9. PROJECT. Useful Formulas for the Divergence. Prove

(a) $\operatorname{div}(k\mathbf{v}) = k \operatorname{div} \mathbf{v}$ (k constant)

(b) $\operatorname{div}(f\mathbf{v}) = f \operatorname{div} \mathbf{v} + \mathbf{v} \bullet \nabla f$

(c) $\operatorname{div}(f\nabla g) = f\nabla^2 g + \nabla f \bullet \nabla g$

(d) $\operatorname{div}(f\nabla g) - \operatorname{div}(g\nabla f) = f\nabla^2 g - g\nabla^2 f$

Verify (b) for $f = e^{xyz}$ and $\mathbf{v} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$. Obtain the answer to Prob. 6 from (b). Verify (c) for $f = x^2 - y^2$ and $g = e^{x+y}$. Give examples of your own for which (a)–(d) are advantageous.

10. CAS EXPERIMENT. Visualizing the Divergence. Graph the given velocity field $\mathbf{v}$ of a fluid flow in a square centered at the origin with sides parallel to the coordinate axes. Recall that the divergence measures outflow minus inflow. By looking at the flow near the sides of the square, can you see whether div $\mathbf{v}$ must be positive or negative or may perhaps be zero? Then calculate div $\mathbf{v}$. First do the given flows and then do some of your own. Enjoy it.

(a) $\mathbf{v} = \mathbf{i}$

(b) $\mathbf{v} = x\mathbf{i}$

(c) $\mathbf{v} = x\mathbf{i} - y\mathbf{j}$

(d) $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$

(e) $\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$

(f) $\mathbf{v} = (x^2 + y^2)^{-1}(-y\mathbf{i} + x\mathbf{j})$

11. Incompressible flow. Show that the flow with velocity vector $\mathbf{v} = y\mathbf{i}$ is incompressible. Show that the particles that at time $t = 0$ are in the cube whose faces are portions of the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ occupy at $t = 1$ the volume 1.

12. Compressible flow. Consider the flow with velocity vector $\mathbf{v} = x\mathbf{i}$. Show that the individual particles have the position vectors $\mathbf{r}(t) = c_1 e^t \mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ with constant c_1, c_2, c_3. Show that the particles that at $t = 0$ are in the cube of Prob. 11 at $t = 1$ occupy the volume e.

13. Rotational flow. The velocity vector $\mathbf{v}(x, y, z)$ of an incompressible fluid rotating in a cylindrical vessel is of the form $\mathbf{v} = \mathbf{w} \times \mathbf{r}$, where $\mathbf{w}$ is the (constant) rotation vector; see Example 5 in Sec. 9.3. Show that div $\mathbf{v} = 0$. Is this plausible because of our present Example 2?

14. Does $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{v}$ imply $\mathbf{u} = \mathbf{v}$ or $\mathbf{u} = \mathbf{v} + \mathbf{k}$ ($\mathbf{k}$ constant)? Give reason.

15–20 LAPLACIAN

Calculate $\nabla^2 f$ by Eq. (3). Check by direct differentiation. Indicate when (3) is simpler. Show the details of your work.

15. $f = \cos^2 x + \sin^2 y$

16. $f = e^{xyz}$

17. $f = \ln(x^2 + y^2)$

18. $f = z - \sqrt{x^2 + y^2}$

19. $f = 1/(x^2 + y^2 + z^2)$

20. $f = e^{2x} \cosh 2y$

9.9 Curl of a Vector Field

The concepts of gradient (Sec. 9.7), divergence (Sec. 9.8), and curl are of fundamental importance in vector calculus and frequently applied in vector fields. In this section we define and discuss the concept of the curl and apply it to several engineering problems.

Let $\mathbf{v}(x, y, z) = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a differentiable vector function of the Cartesian coordinates x, y, z. Then the **curl** *of the vector function* $\mathbf{v}$ or *of the vector field given by* $\mathbf{v}$ is defined by the "symbolic" determinant

$$\text{(1)} \qquad \operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}.$$

This is the formula when x, y, z are *right-handed.* If they are *left-handed*, the determinant has a minus sign in front (just as in (2**) in Sec. 9.3).

Instead of curl $\mathbf{v}$ one also uses the notation rot $\mathbf{v}$. This is suggested by "rotation," an application explored in Example 2. Note that curl $\mathbf{v}$ is a vector, as shown in Theorem 3.

