27–35 **VECTOR SPACE**

Is the given set of vectors a vector space? Give reasons. If your answer is yes, determine the dimension and find a basis. ($v_1, v_2, \cdots$ denote components.)

27. All vectors in R^3 with $v_1 - v_2 + 2v_3 = 0$

28. All vectors in R^3 with $3v_2 + v_3 = k$

29. All vectors in R^2 with $v_1 \geqq v_2$

30. All vectors in R^n with the first $n - 2$ components zero

31. All vectors in R^5 with positive components

32. All vectors in R^3 with $3v_1 - 2v_2 + v_3 = 0$, $4v_1 + 5v_2 = 0$

33. All vectors in R^3 with $3v_1 - v_3 = 0$, $2v_1 + 3v_2 - 4v_3 = 0$

34. All vectors in R^n with $|v_j| = 1$ for $j = 1, \cdots, n$

35. All vectors in R^4 with $v_1 = 2v_2 = 3v_3 = 4v_4$

7.5 Solutions of Linear Systems: Existence, Uniqueness

Rank, as just defined, gives complete information about existence, uniqueness, and general structure of the solution set of linear systems as follows.

A linear system of equations in n unknowns has a unique solution if the coefficient matrix and the augmented matrix have the same rank n, and infinitely many solutions if that common rank is less than n. The system has no solution if those two matrices have different rank.

To state this precisely and prove it, we shall use the generally important concept of a **submatrix** of $\mathbf{A}$. By this we mean any matrix obtained from $\mathbf{A}$ by omitting some rows or columns (or both). By definition this includes $\mathbf{A}$ itself (as the matrix obtained by omitting no rows or columns); this is practical.

THEOREM 1

Fundamental Theorem for Linear Systems

(a) Existence. *A linear system of m equations in n unknowns* $x_1, \cdots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

is **consistent**, *that is, has solutions, if and only if the coefficient matrix* $\mathbf{A}$ *and the augmented matrix* $\tilde{\mathbf{A}}$ *have the same rank. Here,*

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

(b) Uniqueness. *The system* (1) *has precisely one solution if and only if this common rank r of* $\mathbf{A}$ *and* $\tilde{\mathbf{A}}$ *equals n.*

(c) Infinitely many solutions. *If this common rank r is less than n, the system* (1) *has infinitely many solutions. All of these solutions are obtained by determining r suitable unknowns* (*whose submatrix of coefficients must have rank r*) *in terms of the remaining $n - r$ unknowns, to which arbitrary values can be assigned.* (See Example 3 in Sec. 7.3.)

(d) Gauss elimination (Sec. 7.3). *If solutions exist, they can all be obtained by the Gauss elimination.* (This method will automatically reveal whether or not solutions exist; see Sec. 7.3.)

PROOF **(a)** We can write the system (1) in vector form $\mathbf{Ax} = \mathbf{b}$ or in terms of column vectors $\mathbf{c}_{(1)}, \cdots, \mathbf{c}_{(n)}$ of $\mathbf{A}$:

$$\mathbf{c}_{(1)}x_1 + \mathbf{c}_{(2)}x_2 + \cdots + \mathbf{c}_{(n)}x_n = \mathbf{b}. \tag{2}$$

$\widetilde{\mathbf{A}}$ is obtained by augmenting $\mathbf{A}$ by a single column $\mathbf{b}$. Hence, by Theorem 3 in Sec. 7.4, rank $\widetilde{\mathbf{A}}$ equals rank $\mathbf{A}$ or rank $\mathbf{A} + 1$. Now if (1) has a solution $\mathbf{x}$, then (2) shows that $\mathbf{b}$ must be a linear combination of those column vectors, so that $\widetilde{\mathbf{A}}$ and $\mathbf{A}$ have the same maximum number of linearly independent column vectors and thus the same rank.

Conversely, if rank $\widetilde{\mathbf{A}} =$ rank $\mathbf{A}$, then $\mathbf{b}$ must be a linear combination of the column vectors of $\mathbf{A}$, say,

$$\mathbf{b} = \alpha_1\mathbf{c}_{(1)} + \cdots + \alpha_n\mathbf{c}_{(n)} \tag{2*}$$

since otherwise rank $\widetilde{\mathbf{A}} =$ rank $\mathbf{A} + 1$. But (2*) means that (1) has a solution, namely, $x_1 = \alpha_1, \cdots, x_n = \alpha_n$, as can be seen by comparing (2*) and (2).

