Combinatorics 2

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Stars and Bars

put

- Number of ways to n identical objects into k distinct boxes

5 oranges

$$\binom{n+k-1}{n}.$$

$$x_1 + x_2 = 5$$

(0, 5) (5, 0)
(1, 4)
(2, 3)
(3, 2)
(4, 1)

6

$x_1 \geq 0$ $x_2 \geq 0$

Stars and Bars

$o_1 \quad o_2 \quad o_3 \quad o_4 \quad o_5$

- Number of ways to n identical objects into k distinct boxes

$\boxed{o_1\, o_2} \quad \boxed{o_3\, o_4\, o_5}$

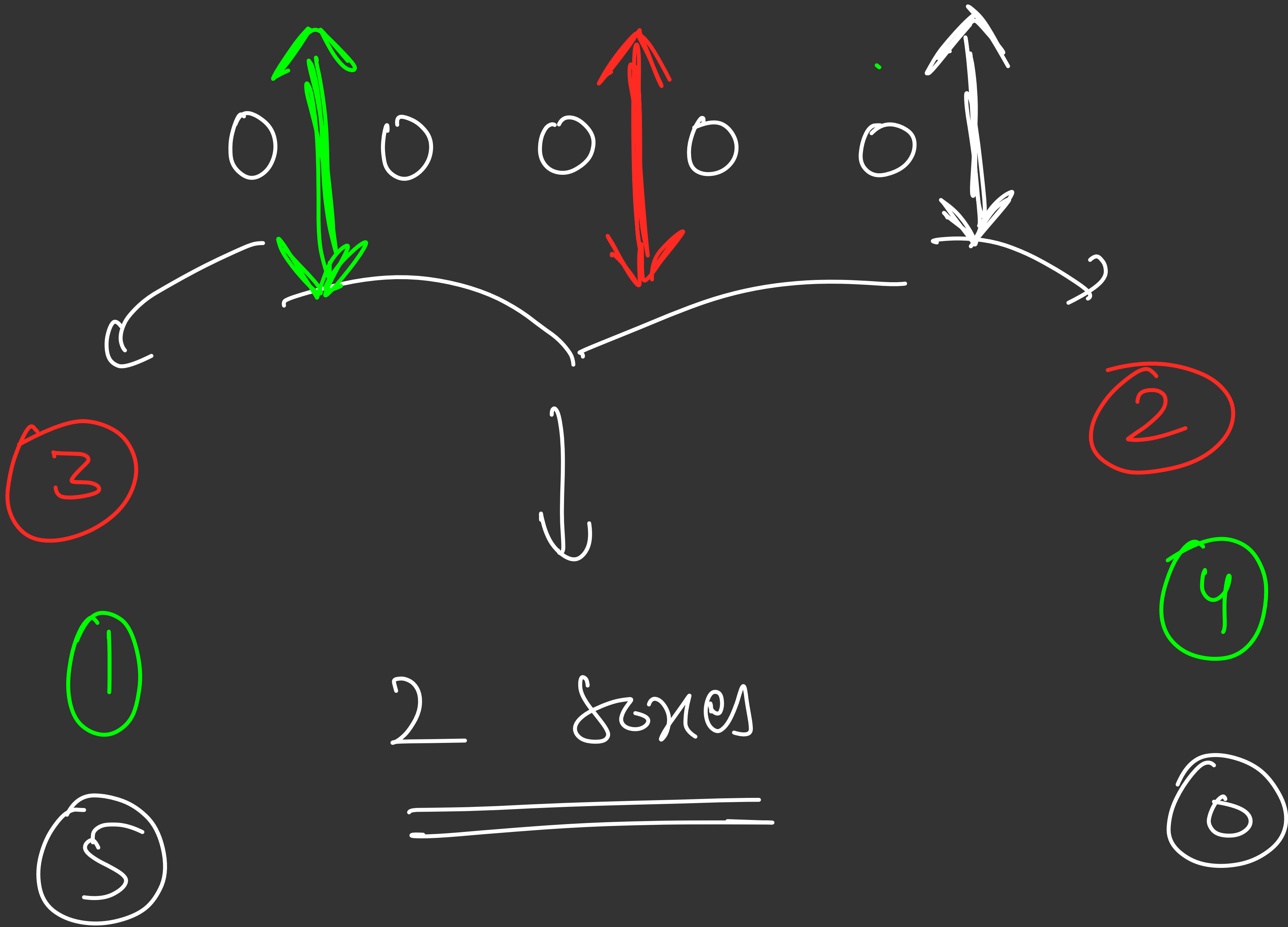
$\boxed{o_2\, o_3} \quad \boxed{o_1\, o_4\, o_5}$

