
CHAPTER THREE

Data Representation

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3-1 Data Types

The term data refers to factual information used for analysis or reasoning. Data itself has no meaning, but becomes information when it is assigned a meaning or interpreted. Information is a collection of facts or data that is communicated. Binary information in digital computers is stored in memory or processor registers. Registers contain either data or control information. Control information is a bit or a group of bits used to specify the sequence of command signals needed for manipulation of the data in other registers. Data are numbers and other binary-coded information that are operated on to achieve required computational results. In this chapter we present the most common types of data found in digital computers and show how the various data types are represented in binary-coded form in computer registers.

The data types found in the registers of digital computers may be classified as being one of the following categories: (1) numbers used in arithmetic computations, (2) letters of the alphabet used in data processing, and (3) other discrete symbols used for specific purposes. All types of data, except binary numbers, are represented in computer registers in binary-coded form. This is because registers are made up of flip-flops and flip-flops are two-state devices that can store only 1's and 0's. The binary number system is the most natural system to use in a digital computer. But sometimes it is convenient to employ different number systems, especially the decimal number system, since it is used by people to perform arithmetic computations.

Number Systems

radix

A number system of *base*, or *radix*, r is a system that uses distinct symbols for r digits. Numbers are represented by a string of digit symbols. To determine the quantity that the number represents, it is necessary to multiply each digit by an integer power of r and then form the sum of all weighted digits. For example, the decimal number system in everyday use employs the radix 10 system. The 10 symbols are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The string of digits 724.5 is interpreted to represent the quantity

decimal

$$7 \times 10^2 + 2 \times 10^1 + 4 \times 10^0 + 5 \times 10^{-1}$$

that is, 7 hundreds, plus 2 tens, plus 4 units, plus 5 tenths. Every decimal number can be similarly interpreted to find the quantity it represents.

binary

The *binary* number system uses the radix 2. The two digit symbols used are 0 and 1. The string of digits 101101 is interpreted to represent the quantity

$$1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 45$$

To distinguish between different radix numbers, the digits will be enclosed in parentheses and the radix of the number inserted as a subscript. For example, to show the equality between decimal and binary forty-five we will write $(101101)_2 = (45)_{10}$.

octal

hexadecimal

Besides the decimal and binary number systems, the *octal* (radix 8) and *hexadecimal* (radix 16) are important in digital computer work. The eight symbols of the octal system are 0, 1, 2, 3, 4, 5, 6, and 7. The 16 symbols of the hexadecimal system are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, and F. The last six symbols are, unfortunately, identical to the letters of the alphabet and can cause confusion at times. However, this is the convention that has been adopted. When used to represent hexadecimal digits, the symbols A, B, C, D, E, F correspond to the decimal numbers 10, 11, 12, 13, 14, 15, respectively.

A number in radix r can be converted to the familiar decimal system by forming the sum of the weighted digits. For example, octal 736.4 is converted to decimal as follows:

$$\begin{aligned} (736.4)_8 &= 7 \times 8^2 + 3 \times 8^1 + 6 \times 8^0 + 4 \times 8^{-1} \\ &= 7 \times 64 + 3 \times 8 + 6 \times 1 + 4/8 = (478.5)_{10} \end{aligned}$$

The equivalent decimal number of hexadecimal F3 is obtained from the following calculation:

$$(F3)_{16} = F \times 16 + 3 = 15 \times 16 + 3 = (243)_{10}$$

conversion

Conversion from decimal to its equivalent representation in the radix r system is carried out by separating the number into its *integer* and *fraction* parts and

converting each part separately. The conversion of a decimal integer into a base r representation is done by successive divisions by r and accumulation of the remainders. The conversion of a decimal fraction to radix r representation is accomplished by successive multiplications by r and accumulation of the integer digits so obtained. Figure 3-1 demonstrates these procedures.

The conversion of decimal 41.6875 into binary is done by first separating the number into its integer part 41 and fraction part .6875. The integer part is converted by dividing 41 by $r = 2$ to give an integer quotient of 20 and a remainder of 1. The quotient is again divided by 2 to give a new quotient and remainder. This process is repeated until the integer quotient becomes 0. The coefficients of the binary number are obtained from the remainders with the first remainder giving the low-order bit of the converted binary number.

