Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/Sound_Noise.
    wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/xnyn.py
```

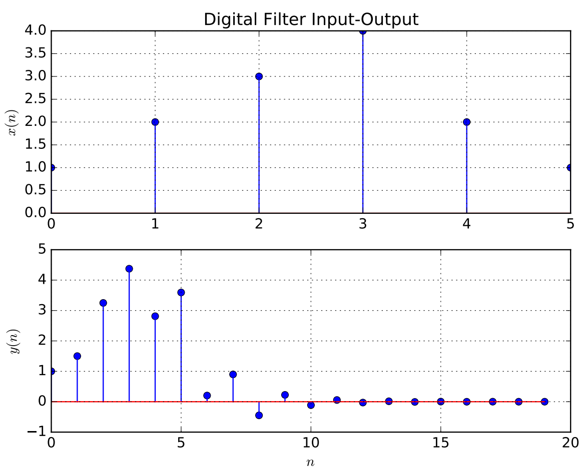

Fig. 3.2

3.3 Repeat the above exercise using a C code.
Solution: Download the C code from the below link,

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/3.3.1.c
```

Then run the follwing command in terminal

```
gcc 3.3.1.c
./a.out
```

Then for the plot **??** download the python file from the below link,

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/3.3.2.py
```

Then run the command

```
python3 3.3.2.py
```

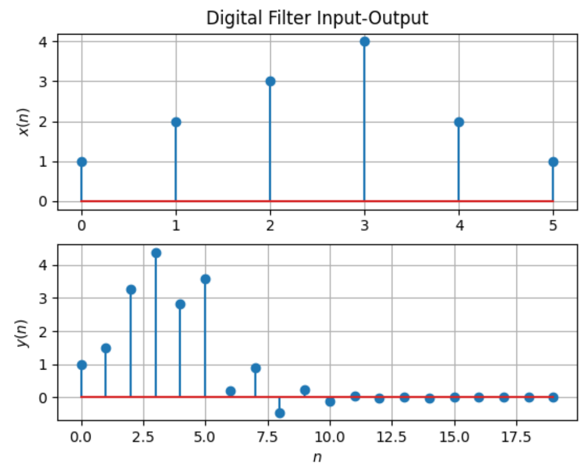

Fig. 3.3: Plot using C code

4 $\mathcal{Z}$-transform

4.1 The $\mathcal{Z}$-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: Given that,

$$X(z) = \mathcal{Z}\{x(n)\} \tag{4.4}$$
$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.5}$$

So,

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1) z^{-n} \tag{4.6}$$

Take $k = n - 1$,

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-(k+1)} \tag{4.7}$$

$$= z^{-1} \sum_{k=-\infty}^{\infty} x(k) z^{-k} \tag{4.8}$$

$$= z^{-1} \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.9}$$

$$= z^{-1} X(z) \tag{4.10}$$

resulting in (4.2) and similarly following the above steps you will get,

$$\mathcal{Z}\{x(n-k)\} = z^{-k} X(n) \tag{4.11}$$

Hence proved.

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1. **Solution:** Now we will find Z transform of the signal $x(n)$,from (**??**),

$$\mathcal{Z}\{x(n)\} = \sum_{n=0}^{5} x(n) z^{-n} \tag{4.12}$$

$$= 1z^0 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + 1z^{-5} \tag{4.13}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.14}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.15}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.11) in (3.2),

$$Y(z) + \frac{1}{2} z^{-1} Y(z) = X(z) + z^{-2} X(z) \tag{4.16}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2} z^{-1}} \tag{4.17}$$

Solution: Now we will rewrite (3.2),

$$y(n) + \frac{1}{2} y(n-1) = x(n) + x(n-2) \tag{4.18}$$

Now since Z-transform is a linear operator we can write that,

$$Y(n) + \frac{1}{2} Y(n-1) = X(n) + X(n-2) \tag{4.19}$$

From (4.11),

$$Y(n) + \frac{z^{-1}}{2} Y(n) = X(n) + z^{-2} X(n) \tag{4.20}$$

$$\implies \frac{Y(n)}{X(n)} = \frac{1 + z^{-2}}{1 + \frac{z^{-1}}{2}} \tag{4.21}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.22}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.23}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.24}$$

Solution: The Z-transform of δn is,

$$\mathcal{Z}\{\delta n\} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} \tag{4.25}$$

$$= \delta(0) z^0 + 0 \ \text{(Using (4.22))} \tag{4.26}$$

$$= 1 \tag{4.27}$$

and the Z-transform of unit-step function $u(n)$ is,

$$U(n) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{4.28}$$

$$= 0 + \sum_{n=0}^{\infty} 1.z^{-n} \tag{4.29}$$

$$= 1 + z^{-1} + z^{-2} + \ldots \tag{4.30}$$

Above is a infinite geometric series with z^{-1} as common ratio , so we can write it as

$$U(n) = \frac{1}{1 - z^{-1}} \because |z| > 1 \tag{4.31}$$

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.32}$$

Solution: The Z- transform will be

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.33}$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \ldots. \tag{4.34}$$

Above is a infinite geometric series with first term 1 and common ratio as $\frac{a}{z}$ and it can be written as,

$$\mathcal{Z}\{a^n u(n)\} = \frac{1}{1-\frac{a}{z}} \because |a| < |z| \tag{4.35}$$

Therefore,

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| > |a| \tag{4.36}$$

4.6 Let

$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{4.37}$$

Plot $\left|H\left(e^{j\omega}\right)\right|$. Comment. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.

Solution: Download the code for the plot 4.6 from the link below

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/dtft.py
```

