9.3 Vector Product (Cross Product)

We shall define another form of multiplication of vectors, inspired by applications, whose result will be a *vector*. This is in contrast to the dot product of Sec. 9.2 where multiplication resulted in a *scalar*. We can construct a vector $\mathbf{v}$ that is perpendicular to two vectors $\mathbf{a}$ and $\mathbf{b}$, which are two sides of a parallelogram on a plane in space as indicated in Fig. 185, such that the length $|\mathbf{v}|$ is numerically equal to the area of that parallelogram. Here then is the new concept.

DEFINITION

Vector Product (Cross Product, Outer Product) of Vectors

The **vector product** or **cross product** $\mathbf{a} \times \mathbf{b}$ (read "**a** cross **b**") of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector $\mathbf{v}$ denoted by

$$\mathbf{v} = \mathbf{a} \times \mathbf{b}$$

I. If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then we define $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{0}$.

II. If both vectors are nonzero vectors, then vector $\mathbf{v}$ has the length

$$|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \gamma, \tag{1}$$

where γ is the angle between $\mathbf{a}$ and $\mathbf{b}$ as in Sec. 9.2.

Furthermore, by design, $\mathbf{a}$ and $\mathbf{b}$ form the sides of a parallelogram on a plane in space. The parallelogram is shaded in blue in Fig. 185. The area of this blue parallelogram is precisely given by Eq. (1), so that the length $|\mathbf{v}|$ of the vector $\mathbf{v}$ is equal to the area of that parallelogram.

III. If $\mathbf{a}$ and $\mathbf{b}$ lie in the same straight line, i.e., $\mathbf{a}$ and $\mathbf{b}$ have the same or opposite directions, then γ is $0°$ or $180°$ so that $\sin \gamma = 0$. In that case $|\mathbf{v}| = 0$ so that $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{0}$.

IV. If cases I and III do not occur, then $\mathbf{v}$ is a nonzero vector. The direction of $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$—precisely in this order (!)—form a right-handed triple as shown in Figs. 185–187 and explained below.

Another term for vector product is outer product.

Remark. Note that I and III completely characterize the exceptional case when the cross product is equal to the zero vector, and II and IV the regular case where the cross product is perpendicular to two vectors.

Just as we did with the dot product, we would also like to express the cross product in components. Let $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$. Then $\mathbf{v} = [v_1, v_2, v_3] = \mathbf{a} \times \mathbf{b}$ has the components

$$v_1 = a_2b_3 - a_3b_2, \qquad v_2 = a_3b_1 - a_1b_3, \qquad v_3 = a_1b_2 - a_2b_1. \tag{2}$$

Here the Cartesian coordinate system is *right-handed*, as explained below (see also Fig. 188). (For a left-handed system, each component of $\mathbf{v}$ must be multiplied by -1. Derivation of (2) in App. 4.)

Right-Handed Triple. A triple of vectors **a**, **b**, **v** is *right-handed* if the vectors in the given order assume the same sort of orientation as the thumb, index finger, and middle finger of the right hand when these are held as in Fig. 186. We may also say that if **a** is rotated into the direction of **b** through the angle $\gamma\,(<\pi)$, then **v** advances in the same direction as a right-handed screw would if turned in the same way (Fig. 187).

