=) 3 dimensional.

4.2. Hausdorff-Besicovich Dimension



$N = r^D$

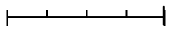
The Hausdorff-Besicovich Dimension is best understood through an example. Consider a line segment that is split equally into N parts. The scaling factor used to scale each of these parts back to the original length of the line segment is called r. Therefore, $N=r^d$ where d is the Hausdorff-Besicovich Dimension.[6] This can be rewritten as the following: D = log (N) / log (r). The advantage of the Hausdorff-Besicovich

[5] Benjamin A. Steinhurst. *"Notions of Dimension"* 2010.

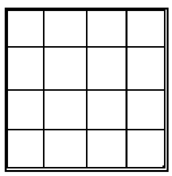
[6] The more abstract accurate and mathematical definition of the Hausdorff-Bescovich dimension was not understand

Dimension over the Lebsque covering dimension is that there is scope for dimensional numbers that are not integers.

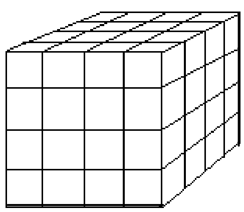
Similarly, you could have a line divides into 4 equal parts, each being 1/4 th of the original length.

 $d = \frac{log(4)}{log(4)} = 1$

A square can be broken into smaller pieces. Each of which is 1/4th the size of the original. In this case it takes 16 of the smaller pieces to create the original.

 $d = \frac{log(16)}{log(4)} = 2$

A cube is can also broken down into smaller cubes, each 1/4 the size of the original. It takes 64 of these smaller cubes to create the original cube

 $d = \frac{log(64)}{log(4)} = 3$

Now, with an understanding of the Hausdorff-Besicovich Dimension, we can properly define a fractal.

5. Fractals

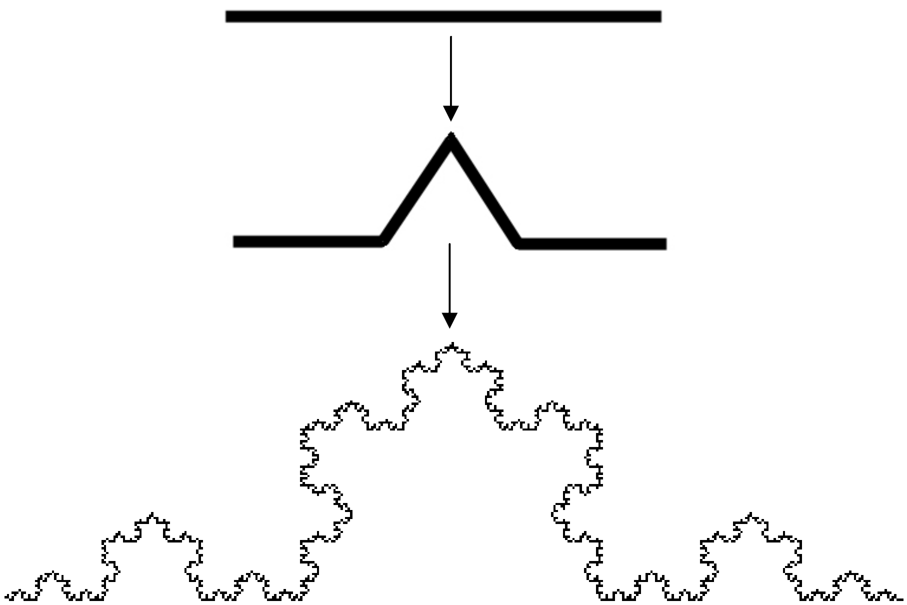
5.1. Introduction and definition

Fractals are mysterious objects that are self-similar at different scales. They are objects that can have an infinite perimeter but cover a finite amount of space.

Definition 5.1.: *A fractal is defined as any object with a Hausdorff-Besicovich dimension number that is strictly greater than its topological dimension.*

An example will be given in the next section to fully understand how a fractional dimension may arise.

Koch Curve



Here, the line is clearly being divided into 4 equal parts. Each of these parts is 1/3 times the original size of the line.

Therefore, $d_H = \frac{\log(4)}{\log(3)} = 1.2618\ldots$

5.2. Area versus the perimeter

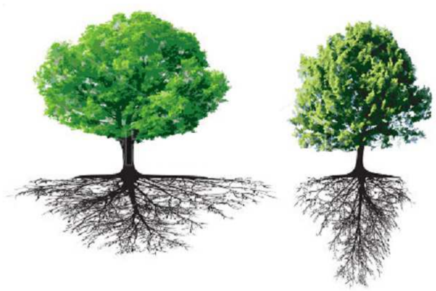
Let us consider the Sierpinski triangle. The construction of the Sierpinski triangle was shown a few pages before. For simplicity sake, let the initial area enclosed by the triangle be equal to 1. We take the triangle and draw a triangle inside it. We now remove this triangle from the original triangle. If the initial area was 1 unit, the area at the first iteration is 3/4 times the original area. Similarly, if the process is repeated n times, the area of the triangle becomes $(3/4)^n$. Now, if 'n' is very large, the area of the triangle goes to 0.
The perimeter of the triangle grows with every iteration, so as the area goes to 0, the perimeter goes to infinity.

The coastline paradox

As mentioned above, the length measures of a fractal increases with the number of iterations. The same idea can be used for borders and the coast. The idea was first recorded by Lewis Fry Richardson in the 1950s. The smaller a scale used to measure the border/coast, the larger the perimeter the fractal is going to be. You can see why in the image on the right.

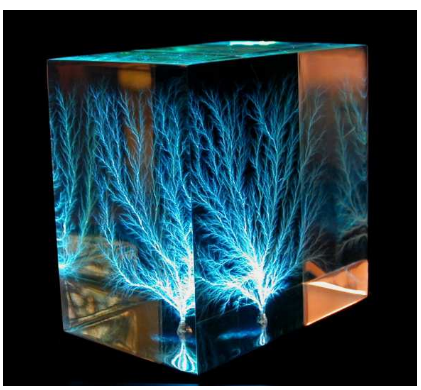
5.3. Real life examples

Tree Roots



Fractals can be found in nature by simply uprooting a small sapling or plant. Roots of plants are self-similar and satisfy other properties required to term something as a fractal. A paper[7] published in the Annals of Botany in 1989 found the fractal dimension for a few root systems. The dimensions they found were around 1.48-1.56.

3D Lichtenberg figures or Electrical treeing



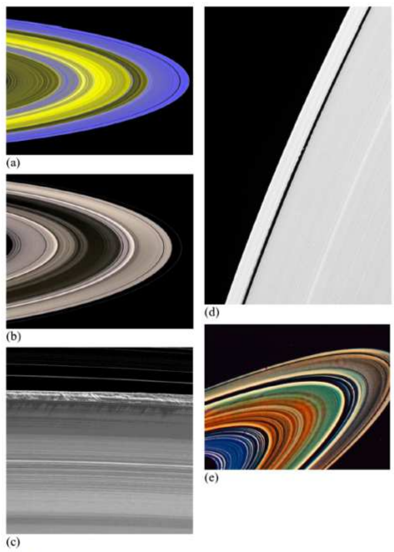
Sometimes, electrical discharges appear on the surface or in the interior of insulating materials. These are called Lichtenberg figures. As shown above, treeing is a 3 dimensional Lichtenberg figure that is a pre-breakdown phenomenon in solid insulation.[8] When humans are struck by lightning, these figures often occur on their skin (figure on right[9])! The self-similar patterns in the branching are what gives these figures its fractal properties.

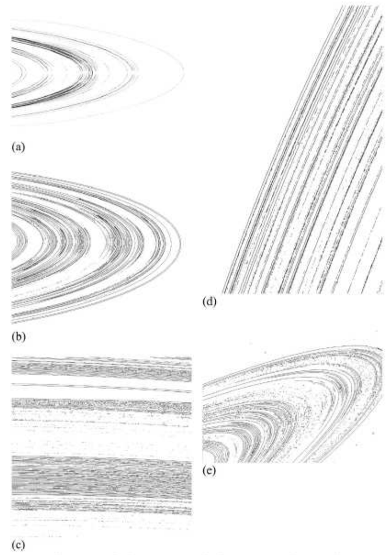
[7] Tatsumi, J., Yamauchi, A., & Kono, Y. (1989). Fractal Analysis of Plant Root Systems. Annals of Botany, 64(5), 499-503. doi:10.1093/oxfordjournals.aob.a087871

[8] https://en.wikipedia.org/wiki/Electrical_treeing

[9] https://piximus.net/others/lichtenberg-figure-human-skin-struck-by-lightning

Edges of Saturn's rings





(a,b,c,d,e): The original images of the Cassini and Voyager missions.

d,e): Images processed, respectively, from Figure 1 (a,b,c,d,e) to capture the ring edges.

With the images taken by the Casini and Voyager missions in the last two decades, Jun Li and Martin Ostoja-Starzewski analysed the edges of the rings of Saturn to find its fractal dimensions in the range 1.63 to 1.78.[10]

5.4. Affine Transformations

The concepts explained at the start of this document can now be used to explain the creation of a fractal. The rotation, translation, and scaling transformation can be combined as an affine transformation to produce fractals.

Definition 5.4.: *An affine transformation of* $\mathbb{R}^n$ *is a map* $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ *of the form* $T(x,y) = A(x,y) + c$ *for all* $(x,y) \in \mathbb{R}^n$ *, where A is a linear transformation on* $\mathbb{R}^n$*.*

Affine transformations preserve co-linearity, meaning points that were on a line before the transformation remain on the line after the transformation. Fractals like the Koch curve fractal (showed earlier) and the Barnsley fern (picture below) can be generated on the computer with the help of affine transformations.