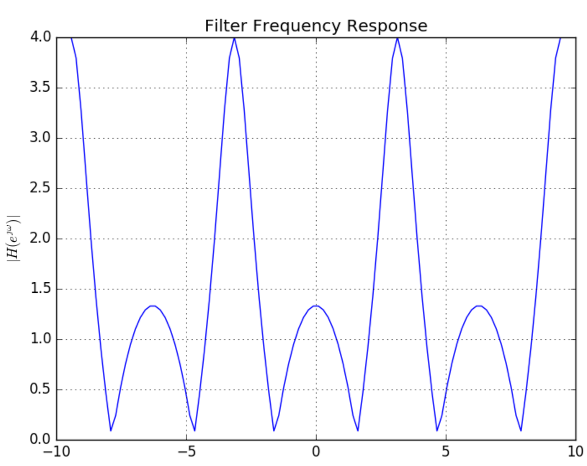

Fig. 4.6: $|H(e^{j\omega})|$

Now using (4.17), we will find $\left|H\left(e^{j\omega}\right)\right|$,

$$H\left(e^{j\omega}\right) = \frac{1+e^{-2j\omega}}{1+\frac{e^{-j\omega}}{2}} \tag{4.38}$$

$$\implies \left|H\left(e^{j\omega}\right)\right| = \frac{\left|1+e^{-2j\omega}\right|}{\left|1+\frac{e^{-j\omega}}{2}\right|} \tag{4.39}$$

$$= \frac{\left|1+e^{2j\omega}\right|}{\left|e^{2j\omega}+\frac{e^{j\omega}}{2}\right|} \tag{4.40}$$

$$= \frac{|1+\cos 2\omega + j\sin 2\omega|}{\left|e^{j\omega}+\frac{1}{2}\right|} \tag{4.41}$$

$$= \frac{\left|4\cos^2(\omega) + 4j\sin(\omega)\cos(\omega)\right|}{|2e^{j\omega}+1|} \tag{4.42}$$

$$= \frac{|4\cos(\omega)|\,|\cos(\omega)+j\sin(\omega)|}{|2\cos(\omega)+1+2j\sin(\omega)|} \tag{4.43}$$

$$\therefore \left|H\left(e^{j\omega}\right)\right| = \frac{|4\cos(\omega)|}{\sqrt{5+4\cos(\omega)}} \tag{4.44}$$

Since $\left|H(e^{j\omega})\right|$ is function of cosine we can say it is periodic.And from the plot 4.6 we can say that it is symmetric about $\omega = 0$ (even function) and it is periodic with period 2π.You can find the same from the theoritical expression $\left|H\left(e^{j\omega}\right)\right|$,

$$H(e^{j\omega}) = H(e^{j(-\omega)})\,(\cos \text{ is an even function}) \tag{4.45}$$

And to find period, the period of $|\cos(\omega)|$ is π and the period of $\sqrt{5+4\cos(\omega)}$ is 2π. So the period of division of both will be,

$$lcm(\pi, 2\pi) = 2\pi \tag{4.46}$$

This gives us the period of $\left|H(e^{j\omega})\right|$ as 2π

4.7 Express $h(n)$ in terms of $H\left(e^{j\omega}\right)$.

Solution: We know that

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \tag{4.47}$$

$$\implies \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \quad (4.48)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n}d\omega \quad (4.49)$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \quad (4.50)$$

$$= \frac{1}{2\pi}\left\{\sum_{k\neq n} h(k)\frac{e^{j\omega(n-k)}}{j(n-k)}\Bigg]_{-\pi}^{\pi} + h(n).2\pi\right\} \quad (4.51)$$

$$= \frac{0 + 2\pi h(n)}{2\pi} \quad (4.52)$$

$$= h(n) \quad (4.53)$$

$$\therefore h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \quad (4.54)$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \quad (5.1)$$

for H(z) in (4.17).
Solution: $H(z)$ is given by

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} = \frac{2 + 2z^{-2}}{2 + z^{-1}} \quad (5.2)$$

$$2z^{-1} - 4 \quad (5.3)$$

$$z^{-1} + 2 \;\big)\; 2z^{-2} + 2 \quad (5.4)$$

$$2z^{-2} + 4z^{-1} \quad (5.5)$$

$$-4z^{-1} + 2 \quad (5.6)$$

$$-4z^{-1} - 8 \quad (5.7)$$

$$10 \quad (5.8)$$

So,

$$H(z) = 2z^{-1} - 4 + \frac{10}{z^{-1} + 2} \quad (5.9)$$

$$= 2z^{-1} - 4 + \frac{5}{\frac{1}{2}z^{-1} + 1} \quad (5.10)$$

$$= 2z^{-1} - 4 + 5\sum_{n=0}^{\infty}\left(-\frac{z^{-1}}{2}\right)^{n} \quad (5.11)$$

$$= 1 - \frac{1}{2}z^{-1} + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n} z^{-n} \quad (5.12)$$

So,h(n) will be given by

$$h(n) = \begin{cases} 5 \times \left(-\frac{1}{2}\right)^{n} & n \geq 2 \\ \left(-\frac{1}{2}\right)^{n} & 2 > n \geq 0 \\ 0 & n < 0 \end{cases} \quad (5.13)$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \quad (5.14)$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.17),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (5.15)$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.16)$$

using (4.32) and (**??**).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

```
wget https://github.com/AvinashNayak27/
digital/blob/master/codes/hn.py
```

on simplfying we get h(n) as

$$\begin{cases} 5 \times \left(-\frac{1}{2}\right)^{n} & n \geq 2 \\ \left(-\frac{1}{2}\right)^{n} & 2 > n \geq 0 \\ 0 & n < 0 \end{cases} \quad (5.17)$$

$$\because 5 \times \left(-\frac{1}{2}\right)^{n} \to 0 \quad \text{for} \quad n \to \infty \quad (5.18)$$

So, we can conclude that h(n) is bounded.

