7.5 INNER PRODUCT SPACES

Prerequisite: Section 6.3, Orthogonal Diagonalization

In $\mathbb{R}^n$ and $\mathbb{C}^n$, we have the dot product along with the operations of vector addition and scalar multiplication. In other vector spaces, we can often create a similar type of product, known as an inner product.

Inner Products

Definition Let $\mathcal{V}$ be a real [complex] vector space with operations $+$ and $\cdot$, and let $\langle\,,\rangle$ be an operation that assigns to each pair of vectors $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ a real [complex] number, denoted $\langle \mathbf{x}, \mathbf{y}\rangle$. Then $\langle\,,\rangle$ is a **real** [**complex**] **inner product** for $\mathcal{V}$ if and only if the following properties hold for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and all $k \in \mathbb{R}$ [$k \in \mathbb{C}$]:

(1) $\langle \mathbf{x}, \mathbf{x}\rangle$ is always real, and $\langle \mathbf{x}, \mathbf{x}\rangle \geq 0$

(2) $\langle \mathbf{x}, \mathbf{x}\rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$

(3) $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{y}, \mathbf{x}\rangle$ $\left[\langle \mathbf{x}, \mathbf{y}\rangle = \overline{\langle \mathbf{y}, \mathbf{x}\rangle}\right]$

(4) $\langle \mathbf{x}+\mathbf{y}, \mathbf{z}\rangle = \langle \mathbf{x}, \mathbf{z}\rangle + \langle \mathbf{y}, \mathbf{z}\rangle$

(5) $\langle k\mathbf{x}, \mathbf{y}\rangle = k\,\langle \mathbf{x}, \mathbf{y}\rangle$.

A vector space together with a real [complex] inner product operation is known as a **real** [**complex**] **inner product space**.

Example 1

Consider the real vector space $\mathbb{R}^n$. Let $\mathbf{x} = [x_1, \ldots, x_n]$ and $\mathbf{y} = [y_1, \ldots, y_n]$ be vectors in $\mathbb{R}^n$. By Theorem 1.5, the operation $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}\cdot\mathbf{y} = x_1y_1 + \cdots + x_ny_n$ (usual real dot product) is a real inner product (verify!). Hence, $\mathbb{R}^n$ together with the dot product is a real inner product space.

Similarly, let $\mathbf{x} = [x_1, \ldots, x_n]$ and $\mathbf{y} = [y_1, \ldots, y_n]$ be vectors in the complex vector space $\mathbb{C}^n$. By Theorem 7.1, the operation $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}\cdot\mathbf{y} = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$ (usual complex dot product) is an inner product on $\mathbb{C}^n$. Thus, $\mathbb{C}^n$ together with the complex dot product is a complex inner product space.

Example 2

Consider the real vector space $\mathbb{R}^2$. For $\mathbf{x} = [x_1, x_2]$ and $\mathbf{y} = [y_1, y_2]$ in $\mathbb{R}^2$, define $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$. We verify the five properties in the definition of an inner product space.

Property (1): $\langle \mathbf{x}, \mathbf{x}\rangle = x_1x_1 - x_1x_2 - x_2x_1 + 2x_2x_2 = x_1^2 - 2x_1x_2 + x_2^2 + x_2^2 = (x_1 - x_2)^2 + x_2^2 \geq 0$.

Property (2): $\langle \mathbf{x}, \mathbf{x}\rangle = 0$ exactly when $x_1 = x_2 = 0$ (that is, when $\mathbf{x} = \mathbf{0}$).

Property (3): $\langle \mathbf{y}, \mathbf{x} \rangle = y_1x_1 - y_1x_2 - y_2x_1 + 2y_2x_2 = x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 = \langle \mathbf{x}, \mathbf{y} \rangle$.

Property (4): Let $\mathbf{z} = [z_1, z_2]$. Then

$$\begin{aligned}
\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle &= (x_1 + y_1)z_1 - (x_1 + y_1)z_2 - (x_2 + y_2)z_1 + 2(x_2 + y_2)z_2 \\
&= x_1z_1 + y_1z_1 - x_1z_2 - y_1z_2 - x_2z_1 - y_2z_1 + 2x_2z_2 + 2y_2z_2 \\
&= (x_1z_1 - x_1z_2 - x_2z_1 + 2x_2z_2) + (y_1z_1 - y_1z_2 - y_2z_1 + 2y_2z_2) \\
&= \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle.
\end{aligned}$$

Property (5): $\langle k\mathbf{x}, \mathbf{y} \rangle = (kx_1)y_1 - (kx_1)y_2 - (kx_2)y_1 + 2(kx_2)y_2 = k(x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2) = k\langle \mathbf{x}, \mathbf{y} \rangle$.

Hence, $\langle , \rangle$ is a real inner product on $\mathbb{R}^2$, and $\mathbb{R}^2$ together with this operation $\langle , \rangle$ is a real inner product space. ■

Example 3

Consider the real vector space $\mathbb{R}^n$. Let $\mathbf{A}$ be a nonsingular $n \times n$ real matrix. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and define $\langle \mathbf{x}, \mathbf{y} \rangle = (\mathbf{A}\mathbf{x}) \cdot (\mathbf{A}\mathbf{y})$ (the usual dot product of $\mathbf{A}\mathbf{x}$ and $\mathbf{A}\mathbf{y}$). It can be shown (see Exercise 1) that $\langle , \rangle$ is a real inner product on $\mathbb{R}^n$, and so $\mathbb{R}^n$ together with this operation $\langle , \rangle$ is a real inner product space. ■

Example 4

Consider the real vector space $\mathcal{P}_n$. Let $\mathbf{p}_1 = a_nx^n + \cdots + a_1x + a_0$ and $\mathbf{p}_2 = b_nx^n + \cdots + b_1x + b_0$ be in $\mathcal{P}_n$. Define $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = a_nb_n + \cdots + a_1b_1 + a_0b_0$. It can be shown (see Exercise 2) that $\langle , \rangle$ is a real inner product on $\mathcal{P}_n$, and so $\mathcal{P}_n$ together with this operation $\langle , \rangle$ is a real inner product space. ■

Example 5

Let $a, b \in \mathbb{R}$, with $a < b$, and consider the real vector space $\mathcal{V}$ of all real-valued continuous functions defined on the interval $[a, b]$ (for example, polynomials, $\sin x$, e^x). Let $\mathbf{f}, \mathbf{g} \in \mathcal{V}$. Define $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b \mathbf{f}(t)\mathbf{g}(t)\,dt$. It can be shown (see Exercise 3) that $\langle , \rangle$ is a real inner product on $\mathcal{V}$, and so $\mathcal{V}$ together with this operation $\langle , \rangle$ is a real inner product space.