Figure 1: Barnsley fern

[10] Li and Ostoja-Starzewski SpringerPlus (2015) 4:158; DOI 10.1186/s40064-015-0926-6

The following are the transformations required to create a Koch curve fractal:

$$T1: \begin{bmatrix} \frac{1}{3}\cos 0 & -\frac{1}{3}\sin 0 \\ \frac{1}{3}\sin 0 & \frac{1}{3}\cos 0 \end{bmatrix} * \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad T2: \begin{bmatrix} \frac{1}{3}\cos(60) & -\frac{1}{3}\sin(60) \\ \frac{1}{3}\sin(60) & \frac{1}{3}\cos(60) \end{bmatrix} * \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$$

$$T3: \begin{bmatrix} \frac{1}{3}\cos(-60) & -\frac{1}{3}\sin(-60) \\ \frac{1}{3}\sin(-60) & \frac{1}{3}\cos(-60) \end{bmatrix} * \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 1/2 \\ \sqrt{3}/6 \end{bmatrix} \qquad T4: \begin{bmatrix} \frac{1}{3}\cos 0 & -\frac{1}{3}\sin 0 \\ \frac{1}{3}\sin 0 & \frac{1}{3}\cos 0 \end{bmatrix} * \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$$

5.5. Generation of fractals

Fractals can be generated on the computer with the help of programming softwares. A web-site (https://code.activestate.com/recipes/langs/python/tags/fractal/) provides the code required to produce various fractals through python. Python is preferred as it is open-source and can easily be procured.

5.5.1. Generating self-similar fractals with Context Free Art

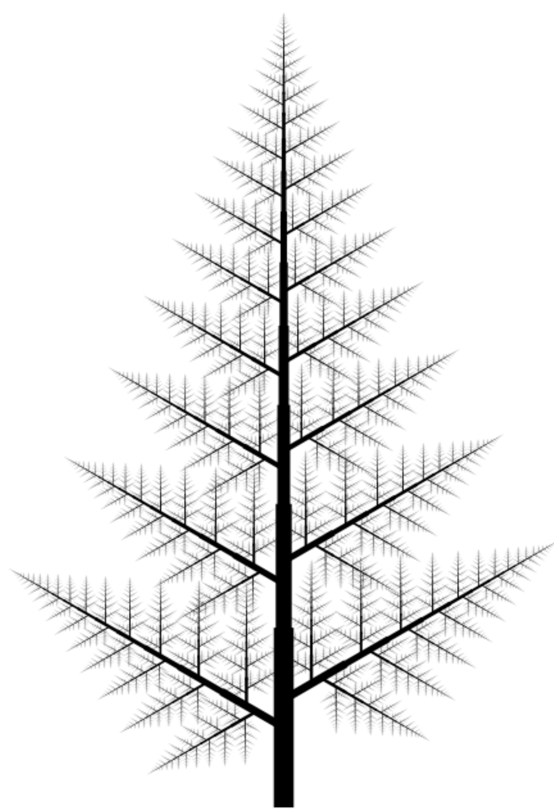
Context Free Art is an open-source software that uses instructions called 'grammar' to generate images. The following sections are about how some fractals were generated with this software.

Tree 101

```
startshape tree
rule main {
  tree { }
}
rule tree {
  line { }
  tree { y 0.8 s 0.8}
  tree { y 0.4 s 0.4 r 60}
  tree { y 0.54 s 0.4 r -60}
}

path line {
  MOVETO { x 0 y 0 }
  LINETO { y 0.9 }
  STROKE { width 0.1 }
}
```

This program makes a shape and then performs transformations on it iteratively to form the fractal on the right.



How it works:

1. ```
   path line {
       MOVETO { x 0 y 0 }
       LINETO { y 0.9 }
       STROKE { width 0.1 }
   }
   ```
   This makes the shape, a rectangle.

2. `tree { y 0.8 s 0.8}` $=T_1$

   This transformation moves the object 0.8 units in the y-direction and scales it by a factor of 0.8.

   $$T_1: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0.8 & 0 \\ 0 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0.8 \end{pmatrix}$$

3. `tree { y 0.4 s 0.4 r 60}`

   This transformation moves it 0.4 units in the y-direction, scales it by 0.4 and rotates it by 60°.

   First $T_2: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0.4 & 0 \\ 0 & 0.4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0.4 \end{pmatrix}$ and then, $R_1: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

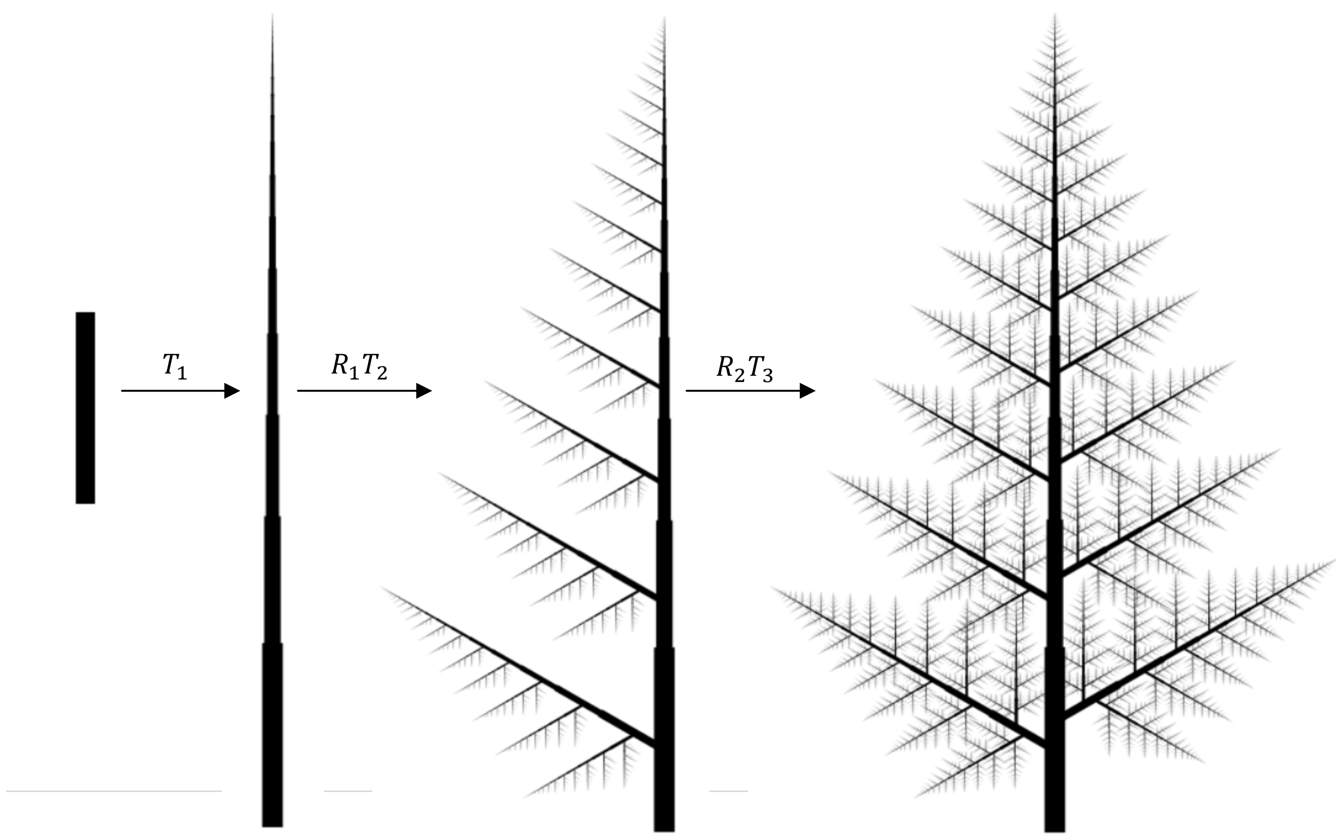
   $$R_1T_2: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0.2 & -0.2\sqrt{3} \\ 0.2\sqrt{3} & 0.2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.2\sqrt{3} \\ 0.2 \end{pmatrix}$$

4. `tree { y 0.54 s 0.4 r -60}`

   This transformation moves it 0.54 units in the y-direction, scales it by 0.4 and rotates it by -60°.

   First $T_3: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0.4 & 0 \\ 0 & 0.4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0.54 \end{pmatrix}$ and then, $R_2: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos 60° & \sin 60° \\ -\sin 60° & \cos 60° \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

   $$R_2T_3: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0.2 & 0.2\sqrt{3} \\ -0.2\sqrt{3} & 0.2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -0.2\sqrt{3} \\ 0.2 \end{pmatrix}$$


   ```

