(a) Line through two points. Derive from $D = 0$ in (12) the familiar formula

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}.$$

(b) Plane. Find the analog of (12) for a plane through three given points. Apply it when the points are (1, 1, 1), (3, 2, 6), (5, 0, 5).

(c) Circle. Find a similar formula for a circle in the plane through three given points. Find and sketch the circle through (2, 6), (6, 4), (7, 1).

(d) Sphere. Find the analog of the formula in (c) for a sphere through four given points. Find the sphere through (0, 0, 5), (4, 0, 1), (0, 4, 1), (0, 0, −3) by this formula or by inspection.

(e) General conic section. Find a formula for a general conic section (the vanishing of a determinant of 6th order). Try it out for a quadratic parabola and for a more general conic section of your own choice.

21–25 CRAMER'S RULE

Solve by Cramer's rule. Check by Gauss elimination and back substitution. Show details.

21. $\begin{aligned} 3x - 5y &= 15.5 \\ 6x + 16y &= 5.0 \end{aligned}$

22. $\begin{aligned} 2x - 4y &= -24 \\ 5x + 2y &= 0 \end{aligned}$

23. $\begin{aligned} 3y - 4z &= 16 \\ 2x - 5y + 7z &= -27 \\ -x \qquad - 9z &= 9 \end{aligned}$

24. $\begin{aligned} 3x - 2y + z &= 13 \\ -2x + y + 4z &= 11 \\ x + 4y - 5z &= -31 \end{aligned}$

25. $\begin{aligned} -4w + x + y \qquad &= -10 \\ w - 4x \qquad + z &= 1 \\ w \qquad - 4y + z &= -7 \\ x + y - 4z &= 10 \end{aligned}$

7.8 Inverse of a Matrix. Gauss–Jordan Elimination

***In this section we consider* square *matrices exclusively*.**

The **inverse** of an $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ is denoted by $\mathbf{A}^{-1}$ and is an $n \times n$ matrix such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{1}$$

where $\mathbf{I}$ is the $n \times n$ unit matrix (see Sec. 7.2).

If $\mathbf{A}$ has an inverse, then $\mathbf{A}$ is called a **nonsingular matrix**. If $\mathbf{A}$ has no inverse, then $\mathbf{A}$ is called a **singular matrix**.

If $\mathbf{A}$ *has an inverse, the inverse is unique.*

Indeed, if both $\mathbf{B}$ and $\mathbf{C}$ are inverses of $\mathbf{A}$, then $\mathbf{AB} = \mathbf{I}$ and $\mathbf{CA} = \mathbf{I}$, so that we obtain the uniqueness from

$$\mathbf{B} = \mathbf{IB} = (\mathbf{CA})\mathbf{B} = \mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C}.$$

We prove next that $\mathbf{A}$ has an inverse (is nonsingular) if and only if it has maximum possible rank n. The proof will also show that $\mathbf{Ax} = \mathbf{b}$ implies $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ provided $\mathbf{A}^{-1}$ exists, and will thus give a motivation for the inverse as well as a relation to linear systems. (But this will ***not*** give a good method of solving $\mathbf{Ax} = \mathbf{b}$ ***numerically*** because the Gauss elimination in Sec. 7.3 requires fewer computations.)

THEOREM 1

Existence of the Inverse

The inverse $\mathbf{A}^{-1}$ *of an* $n \times n$ *matrix* $\mathbf{A}$ *exists if and only if* rank $\mathbf{A} = n$, *thus* (by Theorem 3, Sec. 7.7) *if and only if* det $\mathbf{A} \neq 0$. *Hence* $\mathbf{A}$ *is nonsingular if* rank $\mathbf{A} = n$, *and is singular if* rank $\mathbf{A} < n$.

PROOF Let $\mathbf{A}$ be a given $n \times n$ matrix and consider the linear system

$$\mathbf{Ax} = \mathbf{b}. \tag{2}$$

If the inverse $\mathbf{A}^{-1}$ exists, then multiplication from the left on both sides and use of (1) gives

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

This shows that (2) has a solution $\mathbf{x}$, which is unique because, for another solution $\mathbf{u}$, we have $\mathbf{Au} = \mathbf{b}$, so that $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b} = \mathbf{x}$. Hence $\mathbf{A}$ must have rank n by the Fundamental Theorem in Sec. 7.5.

