GUIDELINES FOR ASYMPTOTIC ANALYSIS-PART 2

$Now\ If\ there\ is\ three\ for\ Loop\ running$:

$$
\begin{aligned}
&for(i = 1; i \leq n; i++)\{\\
&\quad for(i = 1; i \leq n; i++)\{\\
&\quad\quad for(i = 1; i \leq n; i++)\{\\
&\quad\quad\quad c = c + 1;\\
&\quad\quad\}\\
&\quad\}\\
&\}
\end{aligned}
$$

$1st\ Loop\ will\ execute\ in\ 'n'\ time$, $Second\ loop\ will\ execute\ 1\ to\ n\ in\ n\ time$ $and\ third\ loop\ will\ execute\ 1\ to\ n\ in\ n^2\ time\ i.e. considering\ 1st\ and\ 2nd\ Loop$ $or, after\ running\ 2nd\ loop\ 3rd\ loop\ runs\ n\ times\ 1\ to\ n.$

, $hence\ Total\ time\ complexity\ taken: c \times n \times n \times n = cn^3 = O(n^3)$

Evaluation

1. $1st\ loop\ runs\ in\ O(n).$

2. $2nd\ \ loop\ runs\ in\ n\ time\ 1\ to\ n\ i.e.\ \mathrm{O}(\sum_{n=1}^{n} n)$.

$$\mathrm{O}\left(\sum_{n=1}^{n} n\right) = \mathrm{O}(1+2+3+\cdots+n) = \mathrm{O}\left(\frac{n^2+n}{2}\right) = \mathrm{O}(n^2)$$

3. $3rd\ \ loop\ runs\ in\ n^2\ time\ 1\ to\ n\ i.e.\ \mathrm{O}(\sum_{n=1}^{n} n^2)$.

$$\mathrm{O}\left(\sum_{n=1}^{n} n^2\right) = \mathrm{O}(1^2+2^2+3^2+\cdots+n^2)$$

$$= \mathrm{O}\left(\frac{2n^3+3n^2+n}{6}\right) = \mathrm{O}\left(\frac{2n^3}{6}+\frac{3n^2}{6}+\frac{n}{6}\right)$$

$$= \mathrm{O}\left(\frac{1}{6}(2n^3+3n^2+n)\right) = \mathrm{O}(2n^3) = \mathrm{O}(n^3)$$

$Hence\ by\ Addition\ rule\ of\ Asymptotic\ Notation:$

$$\Longrightarrow \mathrm{O}(n) + O(n^2) + O(n^3) = \mathrm{O}\{max(n+n^2+n^3)\}$$

$$= O(n^3)$$

$Now\ If\ there\ is\ four\ \ Loop\ running$:

$$
\begin{aligned}
&for(i = 1; i \leq n; i++)\{ \\
&\quad for(i = 1; i \leq n; i++)\{ \\
&\quad\quad for(i = 1; i \leq n; i++)\{ \\
&\quad\quad\quad for(i = 1; i \leq n; i++)\{ \\
&\quad\quad\quad\quad c = c + 1; \\
&\quad\quad\quad \} \\
&\quad\quad \} \\
&\quad \} \\
&\}
\end{aligned}
$$

$1st\ Loop\ will\ execute\ in\ 'n'\ time$, $Second\ loop\ will\ execute\ 1\ to\ n\ in\ n\ time$ $and\ third\ loop\ will\ execute\ 1\ to\ n\ in\ n^2\ time\ i.e.\ considering\ 1st\ and\ 2nd\ Loop$ $or, after\ running\ 2nd\ loop\ 3rd\ loop\ runs\ n\ times\ 1\ to\ n.$

$Fourth\ loop\ will\ be\ running\ for\ n^3\ times\ 1\ to\ n\ or\ after\ running\ of\ third\ loop$ $1\ to\ n\ of\ n\ times$.