Analogously, the operation $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b \mathbf{f}(t)\overline{\mathbf{g}(t)}\,dt$ makes the complex vector space of all complex-valued continuous functions on $[a, b]$ into a complex inner product space. ■

Of course, not every operation is an inner product. For example, for the vectors $\mathbf{x} = [x_1, x_2]$ and $\mathbf{y} = [y_1, y_2]$ in $\mathbb{R}^2$, consider the operation $\langle \mathbf{x}, \mathbf{y} \rangle = x_1^2 + y_1^2$. Now, with $\mathbf{x} = \mathbf{y} = [1, 0]$, we have $\langle 2\mathbf{x}, \mathbf{y} \rangle = 2^2 + 1^2 = 5$, but $2\langle \mathbf{x}, \mathbf{y} \rangle = 2(1^2 + 1^2) = 4$. Thus, property (5) fails to hold.

The next theorem lists some useful results for inner product spaces.

Theorem 7.12 Let $\mathcal{V}$ be a real [complex] inner product space with inner product $\langle\,,\rangle$. Then for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and all $k \in \mathbb{R}$ [$k \in \mathbb{C}$], we have

(1) $\langle \mathbf{0}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{0} \rangle = 0$

(2) $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$

(3) $\langle \mathbf{x}, k\mathbf{y} \rangle = k \langle \mathbf{x}, \mathbf{y} \rangle \quad [\langle \mathbf{x}, k\mathbf{y} \rangle = \overline{k} \langle \mathbf{x}, \mathbf{y} \rangle]$.

Note the use of $\overline{k}$ in part (3) for complex vector spaces. The proof of this theorem is straightforward, and parts are left for you to do in Exercise 5.

Length, Distance, and Angles in Inner Product Spaces

The next definition extends the concept of the length of a vector to any inner product space.

Definition If $\mathbf{x}$ is a vector in an inner product space, then the **norm (length)** of $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

This definition yields a nonnegative real number for $\|\mathbf{x}\|$, since by definition, $\langle \mathbf{x}, \mathbf{x} \rangle$ is always real and nonnegative for any vector $\mathbf{x}$. Also note that this definition agrees with the earlier definition of length in $\mathbb{R}^n$ based on the usual dot product in $\mathbb{R}^n$. We also have the following result:

Theorem 7.13 Let $\mathcal{V}$ be a real [complex] inner product space, with $\mathbf{x} \in \mathcal{V}$. Let $k \in \mathbb{R}$ [$k \in \mathbb{C}$]. Then, $\|k\mathbf{x}\| = |k|\,\|\mathbf{x}\|$.

The proof of this theorem is left for you to do in Exercise 6.

As before, we say that a vector of length 1 in an inner product space is a **unit vector**. For instance, in the inner product space of Example 4, the polynomial $\mathbf{p} = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}$ is a unit vector since $\|\mathbf{p}\| = \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$.

We define the distance between two vectors in the general inner product space setting as we did for $\mathbb{R}^n$.

Definition Let $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, an inner product space. Then the **distance between x and y** is $\|\mathbf{x} - \mathbf{y}\|$.

Example 6

Consider the real vector space $\mathcal{V}$ of real continuous functions from Example 5, with $a = 0$ and $b = \pi$. That is, $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^{\pi} \mathbf{f}(t)\mathbf{g}(t)\,dt$ for all $\mathbf{f}, \mathbf{g} \in \mathcal{V}$. Let $\mathbf{f} = \cos t$ and $\mathbf{g} = \sin t$. Then the distance between $\mathbf{f}$ and $\mathbf{g}$ is

$$\|\mathbf{f}-\mathbf{g}\| = \sqrt{\langle \cos t - \sin t,\ \cos t - \sin t\rangle} = \sqrt{\int_0^{\pi} (\cos t - \sin t)^2\, dt}$$

$$= \sqrt{\int_0^{\pi} \left(\cos^2 t - 2\cos t \sin t + \sin^2 t\right)\, dt}$$

$$= \sqrt{\int_0^{\pi} (1-\sin 2t)\, dt} = \sqrt{\left(t + \frac{1}{2}\cos 2t\right)\Big|_0^{\pi}} = \sqrt{\pi}.$$

Hence, the distance between $\cos t$ and $\sin t$ is $\sqrt{\pi}$ under this inner product. ■

The next theorem shows that some other familiar results from the ordinary dot product carry over to the general inner product.

Theorem 7.14 Let $\mathbf{x},\mathbf{y} \in \mathcal{V}$, an inner product space, with inner product $\langle\, ,\rangle$. Then

(1) $|\langle \mathbf{x},\mathbf{y}\rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$ — Cauchy-Schwarz Inequality

(2) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. — Triangle Inequality

The proofs of these statements are analogous to the proofs for the ordinary dot product and are left for you to do in Exercise 11.

From the Cauchy-Schwarz Inequality, we have $-1 \leq \langle \mathbf{x},\mathbf{y}\rangle / \left(\|\mathbf{x}\|\,\|\mathbf{y}\|\right) \leq 1$, for any nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in a *real* inner product space. Hence, we can make the following definition:

Definition Let $\mathbf{x},\mathbf{y} \in \mathcal{V}$, a *real* inner product space. Then the **angle between x and y** is the angle θ from 0 to π such that $\cos\theta = \langle \mathbf{x},\mathbf{y}\rangle / \left(\|\mathbf{x}\|\,\|\mathbf{y}\|\right)$.

Example 7

Consider again the inner product space of Example 6, where $\langle \mathbf{f},\mathbf{g}\rangle = \int_0^{\pi} \mathbf{f}(t)\mathbf{g}(t)\, dt$. Let $\mathbf{f} = t$ and $\mathbf{g} = \sin t$. Then $\langle \mathbf{f},\mathbf{g}\rangle = \int_0^{\pi} t\sin t\, dt$. Using integration by parts, we get $(-t\cos t)|_0^{\pi} + \int_0^{\pi} \cos t\, dt = \pi + (\sin t)|_0^{\pi} = \pi$. Also, $\|\mathbf{f}\|^2 = \langle \mathbf{f},\mathbf{f}\rangle = \int_0^{\pi} (\mathbf{f}(t))^2\, dt = \int_0^{\pi} t^2\, dt = (t^3/3)|_0^{\pi} = \pi^3/3$, and so $\|\mathbf{f}\| = \sqrt{\pi^3/3}$. Similarly, $\|\mathbf{g}\|^2 = \langle \mathbf{g},\mathbf{g}\rangle = \int_0^{\pi} (\mathbf{g}(t))^2\, dt = \int_0^{\pi} \sin^2 t\, dt = \int_0^{\pi} \frac{1}{2}(1-\cos 2t)\, dt = \left(\frac{1}{2}t - \frac{1}{4}\sin 2t\right)\Big|_0^{\pi} = \pi/2$, and so $\|\mathbf{g}\| = \sqrt{\pi/2}$. Hence, the cosine of the angle θ between t and $\sin t$ equals $\langle \mathbf{f},\mathbf{g}\rangle / \left(\|\mathbf{f}\|\,\|\mathbf{g}\|\right) = \pi / \left(\sqrt{\pi^3/3}\sqrt{\pi/2}\right) = \sqrt{6}/\pi \approx 0.78$. Hence, $\theta \approx 0.68$ radians $(38.8°)$. ■

Orthogonality in Inner Product Spaces

We next define orthogonal vectors in a general inner product space setting and show that nonzero orthogonal vectors are linearly independent.

