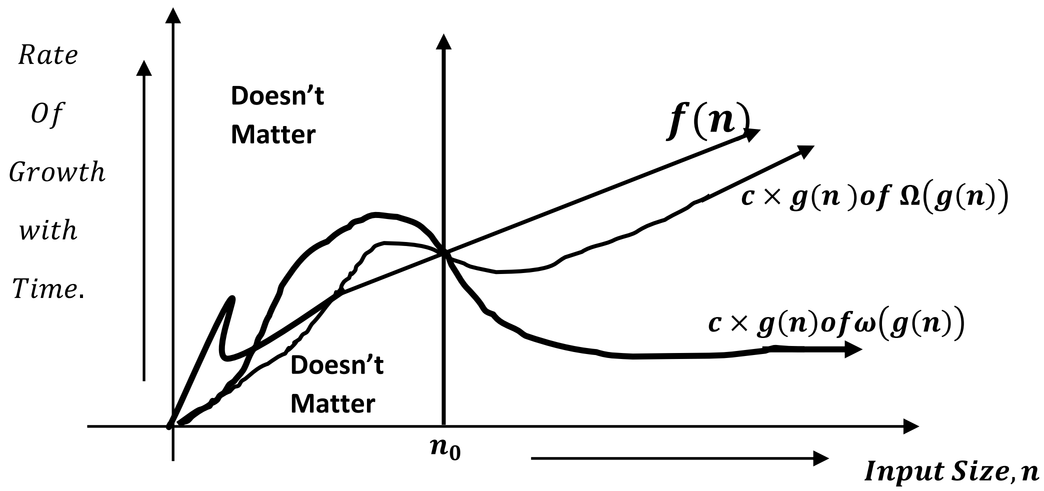
11. LITTLE – OMEGA (ω) NOTATION



DEFINITION: *Let f and g be two functions that map a set of natural numbers, that is $f: \mathbb{N} \to \mathbb{R}$. Let $\omega(g)$ be the set of all functions with a similar rate of growth. The relation $f(n) = \omega(g(n))$ holds, if there exists two positive constants c and n_0 such that :*

$$f(n) > c \times g(n) \text{ for all } n > n_0.$$

Example:1

$$Prove\ f(n) = 5n^2\ is\ in\ \omega(n)$$

SOLUTION:

We know by definition:

$$f(n) > c \times g(n), for\ all\ n > n_0$$

or,

$$0 < c \times g(n) < f(n), for\ all\ n > n_0$$

Given to prove $g(n) = n$.

$\Longrightarrow 5n^2 > c \times n$

We can write it as:

$\Longrightarrow f(n) > 5n$

Hence $c = 5\ \ and$:

$\Longrightarrow 5n^2 > 5n$

$\Longrightarrow n > 1$

Let $n_0 = 2\ as\ n > 1$

$Therefore\ we\ get\ n_0 = 2\ and\ c = 5$

N($n > n_0$)	$f(n) = 5n^2$	$\omega = c \times g(n) = 5 \times n$
2	20	10
3	45	15
4	80	20
5	125	25
6	180	30
7	245	35

Hence $5n^2$ *is in* $\omega(n)$.

If we differentiate between $\Omega(g(n))$ *and* $\omega(g(n))$, *we get*:

So for $\Omega(g(n))$ we see that :

$$0 \leq c \times g(n) \leq f(n), for\ all\ n \geq n_0$$

Therefore, we can tell that:

$$5n^2 \leq 5n^2, where\ c = 5\ and\ n_0 = 1$$

Furthermore, it is also correct that:

$\approx (5n^2 - n^2) \leq 5n^2$

$\approx 4n^2 \leq 5n^2$

Where $c = 4\ and$

$\approx 0 \leq n^2$

$\approx n \times n \geq 0$

$\approx n \geq \frac{0}{n}$

$\approx n \geq 0$

Also, we can say that $n \geq 1, hence\ n_0 = 1$.

As set of Natural Real Numbers.

$N(n \geq n_0)$	$f(n) = 5n^2$	$\Omega = c \times g(n) = 4 \times n^2$
1	5	4
2	20	16
3	45	36
4	80	64
5	125	100
6	180	144
7	245	196

Now plot the graph to see the difference between $\boldsymbol{\omega(g(n))}$ $\boldsymbol{and}$ $\boldsymbol{\Omega(g(n))}$.

Example:2

$$\boldsymbol{Prove\ f(n) = 3n^3 + 2n + 7\ is\ in\ \omega(n)}$$

SOLUTION:

We know by definition:

$$\boldsymbol{f(n) > c \times g(n)\ , for\ all\ n > n_0}$$

or,

$$\boldsymbol{0 < c \times g(n) < f(n), for\ all\ n > n_0}$$

Note we can write: $\boldsymbol{3n < 3n^3 + 2n + 7\ is\ true\ also}$:

$\boldsymbol{\Rightarrow f(n) > 3n + 2n + 7n}$

$\boldsymbol{\Rightarrow f(n) > 12n, is\ true.}$

Therefore:

$\boldsymbol{3n^3 + 2n + 7 > 12n}$

Hence $\boldsymbol{c = 12\ and}$:

$\boldsymbol{3n^3 + 2n + 7 > 12n}$

$\boldsymbol{\Rightarrow 3n^3 - 10n + 7 > 0}$

We can write it as:

$$(n-1)(3n^2+3n-7)<0$$

So we got:

$$\Rightarrow n-1<0$$
$$\Rightarrow -1<-n$$
$$\Rightarrow n>1$$

$$\Rightarrow 3n^2+3n-7<0$$

By Quadratic equation:

$$\Rightarrow n_{1,2}=\frac{\left(-b\pm\sqrt{b^2-4ac}\right)}{2a}$$
$$\Rightarrow \frac{\left(-3\pm\sqrt{3^2-4\times 3\times(-7)}\right)}{2\times 3}$$
$$\Rightarrow \frac{\left(-3\pm\sqrt{9+84}\right)}{6}$$
$$\Rightarrow \frac{\left(-3\pm\sqrt{93}\right)}{6}$$

$$\Rightarrow n_1=\frac{\left(-3-\sqrt{93}\right)}{6},\ n_2=\frac{\left(-3+\sqrt{93}\right)}{6}$$

Therefore, we get,

$$\Rightarrow n<\frac{\left(-3-\sqrt{93}\right)}{6}\ or\ 1<n<\frac{\left(-3+\sqrt{93}\right)}{6}$$

$$\Rightarrow n<-2.11\ (approx) or\ 1<n<1.11(approx)$$

We will take only : $n>1, let\ n_0=2$

$\mathrm{N}(n > n_0)$	$f(n) = 3n^3 + 2n + 7$	$\omega = c \times g(n) = 12n$
2	35	24
3	94	36
4	207	48
5	392	60
6	667	72
7	1050	84

Hence f is $\omega(g)$, or $f(n) = 3n^3 + 2n + 7$ is in $\omega(n)$.

If we differentiate between $\Omega(g(n))$ and $\omega(g(n))$, we get:

So for $\Omega(g(n))$ we see that:

$$0 \leq c \times g(n) \leq f(n), for\ all\ n \geq n_0$$

Therefore, we can tell that:

$3n^3 \leq 3n^3 + 2n + 7, where\ c = 3\ and\ g(n) = n^3\ and\ n_0 = 1$

$n \geq 1$.

$\mathrm{N}(n \geq n_0)$	$f(n) = 3n^3 + 2n + 7$	$\Omega = c \times g(n) = 3n^3$
1	12	3
2	35	24
3	94	81
4	207	192
5	392	375
6	667	648
7	1050	1029

Now plot the graph to see the difference between $\omega(g(n))$ and $\Omega(g(n))$.

LITTLE OMEGA DEFINITION IN LIMITS - LITTLE OH RATIO THEOREM

DEFINITION : $\boldsymbol{The\ relation\ f(n) = \omega\big(g(n)\big) holds\ good}$

$\boldsymbol{if\ and\ only\ if}\ \lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = \infty.$

Example:1

$\boldsymbol{Prove\ f(n) = 5n^2\ is\ in\ \omega(n)}$

SOLUTION:

$$\boldsymbol{We\ have}\ \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{5n^2}{n}$$

$$\lim_{n\to\infty} \frac{5n^2}{n} = \lim_{n\to\infty} 5n = \infty$$

[As by infinity property of limits : (Limits where x tends to ∞ acting upon polynomial function)= $\lim\limits_{n\to\infty}(ax^n + \cdots + bx + c) = \infty$

$, a > 0$]

Hence it satisfies:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty\ \boldsymbol{and\ it\ is\ in}\ \omega\big(g(n)\big).$$

Example:2

$$Prove\ f(n) = 3n^3 + 2n + 7\ is\ in\ \omega(n)$$

SOLUTION:

$$= \lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{(3n^3 + 2n + 7)}{n}.$$

$$= \lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{3n^3}{n} + \frac{2n}{n} + \frac{7}{n} = 3n^2 + 2 + \frac{7}{n}.$$

$$= \lim_{n\to\infty} 3n^2 + \lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{7}{n}$$

$$\approx \lim_{n\to\infty} (3n^2) = \infty$$

[As by infinity property of limits: (Limits where x tends to ∞ acting upon polynomial function) = $\lim_{n\to\infty}(ax^n + \cdots + bx + c) = \infty$

$, a > 0$]

$$\approx \lim_{n\to\infty} (2) = 2$$

[As $\lim_{n\to a} c = c\ , where\ c\ is\ constant.$]

$$\approx \lim_{n\to\infty} \left(\frac{7}{n}\right) = 0$$

[By infinity property ,$\lim_{n\to\infty}\left(\frac{c}{x^a}\right) = 0$]

Hence: $\infty + 2 + 0 = \infty$.

Therefore, it satisfies:

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \infty \textit{ and it is in } \omega\big(g(n)\big).$$

Example:3

Prove $\frac{n^2}{3} = \omega(n)$.

Solution

$$= \lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\left(\frac{\left(\frac{n^2}{3}\right)}{n}\right).$$

$$= \lim_{n\to\infty}\left(\frac{n^2}{3n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{3}\right)$$

$$= \frac{1}{3}\lim_{n\to\infty}(n)$$

[$\lim_{n\to a}\big(c \times f(n)\big) = c \times \lim_{n\to a} f(n), \textit{where c is constant}$]

$$= \frac{1}{3} \times \infty, \left[\lim_{n\to\infty} n = \infty\right]$$

$= \infty$

Hence it satisfies:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \text{ and it is in } \omega(g(n)).$$

One can observe that ω can be helpful in finding a loose lower bound and should not be used as tight bound.

For example: $\frac{n^3}{3} \neq \omega(n^2)$.

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