Probability & Expectation

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What is probability ?

→ likelihood of an event

→ chances of an outcome to occur

→ how many events are favourable $\propto$

compared to the total no. of events

What is Probability

Finding out how likely an event will occur.

$$\text{Probability of an event} = \frac{\text{Total number of favourable outcomes}}{\text{Total number of possible outcomes}}$$

① . Coin

probability (that you will get a HEAD)

H

$$P(\text{event}) \rightarrow \frac{\text{favourable outcomes}}{\text{total outcomes}}$$

H T

$$= \frac{1}{2}$$

(2) Coin $\longrightarrow$ Probability of getting 2 Heads out of 3 coin tosses

HHH $\longrightarrow$

HHT $\longrightarrow$

HTH $\longrightarrow$

HTT $\longrightarrow$

THH $\longrightarrow$

THT $\longrightarrow$

TTH $\longrightarrow$

TTT $\longrightarrow$

$$\frac{3}{8}$$

③ Dia with 6 sides

(1, 2, 3, 4, 5, 6) 

$P(4) \longrightarrow \frac{1}{6}$

$P(\text{even}) \longrightarrow \frac{(2, 4, 6)}{(1, 2, 3, 4, 5, 6)} \longrightarrow \frac{3}{6} \longrightarrow \frac{1}{2}$

(4) Dice → roll the dice twice

what is the probability that the sum of numbers on the 2 dice = 8

(x, y)

$x + y = 8$

1

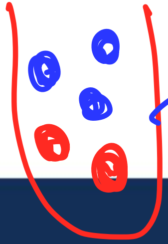
1	0	1
2	1	2
3	1	3
4	1	4
5	1	5
6	1	6

2

6 = 6

$$\frac{5}{36}$$

$\boxed{2 \text{ to } 12}$



$4/6$

$3/5$

1st pick, 2nd pick

DEPENDENT EVENTS

VS

INDEPENDENT EVENTS

Coin (H, T)

Die (1, 2, 3, 4, 5, 6)

Die $\longrightarrow$ 2 times

1st roll

independent

2nd roll

when the events are independent then probability of E_1 and E_2 =

$$P(E_1) \times P(E_2)$$

What is the probability that when you roll a die twice, first time you get an even number and second time you get a prime number

$$P(\underline{1st_{even}} \ \& \ \underline{2nd_{prime}})$$

$$\frac{2,4,6}{1,2,3,4,5,6}$$

2 ✓, 3 ✓, 5 ✓

$$= P(1st_{even}) \times P(2nd_{prime})$$

$$= \frac{3}{6} \times \frac{3}{6} = 1/2 \times 1/2 = 1/4$$

What is the probability of getting even no. on first dice and prime on 2nd dice if the no. that appears on 1st dice is erased.

# →	1st throw	2nd throw
	1 (1/6)	× prime (3/5)
	2 (1/6)	× prime (2/5)
	3 1/6	× 2/5
	4 1/6	× 3/5
	5 1/6	× 2/5
	6 1/6	× 3/5

$$\frac{3\times 3}{30} + \frac{3\times 2}{30}$$

$$= \frac{1}{10} \times 5$$

$$= \frac{1}{2}$$

What is Expectation

If every outcome of an event had a value and a probability to occur, what is the average value we can expect from that event. Expectation is simply a weighted average.

$$E[X] = \sum_i x_i P(x_i)$$

roll a dice . 1 2 3

$$\frac{1+2+3+4+5+6}{6} \qquad \frac{1+2+3}{3}$$

$$\frac{6 \times 7}{2 \times 6} = 3.5$$

5 Red balls → Rs 10

4 yellow balls → Rs 20

2 blue balls → Rs. 30

(10 10 10 10 10 20 20 20 20 30 30) / 11

R R R R R Y Y Y Y B B

=

5 Red balls → Rs 10

4 yellow balls → Rs 20

2 blue balls → Rs. 30

$$\frac{5 \cdot 10 + 20 \cdot 4 + 30 \cdot 2}{11}$$

$$\rightarrow \frac{5}{11} \cdot 10 + \frac{4}{11} \cdot 20 + \frac{2}{11} \cdot 30$$

$$\text{Average} = P(\text{red}) \cdot V(\text{red}) + P(\text{slw}) \cdot V(\text{slw})$$

$$+ P(\text{yellow}) \cdot P(\text{yellow})$$

$\downarrow$

Expectation

Gas Station (5)