Definition A subset $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ of vectors in an inner product space $\mathcal{V}$ with inner product $\langle\,,\rangle$ is **orthogonal** if and only if $\langle \mathbf{x}_i,\mathbf{x}_j\rangle = 0$ for $1 \le i,j \le n$, with $i \ne j$. Also, an orthogonal set of vectors in $\mathcal{V}$ is **orthonormal** if and only if each vector in the set is a unit vector.

The next theorem is the analog of Theorem 6.1, and its proof is left for you to do in Exercise 15.

Theorem 7.15 If $\mathcal{V}$ is an inner product space and T is an orthogonal set of nonzero vectors in $\mathcal{V}$, then T is a linearly independent set.

Example 8

Consider again the inner product space $\mathcal{V}$ of Example 5 of real continuous functions with inner product $\langle \mathbf{f},\mathbf{g}\rangle = \int_a^b \mathbf{f}(t)\mathbf{g}(t)\,dt$, with $a=-\pi$ and $b=\pi$. The set $\{1,\cos t,\sin t\}$ is an orthogonal set in $\mathcal{V}$, since each of the following definite integrals equals zero (verify!):

$$\int_{-\pi}^{\pi}(1)\cos t\,dt, \quad \int_{-\pi}^{\pi}(1)\sin t\,dt, \quad \int_{-\pi}^{\pi}(\cos t)(\sin t)\,dt.$$

Also, note that $\|1\|^2 = \langle 1,1\rangle = \int_{-\pi}^{\pi}(1)(1)\,dt = 2\pi$, $\|\cos t\|^2 = \langle \cos t,\cos t\rangle = \int_{-\pi}^{\pi}\cos^2 t\,dt = \pi$ (why?), and $\|\sin t\|^2 = \langle \sin t,\sin t\rangle = \int_{-\pi}^{\pi}\sin^2 t\,dt = \pi$ (why?). Therefore, the set

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\sin t}{\sqrt{\pi}}\right\}$$

is an orthonormal set in $\mathcal{V}$. ■

Example 8 can be generalized. The set $\{1,\cos t,\sin t,\cos 2t,\sin 2t,\cos 3t,\sin 3t,\ldots\}$ is an orthogonal set (see Exercise 16) and therefore linearly independent by Theorem 7.15. The functions in this set are important in the theory of partial differential equations. It can be shown that every continuously differentiable function on the interval $[-\pi,\pi]$ can be represented as the (infinite) sum of constant multiples of these functions. Such a sum is known as the **Fourier series** of the function.

A basis for an inner product space $\mathcal{V}$ is an **orthogonal** [**orthonormal**] **basis** if the vectors in the basis form an orthogonal [orthonormal] set.

Example 9

Consider again the inner product space $\mathcal{P}_n$ with the inner product of Example 4; that is, if $\mathbf{p}_1 = a_nx^n + \cdots + a_1x + a_0$ and $\mathbf{p}_2 = b_nx^n + \cdots + b_1x + b_0$ are in $\mathcal{P}_n$, then $\langle \mathbf{p}_1,\mathbf{p}_2\rangle = a_nb_n + \cdots + a_1b_1 + a_0b_0$. Now, $\{x^n,x^{n-1},\ldots,x,1\}$ is an orthogonal basis for $\mathcal{P}_n$ with this inner product, since $\langle x^k, x^l \rangle = 0$, for $0 \le k, l \le n$, with $k \ne l$ (why?). Since $\|x^k\| = \sqrt{\langle x^k, x^k \rangle} = 1$, for all k, $0 \le k \le n$ (why?), the set $\{x^n, x^{n-1}, \ldots, x, 1\}$ is also an orthonormal basis for this inner product space. ■

A proof analogous to that of Theorem 6.3 gives us the next theorem (see Exercise 17).

Theorem 7.16 If $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is an orthogonal ordered basis for a subspace $\mathcal{W}$ of an inner product space $\mathcal{V}$, and if $\mathbf{v}$ is any vector in $\mathcal{W}$, then

$$[\mathbf{v}]_B = \left[\frac{\langle \mathbf{v}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}, \frac{\langle \mathbf{v}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}, \ldots, \frac{\langle \mathbf{v}, \mathbf{v}_k \rangle}{\langle \mathbf{v}_k, \mathbf{v}_k \rangle} \right].$$

In particular, if B is an orthonormal ordered basis for $\mathcal{W}$, then $[\mathbf{v}]_B = [\langle \mathbf{v}, \mathbf{v}_1 \rangle, \langle \mathbf{v}, \mathbf{v}_2 \rangle, \ldots, \langle \mathbf{v}, \mathbf{v}_k \rangle]$.

Example 10

Recall the inner product space $\mathbb{R}^2$ in Example 2, with inner product given as follows: if $\mathbf{x} = [x_1, x_2]$ and $\mathbf{y} = [y_1, y_2]$, then $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$. An ordered orthogonal basis for this space is $B = (\mathbf{v}_1, \mathbf{v}_2) = ([2,1],[0,1])$ (verify!). Recall from Example 2 that $\langle \mathbf{x}, \mathbf{x} \rangle = (x_1 - x_2)^2 + x_2^2$. Thus, $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = (2-1)^2 + 1^2 = 2$, and $\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = (0-1)^2 + 1^2 = 2$.

Next, suppose that $\mathbf{v} = [a, b]$ is any vector in $\mathbb{R}^2$. Now, $\langle \mathbf{v}, \mathbf{v}_1 \rangle = \langle [a,b],[2,1] \rangle = (a)(2) - (a)(1) - (b)(2) + 2(b)(1) = a$. Also, $\langle \mathbf{v}, \mathbf{v}_2 \rangle = \langle [a,b],[0,1] \rangle = (a)(0) - (a)(1) - (b)(0) + 2(b)(1) = -a + 2b$. Then,

$$[\mathbf{v}]_B = \left[\frac{\langle \mathbf{v}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}, \frac{\langle \mathbf{v}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \right] = \left[\frac{a}{2}, \frac{-a+2b}{2} \right].$$

Notice that $\frac{a}{2}[2,1] + \left(\frac{-a+2b}{2}\right)[0,1]$ does equal $[a,b] = \mathbf{v}$. ■

The Generalized Gram-Schmidt Process

We can generalize the Gram-Schmidt Process of Section 6.1 to any inner product space. That is, we can replace any linearly independent set of k vectors with an orthogonal set of k vectors that spans the same subspace.

Generalized Gram-Schmidt Process

Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a linearly independent subset of an inner product space $\mathcal{V}$, with inner product $\langle\, , \rangle$. We create a new set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors as follows:

Let $\mathbf{v}_1 = \mathbf{w}_1$.

