Digital Signal Processing
EE3900

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Contents

1. Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{1.2}$$

2. Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

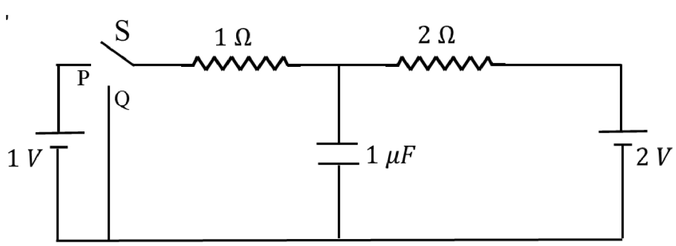

Fig. 2.1.

2. Draw the circuit using latex-tikz.
Solution: The following code yields Fig.**??**

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/Tikz%20Circuits/2.2.tex
```

3. Find q_1.
Solution: Before switching S to Q: At steady state, which achieved when switch S is at P

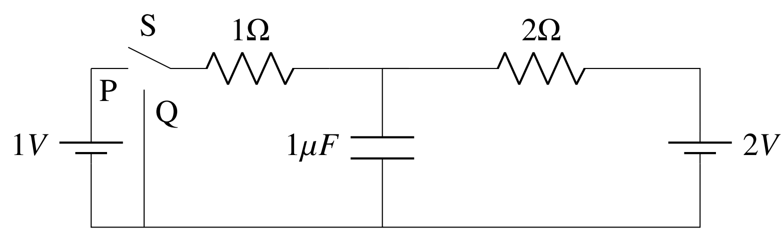

Fig. 2.2. Given Circuit

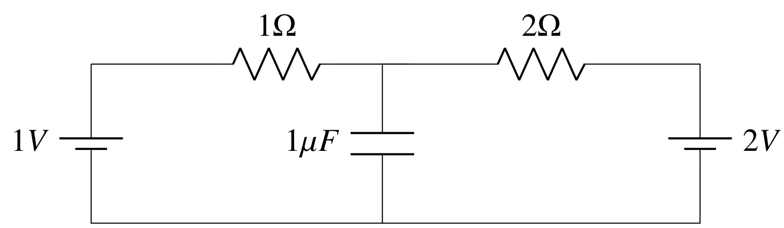

Fig. 2.3. Before switching S to Q

for long time capacoitor behaves as an open switch, hence current through capacitor is 0, Let i be the current flowing in the circuit at steady state. Applying KVL ,

$$1 - i - 2i - 2 = 0 \tag{2.1}$$

$$3i = -1 \Rightarrow i = \frac{-1}{3}A \tag{2.2}$$

Potential Difference across the capacitor at steady state is

$$1 - \left(\frac{-1}{3}\right) = \frac{4}{3}V \tag{2.3}$$

$$q_1 = \frac{4}{3} \cdot 1 \tag{2.4}$$

$$= \frac{4}{3}\mu C \tag{2.5}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: We know that Laplace Transform fo function $f(t)$ is given as $F(s)$,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{2.6}$$

$$\tag{2.7}$$

For $u(t)$, we have,

$$F(s) = \int_0^\infty u(t)e^{-st}\,dt \tag{2.8}$$

Using (**??**),

$$\begin{aligned} F(s) &= \int_0^\infty u(t)e^{-st}\,dt && (2.9)\\ &= \int_0^\infty e^{-st}\,dt && (2.10)\\ &= -\left(0 - \frac{1}{s}\right) && (2.11)\\ &= \frac{1}{s} && (2.12) \end{aligned}$$

