Digital Signal Processing
EE3900

Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution:

```
wget https://github.com/AvinashNayak27/
    fourier/blob/master/Fourier/codes/1.1.py
```

1.2 Show that $x(t)$ is periodic and find its period.
Solution: A signal $x(t)$ is said to be periodic with fundamental period T if

$$x(t + nT) = x(t) \forall n \in \mathbb{Z} \tag{1.2}$$

Let T be fundamental period of $x(t)$. Comparing (1.2) and (1.1), we get

$$A_0 \left|\sin\left(2\pi f_0 t\right)\right| = A_0 \left|\sin\left(2\pi f_0 (t + T)\right)\right| \tag{1.3}$$

$$\left|\sin\left(2\pi f_0 t\right)\right| = \left|\sin\left(2\pi f_0 (t + T)\right)\right| \tag{1.4}$$

$$\left|\sin\left(2\pi f_0 t\right)\right| = \left|\sin\left(2\pi f_0 t + 2\pi f_0 T\right)\right| \tag{1.5}$$

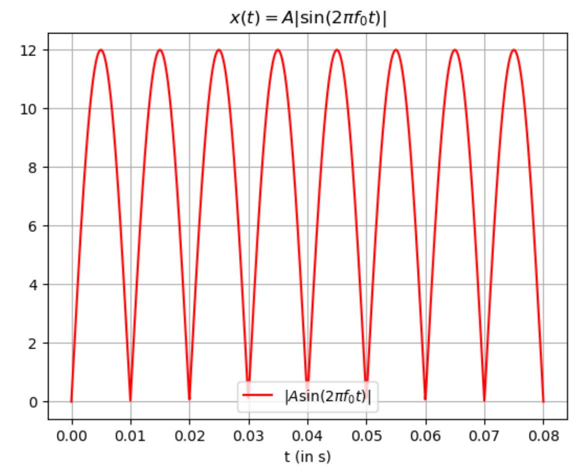


Fig 1.1

As $|sin\theta|$ is periodic with fundamental period $F = \pi$, Hence,

$$\left|\sin\left(t\right)\right| = \left|\sin\left(t + F\right)\right| \tag{1.6}$$

Hence,$2\pi f_0 T = \pi$, therefore, fundamental period(T) is

$$T = \frac{\pi}{2\pi f_0} = \frac{1}{2 f_0} \tag{1.7}$$

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t}\, dt \tag{2.2}$$

Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.3}$$

Mulitply $e^{-j2\pi l f_0 t}$ on both sides of (2.3), we get,

$$x(t)e^{-j2\pi l f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} e^{-j2\pi l f_0 t} \tag{2.4}$$

Integrating (2.4) w.r.t. t from $-T$ to T, and $T = \frac{1}{f_0}$, we get,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-l)f_0 t}\, dt \tag{2.5}$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\, dt \tag{2.6}$$

Consider the following cases.
case-1:$k = l$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^0\, dt \tag{2.7}$$

$$= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.8}$$

case-2: $k \neq l$
Let $n = f_0(k-l)$, here $n \in \mathbb{Z}$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{2n\pi}\, dt \tag{2.9}$$

Here, $2n\pi T = 2f_0(k-l)T\pi$, and $2n\pi T = (k-l)\pi$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.10}$$

$$= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \cos(2n\pi) + j\sin(2n\pi)\, dt \tag{2.11}$$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\, dt = -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.12}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.13}$$

$$= -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.14}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.15}$$

$$= -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.16}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.17}$$

$$\tag{2.18}$$

$$= -\sin((k-l)\pi) + \sin(-(k-l)\pi) \tag{2.19}$$

$$+ j\cos((k-l)\pi) - j\cos(-(k-l)\pi) \tag{2.20}$$

$$\tag{2.21}$$

Since $k - l \in \mathbb{Z}$,$\sin((k-l)\pi) = 0$ and $\sin(-(k-l)\pi) = 0$, simillarly, as $\cos(\theta) = \cos(-\theta)$, we get $\cos((k-l)\pi) - \cos(-(k-l)\pi) = 0$
From (2.18),

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.22}$$

$$= 0 + j0 = 0 \tag{2.23}$$

Hence, we have,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\, dt = \begin{cases} 0 & k \neq l \\ \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt & k = l \end{cases} \tag{2.24}$$