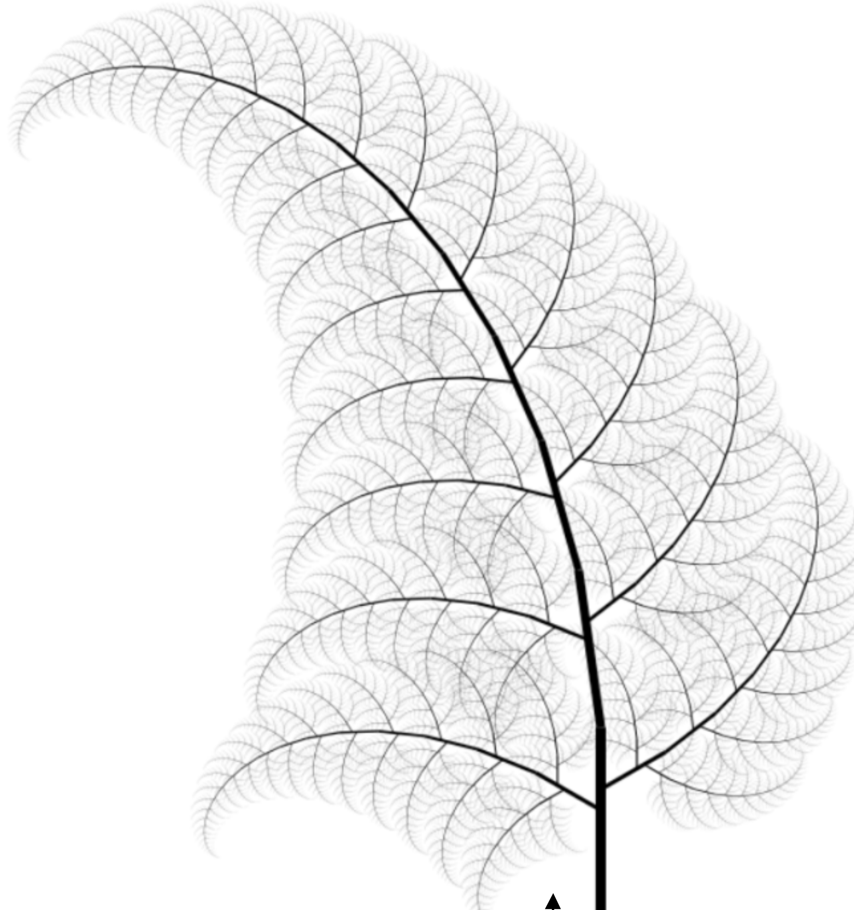
Fern 101

```
startshape tree
rule main {
  fern { }
}
rule fern {
  line { }
  fern { y 0.9 s 0.85 r 8 }
  fern { y 0.5 s 0.4  r 60}
  fern { y 0.6 s 0.4 r -60}

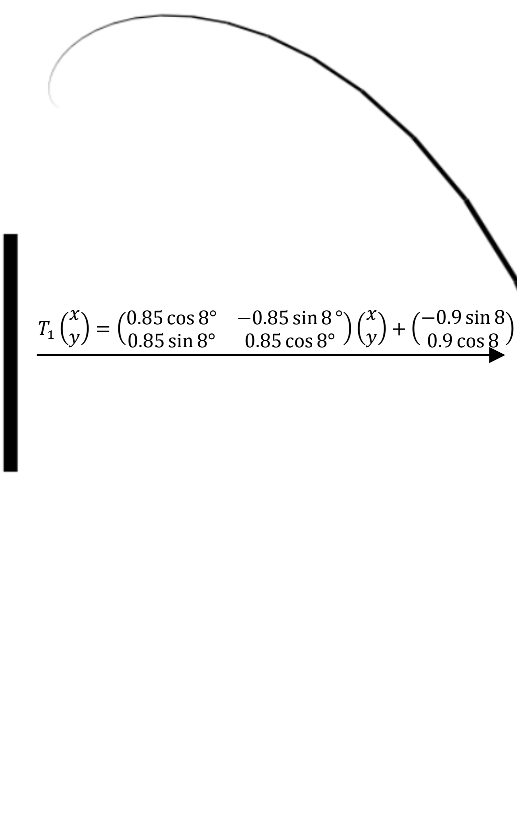
}

path line {
  MOVETO { x 0 y 0 }
  LINETO { y 0.9 }
  STROKE { width 0.05 }
}
```

This program performs iterative transformations on a rectangle to give the fractal on the right.

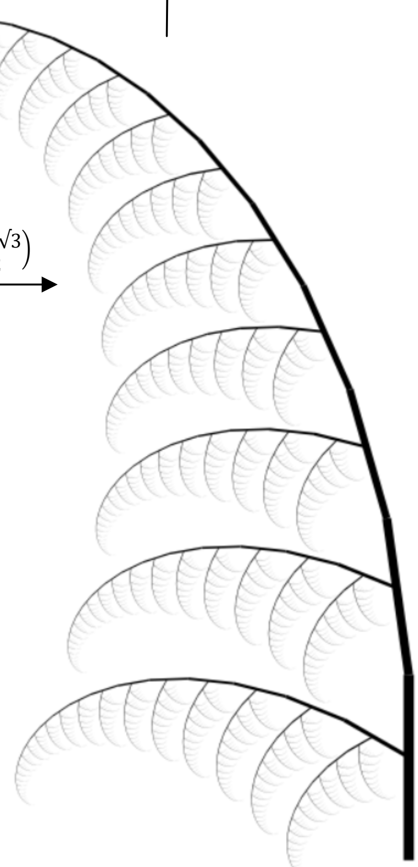


$$T_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0.2 & 0.2\sqrt{3}\\ -0.2\sqrt{3} & 0.2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0.2\sqrt{3}\\0.2\end{pmatrix}$$



$$T_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0.85\cos 8^\circ & -0.85\sin 8^\circ\\ 0.85\sin 8^\circ & 0.85\cos 8^\circ\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-0.9\sin 8\\0.9\cos 8\end{pmatrix}$$

$$T_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0.2 & -0.2\sqrt{3}\\ 0.2\sqrt{3} & 0.2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-0.2\sqrt{3}\\0.2\end{pmatrix}$$



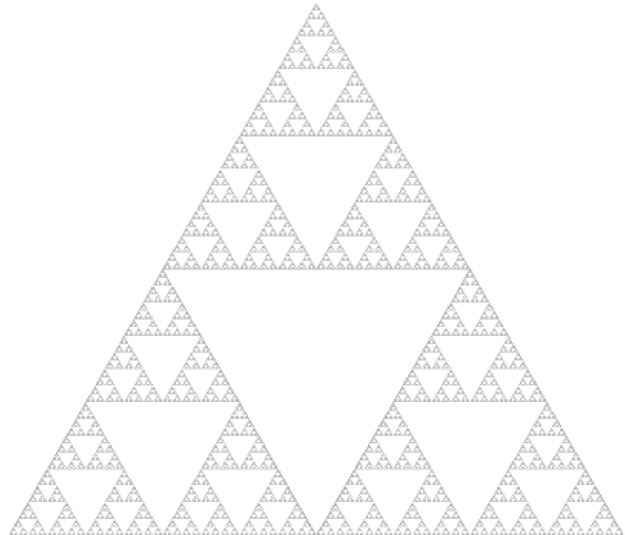
Sierpinski Triangle

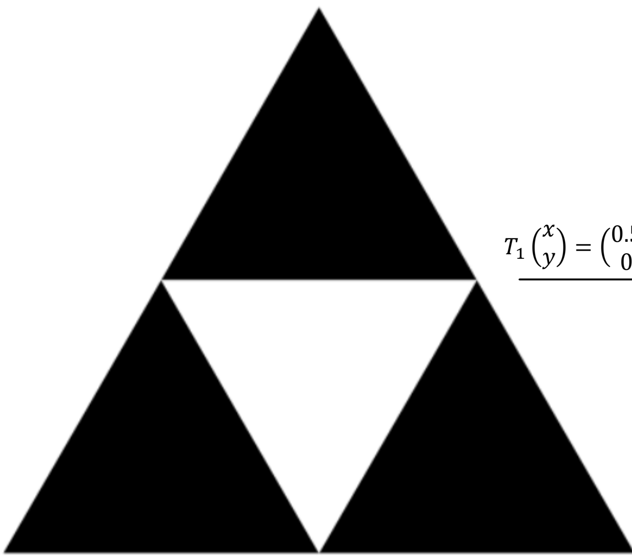
```
startshape SierpinkskiTriangle

rule SierpinkskiTriangle{
  TRIANGLE { }
  SierTri{ }
}

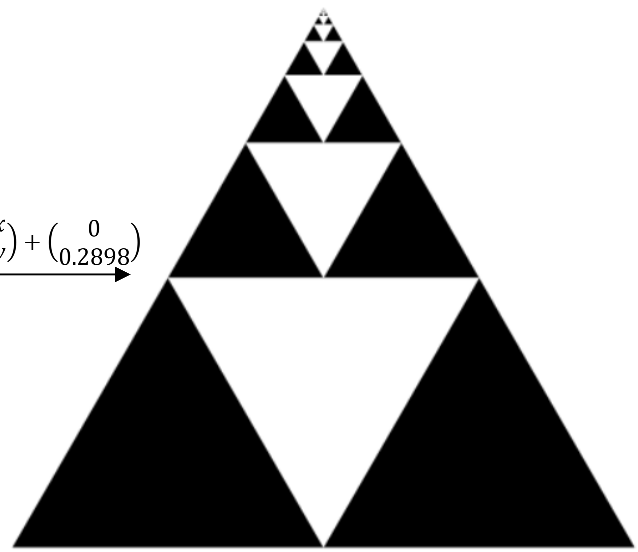
rule SierTri{
  TRIANGLE { b 1 s 0.5 r 180}
  SierTri {s 0.5 y 0.2898}
  SierTri {s 0.5 x 0.2508 y -0.145}
  SierTri {s 0.5 x -0.2508 y -0.145}
}
```

The above program generates the Sierpinski Triangle on the right, by making upside down triangular holes that were scaled by a factor of 0.5 iteratively in different positions.