5.4 Convergent? Justify using the ratio test.

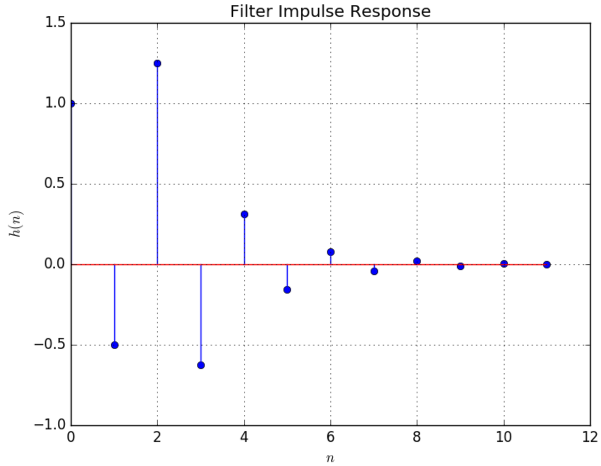

Fig. 5.3: $h(n)$ wrt n

Solution: We can say a given real sequence $\{x_n\}$ is convergent if

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| < 1 \tag{5.19}$$

This is known as Ratio test.
In this case the limit will become,

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{5\left(\frac{-1}{2}\right)^{n+1}}{5\left(\frac{-1}{2}\right)^{n}}\right| \tag{5.20}$$

$$= \lim_{n\to\infty}\left|\frac{-1}{2}\right| \tag{5.21}$$

$$= \frac{1}{2} \tag{5.22}$$

As $\frac{1}{2} < 1$, from root test we can say that $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.23}$$

Is the system defined by (3.2) stable for the impulse response in (5.14)?
Solution: For system of 3.2 ,$h(n)$ is defined in (5.13) So,

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=2}^{\infty} 5\times\left(-\frac{1}{2}\right)^n + \sum_{n=0}^{1}\left(-\frac{1}{2}\right)^n + \sum_{n=-\infty}^{-1} 0 \tag{5.24}$$

$$= 5\times\frac{1}{6}+\frac{1}{2} \tag{5.25}$$

$$= \frac{4}{3} \tag{5.26}$$

Since the sum is finite so the system is stable for impulsive response

5.6 Verify the above result using a python code.
Solution: The above result is verified using the below python code

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/hnverify.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.27}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. **??**.

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/hndef.py
```

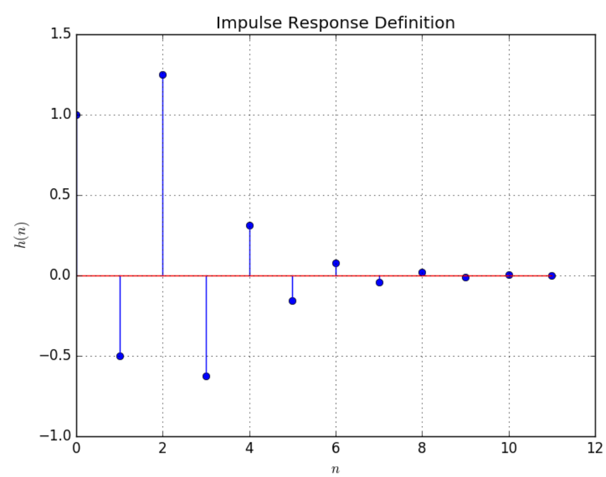

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.28}$$

Comment. The operation in (5.28) is known as *convolution*.

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. **??**.

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/yndef.py
```

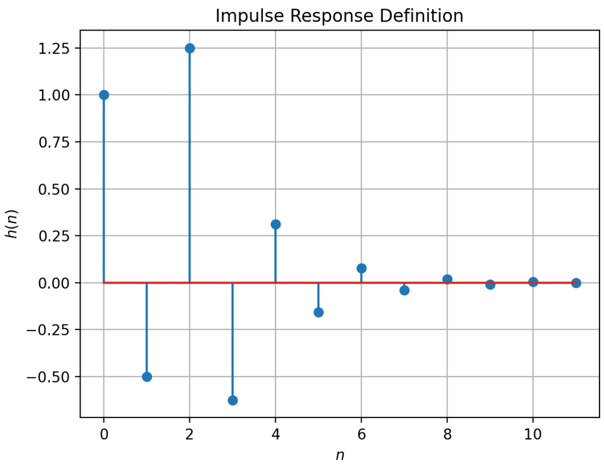

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Toeplitz matrix.
Solution:

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/ynconv.py
```

From (5.28),we express $y(n)$ as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{5.29}$$

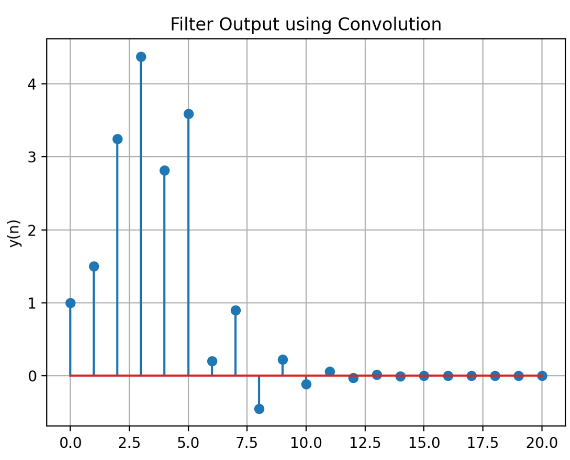

Fig. 5.9: Convolution of $x(n)$ and $h(n)$ using toeplitz matrix

To understand how we can use a Toeplitz matrix, we will see what we are doing in (5.28)

$$y(0) = x(0)\, h(0) \tag{5.30}$$
$$y(1) = x(0)\, h(1) + x(1)\, h(0) \tag{5.31}$$
$$y(2) = x(0)\, h(2) + x(1)\, h(1) + x(2)\, h(0) \tag{5.32}$$

.

.

The same thing can be written as,

$$y(0) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.33}$$

$$y(1) = \begin{pmatrix} h(1) & h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.34}$$

$$y(2) = \begin{pmatrix} h(2) & h(1) & h(0) & 0 & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.35}$$

.

.

Using Toeplitz matrix of $h(n)$ we can simplify it as,

$$y(n) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \\ h(1) & h(0) & 0 & . & . & .0 \\ h(2) & h(1) & h(0) & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & .h(m-1) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.36}$$

Now from (3.1) we will take n

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.37}$$

And from (5.13) we will take some values of n,

$$h(n) = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ . \\ . \end{pmatrix} \tag{5.38}$$

Now using (5.36),

$$y(n) = x(n) * h(n) \tag{5.39}$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & . & .0 \\ -0.5 & 1 & 0 & . & . & .0 \\ 1.25 & -0.5 & 1 & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & . \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.40}$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \\ . \end{pmatrix} \tag{5.41}$$

The above equation (5.41) is the convolution of $x(n)$ and $h(n)$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.42}$$

Solution: From (5.28),

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.43}$$

Replacing $n-k$ with a,we get

$$y(n) = \sum_{n-a=-\infty}^{\infty} x(n-a)h(a) \tag{5.44}$$

$$= \sum_{-a=-\infty}^{\infty} x(n-a)h(a) \tag{5.45}$$

$$= \sum_{a=-\infty}^{\infty} x(n-a)h(a) \tag{5.46}$$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

Solution: Download the below python code for the plot 6.1,

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/dft.py
```

And run the following command in the terminal,

```
python3 dft.py
```

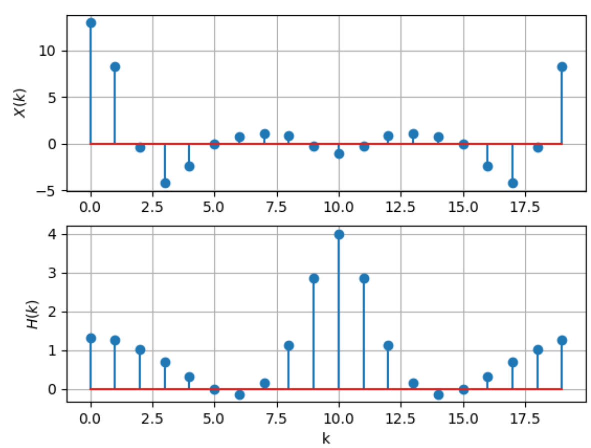

Fig. 6.1: Plot of real part of Discrete Fourier Transforms of $x(n)$ and $h(n)$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: Download the below python code for the plot 6.2,

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/yK.py
```

Then run the following command in the terminal,

```
python3 yK.py
```

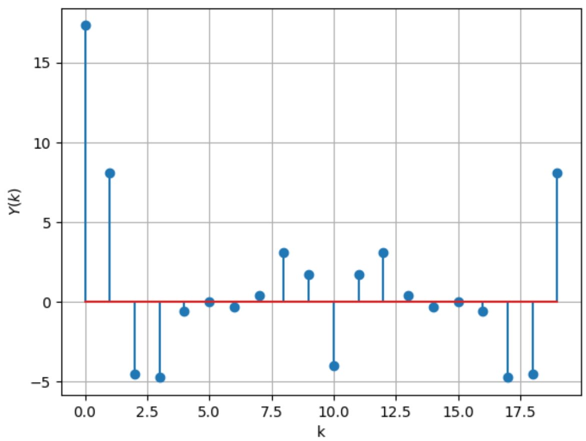

Fig. 6.2: $Y(k)$ as the product of $X(k)$ and $H(k)$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: Download the below python code for the plot **??**,

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/yndft.py
```

Then run the following command,

```
python3 yndft.py
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Download the below python code for the plot **??**,

```
wget https://github.com/AvinashNayak27/
    digital/blob/master/codes/ynIIFT.py
```

Then run the following command,