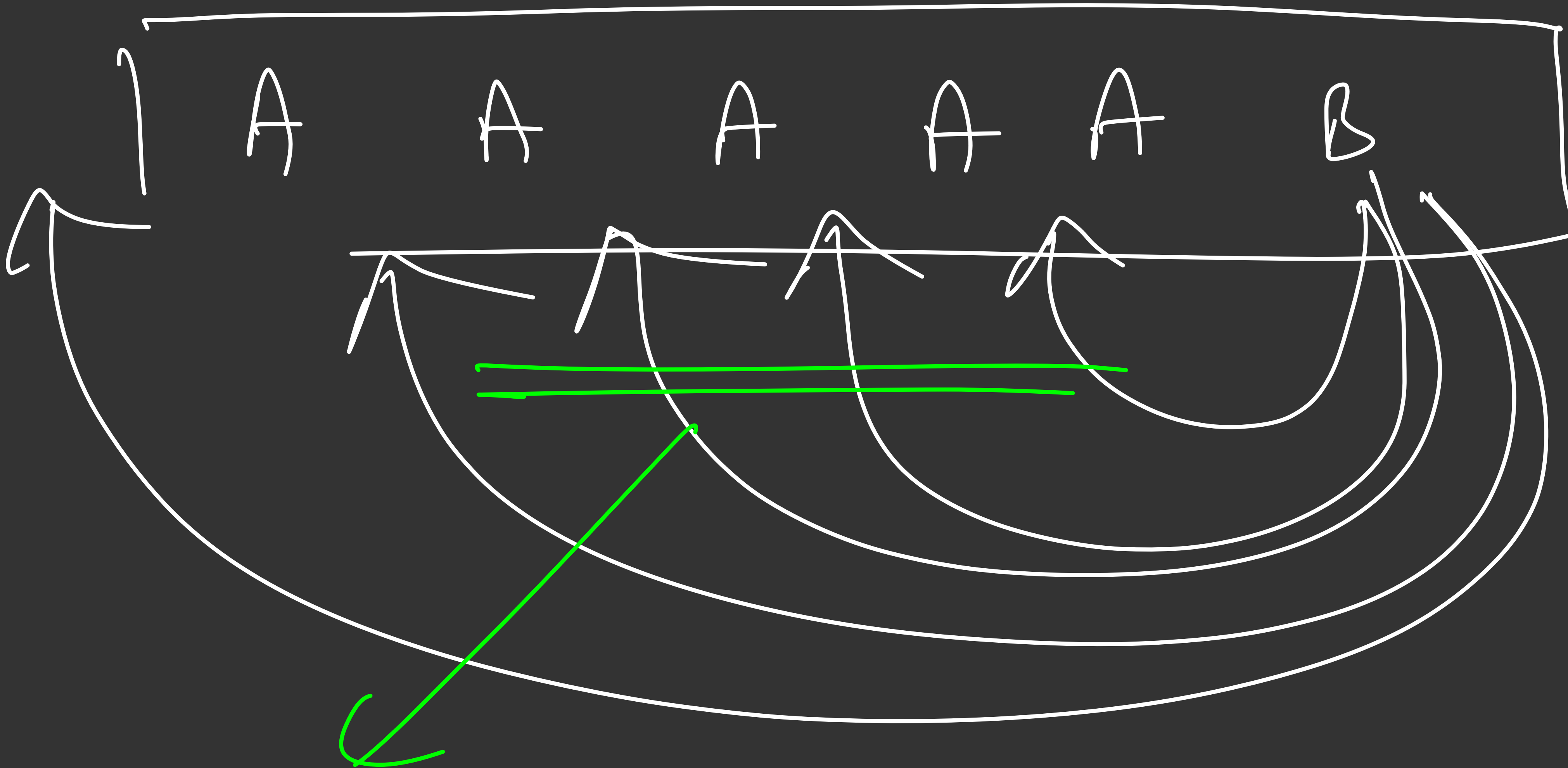
$$C\binom{n+k-1}{n}.$$

$2 \quad 3 \rightarrow 1$

$3 \quad 2 \rightarrow 2$

$C(n+k-1, n)$

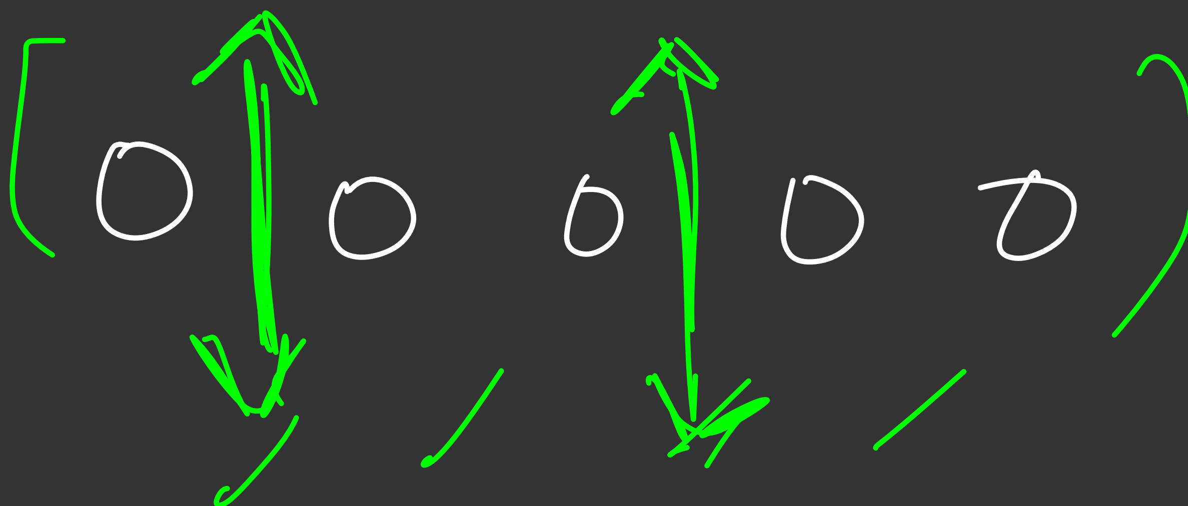

3
1
5
2
4
0
2 zones



5 oranges into 2 boxes

$$\frac{(5+1)!}{5!\,1!} = \frac{6!}{5!} = 6$$

5 oranges → 3 boxes



A A A A A B B

$$\frac{(5+2)!}{2!\,5!} \rightarrow C(5+2,2)$$

$$C(5+2,5)$$

3 orange → 3 boxes

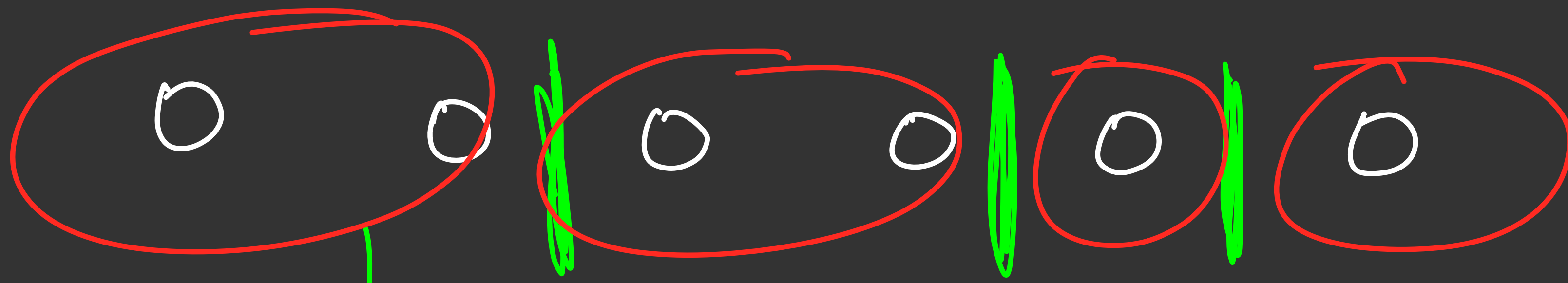
0 0 0 | | → [3, 0, 0]