You want to buy 10L of gas

1. 20 Rs/L → 2/8

2. 30 Rs/L → 2/8

3. 10 Rs/L → 1/8

4. 5 Rs/L → 1/8

5. 40 Rs/L → 2/8

$E(\text{money spent})$

$$\frac{2}{8}\cdot 200 + \frac{2}{8}\cdot 300 + \frac{1}{8}\cdot 100 + \frac{1}{8}\cdot 50$$

$$+\frac{2}{8}\cdot 400$$

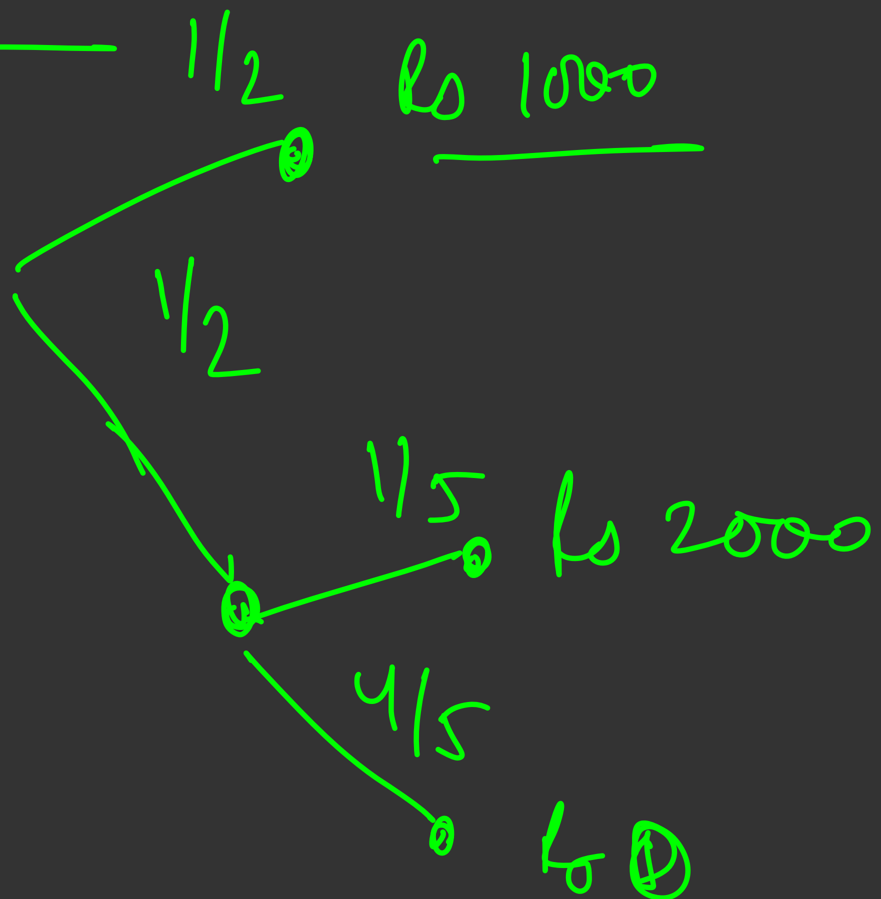
Linearity of Expectation