, $hence\ Total\ time\ complexity\ taken: c \times n \times n \times n \times n = cn^4$
$= O(n^4)$

Evaluation

1. $1st\ loop\ runs\ in\ O(n).$

2. $2nd\ \ loop\ runs\ in\ n\ time\ 1\ to\ n\ i.e.\ O(\sum_{n=1}^{n} n)$.

$$O\left(\sum_{n=1}^{n} n\right) = O(1 + 2 + 3 + \cdots + n) = O\left(\frac{n^2 + n}{2}\right) = O(n^2)$$

3. $\boldsymbol{3rd\ \ loop\ runs\ in\ n^2\ time\ 1\ to\ n\ i.e.}\ \mathbf{O}(\sum_{n=1}^{n} n^2)$.

$$\mathbf{O}\left(\sum_{n=1}^{n} n^2\right) = \mathbf{O}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

$$= \mathbf{O}\left(\frac{2n^3 + 3n^2 + n}{6}\right) = \mathbf{O}\left(\frac{2n^3}{6} + \frac{3n^2}{6} + \frac{n}{6}\right)$$

$$= \mathbf{O}\left(\frac{1}{6}(2n^3 + 3n^2 + n)\right) = \mathbf{O}(2n^3) = \mathbf{O}(n^3)$$

4. $\boldsymbol{4rth\ loop\ runs\ in\ n^3\ times\ 1to\ n\ i.e.}\ \mathbf{O}(\sum_{n=1}^{n} n^3)$

$$\mathbf{O}\left(\sum_{n=1}^{n} n^3\right) = \mathbf{O}(1^3 + 2^3 + 3^3 + \cdots + n^3)$$

$$= \mathbf{O}\left(\frac{(n^4 + 2n^3 + n^2)}{4}\right) = \mathbf{O}\left(\frac{n^4}{4} + \frac{2n^3}{4} + \frac{n^2}{4}\right)$$

$$= \mathbf{O}\left(\frac{1}{4}(n^4 + 2n^3 + n^2)\right) = \mathbf{O}(n^4)$$

$\boldsymbol{Hence\ by\ Addition\ rule\ of\ \ Asymptotic\ Notation}$:

$\Rightarrow \mathbf{O}(n) + O(n^2) + O(n^3) + O(n^4)$

$= \ \mathbf{O}\{max(n + n^2 + n^3 + n^4)\}$

$= O(n^4)$

$$\textit{Deduction of } \sum_{n=1}^{n} n^2 = (1^2 + 2^2 + 3^2 + \cdots + n^2) = \frac{2n^3 + 3n^2 + n}{6}$$

Solution

From Part 1 we know By Growth of Series we got:

$$\sum_{n=1}^{n} n = (1 + 2 + 3 + \cdots + n) = \frac{n(n+1)}{2}$$

$\textit{Now for } (1^2 + 2^2 + 3^2 + \cdots + n^2), \textit{we have}:$

$$\sum_{n=1}^{n} n^2 = \left(1^2 + 2^2 + 3^2 + \cdots + n^2\right)$$

$\textit{Lets take} : \textit{Binomial Series of Expansion}$

$$(a+b)^n = n_{C_0} a^n b^0 + n_{C_1} a^{n-1} b^1 + n_{C_2} a^{n-2} b^2 + \cdots + n_{C_n} a^0 b^n$$

$\textit{Now if we take}: (n+1)^3$

$$= 3_{C_0} \times n^3 \times 1^0 + 3_{C_1} \times n^2 \times 1^1 + 3_{C_2} \times n^1 \times 1^2 + 3_{C_3} \times n^0 \times 1^3$$

$$= 1 \times n^3 \times 1^0 + 3 \times n^2 \times 1^1 + 3 \times n^1 \times 1^2 + 1 \times n^0 \times 1^3$$

$$= n^3 + 3n^2 + 3n + 1$$

$\textit{Now},$

$$\Rightarrow (n+1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1$$

$\textit{Now, putting } n = 1, 2, 3, 4, 5, \ldots, n-1, n \textit{ and Adding it we get}:$

$$\Rightarrow (1 + 1 = 2)^3 - 1^3 = 3 \times 1^2 + 3 \times 1 + 1$$

Hence,

$$2^3 - 1^3 = 3 \times 1^2 + 3 \times 1 + 1$$

$$3^3 - 2^3 = 3 \times 2^2 + 3 \times 2 + 1 \text{ [As we add } 2^3 \; and \; 2^3 gets \; cancelled\text{]}$$