```
python3 ynIIFT.py
```

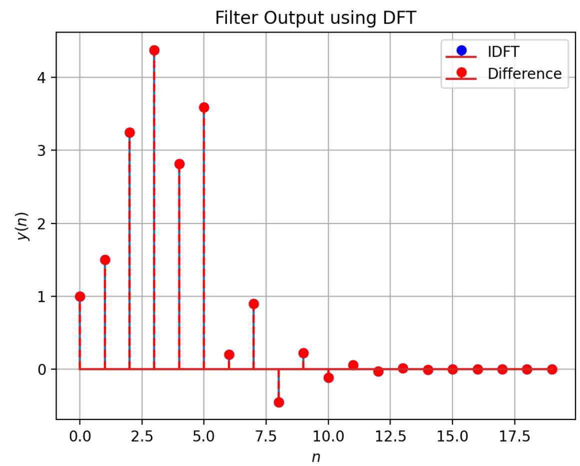

Fig. 6.3: $y(n)$ using IDFT and difference equation

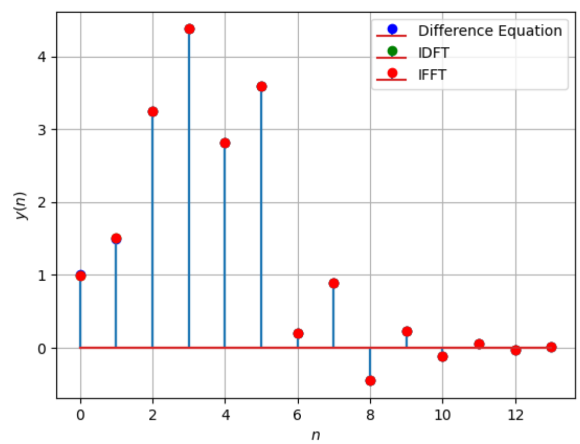

Fig. 6.4: The plot of $y(n)$ using IFFT

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point $DFT matrix$ is defined as

$$\vec{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\vec{F}_N$.

3. Let
$$\vec{I}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^2 & \vec{e}_4^3 & \vec{e}_4^4 \end{pmatrix} \tag{7.4}$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \tag{7.5}$$
4. The 4 point *DFTdiagonalmatrix* is defined as
$$\vec{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution: From (7.2),
$$W_N = e^{-j2\pi/N} \tag{7.8}$$
Consider,
$$W_N^2 = \left(e^{-j2\pi/N}\right)^2 \tag{7.9}$$
$$= e^{-j2\pi/(N/2)} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$
Hence proved.

6. Show that
$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.12}$$
Solution: From the eq (7.5),
$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \tag{7.13}$$
Clearly $\vec{P}_4$ is an elementary matrix of $\vec{I}_4$, so on multiplication with a matrix it will interchange the rows/columns of matrix depending on positions of unit vectors.
From that it follows ,
$$\vec{P}_4^2 = \vec{I}_4 \tag{7.14}$$
So it is similar to prove that,
$$\vec{F}_4\vec{P}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \tag{7.15}$$
Now from (7.3),
$$\vec{F}_2 = \begin{bmatrix} W_2^{0.0} & W_2^{0.1} \\ W_2^{1.0} & W_2^{1.1} \end{bmatrix} \tag{7.16}$$
$$= \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.17}$$
Using the result (7.11), we can write
$$\vec{F}_2 = \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.18}$$
And $\vec{D}_2$ is a diagonal matrix,
$$\vec{D}_2 = diag\left(W_4^0, W_4^1\right) \tag{7.19}$$
$$= diag\,(1, W_4) \tag{7.20}$$
Then,
$$\vec{D}_2\vec{F}_2 = \begin{bmatrix} 1 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.21}$$
$$= \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{bmatrix} \tag{7.22}$$
And for $k \in \mathcal{N}$ and N be a even integer we know that,
$$W_N^{Nk} = 1 \tag{7.23}$$
$$W_N^{Nk+N/2} = -1 \tag{7.24}$$
Using that we can write,
$$-\vec{D}_2\vec{F}_2 = \begin{bmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{bmatrix} \tag{7.25}$$
And from (7.3),
$$\vec{F}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \tag{7.26}$$
And
$$\vec{F}_4\vec{P}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{bmatrix} \tag{7.27}$$
This is same as,
$$\begin{bmatrix} \vec{F}_2 & \vec{D}_2\vec{F}_2 \\ \vec{F}_2 & -\vec{D}_2\vec{F}_2 \end{bmatrix} \tag{7.28}$$
$$\implies \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \tag{7.29}$$
Hence proved.
7. Show that
$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \tag{7.30}$$
Solution: As we saw earlier, it is similar to

prove that

$$\vec{F}_N\vec{P}_N = \begin{bmatrix}\vec{I}_{N/2} & \vec{D}_{N/2}\\ \vec{I}_{N/2} & -\vec{D}_{N/2}\end{bmatrix}\begin{bmatrix}\vec{F}_{N/2} & 0\\ 0 & \vec{F}_{N/2}\end{bmatrix} \quad (7.31)$$

Assuming that N is even, consider LHS

$$\vec{F}_N\vec{P}_N = \begin{bmatrix} W_N^{0\times0} & W_N^{0\times2} & .. & W_N^{0\times1} & W_N^{0\times3}..\\ W_N^{1\times0} & W_N^{1\times2} & .. & W_N^{1\times1} & W_N^{1\times3}..\\ & & .. & & \\ & & .. & & \\ W_N^{N/2\times0} & W_N^{N/2\times2} & .. & W_N^{N/2\times1} & W_N^{N/2\times3}..\\ & & .. & & \\ & & .. & & \\ W_N^{N-1\times0} & W_N^{N-1\times2} & .. & W_N^{N-1\times1} & W_N^{N-1\times3}.. \end{bmatrix} \quad (7.32)$$

On multiplying with $\vec{P}_N$ (permutation matrix), the odd-numbered columns of $\vec{F}_N$ shifted towards left.
Now we can divide the above matrix (7.32), into four sub-matrices as,

$$= \begin{bmatrix} \left[W_N^{n\times2m}\right] & \left[W_N^{n\times(2m+1)}\right]\\ \left[W_N^{(n+\frac{N}{2})\times(2m)}\right] & \left[W_N^{(n+\frac{N}{2})\times(2m+1)}\right]\end{bmatrix} \quad (7.33)$$

$$\text{where}, 0 \le n, m \le \frac{N}{2} - 1$$

$$= \begin{bmatrix} \left[\left(W_N^{n\times m}\right)^2\right] & \left[W_N^{n}\left(W_N^{n\times m}\right)^2\right]\\ \left[W_N^{Nm}\left(W_N^{n\times m}\right)^2\right] & \left[W_N^{Nm+N/2}W_N^{n}\left(W_N^{n\times m}\right)^2\right]\end{bmatrix} \quad (7.34)$$

Using (7.23), (7.24) and (7.11)

$$= \begin{bmatrix} \left[W_{\frac{N}{2}}^{n\times m}\right] & \left[W_N^{n}W_{\frac{N}{2}}^{n\times m}\right]\\ \left[W_{\frac{N}{2}}^{n\times m}\right] & \left[-W_N^{n}W_{\frac{N}{2}}^{n\times m}\right]\end{bmatrix} \quad (7.35)$$

Now from def (7.3) and (7.6), we can write,

