20.16. TIME COMPLEXITY CALCULATION NESTED FOR LOOP (MORE THAN TWO LOOP).

EXAMPLE 2

$$
\begin{aligned}
&\boldsymbol{for(i = 0; i \le n; i++)\{}\\
&\qquad \boldsymbol{for(j = 0; j \le i; j++)\{}\\
&\qquad\quad \boldsymbol{for(k = 0; k \le i; k++)\{}\\
&\qquad\qquad \mathrm{c} = \mathrm{c} + \mathbf{1};\\
&\qquad\quad \}\\
&\qquad \}\\
&\}
\end{aligned}
$$

ANSWER

The loop runs like:

$$f(1) \leq n \text{ , } when\ i = 1$$

$$f(1) \leq 1 \text{ , } when\ j = 1, i = 1$$

$$f(1) \leq 1 \text{ , } when\ k = 1, i = 1$$

$$\mathrm{c} = \mathrm{c} + 1 - - - - -\!\!-\!\!\rightarrow (1)$$

$$T(n) = \sum_{i=1}^{1} 1 = 1\ time$$

$$f(2) \leq n \text{ , } when\ i = 1$$

$$f(1) \leq 2 \text{ , } when\ j = 1, i = 2$$

$$f(1) \leq 2 \text{ , } when\ k = 1, i = 2$$

$$\mathrm{c} = \mathrm{c} + 1 - - - - -\!\!-\!\!\rightarrow (1)$$

$$f(2) \leq 2 \text{ ,} when\ k = 2, i = 2$$

$$\mathrm{c} = \mathrm{c} + 1 - - - - -\!\!-\!\!\rightarrow (2)$$

$$T(n) = \sum_{i=1}^{2} 1 = (1 + 1) = 2\ times$$

$$f(2) \leq 2 \text{ , } when\ j = 1, i = 2$$

$$f(1) \leq 2 \text{ , } when\ k = 1, i = 2$$

$$\mathrm{c} = \mathrm{c} + 1 - - - - -\!\!-\!\!\rightarrow (1)$$

$$f(2) \leq 2 \text{ ,} when\ k = 2, i = 2$$

$$\mathrm{c} = \mathrm{c} + 1 - - - - -\!\!-\!\!\rightarrow (2)$$

$$T(n) = \sum_{i=1}^{2} 1 = (1 + 1) = 2\ times$$

$$T(n) = \sum_{i=1}^{2} 1 + \sum_{i=1}^{2} 1 = (2+2) = 4\ times\ total$$

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$$f(n) \leq n\ , when\ i = n$$

$$f(1) \leq n\ , when\ j = 1, i = n$$

$$f(1) \leq n\ , when\ k = 1, i = n$$

$$\text{c} = \text{c} + 1 - - - - - - \rightarrow (1)$$

$$f(2) \leq n\ , when\ k = 2, i = n$$

$$\text{c} = \text{c} + 1 - \ - - - - - \rightarrow (2)$$

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$$f(n-1) \leq n\ , when\ k = n-1, i = n$$

$$\text{c} = \text{c} + 1 - - - - - - \rightarrow (n-1)$$

$$f(n) \leq n\ , when\ k = n, i = n$$

$$\text{c} = \text{c} + 1 - - - - - - \rightarrow (n)$$

$$T(n) = \sum_{i=1}^{n} 1 = (1 + 1 + \cdots ntimes) = n$$

$$f(2) \leq n\, , when\; j = 2, i = n$$

$$f(1) \leq n\, , when\; k = 1, i = n$$

$$\text{c} = \text{c} + 1 - - - - - - \rightarrow (1)$$

$$f(2) \leq n\, , when\; k = 2, i = n$$

$$\text{c} = \text{c} + 1 - - - - - - \rightarrow (2)$$

.....

$$f(n-1) \leq n\, , when\; k = n-1, i = n$$

$$\text{c} = \text{c} + 1 - \;\; - - - - \rightarrow (n-1)$$

$$f(n) \leq n\, , when\; k = n, i = n$$

$$\text{c} = \text{c} + 1 —— - - - \rightarrow (n)$$

$$T(n) = \sum_{i=1}^{n} 1 = (1 + 1 + \cdots ntimes) = n$$

$$f(n) \leq n\, , when\; j = n, i = n$$

$$f(1) \leq n\, , when\; k = 1, i = n$$

$$\text{c} = \text{c} + 1 - - - - - - \rightarrow (1)$$

$$f(2) \leq n\, , when\; k = 2, i = n$$

$$\text{c} = \text{c} + 1 - - - - - - \rightarrow (2)$$

.....

$$f(n-1) \leq n\, , when\; k = n-1, i = n$$

$$\text{c} = \text{c} + 1 - \;\; - - - - \rightarrow (n-1)$$

$$f(n) \leq n\, , when\; k = n, i = n$$

$$\text{c} = \text{c} + 1 - \;\; - - - - \rightarrow (n)$$

$$T(n) = \sum_{i=1}^{n} 1 = (1 + 1 + \cdots ntimes) = n$$

$$T(n) = \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 1 + \cdots n\ times =$$

$$= (n + n + n + \cdots n\ times) = n^2$$

Hence, we are getting:

$$1 + 4 + 9 + \cdots + n^2 = \sum_{n=1}^{n} n^2$$

i.e.,

$$\sum_{n=1}^{n} n^2 = \quad 1^2 + 2^2 + \cdots + n^2 = \mathrm{O}\left(\frac{2n^3 + 3n^2 + n}{6}\right)$$
$$= O(n^3)$$

$$\textit{Deduction of } \sum_{n=1}^{n} n^2 = (1^2 + 2^2 + 3^2 + \cdots + n^2) = \frac{2n^3 + 3n^2 + n}{6}$$