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$$n^3 - (n-1)^3 = 3 \times (n-1)^2 + 3 \times (n-1) + 1$$

$$(n+1)^3 - n^3 = 3 \times n^2 + 3 \times n + 1 \text{[As we add } n^3 \; and \; n^3 gets \; cancelled\text{]}$$

$$(n+1)^3 - 1 = 3(1^2 + 2^2 + \cdots + (n-1)^2 + n^2) + 3(1 + 2 + \cdots + n) + (1 \times n = n)$$

$$\Rightarrow (n+1)^3 - 1 = 3 \times \sum_{n=1}^{n} n^2 + 3 \times \sum_{n=1}^{n} n + n$$

$$\Rightarrow (n+1)^3 - 1 - 3 \times \sum_{n=1}^{n} n - n = 3 \times \sum_{n=1}^{n} n^2$$

We know: $\sum_{n=1}^{n} n = (1 + 2 + 3 + \cdots + n) = \frac{n(n+1)}{2}$

$$\Rightarrow (n+1)^3 - 1 - 3 \times \left(\frac{n(n+1)}{2}\right) - n = 3 \times \sum_{n=1}^{n} n^2$$

Putting

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

$$\Rightarrow n^3 + 3n^2 + 3n + 1 - 1 - 3 \times \left(\frac{n(n+1)}{2}\right) - n = 3 \times \sum_{n=1}^{n} n^2$$

$$\Rightarrow n^3 + 3n^2 + 3n - n - 3 \times \left(\frac{n(n+1)}{2}\right) = 3 \times \sum_{n=1}^{n} n^2$$

$$\Rightarrow n^3 + 3n^2 + 2n - 3 \times \left(\frac{n(n+1)}{2}\right) = 3 \times \sum_{n=1}^{n} n^2$$

$$\Rightarrow 3 \times \sum_{n=1}^{n} n^2 = n^3 + 3n^2 + 2n - \left(\frac{3n^2 + 3n}{2}\right)$$

$$\Rightarrow 3 \times \sum_{n=1}^{n} n^2 = \frac{2n^3 + 6n^2 + 4n - 3n^2 - 3n}{2}$$

$$\Rightarrow 3 \times \sum_{n=1}^{n} n^2 = \frac{2n^3 + 3n^2 + n}{2}$$

$$\Rightarrow \sum_{n=1}^{n} n^2 = \frac{2n^3 + 3n^2 + n}{6}$$

Therefore, we got for

$$\sum_{n=1}^{n} n^2 = \left(1^2 + 2^2 + 3^2 + \cdots + n^2\right) = \frac{2n^3 + 3n^2 + n}{6}$$

$Deduction\ of\ \sum_{n=1}^{n} n^3 = (1^3 + 2^3 + 3^3 + \cdots + n^3) = \left(\frac{n(n+1)}{2}\right)^2$

$Solution$:

$$\sum_{n=1}^{n} n^2 = \left(1^3 + 2^3 + 3^3 + \cdots + n^3\right)$$

$Lets\ take : Binomial\ Series\ of\ Expansion$

$$(a+b)^n = n_{C_0} a^n b^0 + n_{C_1} a^{n-1} b^1 + n_{C_2} a^{n-2} b^2 + \cdots + n_{C_n} a^0 b^n$$