The fraction part is converted by multiplying it by $r = 2$ to give an integer and a fraction. The new fraction (*without* the integer) is multiplied again by 2 to give a new integer and a new fraction. This process is repeated until the fraction part becomes zero or until the number of digits obtained gives the required accuracy. The coefficients of the binary fraction are obtained from the integer digits with the first integer computed being the digit to be placed next to the binary point. Finally, the two parts are combined to give the total required conversion.

Octal and Hexadecimal Numbers

The conversion from and to binary, octal, and hexadecimal representation plays an important part in digital computers. Since $2^3 = 8$ and $2^4 = 16$, each octal digit corresponds to three binary digits and each hexadecimal digit corresponds to four binary digits. The conversion from binary to octal is easily accomplished by partitioning the binary number into groups of three bits each. The corresponding octal digit is then assigned to each group of bits and the string of digits so obtained gives the octal equivalent of the binary number. Consider, for example, a 16-bit register. Physically, one may think of the

Figure 3-1 Conversion of decimal 41.6875 into binary.

Integer = 41		Fraction = 0.6875	
41		0.6875	
20	1	<u>2</u>	
10	0	1.3750	
5	0	<u>$\times 2$</u>	
2	1	0.7500	
1	0	<u>$\times 2$</u>	
0	1	1.5000	
		<u>$\times 2$</u>	
		1.0000	
$(41)_{10} = (101001)_2$		$(0.6875)_{10} = (0.1011)_2$	
$(41.6875)_{10} = (101001.1011)_2$			

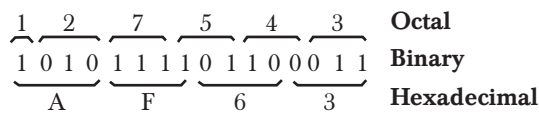


Figure 3-2 Binary, octal, and hexadecimal conversion.

register as composed of 16 binary storage cells, with each cell capable c holding either a 1 or a 0. Suppose that the bit configuration stored in the register is as shown in Fig. 3-2. Since a binary number consists of a string of 1's and 0's, the 16-bit register can be used to store any binary number from 0 to $2^{16} - 1$. For the particular example shown, the binary number stored in the register is the equivalent of decimal 44899. Starting from the low-order bit, we partition the register into groups of three bits each (the sixteenth bit remains in a group by itself). Each group of three bits is assigned its octal equivalent and placed on top of the register. The string of octal digits so obtained represents the octal equivalent of the binary number.

Conversion from binary to hexadecimal is similar except that the bits are divided into groups of four. The corresponding hexadecimal digit for each group of four bits is written as shown below the register of Fig. 3-2. The string of hexadecimal digits so obtained represents the hexadecimal equivalent of the binary number. The corresponding octal digit for each group of three bits is easily remembered after studying the first eight entries listed in Table 3-1. The correspondence between a hexadecimal digit and its equivalent 4-bit code can be found in the first 16 entries of Table 3-2.

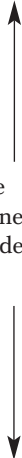
TABLE 3-1 Binary-Coded Octal Numbers

Octal number	Binary-coded octal	Decimal equivalent	
0	000	0	Code for one octal digit
1	001	1	
2	010	2	
3	011	3	
4	100	4	
5	101	5	
6	110	6	
7	111	7	
10	001 000	8	
11	001 001	9	
12	001 010	10	
24	010 100	20	
62	110 010	50	
143	001 100 011	99	
370	011 111 000	248	

Table 3-1 lists a few octal numbers and their representation in registers in binary-coded form. The binary code is obtained by the procedure explained above. Each octal digit is assigned a 3-bit code as specified by the entries of the first eight digits in the table. Similarly, Table 3-2 lists a few hexadecimal numbers and their representation in registers in binary-coded form. Here the binary code is obtained by assigning to each hexadecimal digit the 4-bit code listed in the first 16 entries of the table.