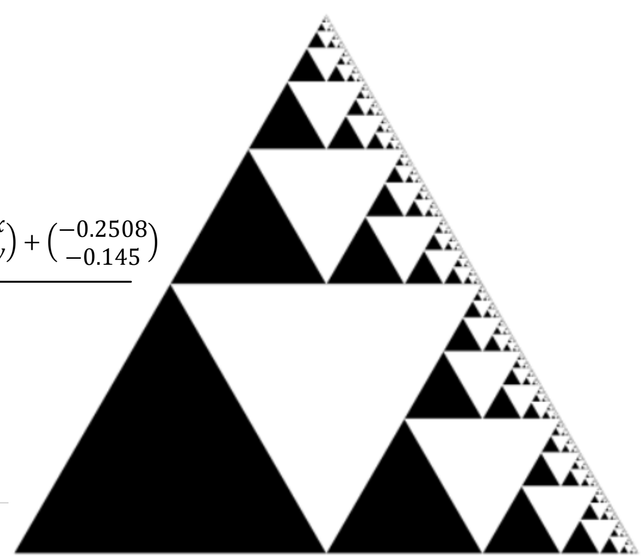




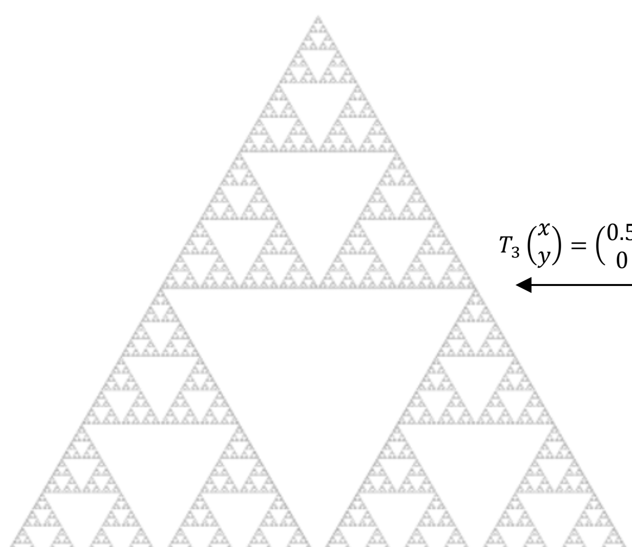
$$T_1\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0.5 & 0\\0 & 0.5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0\\0.2898\end{pmatrix}$$



$$T_2\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0.5 & 0\\0 & 0.5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}0.2508\\-0.145\end{pmatrix}$$



$$T_3\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0.5 & 0\\0 & 0.5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}-0.2508\\-0.145\end{pmatrix}$$



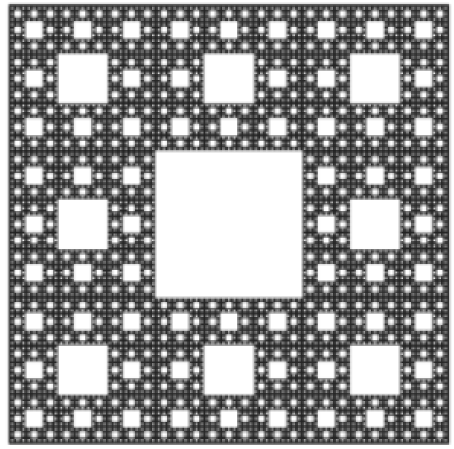
Sierpinski Carpet

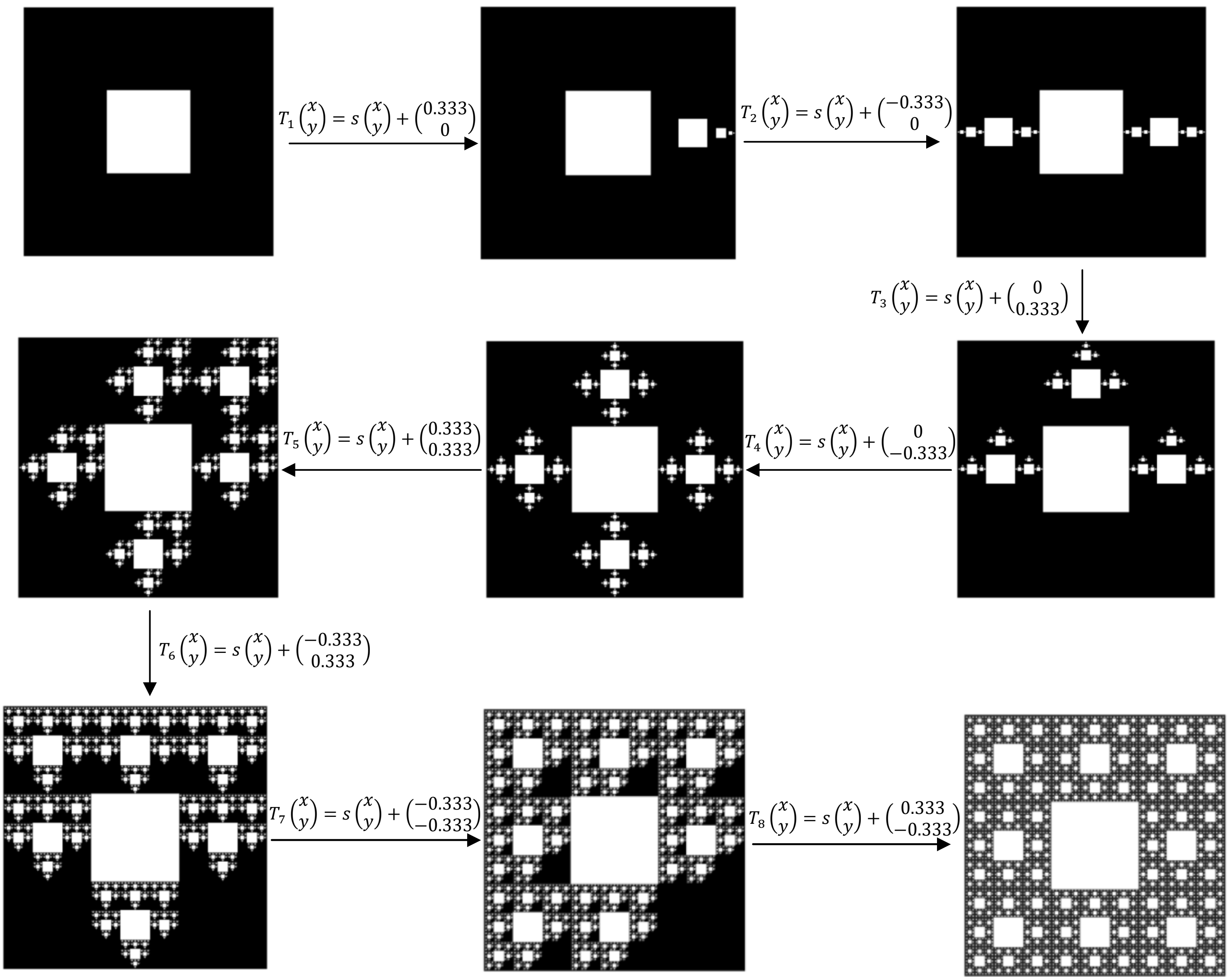
```
startshape SierpinkskiCarpet

rule SierpinkskiCarpet{
   SQUARE [ ]
   SierCar { }
}

rule SierCar {
   SQUARE [ size 0.333 b 1]
   SierCar {x 0.333 size 0.333}
   SierCar {x -0.333 size 0.333}
   SierCar {y 0.333 size 0.333}
   SierCar {y -0.333 size 0.333}
   SierCar {x 0.333 y 0.333 size 0.333}
   SierCar {x -0.333 y 0.333 size 0.333}
   SierCar {x -0.333 y -0.333 size 0.333}
   SierCar {x 0.333 y -0.333 size 0.333}
}
```

Scaling matrix: $s = \begin{pmatrix} 0.333 & 0 \\ 0 & 0.333 \end{pmatrix}$





5.5.2. Iterative programs in computer graphics

Suppose you were playing a game where your character is moving through a forest; for this forest to look more realistic to you, the trees and plants have to look similar but not exactly the same. The more detailed it looked, the more appealing it may be as well. To generate such images, fractal generating programs can be used with probabilistic rules. There could be a rule that the tree would have a 50% chance of having a branch at a certain distance. Trees generated like this may look different and would be generated using only one code. With multiple rules, entire landscapes can be filled with different trees. Similarly, other patterns (such as runes, brick designs and fishes) can be randomly generated.

A Simple Example

```
startshape PROB

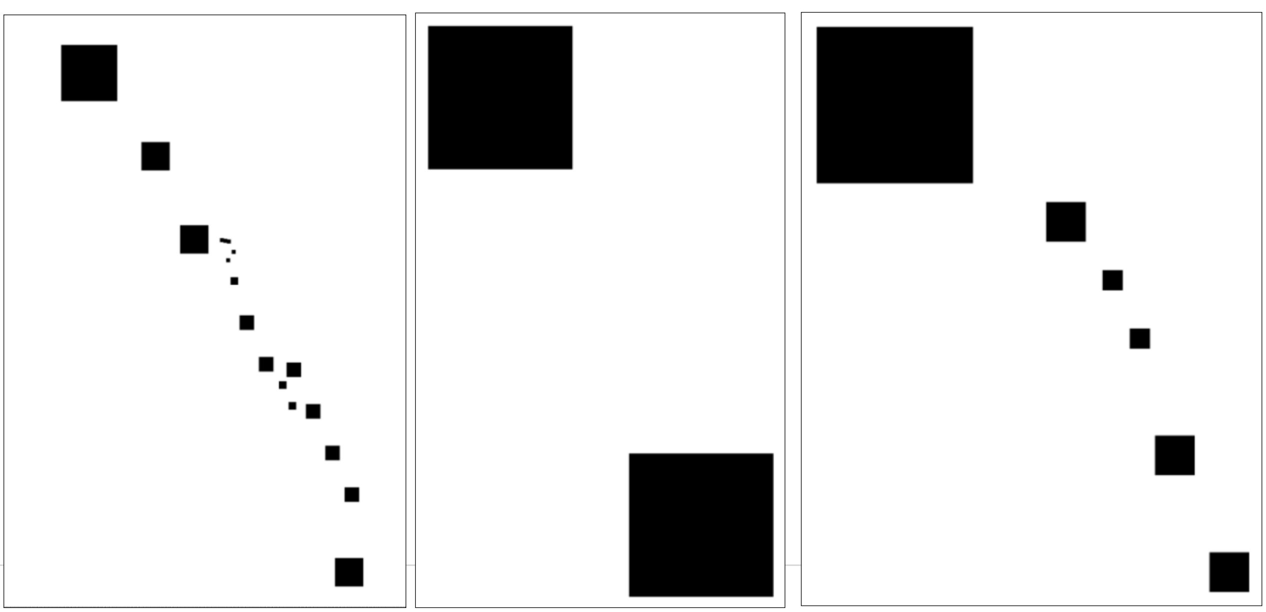
shape PROB
{
STUFF [ x 01]
STUFF [x -0.4 y 3]
}

shape STUFF
rule {
SQUARE[ ]
}
rule 0.5{
PROB [ size 0.5 ]
}
rule 0.1{
PROB [x 0 y -0.2]
}
```

The algorithm of the above program:

1. Makes a square at (0,0)
2. Makes a square at (1, 0) and (-0.4, 3)
3. Makes a square at (0, -0.2), probability of this occurring is 0.1.
4. The squares above have a probability of 0.5 of being 0.5 the size of the original square.

