9.A.1 BIG O NOTATION – EXAMPLES

Example 1

Let $f(n) = 3n^3$ for an algorithm. Prove that $f(n)$ of the algorithm is in $O(n^3)$.

Solution

The definition of the Big -Oh notation is that $f(n) \leq c \times g(n)$.

In order to prove that we know:

$f(n) \in O(g(n))$

or, $f(n) = O(g(n))$

Where $g(n)$ is in n^3.

Hence, we can show that:

$3n^3 \leq c \times n^3$, holds good for a positive number c and sufficiently large values of n.

$f(n) = 3n^3$ *can also be written as*:

$f(n) = 3n^3 + 0$

We can write it as:

$3n^3 + 0 \leq 3n^3 + n^3$

$\approx 3n^3 + 0 \leq 4n^3$ [Note $3n^3 = 3n^3$ *but* $3n^3$ *is always* $\leq 4n^3$]

[As highest degree of polynomial n *is* 3]

$\approx 0 \leq 4n^3 - 3n^3$

$\approx 0 \leq n^3$

Or, Divide n^2 in both side:

$$\approx \frac{0}{n^2} \leq \frac{n^3}{n^2}$$

$$\approx 0 \leq n$$

or, $n \geq 0$

Therefore $n_0 = 0$.

We can again write it as:

$3n^3 + 0 \leq 4 \times n^3$

Hence $c \geq 4$.

Therefore, $f = O(g)$

or in other words, the algorithm is $O(n^3)$.

Hence proved.

Example 2

Let $f(n) = 3n + 8$ for an algorithm. Prove that $f(n)$ of the algorithm is in $O(n)$.

Solution

The definition of the Big -Oh notation is that $f(n) \leq c \times g(n)$.

In order to prove that we know:

$f(n) \in O(g(n))$

or, $f(n) = O(g(n))$

Where $g(n)$ is in n.

Hence, we can show that:

$3n + 8 \leq c \times n$, holds good for a positive number c and sufficiently large values of n.

$3n + 8 \leq 3n + n$

$\approx 3n + 8 \leq 4n$ [$3n$ *is always* $\leq 4n$]

[As highest degree of polynomial n *is* 1]

$\therefore 3n + 8 \leq 4n$

$\approx 8 \leq 4n - 3n$

$\approx 8 \leq n$

$or, n \geq 8$

$\therefore n_0 = 8$

$3n + 8 \leq 4n$

Can be written as:

$3n + 8 \leq 4 \times n$

$Hence$, $c \geq 4$.

Therefore, f is O(g)

or in other words $O(n)$.

Hence proved.

Example 3

Let $f(n) = n^2 + 1$ for an algorithm. Prove that $f(n)$ of the algorithm is in $O(n^2)$.

Solution

The definition of the Big -Oh notation is that $f(n) \leq c \times g(n)$.

In order to prove that we know:

$f(n) \in O(g(n))$

or, $f(n) = O(g(n))$

Where $g(n)$ is in n^2.

Hence, we can show that:

$n^2 + 1 \leq c \times n^2$, holds good for a positive number c and sufficiently large values of n.

$n^2 + 1 \leq n^2 + n^2$

$\approx n^2 + 1 \leq 2n^2 [\, n^2 \; is \; always \; \leq 2n^2]$

[As highest degree of polynomial $n \; is \; 2$]

$\therefore n^2 + 1 \leq 2n^2$

$\approx 1 \leq 2n^2 - n^2$

$\approx 1 \leq n^2$

$\approx -n^2 + 1 \leq 0$

$\approx -(n^2 + 1) \leq 0$

$\approx -(n^2 - (-1)^2) \leq 0$

$As \; we \; know: x^2 - y^2 = (x + y)(x - y)$

$\approx -1 \times (\; (n + (-1)) \;\; (n - (-1)) \;\;) \leq 0$

$$\approx -\mathbf{1} \times (\ (\boldsymbol{n}-\mathbf{1})\ \ (\boldsymbol{n}+\mathbf{1})\ \ ) \leq \mathbf{0}$$

$$\approx (\boldsymbol{n}-\mathbf{1})(\boldsymbol{n}+\mathbf{1}) \leq \frac{\mathbf{0}}{-\mathbf{1}}$$

$$\approx (\boldsymbol{n}-\mathbf{1})(\boldsymbol{n}+\mathbf{1}) \leq \mathbf{0}$$