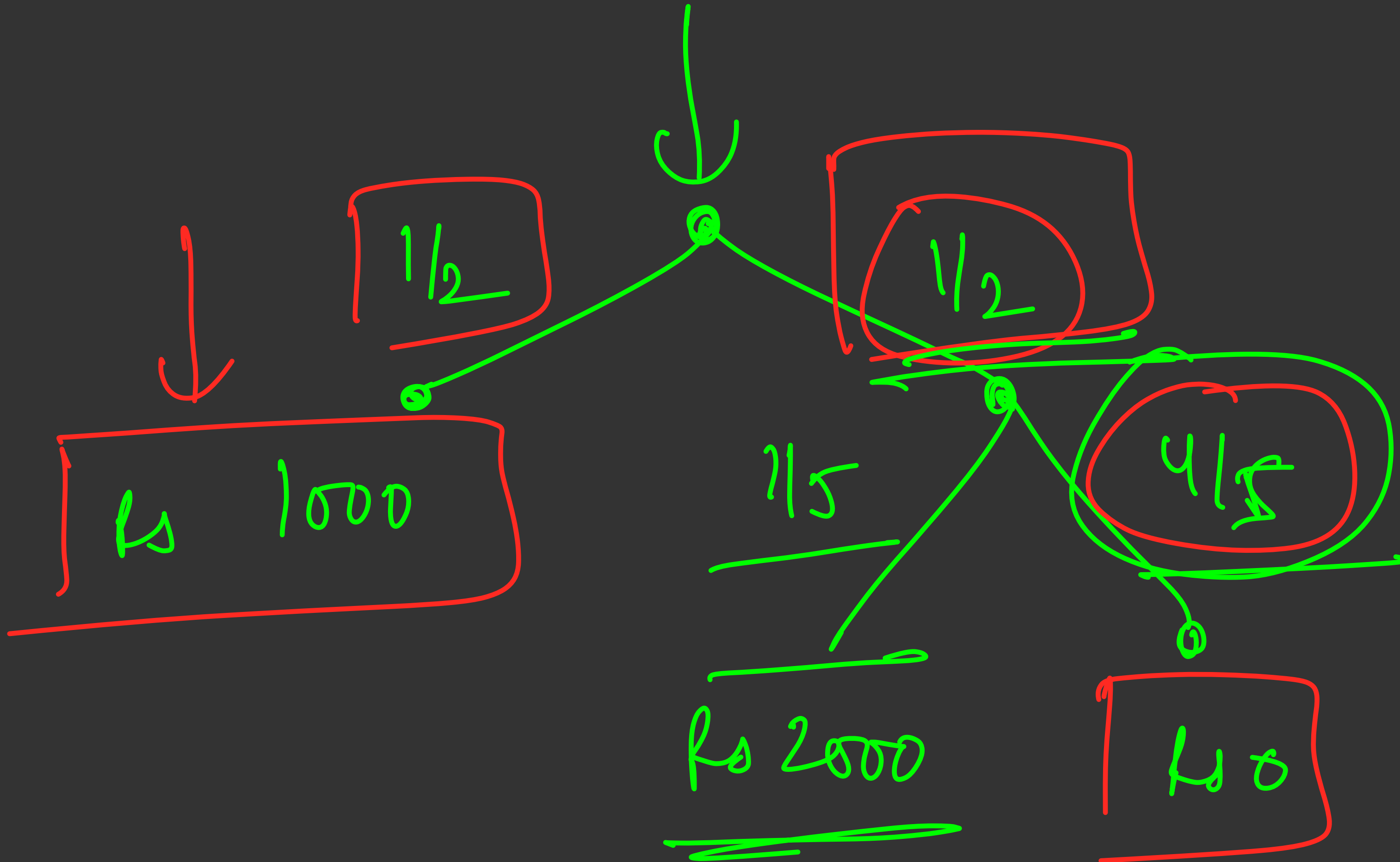
$$E[X+Y] = E[X] + E[Y]$$

$$E[aX+b] = aE[X] + b$$

Casino

2 games





$E(\text{money}) = \text{Rs } 1000 \cdot P(\text{of getting } 1000) + \text{Rs } 0 \cdot P(\text{of getting } 0) + \text{Rs } 2000 \times P(\text{of getting } 2000)$

P(of getting Rs 1000) $\longrightarrow$ P(of playing 1st game)