Let $\mathbf{v}_2 = \mathbf{w}_2 - \left(\frac{\langle \mathbf{w}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\right)\mathbf{v}_1$.

Let $\mathbf{v}_3 = \mathbf{w}_3 - \left(\frac{\langle \mathbf{w}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\right)\mathbf{v}_1 - \left(\frac{\langle \mathbf{w}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\right)\mathbf{v}_2$.

$\vdots$

Let $\mathbf{v}_k = \mathbf{w}_k - \left(\frac{\langle \mathbf{w}_k, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\right)\mathbf{v}_1 - \left(\frac{\langle \mathbf{w}_k, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\right)\mathbf{v}_2 - \cdots - \left(\frac{\langle \mathbf{w}_k, \mathbf{v}_{k-1} \rangle}{\langle \mathbf{v}_{k-1}, \mathbf{v}_{k-1} \rangle}\right)\mathbf{v}_{k-1}$.

A proof similar to that of Theorem 6.4 (see Exercise 21) gives

Theorem 7.17 Let $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be a basis for a finite dimensional inner product space $\mathcal{V}$. Then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ obtained by applying the Generalized Gram-Schmidt Process to B is an orthogonal basis for $\mathcal{V}$.

Hence, every nontrivial finite dimensional inner product space has an orthogonal basis.

Example 11

Recall the inner product space $\mathcal{V}$ from Example 5 of real continuous functions using $a = -1$ and $b = 1$, that is, with inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-1}^{1} \mathbf{f}(t)\mathbf{g}(t)\,dt$. Now, $\{1, t, t^2, t^3\}$ is a linearly independent set in $\mathcal{V}$. We use this set to find four orthogonal vectors in $\mathcal{V}$.

Let $\mathbf{w}_1 = 1$, $\mathbf{w}_2 = t$, $\mathbf{w}_3 = t^2$, and $\mathbf{w}_4 = t^3$. Using the Generalized Gram-Schmidt Process, we start with $\mathbf{v}_1 = \mathbf{w}_1 = 1$ and obtain

$$\mathbf{v}_2 = \mathbf{w}_2 - \left(\frac{\langle \mathbf{w}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\right)\mathbf{v}_1 = t - \left(\frac{\langle t, 1 \rangle}{\langle 1, 1 \rangle}\right)1.$$

Now, $\langle t, 1 \rangle = \int_{-1}^{1} (t)(1)\,dt = \left(t^2/2\right)\Big|_{-1}^{1} = 0$. Hence, $\mathbf{v}_2 = t$. Next,

$$\mathbf{v}_3 = \mathbf{w}_3 - \left(\frac{\langle \mathbf{w}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\right)\mathbf{v}_1 - \left(\frac{\langle \mathbf{w}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\right)\mathbf{v}_2 = t^2 - \left(\frac{\langle t^2, 1 \rangle}{\langle 1, 1 \rangle}\right)1 - \left(\frac{\langle t^2, t \rangle}{\langle t, t \rangle}\right)t.$$

After a little calculation, we obtain $\langle t^2, 1 \rangle = \frac{2}{3}$, $\langle 1, 1 \rangle = 2$, and $\langle t^2, t \rangle = 0$. Hence, $\mathbf{v}_3 = t^2 - \left(\left(\frac{2}{3}\right)/2\right)1 = t^2 - \frac{1}{3}$. Finally,

$$\begin{aligned}\mathbf{v}_4 &= \mathbf{w}_4 - \left(\frac{\langle \mathbf{w}_4, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\right)\mathbf{v}_1 - \left(\frac{\langle \mathbf{w}_4, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\right)\mathbf{v}_2 - \left(\frac{\langle \mathbf{w}_4, \mathbf{v}_3 \rangle}{\langle \mathbf{v}_3, \mathbf{v}_3 \rangle}\right)\mathbf{v}_3 \\ &= t^3 - \left(\frac{\langle t^3, 1 \rangle}{\langle 1, 1 \rangle}\right)1 - \left(\frac{\langle t^3, t \rangle}{\langle t, t \rangle}\right)t - \left(\frac{\langle t^3, t^2 \rangle}{\langle t^2, t^2 \rangle}\right)t^2.\end{aligned}$$

Now, $\langle t^3, 1\rangle = 0$, $\langle t^3, t\rangle = \frac{2}{5}$, $\langle t, t\rangle = \frac{2}{3}$, and $\langle t^3, t^2\rangle = 0$. Hence, $\mathbf{v}_4 = t^3 - \left(\left(\frac{2}{5}\right)/\left(\frac{2}{3}\right)\right)t = t^3 - \frac{3}{5}t$.

Thus, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{1, t, t^2 - \frac{1}{3}, t^3 - \frac{3}{5}t\right\}$ is an orthogonal set of vectors in this inner product space.[3]

■

We saw in Theorem 6.8 that the transition matrix between orthonormal bases of $\mathbb{R}^n$ is an orthogonal matrix. This result generalizes to inner product spaces as follows:

> **Theorem 7.18** Let $\mathcal{V}$ be a finite dimensional real [complex] inner product space, and let B and C be ordered orthonormal bases for $\mathcal{V}$. Then the transition matrix from B to C is an orthogonal [unitary] matrix.

Orthogonal Complements and Orthogonal Projections in Inner Product Spaces

We can generalize the notion of an orthogonal complement of a subspace to inner product spaces as follows:

> **Definition** Let $\mathcal{W}$ be a subspace of a real (or complex) inner product space $\mathcal{V}$. Then the **orthogonal complement** $\mathcal{W}^\perp$ of $\mathcal{W}$ in $\mathcal{V}$ is the set of all vectors $\mathbf{x} \in \mathcal{V}$ with the property that $\langle \mathbf{x}, \mathbf{w}\rangle = 0$, for all $\mathbf{w} \in \mathcal{W}$.

Example 12

Consider again the real vector space $\mathcal{P}_n$, with the inner product of Example 4 — for $\mathbf{p}_1 = a_nx^n + \cdots + a_1x + a_0$ and $\mathbf{p}_2 = b_nx^n + \cdots + b_1x + b_0$, $\langle \mathbf{p}_1, \mathbf{p}_2\rangle = a_nb_n + \cdots + a_1b_1 + a_0b_0$. Example 9 showed that $\left\{x^n, x^{n-1}, \ldots, x, 1\right\}$ is an orthogonal basis for $\mathcal{P}_n$ under this inner product. Now, consider the subspace $\mathcal{W}$ spanned by $\{x, 1\}$. A little thought will convince you that $\mathcal{W}^\perp = \text{span}\left\{x^n, x^{n-1}, \ldots, x^2\right\}$ and so, $\dim(\mathcal{W}) + \dim\left(\mathcal{W}^\perp\right) = 2 + (n-1) = n + 1 = \dim(\mathcal{P}_n)$.