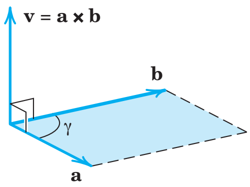


Fig. 185. Vector product

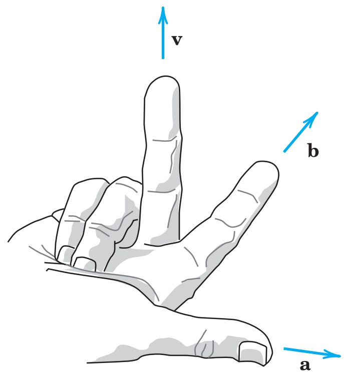


Fig. 186. Right-handed triple of vectors **a, b, v**

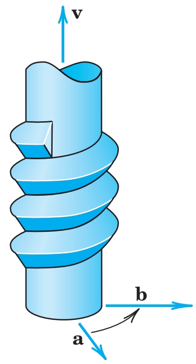
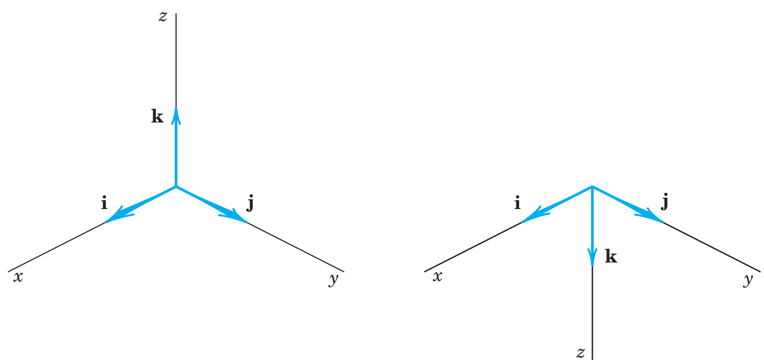


Fig. 187. Right-handed screw

Right-Handed Cartesian Coordinate System. The system is called **right-handed** if the corresponding unit vectors **i**, **j**, **k** in the positive directions of the axes (see Sec. 9.1) form a right-handed triple as in Fig. 188a. The system is called **left-handed** if the sense of **k** is reversed, as in Fig. 188b. In applications, we prefer right-handed systems.



(a) Right-handed (b) Left-handed

Fig. 188. The two types of Cartesian coordinate systems

How to Memorize (2). If you know second- and third-order determinants, you see that (2) can be written

$$(2^*) \qquad v_1 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \qquad v_2 = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} = +\begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \qquad v_3 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

and $\mathbf{v} = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is the expansion of the following symbolic determinant by its first row. (We call the determinant "symbolic" because the first row consists of vectors rather than of numbers.)

$$(2^{**}) \qquad \mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}.$$

For a left-handed system the determinant has a minus sign in front.

EXAMPLE 1

Vector Product

For the vector product $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ of $\mathbf{a} = [1, 1, 0]$ and $\mathbf{b} = [3, 0, 0]$ in right-handed coordinates we obtain from (2)

$$v_1 = 0, \qquad v_2 = 0, \qquad v_3 = 1 \cdot 0 - 1 \cdot 3 = -3.$$

We confirm this by (2**):

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}\mathbf{k} = -3\mathbf{k} = [0, 0, -3].$$

To check the result in this simple case, sketch **a**, **b**, and **v**. Can you see that two vectors in the xy-plane must always have their vector product parallel to the z-axis (or equal to the zero vector)? ■

EXAMPLE 2

Vector Products of the Standard Basis Vectors

$$(3) \qquad \begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k}, & \mathbf{j} \times \mathbf{k} &= \mathbf{i}, & \mathbf{k} \times \mathbf{i} &= \mathbf{j} \\ \mathbf{j} \times \mathbf{i} &= -\mathbf{k}, & \mathbf{k} \times \mathbf{j} &= -\mathbf{i}, & \mathbf{i} \times \mathbf{k} &= -\mathbf{j}. \end{aligned}$$

We shall use this in the next proof. ■

THEOREM 1

General Properties of Vector Products

(a) *For every scalar l,*

$$(4) \qquad (l\mathbf{a}) \times \mathbf{b} = l(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (l\mathbf{b}).$$

(b) *Cross multiplication is distributive with respect to vector addition; that is,*

$$(5) \qquad \begin{aligned} (\alpha) \quad & \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}), \\ (\beta) \quad & (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}). \end{aligned}$$

(c) *Cross multiplication is* ***not commutative*** *but* ***anticommutative;*** *that is,*

$$(6) \qquad \mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) \qquad \text{(Fig. 189).}$$

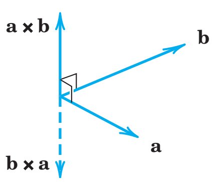


Fig. 189. Anticommutativity of cross multiplication

(d) *Cross multiplication is* ***not associative***; *that is, in general,*

(7) $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

so that the parentheses cannot be omitted.

