"TLE eliminator" v.v. imp for CP

Combinatorics - 1

not so much for interviews

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Binomial Coefficients

Base formula

- Number of ways to choose K items from N items

distinct items

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$c(n, k)$

3 elements A, B, C

choose 2 elements $\rightarrow$ $C(3,2)$

A, B $\longleftrightarrow$ B, A

B, C $\longleftrightarrow$ C, B

A, C

$$\frac{3!}{2!\,(3-2)!} = \frac{3!}{2!\,1!}$$

$$= 3$$

4 element A A B B

choose 2 elements

A A

B B

A B $\longleftrightarrow$ B A

$^4C_2 \rightarrow$

a_1 a_2 a_3

a_1 element of type 1

a_2 element of type 2

a_3 element of type 3

choose k element

$\boxed{1\ 1\ 1}$ $\boxed{2\ 2}$ $\boxed{3\ 3\ 3\ 3}$

a_1 a_2 a_3

choose k element out of it

$k=5$

$k = 5$

$x_1 \quad x_2 \quad x_3$

$$\boxed{x_1 + x_2 + x_3 = 5}$$

such that $0 \le x_1 \le a_1$

$0 \le x_2 \le a_2$

$0 \le x_3 \le a_3$

Requirements to calculate C(n, r)

- Mod Inverse calculation
- Precomputation of Factorials
- Precomputation of Inverse Factorials
- Trick to find Inverse Factorials in O(N)

$C(10^9, 2)$

- Idea to implement nC_r in $O(1)$ time

$$\left\{ \frac{n!}{r!\,(n-r)!} \right\} \quad C(10,4)$$

no of ways mod 10^9+7

no of ways to choose 50 items from 10^6 elements

$C(n, r)$ once α

$n \rightarrow 10^5$

$C(10^5, 4)$

$C(n, r)$

$C(1000, 6)$

$\downarrow$

$r > n$

$C(n, r)$ $r \leq n$

$$\frac{n!}{\boxed{r!}\,\boxed{(n-r)!}}$$

Print $C(n, r)$ mod some prime

$$\frac{n!}{r!\,(n-r)!} \longrightarrow (n!) \cdot \boxed{(r!)}^{-1} \cdot [(n-r)!]^{-1}$$

x mod inverse wrt to 10^9+7

fermat's Theorem $\longrightarrow$ $x^{p-1} \equiv 1 \bmod p$

$$x^{p-2} \equiv x^{-1} \bmod p$$

Suppose you have factorials pre computed

^nC_r

$$\frac{n!}{r!\,(n-r)!} \longrightarrow n! \cdot (r!)^{-1} \cdot [(n-r)!]^{-1}$$

$$[r!]^{p-2} \cdot [(n-r)!]^{p-2}$$

factorials $\longrightarrow$ precomputed

$$1! \quad 2! \quad 3! \quad \xrightarrow{O(n)} \quad n!$$

$$\downarrow$$

$$(1!)^{P-2} \quad (2!)^{P-2} \quad \text{---} \; \underline{O(n)} \; \text{---} \quad (n!)^{P-2}$$

$\boxed{fact[n]}$ $\qquad$ $ifact[n]$

$\downarrow$

$fact(i) = i!$ $\qquad$ $ifact(i) = (i!)^{-1}$

$fact(0) = 1$

for(int i = 1 ; i ≤ n ; i++)

$O(1) \longrightarrow fact(i) = (fact(i-1) \times i) \bmod M$

$O(\log M) \longrightarrow ifact(i) = expo(fact(i), M-2)$

$\log(M)$

$$C(n,r) = \frac{n!}{r! \cdot (n-r)!} = \frac{\text{fact}(n) \cdot \text{ifact}(r)}{} \cdot \text{ifact}(n-r)$$

