There were a few questions regarding the 3rd case of the Master Theorem that I am addressing in this. The 3rd case is when $f(n) = \Omega(n^{\log_b a+\varepsilon})$ for some ε

Let's consider an actual example. Let $f(n) = nlog(n)$ and $log_b\, a = 1$. The question becomes" is there an $\varepsilon > 0$ such that $nlog(n)\ is\ \Omega(n^{1+\varepsilon})$? The answer is no.

We can prove by contradiction. Assume there is an ε such that $nlog(n)\ is\ \Omega(n^{1+\varepsilon})$. It follows by definition of Ω that $\exists\, c, n_0\ s.t. nlog(n) \geq n^{1+\varepsilon}\ \forall\, n \geq n_0 \Leftrightarrow \log(n) \geq n^{\varepsilon}\ \forall\, n \geq n_0 \Leftrightarrow$ $\log(\log(n)) \geq \varepsilon \log(n)\, \forall\, n \geq n_0 \Leftrightarrow \log(n) \geq \varepsilon * n\ \forall\, n \geq \log\,(n_0) \ \Leftrightarrow \log(n)\,/n \geq \varepsilon\ \forall\, n \geq$ $\log\,(n_0)$

The missing link is the fact that $\lim_{n\to\infty} \frac{\log\,(n)}{n} = 0$ (this can be easily proven using l'Hôpital's rule.) This means that $\exists\, m_0 > 0\ s.t.\ \frac{\log(n)}{n} < \ \varepsilon\, \forall\, n \geq m_0$. Pick $n = \max\,(m_0, \log(n_0))$, and you get the contradiction.

Note that I do not expect you to prove it, but I do expect to recognize that this case is not Master Theorem case 3.