Comparing the binary-coded octal and hexadecimal numbers with their binary number equivalent we find that the bit combination in all three representations is exactly the same. For example, decimal 99, when converted to binary, becomes 1100011. The binary-coded octal equivalent of decimal 99 is 001 100 011 and the binary-coded hexadecimal of decimal 99 is 0110 0011. If we neglect the leading zeros in these three binary representations, we find that their bit combination is identical. This should be so because of the straightforward conversion that exists between binary numbers and octal or hexadecimal. The point of all this is that a string of 1's and 0's stored in a register could represent a binary number, but this same string of bits may be interpreted as holding an octal number in binary-coded form (if we divide the bits in groups of three) or as holding a hexadecimal number in binary-coded form (if we divide the bits in groups of four).

TABLE 3-2 Binary-Coded Hexadecimal Numbers

Hexadecimal number	Binary-coded hexadecimal	Decimal equivalent	
0	0000	0	 Code for one hexadecimal digit
1	0001	1	
2	0010	2	
3	0011	3	
4	0100	4	
5	0101	5	
6	0110	6	
7	0111	7	
8	1000	8	
9	1001	9	
A	1010	10	
B	1011	11	
C	1100	12	
D	1101	13	
E	1110	14	
F	1111	15	
14	0001 0100	20	
32	0011 0010	50	
63	0110 0011	99	
F8	1111 1000	248	

The registers in a digital computer contain many bits. Specifying the content of registers by their binary values will require a long string of binary digits. It is more convenient to specify content of registers by their octal or hexadecimal equivalent. The number of digits is reduced by one-third in the octal designation and by one-fourth in the hexadecimal designation. For example, the binary number 1111 1111 1111 has 12 digits. It can be expressed in octals as 7777 (four digits) or in hexadecimal as FFF (three digits). Computer manuals invariably choose either the octal or the hexadecimal designation for specifying contents of registers.

Decimal Representation

The binary number system is the most natural system for a computer, but people are accustomed to the decimal system. One way to solve this conflict is to convert all input decimal numbers into binary numbers, let the computer perform all arithmetic operations in binary and then convert the binary results back to decimal for the human user to understand. However, it is also possible for the computer to perform arithmetic operations directly with decimal numbers provided they are placed in registers in a coded form. Decimal numbers enter the computer usually as binary-coded alphanumeric characters. These codes, introduced later, may contain from six to eight bits for each decimal digit. When decimal numbers are used for internal arithmetic computations, they are converted to a binary code with four bits per digit.



binary code

A binary code is a group of n bits that assume up to 2^n distinct combinations of 1's and 0's with each combination representing one element of the set that is being coded. For example, a set of four elements can be coded by a 2-bit code with each element assigned one of the following bit combinations; 00, 01, 10, or 11. A set of eight elements requires a 3-bit code, a set of 16 elements requires a 4-bit code, and so on. A binary code will have some unassigned bit combinations if the number of elements in the set is not a multiple power of 2. The 10 decimal digits form such a set. A binary code that distinguishes among 10 elements must contain at least four bits, but six combinations will remain unassigned. Numerous different codes can be obtained by arranging four bits in 10 distinct combinations. The bit assignment most commonly used for the decimal digits is the straight binary assignment listed in the first 10 entries of Table 3-3. This particular code is called *binary-coded decimal* and is commonly referred to by its abbreviation BCD. Other decimal codes are sometimes used and a few of them are given in Sec. 3-5.

BCD

It is very important to understand the difference between the *conversion* of decimal numbers into binary and the *binary coding* of decimal numbers. For example, when *converted* to a binary number, the decimal number 99 is represented by the string of bits 1100011, but when represented in BCD, it becomes 1001 1001. The *only* difference between a decimal number represented by the familiar digit symbols 0, 1, 2, . . . , 9 and the BCD symbols 0001, 0010, . . . , 1001 is in the symbols used to represent the digits—the

TABLE 3-3 Binary-Coded Decimal (BCD) Numbers

Decimal number	Binary-coded decimal (BCD) number	
0	0000	 Code for one decimal digit 
1	0001	
2	0010	
3	0011	
4	0100	
5	0101	
6	0110	
7	0111	
8	1000	
9	1001	
10	0001 0000	
20	0010 0000	
50	0101 0000	
99	1001 1001	
248	0010 0100 1000	

number itself is exactly the same. A few decimal numbers and their representation in BCD are listed in Table 3-3.