The following random outp1uts were rendered on the first three attempts. There are infinitely many possible patterns that can emerge from this program.



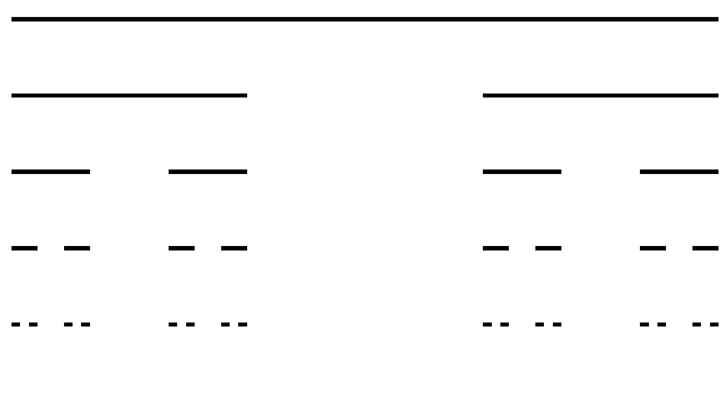
5.6. Some interesting fractals

5.6.1. The cantor set

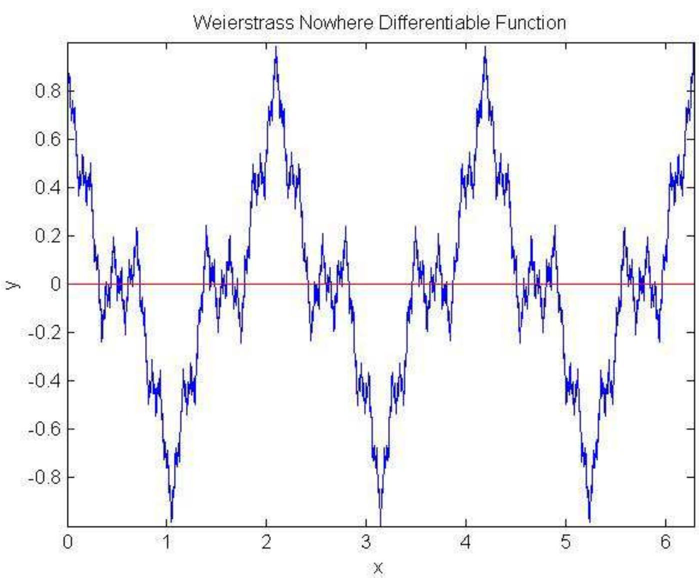
The cantor set was discovered by Georg Cantor in 1883 and is very easy to visualise. The fractal dimension of the cantor set, $d = \frac{log2}{log3} = 0.631 \ldots$

Construction:

Start with a line segment of length 1 unit. Cut the line into 3 equal parts. Delete the middle third of the line. Repeat the step an infinite number of times. Notice that any part of the set is exactly the same as any other part, either scaled up of scaled down. The picture below show the process repeated a few times.



5.6.2. Weierstrass Function



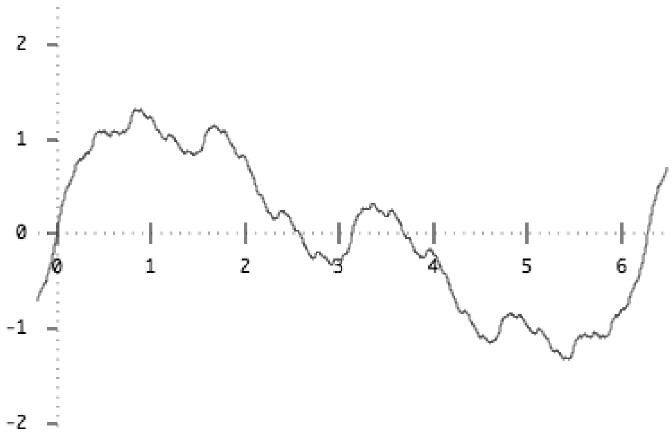
The function on the left[11] has the interesting property of being continuous every ware but differentiable no ware. The graph can be produced with a function that is a sum of sine functions. This is the idea of the Fourier series, which all periodic functions can be written as a sum of different sine functions. The Fourier series can be extended to all functions by saying their period is infinite.

The function that was originally given by Weierstrass in his paper was the following:

$f(x) = \sum a^n \sin(b^n \pi x)$, where $a \in (0,1), b \in \{\pm(2n-1), n \in \mathbb{N}\}$ and $ab > 1 + \frac{3\pi}{2}$

[11] https://www.math.ucdavis.edu/~hunter/m201b/m201b_pics.html

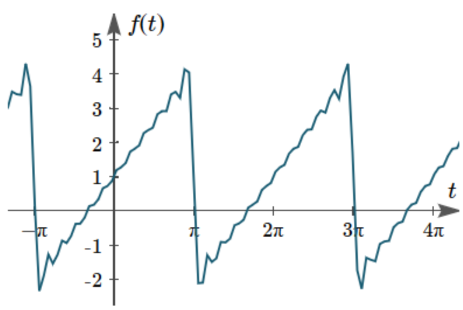
Another Weierstrass function is shown below[12] for different values of a and b



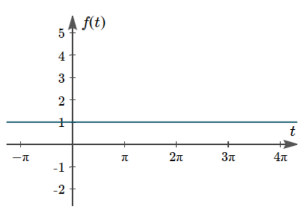
5.6.2.1. *Fourier series*

The Fourier series is a way of representing periodic functions as a sum of sine and cosine functions.

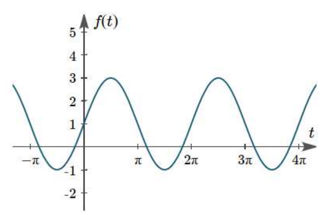
Saw-tooth Signal[13]



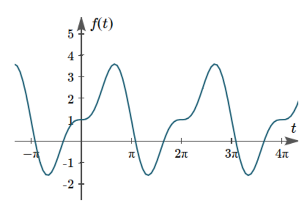
Fourier series approximation of a saw-tooth signal $f(t) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \cdots$



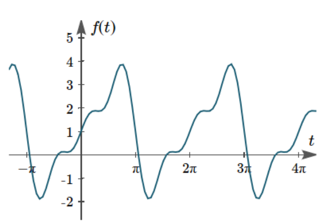
$f(t) = 1$ (first term of the series)



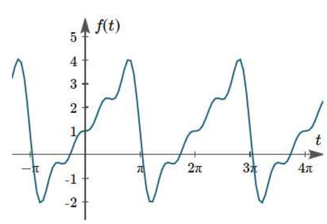
$f(t) = 1 + 2\ \sin t$ (first 2 terms of the series)



$f(t) = 1 + 2\sin t - \sin 2t$ (first 3 terms of the series)



$f(t) = 1 + 2\ \sin t - \sin 2t + \frac{2}{3}\sin 3t$ (first 4 terms)



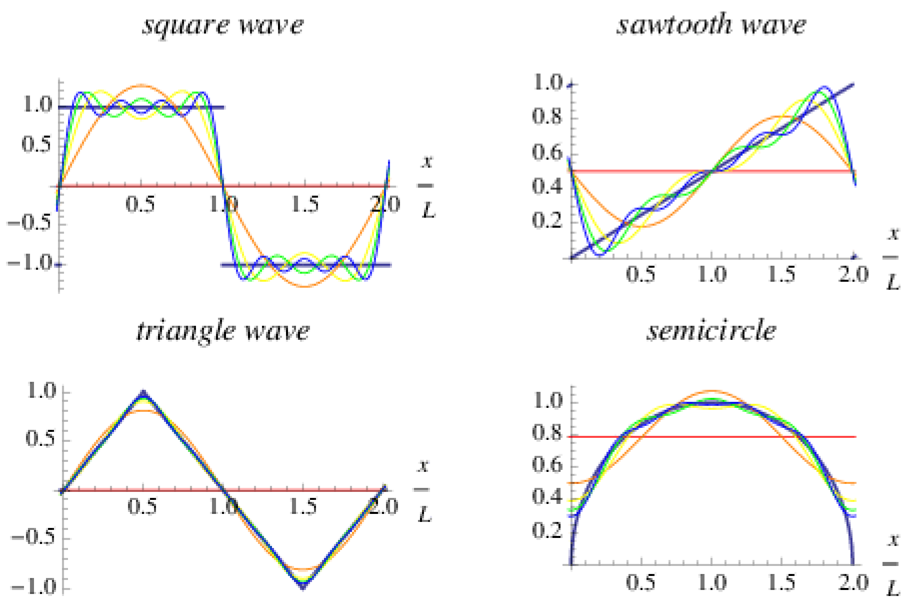
$f(t) = 1 + 2\ \sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t$ (first 5 terms)

As you can see, the infinite series would converge to the Saw tooth curve.