$\longrightarrow \frac{1}{2}$

P(of getting Rs 0) $\longrightarrow$ P(of playing 2nd game and losing 2nd game)

actually play 2nd game $\longrightarrow \frac{1}{2} \times \frac{4}{5}$

$+ \quad \frac{1}{2} + \frac{4}{5}$

not playing 2nd game $\longrightarrow \frac{1}{2} \times 0$

$$f(1000)\cdot 1000 + f(0)\cdot 0 + f(2000)\cdot 2000$$

$$\frac{1}{2}\times 1000 + 0 + \frac{1}{2}\times\frac{1}{5}\times 2000$$

Game 1

1 million dollars

$\boxed{\text{1 million dollars}}$

Game 2

Can win 1 billion dollars with 50% probability

$$\frac{1}{2}\cdot(0) + \frac{1}{2}(1000) = \boxed{\text{500 million dollars}}$$

① Rs 100

② Rs 1 lac

80%

TLE Eliminator

2 lottery systems

①	②
Rs 1000 for sure	Rs 10000 with 50% proba

You have 5 chances to play

0

1/2   1/2

Rs 1 Lac   Rs 0

$$\frac{(0, 1L, 1L, 0, 0, 0, 0, 0, 1L, 1L, 1L)}{n}$$

Rs 50k

The average you will get by performing this event infinite times

$$E(\text{event}_1 + \text{event}_2)$$

$$= E(\text{event}_1) + E(\text{event}_2)$$

Die with 8 sides 1

Die with 6 sides 2

$$E(\text{die}_1 + \text{die}_2)$$

$$\left(\overline{(1, 2, 3, 4, 5, 6, 7, 8)}\right) \quad ① \quad 4.5$$

$$+ \qquad\qquad\qquad\qquad +$$

$$\left(\overline{(1, 2, 3, 4, 5, 6)}\right) \quad ② \quad 3.5$$

$$= 8$$

expectation of sum

= sum of expectation

Problem 1:

What is expected number of throws after which you will get a head

in a fair coin with 2 sides H & T

$E(X)$ $\quad X$ = no. of throws to get a Head,

$$E(X) = \sum x_i \cdot P(x_i)$$

$x_i \in$ occurences

$$E(\underline{\text{no. of throws}}) =$$

$$\lim_{n \to \infty} \left(H \cdot (1/2) + TH \cdot (1/4) + TTH \cdot (1/8) \right.$$

$$\left. - \cdots\cdots \underbrace{T(n-1)}_{T} H \,(1/2 \,)\right)$$

$$\lim_{n\to\infty}\left(1\left(\tfrac{1}{2}\right)+2\cdot\left(\tfrac{1}{4}\right)+3\left(\tfrac{1}{8}\right)+4\ \ldots\right)$$