Alphanumeric Representation

character

ASCII

Many applications of digital computers require the handling of data that consist not only of numbers, but also of the letters of the alphabet and certain special characters. An *alphanumeric character set* is a set of elements that includes the 10 decimal digits, the 26 letters of the alphabet and a number of special characters, such as \$, +, and =. Such a set contains between 32 and 64 elements (if only uppercase letters are included) or between 64 and 128 (if both uppercase and lowercase letters are included). In the first case, the binary code will require six bits and in the second case, seven bits. The standard alphanumeric binary code is the ASCII (American Standard Code for Information Interchange), which uses seven bits to code 128 characters. The binary code for the uppercase letters, the decimal digits, and a few special characters is listed in Table 3-4. Note that the decimal digits in ASCII can be converted to BCD by removing the three high-order bits, 011. A complete list of ASCII characters is provided in Table 11-1.

Binary codes play an important part in digital computer operations. The codes must be in binary because registers can only hold binary information. One must realize that binary codes merely change the symbols, not the meaning of the discrete elements they represent. The operations specified for digital computers must take into consideration the meaning of the

TABLE 3-4 American Standard Code for information Interchange (ASCII)

Character	Binary code	Character	Binary code
A	100 0001	0	011 0000
B	100 0010	1	011 0001
C	100 0011	2	011 0010
D	100 0100	3	011 0011
E	100 0101	4	011 0100
F	100 0110	5	011 0101
G	100 0111	6	011 0110
H	100 1000	7	011 0111
I	100 1001	8	011 1000
J	100 1010	9	011 1001
K	100 1011		
L	100 1100		
M	100 1101	space	010 0000
N	100 1110	.	010 1110
O	100 1111	(010 1000
P	101 0000	+	010 1011
Q	101 0001	\$	010 0100
R	101 0010	*	010 1010
S	101 0011)	010 1001
T	101 0100	—	010 1101
U	101 0101	/	010 1111
V	101 0110	,	010 1100
W	101 0111	=	011 1101
X	101 1000		
Y	101 1001		
Z	101 1010		

bits stored in registers so that operations are performed on operands of the same type. In inspecting the bits of a computer register at random, one is likely to find that it represents some type of coded information rather than a binary number.

Binary codes can be formulated for any set of discrete elements such as the musical notes and chess pieces and their positions on the chessboard. Binary codes are also used to formulate instructions that specify control information for the computer. This chapter is concerned with *data* representation. Instruction codes are discussed in Chap. 5.

3-2 Complements

Complements are used in digital computers for simplifying the subtraction operation and for logical manipulation. There are two types of complements for each base r system: the r 's complement and the $(r - 1)$'s complement.

When the value of the base r is substituted in the name, the two types are referred to as the 2's and 1's complement for binary numbers and the 10's and 9's complement for decimal numbers.

$(r - 1)$'s Complement

9's complement

Given a number N in base r having n digits, the $(r - 1)$'s complement of N is defined as $(r^n - 1) - N$. For decimal numbers $r = 10$ and $r - 1 = 9$, so the 9's complement of N is $(10^n - 1) - N$. Now, 10^n represents a number that consists of a single 1 followed by n 0's. $10^n - 1$ is a number represented by n 9's. For example, with $n = 4$ we have $10^4 = 10000$ and $10^4 - 1 = 9999$. It follows that the 9's complement of a decimal number is obtained by subtracting each digit from 9. For example, the 9's complement of 546700 is $999999 - 546700 = 453299$ and the 9's complement of 12389 is $99999 - 12389 = 87610$.

