20.17. TIME COMPLEXITY CALCULATION NESTED FOR LOOP (MORE THAN TWO LOOP).

EXAMPLE 3

$$
\begin{aligned}
&\boldsymbol{for}(\boldsymbol{i} = \mathbf{0}; \boldsymbol{i} \le \boldsymbol{n}; \boldsymbol{i} + +)\{\\
&\qquad \boldsymbol{for}(\boldsymbol{j} = \mathbf{0}; \boldsymbol{j} \le \boldsymbol{i}; \boldsymbol{j} + +)\{\\
&\qquad\quad \boldsymbol{for}(\boldsymbol{k} = \mathbf{0}; \boldsymbol{k} \le \boldsymbol{j}; \boldsymbol{k} + +)\{\\
&\qquad\qquad \mathrm{c} = \mathrm{c} + \mathbf{1};\\
&\qquad\quad \}\\
&\qquad \}\\
&\}
\end{aligned}
$$

ANSWER

The loop runs like:

$$\boldsymbol{f(1) \le n\,, when\; i = 1}$$

$$\boldsymbol{f(1) \le 1\,, when\; j = 1, i = 1}$$

$$\boldsymbol{f(1) \le 1\,, when\; k = 1, j = 1}$$

$$\mathrm{c} = \mathrm{c} + \mathbf{1} ------\rightarrow (\mathbf{1})$$

$$\boldsymbol{T(n) = \sum_{i=1}^{1} 1 = 1\; time}$$

$$\boldsymbol{f(2) \le n\,, when\; i = 2}$$

$$\boldsymbol{f(1) \le 2\,, when\; j = 1, i = 2}$$

$$\boldsymbol{f(1) \le 1\,, when\; k = 1, j = 1}$$

$$\mathrm{c} = \mathrm{c} + \mathbf{1} ------\rightarrow (\mathbf{1})$$

$$\boldsymbol{T(n) = \sum_{i=1}^{1} 1 = 1\; time}$$

$$\boldsymbol{f(2) \le 2\,, when\; j = 2, i = 2}$$

$$\boldsymbol{f(1) \le 2\,, when\; k = 1, j = 2}$$

$$\mathrm{c} = \mathrm{c} + \mathbf{1} ------\rightarrow (\mathbf{1})$$

$$\boldsymbol{T(n) = \sum_{i=1}^{1} 1 = 1\; time}$$

$$\boldsymbol{f(2) \le 2\,, when\; k = 2, j = 2}$$

$$\mathrm{c} = \mathrm{c} + \mathbf{1} ------\rightarrow (\mathbf{1})$$

$$\boldsymbol{T(n) = \sum_{i=1}^{1} 1 = 1\; time}$$

$$T(n) = \sum_{i=1}^{1} 1 + \sum_{i=1}^{1} 1 + \sum_{i=1}^{1} 1 = 3\ time$$

$$f(3) \le n\ , when\ i = 3$$

$$f(1) \le 3\ , when\ j = 1, i = 3$$

$$f(1) \le 1\ , when\ k = 1, j = 1$$

$$\mathrm{c} = \mathrm{c} + 1 - - - - - - \to (1)$$

$$T(n) = \sum_{i=1}^{1} 1 = 1\ time$$

$$f(2) \le 3\ , when\ j = 2, i = 3$$

$$f(1) \le 2\ , when\ k = 1, j = 2$$

$$\mathrm{c} = \mathrm{c} + 1 - - - - - - \to (1)$$

$$T(n) = \sum_{i=1}^{1} 1 = 1\ time$$

$$f(2) \le 2\ , when\ k = 2, j = 2$$

$$\mathrm{c} = \mathrm{c} + 1 - - - - - - \to (1)$$

$$T(n) = \sum_{i=1}^{1} 1 = 1\ time$$

$$T(n) = \sum_{i=1}^{1} 1 + \sum_{i=1}^{1} 1 = 2\ times$$

$$f(3) \leq 3\ , when\ j = 3, i = 3$$

$$f(1) \leq 3\ , when\ k = 1, j = 3$$

$$\mathrm{c} = \mathrm{c} + 1 - - - - -- \rightarrow (1)$$

$$T(n) = \sum_{i=1}^{1} 1 = 1\ time$$

$$f(2) \leq 3\ , when\ k = 2, j = 3$$

$$\mathrm{c} = \mathrm{c} + 1 - - - - -- \rightarrow (1)$$

$$T(n) = \sum_{i=1}^{1} 1 = 1\ time$$

$$f(2) \leq 3\ , when\ k = 3, j = 3$$

$$\mathrm{c} = \mathrm{c} + 1 - - - - -- \rightarrow (1)$$

$$T(n) = \sum_{i=1}^{1} 1 = 1\ time$$

$$T(n) = \sum_{i=1}^{1} 1 + \sum_{i=1}^{1} 1 + \sum_{i=1}^{1} 1 = 3\ times$$

$$T(n) = \sum_{i=1}^{1} 1 + \sum_{i=1}^{2} 1 + \sum_{i=1}^{3} 1 = 1 + 2 + 3 = 5$$

Hence if we analyse, when

$$\boldsymbol{i = 1, j = 1\ , k\ runs = \sum_{n=1}^{n} n = \frac{n(n+1)}{2} = 1}$$

$$\boldsymbol{i = 2, j = 2\ , k\ runs = \sum_{n=1}^{n} n = \frac{n(n+1)}{2} = 3}$$

$$\boldsymbol{i = 3, j = 3\ , k\ runs = \sum_{n=1}^{n} n = \frac{n(n+1)}{2} = 5}$$

....

$$\boldsymbol{i = n, j = n\ , k\ runs = \sum_{n=1}^{n} n = \frac{n(n+1)}{2}}$$

Hence k runs:

$$\boldsymbol{\sum_{i=1}^{n} \frac{n(n+1)}{2}}$$

$$\boldsymbol{= \sum_{i=1}^{n} \frac{n^2 + n}{2} = \sum_{i=1}^{n} \left(\frac{n^2}{2}\right) + \sum_{i=1}^{n} \frac{n}{2}}$$

$$\sum_{i=1}^{n} \frac{n^2}{2}$$

$$= \frac{1}{2} \times \sum_{i=1}^{n} n^2$$

We know $\sum_{i=1}^{n} n^2 = \frac{2n^3+3n^2+n}{6}$

$$Hence, \frac{1}{2} \times \left(\frac{2n^3 + 3n^2 + n}{6}\right) = \frac{(2n^3 + 3n^2 + n)}{12}$$

Again,

$$\sum_{i=1}^{n} \frac{n}{2} = \frac{1}{2}\sum_{i=1}^{n} n = \frac{1}{2} \times \left(\frac{n(n+1)}{2}\right) = \frac{1}{2} \times \frac{n^2+n}{2} = \frac{n^2+n}{4}$$

$$\sum_{i=1}^{n} \left(\frac{n^2}{2}\right) + \sum_{i=1}^{n} \frac{n}{2} = \frac{(2n^3 + 3n^2 + n)}{12} + \frac{n^2+n}{4}$$

$$\frac{2n^3 + 3n^2 + n + 3n^2 + 3n}{12} = \frac{2n^3 + 6n^2 + 4n}{12}$$

$$= \frac{n^3 + 3n^2 + 2n}{6}$$

$$= O\left(\frac{(n^3 + 3n^2 + 2n)}{6}\right) = O(n^3)$$

We can rewrite the above equation as:

$$\frac{n^3 + 3n^2 + 2n}{6} = \frac{n(n+1)(n+2)}{6}$$