ROC is $Re(s) > 0$ since for $s > 0$, $e^{-st} < \infty$ for $t \to \infty$

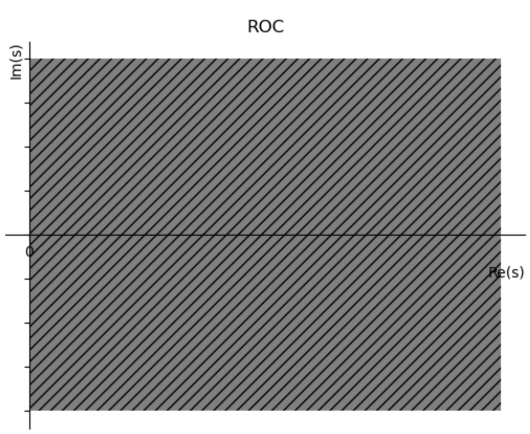

Fig. 2.4.

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \tag{2.13}$$

and find the ROC.
Solution: From **??**,

$$\begin{aligned} F(s) &= \int_0^\infty u(t)e^{-at}e^{-st}\,dt && (2.14)\\ &= \int_0^\infty u(t)e^{-(s+a)t}\,dt && (2.15)\\ &= \int_0^\infty e^{-(s+a)t}\,dt && (2.16)\\ &= -\left(0 - \frac{1}{s+a}\right) && (2.17)\\ &= \frac{1}{s+a} && (2.18) \end{aligned}$$

ROC is

$$Re(s) + a > 0 \Rightarrow Re(s) > -a \tag{2.19}$$

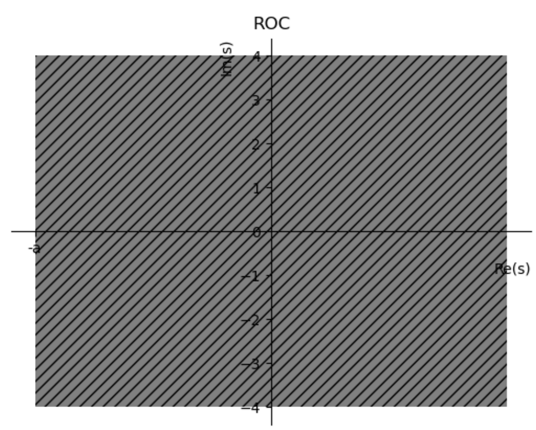

Fig. 2.5.

6. Now consider the following resistive circuit transformed from Fig. **??** where

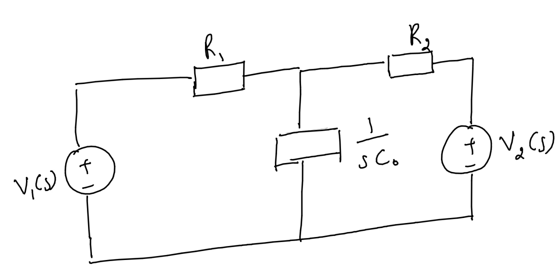

Fig. 2.6.

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{2.20}$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{2.21}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution:

$$R_{eff} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}\Omega \tag{2.22}$$

$$V_{eff} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}V \tag{2.23}$$

$$V_{C_0}(s) = V_S(s)\frac{C_0}{C_0+R_{eff}} \tag{2.24}$$

$$= \left(\frac{4}{3s}\right)\left(\frac{\frac{1}{s}}{\frac{1}{s}+\frac{2}{3}}\right) \tag{2.25}$$

$$= \frac{3+4s}{3s\left(s+\frac{3}{2}\right)} \tag{2.26}$$

7. Find $v_{C_0}(t)$. Plot using python.
Solution: Running the following code gives the plot.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/2.7.py
```