0 0 | | 0 → [2, 0, 1]

0 0 | 0 | → [2, 1, 0]

n oranges → k boxes

↪ (n stars k−1 bars)



4 boxes

n items → 2 boxes

$$x_1 + x_2 + x_3 + x_4 = 6$$

$$\underbrace{x_1 + x_2 + x_3 \cdots x_k}_{\text{bins}} = \underset{\text{items}}{n}$$

put $(k-1)$ partition point among

n items

$$* \; * \;|\; * \; * \;|\; * \;|\; * \; *$$

$$A\,A\,B\,A\,A\,B\,A\,B\,A\,A \longrightarrow \frac{(n+k-1)!}{n!\,(k-1)!}$$

$$\longrightarrow C(n+k-1,\, n)$$

Stars and Bars Application

- What are the number of ways to solve the equation below?

$C(n, r)$

$\Downarrow$

$C(n, n-r)$

$$x_1 + x_2 + \cdots + x_k = n$$

$$x_i \geq 0.$$

$C(n+k-1, n) \longleftrightarrow C(n+k-1, k-1)$

Stars and Bars Application

- What are the number of ways to solve the equation below?

$$x_1 + x_2 + \cdots + x_k = n$$

$$\boxed{x_i \geq 1.}$$

$$\boxed{C(n-1, k-1)}$$

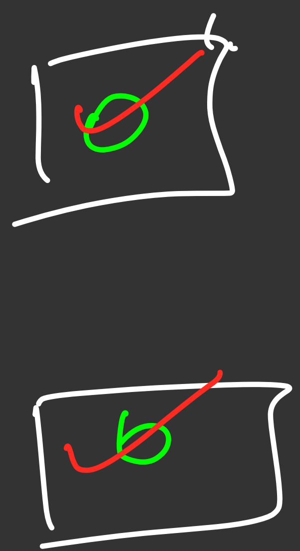
5 oranges

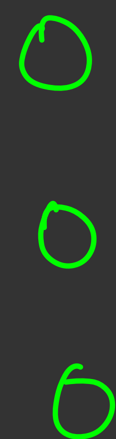
$\downarrow$

2 boxes

each box gets atleast 1 orange

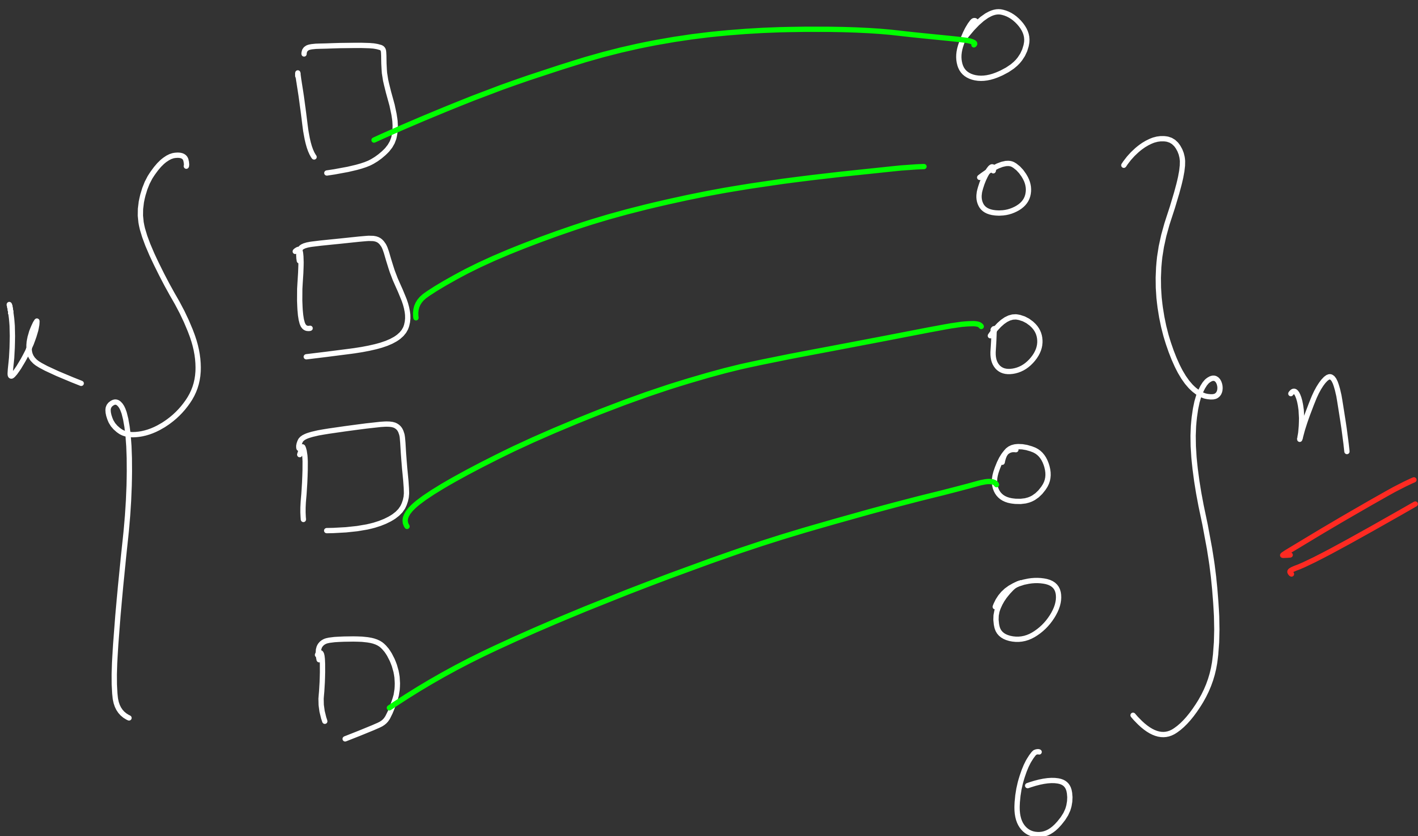
(1, 4)
(2, 3)
(3, 2)
(4, 1)
} 4





$$\begin{cases} 5 \rightarrow 2 \\ x_1 + x_2 = 5 \\ x_1, x_2 \geq 1 \end{cases}$$