---

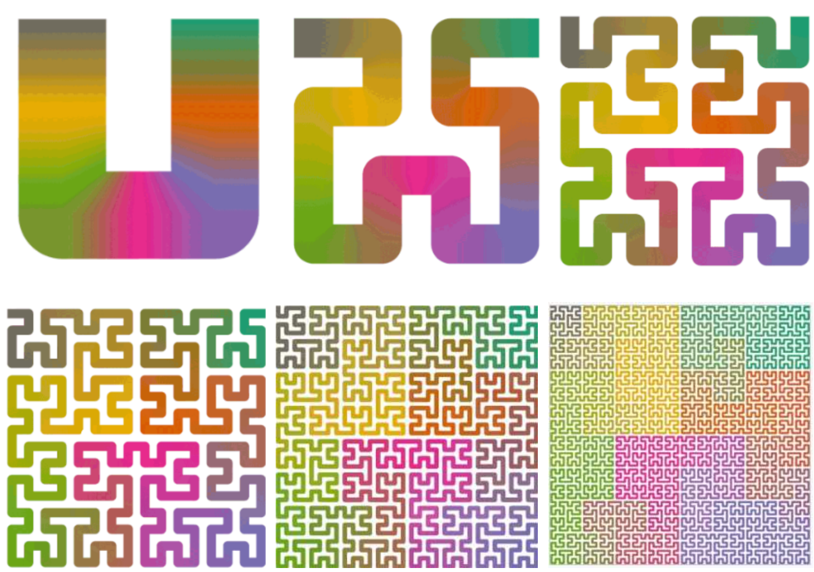
[12] https://sites.math.washington.edu/~conroy/general/weierstrass/weier.htm
[13] https://www.intmath.com/fourier-series/1-overview.php

The following picture[14] shows how curves can be approximated with sine functions.



5.6.3. Hilbert curve

A Hilbert curve is a space filling curve first described by David Hilbert in 1891. It's fractal dimension is 2 ($D = \frac{log(4)}{log(2)} = 2$). The length of the curve is given by $2^n - \frac{1}{2^n}$. They are used as a compact form of data storage. The following images[15] show the sequence of iterations that will converge to the Hilbert Curve.



[14] http://mathworld.wolfram.com/FourierSeries.html

[15] By TimSauder - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=67998181

5.7. Applications of fractals

5.7.1. Computer graphics

Computer programs that produce explicit fractals or contain iterative commands are used to create images that look detailed and realistic. Refer to Iterative programs in computer graphics

5.7.2. Ecological applications of measuring habitat space

Fractal dimensions were used to understand why there seems to be so many more small individual animals in a given habitat than large animals. The authors of a paper reference a study that was done on anthropoids living in vegetation where the total surface area of the habitat increased with taking measurements at a finer scale. The investigating team found the fractal dimension of the habitat to be between 1.3 and 1.5. [16]

5.7.3. Possible application in superconductors

This is highly speculative and may not have any real implications.

This is an idea we came up with during and electromagnetism class, on the 9th of November, while discussing superconductivity. The idea requires some background in Maxwell's equations. The idea is as follows.

Superconductivity happens at very low temperatures when the conductivity $\boldsymbol{\sigma}$ of a material is infinite.

The electric field inside the conductor is $\vec{E} = \frac{\vec{J}}{\sigma} \Longrightarrow \vec{E} = \vec{0}$

Maxwell's equations:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

From Maxwell's third equation, if $\vec{E} = \vec{0}$, then the magnetic field $\vec{B}$ is a constant.
In the case of superconductors, $\vec{B} = \vec{0}$.

Now look at the last equation: if both $\vec{B}$ and $\vec{E}$ are $\vec{0}$, then current density $\vec{J} = \vec{0}$.

Current $I = J \times A$

Since $\vec{J}$ in the "meat" of the conductor is $\vec{0}$, the current inside the superconductor is 0. Therefore, the current flows only on the surface of the conductor.

We just thought that a fractal shaped conductor could be used in a future of increase the amount of charge transmitted in a given time, since many fractals have this interesting ability of having large surface areas with finite volume.

[16] Sugihara. G, May. G.M.(1990). Applications of fractals in ecology. Tree, 5. http://faculty.washington.edu/cet6/pub/Temp/CFR521e/Sugihara_May_1990.pdf

6. Mandelbrot set

6.1. Introduction

$$z_{n+1} = z_n^2 + c$$

This simple looking equation leads to some interesting geometrical (basically the figure/ graph below) and numerical patterns (our favourite constant pi exists hidden away in it, too!).

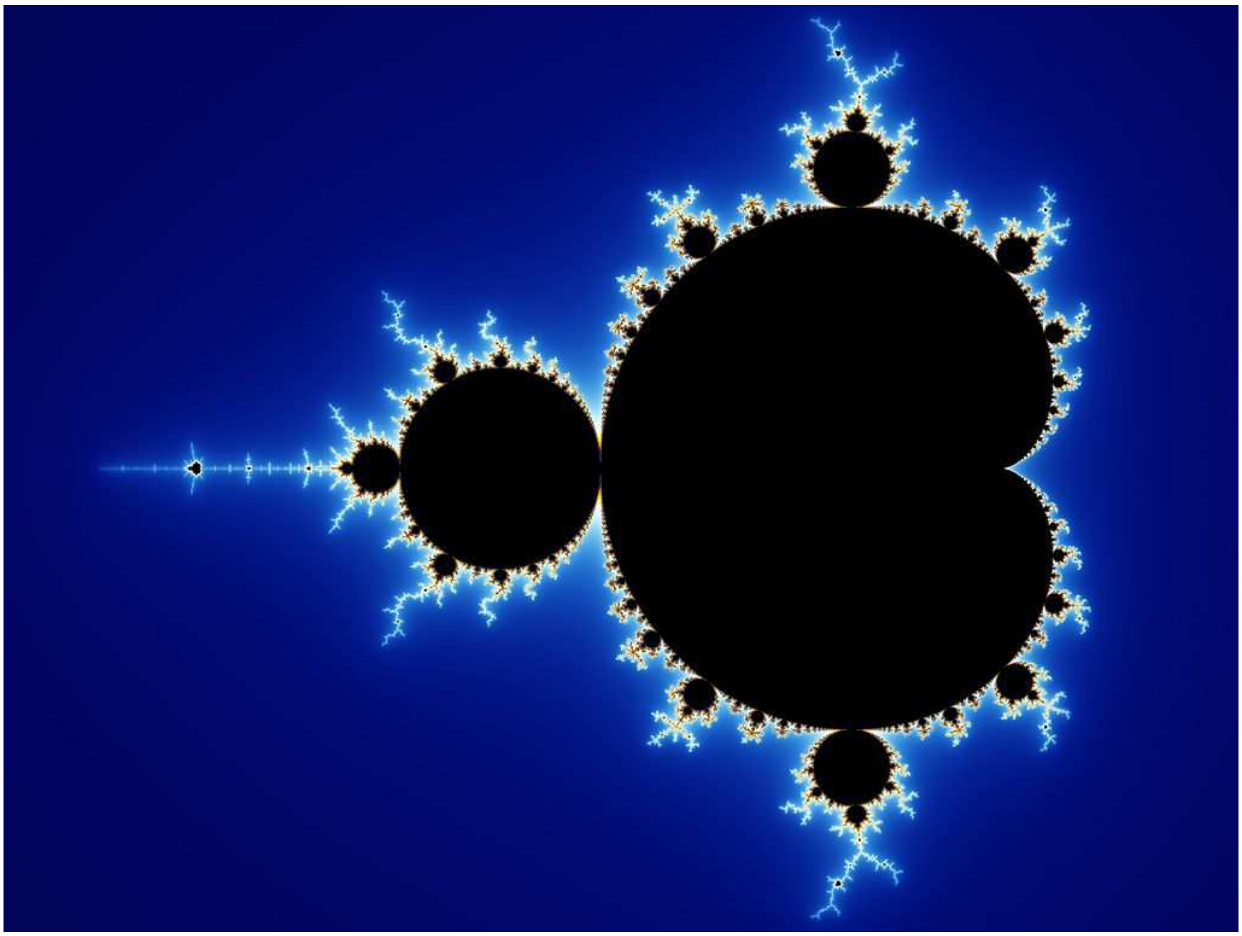


Figure 2: The Mandelbrot set (black) within a continuously colored environment[17]

The Mandelbrot set 'M' is a set of complex numbers '**c**', for which the orbit of 0, under the iteration of $\mathbf{z}^2\mathbf{+c}$, stays bounded.

The orbit of 0, under the iteration z^2+c, is the sequence of complex numbers of the form

$$z_{n+1} = z_n^2 + c$$

[17] Created by Wolfgang Beyer with the program Ultra Fractal 3. - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=321973b

where the first element 'z_0' is 0. If a transformation is defined to be $P_c: z \mapsto z^2 + c$

The $orbit\ of\ 0 = \{0, c, c^2 + c, (c^2 + c)^2 + c, \dots\} = \{0, P_c(0), P_c(P_c(0)), P_c\left(P_c(P_c(0))\right), \dots\}$

Bounded sequence

A complex sequence $\{z_n\}$ is said to be bounded if there exists $\boldsymbol{R} \in \mathbb{R}$ such that the magnitude of z_n is less than $\boldsymbol{R}$ for all values of $\boldsymbol{n} \in \mathbb{Z}$.

6.2. Elements of the Mandelbrot set

1. Let's take different values of c on the real number line and try and find out if their orbit of 0 is bounded.
   i. $c_1 = 0$

$$z_0 = 0$$
$$z_1 = {z_0}^2 + 0 = 0$$
$$z_2 = {z_1}^2 + 0 = 0$$

   And so on. This iteration leads to one value, i.e, 0. Hence, it is bounded and belongs in the Mandelbrot set.

   ii. $c_2 = 1$

$$z_0 = 0$$
$$z_1 = {z_0}^2 + 1 = 1$$
$$z_2 = {z_1}^2 + 1 = 1 + 1 = 2$$
$$z_3 = {z_2}^2 + 1 = 4 + 1 = 5$$

   And so on. This iteration can be seen to lead to increasingly higher values and will tend to infinity. Hence, it is not bounded. 1 does not belong in the Mandelbrot set.

   iii. $c_3 = -1$

$$z_0 = 0$$
$$z_1 = {z_0}^2 + (-1) = -1$$
$$z_2 = {z_1}^2 + (-1) = 1 - 1 = 0$$
$$z_3 = {z_2}^2 + (-1) = 0 - 1 = -1$$
$$z_4 = {z_3}^2 + (-1) = 1 - 1 = 0$$

   And so on. This iteration makes sure that z_n is either -1 or 0, i.e, it is bounded between [-1, 0].

