9.C.2.BIG THETA NOTATION WITH LIMITS-BIG THETA RATIO THEOREM

$$\textit{Definition}: \textit{If the limit } \lim_{n\to\infty}\frac{f(n)}{g(n)} = c \ \textit{ holds good}$$

$$\textit{then } f(n) = \Theta\big(g(n)\big).$$

EXAMPLE : IF $f(n) = n^4 + 3n^3 \ \textit{prove} \ f(n) = \Theta(n^4)$

SOLUTION:

By definition we can say:

$$n^4 \le n^4 + 3n^3 \le 2n^4$$

Hence $g(n) = n^4$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = c$$

$$\Rightarrow \lim_{n\to\infty}\frac{n^4 + 3n^3}{n^4}$$

$$\Rightarrow \lim_{n\to\infty}\frac{n^3(n+3)}{n^3(n)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{n+3}{n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{n}{n} + \frac{3}{n}$$

$$\Rightarrow \lim_{n\to\infty} 1 + \frac{3}{n}$$

We know that :

$$\lim_{x\to a}\big(f(x) \pm g(x)\big) = \lim_{x\to a}\big(f(x)\big) \pm \lim_{x\to a}\big(g(x)\big)$$

Hence

$$\Rightarrow \lim_{n\to\infty}(1) + \lim_{n\to\infty}\left(\frac{3}{n}\right)$$

We know that :

$$\lim_{x\to a}(c) = c\,, where\ c\ is\ constant.$$

And:

$$\lim_{x\to\infty}\left(\frac{c}{x^a}\right) = 0\,, Infinity\ property\ of\ Limit$$

Therefore , from above properties we get:

$$\Longrightarrow \mathbf{1 + 0 = 1}$$

Hence it meets the definition :

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c \ \ , \textit{Hence we can tell}:$$

$$f(n) = n^4 + 3n^3 \ \Longrightarrow \Theta(n^4)$$