EXAMPLE 1 **Curl of a Vector Function**

Let $\mathbf{v} = [yz, \quad 3zx, \quad z] = yz\mathbf{i} + 3zx\mathbf{j} + z\mathbf{k}$ with right-handed x, y, z. Then (1) gives

$$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = -3x\mathbf{i} + y\mathbf{j} + (3z - z)\mathbf{k} = -3x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}.$$ ■

The curl has many applications. A typical example follows. More about the nature and significance of the curl will be considered in Sec. 10.9.

EXAMPLE 2 **Rotation of a Rigid Body. Relation to the Curl**

We have seen in Example 5, Sec. 9.3, that a rotation of a rigid body B about a fixed axis in space can be described by a vector $\mathbf{w}$ of magnitude ω in the direction of the axis of rotation, where ω (>0) is the angular speed of the rotation, and $\mathbf{w}$ is directed so that the rotation appears clockwise if we look in the direction of $\mathbf{w}$. According to (9), Sec. 9.3, the velocity field of the rotation can be represented in the form

$$\mathbf{v} = \mathbf{w} \times \mathbf{r}$$

where $\mathbf{r}$ is the position vector of a moving point with respect to a Cartesian coordinate system *having the origin on the axis of rotation*. Let us choose right-handed Cartesian coordinates such that the axis of rotation is the z-axis. Then (see Example 2 in Sec. 9.4)

$$\mathbf{w} = [0, \quad 0, \quad \omega] = \omega\mathbf{k}, \qquad \mathbf{v} = \mathbf{w} \times \mathbf{r} = [-\omega y, \quad \omega x, \quad 0] = -\omega y\mathbf{i} + \omega x\mathbf{j}.$$

Hence

$$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = [0, \quad 0, \quad 2\omega] = 2\omega\mathbf{k} = 2\mathbf{w}.$$

This proves the following theorem. ■

THEOREM 1 **Rotating Body and Curl**

The curl of the velocity field of a rotating rigid body has the direction of the axis of the rotation, and its magnitude equals twice the angular speed of the rotation.

Next we show how the grad, div, and curl are interrelated, thereby shedding further light on the nature of the curl.

THEOREM 2

Grad, Div, Curl

Gradient fields are ***irrotational***. *That is, if a continuously differentiable vector function is the gradient of a scalar function f, then its curl is the zero vector,*

$$\text{curl}\,(\text{grad}\, f) = \mathbf{0}. \qquad \textbf{(2)}$$

Furthermore, the divergence of the curl of a twice continuously differentiable vector function $\mathbf{v}$ *is zero,*

$$\text{div}\,(\text{curl}\ \mathbf{v}) = 0. \qquad \textbf{(3)}$$

PROOF Both (2) and (3) follow directly from the definitions by straightforward calculation. In the proof of (3) the six terms cancel in pairs. ■

EXAMPLE 3 **Rotational and Irrotational Fields**

The field in Example 2 is not irrotational. A similar velocity field is obtained by stirring tea or coffee in a cup. The gravitational field in Theorem 3 of Sec. 9.7 has curl $\mathbf{p} = \mathbf{0}$. It is an irrotational gradient field. ■

The term "irrotational" for curl $\mathbf{v} = \mathbf{0}$ is suggested by the use of the curl for characterizing the rotation in a field. If a gradient field occurs elsewhere, not as a velocity field, it is usually called **conservative** (see Sec. 9.7). Relation (3) is plausible because of the interpretation of the curl as a rotation and of the divergence as a flux (see Example 2 in Sec. 9.8).

Finally, since the curl is defined in terms of coordinates, we should do what we did for the gradient in Sec. 9.7, namely, to find out whether the curl is a vector. This is true, as follows.

THEOREM 3

Invariance of the Curl

curl $\mathbf{v}$ is a vector. It has a length and a direction that are independent of the particular choice of a Cartesian coordinate system in space.

PROOF The proof is quite involved and shown in App. 4.

We have completed our discussion of vector differential calculus. The companion Chap. 10 on vector integral calculus follows and makes use of many concepts covered in this chapter, including dot and cross products, parametric representation of curves C, along with grad, div, and curl.

PROBLEM SET 9.9

1. **WRITING REPORT. Grad, div, curl.** List the definitions and most important facts and formulas for grad, div, curl, and ∇^2. Use your list to write a corresponding report of 3–4 pages, with examples of your own. No proofs.
2. **(a)** What direction does curl $\mathbf{v}$ have if $\mathbf{v}$ is parallel to the yz-plane? **(b)** If, moreover, $\mathbf{v}$ is independent of x?
3. Prove Theorem 2. Give two examples for (2) and (3) each.