   -1 belongs in the Mandelbrot set.

   iv. On the real number line, what is the greatest value of c which belongs in the Mandelbrot set?

   Suppose c_+ is greatest real positive number that belongs in the Mandelbrot set and is bounded by [n_1, n_2]. We know that $c_+<1$, since anything greater than 1 will tend to infinity.

$$z_0 = 0$$
$$z_1 = {z_0}^2 + (c_+) = c_+$$
$$z_2 = {c_+}^2 + (c_+) = {c_+}^2 + c_+$$
$$z_3 = ({c_+}^2 + c_+)^2 + (c_+) = {c_+}^4 + 2{c_+}^3 + {c_+}^2 + {c_+}^1$$
$$z_4 = ({c_+}^4 + 2{c_+}^3 + {c_+}^2 + {c_+}^1)^2 + c_+ = {c_+}^8 + 4{c_+}^7 + 6{c_+}^6 + 6{c_+}^5 + 5{c_+}^4 + 2{c_+}^3 + {c_+}^2 + {c_+}^1$$

   These are increasing numbers. Since $c_+<1$, the higher powers of c_+ are smaller than the lower powers of c_+, but they are all positive numbers. Hence, z_n shall keep increasing, however small; therefore, no positive real number can belong in the Mandelbrot set. 0 does, hence, it is the largest real number in the Mandelbrot set.[18]

v. What is the smallest real number value of c that belongs in the Mandelbrot set? Suppose this number is c_{-}.

$$z_0 = 0$$
$$z_1 = {z_0}^2 + (c\_) = c\_$$
$$z_2 = c\_^2 + (c\_) > 0, \forall c\_ < -1$$
$$z_3 = \left(c\_^2 + (c\_)\right)^2 + (c\_) = c\_^4 + 2c\_^3 + c\_^2 + c\_^1$$

Consider the elements in this sum: $c\_^4$ and $c\_^2$ are positive, and $2c\_^3$ and $c\_^1$ are negative. We know that $c\_^2 + (c\_) > 0$. What about $c\_^4 + 2c\_^3$? This is positive only if $c\_^4 > |2c\_^3|$. Let us take common integers to find a pattern:

$c\_$	$c\_^4$	$2c\_^3$	$c\_^4 + 2c\_^3$
-1	1	-2	-1
-2	16	-16	0
-3	81	-54	27
-4	256	-128	128
-5	625	-250	375

Notice that after -2, $c\_^4 + 2c\_^3$ increased. The same is true for any real number lesser than -2. Hence, $\forall c\_ < -2,\ z_3 > z_2 > 0$.

$$z_4 = (c\_^4 + 2c\_^3 + c\_^2 + c\_^1)^2 + c\_$$
$$\forall c\_ < -2, z_4 > z_3 > z_2 > 0$$

And so on. This set of z_n is increasing $\forall c\_ < -2$ and hence, any real number smaller than -2 does not belong in the Mandelbrot Set. This implies -2 is a lower bound for M on the real number line. Does this mean -2 is the smallest real number in M, i.e, the greatest lower bound of 'M' is -2?

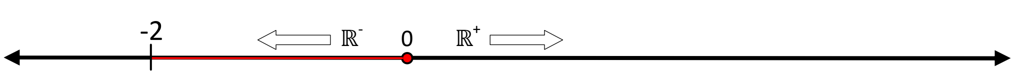
Take $c_4 = -2$,

$$z_0 = 0$$
$$z_1 = {z_0}^2 - 2 = -2$$
$$z_2 = {z_1}^2 - 2 = 4 - 2 = 2$$
$$z_3 = {z_2}^2 - 2 = 4 - 2 = 2$$

And so on. This sequence has converged to a value of 2. Hence, it is bounded and it is in the Mandelbrot set. Therefore, it is the greatest lower bound of 'M' on the real number line.

Visual representation of the above conclusions

The Black *line is the Real Number line, where the numbers in* Red *are in the Mandelbrot set.*



So, this leads to the conclusion of the real number elements in the Mandelbrot set being infinite in number.

2. Let's try the same thing with purely imaginary numbers, i.e, numbers of the form 'a+ib' where a=0 and b≠0.

[18] Is there a change that is small enough to be convergent to a value? The above analysis probably has something wrong in it as points up to 0.25 are seen to be in the graphs of the Mandelbrot set.

i. $c_5 = i$

$$z_0 = 0$$
$$z_1 = {z_0}^2 + (i) = i$$
$$z_2 = i^2 + (i) = -1 + i$$
$$z_3 = (i-1)^2 + (i) = -i$$
$$z_4 = (-i)^2 + (i) = -1 + i$$
$$z_5 = (i-1)^2 + (i) = -i$$
$$z_6 = (-i)^2 + (i) = i - 1$$

And so on. This sequence has only four values: 0, i, -i and 1-i; hence it is bounded. Therefore, '*i*' does belong in the Mandelbrot set.

ii. $c_6 = 2i$

$$z_0 = 0$$
$$z_1 = {z_0}^2 + (2i) = 2i$$
$$z_2 = (2i)^2 + (2i) = -4 + 2i$$
$$z_3 = (2i-4)^2 + (2i) = 12 - 14i$$
$$z_4 = (12-14i)^2 + (2i) = -52 - 168i$$

And so on. This sequence keeps oscillating and increasing in both directions. Therefore, '*2i*' doesn't belong in 'M'.

iii. $c_7 = 1.1i$

$$z_0 = 0$$
$$z_1 = {z_0}^2 + (1.1i) = 1.1i$$
$$z_2 = (1.1i)^2 + (1.1i) = -1.21 + 1.1i$$
$$z_3 = (1.1i - 1.21)^2 + (1.1i) = 2.6741 - 1.562i$$
$$z_4 = (2.6741 - 1.562i)^2 + (2i) = 7.15081081 - 7.2538884i$$

And so on. This sequence keeps increasing in magnitude and hence, 1.1i does not belong in 'M'.

iv. $c_\varepsilon = (1+\varepsilon)i$

Take a small ε, such as 0.01, for $c_7 = (1.01)i$

$\boldsymbol{n}$	$\boldsymbol{z_n}$	$\boldsymbol{\lvert z_n \rvert}$
0	0	0
1	1.01i	1.01
2	$1.01i - 1.0201$	1.435515
3	$0.02050401 - 1.050602i$	1.050802
4	$-0.00042 \ldots + 0.966 \ldots i$	0.967…

And so on. This sequence also keeps decreasing in magnitude. Will its magnitude converge at 0?[19]

v. $c_3 = -i$

$$z_0 = 0$$
$$z_1 = {z_0}^2 + (-i) = -i$$
$$z_2 = (-i)^2 + (-i) = -1 - i$$
$$z_3 = (-i-1)^2 + (-i) = -i$$
$$z_4 = (-i)^2 + (i) = -1 + i$$
$$z_5 = (i-1)^2 + (i) = -i$$
$$z_6 = (-i)^2 + (i) = i - 1$$

[19] The analysis needed to do this was not undertaken by the writers of this report but can be speculated upon

And so on. This sequence has only three values: -i, 1-i and 0; hence, it is bounded and belongs in the Mandelbrot set.

Similarly, other values of complex numbers are used as 'c' and the Mandelbrot set is constructed. Since this is a laborious process, computers are used to find these points.

6.3. Definition

As giving in the introduction above, the Mandelbrot set is a set of complex numbers 'c', for which the orbit of 0, under the iteration of z^2+c, stays bounded.

For $P_c: z \mapsto z^2 + c, the\ orbit\ of\ 0 = \{0, P_c(0), P_c(P_c(0)), P_c\left(P_c(P_c(0))\right), \ldots\}$

Let the following notation for ${P_c}^n(0)$ be defined such that $P_c(0) = {P_c}^1(0), P_c(P_c(0)) = {P_c}^2(0)$ and so on, i.e, ${P_c}^n(0)$ denotes the n ${P_c}^n(0) \leq s$ th iteration of $P_c(0)$. To expand on the definition of the Mandelbrot set, we can set a limit for a specific value of s ∈ $\mathbb{R}^+$.