From (2.5),

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t}\, dt = \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\, dt \tag{2.25}$$

$$= c_k \times \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.26}$$

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.27}$$

$$\therefore c_k = \frac{2}{T} \int_{-\frac{1}{T}}^{\frac{1}{T}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.28}$$

2.2 Find c_k for (1.1)
Solution: We know that,

$$c_k = 2f_0 \int_{0}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.29}$$

when $t \in \left(0, \frac{1}{2f_0}\right)$, $x(t) = A_0 \sin(2\pi f_0 t)$

$$c_k = 2f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \left(\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}\right) e^{-j2\pi k f_0 t}\, dt \tag{2.30}$$

$$= A_0 f_0 \int_{0}^{\frac{1}{2f_0}} \left(\frac{e^{j2\pi(1-k) f_0 t} - e^{j2\pi(-1-k) f_0 t}}{j}\right) dt \tag{2.31}$$

$$= A_0 f_0 \Bigg(\frac{e^{j2\pi(1-k) f_0 t}}{-2\pi(1-k) f_0}\Bigg|_0^{\frac{1}{2f_0}} \tag{2.32}$$

$$- \frac{e^{j2\pi(-1-k) f_0 t}}{-2\pi(-1-k) f_0}\Bigg|_0^{\frac{1}{2f_0}} \Bigg) \tag{2.33}$$

$$= A_0 \left[\frac{e^{j\pi(1-k)} - 1}{2\pi(k-1)} - \frac{e^{-j\pi(1+k)} - 1}{2\pi(k+1)}\right] \tag{2.34}$$

Hence,

$$c_k = \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k = even \\ 0 & k = odd \end{cases} \tag{2.35}$$

2.3 Verify (1.1) using python.
Solution:

```
wget https://github.com/AvinashNayak27/
    fourier/blob/master/Fourier/codes/2.3.py
        python3 2.3.py
```

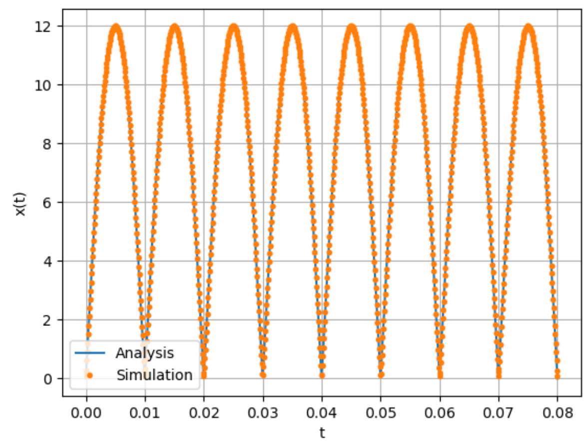


Fig 2.3

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left(a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t\right) \tag{2.36}$$

and obtain the formulae for a_k and b_k.
Solution: Using (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.37}$$

As,

$$e^{j2\pi k f_0 t} = \cos(2\pi k f_0 t) + j \sin(2\pi k f_0 t) \tag{2.38}$$

From (2.1), we have,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \left[\cos(2\pi k f_0 t) + j \sin(2\pi k f_0 t)\right] \tag{2.39}$$

$$= \sum_{k=-\infty}^{\infty} c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t) \tag{2.40}$$

$$= \sum_{k=-\infty}^{-1} \left[c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)\right] \tag{2.41}$$

$$+ c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)\right] \tag{2.42}$$

$$= \sum_{k=1}^{\infty} \left[c_{-k} \cos(2\pi k f_0 t) - j c_{-k} \sin(2\pi k f_0 t)\right] \tag{2.43}$$

$$+ c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + j c_k \sin(2\pi k f_0 t)\right] \tag{2.44}$$

$$= c_0 + \sum_{k=1}^{\infty} \Big((c_k + c_{-k}) \cos(2\pi k f_0 t) \tag{2.45}$$