$$\begin{cases} 3 \rightarrow 2 \\ x_1 + x_2 = 3 \\ x_1, x_2 \geq 0 \end{cases}$$



$$\boxed{n-k} \longrightarrow \boxed{k}$$

$$\boxed{x_1 + x_2 + x_3 \cdots\cdots x_k = n} \quad (1)$$

$$(x_1 - 1) + (x_2 - 1) \cdots\cdots (x_k - 1) = n - k$$

$$(2) \quad \boxed{y_1 + y_2 + y_3 \cdots y_k = n - k}$$

$$y_i = x_i - 1$$

$$\boxed{x_i \geq 1} \quad (1)$$

$$(x_i - 1) \geq 0$$

$$\boxed{y_i \geq 0}$$

$(n-k)$ items into k boxes

n items into r boxes

1 $C(n+r-1, n)$

$C(n-k+k-1, n-k)$

$C(n-1, n-k)$

$C(n-1, n-1-(n-k))$

$C(n-1, k-1)$

Stars and Bars Application

- What are the number of ways to solve the equation below?

$$x_1 + x_2 + \cdots + x_k = n$$

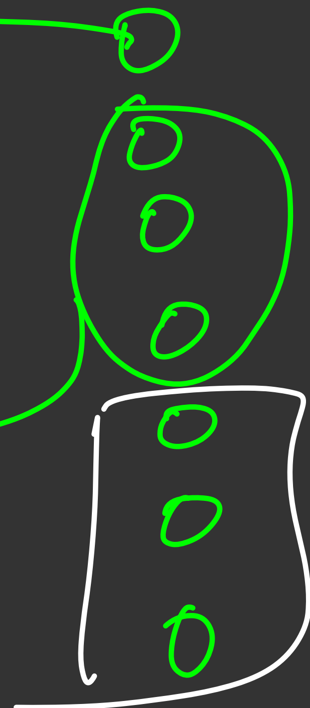
$$x_i \geq a_i .$$

7 oranges $\rightarrow$ 3 boxes

$\square \rightarrow a_1 = 1$

$\square \rightarrow a_2 = 0$

$\square \rightarrow a_3 = 3$



$$x_1 + x_2 + \cdots\cdots x_k = n$$

$$x_i \geq a_i$$

$$(x_1 - a_1) + (x_2 - a_2) \cdots (x_k - a_k) = n - \sum_{i=1}^{k} a_i$$

$$y_1 + y_2 \cdots\cdots y_k = n - \sum_{i=1}^{k} a_i$$

$$y_i = x_i - a_i \longrightarrow \boxed{y_i \geq 0}$$

n items $\longrightarrow$ k boxes

≥ 0 items

$\underline{\boxed{C(n+k-1,\, n)}} \longleftrightarrow C(n+k-1,\, k-1)$

$$\left(n - \sum_{i=1}^{l} a_i\right) \longrightarrow k \text{ boxes}$$

$$C\left(n - \sum_{i=1}^{l} a_i + k - 1,\; n - \sum_{i=1}^{l} a_i\right)$$

$$C\left(n - \sum_{i=1}^{l} a_i + k - 1,\; k-1\right)$$

Let’s solve some problems

- Distributing Apples - Link
- Array - Link

$\underline{\underline{n}} \rightarrow 10$

$1 \le a_i \le n$

2	4
3	4
5	7

$\underline{\underline{1}}\ \underline{\underline{1}}\ \ \underline{\underline{4}}\ \underline{\underline{4}}\ \underline{\underline{4}}\ \ \underline{\underline{7}}\ \underline{\underline{7}}\ \underline{\underline{7}}\ \underline{\underline{7}}\ \underline{\underline{7}}$

7 7 7 7 7 4 4 4 1 1

$c_1 + c_2 + c_3 + c_4 \; --- \; c_n = n$

5 1s 2 3s

$\boxed{11111}$ $\boxed{33}$

5 1s 2 3s 2 4s

$\boxed{11111}$ $\boxed{33}$ $\boxed{44}$

$\rightarrow \boxed{11111} \quad \boxed{44} \quad \boxed{33}$

$(a_i \geq a_{i-1}) \; \forall \; i \; (1 \text{ to } n-1)$

$(a_i \leq a_{i-1}) \; \forall \; i \; (1 \text{ to } n-1)$

Once I choose the count of each number

the order is already fixed.

$$2 \times \left\{ \begin{array}{cccccc} 1 & 1 & 1 & 3 & 3 & 5 \\ 5 & 3 & 3 & 1 & 1 & 1 \end{array} \right.$$

Choosing the count of each element

$$\boxed{c_1 + c_2 + c_3 \; \text{-----} \; c_n = n \qquad c_i \geq 0}$$

n items, k boxes $\rightarrow C(n+k-1, n)$

n items, n boxes $\rightarrow C((n+n-1), n)$

$$\boxed{C(2n-1, n)}$$

Beautiful Array

1 1 1 1 1 2 2

2 2 1 1 1 1 1

= ways to choose element × ways to arrange

$$= \boxed{C(2n-1, n) \times 2} - n$$

1 1 1 1 1 1 1

2 2 2 2 2 2 2

3 — —

n n n n n n

$$\boxed{2\,C(2n-1,n)} - n$$

$n = 2$

$$\boxed{2\,C(4-1,2)} - 2$$

$$(2\cdot 3) - 2 = 4$$

$n = 3 \rightarrow 2\,C(6-1,3) - 3$

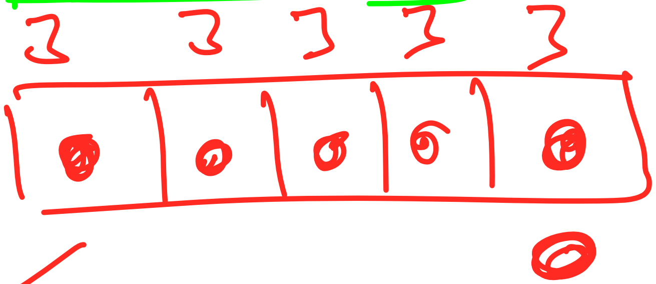
$$2\,C(5,3) - 3$$

$$\left[\cancel{2}\times\frac{5\times4\times\cancel{2}}{\cancel{3}\times\cancel{2}\times1}\right]-3$$

$$20-3=17$$

Problems on independent choices

- Lucky Numbers - Link
- Number of subsets of an array with at least 1 element
- Monotonic Renumeration - Link
- The Intriguing Obsession - Link