Solution

From Part 1 we know By Growth of Series we got:

$$\sum_{n=1}^{n} n = (1 + 2 + 3 + \cdots + n) = \frac{n(n+1)}{2}$$

$\textit{Now for } (1^2 + 2^2 + 3^2 + \cdots + n^2), \textit{we have}:$

$$\sum_{n=1}^{n} n^2 = \left(1^2 + 2^2 + 3^2 + \cdots + n^2\right)$$

$\textit{Lets take : Binomial Series of Expansion}$

$$(a+b)^n = n_{C_0} a^n b^0 + n_{C_1} a^{n-1} b^1 + n_{C_2} a^{n-2} b^2 + \cdots + n_{C_n} a^0 b^n$$

$\textit{Now if we take}: (n+1)^3$

$$= 3_{C_0} \times n^3 \times 1^0 + 3_{C_1} \times n^2 \times 1^1 + 3_{C_2} \times n^1 \times 1^2 + 3_{C_3} \times n^0 \times 1^3$$

$$= 1 \times n^3 \times 1^0 + 3 \times n^2 \times 1^1 + 3 \times n^1 \times 1^2 + 1 \times n^0 \times 1^3$$

$$= n^3 + 3n^2 + 3n + 1$$

$\textit{Now},$

$$\Rightarrow (n+1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1$$

$\textit{Now, putting } n = 1, 2, 3, 4, 5, \ldots, n-1, n \textit{ and Adding it we get}:$

$$\Rightarrow (1 + 1 = 2)^3 - 1^3 = 3 \times 1^2 + 3 \times 1 + 1$$

Hence,

$$2^3 - 1^3 = 3 \times 1^2 + 3 \times 1 + 1$$

$3^3 - 2^3 = 3 \times 2^2 + 3 \times 2 + 1$ **[As we add** 2^3 *and* 2^3 *gets cancelled*]

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$$n^3 - (n-1)^3 = 3 \times (n-1)^2 + 3 \times (n-1) + 1$$

$(n+1)^3 - n^3 = 3 \times n^2 + 3 \times n + 1$**[As we add** n^3 *and* n^3 *gets cancelled*]

$$(n+1)^3 - 1 = 3(1^2 + 2^2 + \cdots + (n-1)^2 + n^2) + 3(1 + 2 + \cdots + n) + (1 \times n = n)$$

$$\Rightarrow (n+1)^3 - 1 = 3 \times \sum_{n=1}^{n} n^2 + 3 \times \sum_{n=1}^{n} n + n$$

$$\Rightarrow (n+1)^3 - 1 - 3 \times \sum_{n=1}^{n} n - n = 3 \times \sum_{n=1}^{n} n^2$$

We know: $\sum_{n=1}^{n} n = (1 + 2 + 3 + \cdots + n) = \frac{n(n+1)}{2}$

$$\Rightarrow (n+1)^3 - 1 - 3 \times \left(\frac{n(n+1)}{2}\right) - n = 3 \times \sum_{n=1}^{n} n^2$$

Putting

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

$$\Rightarrow n^3 + 3n^2 + 3n + 1 - 1 - 3 \times \left(\frac{n(n+1)}{2}\right) - n = 3 \times \sum_{n=1}^{n} n^2$$

$$\Rightarrow \boldsymbol{n^3 + \ 3n^2 + 3n - n - 3 \times \left(\frac{n(n+1)}{2}\right) = 3 \times \sum_{n=1}^{n} n^2}$$

$$\Rightarrow \boldsymbol{n^3 + \ 3n^2 + 2n - 3 \times \left(\frac{n(n+1)}{2}\right) = 3 \times \sum_{n=1}^{n} n^2}$$

$$\Rightarrow \boldsymbol{3 \times \sum_{n=1}^{n} n^2 = \ n^3 + \ 3n^2 + 2n - \left(\frac{3n^2 + 3n}{2}\right)}$$

$$\Rightarrow \boldsymbol{3 \times \sum_{n=1}^{n} n^2 = \frac{2n^3 + 6n^2 + 4n - 3n^2 - 3n}{2}}$$

$$\Rightarrow \boldsymbol{3 \times \sum_{n=1}^{n} n^2 = \frac{2n^3 + 3n^2 + n}{2}}$$

$$\Rightarrow \boldsymbol{\sum_{n=1}^{n} n^2 = \frac{2n^3 + 3n^2 + n}{6}}$$

Therefore, we got for

$$\boldsymbol{\sum_{n=1}^{n} n^2 \ = \left(1^2 + 2^2 + 3^2 + \cdots + n^2\right) = \frac{2n^3 + 3n^2 + n}{6}}$$