BIG OMEGA NOTATION- MATHEMATICAL EXAMPLES AND PROOFS

EXAMPLE 1

1) $\boldsymbol{Find\ lower\ bound\ for\ f(n) = 5n^2}$

SOLUTION: We know by definition of Big Omega Ω notation:

$\boldsymbol{f(n) = \Omega(g(n)), if\ 0 \le c \times g(n) \le f(n)\ for\ all\ n \ge n_0,}$
$\boldsymbol{where\ c\ and\ n_0\ are}$ **constants.**

If there exist c, n_0 Such that:

$$\boldsymbol{0 \le cn^2 \le 5n^2}$$

$$\boldsymbol{\Rightarrow cn^2 \le 5n^2}$$

$$\boldsymbol{\Rightarrow c \le \frac{5n^2}{n^2}}$$

$$\boldsymbol{\Rightarrow c \le 5}$$

and

$$\boldsymbol{we\ see : g(n) = n^2}$$

$$\boldsymbol{f(n) \ge c \times g(n)}$$

$$\boldsymbol{5n^2 \ge c \times n^2 \ge 0}$$

$$We\ know\ c = 5, hence:$$

$$\boldsymbol{5n^2 \ge 5 \times n^2 \ge 0}$$

$$or, n^2 \geq n^2 \geq 0$$

$$or, n \geq n \geq 0$$

$$or, n \geq 0 \textit{ is correct},$$

$or\ we\ may\ say\ n \geq 1$, Based on Analysis given below:

N	$5n^2$	n^2
1	5	1
2	20	4
3	45	9
4	80	16

Hence $n \geq 1$ *is best possible analysis on the,*

baisis of rate of growth.

$$\therefore 5n^2 = \Omega(n^2) \textit{ with } c = 5 \textit{ and } n_0 = 1$$

EXAMPLE 2

2) $\textit{Let } f(n) = n^4 + 3n^3 + 2n + 1\ .\ \textit{Let } g(n) = n^3 + 4.$

$\textit{Prove that } f(n) \textit{ of an algorithm is } \Omega(n^3).$

SOLUTION:

We know by definition of Big Omega Ω notation:

$f(n) = \Omega(g(n)), if\ 0 \leq c \times g(n) \leq f(n)\ for\ all\ n \geq n_0,$
$where\ c\ and\ n_0\ are$ constants.

$$0 \leq c \times (n^3 + 4) \leq n^4 + 3n^3 + 2n + 1$$

$$c \leq \frac{n^4 + 3n^3 + 2n + 1}{n^3 + 4}\ i.e. \frac{f(n)}{g(n)}$$

$$And\ n \leq \sqrt[3]{\frac{n^4 + 3n^3 + 2n + 1}{n^3 + 4}}$$

BUT WITH THE ABOVE EQUATION OF n WE CANNOT DETERMINE, HENCE LET'S GO WITH INPUTS

N	$n^4 + 3n^3 + 2n + 1$	$n^3 + 4$
1	1+3+2+1=6	1+4=5
2	16+24+4+1=45	8+1=9
3	81+81+6+1=169	27+4=31

$$i.e. for\ n = 1, c \leq \frac{6}{5} = 1.2$$

$$i.e. for\ n = 2, c \leq \frac{45}{5} = 5$$

$$i.e. \, for \, n = 3 \, , c \leq \frac{169}{31} = 5.45$$

HENCE: $n \geq 1$

Therefore, we $have \, c$ which has a positive number, for sufficiently large values of n , We can see that $c \geq 1$.

NOTE: $c \, is \, natural \, positive \, number.$

This satisfies the definition:
$f(n) = \Omega(g(n)), if \ 0 \leq c \times g(n) \leq f(n) \ for \ all \ n \geq n_0,$
$where \ c \ and \ n_0 \ are$ constants which imply $f(n)$ grows at a faster rate than a constant time $g(n)$ for a sufficiently large n.

Therefore $\Omega(g(n)) = \Omega(\mathrm{n}^3)$.

EXAMPLE 3

3) $If \ the \ relation \ f(n) = 6n^2 + 7n + 8$ holds, $prove \ that \ f(n) is \ not \ \Omega(n^3)$

SOLUTION

We know by definition of Big Omega Ω notation:

$f(n) = \Omega(g(n)), if\ 0 \leq c \times g(n) \leq f(n)\ for\ all\ n \geq n_0,$
$where\ c\ and\ n_0\ are$ constants.

Note : By the equation we can see that :
$f(n) \geq c \times g(n)$

$$\Rightarrow 6n^2 + 7n + 8 \geq c \times n^3$$

$$\Rightarrow f(n) \leq 6n^2 + 7n^2 + 8n^2\ for\ all\ n \geq 1$$

$$\Longrightarrow f(n) \leq 21n^2$$

Now,

$$\Longrightarrow cn^3 \leq 21n^2$$

$$\Longrightarrow c \leq 21n^{2-3}\ i.e. 21n^{-1}$$

$$\boxed{c \leq \frac{21}{n}}$$

And

$$\Longrightarrow cn^3 - 21n^2 \leq 0$$

$$\Longrightarrow n^2(cn - 21) \leq 0$$

$$\Longrightarrow cn - 21 \leq 0$$

$$\Longrightarrow cn \leq 21$$

$$\Longrightarrow n \leq \frac{21}{c}$$

$$\boxed{n \leq \frac{21}{c}}$$

Now we can see that

$$\boldsymbol{c \le \frac{21}{n}}$$

and

$$\boldsymbol{n \le \frac{21}{c}}$$

c can be $\frac{21}{n}$ or less than $\frac{21}{n}$ as c is a positive number and constant but $\boldsymbol{n} \le \frac{21}{c}$ implies n is smaller than constant c which is not true,

Suppose c is 10 , then n is $\frac{21}{10} \approx 2(\boldsymbol{approx})$

Hence we can say that this cannot be proved as there is no positive number $'\boldsymbol{c}'$ for which this condition holds good.

Therefore: $\boldsymbol{f(n) \notin \Omega(n^3)}$

EXAMPLE 4

4) $If\ the\ relation\ f(n) = 100n + 5$ holds, $prove\ that\ f(n) is\ not\ \Omega(n^2)$

SOLUTION

We know by definition of Big Omega Ω notation:

$f(n) = \Omega(g(n)), if\ 0 \leq c \times g(n) \leq f(n)\ for\ all\ n \geq n_0,$
$where\ c\ and\ n_0\ are$ constants.

Note : By the equation we can see that :
$f(n) \geq c \times g(n)$

$$\Rightarrow 100n + 5 \geq c \times n^2$$

$$\Rightarrow f(n) \leq 100n + 5n\ for\ all\ n \geq 1$$

$$\Longrightarrow f(n) \leq 105n$$

$Now,$

$$\Longrightarrow cn^2 \leq 105n$$

$$\Longrightarrow cn^2 - 105n \leq 0$$

$$\Longrightarrow n(cn - 105) \leq 0$$

$$\Longrightarrow (cn - 105) \leq 0$$

$$\Longrightarrow cn \leq 105$$

$$\Longrightarrow n \leq \frac{105}{c}$$

$\Longrightarrow$ ***Contradiction**: **n cannot be smaller than a constant.***

EXAMPLE 5

$1.\ 2n = \Omega(n)$

$2.\ n^3 = \Omega(n^3)$

$3.\ logn = \Omega(logn)$