4–8 CALCULUTION OF CURL

Find curl **v** for **v** given with respect to right-handed Cartesian coordinates. Show the details of your work.

4. $\mathbf{v} = [2y^2, 5x, 0]$

5. $\mathbf{v} = xyz[x, y, z]$

6. $\mathbf{v} = (x^2 + y^2 + z^2)^{-3/2}[x, y, z]$

7. $\mathbf{v} = [0, 0, e^{-x} \sin y]$

8. $\mathbf{v} = [e^{-z^2}, e^{-x^2}, e^{-y^2}]$

9–13 FLUID FLOW

Let **v** be the velocity vector of a steady fluid flow. Is the flow irrotational? Incompressible? Find the streamlines (the paths of the particles). *Hint.* See the answers to Probs. 9 and 11 for a determination of a path.

9. $\mathbf{v} = [0, 3z^2, 0]$

10. $\mathbf{v} = [\sec x, \csc x, 0]$

11. $\mathbf{v} = [y, -2x, 0]$

12. $\mathbf{v} = [-y, x, \pi]$

13. $\mathbf{v} = [x, y, -z]$

14. PROJECT. Useful Formulas for the Curl. Assuming sufficient differentiability, show that

(a) $\operatorname{curl}(\mathbf{u} + \mathbf{v}) = \operatorname{curl}\mathbf{u} + \operatorname{curl}\mathbf{v}$

(b) $\operatorname{div}(\operatorname{curl}\mathbf{v}) = 0$

(c) $\operatorname{curl}(f\mathbf{v}) = (\operatorname{grad} f) \times \mathbf{v} + f \operatorname{curl}\mathbf{v}$

(d) $\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$

(e) $\operatorname{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \operatorname{curl}\mathbf{u} - \mathbf{u} \cdot \operatorname{curl}\mathbf{v}$

15–20 DIV AND CURL

With respect to right-handed coordinates, let $\mathbf{u} = [y, z, x]$, $\mathbf{v} = [yz, zx, xy]$, $f = xyz$, and $g = x + y + z$. Find the given expressions. Check your result by a formula in Proj. 14 if applicable.

15. $\operatorname{curl}(\mathbf{u} + \mathbf{v})$, $\operatorname{curl}\mathbf{v}$

16. $\operatorname{curl}(g\mathbf{v})$

17. $\mathbf{v} \cdot \operatorname{curl}\mathbf{u}$, $\mathbf{u} \cdot \operatorname{curl}\mathbf{v}$, $\mathbf{u} \cdot \operatorname{curl}\mathbf{u}$

18. $\operatorname{div}(\mathbf{u} \times \mathbf{v})$

19. $\operatorname{curl}(g\mathbf{u} + \mathbf{v})$, $\operatorname{curl}(g\mathbf{u})$

20. $\operatorname{div}(\operatorname{grad}(fg))$

CHAPTER 9 REVIEW QUESTIONS AND PROBLEMS

1. What is a vector? A vector function? A vector field? A scalar? A scalar function? A scalar field? Give examples.

2. What is an inner product, a vector product, a scalar triple product? What applications motivate these products?

3. What are right-handed and left-handed coordinates? When is this distinction important?

4. When is a vector product the zero vector? What is orthogonality?

5. How is the derivative of a vector function defined? What is its significance in geometry and mechanics?

6. If $\mathbf{r}(t)$ represents a motion, what are $\mathbf{r}'(t)$, $|\mathbf{r}'(t)|$, $\mathbf{r}''(t)$, and $|\mathbf{r}''(t)|$?

7. Can a moving body have constant speed but variable velocity? Nonzero acceleration?

8. What do you know about directional derivatives? Their relation to the gradient?

9. Write down the definitions and explain the significance of grad, div, and curl.

10. Granted sufficient differentiability, which of the following expressions make sense? $f \operatorname{curl}\mathbf{v}$, $\mathbf{v} \operatorname{curl} f$, $\mathbf{u} \times \mathbf{v}$, $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$, $f \cdot \mathbf{v}$, $f \cdot (\mathbf{v} \times \mathbf{w})$, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$, $\mathbf{v} \times \operatorname{curl}\mathbf{v}$, $\operatorname{div}(f\mathbf{v})$, $\operatorname{curl}(f\mathbf{v})$, and $\operatorname{curl}(f \cdot \mathbf{v})$.