Using (**??**),

$$\frac{3+4s}{3s\left(s+\frac{3}{2}\right)} = \frac{2}{3s} + \frac{2}{3(\frac{3}{2}+s)} \tag{2.27}$$

Apply inverse Laplacian Transform,

$$V_{C_0}(s) \xleftrightarrow{\mathcal{L}^{-\infty}} V_{C_0}(t) \tag{2.28}$$

$$\mathcal{L}^{-1}\left[V_{C_0}(s)\right] = \mathcal{L}^{-1}\left[\frac{2}{3s} + \frac{2}{3(\frac{3}{2}+s)}\right] \tag{2.29}$$

$$= \mathcal{L}^{-1}\left[\frac{2}{3s}\right] - \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{\frac{3}{2}+s}\right] \tag{2.30}$$

Since,

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = u(t) \tag{2.31}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}u(t) \tag{2.32}$$

Using the above equations,

$$V_{C_0}(t) = \frac{2}{3}\left(1+e^{\frac{-3}{2}t}\right)u(t) \tag{2.33}$$

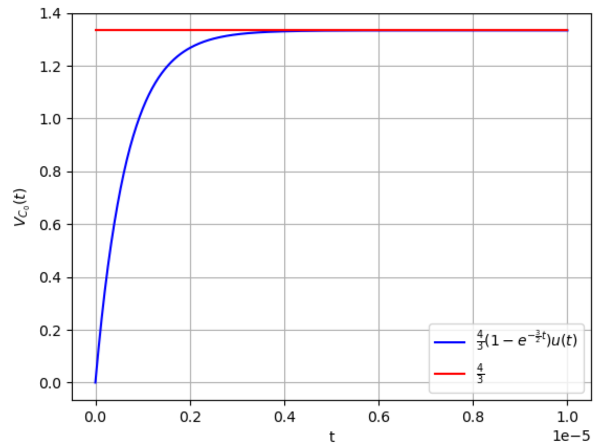

Fig. 2.7. Plot of $V_{C_0}(t)$

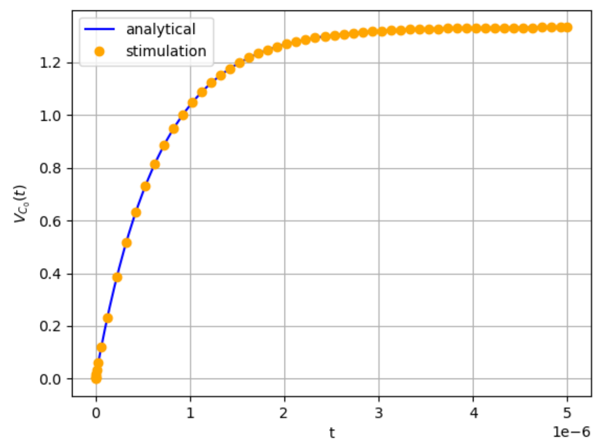

Fig. 2.8.

8. Verify your result using ngspice.
Solution:
9. Obtain Fig. **??** using the equivalent differential equation
Solution: Results obtained can be verified by running the following code.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/2.8.cir
```

And is plotted using the below code.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/2.8.py
```

Using Kirchoff's junction law

$$\frac{v_c(t)-v_1(t)}{R_1} + \frac{v_c(t)-v_2(t)}{R_2} + \frac{dq}{dt} = 0 \tag{2.34}$$

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)-V_1(s)}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+\left(sQ(s)-q(0^-)\right)=0 \tag{2.35}$$

But $q(0^-)=0$ and

$$q(t)=C_0v_c(t) \tag{2.36}$$
$$\implies Q(s)=C_0V_c(s) \tag{2.37}$$

Thus

$$\frac{V_c(s)-V_1(s)}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+sC_0V_c(s)=0 \tag{2.38}$$
$$\implies \frac{V_c(s)-V_1(s)}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+\frac{V_c(s)-0}{\frac{1}{sC_0}}=0 \tag{2.39}$$

which is the same equation as the one we obtained from Fig. **??**

3. Initial Conditions

1. Find q_2 in Fig. **??**.
Solution: At steady state capacitor behaves as an open switch. Hence $V_{C_0}=V_{1\Omega}$.
Let i be the current in the circuit. Using KVL,

$$2-2i-i=0 \implies i=\frac{2}{3} \tag{3.1}$$
$$V_{1\Omega}=i\times 1=\frac{2}{3}V \tag{3.2}$$
$$V_{C_0}=\frac{q_2}{C_0}=V_{1\Omega}=\frac{2}{3} \tag{3.3}$$
$$\implies q_2=\frac{2}{3}\mu C \tag{3.4}$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1,R_2,C_0 for the passive elements. Use latex-tikz.
Solution:

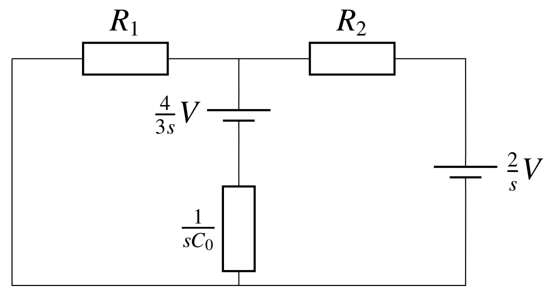

Fig. 3.1. After switching S to Q

3. $V_{C_0}(s)=?$
Solution: Let voltage across capacitor be V. Using KCL at node in Fig. **??**

$$\frac{V-0}{R_1}+\frac{V-\frac{2}{s}}{R_2}+sC_0\left(V-\frac{4}{3s}\right)=0 \tag{3.5}$$
$$\implies V_{C_0}(s)=\frac{\frac{2}{sR_2}+\frac{4C_0}{3}}{\frac{1}{R_1}+\frac{2}{R_2}+sC_0} \tag{3.6}$$

4. $v_{C_0}(t)=?$ Plot using python.
Solution: Running the following code gives the plot.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/3.4.py
```

From (**??**),

$$V_{C_0}(s)=\frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right)+\frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s}-\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \tag{3.7}$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t)=\frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t)+\frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1-e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{3.8}$$

Substituting values gives

$$v_{C_0}(t)=\frac{2}{3}\left(1+e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{3.9}$$

5. Verify your result using ngspice.
Solution: Results obtained can be verified by running the following code.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/3.5.cir
```