$$+ j(c_k - c_{-k}) \sin(2\pi k f_0 t) \Big) \tag{2.46}$$

Substituting $a_k = c_k + c_{-k}$ and $b_k = j(c_k - c_{-k})$, we get,

$$x(t) = c_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.47}$$

$$= \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.48}$$

$$\therefore a_k = \begin{cases} c_k + c_{-k} & k \neq 0 \\ c_0 & k = 0 \end{cases} \tag{2.49}$$

$$b_k = j\left(c_k - c_{-k}\right) \tag{2.50}$$

Using (2.2),

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.51}$$

$$c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi k f_0 t}\, dt \tag{2.52}$$

$$a_k = c_k + c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \left[e^{-j2\pi k f_0 t} + e^{j2\pi k f_0 t}\right] dt \tag{2.53}$$

$$= 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \cos\left(2\pi k f_0 t\right)\, dt \tag{2.54}$$

Similarly, for b_k, we get,

$$b_k = -j\left\{2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \sin\left\{2\pi k f_0 t\right\}\, dt\right\} \tag{2.55}$$

2.5 Find a_k and b_k for (1.1)
Solution: Using (2.49) and (2.50) with (2.35),

$$a_k = c_k + c_{-k} = \begin{cases} \frac{4A_0}{\pi\left(1-k^2\right)} & k = even \\ \frac{2A_0}{\pi} & k = 0 \\ 0 & k = odd \end{cases} \tag{2.56}$$

$$b_k = j\left(c_k - c_{-k}\right) = 0 \tag{2.57}$$

2.6 Verify (2.36) using python.
Solution:

```
wget https://github.com/AvinashNayak27/
    fourier/blob/master/Fourier/codes/2.6.py
        python3 2.3.py
```

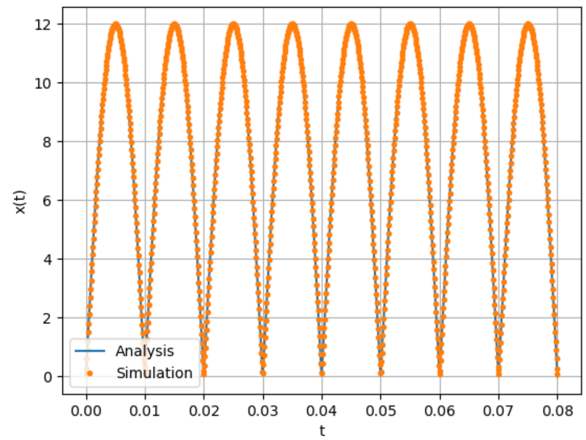


Fig 2.6

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.3}$$

3.3 Show that

$$g(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} G(f) e^{-j2\pi f t_0} \tag{3.4}$$

Solution: Let us consider $x = t - t_0$. Fourier transform of $g(t - t_0)$ is given as

$$g(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f t}\, dt \tag{3.5}$$

$$= \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f((t-t_0)+t_0)}\, du \tag{3.6}$$

$$= \int_{-\infty}^{\infty} g(x) e^{-j2\pi f(x+t_0)}\, dt \tag{3.7}$$

$$= \int_{-\infty}^{\infty} g(x) e^{-j2\pi f(x+t_0)}\, d(x - t_0) \tag{3.8}$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi f t_0} g(x) e^{-j2\pi f x}\, d(x) \tag{3.9}$$

$$= e^{-j2\pi f t_0} \left\{\int_{-\infty}^{\infty} g(x) e^{-j2\pi f x}\, d(x)\right\} \tag{3.10}$$

Using (3.3) in equation (3.10), we get,

$$g(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-\jmath 2\pi f t_0} \tag{3.11}$$

3.4 Show that

$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.12}$$

Solution: Let $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$, then

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{\jmath 2\pi f t}\, df \tag{3.13}$$

Consider $g(-k)$,

$$g(-k) = \int_{-\infty}^{\infty} G(f)e^{\jmath 2\pi f k}\, df \tag{3.14}$$

Let $f = t$,then,

$$g(-k) = \int_{-\infty}^{\infty} G(t)e^{\jmath 2\pi t k}\, dt \tag{3.15}$$

Substituting $k = f$ and in the (3.15), we get,

$$g(-f) = \int_{-\infty}^{\infty} G(t)e^{\jmath 2\pi f t}\, dt \tag{3.16}$$

Comparing (3.16) with (3.3), we can say that,

$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.17}$$

3.5 $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} ?$
Solution: From (3.3), fourier transform of $\delta(t)$ is,