■

The following properties of orthogonal complements are the analogs to Theorems 6.11 and 6.12 and Corollaries 6.13 and 6.14 and are proved in a similar manner (see Exercise 22):

[3] The polynomials 1, t, $t^2 - \frac{1}{3}$, and $t^3 - \frac{3}{5}t$ from Example 11 are multiples of the first four **Legendre polynomials**: $1, t, \frac{3}{2}t^2 - \frac{1}{2}, \frac{5}{2}t^3 - \frac{3}{2}t$. All Legendre polynomials equal 1 when $t = 1$. To find the complete set of Legendre polynomials, we can continue the Generalized Gram-Schmidt Process with t^4, t^5, t^6, and so on, and take appropriate multiples so that the resulting polynomials equal 1 when $t = 1$. These polynomials form an (infinite) orthogonal set for the inner product space of Example 11.

Theorem 7.19 Let $\mathcal{W}$ be a subspace of a real (or complex) inner product space $\mathcal{V}$. Then

(1) $\mathcal{W}^{\perp}$ is a subspace of $\mathcal{V}$.

(2) $\mathcal{W} \cap \mathcal{W}^{\perp} = \{\mathbf{0}\}$.

(3) $\mathcal{W} \subseteq \left(\mathcal{W}^{\perp}\right)^{\perp}$.

Furthermore, if $\mathcal{V}$ is finite dimensional, then

(4) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal basis for $\mathcal{W}$ contained in an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ for $\mathcal{V}$, then $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for $\mathcal{W}^{\perp}$.

(5) $\dim(\mathcal{W}) + \dim\left(\mathcal{W}^{\perp}\right) = \dim(\mathcal{V})$.

(6) $\left(\mathcal{W}^{\perp}\right)^{\perp} = \mathcal{W}$.

Note that if $\mathcal{V}$ is not finite dimensional, $\left(\mathcal{W}^{\perp}\right)^{\perp}$ is not necessarily equal to $\mathcal{W}$, although it is always true that $\mathcal{W} \subseteq \left(\mathcal{W}^{\perp}\right)^{\perp}$.[4]

The next theorem is the analog of Theorem 6.15. It holds for any inner product space $\mathcal{V}$ where the subspace $\mathcal{W}$ is finite dimensional. The proof is left for you to do in Exercise 25.

Theorem 7.20 (Projection Theorem) Let $\mathcal{W}$ be a finite dimensional subspace of an inner product space $\mathcal{V}$. Then every vector $\mathbf{v} \in \mathcal{V}$ can be expressed in a unique way as $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in \mathcal{W}$ and $\mathbf{w}_2 \in \mathcal{W}^{\perp}$.

As before, we define the **orthogonal projection** of a vector $\mathbf{v}$ onto a subspace $\mathcal{W}$ as follows:

Definition If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for $\mathcal{W}$, a subspace of an inner product space $\mathcal{V}$, then the vector $\mathbf{proj}_{\mathcal{W}}\mathbf{v} = \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \cdots + \langle \mathbf{v}, \mathbf{v}_k \rangle \mathbf{v}_k$ is called the **orthogonal projection of v onto** $\mathcal{W}$. If $\mathcal{W}$ is the trivial subspace of $\mathcal{V}$, then $\mathbf{proj}_{\mathcal{W}}\mathbf{v} = \mathbf{0}$.

It can be shown that the formula for $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ yields the unique vector $\mathbf{w}_1$ in the Projection Theorem. Therefore, the choice of orthonormal basis in the definition

[4] The following is an example of a subspace $\mathcal{W}$ of an infinite dimensional inner product space such that $\mathcal{W} \neq \left(\mathcal{W}^{\perp}\right)^{\perp}$. Let $\mathcal{V}$ be the inner product space of Example 5 with $a = 0$, $b = 1$, and let $\mathbf{f}_n(x) = \begin{cases} 1, & \text{if } x > \frac{1}{n} \\ nx, & \text{if } 0 \le x \le \frac{1}{n} \end{cases}$. Let $\mathcal{W}$ be the subspace of $\mathcal{V}$ spanned by $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \ldots\}$. It can be shown that $\mathbf{f}(x) = 1$ is not in $\mathcal{W}$, but $\mathbf{f}(x) \in \left(\mathcal{W}^{\perp}\right)^{\perp}$. Hence, $\mathcal{W} \neq \left(\mathcal{W}^{\perp}\right)^{\perp}$.

does not matter because any choice leads to the same vector for $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$. Hence, the Projection Theorem can be restated as follows:

> If $\mathcal{W}$ is a finite dimensional subspace of an inner product space $\mathcal{V}$, and if $\mathbf{v} \in \mathcal{V}$, then $\mathbf{v}$ can be expressed as $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{v} \in \mathcal{W}$ and $\mathbf{w}_2 = \mathbf{v} - \mathbf{proj}_{\mathcal{W}}\mathbf{v} \in \mathcal{W}^{\perp}$.

Example 13

Consider again the real vector space $\mathcal{V}$ of real continuous functions in Example 8, where $\langle \mathbf{f}, \mathbf{g} \rangle = \int_{-\pi}^{\pi} \mathbf{f}(t)\mathbf{g}(t)\,dt$. Notice from that example that the set $\{1/\sqrt{2\pi}, (\sin t)/\sqrt{\pi}\}$ is an orthonormal (and hence, linearly independent) set of vectors in $\mathcal{V}$. Let $\mathcal{W} = \text{span}(\{1/\sqrt{2\pi}, (\sin t)/\sqrt{\pi}\})$ in $\mathcal{V}$. Then any continuous function $\mathbf{f}$ in $\mathcal{V}$ can be expressed uniquely as $\mathbf{f}_1 + \mathbf{f}_2$, where $\mathbf{f}_1 \in \mathcal{W}$ and $\mathbf{f}_2 \in \mathcal{W}^{\perp}$.

We illustrate this decomposition for the function $\mathbf{f} = t + 1$. Now,

$$\mathbf{f}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{f} = c_1\left(\frac{1}{\sqrt{2\pi}}\right) + c_2\left(\frac{\sin t}{\sqrt{\pi}}\right),$$

where $c_1 = \left\langle (t+1), 1/\sqrt{2\pi} \right\rangle$ and $c_2 = \left\langle (t+1), (\sin t)/\sqrt{\pi} \right\rangle$. Then

$$c_1 = \int_{-\pi}^{\pi} (t+1)\left(\frac{1}{\sqrt{2\pi}}\right) dt = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} (t+1)\,dt$$
$$= \frac{1}{\sqrt{2\pi}}\left(\frac{t^2}{2} + t\right)\Bigg|_{-\pi}^{\pi} = \frac{2\pi}{\sqrt{2\pi}} = \sqrt{2\pi}.$$

Also,

$$c_2 = \int_{-\pi}^{\pi} (t+1)\left(\frac{\sin t}{\sqrt{\pi}}\right) dt = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} (t+1)\sin t\,dt$$
$$= \frac{1}{\sqrt{\pi}}\left(\int_{-\pi}^{\pi} t \sin t\,dt + \int_{-\pi}^{\pi} \sin t\,dt\right).$$