PROOF Equation (4) follows directly from the definition. In (5α), formula (2*) gives for the first component on the left

$$\begin{vmatrix} a_2 & a_3 \\ b_2 + c_2 & b_3 + c_3 \end{vmatrix} = a_2(b_3 + c_3) - a_3(b_2 + c_2)$$

$$= (a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2)$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}.$$

By (2*) the sum of the two determinants is the first component of $(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$, the right side of (5$\alpha$). For the other components in (5α) and in 5(β), equality follows by the same idea.

Anticommutativity (6) follows from (2**) by noting that the interchange of Rows 2 and 3 multiplies the determinant by -1. We can confirm this geometrically if we set $\mathbf{a} \times \mathbf{b} = \mathbf{v}$ and $\mathbf{b} \times \mathbf{a} = \mathbf{w}$; then $|\mathbf{v}| = |\mathbf{w}|$ by (1), and for $\mathbf{b}, \mathbf{a}, \mathbf{w}$ to form a *right-handed* triple, we must have $\mathbf{w} = -\mathbf{v}$.

Finally, $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$, whereas $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$ (see Example 2). This proves (7). ■

Typical Applications of Vector Products

EXAMPLE 3 **Moment of a Force**

In mechanics the moment m of a force $\mathbf{p}$ about a point Q is defined as the product $m = |\mathbf{p}|d$, where d is the (perpendicular) distance between Q and the line of action L of $\mathbf{p}$ (Fig. 190). If $\mathbf{r}$ is the vector from Q to any point A on L, then $d = |\mathbf{r}| \sin \gamma$, as shown in Fig. 190, and

$$m = |\mathbf{r}||\mathbf{p}| \sin \gamma.$$

Since γ is the angle between $\mathbf{r}$ and $\mathbf{p}$, we see from (1) that $m = |\mathbf{r} \times \mathbf{p}|$. The vector

(8) $$\mathbf{m} = \mathbf{r} \times \mathbf{p}$$

is called the **moment vector** or **vector moment** of $\mathbf{p}$ about Q. Its magnitude is m. If $\mathbf{m} \neq \mathbf{0}$, its direction is that of the axis of the rotation about Q that $\mathbf{p}$ has the tendency to produce. This axis is perpendicular to both $\mathbf{r}$ and $\mathbf{p}$. ■

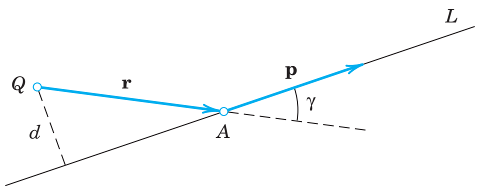


Fig. 190. Moment of a force $\mathbf{p}$

EXAMPLE 4 Moment of a Force

Find the moment of the force $\mathbf{p}$ about the center Q of a wheel, as given in Fig. 191.

Solution. Introducing coordinates as shown in Fig. 191, we have

$$\mathbf{p} = [1000 \cos 30°, \quad 1000 \sin 30°, \quad 0] = [866, \quad 500, \quad 0], \qquad \mathbf{r} = [0, \quad 1.5, \quad 0].$$

(Note that the center of the wheel is at $y = -1.5$ on the y-axis.) Hence (8) and (2**) give

$$\mathbf{m} = \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1.5 & 0 \\ 866 & 500 & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + \begin{vmatrix} 0 & 1.5 \\ 866 & 500 \end{vmatrix} \mathbf{k} = [0, 0, -1299].$$