The definition of the Mandelbrot set can be as follows:

$$M = \{c: \forall c \in \mathbb{C}\ s.t\ \exists s \in \mathbb{R}^+, |\ {P_c}^n(0)| \leq s\ \forall n \in\ \mathbb{N}\}$$

6.4. Visualizing the entire Mandelbrot set

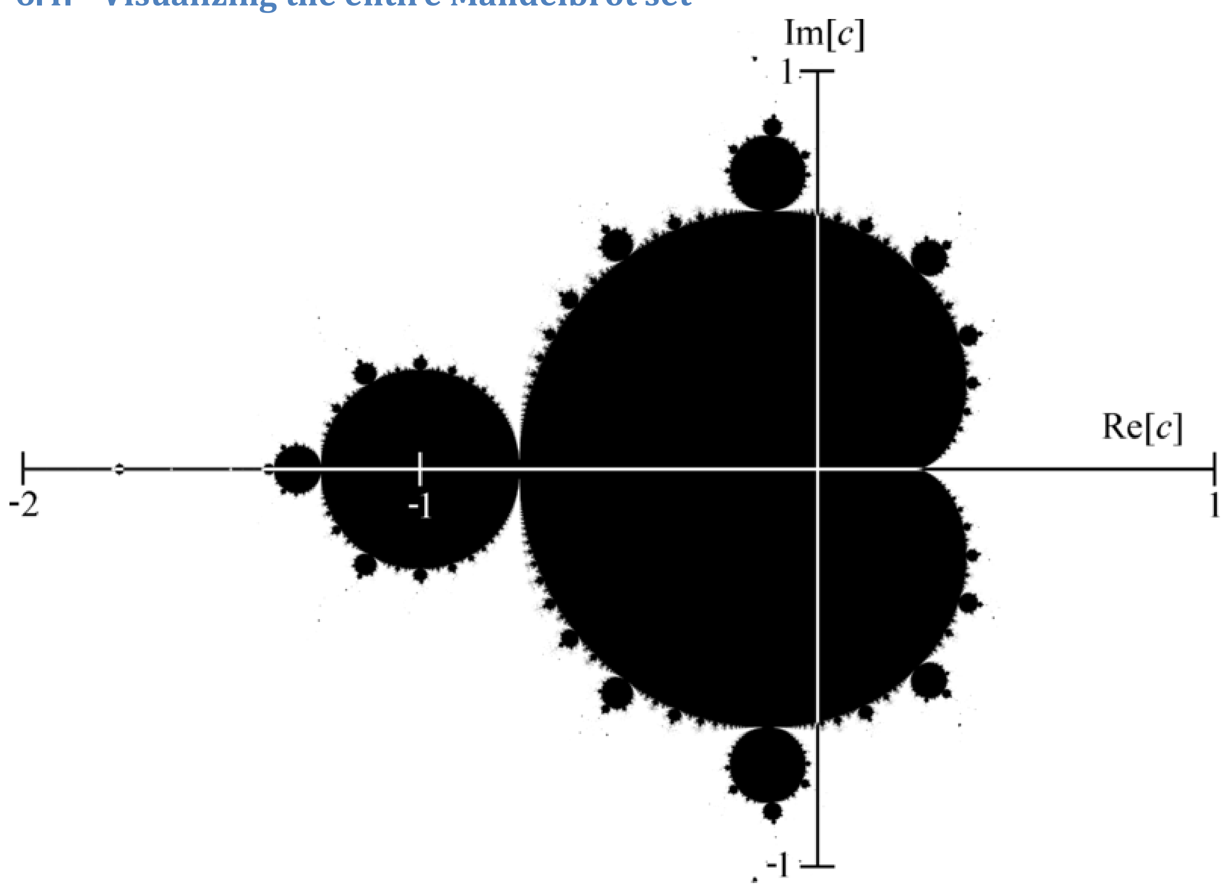


Figure 3: A mathematician's depiction of the Mandelbrot set M.[20]

[20] By Connelly (talk · contribs), Public Domain, https://commons.wikimedia.org/w/index.php?curid=16088

The above graph is all the elements of the Mandelbrot set plotted in black, within the [-2, 1] on the real line and [-1,1] on the imaginary line. There are other ways of visualizing the set, which involve making different values of c in different colours depending on how fast $P_c{}^n(0)$ exceeds *s*; usually, the value of *s* is taken to be 2. This method of visualising the set can result in some interesting patterns, which reveal the figure's self-similarity at different scales and at different locations. It is not quasi-self-similar as these different versions may be slightly different in different areas. With the use of fast computers and algorithms, people have rendered this set to a resolution of more than 750 million iterations! Zoom sequences how the set behaves at greater resolutions of the graph. One example is as given below:

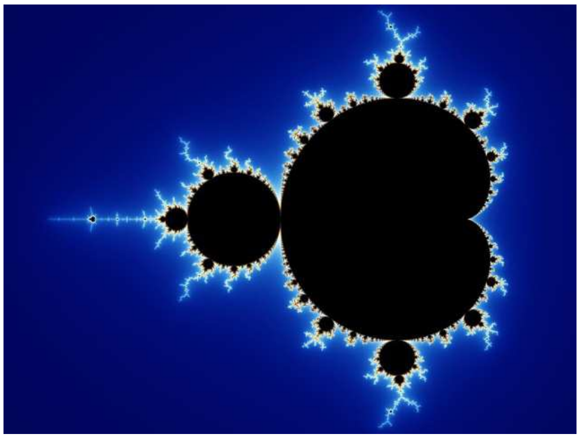


Figure 3: Mandelbrot set with coloured environment (By Created by Wolfgang Beyer with the program Ultra Fractal 3. - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=32197)

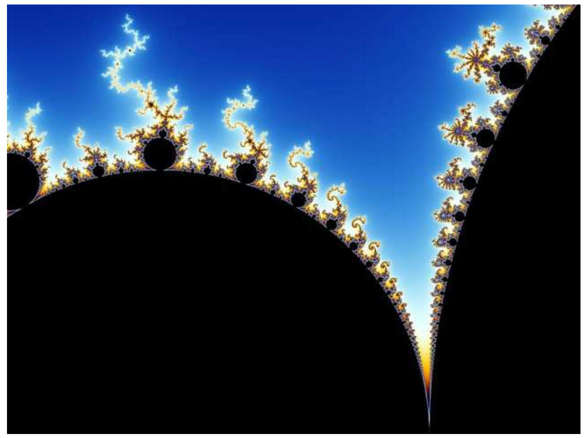


Figure 4: Gap between the "head" and the "body", also called the "seahorse valley" (CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=321975)

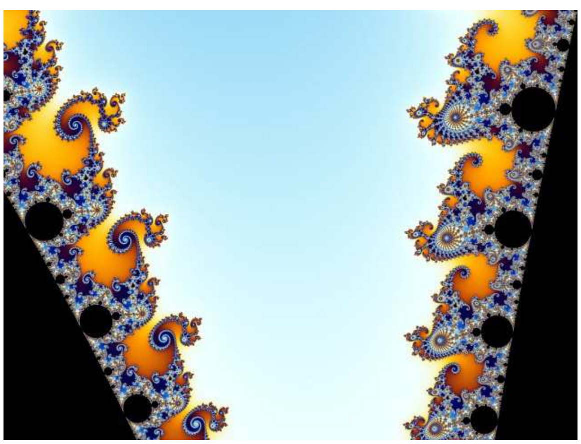


Figure 5: Double-spirals on the left, "seahorses" on the right (CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=321977)

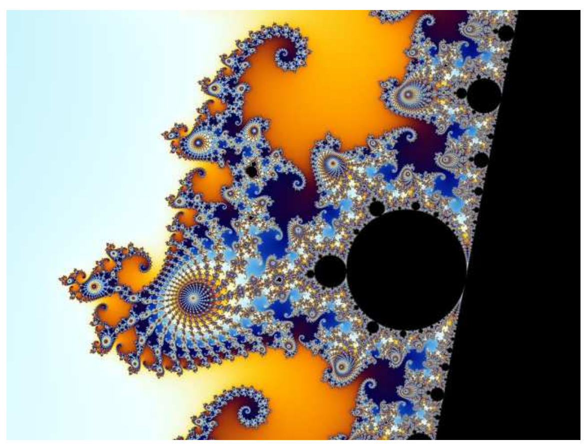


Figure 6: Zoom on the sea-horses (CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=322015)

6.5. Fractal nature of the Mandelbrot Set

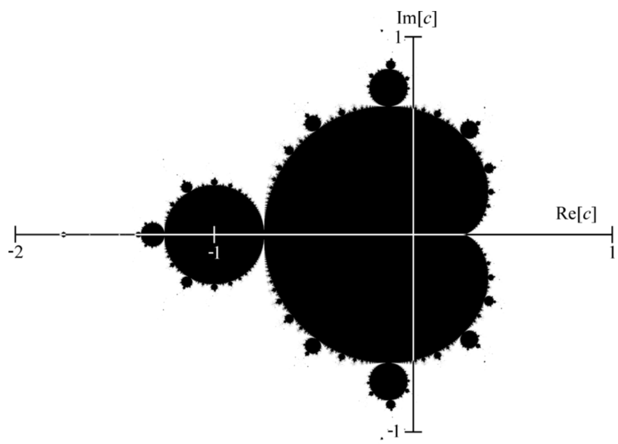
Intuitively, the boundary of the Mandelbrot Set would be considered to have 1 dimension. The Hausdorff dimension of its boundary is found to be 2 (Shishikura)!

6.6. Pi in the Mandelbrot set

Yes, the π is in the Mandelbrot set. This was shown in 1991 by a computer science graduate student, Dave Boll, while he was trying to prove that there was only one point, c=(-0.75, 0), connecting the disc and the cardioid on its right. To do this, Boll took the points (-0.75, ε), for different values of ε being a small number and observed the number of iterations it took for the magnitude of the orbit to become greater than 2.

The following was the result:

ε (ε_n)	No. of iterations (n)	$E_n n$	π with ε_n no. of significant figures
1	3	3	3
0.1	33	3.3	3.1
0.01	315	3.15	3.14
0.001	3143	3.143	3.142
0.0001	31417	3.1417	3.1416
0.00001	314160	3.14160	3.14159
0.000001	3141593	3.141593	3.141593
0.0000001	31415928	3.1415928	3.1415926



The product of ε and the number of iterations is π± ε! In a slightly similar pattern, pi appears in the point (0.25+ ε, 0). This has been explained and proved in the paper "π in the Mandelbrot Set"[21] by Aaron Klebanoff.

[21] https://pdfs.semanticscholar.org/dbed/13dae724fed20356b81be91c63fc13b1e1b8.pdf

Closing Remarks and Conclusion

Fractals are everywhere in nature (to a certain degree of observation, of course) and they are fascinating objects. To study such objects, a fair base in the areas of linear algebra and topology is needed; the lack of which has lead to the absence of an intuitive and original way of explaining what a Lebsque covering dimension is. The generation of fractals is done by superposing iterations of a mapping that is usually affine (in the case of self-similar fractals). There are higher degrees to which fractals can be analysed which may include more linear algebra, but for the scope of the project and the time constraints, they were not studied. The Mandelbrot Set, which has no linear algebra at face value, is nonetheless a fascinating fractal that has many more bits of analysis that has been/ can be done.

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