Precomputation $\longrightarrow$ $O(n)$ + $O(n \log P)$

$O(n)$ → fact

$O(n \log P)$ → ifact

$O(n)$

$$\text{fact}(n) = n!$$

$$\text{ifact}(n) = (n!)^{-1} = \underline{\underline{\text{expo}(n!, M-2)}}$$

$\log M$ time

$\underline{\underline{\text{ifact}(n)}}$, $\underline{\underline{\text{ifact}(n-1)}}$

$$\text{fact}(i) = i \times \text{fact}(i-1)$$

$$\text{ifact}(i) = \left(\frac{1}{i!}\right)$$

$$\text{ifact}(i-1) = \left(\frac{1}{(i-1)!}\right)$$

$$\boxed{ifact(i-1) = ifact(i) \times i}$$

$$\frac{1}{(i-1)!} = \frac{1}{(i)!} \times i = \frac{1}{(i-1)!}$$

$fact(n)$, $ifact(n)$

$$ifact(n-1) = ifact(n) \times n$$

$$ifact(n-2) = ifact(n-1) \times (n-1)$$

Efficient Implementation

```
ll combination1(ll n, ll r, ll m, vector<ll>& fact, vector<ll>& ifact){
    return mod_mul(fact[n], mod_mul(ifact[r], ifact[n - r], m), m);
}
void solve(){
    int n = 10;
    vector<ll> fact(n + 1);
    vector<ll> ifact(n + 1);
    fact[0] = 1;
    for(int i = 1; i <= n; i++){
        fact[i] = mod_mul(fact[i - 1], i, MOD);
        [illegible]
    }
    ifact[n] = mminvprime(fact[n], MOD);
    for(int i = n - 1; i >= 0; i--){
        ifact[i] = mod_mul(ifact[i + 1], i + 1, MOD);
    }

    cout << combination1(8, 6, MOD, fact, ifact) << endl;
}
```

$O(1)$

$\log M$

mod_mul(a, b, m)

$(a \times b)\% m$

mminvprime(x, m)

(x^{m-2})

$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

$$\frac{5!}{2!\,3!} \rightarrow \frac{\cancel{3!}(4)(5)}{2!\,\cancel{3!}} = \frac{4\times 5}{2\times 1}$$

$$C(7,3) \rightarrow \frac{7!}{3!\,4!} = \frac{(4!)(5)(6)(7)}{3!\,4!} = \frac{5\times 6\times 7}{3\times 2\times 1}$$

$$\frac{n!}{r!\,(n-r)!} = \frac{(n-r)! \times (n-r+1).(n-r+2) \ldots n}{r!\,(n-r)!}$$

$$= \frac{(n-r+1)(n-r+2) \ldots n}{r\,(r-1).(r-2) \ldots 1}$$

$$= \frac{n \times (n-1) \times (n-2) \ldots (n-r+1)}{r \times (r-1) \times (r-2) \ldots (1)}$$

$C(n, r) \longrightarrow$ individually

$$\frac{n \times (n-1) \times (n-2) \; --- (n-r+1)}{r \times (r-1) \times (r-2) \; --- (1)}$$

$O(r) + O(r) + O(\log P)$

Problem 1

${}^nC_r$ for some general n, r

$1 \le n \le 10^6$

many time

$O(1)$ time

Problem 2

Calculate ${}^nC_r$ just once

$$\frac{n \times (n-1) \ldots (n-r+1)}{r \times (r-1) \ldots 1}$$

$O(r)$

$$^nC_r = \frac{\boxed{n \times (n-1) \cdots n-r+1)}}{\boxed{r \times (r-1) \cdots 1}}$$

num $\rightarrow O(r)$

den $\rightarrow O(r)$

num . $\underline{\underline{\text{den}^{-1}}}$

$\downarrow O(\log p)$

$\underline{O(r) + O(r)} + \underline{O(\log p)}$

$$(a \times b) \bmod m$$

$$[(a \% m) * (b \% m)] \% m$$

$$\frac{5 \times 4 \times 3}{3 \times 2 \times 1}$$

Important Binomial Results

2

1

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

$$C(n,k) = C(n,n-k)$$

$$\frac{n!}{k!\,(n-k)!}$$

$$\frac{n!}{(n-(n-k))!\,(n-k)!}$$

$$\frac{n!}{k!\,(n-k)!}$$

Given n items choose k items ①

Given n items don't choose $(n-k)$ items ②

$$A, B, C \longleftrightarrow A, B, C$$

$$\boxed{{}^{n}C_{r}} = \boxed{{}^{n-1}C_{r-1}} + \boxed{{}^{n-1}C_{r}}$$

Chose r items from n

Choose r-1 items from n-1

chose r items from n-1

1 2 3 4 5 6

chose 3 items from 6

chose 3 items from 5

+

chose 2 item from 5

$$\left[\sum_{k=0}^{n} {}^nC_k\right] = \underline{\underline{2^n}}$$

$${}^nC_0 + {}^nC_1 + {}^nC_2 \text{------} + {}^nC_n$$

Array = [1, 2, 3]