The very last integral equals zero. Using integration by parts on the other integral, we obtain

$$c_2 = \frac{1}{\sqrt{\pi}}\left((-t\cos t)\big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos t\,dt\right) = \left(\frac{1}{\sqrt{\pi}}\right)2\pi = 2\sqrt{\pi}.$$

Hence,

$$\mathbf{f}_1 = c_1\left(\frac{1}{\sqrt{2\pi}}\right) + c_2\left(\frac{\sin t}{\sqrt{\pi}}\right) = \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\right) + 2\sqrt{\pi}\left(\frac{\sin t}{\sqrt{\pi}}\right) = 1 + 2\sin t.$$

Then by the Projection Theorem, $\mathbf{f}_2 = \mathbf{f} - \mathbf{f}_1 = (t+1) - (1 + 2\sin t) = t - 2\sin t$ is orthogonal to $\mathcal{W}$. We check that $\mathbf{f}_2 \in \mathcal{W}^{\perp}$ by showing that $\mathbf{f}_2$ is orthogonal to both $1/\sqrt{2\pi}$ and $(\sin t)/\sqrt{\pi}$.

$$\left\langle \mathbf{f}_2, \frac{1}{\sqrt{2\pi}} \right\rangle = \int_{-\pi}^{\pi} (t - 2\sin t)\left(\frac{1}{\sqrt{2\pi}}\right) dt = \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{t^2}{2} + 2\cos t\right)\Bigg|_{-\pi}^{\pi} = 0.$$

Also,

$$\left\langle \mathbf{f}_2, \frac{\sin t}{\sqrt{\pi}} \right\rangle = \int_{-\pi}^{\pi} (t - 2\sin t)\left(\frac{\sin t}{\sqrt{\pi}}\right) dt = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} t \sin t \, dt - \frac{2}{\sqrt{\pi}} \int_{-\pi}^{\pi} \sin^2 t \, dt,$$

which equals $2\sqrt{\pi} - 2\sqrt{\pi} = 0$. ∎

New Vocabulary

angle between vectors (in an inner product space)
Cauchy-Schwarz Inequality (in an inner product space)
complex inner product (on a complex vector space)
complex inner product space
distance between vectors (in an inner product space)
Fourier series
Generalized Gram-Schmidt Process (in an inner product space)
Legendre polynomials
norm (length) of a vector (in an inner product space)
orthogonal basis (in an inner product space)
orthogonal complement (of a subspace in an inner product space)
orthogonal projection (of a vector onto a subspace of an inner product space)
orthogonal set of vectors (in an inner product space)
orthonormal basis (in an inner product space)
orthonormal set of vectors (in an inner product space)
real inner product (on a real vector space)
real inner product space
Triangle Inequality (in an inner product space)
unit vector (in an inner product space)

Highlights

- Real and complex inner products are generalizations of the real and complex dot products, respectively.
- An inner product space is a vector space that possesses three operations: vector addition, scalar multiplication, and inner product.
- For vectors $\mathbf{x}, \mathbf{y}$ and scalar k in a real inner product space, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$, and $\langle \mathbf{x}, k\mathbf{y} \rangle = k \langle \mathbf{x}, \mathbf{y} \rangle$.
- For vectors $\mathbf{x}$, $\mathbf{y}$ and scalar k in a real or complex inner product space, $\langle k\mathbf{x}, \mathbf{y} \rangle = k \langle \mathbf{x}, \mathbf{y} \rangle$.
- For vectors $\mathbf{x}, \mathbf{y}$ and scalar k in a complex inner product space, $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$, $\langle \mathbf{x}, k\mathbf{y} \rangle = \overline{k} \langle \mathbf{x}, \mathbf{y} \rangle$, and $\|k\mathbf{x}\| = |k| \, \|\mathbf{x}\|$.
- The length of a vector $\mathbf{x}$ in an inner product space is $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, and the distance between vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space is $\|\mathbf{x} - \mathbf{y}\|$.

- The angle θ between two vectors in a real inner product space is defined as the angle between 0 and π such that $\cos\theta = \langle \mathbf{x}, \mathbf{y} \rangle / \left(\|\mathbf{x}\|\,\|\mathbf{y}\|\right)$.
- Orthogonal and orthonormal sets of vectors, and orthogonal complements of subspaces, are defined for inner product spaces analogously as for real vector spaces.
- An orthogonal set of nonzero vectors in an inner product space is a linearly independent set.
- If $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is an orthogonal ordered basis for a subspace $\mathcal{W}$ of an inner product space $\mathcal{V}$, and if $\mathbf{v}$ is any vector in $\mathcal{W}$, then $[\mathbf{v}]_B = \left[\frac{\langle \mathbf{v}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}, \frac{\langle \mathbf{v}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}, \ldots, \frac{\langle \mathbf{v}, \mathbf{v}_k \rangle}{\langle \mathbf{v}_k, \mathbf{v}_k \rangle}\right]$.
- The Generalized Gram-Schmidt Process can be used to find an orthogonal basis for any subspace spanned by a finite linearly independent subset.
- If $\mathcal{W}$ is a finite dimensional subspace of an inner product space $\mathcal{V}$, then every vector $\mathbf{v}$ in $\mathcal{V}$ can be expressed uniquely as the sum of vectors $\mathbf{w}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{v} \in \mathcal{W}$ and $\mathbf{w}_2 = \mathbf{v} - \mathbf{proj}_{\mathcal{W}}\mathbf{v} \in \mathcal{W}^{\perp}$.
- The transition matrix from one ordered orthonormal basis to another in a real [complex] inner product space is an orthogonal [unitary] matrix.

EXERCISES FOR SECTION 7.5

1. **(a)** Let $\mathbf{A}$ be a nonsingular $n \times n$ real matrix. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define an operation $\langle \mathbf{x}, \mathbf{y} \rangle = (\mathbf{A}\mathbf{x}) \cdot (\mathbf{A}\mathbf{y})$ (dot product). Prove that this operation is a real inner product on $\mathbb{R}^n$.

★(b) For the inner product in part (a) with $\mathbf{A} = \begin{bmatrix} 5 & 4 & 2 \\ -2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$, find $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\|\mathbf{x}\|$, for $\mathbf{x} = [3, -2, 4]$ and $\mathbf{y} = [-2, 1, -1]$.

2. Define an operation $\langle\,,\rangle$ on $\mathcal{P}_n$ as follows: if $\mathbf{p}_1 = a_n x^n + \cdots + a_1 x + a_0$ and $\mathbf{p}_2 = b_n x^n + \cdots + b_1 x + b_0$, let $\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = a_n b_n + \cdots + a_1 b_1 + a_0 b_0$. Prove that this operation is a real inner product on $\mathcal{P}_n$.

3. **(a)** Let a and b be fixed real numbers with $a < b$, and let $\mathcal{V}$ be the set of all real continuous functions on $[a, b]$. Define $\langle\,,\rangle$ on $\mathcal{V}$ by $\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b \mathbf{f}(t)\mathbf{g}(t)\,dt$. Prove that this operation is a real inner product on $\mathcal{V}$.