Runningn the below code plots the figure **??**, and verifies our result.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/3.5.py
```

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: From the initial conditions,

$$v_{C_0}(0-)=\frac{q_1}{C}=\frac{4}{3}V \tag{3.10}$$

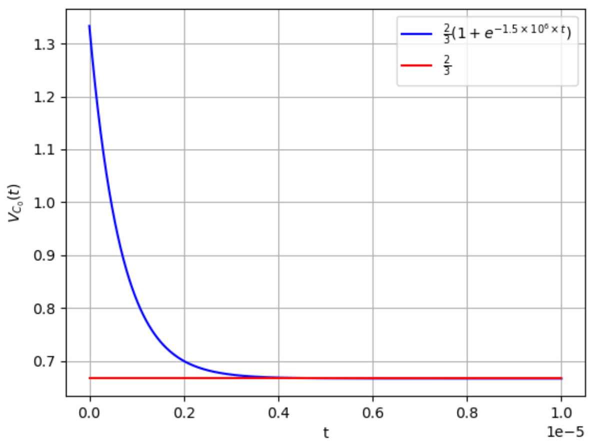

Fig. 3.2. Plot of $V_{C_0}(t)$

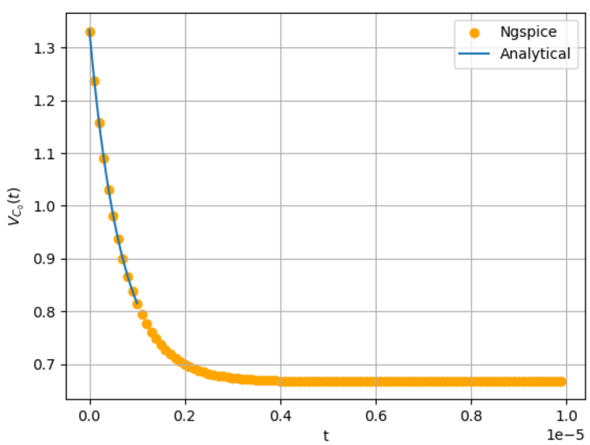

Fig. 3.3. ngspice plot of $V_{C_0}(t)$

Using (**??**),

$$v_{C_0}(0+) = \lim_{t\to 0+} v_{C_0}(t) = \frac{4}{3}V \tag{3.11}$$

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3}V \tag{3.12}$$

7. Obtain Fig. **??** using the equivalent differential equation
Solution: Using Kirchoff's junction law

$$\frac{v_c(t)-0}{R_1} + \frac{v_c(t)-v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{3.13}$$

where $q(t)$ is the charge on the capacitor. On taking the Laplace transform on both sides of the equation (**??**), we get,

$$\frac{V_c(s)-0}{R_1} + \frac{V_c(s)-V_2(s)}{R_2} + sQ(s) - q(0^-) = 0 \tag{3.14}$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \tag{3.15}$$

$$\implies Q(s) = C_0 V_c(s) \tag{3.16}$$

Thus

$$\frac{V_c(s)-0}{R_1} + \frac{V_c(s)-V_2(s)}{R_2} + \left(sC_0 V_c(s) - \frac{4}{3}C_0\right) = 0 \tag{3.17}$$

$$\implies \frac{V_c(s)-0}{R_1} + \frac{V_c(s)-V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \tag{3.18}$$

which is the same equation as the one we obtained from Fig. **??**

4. Bilinear Transform

4.1. In Fig. **??**, consider the case when S is switched to Q right in the beginning. Formulate the differential equation
Solution: The differential equation is the same as before

$$\frac{v_c(t)-0}{R_1} + \frac{v_c(t)-v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{4.1}$$

$$\text{i.e., } \frac{v_c(t)}{R_1} + \frac{v_c(t)-v_2(t)}{R_2} + C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = 0 \tag{4.2}$$

but with a different initial condition

$$q(0^-) = q(0) = 0 \tag{4.3}$$

4.2. Find $H(s)$ considering the ouput voltage at the capacitor
Solution: On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)}{R_1} + \frac{V_c(s)-V_2(s)}{R_2} + sQ(s) - 0 = 0 \tag{4.4}$$

$$\implies V_c(s)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + sC_0 V_c(s) = \frac{V_2(s)}{R_2} \tag{4.5}$$

$$\implies \frac{V_c(s)}{V_2(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{4.6}$$

The transfer function is thus

$$H(s) = \frac{\frac{1}{R_2C_0}}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \tag{4.7}$$

On substituting the values, we get

$$H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \tag{4.8}$$

4.3. Plot $H(s)$. What kind of filter is it?
Solution: Download the following Python code that plots Fig. **??**

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/4.3.py
```

