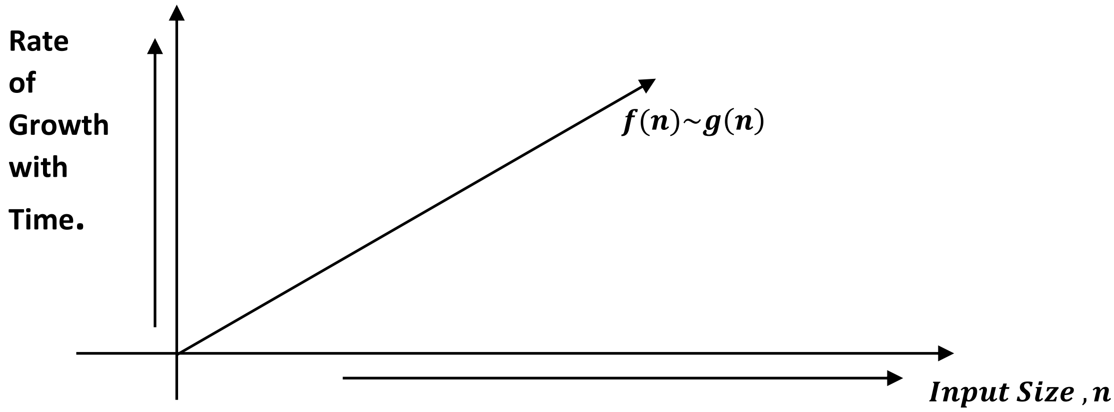
12. Tilde Notation (~)

DEFINITION: The notation is useful when the function $f(n)$ and $g(n)$ growth at the same rate. It is written as:

$$f(n) \sim g(n)$$



The above definition suggests that $\frac{f(n)}{g(n)}$ approaches to 1 as N grows. We can observe it from table:

f(n)	$\sim g(n)$(Tilde approximation)	Order of Growth
$\frac{n^3}{6} - \frac{n^2}{2} + \frac{n}{3}$	$\sim \frac{n^3}{6}$	n^3
$\frac{n^2}{2} - \frac{n}{2}$	$\sim \frac{n^2}{2}$	n^2
$logN + 1$	$\sim \log(N)$	$\log(N)$
3	~3	1

Hence, we write it as follows:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

Example 1

$$\boldsymbol{Prove}\ \left(\frac{n^3}{6} - \frac{n^2}{2} + \frac{n}{3}\right) \sim \frac{n^3}{6}$$

Solution:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{n^3}{6} - \frac{n^2}{2} + \frac{n}{3}\right)}{\frac{n^3}{6}}$$

We know that $\lim_{x\to a}[c \times f(x)] = c \times \lim_{x\to a} f(x)$, $\boldsymbol{hence}$:

$$= \frac{1}{\frac{1}{6}} \lim_{n\to\infty} \frac{\left(\frac{n^3}{6} - \frac{n^2}{2} + \frac{n}{3}\right)}{n^3}$$

$$= 6 \times \lim_{n\to\infty} \frac{\left(\frac{n^3}{6} - \frac{n^2}{2} + \frac{n}{3}\right)}{n^3}$$

$$= 6 \times \lim_{n\to\infty} \frac{\left(\frac{n^3 - 3n^2 + 2n}{6}\right)}{n^3}$$

$$= 6 \times \frac{1}{6} \times \lim_{n\to\infty} \frac{(n^3 - 3n^2 + 2n)}{n^3}$$

$$= 6 \times \frac{1}{6} \times \lim_{n\to\infty} \left(\frac{n^3}{n^3} - \frac{3n^2}{n^3} + \frac{2n}{n^3} \right)$$

$$= 6 \times \frac{1}{6} \times \lim_{n\to\infty} \left(1 - \frac{3}{n} + \frac{2}{n^2} \right)$$

As per ,

$$\lim_{x\to\infty} \left(\frac{c}{x^a}\right) = 0 \text{ , } Infinity\ property\ of\ Limit\ and\ \lim_{n\to a} c = c, where\ c\ is\ constant.$$

