Pingala Series

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Contents

Abstract—**This manual provides a simple introduction to Transforms**

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

Solution: : Run the below python code

```
wget https://github.com/AvinashNayak27/
    Pingala/tree/main/pingala/codes/1.py
```

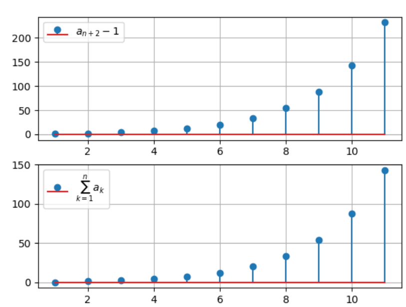

Fig. 1.1

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

Solution: : Run the below python code

```
wget https://github.com/AvinashNayak27/
    Pingala/tree/main/pingala/codes/1.py
```

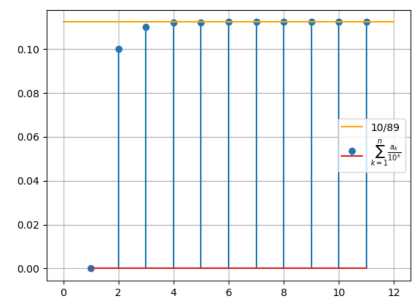

Fig. 1.2

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

Solution: : Run the below python code

```
wget https://github.com/AvinashNayak27/
    Pingala/tree/main/pingala/codes/1.py
```

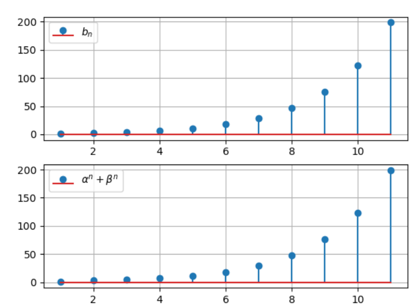

Fig. 1.3

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: : Run the below python code

wget https://github.com/AvinashNayak27/
Pingala/tree/main/pingala/codes/1.py

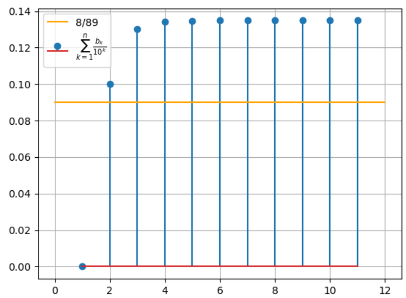

Fig. 1.4

From the figure we can see that $b(n)$ doesn't converge to $\frac{8}{89}$, it converges to $\frac{12}{89}$

2 Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n) \tag{2.2}$$
$$x(0) = x(1) = 1, n \geq 0 \tag{2.3}$$

Generate a stem plot for $x(n)$.
Solution: Run the below python code

wget https://github.com/AvinashNayak27/
Pingala/tree/main/pingala/codes/2.py

Use the following command in the terminal to run the code

2.3 Find $X^+(z)$.
Solution: : Applying positive Z-transform (linear operator) on both sides of equation 2.3

$$\sum_{k=0}^{\infty} x(k+2)z^{-k} = \sum_{k=0}^{\infty} x(k+1)z^{-k} + \sum_{k=0}^{\infty} x(k)z^{-k} \tag{2.4}$$