1's complement

For binary numbers, $r = 2$ and $r - 1 = 1$, so the 1's complement of N is $(2^n - 1) - N$. Again, 2^n is represented by a binary number that consists of a 1 followed by n 0's. $2^n - 1$ is a binary number represented by n 1's. For example, with $n = 4$, we have $2^4 = (10000)_2$ and $2^4 - 1 = (1111)_2$. Thus the 1's complement of a binary number is obtained by subtracting each digit from 1. However, the subtraction of a binary digit from 1 causes the bit to change from 0 to 1 or from 1 to 0. Therefore, the 1's complement of a binary number is formed by changing 1's into 0's and 0's into 1's. For example, the 1's complement of 1011001 is 0100110 and the 1's complement of 0001111 is 1110000.

The $(r - 1)$'s complement of octal or hexadecimal numbers are obtained by subtracting each digit from 7 or F (decimal 15) respectively.

(r) 's Complement

10's complement

The r 's complement of an n -digit number N in base r is defined as $r^n - N$ for $N \neq 0$ and 0 for $N = 0$. Comparing with the $(r - 1)$'s complement, we note that the r 's complement is obtained by adding 1 to the $(r - 1)$'s complement since $r^n - N = [(r^n - 1) - N] + 1$. Thus the 10's complement of the decimal 2389 is $7610 + 1 = 7611$ and is obtained by adding 1 to the 9's complement value. The 2's complement of binary 101100 is $010011 + 1 = 010100$ and is obtained by adding 1 to the 1's complement value.

2's complement

Since 10^n is a number represented by a 1 followed by n 0's, then $10^n - N$, which is the 10's complement of N , can be formed also by leaving all least significant 0's unchanged, subtracting the first nonzero least significant digit from 10, and then subtracting all higher significant digits from 9. The 10's complement of 246700 is 753300 and is obtained by leaving the two zeros unchanged, subtracting 7 from 10, and subtracting the other three digits from 9. Similarly, the 2's complement can be formed by leaving all least significant 0's and the first 1 unchanged, and then replacing 1's by 0's and 0's by 1's in all other higher, significant bits. The 2's complement of 1101100 is 0010100 and is obtained by leaving the two low-order 0's and the first 1 unchanged, and then replacing 1's by 0's and 0's by 1's in the other four most significant bits.

In the definitions above it was assumed that the numbers do not have a radix point. If the original number N contains a radix point, it should be removed temporarily to form the r 's or $(r - 1)$'s complement. The radix point is then restored to the complemented number in the same relative position. It is also worth mentioning that the complement of the complement restores the number to its original value. The r 's complement of N is $r^n - N$. The complement of the complement is $r^n - (r^n - N) = N$ giving back the original number.

Subtraction of Unsigned Numbers

The direct method of subtraction taught in elementary schools uses the borrow concept. In this method we borrow a 1 from a higher significant position when the minuend digit is smaller than the corresponding subtrahend digit. This seems to be easiest when people perform subtraction with paper and pencil. When subtraction is implemented with digital hardware, this method is found to be less efficient than the method that uses complements.

subtraction

The subtraction of two n -digit unsigned numbers $M - N$ ($N \neq 0$) in base r can be done as follows:

1. Add the minuend M to the r 's complement of the subtrahend N . This performs $M + (r^n - N) = M - N + r^n$.
2. If $M \geq N$, the sum will produce an end carry r^n which is discarded, and what is left is the result $M - N$.
3. If $M < N$, the sum does not produce an end carry and is equal to $r^n - (N - M)$, which is the r 's complement of $(N - M)$. To obtain the answer in a familiar form, take the r 's complement of the sum and place a negative sign in front.

end carry

Consider, for example, the subtraction $72532 - 13250 = 59282$. The 10's complement of 13250 is 86750. Therefore:

$$\begin{array}{r}
 M = 72532 \\
 10\text{'s complement of } N = +86750 \\
 \text{Sum} = 159282 \\
 \text{Discard end carry } 10^5 = -100000 \\
 \text{Answer} = 59282
 \end{array}$$

Now consider an example with $M < N$. The subtraction $13250 - 72532$ produces negative 59282. Using the procedure with complements, we have

$$\begin{array}{r}
 M = 13250 \\
 10\text{'s complement of } N = +27468 \\
 \text{Sum} = 40718
 \end{array}$$