MATHEMATICAL INDUCTION
IN ASYMPTOTIC NOTATION

EXAMPLE 1

Q 1) If $f(n) = n^4 + 3n^3$ *is in* $\Theta(n^4)$, *Prove that*

$n^4 \leq n^4 + 3n^3 \leq 2n^4$ *through Induction.*

Solution:

We know about Theta notation:

$= c_1 g(n) \leq f(n) \leq c_2 g(n), for\ all\ n \geq n_0$

Let $c_1 = 1$, $c_2 = 2$

We know in Mathematical Induction:

Suppose $P(n)$ *is a mathematical relation which is to be proved, for positive integral values of* n. *If we can prove that*:

$1.\ P(1)$ *is true*.

$2.\ P(m)$ *is true* .

$3.$ *If* $P(m)$ *is true then* $P(m+1)$ *is true*.

Then $P(n)$ *is true for any positive integral values of process is known as* "*method of mathematical induction*".

So, as per the process of induction:

For $c_1 = 1$

$$\Rightarrow (1) \times n^4 \leq n^4 + 3n^3$$

$$\Rightarrow n^4 \leq n^4 + 3n^3$$

$$\Rightarrow 0 \leq 3n^3$$

$$\Rightarrow 0 \leq 3n^2 \times n$$

$$\Rightarrow 0 \leq n$$

$$\Rightarrow n \geq 0$$

$$\Rightarrow 0 \leq n$$

For $c_2 = 2$

$$\Rightarrow n^4 + 3n^3 \geq 2n^4$$

$$\Rightarrow n^4 - 2n^4 \geq -3n^3$$

$$\Rightarrow -n^4 \geq -3n^3$$

$$\Rightarrow -\frac{n^4}{n^3} \geq -\frac{3n^3}{n^3}$$

$$\Rightarrow -n \leq -3$$

$\Rightarrow n \leq 3$

$Hence\ , 0 \leq n \leq 3$

Hence choice for $n_0\ as\ n_0 \leq n$

Therefore $n_0 = 3$.

Now let us apply Mathematical Induction:

Here instead of checking $P(1)\ we\ will\ check\ P(n_0)$:

For $P(3) =$

$(1)\ \times (3^4) \leq 3^4 + 3 \times 3^3 \leq 2 \times 3^4$

$81 \leq 162 \leq 162\ satisfies\ at\ P(3)$.

Hence $P(3)\ is\ true.$

$At\ P(m) we\ have:$

$\Rightarrow 1 \times m^4 \leq m^4 + 3 \times m^3 \leq 2 \times m^4$

$\Rightarrow m^4 \leq m^4 + 3m^3 \leq 2m^4$

Hence $P(m)\ is\ true.$

$At\ P(m+1) we\ have:$

$$\Longrightarrow \mathbf{1} \times (\boldsymbol{m} + \mathbf{1})^{\mathbf{4}} \leq (\boldsymbol{m} + \mathbf{1})^{\mathbf{4}} + \mathbf{3} \times (\boldsymbol{m} + \mathbf{1})^{\mathbf{3}} \leq \mathbf{2} \times (\boldsymbol{m} + \mathbf{1})^{\mathbf{4}}$$

$$\Longrightarrow (\boldsymbol{m} + \mathbf{1})^{\mathbf{4}} \leq (\boldsymbol{m} + \mathbf{1})^{\mathbf{4}} + \mathbf{3}(\boldsymbol{m} + \mathbf{1})^{\mathbf{3}} \leq \mathbf{2}(\boldsymbol{m} + \mathbf{1})^{\mathbf{4}}$$

Hence $\boldsymbol{P}(\boldsymbol{m} + \mathbf{1})$ $\boldsymbol{is\ true}$.

Thus, we prove that:

1. $\boldsymbol{P}(\boldsymbol{n_0})$ $\boldsymbol{is\ true}$.
2. $\boldsymbol{P}(\boldsymbol{m})$ $\boldsymbol{true\ \ then\ P(m + 1) is\ also\ true}$.

Hence Asymptotic notation is in:

$$\boldsymbol{n^4 \leq n^4 + 3n^3 \leq 2n^4}$$

EXAMPLE 2

Q 2) If $f(n) = n^2 + 1 = O(n^2)$, $prove\ n^2 + 1 \leq 2n^2$

$by\ Induction.$

Solution:

The definition of the Big -Oh notation is that $f(n) \leq c \times g(n)$.

