Lentati+2013 method for red noise analysis

October 21, 2021

Suppose you are given some data D consisting of n_D data points given by values y_i given at times t_i with uncertainties σ_i. Let T be the total time span of the data. We assume that the data is made of some deterministic signal $f(t)$, red noise $R(t)$ and white noise $W(t)$.

$$y_i = f(t_i; \boldsymbol{\epsilon}) + R(t_i; \boldsymbol{\psi}) + W_i(\boldsymbol{\alpha})$$

where $\boldsymbol{\epsilon}$ contains deterministic signal parameters, $\boldsymbol{\psi}$ contains red noise parameters and α contains white noise parameters (EFACs (E_f) and EQUADs (E_q)). Let $\Theta := \{\boldsymbol{\epsilon}, \boldsymbol{\psi}, \boldsymbol{\alpha}\}$. Our aim is to estimate Θ.

Given some parameters $\boldsymbol{\Theta}$, we can define the residuals

$$r_i := y_i - f(t_i; \boldsymbol{\epsilon})$$

and the white noise covariance matrix

$$N_{ij} := N_{(i)}\delta_{ij}$$

where

$$N_i = (E_f \sigma_i)^2 + E_q^2 \,.$$

Note that N_{ij} is a diagonal matrix.

Now, let us write the red noise as a Fourier sum with n_r frequency components where $n_r \ll n_D$.

$$R(t_i; \psi) = a_p F_{pi} \,.$$

Here, F_{pi} is a $2n_r \times n_D$ matrix containing the following elements.

$$F_{pi} = \begin{cases} \frac{1}{T} \sin\left(\frac{2\pi(p+1)t_i}{T}\right) & 0 \le p < n_r \\ \frac{1}{T} \cos\left(\frac{2\pi(p+1-n_r)t_i}{T}\right) & n_r \le p < 2n_r \end{cases} .$$

Note that we have excluded the zero-frequency DC term. This will be modelled as an offset in the signal.

We can now define the $2n_r \times 2n_r$ correlation matrix ψ_{ij}as

$$\psi_{pq} := \psi_{(p)}\delta_{pq}$$

where $\psi_{(p)} := \langle a_p a_p^* \rangle$. We want to estimate the power spectral densities $\psi_{(p)}$ along with the signal parameters. Note that $\psi_{(n_r+p)} = \psi_{(p)}$ because we don't care about the phase of the sines and the cosines.

We need to define two more objects: the $2n_r \times 2n_r$ matrix Σ_{pq} and a vector d_p with $2n_r$ elements.

$$\Sigma_{pq} := F_{pi} N_{ij}^{-1} F_{qj} + \psi_{pq}^{-1}$$

$$d_p := F_{pi} N_{ij}^{-1} r_j$$

Now, we can write the likelihood function as

$$\mathcal{L}[\mathbf{\Theta}|D] \propto \frac{1}{\sqrt{\det(\psi)\det(N)\det(\Sigma)}} \times \exp\left[-\frac{1}{2}\left(r_i N_{ij}^{-1} r_j - d_p \Sigma_{pq}^{-1} d_q\right)\right]$$

Here,

$$\det(\psi) = \prod_{p=0}^{n_r-1} |\psi_{(p)}|^2$$

$$\det(N) = \prod_{i=0}^{n_D-1} N_{(i)}$$

$$r_i N_{ij}^{-1} r_j = \sum_{i=0}^{n_D-1} \frac{(r_i)^2}{(E_f\sigma_i)^2 + E_q^2}$$

$$\Sigma_{pq} := \sum_{i=0}^{n_D-1} \frac{F_{pi}F_{qi}}{(E_f\sigma_i)^2 + E_q^2} + \frac{1}{\psi_{(p)}}\delta_{pq}$$

Note that we have gotten rid of the $n_D \times n_D$matrix inversion and determinant calculation $(O(n_D^3))$that were giving us problem. Instead, we now have a $n_r \times n_r$matrix inversion and determinant calculation which is much easier because $n_r \ll n_D$. The computations required now are at worst $O(n_D n_r^2)$.