$$= \begin{bmatrix}\vec{F}_{\frac{N}{2}} & \vec{D}_{\frac{N}{2}}\vec{F}_{\frac{N}{2}}\\ \vec{F}_{\frac{N}{2}} & -\vec{D}_{\frac{N}{2}}\vec{F}_{\frac{N}{2}}\end{bmatrix} \quad (7.36)$$

$$\implies \vec{F}_N\vec{P}_N = \begin{bmatrix}\vec{I}_{N/2} & \vec{D}_{N/2}\\ \vec{I}_{N/2} & -\vec{D}_{N/2}\end{bmatrix}\begin{bmatrix}\vec{F}_{N/2} & 0\\ 0 & \vec{F}_{N/2}\end{bmatrix} \quad (7.37)$$

Hence proved.
Note : If we want to do the above matrix decomposition recursively the value of N should in the form of 2^k.

8. Find

$$\vec{P}_4\vec{x} \quad (7.38)$$

Solution: Let $\vec{x}$,

$$\vec{x} = \begin{bmatrix}x(0)\\x(1)\\x(2)\\x(3)\end{bmatrix} \quad (7.39)$$

and $\vec{P}_4$ is 4 - point permutation matrix.
So,

$$\vec{P}_4\vec{x} = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x(0)\\x(1)\\x(2)\\x(3)\end{bmatrix} \quad (7.40)$$

$$= \begin{bmatrix}x(0)\\x(2)\\x(1)\\x(3)\end{bmatrix} \quad (7.41)$$

9. Show that

$$\vec{X} = \vec{F}_N\vec{x} \quad (7.42)$$

where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution: From (7.1),

$$X(k) = \sum_{n=0}^{N-1} x(n)W^{kn} \quad (7.43)$$

Now we will try to convert the above expression into matrix equations,

$$X(0) = \sum_{n=0}^{N-1} x(n)W^{0.n} \quad (7.44)$$

$$= \begin{pmatrix}W^{0.0}\\W^{0.1}\\W^{0.2}\\W^{0.(N-1)}\end{pmatrix}^T \begin{pmatrix}x(0)\\x(1)\\x(2)\\.\\.\\x(N-1)\end{pmatrix} \quad (7.45)$$

$$X(1) = \begin{pmatrix} W^{1.0} \\ W^{1.1} \\ W^{1.2} \\ W^{1.(N-1)} \end{pmatrix}^T \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{pmatrix} \tag{7.46}$$

.

.

$$X(N-1) = \begin{pmatrix} W^{(N-1)\times 0} \\ W^{(N-1)\times 1} \\ W^{(N-1)\times 2} \\ W^{(N-1)\times(N-1)} \end{pmatrix}^T \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{pmatrix} \tag{7.47}$$

$\vec{X} =$

$$\begin{bmatrix} W_N^{0\times 0} & W_N^{0\times 1} & .. & W_N^{0\times N-1} \\ .. & .. & .. & .. \\ W_N^{N-1\times 0} & W_N^{N-1\times 1} & .. & W_N^{N-1\times N-1} \end{bmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(N-1) \end{pmatrix} \tag{7.48}$$

From def (7.3),

$$\vec{X} = \vec{F}_N \vec{x} \tag{7.49}$$

Hence proved.

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.50}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.51}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.52}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.53}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.54}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.55}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.56}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.57}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.58}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.59}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.60}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.61}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.62}$$

Solution: The 8-point FFT can be expressed as,

$$X(k) = \sum_0^7 x(n) e^{\frac{-2\pi kn}{8}} \tag{7.63}$$

$$= \sum_0^3 x(2n) e^{\frac{-2\pi kn}{4}} + \sum_1^3 e^{\frac{-2\pi k(2n+1)}{8}} \tag{7.64}$$

$$= \sum_0^3 x(2n) e^{\frac{-2\pi kn}{4}} + e^{\frac{-2\pi k}{8}} \sum_1^3 x(2n) e^{\frac{-2\pi kn}{4}} \tag{7.65}$$

Call these 4 - point FFTs as X_1 and X_2,

$$X(k) = X_1(k) + W_8^k X_2(k) \tag{7.66}$$

Now consider,

$$X(k+4) = X_1(k+4) + W_8^{k+4} X_2(k+4) \quad (7.67)$$
$$= X_1(k) - W_8^k X_2(k) \quad (7.68)$$

Since the twiddle factors along with X_1 and X_2 are of 4-point $X_1(k+4) = X_1(k)$ and $X_2(k+4) = X_2(k)$.
With that (7.68) we can see how (7.50) and (7.51) are derived.
Now consider these 4-point FFTs,

$$X_1(k) = \sum_0^1 x(4n) e^{\frac{-j2\pi nk}{2}} + e^{\frac{-j2\pi k}{4}} \sum_0^1 x(4n+2) e^{\frac{-j2\pi nk}{2}} \quad (7.69)$$
$$= X_3(k) + W_4^k X_4(k) \quad (7.70)$$

where, $X_3(k)$ and $X_4(k)$ are 2-point FFTs of $x_1(n) = x_1(4n)$ and $x_2(n) = x(4n+2)$.
And you can see that,

$$X_1(k+2) = X_3(k) - W_4^k X_4(k) \quad (7.71)$$

With that we can see how we got (7.10) and (7.53).
And similarly we can write the 2-point FFTs from $X_2(k)$ as $X_5(k)$ and $X_6(k)$ of subsequences $x(4n+1)$ and $x(4n+3)$.
With that we can get (7.54) and (7.55).
Mathematically we can write these 2-point FFTs as,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \quad (7.72)$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \quad (7.73)$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \quad (7.74)$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \quad (7.75)$$

where, the subsequences required for each 2-point FFT can be obtained from (7.56) , (7.57) and (7.58).

11. For

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad (7.76)$$

compute the DFT using (7.42)
Solution: Download the below python code,