Run the codes by executing

```
python 4.3.py
```

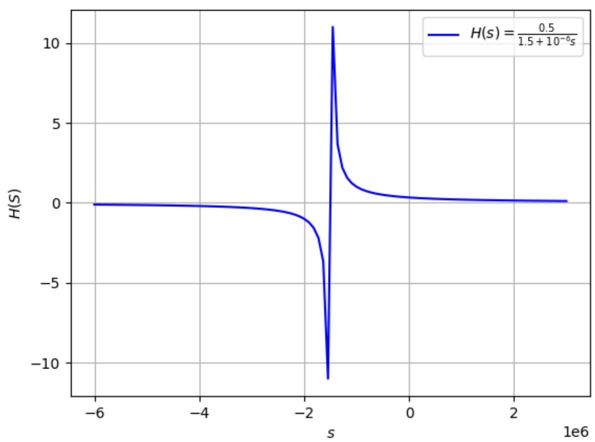

Fig. 4.1. Plot of $H(s)$

Consider the frequency-domain transfer function by putting $s = \jmath\omega$

$$H(\jmath\omega) = \frac{5 \times 10^5}{\jmath\omega + 1.5 \times 10^6} \tag{4.9}$$

$$\implies \left|H(\jmath\omega)\right| = \frac{5 \times 10^5}{\sqrt{\omega^2 + 2.25 \times 10^{12}}} \tag{4.10}$$

As ω increases, $\left|H(\jmath\omega)\right|$ decreases
In other words, the amplitude of high-frequency signals gets diminished and they get filtered out
Therefore, this is a low-pass filter

4.4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.11}$$

Solution:

$$\frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = 0 \tag{4.12}$$

$$\implies C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = \frac{2u(t) - v_c(t)}{R_2} - \frac{v_c(t)}{R_1} \tag{4.13}$$

$$\implies v_c(t)|_{t=n}^{n+1} = \int_n^{n+1}\left(\frac{2u(t) - v_c(t)}{R_2C_0} - \frac{v_c(t)}{R_1C_0}\right)\mathrm{d}t \tag{4.14}$$

By the trapezoidal rule of integration

$$\int_a^b f(t)\mathrm{d}t \approx \frac{b-a}{2}(f(a) + f(b)) \tag{4.15}$$

Consider $y(t) = v_c(t)$

$$y(n+1) - y(n) = \frac{1}{R_2C_0}(u(n) + u(n+1)) - \frac{1}{2}(y(n+1) + y(n))\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right) \tag{4.16}$$

Thus, the difference equation is

$$y(n+1)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = y(n)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}(u(n) + u(n+1)) \tag{4.17}$$

4.5. Find $H(z)$
Solution: Let $\mathcal{Z}\{y(n)\} = Y(z)$
On taking the Z-transform on both sides of the difference equation

$$zY(z)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = Y(z)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}\left(\frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}}\right) \tag{4.18}$$

$$Y(z)\left(z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = \frac{1}{R_2C_0}\frac{1+z}{1-z^{-1}} \tag{4.19}$$

Also

$$v_2(t) = 2 \qquad \forall t \geq 0 \tag{4.20}$$

$$\implies x(n) = 2u(n) \tag{4.21}$$

$$\implies X(z) = \frac{2}{1-z^{-1}} \qquad |z| > 1 \tag{4.22}$$

Thus, the transfer function in z-domain is

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.23}$$
$$= \frac{\frac{1+z}{2R_2C_0}}{z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} \tag{4.24}$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \tag{4.25}$$

On substituting the values

$$H(z) = \frac{2.5\times10^5(1+z^{-1})}{7.5\times10^5 + 1 + (7.5\times10^5 - 1)z^{-1}} \tag{4.26}$$

with the ROC being

$$|z| > \max\left(1, \left|\frac{7.5\times10^5 - 1}{7.5\times10^5 + 1}\right|\right) \tag{4.27}$$
$$\implies |z| > 1 \tag{4.28}$$