11–19 ALGEBRAIC OPERATIONS FOR VECTORS

Let $\mathbf{a} = [4, 7, 0]$, $\mathbf{b} = [3, -1, 5]$, $\mathbf{c} = [-6, 2, 0]$, and $\mathbf{d} = [1, -2, 8]$. Calculate the following expressions. Try to make a sketch.

11. $\mathbf{a} \cdot \mathbf{c}$, $\quad 3\mathbf{b} \cdot 8\mathbf{d}$, $\quad 24\mathbf{d} \cdot \mathbf{b}$, $\quad \mathbf{a} \cdot \mathbf{a}$

12. $\mathbf{a} \times \mathbf{c}$, $\quad \mathbf{b} \times \mathbf{d}$, $\quad \mathbf{d} \times \mathbf{b}$, $\quad \mathbf{a} \times \mathbf{a}$

13. $\mathbf{b} \times \mathbf{c}$, $\quad \mathbf{c} \times \mathbf{b}$, $\quad \mathbf{c} \times \mathbf{c}$, $\quad \mathbf{c} \cdot \mathbf{c}$

14. $5(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, $\quad \mathbf{a} \cdot (5\mathbf{b} \times \mathbf{c})$, $\quad (5\mathbf{a}\ \mathbf{b}\ \mathbf{c})$, $\quad 5(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$

15. $6(\mathbf{a} \times \mathbf{b}) \times \mathbf{d}$, $\quad \mathbf{a} \times 6(\mathbf{b} \times \mathbf{d})$, $\quad 2\mathbf{a} \times 3\mathbf{b} \times \mathbf{d}$

16. $(1/|\mathbf{a}|)\mathbf{a}$, $\quad (1/|\mathbf{b}|)\mathbf{b}$, $\quad \mathbf{a} \cdot \mathbf{b}/|\mathbf{b}|$, $\quad \mathbf{a} \cdot \mathbf{b}/|\mathbf{a}|$

17. $(\mathbf{a}\ \mathbf{b}\ \mathbf{d})$, $\quad (\mathbf{b}\ \mathbf{a}\ \mathbf{d})$, $\quad (\mathbf{b}\ \mathbf{d}\ \mathbf{a})$

18. $|\mathbf{a} + \mathbf{b}|$, $\quad |\mathbf{a}| + |\mathbf{b}|$

19. $\mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{a}$, $\quad (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{c}$, $\quad |\mathbf{a} \times \mathbf{b}|$

20. Commutativity. When is $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$? When is $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$?

21. Resultant, equilibrium. Find **u** such that **u** and **a**, **b**, **c**, **d** above and **u** are in equilibrium.

22. Resultant. Find the most general **v** such that the resultant of **v**, **a**, **b**, **c** (see above) is parallel to the *yz*-plane.

23. Angle. Find the angle between **a** and **c**. Between **b** and **d**. Sketch **a** and **c**.

24. Planes. Find the angle between the two planes P_1: $4x - y + 3z = 12$ and P_2: $x + 2y + 4z = 4$. Make a sketch.

25. Work. Find the work done by $\mathbf{q} = [5, 2, 0]$ in the displacement from (1, 1, 0) to (4, 3, 0).

26. Component. When is the component of a vector **v** in the direction of a vector **w** equal to the component of **w** in the direction of **v**?

27. Component. Find the component of $\mathbf{v} = [4, 7, 0]$ in the direction of $\mathbf{w} = [2, 2, 0]$. Sketch it.

28. Moment. When is the moment of a force equal to zero?

29. Moment. A force $\mathbf{p} = [4, 2, 0]$ is acting in a line through $(2, 3, 0)$. Find its moment vector about the center $(5, 1, 0)$ of a wheel.

30. Velocity, acceleration. Find the velocity, speed, and acceleration of the motion given by $\mathbf{r}(t) = [3 \cos t, 3 \sin t, 4t]$ $(t = \text{time})$ at the point $P: (3/\sqrt{2}, 3/\sqrt{2}, \pi)$.