$$n\left(\tfrac{1}{2^n}\right)$$

$n \to 1000$

$\tfrac{1}{2^{1000}}$

2

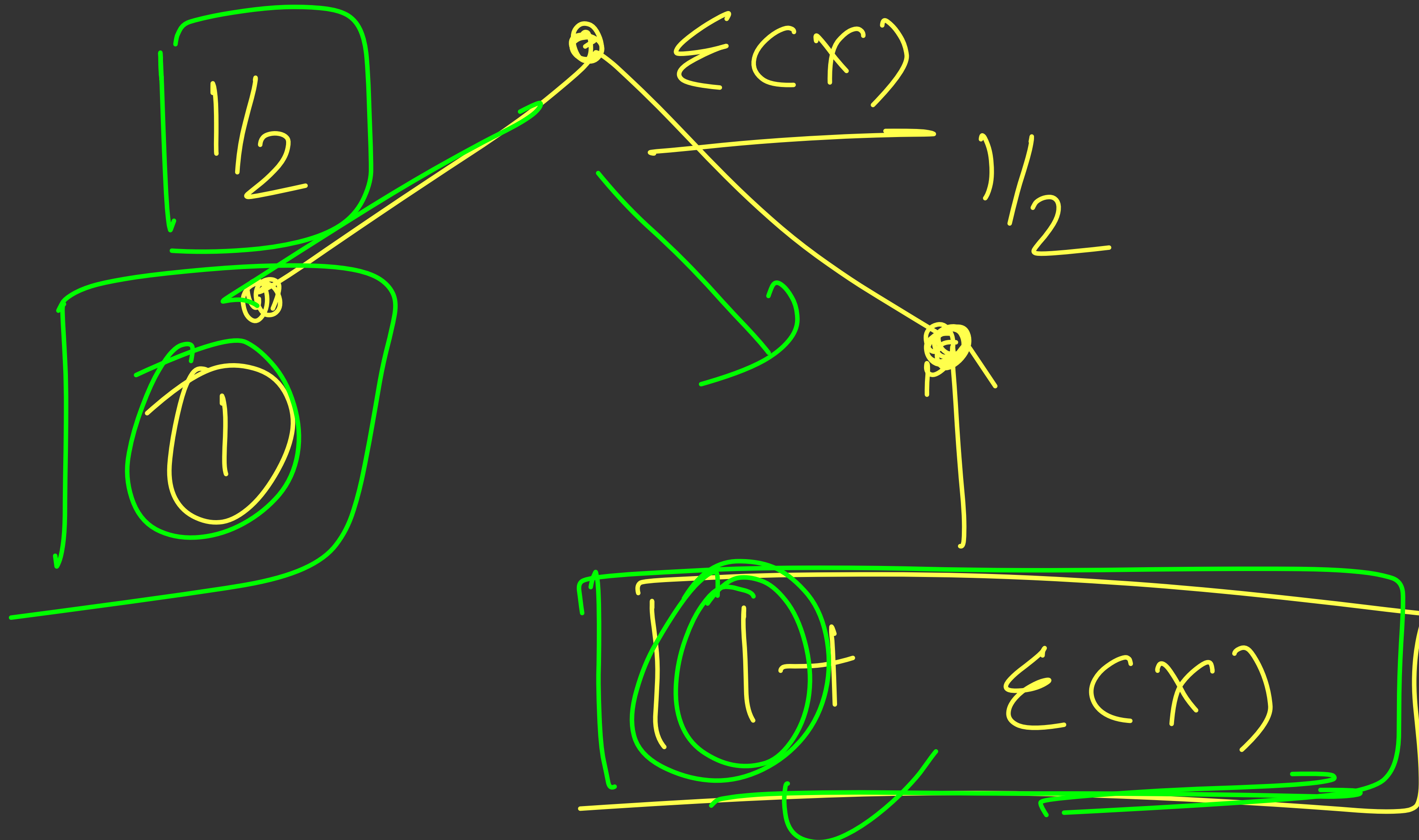
AGP

Arithmetic Geometric progression

2 5 8 11 14 17 ...

Arithme

2 6 18 54 ...

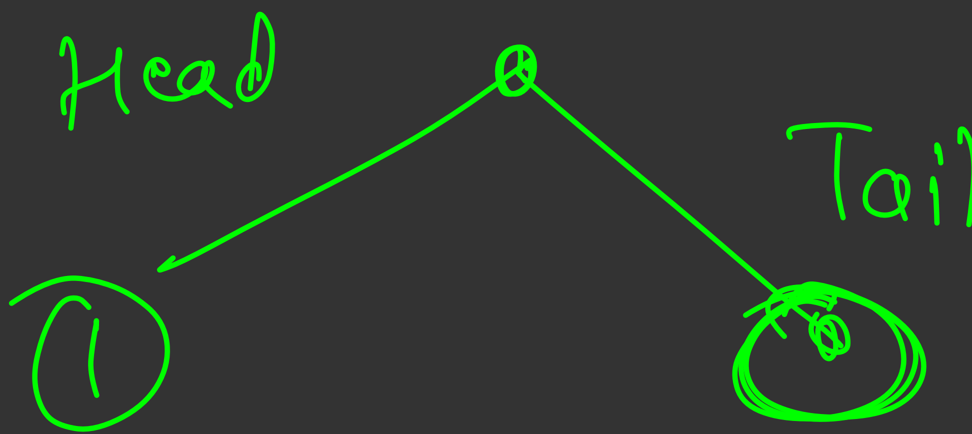


$E(X)$ = no. of tries req to get a head

$$E(x) = \frac{1}{2}(1) + \frac{1}{2}(1 + E(x))$$

$$E(x) = 2$$

$$\text{Coin} \longrightarrow E(x)$$



$$1 + E(x)$$

Problem 2:

What is expected number of throws after which you will get 2 consecutive heads.