$\{\ \} \rightarrow {}^3C_0$

$\{1, 2\}$, $\{1, 3\}$, $\{2, 3\} \rightarrow {}^3C_2$

$\{1, 2, 3\} \rightarrow {}^3C_3$

$\{1\}$, $\{2\}$, $\{3\} \rightarrow {}^3C_1$

①

$C(1000, 998) \longrightarrow$ ② $C(1000, 2)$

$0 \leq k \leq 10$

$C(10^9, 10^9 - k) \longrightarrow C(10^9, k)$

Important Binomial Results

- K items always included - C(n - k, r - k)
- K items are never included - C(n - k, r)
- Value of C(n, r) is greatest when r = n/2

1 2 3 4 - - - - 100

choose 10 element from 100 elements such that

1 , 4 , 96 are always present

$$\binom{\text{remaining element}}{\text{remaining to be chosen}} \to {}^{97}C_{7}$$

Choose 10 items from 100

such that 7 9

8 76 are never included

$$\boxed{C^{96}_{10}}$$

$^nC_r \rightarrow$

$\boxed{^5C_r}$

$^5C_0 \quad ^5C_1 \quad ^5C_2$

$^5C_3 \quad ^5C_4 \quad ^5C_5$

1 5 10 10 5 1

$${}^{n}C_{0} = {}^{n}C_{n}$$

$${}^{n}C_{1} = {}^{n}C_{n-1}$$

$${}^{6}C_{r}$$

$${}^{6}C_{0} \quad {}^{6}C_{1} \quad {}^{6}C_{2} \quad \underline{\underline{{}^{6}C_{3}}} \quad {}^{6}C_{4} \quad {}^{6}C_{5} \quad {}^{6}C_{6}$$

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

Q1 how many ways are there to choose equal no. of odd elements and even elements from an array of n elements

$1 \le n \le 10^6$ mod 10^9+7

$a_1 \quad a_2 \quad - \quad - - - - \quad a_n$

3	2	5	1	6

$^3C_1 \cdot {^2C_1} = 6$

$^3C_2 = 3$, $^2C_2 = 1$

$(3,2)$ $(3,6)$ $\Rightarrow$ $(3,5,2,6)$

$(5,2)$ $(5,6)$ $(3,1,2,6)$

$(1,2)$ $(1,6)$ $(5,1,2,6)$

$$2^{(\min(\text{odd},\ \text{even}))} \rightarrow 2^2 \rightarrow 4$$

if I choose 1 odd
element 1 even

odd_count $C(odd, 1)$
even_count $C(even, 1)$

$C(odd, 1) \times C(even, 1)$

```
odd = 0 , even = 0

for ( auto i : arr )
    if ( i % 2 == 0 )
        even ++
    else
        odd ++
```

$O(n)$

```
ans = 0
for ( int i = 1 ; i <= min(odd, even) ;
                                   i ++ )
{
```

$$\left\{ \; ans \mathrel{+}= \underline{C(odd, i)} \times \underline{C(even, i)} \right.$$