[From Quadratic Inequalities]

$$Say\ n = 1$$

$$\approx (1-1) \times (1+1) \leq 0$$

$$\approx 0 \times 2 \leq 0$$

$$\approx 0 \leq 0[True]$$

Also,

$$\approx 0 \geq 0[True]$$

$$Say\ n = 2$$

$$\approx (2-1) \times (2+1) \leq 0$$

$$\approx 1 \times 3 \leq 0$$

$$\approx 3 \leq 0[False]$$

But,

$$\approx 3 \geq 0[True]$$

Hence, we can say $n \geq 1$

$$Say\ n = 0$$

$\approx (0 - 1) \times (0 + 1) \leq 0$

$\approx -1 \times 1 \leq 0$

$\approx -1 \leq 0[True]$

$$Say\ n = -1$$

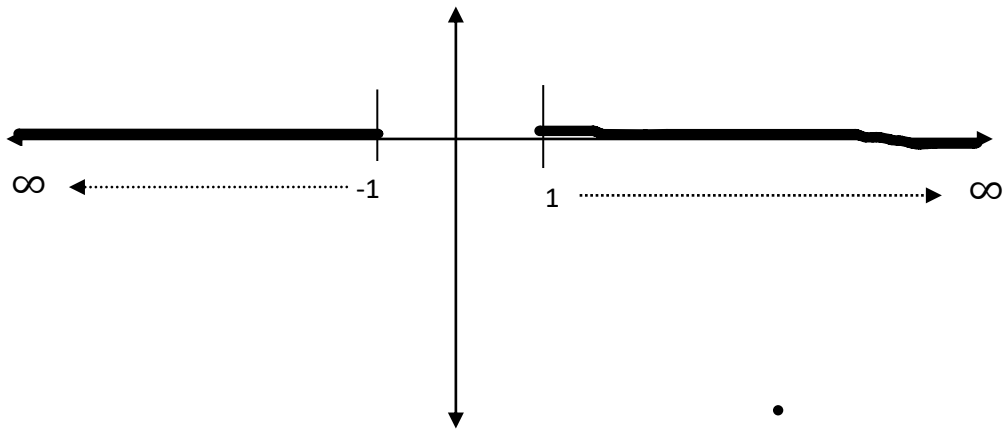
$\approx (-1 - 1) \times (-1 + 1) \leq 0$

$\approx -2 \times 0 \leq 0$

$\approx 0 \leq 0[True]$

That implies $n \leq -1$

$$\therefore \mathbf{-1 \geq n \geq 1}$$



$(-\infty, -1] \cup [1, \infty)$ *and they are in Sem − open and semi closed intervals*.

Hence now we get:

$$n^2 + 1 \leq 2n^2 \; for \; all \; n \geq 1$$

As from the definition of Big – O notation, all should be positive numbers but not negative numbers.

Therefore, we get $n_0 = 1$.

And,

$n^2 + 1 \leq 2 \times n^2$ *in terms of* $\boldsymbol{f(n) \leq c \times g(n)}$

Hence:

$\boldsymbol{c \geq 2}$.

Therefore, f is O(g)

or in other words $\boldsymbol{O(n^2)}$.

Hence proved.

Example 4

Let $\boldsymbol{f(n) = n^4 + 100n^2 + 50}$ for an algorithm. Prove that $\boldsymbol{f(n)}$ of the algorithm is in $\boldsymbol{O(n^4)}$.

Solution

The definition of the Big -Oh notation is that $\boldsymbol{f(n) \leq c \times g(n)}$.

In order to prove that we know:

$\boldsymbol{f(n) \in O(g(n))}$

or, $\boldsymbol{f(n) = O(g(n))}$

Where $\boldsymbol{g(n)}$ is in $\boldsymbol{n^4}$.