E(2 consecutive Heads)

T H T T T H T H H

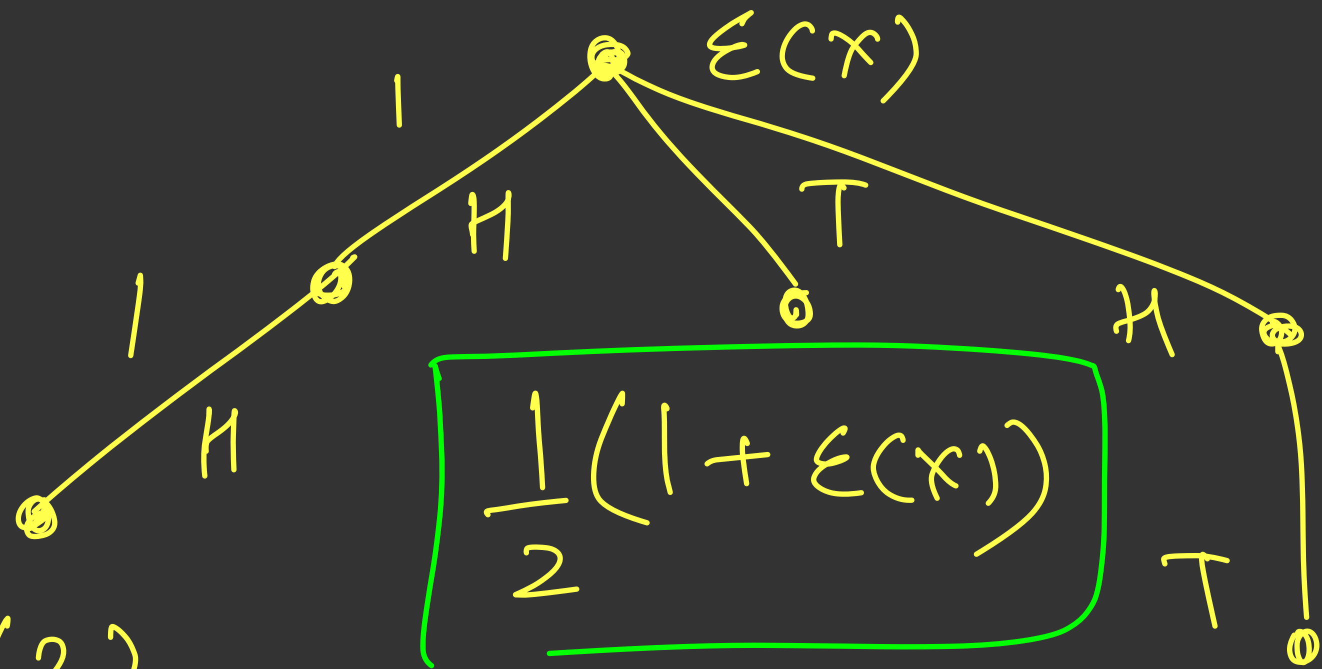
$E(2\text{ case}) = E(H) + 1$

2

$\rightarrow$ HT

$\rightarrow$ HH

THHH



$\underline{\frac{1}{4}(2)}$

$\boxed{\frac{1}{2}(1+E(x))}$

$\frac{1}{4}(2+E(x))$

$$E(x) = \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot(2+E(x)) + \frac{1}{2}(1+E(x))$$