★(b) For the inner product of part (a) with $a = 0$ and $b = \pi$, find $\langle \mathbf{f}, \mathbf{g} \rangle$ and $\|\mathbf{f}\|$, for $\mathbf{f} = e^t$ and $\mathbf{g} = \sin t$.

4. Define $\langle\,,\rangle$ on the real vector space $\mathcal{M}_{mn}$ by $\langle \mathbf{A}, \mathbf{B} \rangle = \mathrm{trace}(\mathbf{A}^T\mathbf{B})$. Prove that this operation is a real inner product on $\mathcal{M}_{mn}$. (Hint: Refer to Exercise 14 in Section 1.4 and Exercise 26 in Section 1.5.)

5. **(a)** Prove part (1) of Theorem 7.12. (Hint: $\mathbf{0} = \mathbf{0} + \mathbf{0}$. Use property (4) in the definition of an inner product space.)

(b) Prove part (3) of Theorem 7.12. (Be sure to give a proof for both real and complex inner product spaces.)

►**6.** Prove Theorem 7.13.

7. Let $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, a real inner product space.

(a) Prove that $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\langle\mathbf{x},\mathbf{y}\rangle + \|\mathbf{y}\|^2$.

(b) Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal in $\mathcal{V}$ if and only if $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

(c) Show that $\frac{1}{2}\left(\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2\right) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

8. The following formulas show how the value of the inner product can be derived from the norm (length):

(a) Let $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, a real inner product space. Prove the following (real) **Polarization Identity:**

$$\langle\mathbf{x},\mathbf{y}\rangle = \frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\right).$$

(b) Let $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, a complex inner product space. Prove the following **Complex Polarization Identity:**

$$\langle\mathbf{x},\mathbf{y}\rangle = \frac{1}{4}\left(\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\right) + i\left(\|\mathbf{x}+i\mathbf{y}\|^2 - \|\mathbf{x}-i\mathbf{y}\|^2\right)\right).$$

9. Consider the inner product space $\mathcal{V}$ of Example 5, with $a = 0$ and $b = \pi$.

⋆**(a)** Find the distance between $\mathbf{f} = t$ and $\mathbf{g} = \sin t$ in $\mathcal{V}$.

(b) Find the angle between $\mathbf{f} = e^t$ and $\mathbf{g} = \sin t$ in $\mathcal{V}$.

10. Consider the inner product space $\mathcal{V}$ of Example 3, using

$$\mathbf{A} = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & -1 & -1 \end{bmatrix}.$$

(a) Find the distance between $\mathbf{x} = [2,-1,3]$ and $\mathbf{y} = [5,-2,2]$ in $\mathcal{V}$.

⋆**(b)** Find the angle between $\mathbf{x} = [2,-1,3]$ and $\mathbf{y} = [5,-2,2]$ in $\mathcal{V}$.

11. Let $\mathcal{V}$ be an inner product space.

(a) Prove part (1) of Theorem 7.14. (Hint: Modify the proof of Theorem 1.6.)

(b) Prove part (2) of Theorem 7.14. (Hint: Modify the proof of Theorem 1.7.)

12. Let f and g be continuous real-valued functions defined on a closed interval $[a,b]$. Show that

$$\left(\int_a^b f(t)g(t)\,dt\right)^2 \leq \int_a^b \left(f(t)\right)^2\,dt \int_a^b \left(g(t)\right)^2\,dt.$$

(Hint: Use the Cauchy-Schwarz Inequality in an appropriate inner product space.)

13. A **metric space** is a set in which every pair of elements x, y has been assigned a real number distance d with the following properties:

(i) $d(x,y) = d(y,x)$.

(ii) $d(x,y) \geq 0$, with $d(x,y) = 0$ if and only if $x = y$.

(iii) $d(x,y) \leq d(x,z) + d(z,y)$, for all z in the set.

Prove that every inner product space is a metric space with $d(\mathbf{x},\mathbf{y})$ taken to be $\|\mathbf{x}-\mathbf{y}\|$ for all vectors $\mathbf{x}$ and $\mathbf{y}$ in the space.

14. Determine whether the following sets of vectors are orthogonal:

★**(a)** $\{t^2,\ t+1,\ t-1\}$ in $\mathcal{P}_3$, under the inner product of Example 4

(b) $\{[15,9,19],[-2,-1,-2],[-12,-9,-14]\}$ in $\mathbb{R}^3$, under the inner product of Example 3, with

$$\mathbf{A} = \begin{bmatrix} -3 & 1 & 2 \\ 0 & -2 & 1 \\ 2 & -1 & -1 \end{bmatrix}$$

★**(c)** $\{[5,-2],[3,4]\}$ in $\mathbb{R}^2$, under the inner product of Example 2

(d) $\{3t^2-1,\ 4t,\ 5t^3-3t\}$ in $\mathcal{P}_3$, under the inner product of Example 11

15. Prove Theorem 7.15. (Hint: Modify the proof of Result 7 in Section 1.3.)

16. **(a)** Show that $\int_{-\pi}^{\pi} \cos mt\,dt = 0$ and $\int_{-\pi}^{\pi} \sin nt\,dt = 0$, for all integers $m,n \geq 1$.

(b) Show that $\int_{-\pi}^{\pi} \cos mt \cos nt\,dt = 0$ and $\int_{-\pi}^{\pi} \sin mt \sin nt\,dt = 0$, for any *distinct* integers $m,n \geq 1$. (Hint: Use trigonometric identities.)

(c) Show that $\int_{-\pi}^{\pi} \cos mt \sin nt\,dt = 0$, for any integers $m,n \geq 1$.

(d) Conclude from parts (a), (b), and (c) that $\{1, \cos t, \sin t, \cos 2t, \sin 2t, \cos 3t, \sin 3t, \ldots\}$ is an orthogonal set of real continuous functions on $[-\pi,\pi]$, as claimed after Example 8.

17. Prove Theorem 7.16. (Hint: Modify the proof of Theorem 6.3.)

18. Let $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ be an orthonormal basis for a complex inner product space $\mathcal{V}$. Prove that for all $\mathbf{v},\mathbf{w} \in \mathcal{V}$,

$$\langle \mathbf{v},\mathbf{w}\rangle = \langle \mathbf{v},\mathbf{v}_1\rangle\,\overline{\langle \mathbf{w},\mathbf{v}_1\rangle} + \langle \mathbf{v},\mathbf{v}_2\rangle\,\overline{\langle \mathbf{w},\mathbf{v}_2\rangle} + \cdots + \langle \mathbf{v},\mathbf{v}_k\rangle\,\overline{\langle \mathbf{w},\mathbf{v}_k\rangle}.$$

(Compare this with Exercise 9(a) in Section 6.1.)

★19. Use the Generalized Gram-Schmidt Process to find an orthogonal basis for $\mathcal{P}_2$ containing $t^2 - t + 1$ under the inner product of Example 11.