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \delta(t)e^{-\jmath 2\pi f t}\, dt \tag{3.18}$$

$$= \int_{-\infty}^{\infty} \delta(0)e^{-\jmath 2\pi f 0}\, dt \tag{3.19}$$

$$= \int_{-\infty}^{\infty} \delta(0)\, dt \tag{3.20}$$

$$= 1 \tag{3.21}$$

Hence, $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$

3.6 $e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} ?$
Solution: Suppose $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$. Hence,

$$g(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi f t} \tag{3.22}$$

$$g(t)e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi f t}e^{-j2\pi f_0 t} \tag{3.23}$$

$$g(t)e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-j2\pi f_0 f} \tag{3.24}$$

$$\tag{3.25}$$

From (3.16),

$$g(t-f_0) \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-\jmath 2\pi f t f_0} \tag{3.26}$$

$$g(t)e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-j2\pi f_0 f} \tag{3.27}$$

$$\tag{3.28}$$

From (3.13),

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1 \tag{3.29}$$

$$1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(-f) = \delta(f) \tag{3.30}$$

Hence,

$$g(t-f_0) \overset{\mathcal{F}}{\longleftrightarrow} \delta((f+f_0)) \tag{3.31}$$

$$g(t)e^{\jmath 2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi (f-f_0)t}\, dt \tag{3.32}$$

$$= G(f-f_0) \tag{3.33}$$

Hence,

$$e^{-\jmath 2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(-(f+f_0)) = \delta(f+f_0) \tag{3.34}$$

3.7 $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow} ?$
Solution: We know that

$$\cos(2\pi f_0 t) = \frac{1}{2}\left(e^{\jmath 2\pi f_0 t} + e^{-\jmath 2\pi f_0 t}\right) \tag{3.35}$$

$$\tag{3.36}$$

Hence,

$$\mathcal{F}(\cos(2\pi f_0 t)) = \mathcal{F}(\frac{1}{2}\left(e^{\jmath 2\pi f_0 t} + e^{-\jmath 2\pi f_0 t}\right)) \tag{3.37}$$

$$\mathcal{F}(\cos(2\pi f_0 t)) = \frac{1}{2}\mathcal{F}(\left(e^{\jmath 2\pi f_0 t}\right)) + \frac{1}{2}\mathcal{F}(e^{-\jmath 2\pi f_0 t}) \tag{3.38}$$

$$= \frac{1}{2}\mathcal{F}(\left(e^{\jmath 2\pi f_0 t}\right)) + \frac{1}{2}\mathcal{F}(e^{-\jmath 2\pi f_0 t}) \tag{3.39}$$

$$= \frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right) \tag{3.40}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.
Solution: As obtained earlier, from equation

(2.35),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{3.41}$$

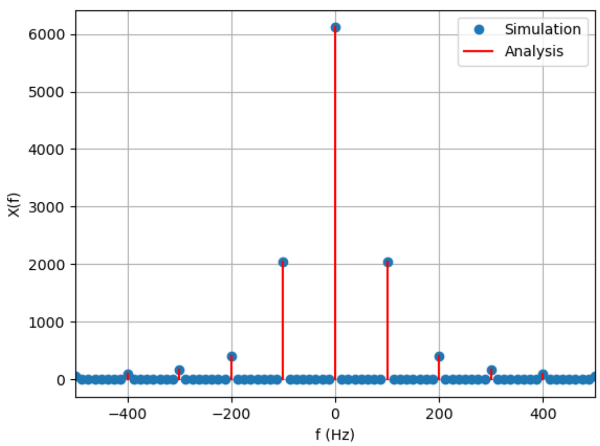
$$e^{j2\pi k f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} = \delta f - k f_0 \tag{3.42}$$

Hence, from the value of c_k,

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \delta\left(f + k f_0\right) c_k \tag{3.43}$$

$$\implies x(t) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \frac{2A_0}{\pi} \frac{\delta\left(f + 2k f_0\right)}{1 - 4k^2} \tag{3.44}$$

Fourier transform of $x(t)$ is verified in the following figure. The figure is plotted using the below python code.