1 to 3

Coin → H, T

Dice → 1, 2, 3, 4, 5, 6

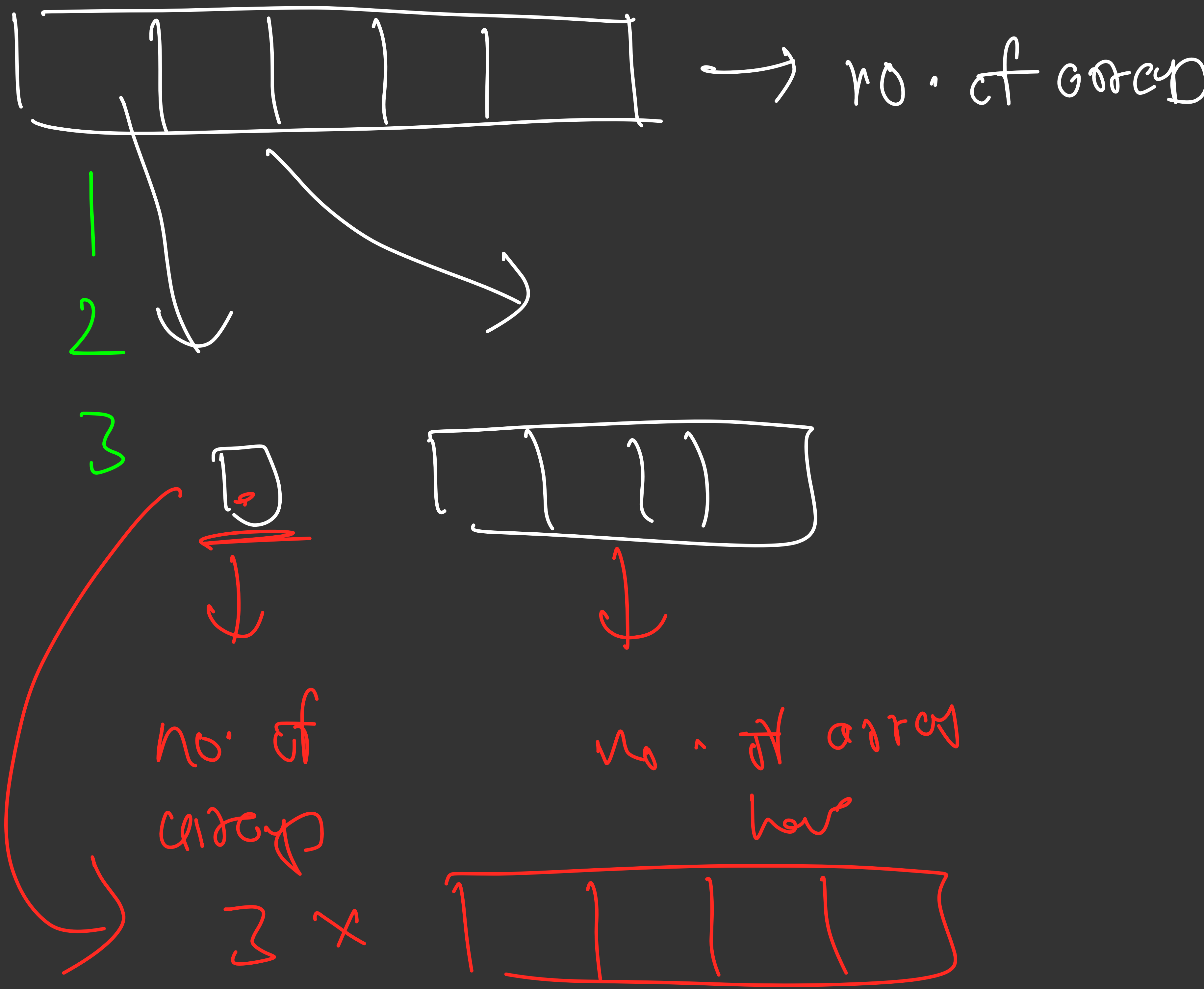
What are the different ways in which you can get an even number of dice and an H on the coin

1×3

P_1 $\quad\quad\quad\quad\quad\quad\quad\quad$ P_2

A_1 AND A_2

$\boxed{x} \times \boxed{y}$



no. of array
1
2
3
no. of arrays
no. of array here
3 ×

3^5
array
(1 to 3)
3^4
3^3
3^2
3
3
3
3
0

n

$$2^1 + 2^2 + 2^3 \cdots\cdots 2^n = x$$

$$2^0 + 2^1 + 2^2 + 2^3 \cdots\cdots 2^n = x + 2^0$$

$$x = \left(2^0 + 2^1 + 2^2 \cdots\cdots 2^n\right) - 2^0$$

$$= \boxed{\left(2^0 + 2^1 + 2^2 \cdots\cdots 2^n\right) - 1}$$

$16 \rightarrow 2^4$

$1\ 0\ 0\ 0\ 0$

$2^3\ 2^2\ 2^1\ 2^0$

$16 \rightarrow 10000 \rightarrow 2^4$

$15 \rightarrow 01111 \rightarrow \boxed{2^0 + 2^1 + 2^2 + 2^3}$

$2^n = 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots n$ zeros

$2^n - 1 = 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \ldots n$ ones

$$\boxed{2^n - 1} = \boxed{2^{n-1} + 2^{n-2} \ldots\ldots 2^0}$$

$$\boxed{2^0 + 2^1 + 2^2 \ldots\ldots 2^n} \longrightarrow \boxed{2^{n+1} - 1}$$

$$2^0 + 2^1 + 2^2 + 2^3 - - - \cdot 2^n = x + 2^0$$

$$x = \left(2^0 + 2^1 + 2^2 \cdots 2^n\right) - 2^0$$

$$= \boxed{\left(2^0 + 2^1 + 2^2 \cdots 2^n\right) - 1}$$

$$x = \left(2^{n+1} - 1\right) - 1 = 2^{n+1} - 2$$

$$\longrightarrow 2 \cdot \left(2^n - 1\right)$$

1	2	3	4
✓	✗	✗	✓
✗	✗	———	✗

$\longrightarrow \underline{\underline{2^4}} \longrightarrow \underline{\underline{2^4}}$

