$\boldsymbol{Amortized\ Time\ Complexity -}$
$\boldsymbol{Aggregate\ Method}$

$\boldsymbol{This\ is\ also\ known\ as\ the}$ "$summation\ method$". $The\ brute\ force\ procedure\ is\ to\ calculate\ T(m)$ $worst - case\ time\ for\ a\ sequence\ of\ all$ $operations\ O_1, O_2, \ldots, O_m. The\ amortized\ time$ $can\ be\ calculated\ as\ follows$:

$Amortized\ cost\ per\ operation =$

$$\frac{Cost\ of\ m\ operations}{m} = \frac{T(m)}{m}$$

$\boldsymbol{As\ such\ operation\ is\ possible\ only\ in\ Amortized}$ $\boldsymbol{operations\ in\ dynamically\ , the\ table\ drawn}$ $\boldsymbol{according\ to\ cost\ is\ known\ as}$ $\boldsymbol{\underline{Dynamic\ Tables.}}$

$\underline{Insertion\ 1}$

$Item\ no. 1\ inserted$

$Dynamic\ Table\ size: 1$

$Total\ cost(C_1) = 1$

$Cost\ per\ operation : \frac{(Cost\ of\ 1\ operation) = 1}{1}$

$= 1$

$Cost\ of\ Doubling\ and\ Copying = 0$

$\underline{Insertion\ 2}$

$Item\ no. 2\ inserted$

$Dynamic\ Table\ size\ increases\ to: 2$

$Total\ cost(C_2) = 2[As\ 1st\ element\ is\ copied\ to\ the\ table$

$making\ table\ increase\ its\ size]$

$Cost\ per\ operation : \frac{(Cost\ of\ 2\ operation) = 2}{2} = 1$

$Cost\ of\ Doubling\ and\ Copying = 2^0 = 1\ [As\ 1\ element$

$is\ copied\ ]$

$\underline{Insertion\ 3}$

$Item\ no.3\ inserted$

$Dynamic\ Table\ size\ increases\ to: 4$

$Total\ cost(C_1) = 3$

$[As\ 1st\ element\ and\ 2nd\ element\ is\ copied\ to\ the\ table$

$and\ 3rd\ element\ is\ inserted\ making\ table\ increase\ its\ size\ ]$

$$Cost\ per\ operation : \frac{(Cost\ of\ 3\ operation) = 3}{3} = 1$$

$Cost\ of\ Doubling\ and\ Copying = 2^1 = 2\ [As\ 2\ element$

$is\ copied\ ]$

$\underline{Insertion\ 4}$

$Item\ no.4\ inserted$

$Dynamic\ Table\ size\ remain\ to: 4$

$Total\ cost(C_1) = 1$

$[As\ 1\ element\ is\ inserted\ and\ nothing\ change\ s]$

]

$$Cost\ per\ operation : \frac{(Cost\ of\ 1\ operation) = 1}{1} = 1$$

$Cost\ of\ Doubling\ and\ Copying = 0.$

$Analysizing\ above\ we\ can\ put\ them\ in\ tables$:

$Item\ No.\ (i)$	$Table\ Size$	$Total\ Cost_i(C_i)$	$Cost\ Of\ Operation$	$Cost\ of\ Doubling\ And\ Copying$
1	1	1	1	0
2	2	2(1+1)	1	$1 = 2^0$
3	4	3(2+1)	1	$2 = 2^1$
4	4	1	1	0
5	8	5(4+1)	1	$4 = 2^2$
6	8	1	1	0
7	8	1	1	0
8	8	1	1	0
9	16	9(8+1)	1	$8 = 2^3$
10	16	1	1	0
11	16	1	1	0
12	16	1	1	0
13	16	1	1	0
14	16	1	1	0
15	16	1	1	0
16	16	1	1	0
17	32	17(16+1)	1	$16 = 2^4$

$\boldsymbol{Here\ we\ see\ worst\ case\ for\ time\ complexity\ for\ single\ insertion\ (\ whenever\ there\ is\ an\ overflow) = O(n)}$

$\boldsymbol{Hence\ worst\ case\ time\ complexity\ of\ n^{th}\ insertion}$:
$\boldsymbol{n \times O(n) = O(n^2)}$

$$\boldsymbol{Cost\ of\ i^{th}\ insertion = \begin{cases} i, when\ i-1\ is\ an\ exact\ power\ of\ 2 \\ \\ 1\ otherwise \end{cases}}$$