Fourier Transform of $x(t)$.

```
wget https://github.com/AvinashNayak27/
    fourier/blob/master/Fourier/codes/3.8.py
```

3.9 Show that

$$\text{rect}\, t \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\, f \tag{3.45}$$

Verify using python.
Solution: We know that,

$$rect(t) = \begin{cases} 0 & t < \frac{-1}{2} \\ 1 & \frac{-1}{2} < t < \frac{1}{2} \\ 0 & t > \frac{1}{2} \end{cases} \tag{3.46}$$

$$\text{sinc}(f) = \frac{\sin \pi f}{\pi f} \tag{3.47}$$

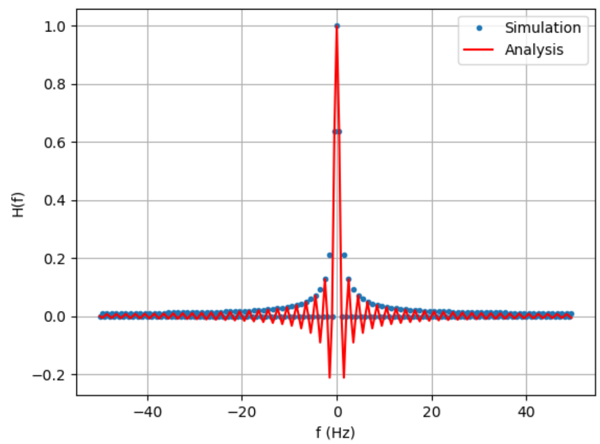
Applying fourier transform we get,

$$\text{rect}\, t \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \text{rect}\, t e^{-\jmath 2\pi f t}\, dt \tag{3.48}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\jmath 2\pi f t}\, dt \tag{3.49}$$

$$= \frac{e^{\jmath\pi f} - e^{-\jmath\pi f}}{\jmath 2\pi f} = \frac{\sin \pi f}{\pi f} = \text{sinc}\, f \tag{3.50}$$

The below python code plots the figure



Fourier Transform of rect (t).

```
wget https://github.com/
    AvinashNayak27/fourier/blob/
    master/Fourier/codes/3.9.py
```

3.10 $\text{sinc}\, t \overset{\mathcal{F}}{\longleftrightarrow}$? Verify using python.
Solution: From (3.10), we have

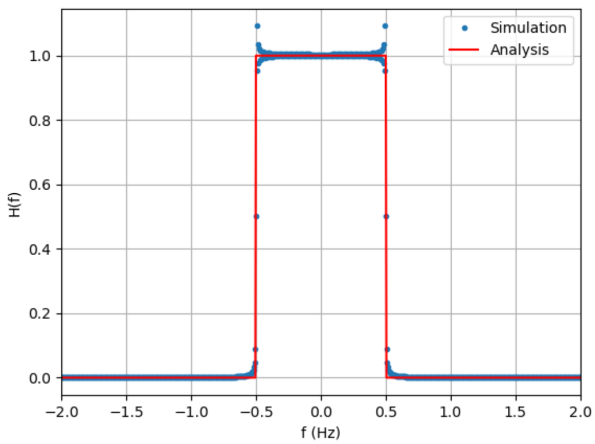
$$\text{sinc}\, t \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}(-f) = \text{rect}\, f \tag{3.51}$$

Since rect f is an even function. The below python code plots the figure

```
wget https://github.com/AvinashNayak27/
    fourier/blob/master/Fourier/codes/3.10.py
```

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution: The function $H(f)$ is a low pass filter which filters out even harmonics and leaves the zero frequency component behind. The rectangular function represents an ideal



Fourier Transform of rect(t).

low pass filter. Suppose the cutoff frequency is $f_c = 50$ Hz, then

$$H(f) = \text{rect}\left(\frac{f}{2f_c}\right) = \begin{cases} 1 & |f| < f_c \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Multiplying by a scaling factor to get DC 5V,

$$H(f) = \frac{\pi V_0}{2A_0}\text{rect}\left(\frac{f}{2f_c}\right) \tag{4.2}$$

where $V_0 = 5$ V.