$\underline{O(n)} \longrightarrow \underline{O(n)}$

Arrangement

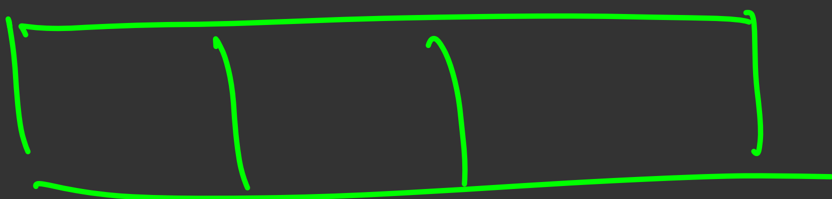
- Arranging distinct elements

$$n!$$

- Arranging similar elements

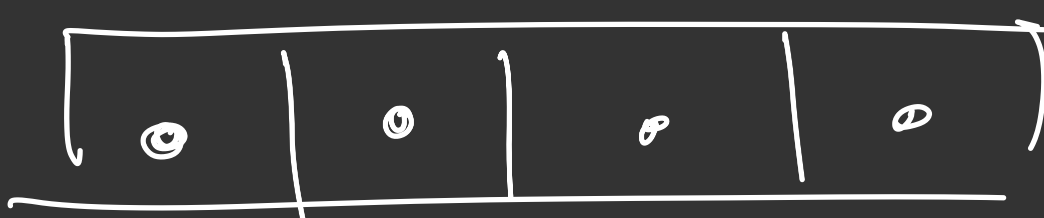
$$\frac{(a_1+a_2+a_3+\ldots+a_n)!}{a_1!\cdot a_2!\cdot\ldots\cdot a_n!}$$

A B C 6 → 3!



3 2 1 → 3!

A B C D



4 3 2 1 → 4!

A A B B B C C C C

A A B C B B C C C
1 2

A A B C B B C C C
2 1

A B C D E E E E

1 2 3 4

A B E C E D E E

$$\frac{8!}{4!}$$

$4!$

Assume all Es are different

permute all of them

n Es $\boxed{n!}$

total wage

permitting non Eg

permitting Eg

$A_1 \; A_2 \; B$

$A \; A \; B$

$A \; B \; A$

$B \; A \; A$

$A_1 \; A_2 \; B$

$A_2 \; A_1 \; B$

$A_1 \; B \; A_2$

$A_2 \; B \; A_1$

$B \; A_1 \; A_2$

$B \; A_2 \; A_1$

$$A \quad A \quad \boxed{B \quad B \quad B} \quad \boxed{C \quad C \quad C \quad C}$$

$$\frac{9!}{2!\;3!\;4!}$$

Let’s solve some problems

- Creating Strings 2 - Link
- Kth String in Dictionary
- Unique Paths - Link
- Arrange N different items such that K of them always come together

