Digital Signal Processing
EE3900

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Contents

1. Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \tag{1.2}$$

2. Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

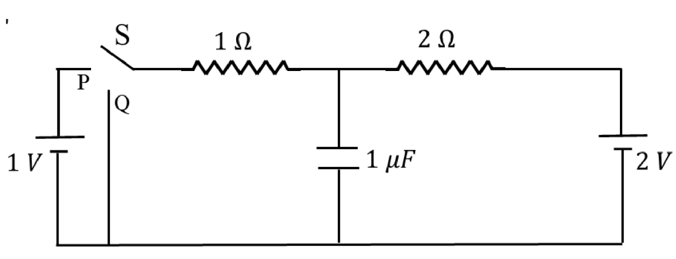

Fig. 2.1.

2. Draw the circuit using latex-tikz.
Solution: The following code yields Fig.2.2

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/Tikz%20Circuits/2.2.tex
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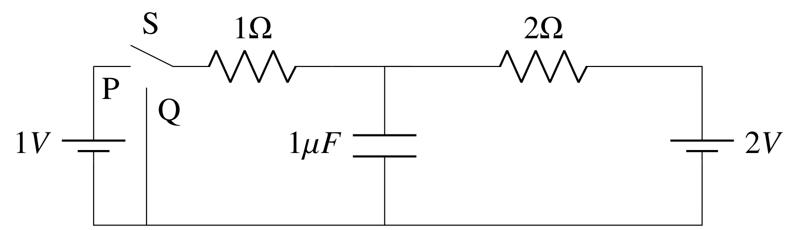

Fig. 2.2. Given Circuit

3. Find q_1.
Solution: Before switching S to Q: At steady state, which achieved when switch S is at P for long time capacoitor behaves as an open switch, hence current through capacitor is 0, Let i be the current flowing in the circuit at steady state. Applying KVL ,

$$1 - i - 2i - 2 = 0 \tag{2.1}$$

$$3i = -1 \Rightarrow i = \frac{-1}{3}A \tag{2.2}$$

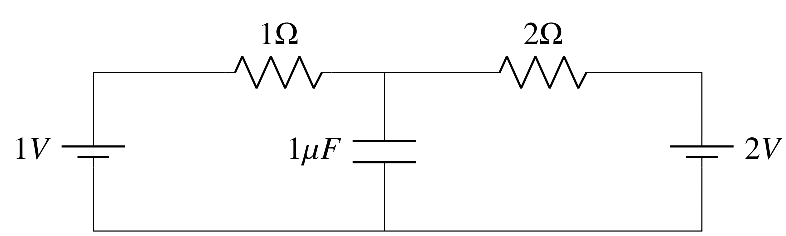

Fig. 2.3. Before switching S to Q

Potential Difference across the capacitor at steady state is

$$1 - \left(\frac{-1}{3}\right) = \frac{4}{3}V \tag{2.3}$$

$$q_1 = \frac{4}{3} \cdot 1 \tag{2.4}$$

$$= \frac{4}{3}\mu C \tag{2.5}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: We know that Laplace Transform fo function $f(t)$ is given as $F(s)$,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{2.6}$$