31. Tetrahedron. Find the volume if the vertices are $(0, 0, 0)$, $(3, 1, 2)$, $(2, 4, 0)$, $(5, 4, 0)$.

32–40 GRAD, DIV, CURL, ∇^2, $D_v f$

Let $f = xy - yz$, $\mathbf{v} = [2y, 2z, 4x + z]$, and $\mathbf{w} = [3z^2, x^2 - y^2, y^2]$. Find:

32. grad f and f grad f at P: $(2, 7, 0)$

33. div $\mathbf{v}$, div $\mathbf{w}$

34. curl $\mathbf{v}$, curl $\mathbf{w}$

35. div (grad f), $\nabla^2 f$, $\nabla^2(xyf)$

36. (curl $\mathbf{w}$) $\cdot$ $\mathbf{v}$ at $(4, 0, 2)$

37. grad (div $\mathbf{w}$)

38. $D_v f$ at P: $(1, 1, 2)$

39. $D_w f$ at P: $(3, 0, 2)$

40. $\mathbf{v} \cdot ((\text{curl } \mathbf{w}) \times \mathbf{v})$

SUMMARY OF CHAPTER 9
Vector Differential Calculus. Grad, Div, Curl

All vectors of the form $\mathbf{a} = [a_1, a_2, a_3] = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ constitute the **real vector space** R^3 with componentwise vector addition

$$[a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1 + b_1, a_2 + b_2, a_3 + b_3] \tag{1}$$

and componentwise scalar multiplication (c a scalar, a real number)

$$c[a_1, a_2, a_3] = [ca_1, ca_2, ca_3] \qquad \text{(Sec. 9.1).} \tag{2}$$

For instance, the *resultant* of forces $\mathbf{a}$ and $\mathbf{b}$ is the sum $\mathbf{a} + \mathbf{b}$.

The **inner product** or **dot product** of two vectors is defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \gamma = a_1b_1 + a_2b_2 + a_3b_3 \qquad \text{(Sec. 9.2)} \tag{3}$$

where γ is the angle between $\mathbf{a}$ and $\mathbf{b}$. This gives for the **norm** or **length** $|\mathbf{a}|$ of $\mathbf{a}$

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2} \tag{4}$$

as well as a formula for γ. If $\mathbf{a} \cdot \mathbf{b} = 0$, we call $\mathbf{a}$ and $\mathbf{b}$ **orthogonal**. The dot product is suggested by the *work* $W = \mathbf{p} \cdot \mathbf{d}$ done by a force $\mathbf{p}$ in a displacement $\mathbf{d}$.

The **vector product** or **cross product** $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is a vector of length

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \gamma \qquad \text{(Sec. 9.3)} \tag{5}$$

and perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$ form a *right-handed* triple. In terms of components with respect to right-handed coordinates,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{(Sec. 9.3).} \tag{6}$$

The vector product is suggested, for instance, by moments of forces or by rotations. CAUTION! This multiplication is *anti*commutative, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, and is *not* associative.

An (oblique) box with edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ has volume equal to the absolute value of the **scalar triple product**

$$(\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \tag{7}$$

Sections 9.4–9.9 extend differential calculus to vector functions

$$\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)] = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}$$

and to vector functions of more than one variable (see below). The derivative of $\mathbf{v}(t)$ is

$$\mathbf{v}' = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = [v_1', v_2', v_3'] = v_1'\mathbf{i} + v_2'\mathbf{j} + v_3'\mathbf{k}. \tag{8}$$

Differentiation rules are as in calculus. They imply (Sec. 9.4)

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}', \qquad (\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'.$$

Curves C in space represented by the position vector $\mathbf{r}(t)$ have $\mathbf{r}'(t)$ as a **tangent vector** (the **velocity** in mechanics when t is time), $\mathbf{r}'(s)$ (s arc length, Sec. 9.5) as the *unit tangent vector,* and $|\mathbf{r}''(s)| = \kappa$ as the *curvature* (the *acceleration* in mechanics).

Vector functions $\mathbf{v}(x, y, z) = [v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)]$ represent vector fields in space. Partial derivatives with respect to the Cartesian coordinates x, y, z are obtained componentwise, for instance,

$$\frac{\partial \mathbf{v}}{\partial x} = \left[\frac{\partial v_1}{\partial x}, \frac{\partial v_2}{\partial x}, \frac{\partial v_3}{\partial x}\right] = \frac{\partial v_1}{\partial x}\mathbf{i} + \frac{\partial v_2}{\partial x}\mathbf{j} + \frac{\partial v_3}{\partial x}\mathbf{k} \qquad \text{(Sec. 9.6).}$$

The **gradient** of a scalar function f is

$$\operatorname{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \qquad \text{(Sec. 9.7).} \tag{9}$$

The **directional derivative** of f in the direction of a vector $\mathbf{a}$ is

$$D_{\mathbf{a}} f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|} \mathbf{a} \cdot \nabla f \qquad \text{(Sec. 9.7).} \tag{10}$$

The **divergence** of a vector function $\mathbf{v}$ is

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}. \qquad \text{(Sec. 9.8).} \tag{11}$$