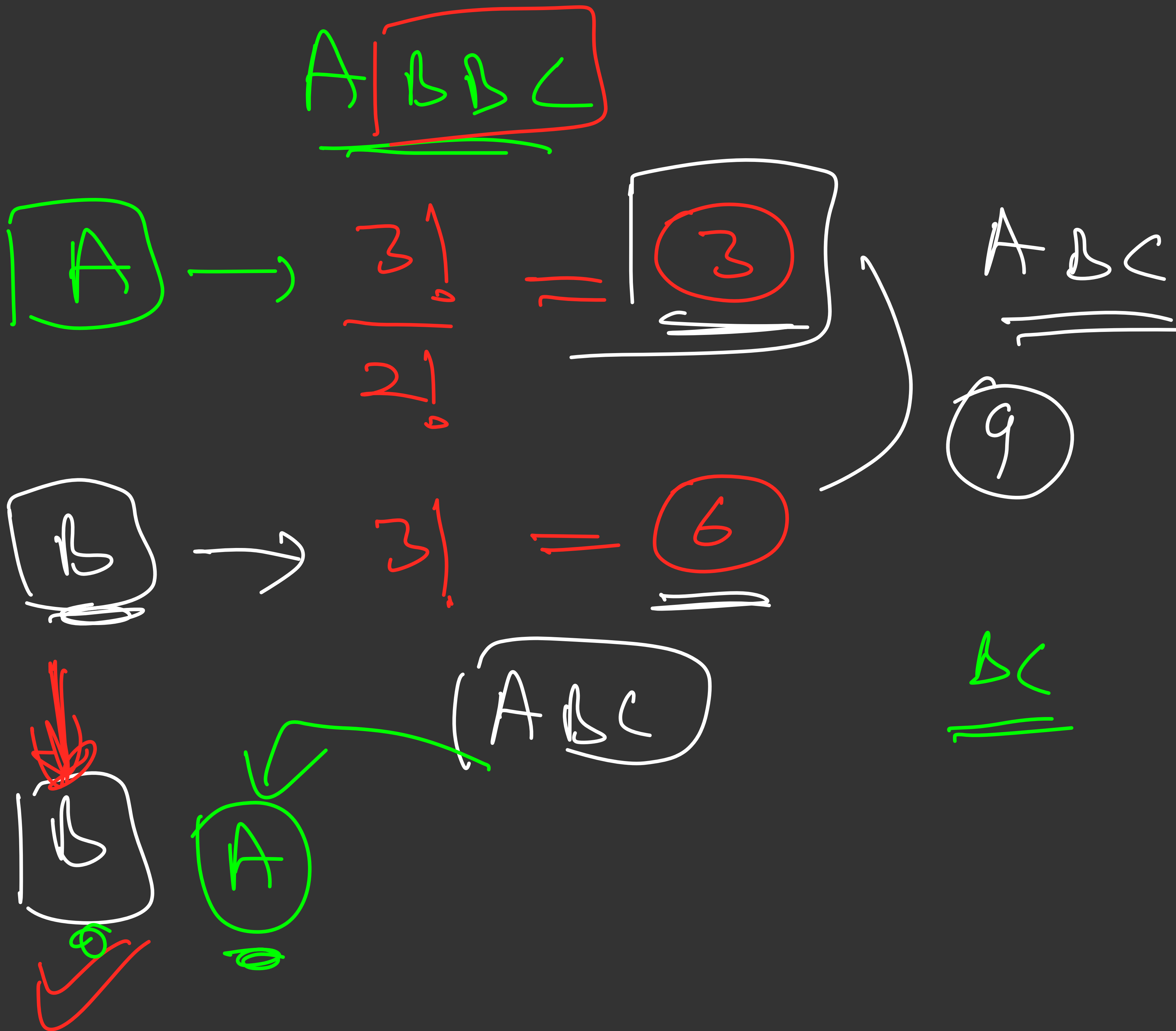
A A B C D D A X Y

100th lexicographically min string

A B B C

$$\frac{4!}{2!} = \frac{4 \times 3 \times 2!}{2!} = 12$$

8th std



ABBC
A
3!
2!
3
ABC
9
B
3!
6
ABC
BC
2!
B
A

A B B C

$\frac{3!}{2!}$

A → A B B C

A B C B

A C B B

0th

5th

A B C → 3!

B → B A B C

B A C B

B B A C

B B C A

B C A B

A B C

3rd string

1st string

$O(n^2)$

$O(n)$

$n!$

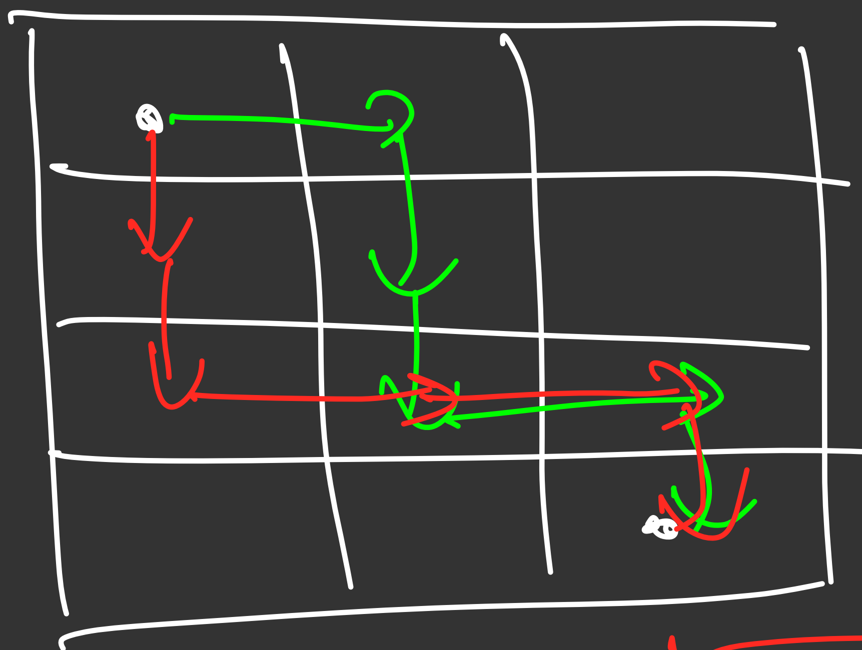
A

$(n-1)!$ ✓

a! b! c! ... z!