$$E(x) = \frac{3}{2} + \frac{3}{4}\cdot E(x) = \frac{6}{4} + \frac{3}{4}(Ex)$$

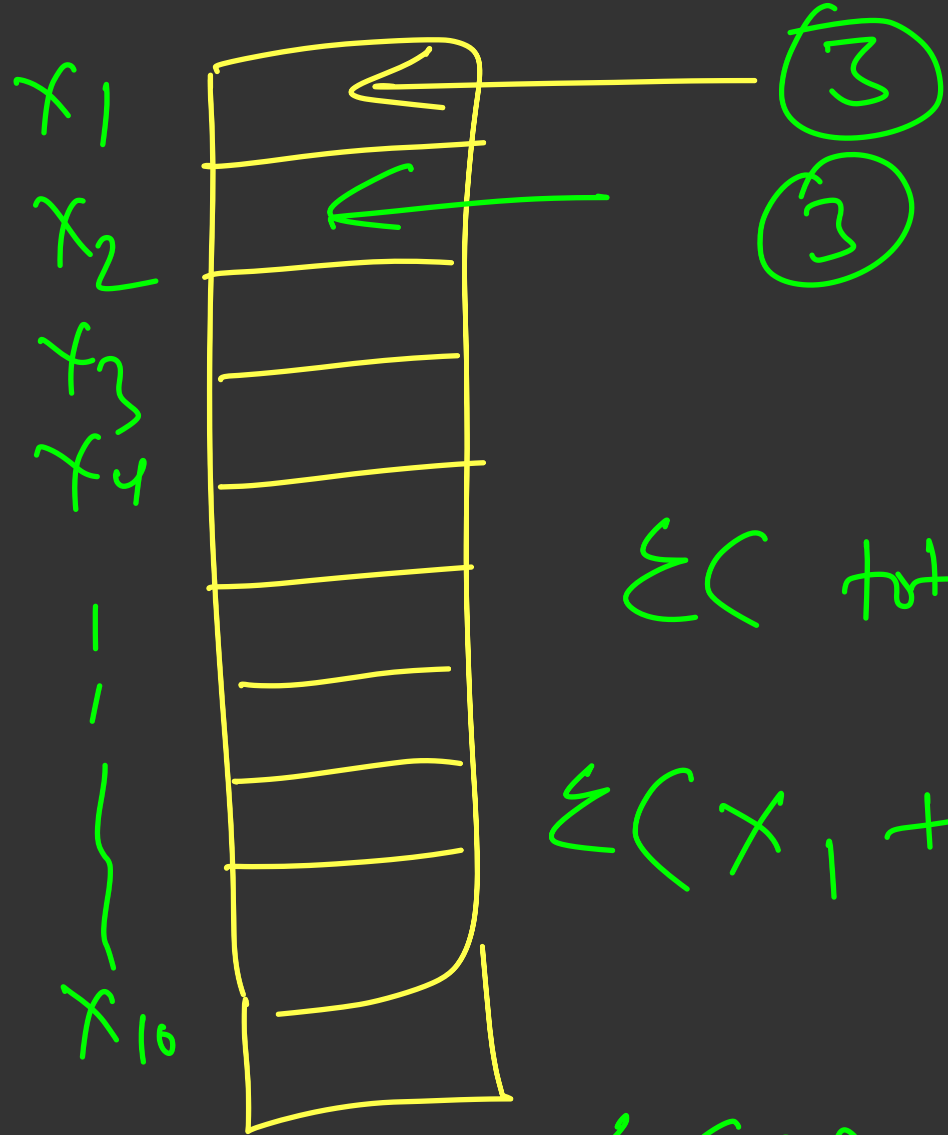
$$\frac{1}{4}(E(x)) = \frac{6}{4} \implies \underline{E(x) = 6}$$

Problem 3: More Cool Problems

What is expected number of interviews we will have to take to hire 10 candidates if the probability of getting hired for a candidate is 1/3