4.2 Find $h(t)$.

Solution: Suppose $g(t) \xleftrightarrow{\mathcal{F}} G(f)$. Then, for some nonzero $a \in \mathbb{R}$, let $u = at$,

$$g(at) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(at)e^{-\jmath 2\pi f t}\, dt \tag{4.3}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} g(u)e^{\left(-\jmath 2\pi \frac{f}{a} t\right)}\, dt \tag{4.4}$$

$$= \frac{1}{a}G\left(\frac{f}{a}\right) \tag{4.5}$$

Using (4.5), from (4.2),

$$h(t) \xleftrightarrow{\mathcal{F}} \frac{\pi V_0}{2A_0}\text{rect}\left(\frac{f}{2f_c}\right) \tag{4.6}$$

$$h(t) \xleftrightarrow{\mathcal{F}} \frac{\pi V_0 2f_c}{2A_0}\frac{1}{2f_c}\text{rect}\left(\frac{f}{2f_c}\right) \tag{4.7}$$

$$h(t) \xleftrightarrow{\mathcal{F}} \frac{2\pi f_c V_0}{A_0}\text{rect}\left(\frac{f}{2f_c}\right) \tag{4.8}$$

$$h(t) = \frac{2\pi V_0}{A_0} f_c \,\text{sinc}\,(2f_c t) \tag{4.9}$$

4.3 Verify your result using convolution.

Solution: Fourier transform of $x(t)$ and $h(t)$ respectively is

$$X(f) = \sum_{k=-\infty}^{\infty} \frac{2A_0}{\pi}\frac{\delta(f + 2kf_0)}{1 - 4k^2} \tag{4.10}$$

$$H(f) = \frac{\pi V_0}{2A_0}\text{rect}\left(\frac{f}{2f_c}\right) \tag{4.11}$$

$$X(f) \times H(f) = \sum_{k=-\infty}^{\infty} V_0\frac{\delta(f + 2kf_0)}{1 - 4k^2} \times \text{rect}\left(\frac{f}{2f_c}\right) \tag{4.12}$$

$$X(f) \times H(f) = \sum_{k=0}^{0} V_0\frac{\delta(f + 2kf_0)}{1 - 4k^2} \tag{4.13}$$

Hence,

$$X(f) \times H(f) = V_0\frac{\delta(f)}{1 - 4 \times 0} \tag{4.14}$$

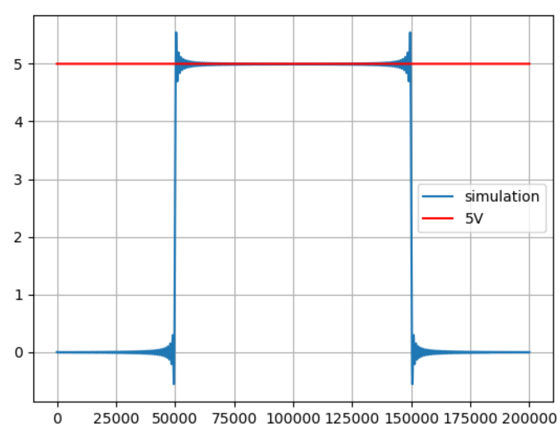
$$X(f) \times H(f) = V_0\delta(f) \tag{4.15}$$

Since $1 \xleftrightarrow{\mathcal{F}} \delta(0)$, Hence,

$$V_0\delta(t) \xleftrightarrow{\mathcal{F}^{-1}} V_0 \times 1 \tag{4.16}$$

$$\implies H(t) \circledast x(t) = V_0 \tag{4.17}$$

Hence verified. The following python code



Convolution of the $x(t)$ and $h(t)$.

plots the figure

```
wget https://github.com/AvinashNayak27/
    fourier/blob/master/Fourier/codes/4.3.py
```

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$.
Solution: The Butterworth filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1+\left(\frac{f}{f_c}\right)^{2n}\right)} \tag{5.1}$$

where n is the order of the filter and f_c is the cutoff frequency. The attenuation at frequency f is given by

$$A = -10\log_{10}|H(f)|^2 \tag{5.2}$$
$$= -20\log_{10}|H(f)| \tag{5.3}$$

We consider the following design parameters for our lowpass analog Butterworth filter:

a) Passband edge, $f_p = 50$ Hz
b) Stopband edge, $f_s = 100$ Hz
c) Passband attenuation, $A_p = -1$ dB
d) Stopband attenuation, $A_s = -20$ dB