This moment vector $\mathbf{m}$ is normal, i.e., perpendicular to the plane of the wheel. Hence it has the direction of the axis of rotation about the center Q of the wheel that the force $\mathbf{p}$ has the tendency to produce. The moment $\mathbf{m}$ points in the negative z-direction, This is, the direction in which a right-handed screw would advance if turned in that way. ■

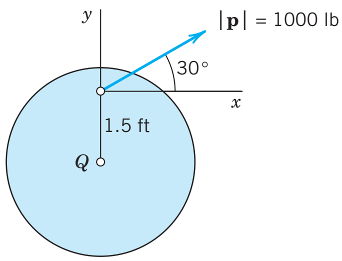


Fig. 191. Moment of a force $\mathbf{p}$

EXAMPLE 5 Velocity of a Rotating Body

A rotation of a rigid body B in space can be simply and uniquely described by a vector $\mathbf{w}$ as follows. The direction of $\mathbf{w}$ is that of the axis of rotation and such that the rotation appears clockwise if one looks from the initial point of $\mathbf{w}$ to its terminal point. The length of $\mathbf{w}$ is equal to the **angular speed** $\omega (>0)$ of the rotation, that is, the linear (or tangential) speed of a point of B divided by its distance from the axis of rotation.

Let P be any point of B and d its distance from the axis. Then P has the speed ωd. Let $\mathbf{r}$ be the position vector of P referred to a coordinate system with origin 0 on the axis of rotation. Then $d = |\mathbf{r}| \sin \gamma$, where γ is the angle between $\mathbf{w}$ and $\mathbf{r}$. Therefore,

$$\omega d = |\mathbf{w}||\mathbf{r}| \sin \gamma = |\mathbf{w} \times \mathbf{r}|.$$

From this and the definition of vector product we see that the velocity vector $\mathbf{v}$ of P can be represented in the form (Fig. 192)

$$\mathbf{v} = \mathbf{w} \times \mathbf{r}. \tag{9}$$

This simple formula is useful for determining $\mathbf{v}$ at any point of B. ■

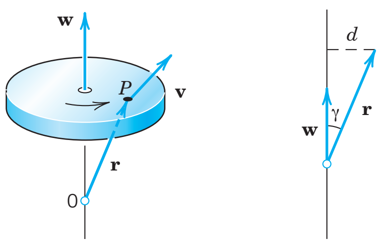


Fig. 192. Rotation of a rigid body

Scalar Triple Product

Certain products of vectors, having three or more factors, occur in applications. The most important of these products is the scalar triple product or mixed product of three vectors **a**, **b**, **c**.

$$(\mathbf{a}\quad \mathbf{b}\quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}). \tag{10*}$$

The scalar triple product is indeed a scalar since (10*) involves a dot product, which in turn is a scalar. We want to express the scalar triple product in components and as a third-order determinant. To this end, let $\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, and $\mathbf{c} = [c_1, c_2, c_3]$. Also set $\mathbf{b} \times \mathbf{c} = \mathbf{v} = [v_1, v_2, v_3]$. Then from the dot product in components [formula (2) in Sec. 9.2] and from (2*) with **b** and **c** instead of **a** and **b** we first obtain

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \mathbf{a} \cdot \mathbf{v} = a_1 v_1 + a_2 v_2 + a_3 v_3 \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \end{aligned}$$

The sum on the right is the expansion of a third-order determinant by its first row. Thus we obtain the desired formula for the scalar triple product, that is,

$$(\mathbf{a}\quad \mathbf{b}\quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{10}$$

The most important properties of the scalar triple product are as follows.

THEOREM 2

Properties and Applications of Scalar Triple Products

(a) *In* (10) *the dot and cross can be interchanged:*

$$(\mathbf{a}\quad \mathbf{b}\quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \tag{11}$$

(b) Geometric interpretation. *The absolute value* $|(\mathbf{a}\quad \mathbf{b}\quad \mathbf{c})|$ *of* (10) *is the volume of the parallelepiped* (oblique box) *with* **a**, **b**, **c** *as edge vectors* (Fig. 193).