$$\tag{2.7}$$

For $u(t)$, we have,

$$F(s) = \int_0^\infty u(t)e^{-st}\,dt \tag{2.8}$$

Using (1.1),

$$F(s) = \int_0^\infty u(t)e^{-st}\,dt \tag{2.9}$$

$$= \int_0^\infty e^{-st}\,dt \tag{2.10}$$

$$= -\left(0 - \frac{1}{s}\right) \tag{2.11}$$

$$= \frac{1}{s} \tag{2.12}$$

ROC is $Re(s) > 0$ since for $s > 0$, $e^{-st} < \infty$ for $t \to \infty$

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \tag{2.13}$$

and find the ROC.
Solution: From 2.6,

$$F(s) = \int_0^\infty u(t)e^{-at}e^{-st}\,dt \tag{2.14}$$

$$= \int_0^\infty u(t)e^{-(s+a)t}\,dt \tag{2.15}$$

$$= \int_0^\infty e^{-(s+a)t}\,dt \tag{2.16}$$

$$= -\left(0 - \frac{1}{s+a}\right) \tag{2.17}$$

$$= \frac{1}{s+a} \tag{2.18}$$

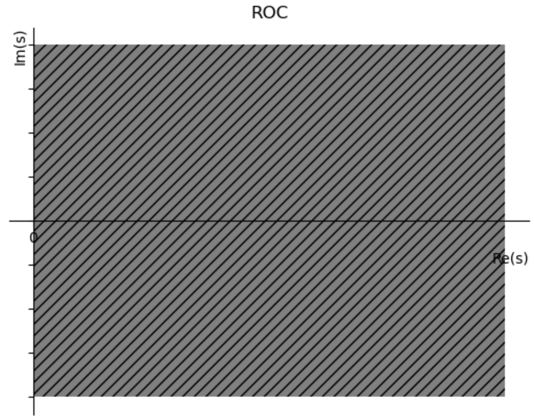

Fig. 2.4.

ROC is

$$Re(s) + a > 0 \Rightarrow Re(s) > -a \tag{2.19}$$

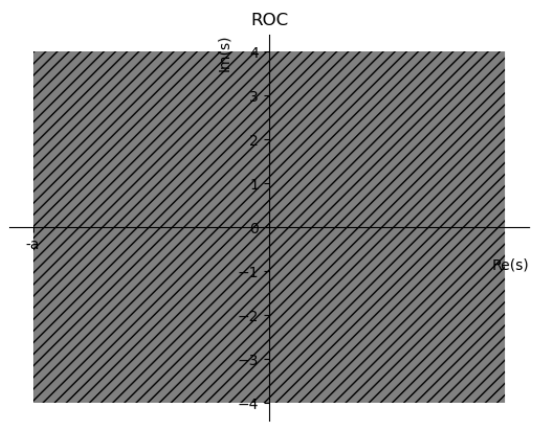

Fig. 2.5.

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{2.20}$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{2.21}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution:

$$R_{eff} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}\Omega \tag{2.22}$$

$$V_{eff} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}V \tag{2.23}$$

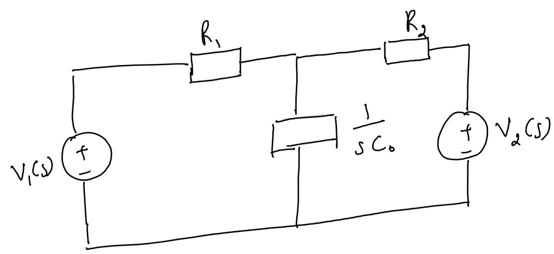

Fig. 2.6.

$$V_{C_0}(s) = V_S(s)\frac{C_0}{C_0 + R_{eff}} \tag{2.24}$$
$$= \left(\frac{4}{3s}\right)\left(\frac{\frac{1}{s}}{\frac{1}{s}+\frac{2}{3}}\right) \tag{2.25}$$
$$= \frac{3+4s}{3s\left(s+\frac{3}{2}\right)} \tag{2.26}$$