Conversely, let rank $\mathbf{A} = n$. Then by the same theorem, the system (2) has a unique solution $\mathbf{x}$ for any $\mathbf{b}$. Now the back substitution following the Gauss elimination (Sec. 7.3) shows that the components x_j of $\mathbf{x}$ are linear combinations of those of $\mathbf{b}$. Hence we can write

$$\mathbf{x} = \mathbf{Bb} \tag{3}$$

with $\mathbf{B}$ to be determined. Substitution into (2) gives

$$\mathbf{Ax} = \mathbf{A}(\mathbf{Bb}) = (\mathbf{AB})\mathbf{b} = \mathbf{Cb} = \mathbf{b} \qquad (\mathbf{C} = \mathbf{AB})$$

for any $\mathbf{b}$. Hence $\mathbf{C} = \mathbf{AB} = \mathbf{I}$, the unit matrix. Similarly, if we substitute (2) into (3) we get

$$\mathbf{x} = \mathbf{Bb} = \mathbf{B}(\mathbf{Ax}) = (\mathbf{BA})\mathbf{x}$$

for any $\mathbf{x}$ (and $\mathbf{b} = \mathbf{Ax}$). Hence $\mathbf{BA} = \mathbf{I}$. Together, $\mathbf{B} = \mathbf{A}^{-1}$ exists. ■

Determination of the Inverse by the Gauss–Jordan Method

To actually determine the inverse $\mathbf{A}^{-1}$ of a nonsingular $n \times n$ matrix $\mathbf{A}$, we can use a variant of the Gauss elimination (Sec. 7.3), called the **Gauss–Jordan elimination**.[3] The idea of the method is as follows.

Using $\mathbf{A}$, we form n linear systems

$$\mathbf{Ax}_{(1)} = \mathbf{e}_{(1)}, \quad \cdots, \quad \mathbf{Ax}_{(n)} = \mathbf{e}_{(n)}$$

where the vectors $\mathbf{e}_{(1)}, \cdots, \mathbf{e}_{(n)}$ are the columns of the $n \times n$ unit matrix $\mathbf{I}$; thus, $\mathbf{e}_{(1)} = [1 \quad 0 \quad \cdots \quad 0]^{\mathsf{T}}$, $\mathbf{e}_{(2)} = [0 \quad 1 \quad 0 \quad \cdots \quad 0]^{\mathsf{T}}$, etc. These are n vector equations in the unknown vectors $\mathbf{x}_{(1)}, \cdots, \mathbf{x}_{(n)}$. We combine them into a single matrix equation

[3] WILHELM JORDAN (1842–1899), German geodesist and mathematician. He did important geodesic work in Africa, where he surveyed oases. [See Althoen, S.C. and R. McLaughlin, Gauss–Jordan reduction: A brief history. *American Mathematical Monthly*, Vol. **94**, No. 2 (1987), pp. 130–142.]

We do ***not recommend*** it as a method for solving systems of linear equations, since the number of operations in addition to those of the Gauss elimination is larger than that for back substitution, which the Gauss–Jordan elimination avoids. See also Sec. 20.1.