$\boldsymbol{Cost\ of\ 5\ is\ 5\ when\ i = 5\ and\ i - 1 = 2^2 = 4}$

$Therefore\ we\ can\ write$:

$$\boldsymbol{Cost(C)\ of\ n\ iterations = \sum_{i=1}^{n} C_i}$$

$$\boldsymbol{= n + \sum_{i=0}^{log_2(n-1)} 2^i}$$

$\boldsymbol{\sum_{i=0}^{log_2(n-1)} 2^i = The\ series\ some\ thing\ stands\ like\ this}$:

$\boldsymbol{= 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{log_2(n-1)}}$

$\boldsymbol{= 1 + 2 + 4 + 8 + \cdots + 2^{log_2(n-1)}}$

$\boldsymbol{This\ is\ a\ geometric\ progression}$:

$\boldsymbol{Hence\ we\ have\ geometric\ sum\ rule}$:

$$= a_1 \times \frac{(1 - r^n)}{1 - r}$$

$Here\ a_1 = 1\ , r = 2, n = log_2(n-1) + 1$

$$= 1 \times \frac{1 - 2^{log_2(n-1)+1}}{1 - 2}$$

$$= \frac{1 - 2^{log_2(n-1)+1}}{1 - 2}$$

$$= \frac{1 - (2^{log_2(n-1)} \times 2)}{-1} \left[\Rightarrow a^{b+c} = a^b \times a^c\right]$$

$$= \frac{1 - \left((n-1) \times 2\right)}{-1} \left[\Rightarrow a^{log_a(b)} = b\right]$$

$$= \frac{1 - (2n - 2)}{-1}$$

$$= \frac{1 - 2n + 2}{-1}$$

$$= \frac{-2n + 3}{-1}$$

$$= 2n - 3$$

$Now\ , coming\ to\ the\ equation\ we\ have: n + 2n - 3 = 3n - 3$

$O(3n - 3) = O(3n) = O(n)$

$\underline{\textit{Alternative way}(2):}$

$\textit{If we see the series} = 2^0 + 2^1 + 2^2 + 2^3 + \cdots$

$\textit{It looks like} := 2^{log_2 1} + 2^{log_2 2} + 2^{log_2 4} + 2^{log_2 8} + \cdots + 2^{log_2 2n}$

$\textit{where}, log_2 1 = 0; log_2 2 = 1; log_2 4 = 2\ ; \ldots .\ etc.$

$\textit{Now , if we leave the 1st i.e.}\ 2^{log_2 1} = 2^0 = 1.$

$$\textit{We get a series} = \sum_{i=1}^{n} 2^{log_2 2i}$$

$$= 2^{log_2 1} + \sum_{i=1}^{n} 2^{log_2 2i}$$

$$= 1 + \sum_{i=1}^{n} 2^{log_2 2i}$$

$\textit{This is a geometric progression}:$

$\textit{Hence we have geometric sum rule}:$

$$= a_1 \times \frac{(1 - r^n)}{1 - r}$$

$\textit{Where}\ a_1 = 2\ , r = 2\ , r^n = 2^{log_2 2n}$

$$= 2 \times \frac{\left(1 - 2^{log_2 2n}\right)}{1 - 2}$$

$$\Rightarrow 2^{log_2 2n} = 2n \quad , \textit{Hence} = 2 \times \frac{(1 - 2n)}{1 - 2}$$

$$\Rightarrow 2 \times \frac{1 - 2n}{-1}$$

$$\Rightarrow 2 \times \frac{-1(2n - 1)}{-1}$$

$$\Rightarrow 2 \times (2n - 1)$$

$$\Rightarrow 4n - 2$$

$$\Rightarrow \textit{Now}, 1 + 4n - 2 = 4n - 1 = O(4n) = O(n)$$

$\underline{\boldsymbol{Alternative\ way(3):}}$

$\boldsymbol{When\ we\ have\ a\ series\ like: 2^0 + 2^1 + 2^2 + \cdots like\ this:}$