Hence:

$$\lim_{n\to\infty} 1 = 1 \text{ , } \lim_{n\to\infty} \frac{3}{n} = 0 \text{ , } \lim_{n\to\infty} \frac{2}{n^2} = 0$$

$$= 6 \times \frac{1}{6} \times (1 - 0 + 0)$$

$$= 1 \times 1$$

$$= 1$$

Hence, $\left(\frac{n^3}{6} - \frac{n^2}{2} + \frac{n}{3}\right) \sim \frac{n^3}{6}$ and rate growth is n^3

Example 2

$$Prove\ \frac{n^2}{2} - \frac{n}{2} \sim \frac{n^2}{2}$$

Solution

$$\lim_{n\to\infty} \frac{\frac{\left(\frac{n^2}{2} - \frac{n}{2}\right)}{n^2}}{2}$$

We know that $\lim_{x\to a}[c \times f(x)] = c \times \lim_{x\to a} f(x)$, $hence$:

$$= \frac{1}{\frac{1}{2}} \times \lim_{n\to\infty} \left(\frac{\left(\frac{n^2}{2} - \frac{n}{2}\right)}{n^2} \right)$$

$$= 2 \times \lim_{n\to\infty} \left(\frac{\left(\frac{n^2}{2} - \frac{n}{2}\right)}{n^2} \right)$$

We know that $\lim\limits_{x \to a}[c \times f(x)] = c \times \lim\limits_{x \to a} f(x)$, *hence*:

$$= 2 \times \frac{1}{2} \times \lim_{n \to \infty} \left(\frac{n^2 - n}{n^2} \right)$$

$$= 2 \times \frac{1}{2} \times \lim_{n \to \infty} \left(\frac{n^2}{n^2} - \frac{n}{n^2} \right)$$

$$= 2 \times \frac{1}{2} \times \lim_{n \to \infty} \left(1 - \frac{1}{n} \right)$$

As per,

$$\lim_{x \to \infty} \left(\frac{c}{x^a} \right) = 0\text{, Infinity property of Limit and } \lim_{n \to a} c = c\text{, where } c \text{ is constant.}$$

Hence:

$$= 2 \times \frac{1}{2} \times (1 - 0)$$

$$= 1 \times 1$$

$$= 1$$

Hence, $\frac{n^2}{2} - \frac{n}{2} \sim \frac{n^2}{2}$ and rate growth is n^2

Example 3

$$\boldsymbol{Prove\ logN + 1 \sim \log(N)}$$

Solution

$$= \lim_{n\to\infty} \left(\frac{\log n + 1}{\log n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{\log n}{\log n} + \frac{1}{\log n}\right)$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{\log n}\right)$$

We know that:

$$= \lim_{x\to a}\big(f(x) \pm g(x)\big) = \lim_{x\to a}\big(f(x)\big) \pm \lim_{x\to a}\big(g(x)\big)$$

$$= \lim_{n\to\infty}(1) + \lim_{n\to\infty}\frac{1}{logn}$$

We know, $\lim_{n\to a} c = c, where\ c\ is\ constant.$

$$= 1 + \lim_{n\to\infty}\frac{1}{logn}$$

We know, $\lim_{n \to a} \left(\frac{f(n)}{g(n)}\right) = \frac{\lim_{n \to a} f(n)}{\lim_{n \to a} g(n)}$.

Hence:

$$= 1 + \frac{\lim_{n \to \infty}(1)}{\lim_{n \to \infty}(\log n)}$$

We know, $\lim_{n \to a} c = c, where\ c\ is\ constant\ \ and\ \lim_{n \to \infty}\big(log(n)\big) = \infty\ , Hence$:

$$= 1 + \frac{1}{\infty}$$

And infinity property $\frac{c}{\infty} = 0\ \ , where\ c\ is\ constant.$

$$= 1 + 0$$

$$= 1$$

$Hence\ logN + 1 \sim \log(N)\ and\ rate\ of\ growth\ is\ log(n).$

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