$O(26)$

$O(n \cdot 26)$



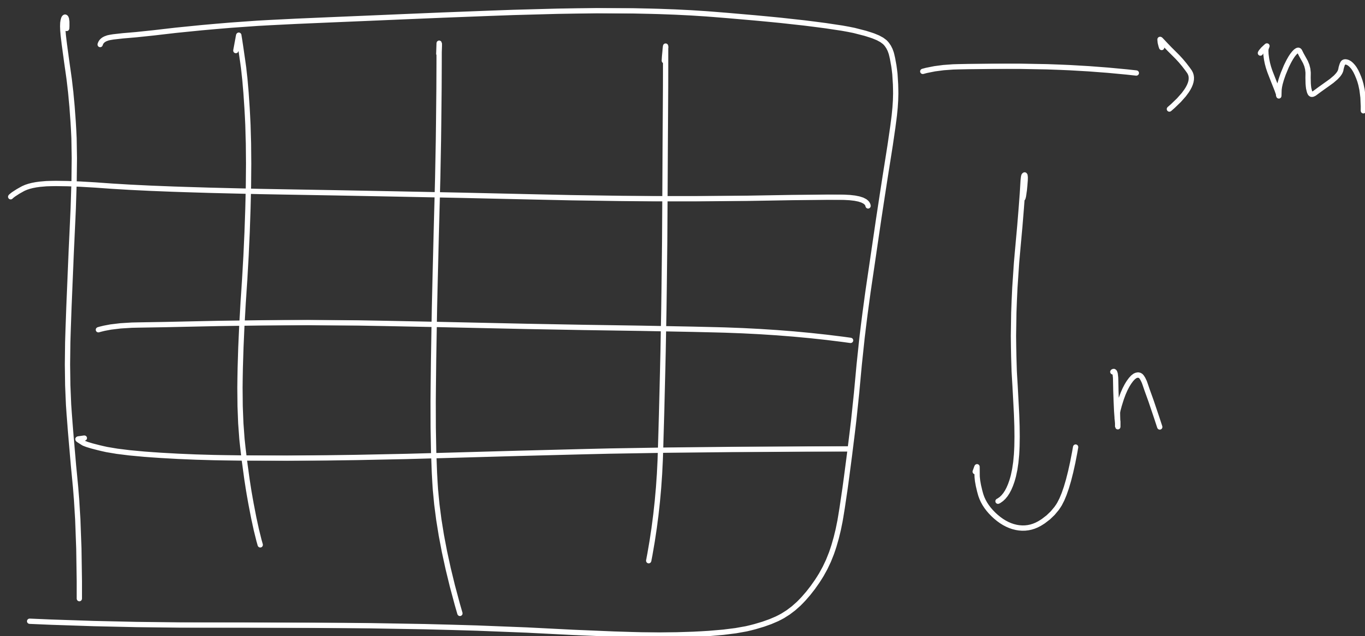
D D D R R

R D D R D

D D R R D

R R D D D

D L D R D



$(m-1) \rightarrow \qquad (n-1) \downarrow$

$$\rightarrow \quad \underline{C(n-1+m-1,\ m-1)}$$

$(m-1)\ R \qquad (n-1)\ D_s$

$$\frac{(n+m-2)}{(n-1)!\,(m-1)!}$$

1 2 3 4 5 6 7

1 2 3 5 6 4 7

1 $\boxed{3\ 6\ 5}$ 4 7 2

1 $\boxed{6\ 5\ 3}$ 4 7 2

1 2 4 1 3 5 6

1 2 3 4 5

$$\boxed{5!} \times \boxed{3!} \rightarrow$$

$\boxed{A\ A\ B\ B}\ C\ D\ \boxed{E\ F\ F\ F}$

No of way to arrange such that

$\boxed{\text{As and Bs come together}}$

and $\boxed{\text{Es and Fs come together}}$

$$(4!) \times \left(\frac{4!}{2!\,2!}\right) \times \left(\frac{4!}{3!\,1!}\right)$$