$\boldsymbol{We\ get\ a\ geometric\ sequence =}$

$$\boldsymbol{1 + 2 + 4 + 8 + \cdots + \frac{n}{4} + \frac{n}{2} + n}$$

$$\boldsymbol{= n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots + 4 + 2 + 1}$$

$$\boldsymbol{= \frac{n}{2^0} + \frac{n}{2^1} + \frac{n}{2^2} + \frac{n}{2^3} + \cdots + 2^2 + 2^1 + 2^0}$$

$$\boldsymbol{= \sum_{i=0}^{n} \left(\frac{n}{2^i}\right)}$$

$$\boldsymbol{= n + \sum_{i=1}^{n} \left(\frac{n}{2^i}\right)}$$

$\boldsymbol{A\ geometric\ sequence\ has\ a\ constant\ ratio\ r\ and\ is}$

$\boldsymbol{defined\ by\ a_n = a_1 \times r^{n-1}}$

$$\boldsymbol{\sum_{i=1}^{n} \left(\frac{n}{2^i}\right)}$$

$$\boldsymbol{a_i = \frac{n}{2^i}\ , a_{i+1} = \frac{n}{2^{(i+1)}}}$$

$\boldsymbol{Computing\ the\ adjacent\ ratio: r = \frac{a_{i+1}}{a_i}}$

$$r = \frac{\frac{n}{2^{(i+1)}}}{\frac{n}{2^{i}}} = \frac{n}{2^{(i+1)}} \times \frac{2^{i}}{n} = \frac{1}{2^{(i+1-i)}} = \frac{1}{2}$$

$when\ i = n\ , then:$

$$a_i = a_1 r^{i-1}$$

$$a_1 = \frac{n}{2}\ and\ r = \frac{1}{2}\ , hence:$$

$$a_i = \frac{n}{2^{i}}\left(\frac{1}{2}\right)^{i-1}$$

$$a_i = \frac{n}{2^{i}} \times \frac{1}{2^{i-1}}$$

$$a_i = \frac{n}{2^{i}} \times \frac{1}{2^{i-1}}$$

$$a_i = \frac{n}{2^{i-1+1}} = \frac{n}{2^{i}}$$

$Geometric\ sequence\ sum\ formula:$

$$S_n = a_1 \times \frac{1 - r^{i}}{1 - r}, where\ r \neq 1$$

$Now\ put\ the\ values:$

$$i = n, a_1 = \frac{n}{2}, r = \frac{1}{2}$$

$$= \frac{n}{2} \times \frac{1 - \left(\frac{1}{2}\right)^{n}}{1 - \frac{1}{2}}$$

$$= \frac{n\left(1-\left(\frac{1}{2}\right)^{n}\right)}{2\left(1-\frac{1}{2}\right)}$$

$$= \frac{n\left(1-\left(\frac{1}{2}\right)^{n}\right)}{2\left(\frac{1}{2}\right)}$$

$$= n\left(1-\left(\frac{1}{2}\right)^{n}\right)$$

$$Hence, \sum_{i=1}^{n}\left(\frac{n}{2^{i}}\right) = n\left(1-\left(\frac{1}{2}\right)^{n}\right)$$

$$And\ , n + \sum_{i=1}^{n}\left(\frac{n}{2^{i}}\right) = n + n\left(1-\left(\frac{1}{2}\right)^{n}\right)$$

$$= n + n - n\left(\frac{1}{2}\right)^{n}$$

$$= 2n - n\left(\frac{1}{2}\right)^{n}$$

$$Hence\ , we\ have\ O\left(2n - n\left(\frac{1}{2}\right)^{n}\right) = O(2n) = O(n)$$

$$Therefore\ Amortized\ cost = \frac{O(n)}{n} = O(1)$$

$We\ can\ write\ \Theta(1) as\ it\ is\ just\ giving\ us\ the\ Average\ case\ complexity.$

$Example\ 2: Consider\ the\ following\ sequence\ of\ push\ and$

$pop\ operation\ operations\ is\ 1. If\ the\ sequence\ is\ given\ as\ follows:$

$1\ push\quad 1push\quad 1push\quad 3\ pops\quad 2push\quad 2pops\quad 2push\quad 2pops$

$then, what\ is\ the\ amortized\ cost?$

$Solution:$

$As\ per\ the\ cost\ given\ , the\ cost\ of\ the\ sequence\ would\ be\ as\ follows:$

$t_i = 1\quad 1\quad 1\quad 3\quad 2\quad 2\quad 2\quad 2$

$$Cost = \frac{1+1+1+3+2+2+2+2}{(No.of\ push\ pops)8} = \frac{14}{8} = 1.75$$

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