7. Find $v_{C_0}(t)$. Plot using python.
Solution: Running the following code gives the plot.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/2.7.py
```

Using (2.26),

$$\frac{3+4s}{3s\left(s+\frac{3}{2}\right)} = \frac{2}{3s} + \frac{2}{3(\frac{3}{2}+s)} \tag{2.27}$$

Apply inverse Laplacian Transform,

$$V_{C_0}(s) \xleftrightarrow{\mathcal{L}^{-\infty}} V_{C_0}(t) \tag{2.28}$$
$$\mathcal{L}^{-1}\left[V_{C_0}(s)\right] = \mathcal{L}^{-1}\left[\frac{2}{3s} + \frac{2}{3(\frac{3}{2}+s)}\right] \tag{2.29}$$
$$= \mathcal{L}^{-1}\left[\frac{2}{3s}\right] - \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{\frac{3}{2}+s}\right] \tag{2.30}$$

Since,

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = u(t) \tag{2.31}$$
$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}u(t) \tag{2.32}$$

Using the above equations,

$$V_{C_0}(t) = \frac{2}{3}\left(1 + e^{\frac{-3}{2}t}\right)u(t) \tag{2.33}$$

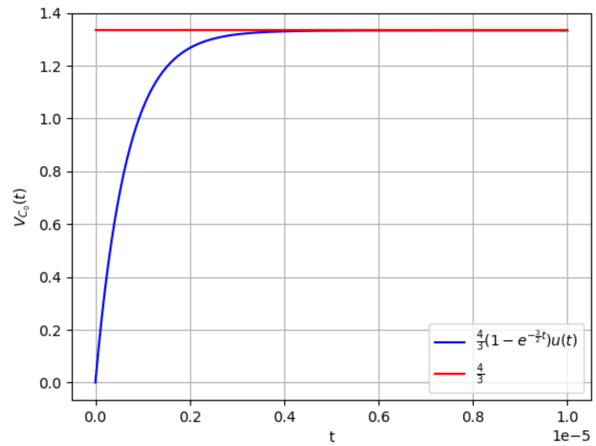

Fig. 2.7. Plot of $V_{C_0}(t)$

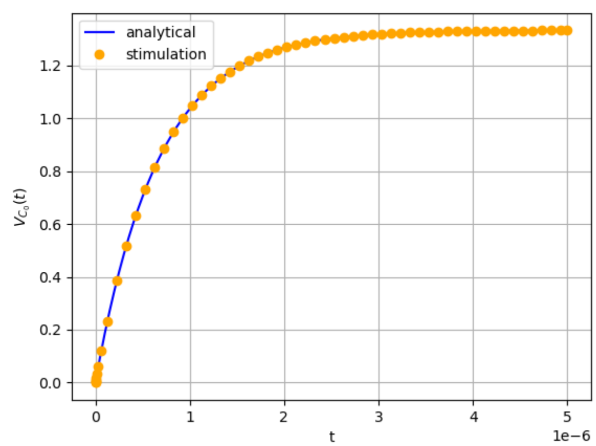

Fig. 2.8.

8. Verify your result using ngspice.
Solution:
9. Obtain Fig. 2.7 using the equivalent differential equation
Solution: Results obtained can be verified by running the following code.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/2.8.cir
```

And is plotted using the below code.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/2.8.py
```

Using Kirchoff's junction law

$$\frac{v_c(t) - v_1(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (2.34)$$

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sQ(s) - q(0^-)\right) = 0 \quad (2.35)$$

But $q(0^-) = 0$ and

$$q(t) = C_0 v_c(t) \quad (2.36)$$
$$\implies Q(s) = C_0 V_c(s) \quad (2.37)$$

Thus

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sC_0 V_c(s) = 0 \quad (2.38)$$

$$\implies \frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - 0}{\frac{1}{sC_0}} = 0 \quad (2.39)$$

which is the same equation as the one we obtained from Fig. 2.7

3. Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: At steady state capacitor behaves as an open switch. Hence $V_{C_0} = V_{1\Omega}$.
Let i be the current in the circuit. Using KVL,

$$2 - 2i - i = 0 \implies i = \frac{2}{3} \quad (3.1)$$
$$V_{1\Omega} = i \times 1 = \frac{2}{3}V \quad (3.2)$$
$$V_{C_0} = \frac{q_2}{C_0} = V_{1\Omega} = \frac{2}{3} \quad (3.3)$$
$$\implies q_2 = \frac{2}{3}\mu C \quad (3.4)$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution:

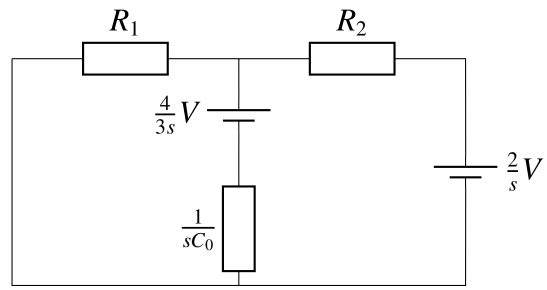

Fig. 3.1. After switching S to Q

3. $V_{C_0}(s) = ?$
Solution: Let voltage across capacitor be V. Using KCL at node in Fig. 3.1

$$\frac{V - 0}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \quad (3.5)$$
$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \quad (3.6)$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution: Running the following code gives the plot.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/3.4.py
```