$\mathbf{AX} = \mathbf{I}$, with the unknown matrix $\mathbf{X}$ having the columns $\mathbf{x}_{(1)}, \cdots, \mathbf{x}_{(n)}$. Correspondingly, we combine the n augmented matrices $[\mathbf{A} \quad \mathbf{e}_{(1)}], \cdots, [\mathbf{A} \quad \mathbf{e}_{(n)}]$ into one wide $n \times 2n$ "augmented matrix" $\widetilde{\mathbf{A}} = [\mathbf{A} \quad \mathbf{I}]$. Now multiplication of $\mathbf{AX} = \mathbf{I}$ by $\mathbf{A}^{-1}$ from the left gives $\mathbf{X} = \mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}$. Hence, to solve $\mathbf{AX} = \mathbf{I}$ for $\mathbf{X}$, we can apply the Gauss elimination to $\widetilde{\mathbf{A}} = [\mathbf{A} \quad \mathbf{I}]$. This gives a matrix of the form $[\mathbf{U} \quad \mathbf{H}]$ with upper triangular $\mathbf{U}$ because the Gauss elimination triangularizes systems. The Gauss–Jordan method reduces $\mathbf{U}$ by further elementary row operations to diagonal form, in fact to the unit matrix $\mathbf{I}$. This is done by eliminating the entries of $\mathbf{U}$ above the main diagonal and making the diagonal entries all 1 by multiplication (see Example 1). Of course, the method operates on the entire matrix $[\mathbf{U} \quad \mathbf{H}]$, transforming $\mathbf{H}$ into some matrix $\mathbf{K}$, hence the entire $[\mathbf{U} \quad \mathbf{H}]$ to $[\mathbf{I} \quad \mathbf{K}]$. This is the "augmented matrix" of $\mathbf{IX} = \mathbf{K}$. Now $\mathbf{IX} = \mathbf{X} = \mathbf{A}^{-1}$, as shown before. By comparison, $\mathbf{K} = \mathbf{A}^{-1}$, so that we can read $\mathbf{A}^{-1}$ directly from $[\mathbf{I} \quad \mathbf{K}]$.

The following example illustrates the practical details of the method.

EXAMPLE 1 Finding the Inverse of a Matrix by Gauss–Jordan Elimination

Determine the inverse $\mathbf{A}^{-1}$ of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Solution. We apply the Gauss elimination (Sec. 7.3) to the following $n \times 2n = 3 \times 6$ matrix, where BLUE always refers to the previous matrix.

$$[\mathbf{A} \quad \mathbf{I}] = \left[\begin{array}{rrr|rrr} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \text{Row 2} + 3 \text{ Row 1} \\ \text{Row 3} - \text{Row 1} \end{array}$$

$$\left[\begin{array}{rrr|rrr} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} - \text{Row 2} \end{array}$$

This is $[\mathbf{U} \quad \mathbf{H}]$ as produced by the Gauss elimination. Now follow the additional Gauss–Jordan steps, reducing $\mathbf{U}$ to $\mathbf{I}$, that is, to diagonal form with entries 1 on the main diagonal.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} -\text{Row 1} \\ 0.5 \text{ Row 2} \\ -0.2 \text{ Row 3} \end{array}$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \text{Row 1} + 2 \text{ Row 3} \\ \text{Row 2} - 3.5 \text{ Row 3} \\ \\ \end{array}$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \text{Row 1} + \text{Row 2} \\ \\ \\ \end{array}$$

The last three columns constitute $\mathbf{A}^{-1}$. Check:

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Similarly, $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. ■

Formulas for Inverses

Since finding the inverse of a matrix is really a problem of solving a system of linear equations, it is not surprising that Cramer's rule (Theorem 4, Sec. 7.7) might come into play. And similarly, as Cramer's rule was useful for theoretical study but not for computation, so too is the explicit formula (4) in the following theorem useful for theoretical considerations but not recommended for actually determining inverse matrices, except for the frequently occurring 2×2 case as given in (4*).

THEOREM 2

Inverse of a Matrix by Determinants

The inverse of a nonsingular $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ is given by

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[C_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}, \tag{4}$$

where C_{jk} is the cofactor of a_{jk} in det $\mathbf{A}$ (see Sec. 7.7). (CAUTION! Note well that in $\mathbf{A}^{-1}$, the cofactor C_{jk} occupies the same place as a_{kj} (not a_{jk}) does in $\mathbf{A}$.)

In particular, the inverse of

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \textit{is} \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}. \tag{4*}$$

PROOF We denote the right side of (4) by $\mathbf{B}$ and show that $\mathbf{BA} = \mathbf{I}$. We first write

$$\mathbf{BA} = \mathbf{G} = [g_{kl}] \tag{5}$$

and then show that $\mathbf{G} = \mathbf{I}$. Now by the definition of matrix multiplication and because of the form of $\mathbf{B}$ in (4), we obtain (CAUTION! C_{sk}, not C_{ks})

$$g_{kl} = \sum_{s=1}^{n} \frac{C_{sk}}{\det \mathbf{A}} a_{sl} = \frac{1}{\det \mathbf{A}}(a_{1l}C_{1k} + \cdots + a_{nl}C_{nk}). \tag{6}$$