Hence, we can show that:

$$n^4 + 100n^2 + 50 \leq c \times n^4$$

$$\approx n^4 + 100n^2 + 50 \leq n^4 + n^4$$

$$\approx n^4 + 100n^2 + 50 \leq 2n^4 \; [n^4 \; is \; always \leq 2n^4]$$

[As highest degree of polynomial n is 4]

$Now,$

$$\approx -2n^4 + n^4 + 100n^2 + 50 \leq 0$$

$$\approx -n^4 + 100n^2 + 50 \leq 0$$

We can write the above equation as:

$$\approx -n^4 + 100n^2 + 50 = 0 \; and \; -n^4 + 100n^2 + 50 < 0$$

Taking the equation:

$$\approx -n^4 + 100n^2 + 50 = 0$$

Rewriting the equation, $u = n^2 \; and \; u^2 = n^4$:

$$\approx -u^2 + 100u + 50 = 0$$

Solving with quadratic equation formula:

Quadratic Equation of the form $ax^2 + bx + c = 0$:

$$x_{1,2} = \frac{\left(-b \pm \sqrt{b^2 - 4ac}\right)}{2a}$$

For a $=-1$, b $= 100$, c $= 50$

$$u_{1,2} = \frac{\left(-100 \pm \sqrt{100^2 - 4(-1)(50)}\right)}{2(-1)}$$

$$u_{1,2} = \frac{\left(-100 \pm \sqrt{100^2 - 4(-1)(50)} \right)}{-2}$$

$$u_{1,2} = \frac{\left(-100 \pm \sqrt{100^2 + 200} \right)}{-2}$$

$$u_{1,2} = \frac{\left(-100 \pm \sqrt{10000 + 200} \right)}{-2}$$

$$u_{1,2} = \frac{\left(-100 \pm \sqrt{10200} \right)}{-2}$$

$$Using\ prime\ factorization\ of\ 10200 = 2^3 \times 3 \times 5^2 \times 17$$

$$u_{1,2} = \frac{\left(-100 \pm \sqrt{2^3 \times 3 \times 5^2 \times 17} \right)}{-2}$$

Applying exponent rule: $a^{b+c} = a^b \times a^c$

$$u_{1,2} = \frac{\left(-100 \pm \sqrt{2^2 \times 2 \times 3 \times 5^2 \times 17} \right)}{-2}$$

Applying radical rule: $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

$$u_{1,2} = \frac{\left(-100 \pm \sqrt{2^2} \times \sqrt{5^2} \times \sqrt{2 \times 3 \times 17} \right)}{-2}$$

Applying radical rule: $\sqrt[n]{a^n} = a$

$$u_{1,2} = \frac{\left(-100 \pm 2 \times 5 \times \sqrt{2 \times 3 \times 17}\right)}{-2}$$

$$u_{1,2} = \frac{\left(-100 \pm 10 \times \sqrt{2 \times 3 \times 17}\right)}{-2}$$

$$u_{1,2} = \frac{\left(-100 \pm 10\sqrt{102}\right)}{-2}$$

$$u = \frac{\left(-100 + 10\sqrt{102}\right)}{-2}$$

$$= \frac{10\left(-10 + \sqrt{102}\right)}{-2}$$

$$= -5\left(-10 + \sqrt{102}\right) - -i$$

$$u = \frac{\left(-100 - 10\sqrt{102}\right)}{-2}$$

$$= \frac{-10\left(10 + \sqrt{102}\right)}{-2}$$

$$= 5\left(10 + \sqrt{102}\right) - -ii$$

Substituting back $u = n^2$ and solving for n,

$n^2 = -5\left(-10 + \sqrt{102}\right)$

$\left(g(x)\right)^2$ *cannot be negative for* $x \in R$ *, hence no solution.*

$n^2 = 5\left(10 + \sqrt{102}\right)$

We know $\left(g(x)\right)^2 = f(a)$ *the solutions are* $\sqrt{f(a)}$, $-\sqrt{f(a)}$

$\therefore n = \sqrt{5\left(10 + \sqrt{102}\right)}$ and $n = -\sqrt{5\left(10 + \sqrt{102}\right)}$