4.6. How can you obtain $H(z)$ from $H(s)$?
Solution: The Z-transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{4.29}$$

where T is the step size of the trapezoidal rule (1 in our case)
This is known as the bilinear transform
Thus

$$H(z) = \frac{\frac{1}{R_2C_0}}{2\frac{1-z^{-1}}{1+z^{-1}} + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \tag{4.30}$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 - z^{-1} + \left(\frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)(1+z^{-1})} \tag{4.31}$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \tag{4.32}$$
$$= \frac{2.5\times10^5(1+z^{-1})}{7.5\times10^5 + 1 + (7.5\times10^5 - 1)z^{-1}} \tag{4.33}$$

which is the same as what we obtained earlier

4.7. Find $y(n)$ from $H(z)$ and verify whether $y(n) = y(t)|_{t=n}$

Solution:

$$Y(z) = H(z)X(z) \tag{4.34}$$
$$= \frac{2.5\times10^5(1+z^{-1})}{7.5\times10^5 + 1 + (7.5\times10^5 - 1)z^{-1}}\frac{2}{1-z^{-1}} \tag{4.35}$$
$$= \frac{\frac{2}{3}}{1-z^{-1}} - \frac{\frac{2}{3}}{7.5\times10^5 + 1 + (7.5\times10^5 - 1)z^{-1}} \tag{4.36}$$

On taking the inverse Z-transform by considering the ROC to be $|z| > 1$, we get

$$y(n) = \frac{2}{3}u(n) - \frac{2}{3}\frac{1}{7.5\times10^5 + 1}\left(-\frac{7.5\times10^5 - 1}{7.5\times10^5 + 1}\right)^n u(n) \tag{4.37}$$
$$= \frac{2}{3}\left(1 - \frac{(1 - 7.5\times10^5)^n}{(1 + 7.5\times10^5)^{n+1}}\right)u(n) \tag{4.38}$$

If we are sampling the signal at intervals of $T \ll 10^{-5}$, say $[parse-numbers = false]10^{-7}$, i.e., $n = 10^{-7}, 2\times10^{-7}, \ldots$

$$y(n) \approx \frac{2}{3}\left(1 - \frac{1 - 7.5\times10^5 n}{1 + 7.5\times10^5 n}\right)u(n) \tag{4.39}$$

Now

$$Y(s) = H(s)X(s) \tag{4.40}$$
$$= \frac{5\times10^5}{s + 1.5\times10^6}\frac{2}{s} \tag{4.41}$$
$$= \frac{10^6}{1.5\times10^6}\left(\frac{1}{s} - \frac{1}{s + 1.5\times10^6}\right) \tag{4.42}$$

On taking the inverse Laplace transform by considering the ROC to be $\Re(s) > 0$, we get

$$y(t) = \frac{2}{3}\left(1 - e^{-1.5\times10^6 t}\right)u(t) \tag{4.43}$$

But

$$e^{-1.5\times10^6 t} = \frac{e^{-0.75\times10^6 t}}{e^{0.75\times10^6 t}} \tag{4.44}$$
$$\approx \frac{1 - 7.5\times10^5 t}{1 + 7.5\times10^5 t} \quad \text{when } t \ll 10^{-6} \tag{4.45}$$

Therefore

$$y(t) \approx \frac{2}{3}\left(1 - \frac{1 - 7.5\times10^5 t}{1 + 7.5\times10^5 t}\right)u(t) \tag{4.46}$$
$$\therefore y(n) = y(t)|_{t=n} \tag{4.47}$$

Download the following codes for simulation and plotting Fig. **??** respectively

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/4.7.cir
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/4.7.py
```

Run the codes by executing

```
ngspice 4.7.cir
python 4.7.py
```

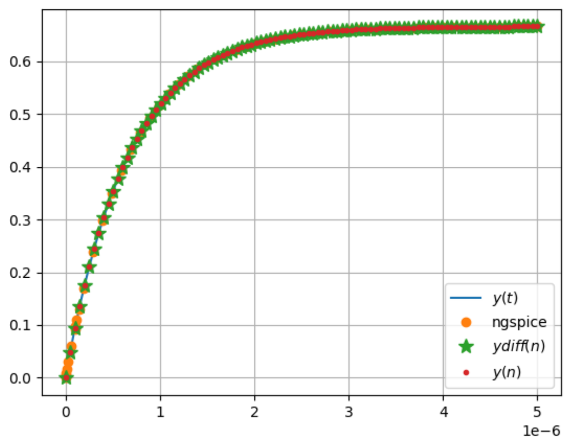

Fig. 4.2. Plots of $y(t)$ and $y(n)$