$\{1, 4\}$

$\underline{\underline{2^4}}$

$$[1, 2, 3]$$

$\rightarrow \boxed{\{\}}$ $\rightarrow \{1, 3\}$

$\rightarrow \{1, 2\}$ $\rightarrow \{1\}$

$\rightarrow \{2\}$

$\rightarrow \{2, 3\}$ $\rightarrow \{3\}$

$\rightarrow \{1, 2, 3\}$

# of subset

$$= {}^nC_0 + \boxed{{}^nC_1 + {}^nC_2 - - - - {}^nC_n}$$

$$= 2^n$$

$$2^n - {}^nC_0 \longrightarrow 2^n - 1$$

a → 1 2 2 3 1 3 4 5 4 5

b → 0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 1 1 1 1

$a \rightarrow$ x y y y y x y Z y y z

$b \rightarrow$ 5 5 5 5 5 5 5 5 5

1st x $\longleftrightarrow$ last x

1st y $\longleftrightarrow$ last y

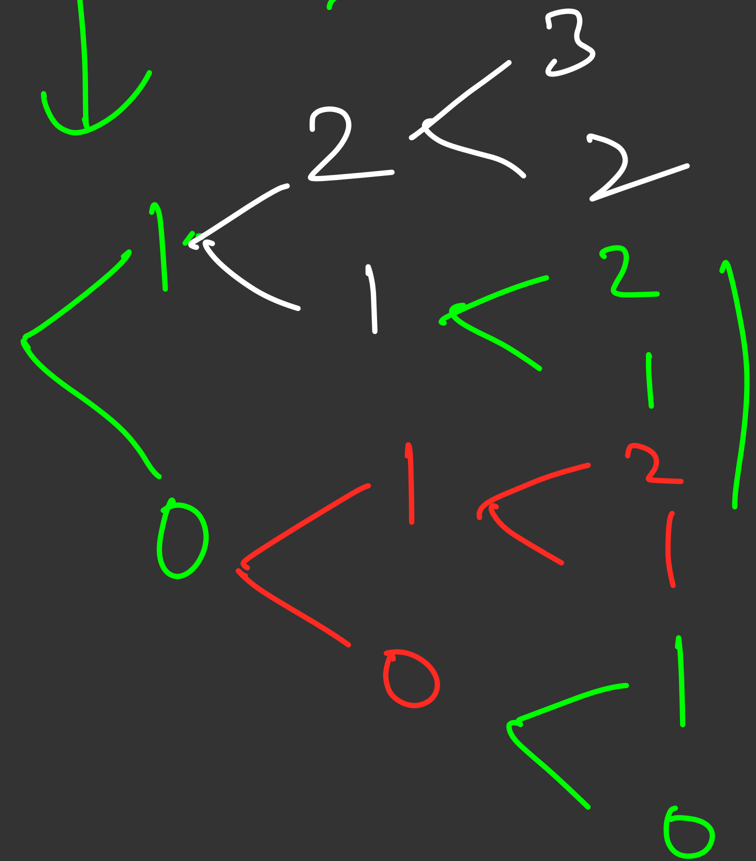
lst 1 $\longleftrightarrow$ last 1

lst 2 $\longleftrightarrow$ last 2

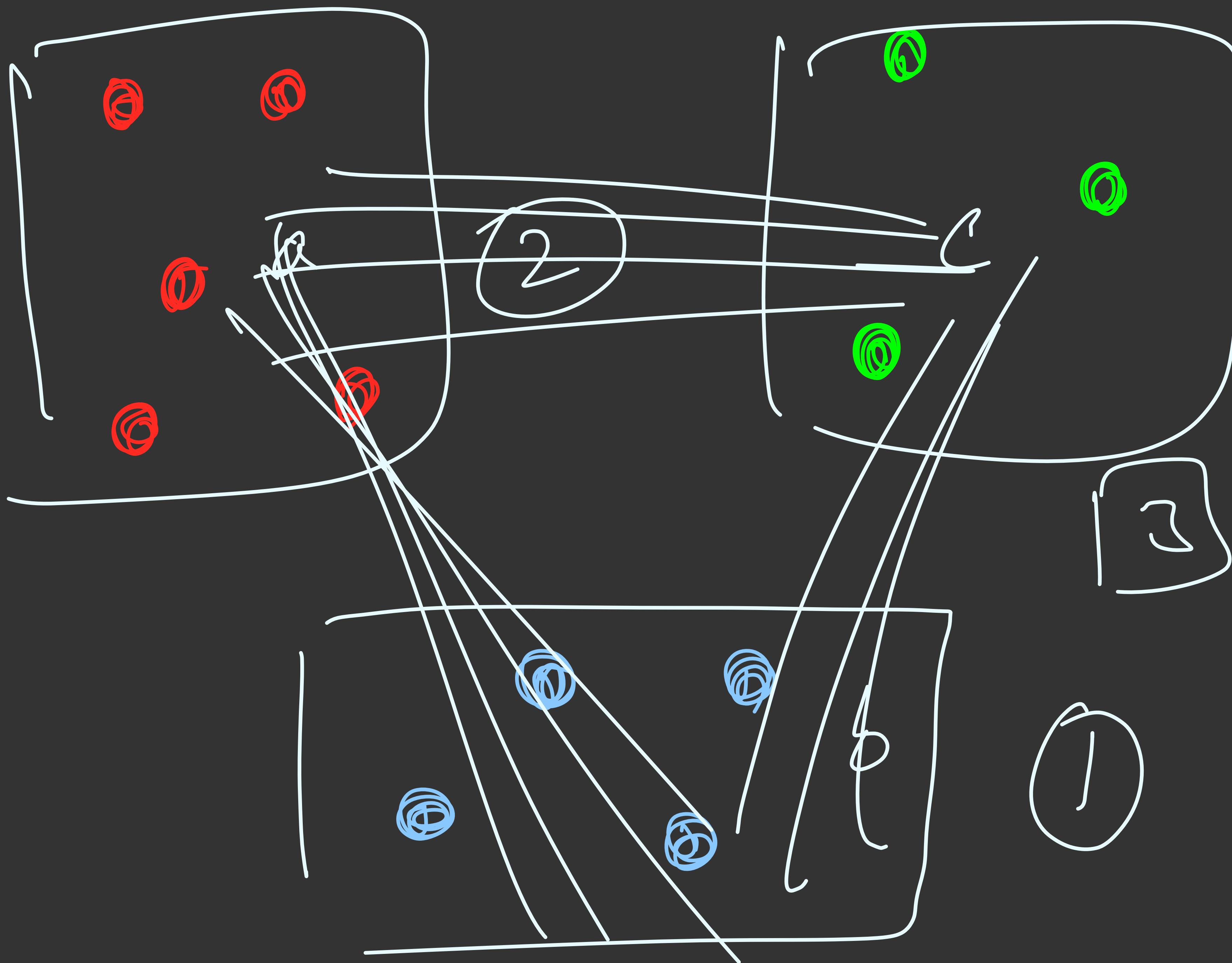
lst 3 $\longleftrightarrow$ last

a → 6 6 7 1 2 2 3 2 1 1 4 5 5 4

b → 0 0



$2^{\text{components} - 1}$

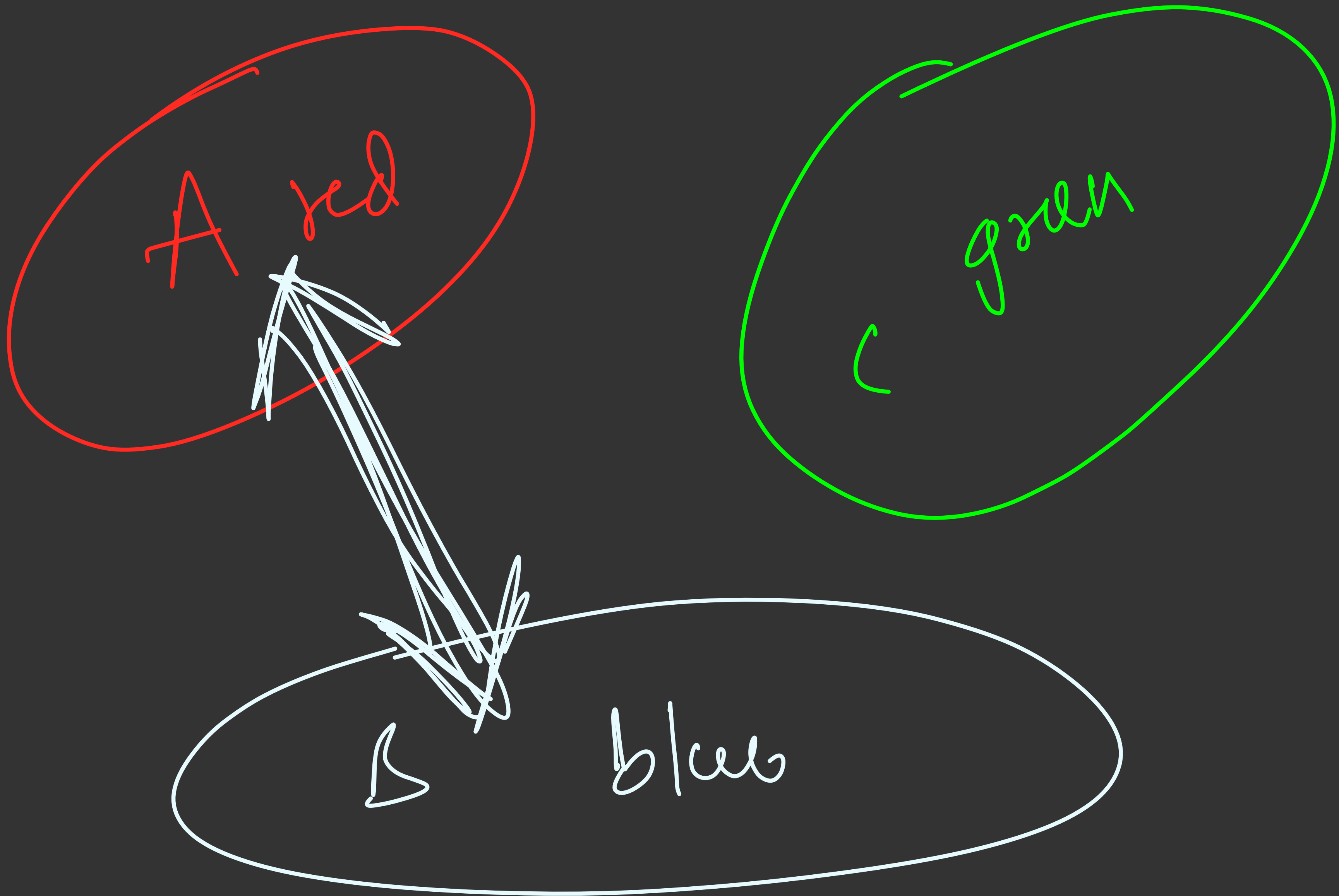

2
3
1

$\#(X, Y) \rightarrow$ # of way of connect bridge b/w X comp and Y comp

Red, Blue, Green

$\#(\text{Red}, \text{Blue}) \times \#(\text{Blue}, \text{Green})$

$\times \ \#(\text{Red}, \text{Green})$


A red
C green
B blue

A

$0 \longrightarrow AC_0 \times BC_0$

