PROBLEM SET 9.7

1–6 CALCULATION OF GRADIENTS

Find grad f. Graph some level curves $f = \text{const}$. Indicate ∇f by arrows at some points of these curves.

1. $f = (x + 1)(2y - 1)$

2. $f = 9x^2 + 4y^2$

3. $f = y/x$

4. $(y + 6)^2 + (x - 4)^2$

5. $f = x^4 + y^4$

6. $f = (x^2 - y^2)/(x^2 + y^2)$

7–10 USEFUL FORMULAS FOR GRADIENT AND LAPLACIAN

Prove and illustrate by an example.

7. $\nabla(f^n) = nf^{n-1}\nabla f$

8. $\nabla(fg) = f\nabla g + g\nabla f$

9. $\nabla(f/g) = (1/g^2)(g\nabla f - f\nabla g)$

10. $\nabla^2(fg) = g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g$

11–15 USE OF GRADIENTS. ELECTRIC FORCE

The force in an electrostatic field given by $f(x, y, z)$ has the direction of the gradient. Find ∇f and its value at P.

11. $f = xy, \quad P: (-4, 5)$

12. $f = x/(x^2 + y^2), \quad P: (1, 1)$

13. $f = \ln(x^2 + y^2), \quad P: (8, 6)$

14. $f = (x^2 + y^2 + z^2)^{-1/2} \quad P: (12, 0, 16)$

15. $f = 4x^2 + 9y^2 + z^2, \quad P: (5, -1, -11)$

16. For what points $P: (x, y, z)$ does ∇f with $f = 25x^2 + 9y^2 + 16z^2$ have the direction from P to the origin?

17. Same question as in Prob. 16 when $f = 25x^2 + 4y^2$.

18–23 VELOCITY FIELDS

Given the velocity potential f of a flow, find the velocity $\mathbf{v} = \nabla f$ of the field and its value $\mathbf{v}(P)$ at P. Sketch $\mathbf{v}(P)$ and the curve $f = \text{const}$ passing through P.

18. $f = x^2 - 6x - y^2, \quad P: (-1, 5)$

19. $f = \cos x \cosh y, \quad P: (\frac{1}{2}\pi, \ln 2)$

20. $f = x(1 + (x^2 + y^2)^{-1}), \quad P: (1, 1)$

21. $f = e^x \cos y, \quad P: (1, \frac{1}{2}\pi)$

22. At what points is the flow in Prob. 21 directed vertically upward?

23. At what points is the flow in Prob. 21 horizontal?

24–27 HEAT FLOW

Experiments show that in a temperature field, heat flows in the direction of maximum decrease of temperature T. Find this direction in general and at the given point P. Sketch that direction at P as an arrow.

24. $T = 3x^2 - 2y^2, \quad P: (2.5, 1.8)$

25. $T = z/(x^2 + y^2), \quad P: (0, 1, 2)$

26. $T = x^2 + y^2 + 4z^2, \quad P: (2, -1, 2)$

27. **CAS PROJECT. Isotherms.** Graph some curves of constant temperature ("isotherms") and indicate directions of heat flow by arrows when the temperature equals **(a)** $x^3 - 3xy^2$, **(b)** $\sin x \sinh y$, and **(c)** $e^x \cos y$.

28. **Steepest ascent.** If $z(x, y) = 3000 - x^2 - 9y^2$ [meters] gives the elevation of a mountain at sea level, what is the direction of steepest ascent at $P: (4, 1)$?

29. **Gradient.** What does it mean if $|\nabla f(P)| > |\nabla f(Q)|$ at two points P and Q in a scalar field?

9.8 Divergence of a Vector Field

Vector calculus owes much of its importance in engineering and physics to the gradient, divergence, and curl. From a scalar field we can obtain a vector field by the gradient (Sec. 9.7). Conversely, from a vector field we can obtain a scalar field by the divergence or another vector field by the curl (to be discussed in Sec. 9.9). These concepts were suggested by basic physical applications. This will be evident from our examples.