Let $c = 2\ and\ given\ g(n) = n^2\ and\ f(n) = n^2 + 1$.

We know in Mathematical Induction:

$Suppose\ P(n)\ is\ a\ mathematical\ relation\ which\ is\ to\ be\ proved,$

$for\ positive\ integral\ values\ of\ n. If\ we\ can\ prove\ that:$

$1. P(1)\ is\ true.$

$2. P(m)\ is\ true\ .$

$3. If\ \ P(m)\ is\ true\ then\ P(m+1)\ is\ true.$

Then $P(n)\ is\ true\ for\ any\ positive\ integral\ values$

$of\ process\ is\ known\ as$ "$method\ of\ mathematical\ induction$".

$\Rightarrow n^2 + 1 \leq 2n^2$

$\Rightarrow 1 \leq 2n^2 - n^2$

$\Rightarrow n^2 \geq 1$

$\Rightarrow n \geq 1$

Hence $n_0 = 1$, $Therefore$:

Now let us apply Mathematical Induction:

Here instead of checking $\boldsymbol{P(1)}$ *we will check* $\boldsymbol{P(n_0)}$:

For P(1) we get:

$\Rightarrow \mathbf{1^2 + 1 \leq 2 \times 1^2}$

$\Rightarrow \mathbf{2 \leq 2}$

Hence $\boldsymbol{P(1)}$ ***is true*** .

For P(m) we get:

$\Rightarrow \boldsymbol{m^2 + m \leq 2 \times m^2}$

$\Rightarrow \boldsymbol{m^2 + m \leq 2m^2}$

Hence $\boldsymbol{P(m)}$ ***is true then for*** :

For P(m+1) we get:

$\Rightarrow \boldsymbol{(m+1)^2 + (m+1) \leq 2(m+1)^2}$

Hence $\boldsymbol{P(m)}$ ***is true then for*** $\boldsymbol{P(m+1)}$***is also true*** :

Therefore , f(n)is in $\boldsymbol{n^2 + 1 \leq 2n^2}$ ***is true .***

EXAMPLE 3

Q 2) If $\boldsymbol{f(n) = n^2 + 1 = \Omega(n^2)}$, $\boldsymbol{prove\ n^2 \leq n^2 + 1}$

$\boldsymbol{by\ Induction.}$

Solution:

The definition of the Big -Omega notation is that $\boldsymbol{f(n) \geq c \times g(n)}$.

Let $\boldsymbol{c = 1\ and\ given\ g(n) = n^2\ and\ f(n) = n^2 + 1}$.

And $\boldsymbol{n_0 = 1}$.

We know in Mathematical Induction:

$\boldsymbol{Suppose\ P(n)\ is\ a\ mathematical\ relation\ which\ is\ to\ be\ proved,}$

$\boldsymbol{for\ positive\ integral\ values\ of\ n. If\ we\ can\ prove\ that}$:

$\boldsymbol{1. P(1)\ is\ true.}$

$\boldsymbol{2. P(m)\ is\ true\ .}$

$\boldsymbol{3. If\ P(m)\ is\ true\ then\ P(m+1)\ is\ true.}$

Then $\boldsymbol{P(n)\ is\ true\ for\ any\ positive\ integral\ values}$

$\boldsymbol{of\ process\ is\ known\ as}$ "$\boldsymbol{method\ of\ mathematical\ induction}$".

Here instead of checking $\boldsymbol{P(1)\ we\ will\ check\ P(n_0)}$:

For P (1) we get:

$\boldsymbol{\Rightarrow 1^2 \leq 1^2 + 1}$

$\boldsymbol{\Rightarrow 1 \leq 2}$

Hence $\boldsymbol{P(1)\ is\ true\ .}$

For P(m) we get:

$\Longrightarrow \boldsymbol{m^2 \leq m^2 + 1}$

Hence $\boldsymbol{P(m)\ is\ true\ then\ for}$:

For P(m+1) we get:

$\Longrightarrow \boldsymbol{(m+1)^2 \leq (m+1)^2 + 1}$

Hence $\boldsymbol{P(m)\ is\ true\ then\ for\ P(m+1) is\ also\ true}$:

$$\boldsymbol{Therefore\ , f(n) is\ \ in\ n^2 \leq n^2 + 1\ is\ true\ .}$$