$1 \longrightarrow AC_1 \times BC_1$

$2 \longrightarrow AC_2 \times BC_2$

B Blue

1 2 3 4

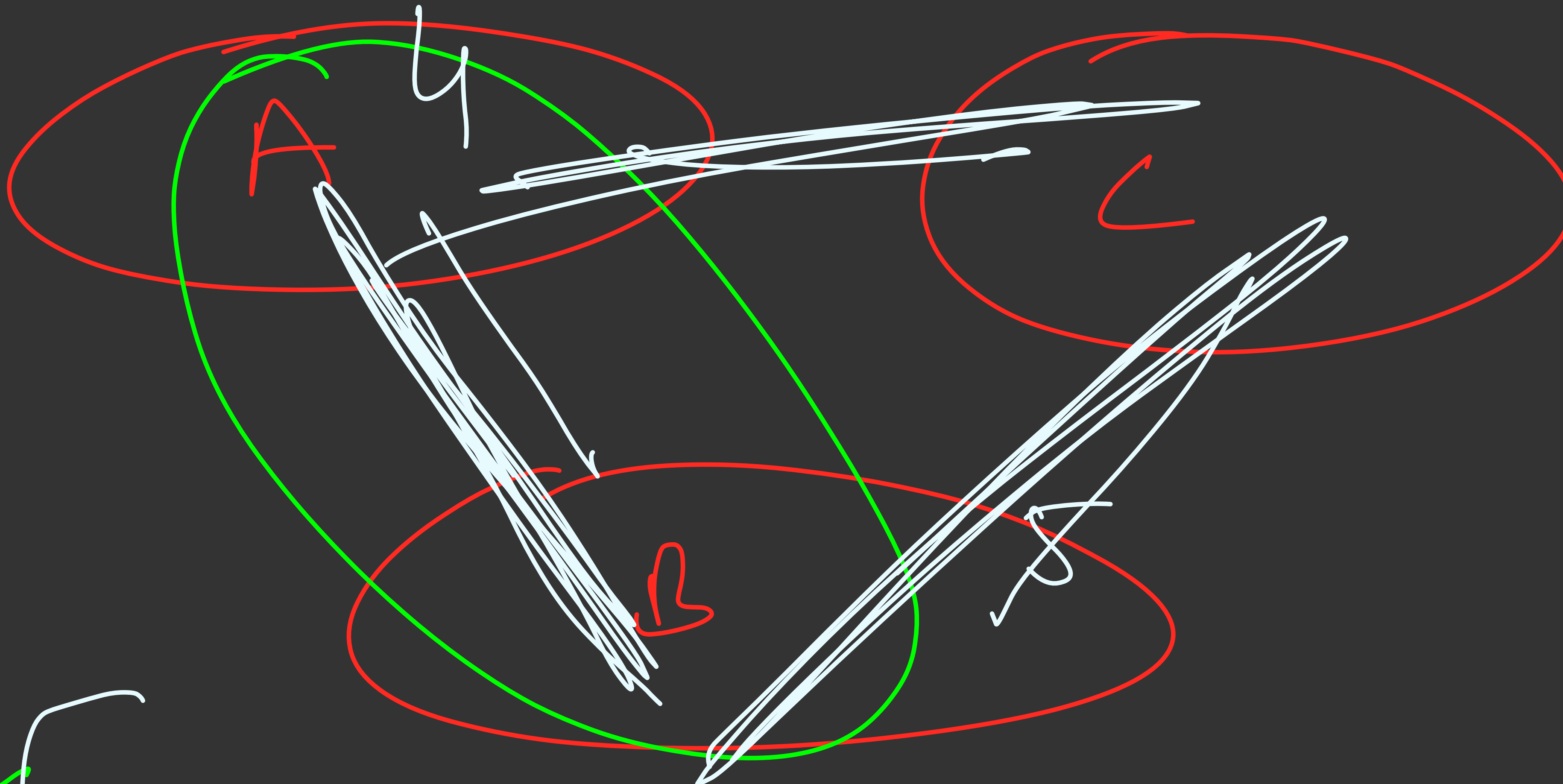
4 Red

$$ {}^4C_2 \times {}^3C_2 \times 2! $$

$$ {}^4C_3 \times {}^3C_3 \times 3! $$

5 6 7

3 Blue



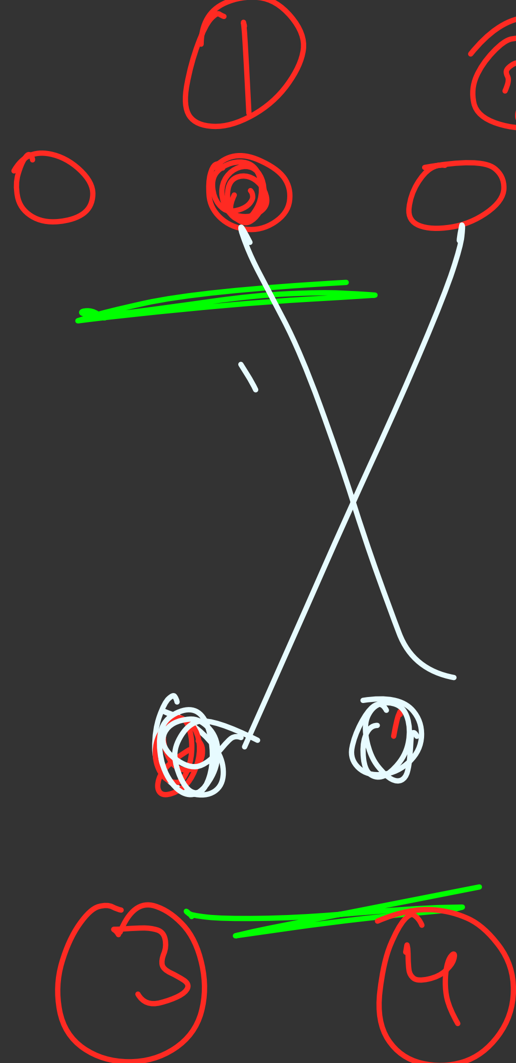
$$\left[{}^{A}C_{0} \times {}^{B}C_{0} \times 0! \right] + \left[{}^{A}C_{1} \times {}^{B}C_{1} \times 1! \right]$$

$O(1) \quad O(1) \quad O(1)$

— . . .

$$\left[{}^{A}C_{\min(A,B)} \; {}^{B}C_{\min(A,B)} \; (\min(A,B))! \right]$$

$$1 \le A, B, C \le 5000$$



2 red

2 blue

$$^3C_2 \times {}^2C_2$$