$$z^2(X^+(z) - x(0) - z^{-1}x(1)) = z(X^+(z) - x(0)) + X^+(z) \tag{2.5}$$

$$\implies X^+(z) = \frac{z^2}{z^2 - z - 1} \tag{2.6}$$

$$\implies X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.7}$$

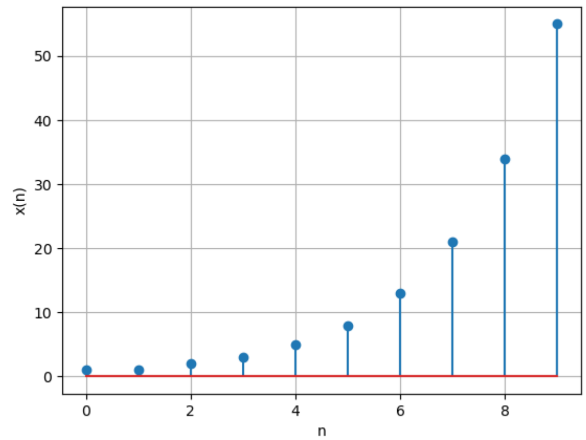

Fig. 2.2

2.4 Find $x(n)$.
Solution: :

$$\frac{1}{1 - z^{-1} - z^{-2}} = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \tag{2.8}$$

where α, β are the roots of equation

$$z^2 - z - 1 = 0 \tag{2.9}$$

Hence,

$$X^+(z) = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \tag{2.10}$$

$$\implies X^+(z) = \frac{1}{\alpha - \beta}\left(\frac{\alpha}{1 - \alpha z^{-1}} - \frac{\beta}{1 - \beta z^{-1}}\right) \tag{2.11}$$

As

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{2.12}$$

$$\implies a^{n+1} u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{a}{1 - az^{-1}} \quad |z| > |a| \tag{2.13}$$

$$\therefore x(n) = \frac{1}{\alpha - \beta}(\alpha^{n+1} - \beta^{n+1}) \tag{2.14}$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \geq 0 \tag{2.15}$$

Solution: Run the below python code

```
wget https://github.com/AvinashNayak27/
    Pingala/tree/main/pingala/codes/2.py
```

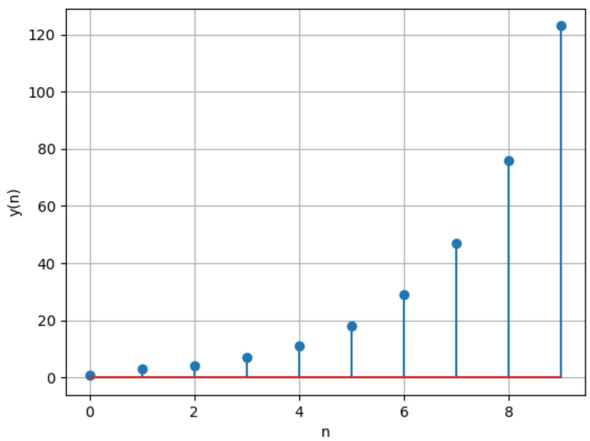

Fig. 2.5

2.6 Find $Y^+(z)$.
Solution: :
Taking +ve Z-transform on both sides of the equation 2.15, we get

$$\sum_{k=0}^{\infty} y(k)z^{-k} = \sum_{k=0}^{\infty} x(k+1)z^{-k} + \sum_{k=0}^{\infty} x(k-1)z^{-k} \tag{2.16}$$

$$Y^+(z) = z(X^+(z) - x(0)) + z^{-1}X^+(z) \tag{2.17}$$

$$\because x(-1) = 0$$

$$Y^+(z) = \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \tag{2.18}$$

$$\therefore Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.19}$$

2.7 Find $y(n)$.
Solution: : $y(n)$ is coefficient of z^{-n} in $Y^+(z)$

$$Y^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} + \frac{2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.20}$$

Applying inverse Z-transform to 2.20, we get

$$y(k) = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} + 2\frac{\alpha^k - \beta^k}{\alpha - \beta} \tag{2.21}$$

$$= \frac{\alpha^{k+2} + \alpha^k - \beta^k - \beta^{k+2}}{\alpha - \beta} \tag{2.22}$$

$$= \frac{\alpha^{k+2} - \beta^{k+2} + \alpha\beta^{k+1} - \beta\alpha^{k+1}}{\alpha - \beta}[\because \alpha\beta = -1] \tag{2.23}$$

$$\therefore y(k) = \alpha^{k+1} + \beta^{k+1}$$

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{3.1}$$

Solution:

$$x(k) = a(k+1) \tag{3.2}$$

$$\implies \sum_{k=0}^{n-1} x(k) = \sum_{k=0}^{n-1} a(k+1) \tag{3.3}$$

$$\implies \sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) \tag{3.4}$$

$$x(n) * u(n-1) = \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \tag{3.5}$$

$$u(n-1-k) = \begin{cases} 0 & k > n-1 \\ 1 & k >= n-1 \end{cases} \tag{3.6}$$

$$x(k) = 0 \quad \forall k < 0 \tag{3.7}$$

$$\implies x(n) * u(n-1) = \sum_{k=0}^{n-1} x(k) \tag{3.8}$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \tag{3.9}$$

can be expressed as

$$[x(n+1) - 1]\, u(n) \tag{3.10}$$

Solution: :

$$x(k) = a(k+1) \tag{3.11}$$

$$\implies x(k+1) = a(k+2) \tag{3.12}$$

$$a(k+2) - 1 = x(k+1) - 1 \tag{3.13}$$

$$\therefore a_{n+2} - 1 = [x(k+1) - 1]u(k) \quad [\forall n >= 1] \tag{3.14}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.15)$$

Solution: :

$$X^+(z) = \sum_{k=0}^{\infty} x(k) z^{-k} = z \sum_{k=1}^{\infty} a(k) z^{-k} \quad (3.16)$$

$$z = 10 \quad (3.17)$$

$$\implies 10 \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = X^+(10) \quad (3.18)$$

$$\implies \sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+(10) \quad (3.19)$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \quad (3.20)$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \quad (3.21)$$

and find $W(z)$.
Solution: : Applying Z-transform on 3.21

$$W(z) = \sum_{n=-\infty}^{\infty} \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) z^{-n} \quad (3.22)$$

$$= \sum_{n=0}^{\infty} \left(\alpha^{n+1} + \beta^{n+1}\right) z^{-n} \quad (3.23)$$

$$= \alpha \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \beta \sum_{n=0}^{\infty} (\beta z^{-1})^n \quad (3.24)$$

$$\text{ROC:} |z| > \max(\alpha, \beta) \quad (3.25)$$

$$= \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\beta}{1 - \beta z^{-1}} \quad (3.26)$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (3.27)$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.28)$$

Solution: :

$$y(k) = b(k+1) \quad (3.29)$$

$$\sum_{k=0}^{\infty} y(k) z^{-k} = \sum_{k=0}^{\infty} b(k+1) z^{-k} = Y^+(z) \quad (3.30)$$

$$\sum_{k=0}^{\infty} y(k) z^{-k} = z \sum_{k=1}^{\infty} b(k) z^{-k} = Y^+(z) \quad (3.31)$$

$$z = 10 \quad (3.32)$$

$$\implies \sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \quad (3.33)$$

3.6 Solve the JEE 2019 problem.
Solution: :

$$X^+(z) = z \sum_{k=1}^{\infty} a(k) z^{-k} = \frac{1}{1 - z^{-1} - z^{-2}} \quad (3.34)$$

$$z = 10 \quad (3.35)$$

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10\left(1 - \frac{1}{10} - \frac{1}{100}\right)} \quad (3.36)$$

$$= \frac{10}{89} \quad (3.37)$$

$$y(k) = \alpha^{k+1} + \beta^{k+1} \quad (3.38)$$

$$y(k) = b(k+1) \quad (3.39)$$

$$\implies b(k) = \alpha^k + \beta^k \quad (3.40)$$

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} Y^+(10) \quad (3.41)$$

$$= \frac{1}{10} \left[\frac{1 + \frac{2}{10}}{1 - \frac{1}{10} - \frac{1}{100}} \right] \quad (3.42)$$

$$\because Y^+(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \quad (3.43)$$

$$= \frac{12}{89} \quad (3.44)$$

Run the following code to get the expressions of $x(n)$ and $y(n)$

```
wget https://github.com/AvinashNayak27/
    Pingala/tree/main/pingala/codes/xnyn.py
```