We are required to find a desriable order n and cutoff frequency f_c for the filter. From (5.3),

$$A_p = -10\log_{10}\left[1+\left(\frac{f_p}{f_c}\right)^{2n}\right] \tag{5.4}$$

$$A_s = -10\log_{10}\left[1+\left(\frac{f_s}{f_c}\right)^{2n}\right] \tag{5.5}$$

Thus,

$$\left(\frac{f_p}{f_c}\right)^{2n} = 10^{-\frac{A_p}{10}} - 1 \tag{5.6}$$

$$\left(\frac{f_s}{f_c}\right)^{2n} = 10^{-\frac{A_s}{10}} - 1 \tag{5.7}$$

Therefore, on dividing the above equations and solving for n,

$$n = \frac{\log\left(10^{-\frac{A_s}{10}}-1\right) - \log\left(10^{-\frac{A_p}{10}}-1\right)}{2\left(\log f_s - \log f_p\right)} \tag{5.8}$$

In this case, making appropriate susbstitutions gives $n = 4.29$. Hence, we take $n = 5$. Solving for f_c in (5.6) and (5.7),

$$f_{c1} = f_p\left[10^{-\frac{A_p}{10}} - 1\right]^{-\frac{1}{2n}} = 57.23\,\text{Hz} \tag{5.9}$$

$$f_{c2} = f_s\left[10^{-\frac{A_s}{10}} - 1\right]^{-\frac{1}{2n}} = 63.16\,\text{Hz} \tag{5.10}$$

Hence, we take $f_c = \sqrt{f_{c1}f_{c2}} = 60\,\text{Hz}$ approximately.

5.2 Design a Chebyshev filter for $H(f)$. **Solution:** The Chebyshev filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1+\epsilon^2 C_n^2\left(\frac{f}{f_c}\right)\right)} \tag{5.11}$$

where

a) n is the order of the filter
b) ϵ is the ripple
c) f_c is the cutoff frequency
d) $C_n = \cosh^{-1}(n\cosh x)$ denotes the n$^{\text{th}}$ order Chebyshev polynomial, given by

$$c_n(x) = \begin{cases} \cos\left(n\cos^{-1}x\right) & |x| \le 1 \\ \cosh\left(n\cosh^{-1}x\right) & \text{otherwise} \end{cases} \tag{5.12}$$

We are given the following specifications:

a) Passband edge (which is equal to cutoff frequency), $f_p = f_c$
b) Stopband edge, f_s
c) Attenuation at stopband edge, A_s
d) Peak-to-peak ripple δ in the passband. It is given in dB and is related to ϵ as

$$\delta = 10\log_{10}\left(1+\epsilon^2\right) \tag{5.13}$$

and we must find a suitable n and ϵ. From (5.13),

$$\epsilon = \sqrt{10^{\frac{\delta}{10}} - 1} \tag{5.14}$$

At $f_s > f_p = f_c$, using (5.12), A_s is given by

$$A_s = -10\log_{10}\left[1+\epsilon^2 c_n^2\left(\frac{f_s}{f_p}\right)\right] \tag{5.15}$$

$$\implies c_n\left(\frac{f_s}{f_p}\right) = \frac{\sqrt{10^{-\frac{A_s}{10}}-1}}{\epsilon} \tag{5.16}$$

$$\implies n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{A_s}{10}}-1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{f_s}{f_p}\right)} \tag{5.17}$$

We consider the following specifications:

a) Passband edge/cutoff frequency, $f_p = f_c = 60\,\text{Hz}$.
b) Stopband edge, $f_s = 100\,\text{Hz}$.
c) Passband ripple, $\delta = 0.5\,\text{dB}$
d) Stopband attenuation, $A_s = -20\,\text{dB}$

$\epsilon = 0.35$ and $n = 3.68$. Hence, we take $n = 4$ as the order of the Chebyshev filter.

