Simulation Lab (MC 503)

Numerical integration

If a function $y = f(x)$ is not known explicitly but we are given only a set of numerical values of the function corresponding to some values of x, the numerical integration is a process of finding the numerical value of a integral:

$$I = \int_a^b f(x)\ dx \tag{1}$$

The numerical methods give us, in general, an approximate value of the above definite integral [see eq.(1)].
Here we follow some of them. Such as

- Simpson's 1/3 rule.

- Trapezoidal rule.

Let us consider the above integral [eq.(1)]. We divide the range of the integration $[a, b]$ into n equal sub- intervals, each of width h, by the points $a = x_0, x_1 = x_0+h, x_2 = x_0+2h, \cdots x_r = x_0 + rh, \cdots x_{n-1} = x_0 + (n-1)h, x_n = x_0 + nh = b$.

$\therefore b - a = x_n - x_0 = nh$

$$\therefore\ I = \int_a^b f(x)\ dx = \int_{x_0}^{x_0+nh} f(x)\ dx \tag{2}$$

1 Simpson's $1/3$ rule.

First we consider $n = 2$. So, there are only three functional values, $y_0 = f(x_0) = f(a), y_1 = f(x_0 + h), y_2 = f(x_0 + 2h) = f(x_n) = f(b)$

$$I_S^{2\ quadrature} = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] \tag{3}$$

Equation (3) is called *Simpson's 1/3 rule* for numerical integration.
The error committed in this rule is given by

$$E_S \simeq -\frac{h}{90}\left[y_{-1} - 4y_0 + 6y_1 - 4y_2 + y_3\right] \tag{4}$$

Now we assume $n = 2m$. Then the functional values look like $a = x_0, x_0+h, x_0+2h, \cdots, x_0+(2m-1)h, x_0+2mh = b = x_n$. Then

$$I_S^{2m\ quadrature} = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + \cdots + y_{n-3} + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-4} + y_{n-2})\right] \tag{5}$$

Error can be formulated as

$$\begin{aligned} E_S^{2m\ quadrature} &\simeq -\frac{h}{90}[y_{-1} + y_{n+1} - 4(y_0 + y_n) + 7(y_1 + y_{n-1}) \\ &-8(y_2 + y_4 + y_{n-2}) + 8(y_3 + y_5 + \cdots + y_{n-3})] \quad for\ (n \geq 6) \end{aligned} \tag{6}$$

for $n = 4$ it looks like $-\frac{h}{90}[y_{-1} + y_5 - 4(y_0 + y_4) + 7(y_1 + y_3) - 8y_2]$

Remark 1. *The Simpson's 1/3 rule is applicable only when the number of sub-intervals $n(= 2m)$ is even.*

2 Trapezoidal rule

It can be solved by

$$I_T = \frac{h}{2}\left[y_0 + y_n + 2\sum_{i=1}^{n-1} y_i\right] \tag{7}$$

Error can be approximated as

$$E_T \simeq -\frac{h}{12}[y_{-1} + y_n - (y_0 + y_{n-1})] \tag{8}$$

Remark 2. *This rule is applicable to any number of sub-intervals n, even or odd. The accuracy of the computed result can be improved by increasing n.*

3 Exercise (Assignment 3)

1. For $n = 10$, evaluate $\int_0^1 (4x - 3x^2)\ dx$ using Simpson's 1/3 rule and Trapezoidal rule respectively.

2. For $h = 1$, evaluate $\int_0^5 \frac{dx}{1+x}$ using Trapezoidal rule.

3. For $n = 6$, evaluate $\int_0^5 \frac{x\ dx}{1+x}$ using Simpson's 1/3 rule and Trapezoidal rule respectively. Correct upto three significant digits.

4. For $n = 4$, evaluate $\int\limits_{1.2}^{1.6} (x + \frac{1}{x})\ dx$ using Simpson's 1/3 rule and Trapezoidal rule respectively. Correct upto two significant digits.

5. For $n = 6$, evaluate $\int\limits_{0}^{0.6} e^x\ dx$ using Simpson's 1/3 rule and Trapezoidal rule respectively. Correct upto 5 significant digits.

6. For $n = 6$, evaluate $\int\limits_{0}^{\pi/2} e^x\ \sin x\ dx$ using Simpson's 1/3 rule and Trapezoidal rule respectively. Correct upto five decimal places.