(b) If rank $\mathbf{A} = n$, the n column vectors in (2) are linearly independent by Theorem 3 in Sec. 7.4. We claim that then the representation (2) of $\mathbf{b}$ is unique because otherwise

$$\mathbf{c}_{(1)}x_1 + \cdots + \mathbf{c}_{(n)}x_n = \mathbf{c}_{(1)}\widetilde{x}_1 + \cdots + \mathbf{c}_{(n)}\widetilde{x}_n.$$

This would imply (take all terms to the left, with a minus sign)

$$(x_1 - \widetilde{x}_1)\mathbf{c}_{(1)} + \cdots + (x_n - \widetilde{x}_n)\mathbf{c}_{(n)} = \mathbf{0}$$

and $x_1 - \widetilde{x}_1 = 0, \cdots, x_n - \widetilde{x}_n = 0$ by linear independence. But this means that the scalars $x_1, \cdots, x_n$ in (2) are uniquely determined, that is, the solution of (1) is unique.

(c) If rank $\mathbf{A} =$ rank $\widetilde{\mathbf{A}} = r < n$, then by Theorem 3 in Sec. 7.4 there is a linearly independent set K of r column vectors of $\mathbf{A}$ such that the other $n - r$ column vectors of $\mathbf{A}$ are linear combinations of those vectors. We renumber the columns and unknowns, denoting the renumbered quantities by ˆ, so that $\{\hat{\mathbf{c}}_{(1)}, \cdots, \hat{\mathbf{c}}_{(r)}\}$ is that linearly independent set K. Then (2) becomes

$$\hat{\mathbf{c}}_{(1)}\hat{x}_1 + \cdots + \hat{\mathbf{c}}_{(r)}\hat{x}_r + \hat{\mathbf{c}}_{(r+1)}\hat{x}_{r+1} + \cdots + \hat{\mathbf{c}}_{(n)}\hat{x}_n = \mathbf{b},$$

$\hat{\mathbf{c}}_{(r+1)}, \cdots, \hat{\mathbf{c}}_{(n)}$ are linear combinations of the vectors of K, and so are the vectors $\hat{x}_{r+1}\hat{\mathbf{c}}_{(r+1)}, \cdots, \hat{x}_n\hat{\mathbf{c}}_{(n)}$. Expressing these vectors in terms of the vectors of K and collecting terms, we can thus write the system in the form

$$\hat{\mathbf{c}}_{(1)}y_1 + \cdots + \hat{\mathbf{c}}_{(r)}y_r = \mathbf{b} \tag{3}$$

with $y_j = \hat{x}_j + \beta_j$, where β_j results from the $n - r$ terms $\hat{\mathbf{c}}_{(r+1)}\hat{x}_{r+1}, \cdots, \hat{\mathbf{c}}_{(n)}\hat{x}_n$; here, $j = 1, \cdots, r$. Since the system has a solution, there are $y_1, \cdots, y_r$ satisfying (3). These scalars are unique since K is linearly independent. Choosing $\hat{x}_{r+1}, \cdots, \hat{x}_n$ fixes the β_j and corresponding $\hat{x}_j = y_j - \beta_j$, where $j = 1, \cdots, r$.

(d) This was discussed in Sec. 7.3 and is restated here as a reminder. ■

The theorem is illustrated in Sec. 7.3. In Example 2 there is a unique solution since rank $\tilde{\mathbf{A}}$ = rank $\mathbf{A} = n = 3$ (as can be seen from the last matrix in the example). In Example 3 we have rank $\tilde{\mathbf{A}}$ = rank $\mathbf{A} = 2 < n = 4$ and can choose x_3 and x_4 arbitrarily. In Example 4 there is no solution because rank $\mathbf{A} = 2 <$ rank $\tilde{\mathbf{A}} = 3$.

Homogeneous Linear System

Recall from Sec. 7.3 that a linear system (1) is called **homogeneous** if all the b_j's are zero, and **nonhomogeneous** if one or several b_j's are not zero. For the homogeneous system we obtain from the Fundamental Theorem the following results.

THEOREM 2

Homogeneous Linear System

A homogeneous linear system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
\tag{4}
$$

always has the **trivial solution** $x_1 = 0, \cdots, x_n = 0$. *Nontrivial solutions exist if and only if rank* $\mathbf{A} < n$. *If* rank $\mathbf{A} = r < n$, *these solutions, together with* $\mathbf{x} = \mathbf{0}$, *form a vector space* (see Sec. 7.4) *of dimension* $n - r$ called the **solution space** of (4).