To begin, let $\mathbf{v}(x, y, z)$ be a differentiable vector function, where x, y, z are Cartesian coordinates, and let v_1, v_2, v_3 be the components of $\mathbf{v}$. Then the function

$$\text{div } \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \qquad (1)$$

is called the **divergence** of $\mathbf{v}$ or the *divergence of the vector field defined by* $\mathbf{v}$. For example, if

$$\mathbf{v} = [3xz, 2xy, -yz^2] = 3xz\mathbf{i} + 2xy\mathbf{j} - yz^2\mathbf{k}, \qquad \text{then} \qquad \operatorname{div} \mathbf{v} = 3z + 2x - 2yz.$$

Another common notation for the divergence is

$$\begin{aligned}\operatorname{div} \mathbf{v} = \nabla \bullet \mathbf{v} &= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial \mathbf{y}}, \frac{\partial}{\partial \mathbf{z}}\right] \bullet [v_1, v_2, v_3] \\ &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \bullet (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},\end{aligned}$$

with the understanding that the "product" $(\partial/\partial x)v_1$ in the dot product means the partial derivative $\partial v_1/\partial x$, etc. This is a convenient notation, but nothing more. Note that $\boldsymbol{\nabla} \bullet \mathbf{v}$ means the scalar div $\mathbf{v}$, whereas ∇f means the vector grad f defined in Sec. 9.7.

In Example 2 we shall see that the divergence has an important physical meaning. Clearly, the values of a function that characterizes a physical or geometric property must be independent of the particular choice of coordinates. In other words, these values must be invariant with respect to coordinate transformations. Accordingly, the following theorem should hold.

THEOREM 1

Invariance of the Divergence

The divergence div $\mathbf{v}$ *is a scalar function, that is, its values depend only on the points in space (and, of course, on* $\mathbf{v}$*) but not on the choice of the coordinates in* (1)*, so that with respect to other Cartesian coordinates* x^*, y^*, z^* *and corresponding components* v_1^*, v_2^*, v_3^* *of* $\mathbf{v}$,

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1^*}{\partial x^*} + \frac{\partial v_2^*}{\partial y^*} + \frac{\partial v_3^*}{\partial z^*}. \tag{2}$$

We shall prove this theorem in Sec. 10.7, using integrals.

Presently, let us turn to the more immediate practical task of gaining a feel for the significance of the divergence. Let $f(x, y, z)$ be a twice differentiable scalar function. Then its gradient exists,

$$\mathbf{v} = \operatorname{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and we can differentiate once more, the first component with respect to x, the second with respect to y, the third with respect to z, and then form the divergence,

$$\operatorname{div} \mathbf{v} = \operatorname{div} (\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Hence we have the basic result that *the divergence of the gradient is the Laplacian* (Sec. 9.7),

$$\text{div}\,(\text{grad}\, f) = \nabla^2 f. \tag{3}$$

EXAMPLE 1 Gravitational Force. Laplace's Equation

The gravitational force $\mathbf{p}$ in Theorem 3 of the last section is the gradient of the scalar function $f(x, y, z) = c/r$, which satisfies Laplaces equation $\nabla^2 f = 0$. According to (3) this implies that div $\mathbf{p} = 0$ $(r > 0)$. ■

The following example from hydrodynamics shows the physical significance of the divergence of a vector field. We shall get back to this topic in Sec. 10.8 and add further physical details.

EXAMPLE 2 Flow of a Compressible Fluid. Physical Meaning of the Divergence

We consider the motion of a fluid in a region R having no **sources** or **sinks** in R, that is, no points at which fluid is produced or disappears. The concept of **fluid state** is meant to cover also gases and vapors. Fluids in the restricted sense, or liquids, such as water or oil, have very small compressibility, which can be neglected in many problems. In contrast, gases and vapors have high compressibility. Their density ρ (= mass per unit volume) depends on the coordinates x, y, z in space and may also depend on time t. We assume that our fluid is compressible. We consider the flow through a rectangular box B of small edges Δx, Δy, Δz parallel to the coordinate axes as shown in Fig. 218. (Here Δ is a standard notation for small quantities and, of course, has nothing to do with the notation for the Laplacian in (11) of Sec. 9.7.) The box B has the volume $\Delta V = \Delta x\, \Delta y\, \Delta z$. Let $\mathbf{v} = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be the velocity vector of the motion. We set

$$\mathbf{u} = \rho\mathbf{v} = [u_1, u_2, u_3] = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \tag{4}$$

and assume that $\mathbf{u}$ and $\mathbf{v}$ are continuously differentiable vector functions of x, y, z, and t, that is, they have first partial derivatives which are continuous. Let us calculate the change in the mass included in B by considering the **flux** across the boundary, that is, the total loss of mass leaving B per unit time. Consider the flow through the left of the three faces of B that are visible in Fig. 218, whose area is $\Delta x\, \Delta z$. Since the vectors $v_1\mathbf{i}$ and $v_3\,\mathbf{k}$ are parallel to that face, the components v_1 and v_3 of $\mathbf{v}$ contribute nothing to this flow. Hence the mass of fluid entering through that face during a short time interval Δt is given approximately by