Now we can easily understand,

$\approx -n^4 + 100n^2 + 50 = 0$

putting n= $-\sqrt{5(10+\sqrt{102})}$ in the above equation we will get a negative value while putting n= $\sqrt{5(10+\sqrt{102})}$ in the above equation we will get a positive value,

Hence:

$$n \leq -\sqrt{5(10+\sqrt{102})}$$

or

$$n \geq \sqrt{5(10+\sqrt{102})}$$

Now what does $\sqrt{5(10+\sqrt{102})}$ $stand\ for: 10.028484537$

As by definition: the function f and g should be set of natural numbers and it should grow by time (growth rate) we take $n \geq 11$.

Or by analysis:

$\approx -n^4 + 100n^2 + 50 \leq 0$

$\approx 100n^2 + 50 \leq n^4$

$or, n^4 \geq 100n^2 + 50$

if we take $n = 10$

$\approx 10^4 \geq 100 \times 10^2 + 50$

$\approx 10000 \geq 10000 + 50$

$\approx 10000 \geq 10050[Not\ True]$

if we take $n = 11$

$\approx 11^4 \geq 100 \times 11^2 + 50$

$\approx 14641 \geq 1210 + 50$

$\approx 14641 \geq 1260[\ True]$

Hence, we confirm in both the ways that:

$n^4 + 100n^2 + 50 \leq 2n^4\ for\ all\ n \geq 11$

Therefore, we get $n_0 = 11$.

And,

$n^4 + 100n^2 + 50 \leq 2 \times n^4\ in\ terms\ of\ f(n) \leq c \times g(n)$

Hence:

$c \geq 2$.

Therefore, f is O(g)

or in other words $O(n^4)$.

Hence proved.

Example 5

Let $f(n) = n$ for an algorithm. Let $g(n) = n$. Prove that $f(n)\ of\ this\ algorithm\ is\ in\ O(n)$.

Solution

$f(n) \leq c \times g(n)$

$\Rightarrow n \leq 1 \times n, for\ all\ n \geq 1$

$\Rightarrow n = O(n), c \geq 1\ and\ n_0 = 1$

Example 6

Let $f(n) = 410$ for an algorithm. Let $g(n) = 410$. Prove that $f(n)$ *of this algorithm is in* $O(1)$.

Solution

$f(n) \leq c \times g(n)$

$\Rightarrow 410 \leq 1 \times 410, for\ all\ n \geq 1$

$\Rightarrow 410 = O(1), c \geq 1\ and\ n_0 = 1$

Example 7

Find upper bound for $f(n) = 2n^3 - 2n^2$. Prove that $f(n)$ *of this algorithm is in* $O(n^3)$.

Solution

The definition of Big-Oh notation is that $f(n) \leq c \times g(n)$.

$\Rightarrow f(n) \leq (2n^3 + 2n^3) - 2n^3$

[*As growth rate is doubled* $\begin{pmatrix} (2n^3 + n^3) - 2n^3 = n^3 \leq 2n^3 \\ hence\ we\ need\ ? \geq 2n^3 \end{pmatrix}$,

$Therefore, (2n^3 + 2n^3) - 2n^3 \geq 2n^3$]

$\Rightarrow f(n) \leq (4n^3) - 2n^3$

$\Rightarrow f(n) \leq 2n^3$

And we can say:

$2n^3 - 2n^2 \leq 2n^3$

Deducing it to $f(n) \leq c \times g(n), we\ get$:

$\Rightarrow 2n^3 - 2n^2 \leq 2 \times n^3$

Hence c = 2 and $g(n) = n^3$

By Inequality deduction:

$\Rightarrow -2n^2 \leq 2n^3 - 2n^3$

$\Rightarrow -2n^2 \leq 0$

$\Rightarrow n^2 \leq 0$

$\Rightarrow n \leq 0$

More specifically:

N	$2n^3 - 2n^2$	$2n^3$
0	0	0
1	0	1
2	8	16

Hence it starts from 1, hence $n \geq 1$

Therefore $n_0 = 1$.

Hence $f = O(g)$

or $f(n) = O(n^3)$

NO UNIQUENESS in Above Method

There is no unique set of values for n_0 and c in proving the asymptotic bounds.