1 2

1/3 1/3



$P(\text{hire}) = 1/3$

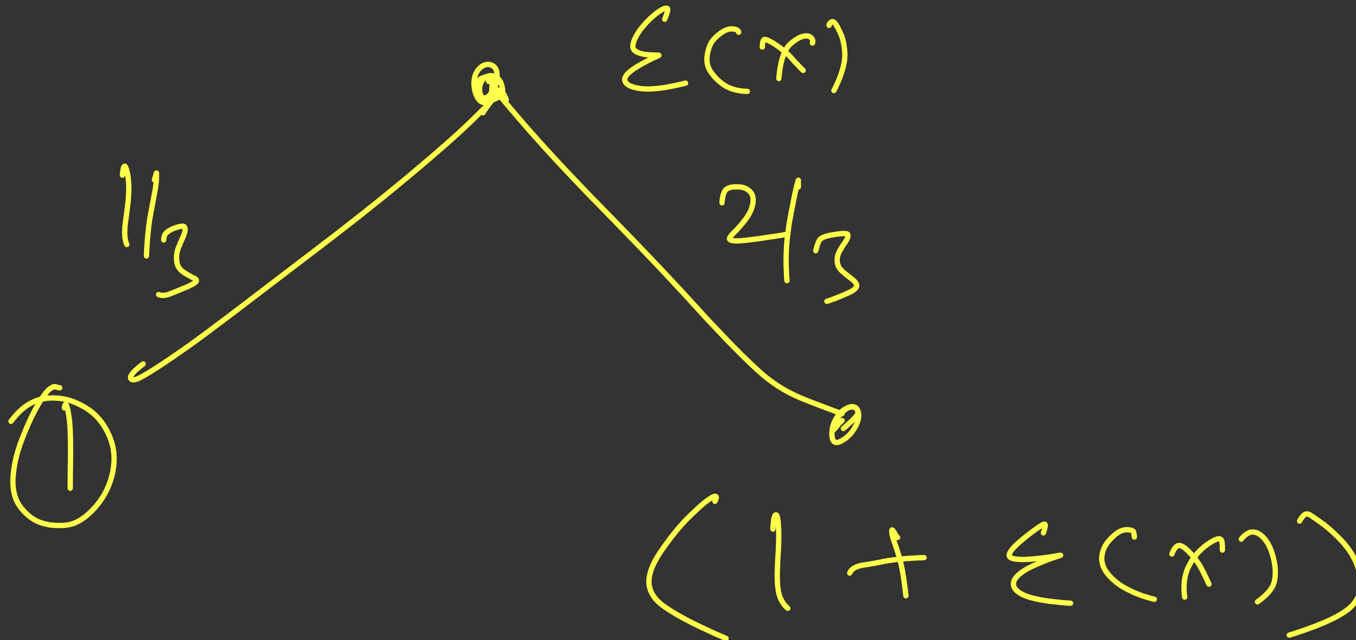
$E(\text{total interview})$

$E(x_1 + x_2 \; ---- \; x_{10})$

$\downarrow$

$E(x_1) + E(x_2) \; --- \; E(x_{10})$

$3 + 3 \; --- \; 3 = \underline{\underline{30}}$



$$E(x) = 1/3 + 2/3\,(1 + E(x))$$

$$1/3\,(E(x)) = 1 \Rightarrow E(x) = 3$$

Bernoulli Try

$$1/p_{yes}$$

$$E(\underline{\text{no. of tries to get}} \text{ a Yes}) = \frac{1}{p_{yes}}$$

$$E(x) = \underline{\underline{p(1)}} + (1-p)(1+E(x))$$

$$E(x) = \frac{1}{p}$$

Expected Value or

Expected no. of dies

$$\boxed{E(x) = \sum x_i \cdot p(x_i)}$$

$$\boxed{E(x_1 + x_2) = E(x_1) + E(x_2)}$$

$$P(\text{event}) = \frac{\text{fav out}}{\text{possible outc}}$$

Problems based on Bounds

(monte carlo algorithms)

Majority Element Problem

Given an array, find the majority element in it. Given that the majority element always exists. Majority Element = element occuring more than n/2 times in the array.

(las vegas algorithms)

$$1 <= n <= 10^5$$
$$1 <= arr[i] < 10^{18}$$

| 10 2 9 10 10 10 8 10

10

$$P(\text{winning}) = \frac{\text{fav out}}{\text{total out}} = \frac{> n/2}{n}$$

$$> 1/2$$

$$P(\text{losing}) = < 1/2$$

What is the probability that I play 10 times and I lose every single time

$< 1/2 \cdot < 1/2$

$< 1/2^{10}$

< 0.001

Problems based on Bounds

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Given an array, find the majority element in it. Given that the majority element always exists. Majority Element = element occuring more than n/2 times in the array.

$$1 <= n <= 10^5$$
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