(c) Linear independence. *Three vectors in* R^3 *are linearly independent if and only if their scalar triple product is not zero.*

PROOF **(a)** Dot multiplication is commutative, so that by (10)

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

From this we obtain the determinant in (10) by interchanging Rows 1 and 2 and in the result Rows 2 and 3. But this does not change the value of the determinant because each interchange produces a factor -1, and $(-1)(-1) = 1$. This proves (11).

(b) The volume of that box equals the height $h = |\mathbf{a}||\cos\gamma|$ (Fig. 193) times the area of the base, which is the area $|\mathbf{b} \times \mathbf{c}|$ of the parallelogram with sides $\mathbf{b}$ and $\mathbf{c}$. Hence the volume is

$$|\mathbf{a}||\mathbf{b} \times \mathbf{c}|\,|\cos\gamma| = |\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})| \qquad \text{(Fig. 193)}$$

as given by the absolute value of (11).

(c) Three nonzero vectors, whose initial points coincide, are linearly independent if and only if the vectors do not lie in the same plane nor lie on the same straight line.

This happens if and only if the triple product in (b) is not zero, so that the independence criterion follows. (The case of one of the vectors being the zero vector is trivial.)

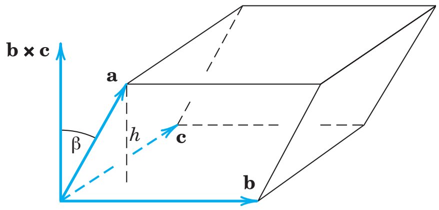


Fig. 193. Geometric interpretation of a scalar triple product

EXAMPLE 6 **Tetrahedron**

A tetrahedron is determined by three edge vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, as indicated in Fig. 194. Find the volume of the tetrahedron in Fig. 194, when $\mathbf{a} = [2, 0, 3]$, $\mathbf{b} = [0, 4, 1]$, $c = [5, 6, 0]$.

Solution. The volume V of the parallelepiped with these vectors as edge vectors is the absolute value of the scalar triple product

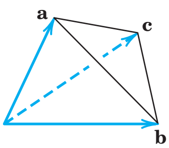


Fig. 194. Tetrahedron

$$(\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 4 & 1 \\ 5 & 6 & 0 \end{vmatrix} = 2\begin{vmatrix} 4 & 1 \\ 6 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 5 & 6 \end{vmatrix} = -12 - 60 = -72.$$

Hence $V = 72$. The minus sign indicates that if the coordinates are right-handed, the triple $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is left-handed. The volume of a tetrahedron is $\frac{1}{6}$ of that of the parallelepiped (can you prove it?), hence 12.

Can you sketch the tetrahedron, choosing the origin as the common initial point of the vectors? What are the coordinates of the four vertices? ■

This is the end of vector *algebra* (in space R^3 and in the plane). Vector *calculus* (differentiation) begins in the next section.

PROBLEM SET 9.3

1–10 GENERAL PROBLEMS

1. Give the details of the proofs of Eqs. (4) and (5).
2. What does $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ with $\mathbf{a} \neq \mathbf{0}$ imply?
3. Give the details of the proofs of Eqs. (6) and (11).
4. **Lagrange's identity for $|\mathbf{a} \times \mathbf{b}|$.** Verify it for $\mathbf{a} = [3, 4, 2]$ and $\mathbf{b} = [1, 0, 2]$. Prove it, using $\sin^2\gamma = 1 - \cos^2\gamma$. The identity is

$$\textbf{(12)} \qquad |\mathbf{a} \times \mathbf{b}| = \sqrt{(\mathbf{a} \bullet \mathbf{a})\,(\mathbf{b} \bullet \mathbf{b}) - (\mathbf{a} \bullet \mathbf{b})^2}.$$