Example 13

Theorem 5.28 shows the operator on $\mathbb{R}^4$ in Example 12 is not diagonalizable because the geometric multiplicity of $\lambda_1 = 3$ is 2, while its algebraic multiplicity is 3.

Example 14

Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation about the z-axis through an angle of $\frac{\pi}{3}$. Then the matrix for L with respect to the standard basis is

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

as described in Table 5.1. Using $\mathbf{A}$, we calculate $p_L(x) = x^3 - 2x^2 + 2x - 1 = (x-1)(x^2 - x + 1)$, where the quadratic factor has no real roots. Therefore, $\lambda = 1$ is the only eigenvalue, and its algebraic multiplicity is 1. Hence, by Theorem 5.28, L is not diagonalizable because the sum of the algebraic multiplicities of its eigenvalues equals 1, which is less than $\dim(\mathbb{R}^3) = 3$.

The Cayley-Hamilton Theorem

We conclude this section with an interesting relationship between a matrix and its characteristic polynomial. If $p(x) = a_n x^n + a_{n-1} x^{n-1} \cdots + a_1 x + a_0$ is any polynomial and $\mathbf{A}$ is an $n \times n$ matrix, we define $p(\mathbf{A})$ to be the $n \times n$ matrix given by $p(\mathbf{A}) = a_n \mathbf{A}^n + a_{n-1} \mathbf{A}^{n-1} \cdots + a_1 \mathbf{A} + a_0 \mathbf{I}_n$.

Theorem 5.29 (Cayley-Hamilton Theorem) Let $\mathbf{A}$ be an $n \times n$ matrix, and let $p_{\mathbf{A}}(x)$ be its characteristic polynomial. Then $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{O}_n$.

The Cayley-Hamilton Theorem is an important result in advanced linear algebra. We have placed its proof in Appendix A for the interested reader.

Example 15

Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$. Then $p_{\mathbf{A}}(x) = x^2 - 2x - 11$ (verify!). The Cayley-Hamilton Theorem states that $p_{\mathbf{A}}(\mathbf{A}) = \mathbf{O}_2$. To check this, note that

$$p_{\mathbf{A}}(\mathbf{A}) = \mathbf{A}^2 - 2\mathbf{A} - 11\mathbf{I}_2 = \begin{bmatrix} 17 & 4 \\ 8 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ 8 & -2 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$