From (3.6),

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) + \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \quad (3.7)$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \quad (3.8)$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)t}\right)u(t) \quad (3.9)$$

5. Verify your result using ngspice.
Solution: Results obtained can be verified by running the following code.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/3.5.cir
```

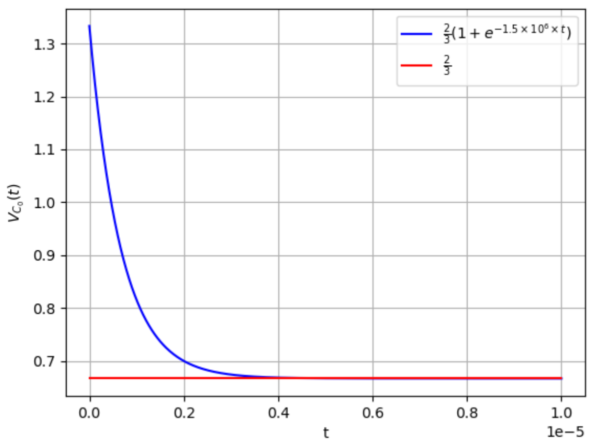

Fig. 3.2. Plot of $V_{C_0}(t)$

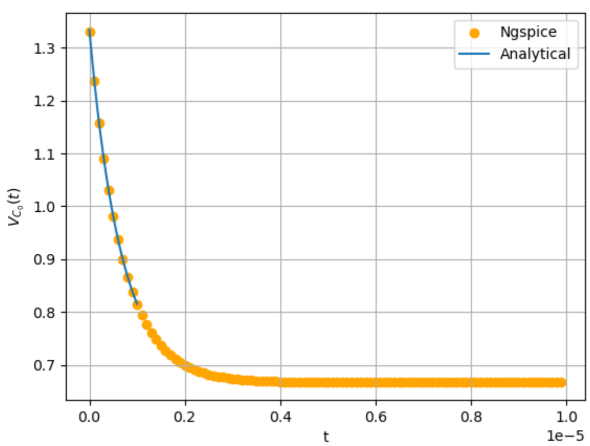

Fig. 3.3. ngspice plot of $V_{C_0}(t)$

Runningn the below code plots the figure 3.3, and verifies our result.

```
wget https://github.com/AvinashNayak27/
    Spice/blob/master/codes/3.5.py
```

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3}V \tag{3.10}$$

Using (3.9),

$$v_{C_0}(0+) = \lim_{t\to 0+} v_{C_0}(t) = \frac{4}{3}V \tag{3.11}$$

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3}V \tag{3.12}$$

7. Obtain Fig. 3.2 using the equivalent differential equation

Solution: Using Kirchoff's junction law

$$\frac{v_c(t)-0}{R_1} + \frac{v_c(t)-v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{3.13}$$

where $q(t)$ is the charge on the capacitor. On taking the Laplace transform on both sides of the equation (3.13), we get,

$$\frac{V_c(s)-0}{R_1} + \frac{V_c(s)-V_2(s)}{R_2} + sQ(s) - q(0^-) = 0 \tag{3.14}$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \tag{3.15}$$

$$\implies Q(s) = C_0 V_c(s) \tag{3.16}$$

Thus

$$\frac{V_c(s)-0}{R_1} + \frac{V_c(s)-V_2(s)}{R_2} + \left(sC_0V_c(s) - \frac{4}{3}C_0\right) = 0 \tag{3.17}$$

$$\implies \frac{V_c(s)-0}{R_1} + \frac{V_c(s)-V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \tag{3.18}$$

which is the same equation as the one we obtained from Fig. 3.2