20. Use the Generalized Gram-Schmidt Process to find an orthogonal basis for $\mathbb{R}^3$ containing $[-9, -4, 8]$ under the inner product of Example 3 with the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 3 \\ 2 & -1 & 2 \end{bmatrix}.$$

21. Prove Theorem 7.17. (Hint: Modify the proof of Theorem 6.4.)

22. (a) Prove parts (1) and (2) of Theorem 7.19. (Hint: Modify the proof of Theorem 6.11.)

▶(b) Prove parts (4) and (5) of Theorem 7.19. (Hint: Modify the proofs of Theorem 6.12 and Corollary 6.13.)

(c) Prove part (3) of Theorem 7.19.

▶(d) Prove part (6) of Theorem 7.19. (Hint: Use part (5) of Theorem 7.19 to show that $\dim(\mathcal{W}) = \dim\left(\left(\mathcal{W}^{\perp}\right)^{\perp}\right)$. Then use part (c) and apply Theorem 4.16, or its complex analog.)

★23. Find $\mathcal{W}^{\perp}$ if $\mathcal{W} = \text{span}\left(\left\{t^3 + t^2,\ t - 1\right\}\right)$ in $\mathcal{P}_3$ with the inner product of Example 4.

24. Find an orthogonal basis for $\mathcal{W}^{\perp}$ if $\mathcal{W} = \text{span}(\{(t-1)^2\})$ in $\mathcal{P}_2$, with the inner product $\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 \mathbf{f}(t)\mathbf{g}(t)\,dt$, for all $\mathbf{f}, \mathbf{g} \in \mathcal{P}_2$.

▶25. Prove Theorem 7.20. (Hint: Choose an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for $\mathcal{W}$. Then define $\mathbf{w}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{v} = \langle \mathbf{v}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \cdots + \langle \mathbf{v}, \mathbf{v}_k \rangle \mathbf{v}_k$. Let $\mathbf{w}_2 = \mathbf{v} - \mathbf{w}_1$, and prove $\mathbf{w}_2 \in \mathcal{W}^{\perp}$. Finally, see the proof of Theorem 6.15 for uniqueness.)

★26. In the inner product space of Example 8, decompose $\mathbf{f} = \frac{1}{k}e^t$, where $k = e^{\pi} - e^{-\pi}$, as $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in \mathcal{W} = \text{span}(\{\cos t, \sin t\})$ and $\mathbf{w}_2 \in \mathcal{W}^{\perp}$. Check that $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 0$. (Hint: First find an orthonormal basis for $\mathcal{W}$.)

27. Decompose $\mathbf{v} = 4t^2 - t + 3$ in $\mathcal{P}_2$ as $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 \in \mathcal{W} = \text{span}(\{2t^2 - 1, t + 1\})$ and $\mathbf{w}_2 \in \mathcal{W}^{\perp}$, under the inner product of Example 11. Check that $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 0$. (Hint: First find an orthonormal basis for $\mathcal{W}$.)

28. **Bessel's Inequality:** Let $\mathcal{V}$ be a real inner product space, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an orthonormal set in $\mathcal{V}$. Prove that for any vector $\mathbf{v} \in \mathcal{V}$, $\sum_{i=1}^{k} \langle \mathbf{v}, \mathbf{v}_i \rangle^2 \leq \|\mathbf{v}\|^2$. (Hint: Let $\mathcal{W} = \text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_k\})$. Now, $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1 = \mathbf{proj}_{\mathcal{W}}\mathbf{v} \in \mathcal{W}$ and $\mathbf{w}_2 \in \mathcal{W}^{\perp}$. Expand $\langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{w}_1 + \mathbf{w}_2, \mathbf{w}_1 + \mathbf{w}_2 \rangle$. Show that $\|\mathbf{v}\|^2 \geq \|\mathbf{w}_1\|^2$, and use the definition of $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$.)

29. Let $\mathcal{W}$ be a finite dimensional subspace of an inner product space $\mathcal{V}$. Consider the mapping $L: \mathcal{V} \to \mathcal{W}$ given by $L(\mathbf{v}) = \mathbf{proj}_{\mathcal{W}}\mathbf{v}$.

(a) Prove that L is a linear transformation.

⋆(b) What are the kernel and range of L?

(c) Show that $L \circ L = L$.

⋆30. True or False:

(a) If $\mathcal{V}$ is a complex inner product space, then for all $\mathbf{x} \in \mathcal{V}$ and all $k \in \mathbb{C}$, $\|k\mathbf{x}\| = \overline{k}\|\mathbf{x}\|$.

(b) In a complex inner product space, the distance between two distinct vectors can be a pure imaginary number.

(c) Every linearly independent set of unit vectors in an inner product space is an orthonormal set.

(d) It is possible to define more than one inner product on the same vector space.

(e) The uniqueness proof of the Projection Theorem shows that if $\mathcal{W}$ is a subspace of $\mathbb{R}^n$, then $\mathbf{proj}_{\mathcal{W}}\mathbf{v}$ is independent of the particular inner product used on $\mathbb{R}^n$.

REVIEW EXERCISES FOR CHAPTER 7

1. Let $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z} \in \mathbb{C}^3$ be given by $\mathbf{v} = [i, 3-i, 2+3i]$, $\mathbf{w} = [-4-4i,\ 1+2i,\ 3-i]$, and $\mathbf{z} = [2+5i,\ 2-5i,\ -i]$.

⋆(a) Compute $\mathbf{v} \cdot \mathbf{w}$.

⋆(b) Compute $(1+2i)(\mathbf{v} \cdot \mathbf{z})$, $((1+2i)\mathbf{v}) \cdot \mathbf{z}$, and $\mathbf{v} \cdot ((1+2i)\mathbf{z})$.

(c) Explain why not all of the answers to part (b) are identical.

(d) Compute $\mathbf{w} \cdot \mathbf{z}$ and $\mathbf{w} \cdot (\mathbf{v} + \mathbf{z})$.

2. **(a)** Compute $\mathbf{H} = \mathbf{A}^*\mathbf{A}$, where $\mathbf{A} = \begin{bmatrix} 1-i & 2+i & 3-4i \\ 0 & 5-2i & -2+i \end{bmatrix}$ and show that $\mathbf{H}$ is Hermitian.

(b) Show that $\mathbf{AA}^*$ is also Hermitian.

3. Prove that if $\mathbf{A}$ is a skew-Hermitian $n \times n$ matrix and $\mathbf{w}$, $\mathbf{z} \in \mathbb{C}^n$, then $(\mathbf{Az}) \cdot \mathbf{w} = -\mathbf{z} \cdot (\mathbf{Aw})$.

4. In each part, solve the given system of linear equations.

⋆(a) $$\begin{cases} (i)w + (1+i)z = -1+2i \\ (1+i)w + (5+2i)z = 5-3i \\ (2-i)w + (2-5i)z = 1-2i \end{cases}$$