$$(\rho v_2)_y\, \Delta x\, \Delta z\, \Delta t = (u_2)_y\, \Delta x\, \Delta z\, \Delta t,$$

where the subscript y indicates that this expression refers to the left face. The mass of fluid leaving the box B through the opposite face during the same time interval is approximately $(u_2)_{y+\Delta y}\, \Delta x\, \Delta z\, \Delta t$, where the subscript $y + \Delta y$ indicates that this expression refers to the right face (which is not visible in Fig. 218). The difference

$$\Delta u_2\, \Delta x\, \Delta z\, \Delta t = \frac{\Delta u_2}{\Delta y}\, \Delta V\, \Delta t \qquad [\Delta u_2 = (u_2)_{y+\Delta y} - (u_2)_y]$$

is the approximate loss of mass. Two similar expressions are obtained by considering the other two pairs of parallel faces of B. If we add these three expressions, we find that the total loss of mass in B during the time interval Δt is approximately

$$\left(\frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta y} + \frac{\Delta u_3}{\Delta z}\right)\Delta V\, \Delta t,$$

where

$$\Delta u_1 = (u_1)_{x+\Delta x} - (u_1)_x \qquad \text{and} \qquad \Delta u_3 = (u_3)_{z+\Delta z} - (u_3)_z.$$

This loss of mass in B is caused by the time rate of change of the density and is thus equal to

$$-\frac{\partial \rho}{\partial t}\, \Delta V\, \Delta t.$$

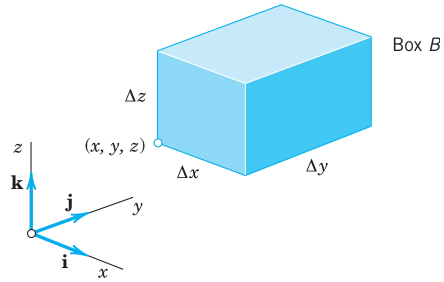

Fig. 218. Physical interpretation of the divergence

If we equate both expressions, divide the resulting equation by $\Delta V \Delta t$, and let Δx, Δy, Δz, and Δt approach zero, then we obtain

$$\operatorname{div} \mathbf{u} = \operatorname{div} (\rho \mathbf{v}) = -\frac{\partial \rho}{\partial t}$$

or

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \mathbf{v}) = 0. \tag{5}$$

This important relation is called the *condition for the conservation of mass* or the **continuity equation** *of a compressible fluid flow.*

If the flow is **steady**, that is, independent of time, then $\partial \rho / \partial t = 0$ and the continuity equation is

$$\operatorname{div} (\rho \mathbf{v}) = 0. \tag{6}$$

If the density ρ is constant, so that the fluid is incompressible, then equation (6) becomes

$$\operatorname{div} \mathbf{v} = 0. \tag{7}$$

This relation is known as the **condition of incompressibility**. It expresses the fact that the balance of outflow and inflow for a given volume element is zero at any time. Clearly, the assumption that the flow has no sources or sinks in R is essential to our argument. $\mathbf{v}$ is also referred to as **solenoidal**.

From this discussion you should conclude and remember that, roughly speaking, ***the divergence measures outflow minus inflow***. ■

Comment. The **divergence theorem** of Gauss, an integral theorem involving the divergence, follows in the next chapter (Sec. 10.7).

PROBLEM SET 9.8

1–6 CALCULATION OF THE DIVERGENCE

Find div $\mathbf{v}$ and its value at P.

1. $\mathbf{v} = [x^2, 4y^2, 9z^2]$, P: $(-1, 0, \frac{1}{2})$

2. $\mathbf{v} = [0, \cos xyz, \sin xyz]$, P: $(2, \frac{1}{2}\pi, 0]$

3. $\mathbf{v} = (x^2 + y^2)^{-1}[x, y]$

4. $\mathbf{v} = [v_1(y, z), v_2(z, x), v_3(x, y)]$, P: $(3, 1, -1)]$

5. $\mathbf{v} = x^2y^2z^2[x, y, z]$, P: $(3, -1, 4)$

6. $\mathbf{v} = (x^2 + y^2 + z^2)^{-3/2}[x, y, z]$

7. For what v_3 is $\mathbf{v} = [e^x \cos y, e^x \sin y, v_3]$ solenoidal?

8. Let $\mathbf{v} = [x, y, v_3]$. Find a v_3 such that **(a)** div $\mathbf{v} > 0$ everywhere, **(b)** div $\mathbf{v} > 0$ if $|z| < 1$ and div $\mathbf{v} < 0$ if $|z| > 1$.