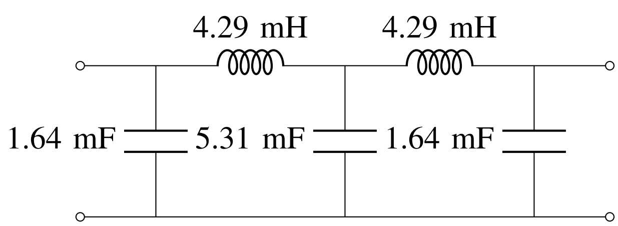
5.3 Design a circuit for your Butterworth filter.
Solution: Looking at the table of normalized element values L_k, C_k, of the Butterworth filter for order 5, and noting that de-normalized values L'_k and C'_k are given by

$$C'_k = \frac{C_k}{\omega_c} \qquad L'_k = \frac{L_k}{\omega_c} \tag{5.18}$$

De-normalizing these values, taking $f_c = 60$ Hz,

$$C'_1 = C'_5 = 1.64\,\text{mF} \tag{5.19}$$
$$L'_2 = L'_4 = 4.29\,\text{mH} \tag{5.20}$$
$$C'_3 = 5.31\,\text{mF} \tag{5.21}$$

The L-C network is shown in Fig.



L-C Butterworth Filter

Below python code plots the figure

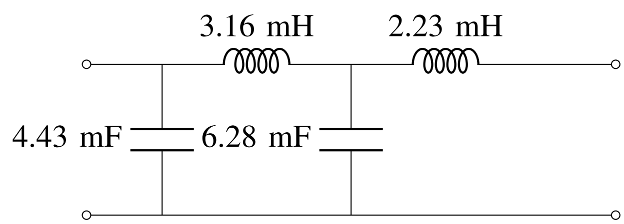
```
wget https://github.com/AvinashNayak27/
    fourier/blob/master/Fourier/codes/5.3.py
```

5.4 Design a circuit for your Chebyshev filter.

Solution: Looking at the table of normalized element values of the Chebyshev filter for order 3 and 0.5 dB ripple, and de-nomrmalizing those values, taking $f_c = 50$ Hz,

$$C'_1 = 4.43\,\text{mF} \tag{5.22}$$
$$L'_2 = 3.16\,\text{mH} \tag{5.23}$$
$$C'_3 = 6.28\,\text{mF} \tag{5.24}$$
$$L'_4 = 2.23\,\text{mH} \tag{5.25}$$

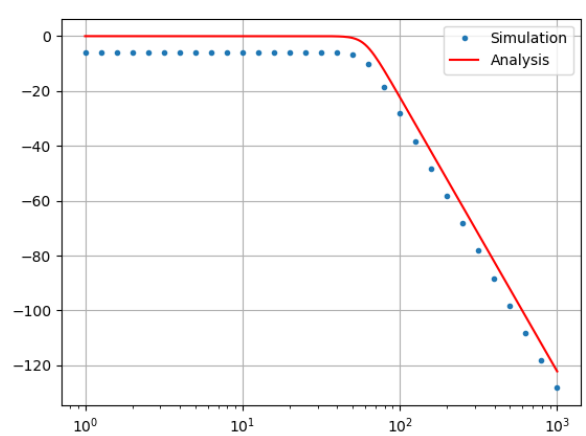
The L-C network is shown in Fig. Below



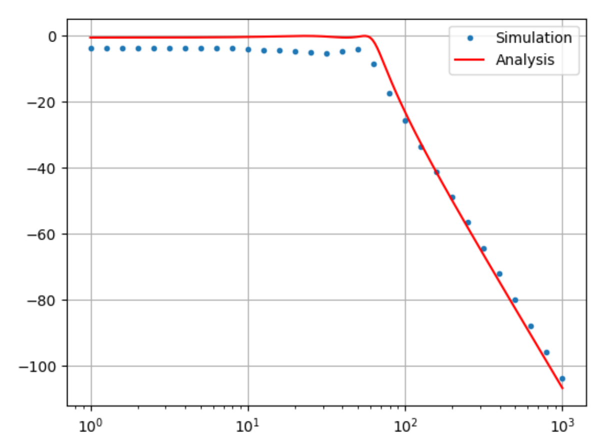
L-C Chebyshev Filter

python code plots the figure

```
wget https://github.com/AvinashNayak27/
    fourier/blob/master/Fourier/codes/5.4.py
```



Simulation of Chebyshev filter.



Simulation of Chebyshev filter.