```
wget https://github.com/AvinashNayak27
/digital/tree/master/codes/xkDFT.py
```

Then run the following command on terminal,

```
python3 xkDFT.py
```

The plot of DFT can be seen in Fig 7.11

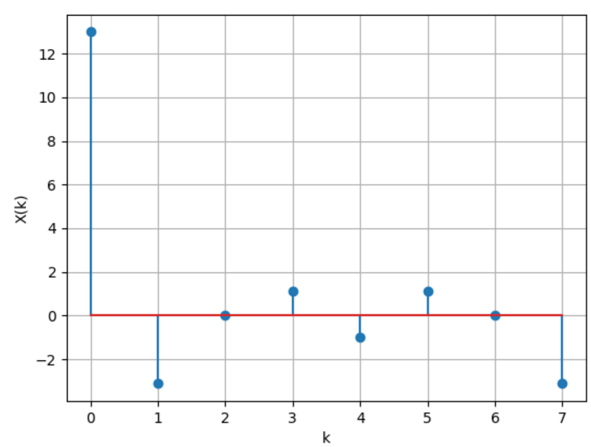

Fig. 7.11: DFT using DFT matrix

12. Repeat the above exercise using the FFT after zero padding $\vec{x}$.
Solution: Download the below python code,

```
wget https://github.com/AvinashNayak27/
digital/tree/master/codes/xkFFT.py
```

Then run the following command on terminal,

```
python3 xkFFT.py
```

The plot of DFT can be seen in Fig 7.13
13. Write a C program to compute the 8-point FFT.
Solution: Download the C code from the following link

```
wget https://github.com/AvinashNayak27/
digital/tree/master/codes/xkFFT.c
```

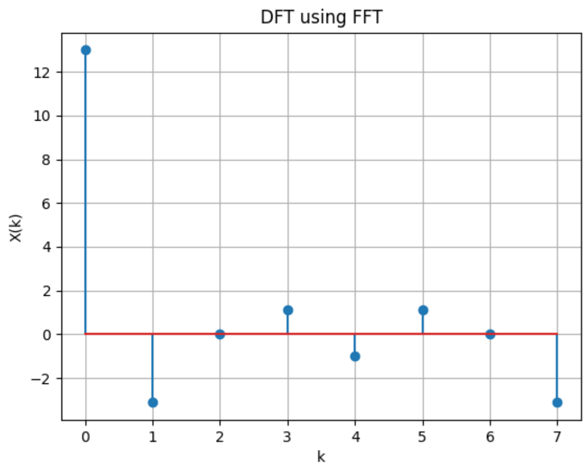

Fig. 7.12: FFT using Matrix decompostion

Then run the following command,

```
gcc xkFFT.c
./a.out
```

Download the below python code which uses fft.dat file from the C code.

```
wget https://github.com/AvinashNayak27/
    digital/tree/master/codes/xk8pointFFT
    .py
```

Then run the following command for the plot,

```
python3 xk8pointFFT.py
```

You will get output of DFT of $x(n)$.

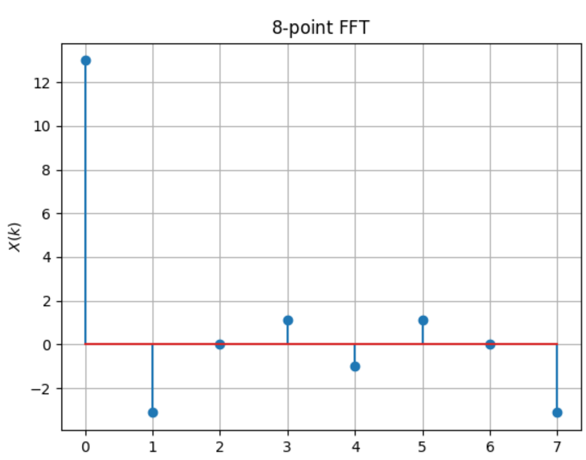

Fig. 7.13: FFT using C code

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a(m)\, z^{-m} Y(z) = \sum_{k=0}^{N} b(k)\, z^{-k} X(z) \quad (8.2)$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \quad (8.3)$$

For obtaining the discrete Fourier transform, put $z = \jmath^{\frac{2\pi i}{I}}$ where I is the length of the input signal and $i = 0, 1, \ldots, I-1$
Download the following Python code that does the above