Let us consider, $100n + 5 = O(n)$. For this function there are multiple n_0 and c values possible.

Solution1:

$100n+5 \leq 100n+n$

$\approx 100n+5 \leq 101n$, for all $n \geq 5, n_0 = 5$ and $c \geq 101$ is a solution.

Solution2:

$100n+5 \leq 100n+5n$

$\approx 100n+5 \leq 105n$, for all $n \geq 1, n_0 = 1$ and $c \geq 105$ is also a solution.

Example 8

Let $f(n) = 3n^3 + 2n^2 + 3$ for an algorithm. Let $g(n) = n^3$. Prove that $f(n)$ *of this algorithm is in* $O(n^3)$.

Solution

The definition of Big-Oh notation is that $f(n) \leq c \times g(n)$. Therefore, one must show that $3n^3 + 2n^2 + 3 \leq cn^3$ holds good for a positive number c and for sufficiently large values of n.

$f(n) = 3n^3 + 2n^2 + 3$

$f(n) \leq 3n^3 + 2n^3 + 3$ (*as growth of functions* n^2 *to* n^3)

$f(n) \leq 3n^3 + 2n^3 + 3n^3$ (3 *is less than* n^3)

$f(n) \leq 8n^3$

It can be observed that $c = 3 + 2 + 3 = 8$ (one can approximate $2n^2$ and 3 to $2n^3$ *and* $3n^3$ *respectively*) . This condition holds good for any values of $c \geq 8$.

Let the polynomial be

$$f(n) = \sum_{i=0}^{m} a_i n^i$$

whose degree is m. Then one can show that $f(n) = O(n^m)$.

$|f(n)| \leq |a_m|n^m + |a_{m-1}|n^{m-1} + \cdots + |a_1|n + |a_0|$

$\approx |f(n)| \leq |a_m|n^m + |a_{m-1}|n^m + \cdots + |a_1|n^m + |a_0|n^m$ *for all* $n \geq 1$

$$\approx \left(\sum_{i=0}^{m} |a_i|\right) n^m$$

$\approx c \times n^m$

$\approx O(n^m)$

Hence the above algorithm has $O(n^3)$.

This is another way we can prove the algorithm has *the complexity* .

Example 9

Let $f(n) = \frac{(2x^3+13\log_2 x)}{7n^2}$ *for an algorithm A. Prove that* $f(n)$ *of algorithm A is* $O(n)$.

Solution

It can be observed that $logx < x$ is always true. Therefore, one can argue that $13\log_2 x \leq 13x$ and as $13x \leq 13x^3$ always , one can rewrite $f(n)$ as follows:

$$f(n) \leq \frac{2x^3 + 13x^3}{7n^2}$$

$$f(n) \leq \frac{15x^3}{7n^2}$$

$$\cong 2n^{3-2}$$

$\cong 2n$ for all n > 1

$$\therefore f(n) = O(n)$$

Example 10

Consider an Algorithm that is assume to run in time $\mathbf{O}(n^2)$ $\boldsymbol{and\ that\ only\ five\ seconds\ to\ compute}$

results for an instance of size 30. How long will the algorithm take to compute the results if the instance size is increased to 50?

Solution

Recollect that $\mathbf{O}(n^2)$ implies that $f(n) \leq c \times g(n), \boldsymbol{that}$ is,

f(n)$\leq c \times n^2$. Here c is a constant that depends on the machine. As mentioned in the problem, the algorithm takes five seconds to compute 30 instances. Therefore, one can calculate the value of c as follows:

$cn^2\ steps = 5\ seconds$

$c(30)^2 = 900c\ steps.$

$$\therefore c = \frac{5}{900}$$

Once the values of c are decided , time for all other instances can easily be calculated . if the algorithm instance is increased to 50 , then the time taken by the program can be calculated as follows:

$c(50)^2 = 2500c$

Substituting the value of c in this equation, one gets $2500 \times \frac{5}{900} = \frac{125}{9}$

$= 13.88\ seconds$.

The calculation shows that the algorithm requires 13.88 seconds to execute 50 instances.

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