$Now\ if\ we\ take: (n+1)^4$

$$= 4_{C_0} \times n^4 \times 1^0 + 4_{C_1} \times n^3 \times 1^1 + 4_{C_2} \times n^2 \times 1^2 + 4_{C_3} \times n^1 \times 1^3 + 4_{C_4} \times n^0 \times 1^4$$

$$= 1 \times n^4 \times 1 + 4 \times n^3 \times 1 + 6 \times n^2 \times 1 + 4 \times n^1 \times 1 + 1 \times 1 \times 1$$

$$= n^4 + 4n^3 + 6n^2 + 4n + 1$$

$$= (n+1)^4 - n^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 - n^4 = 4n^3 + 6n^2 + 4n + 1$$

$Now, putting\ n = 1, 2, 3, 4, 5, \ldots, n-1, n\ and\ Adding\ it\ we\ get$:

$$\Rightarrow (1 + 1 = 2)^4 - 1^4 = 4 \times 1^3 + 6 \times 1^2 + 4 \times 1 + 1$$

Hence,

$\mathbf{2^4 - 1^3 = 4 \times 1^3 + 6 \times 1^2 + 4 \times 1 + 1}$

$\mathbf{3^4 - 2^4 = 4 \times 2^3 + 6 \times 2^2 + 4 \times 2 + 1}$**[As we add** $\mathbf{2^4}$ ***and*** $\mathbf{2^4}$ ***gets cancelled*****]**

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$\mathbf{n^4 - (n-1)^4 = 4 \times (n-1)^3 + 6 \times (n-1)^2 + 4 \times (n-1) + 1}$

$\mathbf{(n+1)^4 - n^4 = 4 \times n^3 + 6 \times (n)^2 + 4 \times (n) + 1}$**[As we add** $\mathbf{n^4}$ ***and*** $\mathbf{n^4}$ ***gets cancelled*****]**

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$$\mathbf{(n+1)^4 - 1^4 = 4(1^3 + 2^3 + \cdots + (n-1)^3 + n^3) + 6(1^2 + 2^2 + \cdots + (n-1)^2 + n^2) + 4(1 + 2 + 3 + \cdots + n) + (1 \times n = n)}$$

$$\mathbf{(n+1)^4 - 1^4 = 4 \times \sum_{n=1}^{n} n^3 + 6 \times \sum_{n=1}^{n} n^2 + 4 \times \sum_{n=1}^{n} n + n}$$

$$\mathbf{\Rightarrow 4 \times \sum_{n=1}^{n} n^3 = (n+1)^4 - 1^4 - 6 \times \sum_{n=1}^{n} n^2 - 4 \times \sum_{n=1}^{n} n - n}$$

$$\mathbf{\Rightarrow 4 \times \sum_{n=1}^{n} n^3 = (n+1)^4 - 1^4 - \left(6 \times \frac{2n^3 + 3n^2 + n}{6}\right) - \left(4 \times \left(\left(\frac{n^2+n}{2}\right)\right)\right) - n}$$

$$\mathbf{\Rightarrow 4 \times \sum_{n=1}^{n} n^3 = (n+1)^4 - 1^4 - (2n^3 + 3n^2 + n) - \left(2(n^2 + n)\right) - n}$$

$$\mathbf{\Rightarrow 4 \times \sum_{n=1}^{n} n^3 = n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 - (2n^3 + 3n^2 + n) - (2n^2 + 2n) - n}$$

$$\mathbf{\Rightarrow 4 \times \sum_{n=1}^{n} n^3 = n^4 + 4n^3 + 6n^2 + 4n - (2n^3 + 3n^2 + n) - (2n^2 + 2n) - n}$$

$$\Rightarrow 4 \times \sum_{n=1}^{n} n^3 = n^4 + 4n^3 + 6n^2 + 4n - 2n^3 - 3n^2 - n - 2n^2 - 2n - n$$

$$\Rightarrow 4 \times \sum_{n=1}^{n} n^3 = n^4 + 2n^3 + n^2$$

$$\Rightarrow \sum_{n=1}^{n} n^3 = \frac{n^4 + 2n^3 + n^2}{4}$$