In particular, if $\mathbf{x}_{(1)}$ *and* $\mathbf{x}_{(2)}$ *are solution vectors of* (4), *then* $\mathbf{x} = c_1\mathbf{x}_{(1)} + c_2\mathbf{x}_{(2)}$ *with any scalars* c_1 *and* c_2 *is a solution vector of* (4). (This ***does not hold*** for nonhomogeneous systems. Also, the term *solution space is used for homogeneous* systems only.)

PROOF The first proposition can be seen directly from the system. It agrees with the fact that $\mathbf{b} = \mathbf{0}$ implies that rank $\tilde{\mathbf{A}}$ = rank $\mathbf{A}$, so that a homogeneous system is always ***consistent***. If rank $\mathbf{A} = n$, the trivial solution is the unique solution according to (b) in Theorem 1. If rank $\mathbf{A} < n$, there are nontrivial solutions according to (c) in Theorem 1. The solutions form a vector space because if $\mathbf{x}_{(1)}$ and $\mathbf{x}_{(2)}$ are any of them, then $\mathbf{A}\mathbf{x}_{(1)} = \mathbf{0}$, $\mathbf{A}\mathbf{x}_{(2)} = \mathbf{0}$, and this implies $\mathbf{A}(\mathbf{x}_{(1)} + \mathbf{x}_{(2)}) = \mathbf{A}\mathbf{x}_{(1)} + \mathbf{A}\mathbf{x}_{(2)} = \mathbf{0}$ as well as $\mathbf{A}(c\mathbf{x}_{(1)}) = c\mathbf{A}\mathbf{x}_{(1)} = \mathbf{0}$, where c is arbitrary. If rank $\mathbf{A} = r < n$, Theorem 1 (c) implies that we can choose $n - r$ suitable unknowns, call them $x_{r+1}, \cdots, x_n$, in an arbitrary fashion, and every solution is obtained in this way. Hence a basis for the solution space, briefly called a **basis of solutions** of (4), is $\mathbf{y}_{(1)}, \cdots, \mathbf{y}_{(n-r)}$, where the basis vector $\mathbf{y}_{(j)}$ is obtained by choosing $x_{r+j} = 1$ and the other $x_{r+1}, \cdots, x_n$ zero; the corresponding first r components of this solution vector are then determined. Thus the solution space of (4) has dimension $n - r$. This proves Theorem 2. ■

The solution space of (4) is also called the **null space** of $\mathbf{A}$ because $\mathbf{Ax} = \mathbf{0}$ for every $\mathbf{x}$ in the solution space of (4). Its dimension is called the **nullity** of $\mathbf{A}$. Hence Theorem 2 states that

$$\text{(5)} \qquad \operatorname{rank} \mathbf{A} + \operatorname{nullity} \mathbf{A} = n$$

where n is the number of unknowns (number of columns of $\mathbf{A}$).

Furthermore, by the definition of rank we have rank $\mathbf{A} \leqq m$ in (4). Hence if $m < n$, then rank $\mathbf{A} < n$. By Theorem 2 this gives the practically important

THEOREM 3

Homogeneous Linear System with Fewer Equations Than Unknowns

A homogeneous linear system with fewer equations than unknowns always has nontrivial solutions.

Nonhomogeneous Linear Systems

The characterization of all solutions of the linear system (1) is now quite simple, as follows.

THEOREM 4

Nonhomogeneous Linear System

If a nonhomogeneous linear system (1) *is consistent, then all of its solutions are obtained as*

$$\text{(6)} \qquad \mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h$$

where $\mathbf{x}_0$ *is any* (fixed) *solution of* (1) *and* $\mathbf{x}_h$ *runs through all the solutions of the corresponding homogeneous system* (4).

PROOF The difference $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_0$ of any two solutions of (1) is a solution of (4) because $\mathbf{Ax}_h = \mathbf{A}(\mathbf{x} - \mathbf{x}_0) = \mathbf{Ax} - \mathbf{Ax}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$. Since $\mathbf{x}$ is any solution of (1), we get all the solutions of (1) if in (6) we take any solution $\mathbf{x}_0$ of (1) and let $\mathbf{x}_h$ vary throughout the solution space of (4). ■

This covers a main part of our discussion of characterizing the solutions of systems of linear equations. Our next main topic is determinants and their role in linear equations.

7.6 For Reference: Second- and Third-Order Determinants

We created this section as a quick general reference section on second- and third-order determinants. It is completely independent of the theory in Sec. 7.7 and suffices as a reference for many of our examples and problems. Since this section is for reference, ***go on to the next section, consulting this material only when needed***.

A **determinant of second order** is denoted and defined by

$$\text{(1)} \qquad D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

So here we have ***bars*** (whereas a matrix has ***brackets***).