```
wget https://github.com/AvinashNayak27
    /digital/tree/master/codes/8.1.py
```

Run the code by executing

```
python3 8.1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b
Solution: The polynomial coefficients obtained are

$$\vec{a} = \begin{pmatrix} 1.000 \\ -2.519 \\ 2.561 \\ -1.206 \\ 0.220 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 0.003 \\ 0.014 \\ 0.021 \\ 0.014 \\ 0.003 \end{pmatrix} \quad (8.4)$$

The difference equation is then given by

$$\vec{a}^{\top} \vec{y} = \vec{b}^{\top} \vec{x} \quad (8.5)$$

where

$$\vec{y} = \begin{pmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{pmatrix} \tag{8.6}$$

We have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{8.7}$$

By using partial fraction decomposition, we can write this as

$$H(z) = \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \tag{8.8}$$

On taking the inverse Z-transform on both sides by using (4.32)

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \tag{8.9}$$

$$\frac{1}{1 - p(i) z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} (p(i))^n u(n) \tag{8.10}$$

$$z^{-j} \overset{\mathcal{Z}}{\rightleftharpoons} \delta(n-j) \tag{8.11}$$

Thus

$$h(n) = \sum_i r(i)\, (p(i))^n\, u(n) + \sum_j k(j) \delta(n-j) \tag{8.12}$$

Download the following Python code

```
wget https://github.com/AvinashNayak27
/digital/tree/master/codes/8.2.py
```

Run the code by executing

```
python3 8.2.py
```

The above code outputs the values of $r(i), p(i), k(i)$

$$h(n) = \Re\left((0.24 - 0.71\jmath)(0.56 + 0.14\jmath)^n\right) u(n) + \Re\left((0.24 + 0.71\jmath)(0.56 - 0.14\jmath)^n\right) u(n) + 0.016\delta(n) \tag{8.13}$$

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=4 and cutoff-frequency=4kHz.

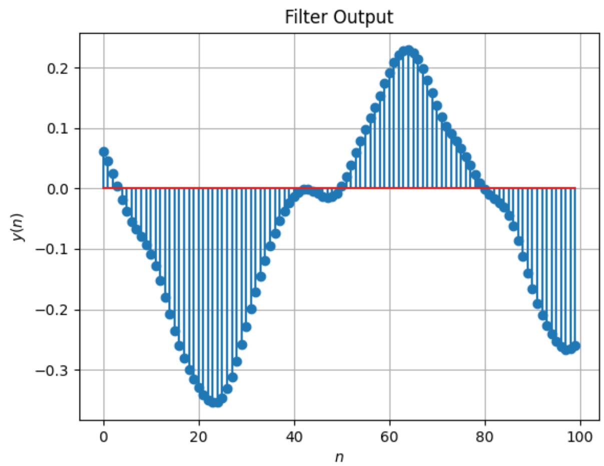

Fig. 8.2: Plot of $y(n)$

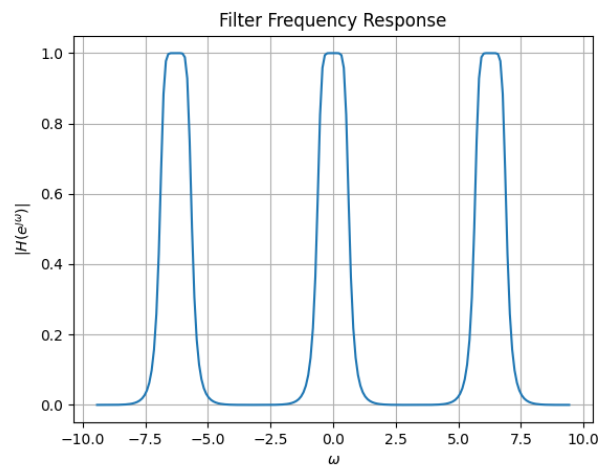

Fig. 8.2: Plot of $\left|H(e^{j\omega})\right|$

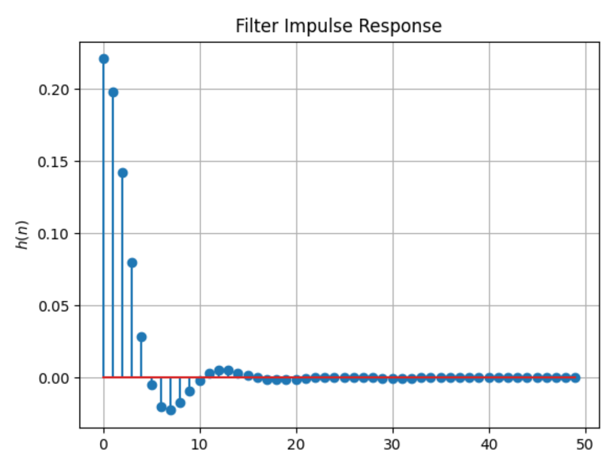

Fig. 8.2: Plot of $h(n)$

8.5 Modifying the code with different input parameters and to get the best possible output. **Solution:** Order: 10 Cutoff frequency: 3000 Hz