38. TEAM PROJECT. Geometric Applications. To increase your skill in dealing with vectors, use vectors to prove the following (see the figures).

(a) The diagonals of a parallelogram bisect each other.

(b) The line through the midpoints of adjacent sides of a parallelogram bisects one of the diagonals in the ratio 1 : 3.

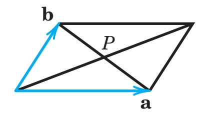
(c) Obtain (b) from (a).

(d) The three medians of a triangle (the segments from a vertex to the midpoint of the opposite side) meet at a single point, which divides the medians in the ratio 2 : 1.

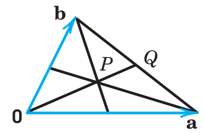
(e) The quadrilateral whose vertices are the midpoints of the sides of an arbitrary quadrilateral is a parallelogram.

(f) The four space diagonals of a parallelepiped meet and bisect each other.

(g) The sum of the vectors drawn from the center of a regular polygon to its vertices is the zero vector.



Team Project 38(a)



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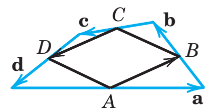

Team Project 38(e)

9.2 Inner Product (Dot Product)

Orthogonality

The inner product or dot product can be motivated by calculating work done by a constant force, determining components of forces, or other applications. It involves the length of vectors and the angle between them. The inner product is a kind of multiplication of two vectors, defined in such a way that the outcome is a scalar. Indeed, another term for inner product is scalar product, a term we shall not use here. The definition of the inner product is as follows.

DEFINITION

Inner Product (Dot Product) of Vectors

The **inner product** or **dot product** $\mathbf{a} \cdot \mathbf{b}$ (read "**a** dot **b**") of two vectors **a** and **b** is the product of their lengths times the cosine of their angle (see Fig. 178),

$$\text{(1)} \qquad \begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos \gamma \qquad && \text{if} \quad \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0} \\ \mathbf{a} \cdot \mathbf{b} &= 0 && \text{if} \quad \mathbf{a} = \mathbf{0} \text{ or } \mathbf{b} = \mathbf{0}. \end{aligned}$$

The angle γ, $0 \leqq \gamma \leqq \pi$, between **a** and **b** is measured when the initial points of the vectors coincide, as in Fig. 178. In components, $\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, and

$$\text{(2)} \qquad \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The second line in (1) is needed because γ is undefined when $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$. The derivation of (2) from (1) is shown below.

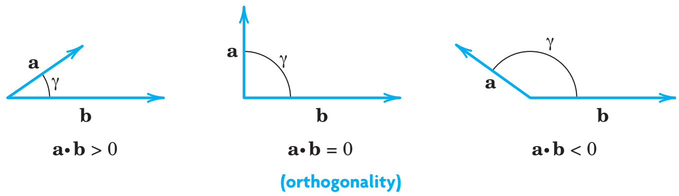

Fig. 178. Angle between vectors and value of inner product

Orthogonality. Since the cosine in (1) may be positive, 0, or negative, so may be the inner product (Fig. 178). The case that the inner product is zero is of particular practical interest and suggests the following concept.

A vector $\mathbf{a}$ is called **orthogonal** to a vector $\mathbf{b}$ if $\mathbf{a} \cdot \mathbf{b} = 0$. Then $\mathbf{b}$ is also orthogonal to $\mathbf{a}$, and we call $\mathbf{a}$ and $\mathbf{b}$ **orthogonal vectors**. Clearly, this happens for nonzero vectors if and only if $\cos\gamma = 0$; thus $\gamma = \pi/2$ (90°). This proves the important

THEOREM 1

Orthogonality Criterion

The inner product of two nonzero vectors is 0 *if and only if these vectors are perpendicular.*

Length and Angle. Equation (1) with $\mathbf{b} = \mathbf{a}$ gives $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$. Hence

(3)
$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

From (3) and (1) we obtain for the angle γ between two nonzero vectors

(4)
$$\cos\gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\mathbf{a} \cdot \mathbf{a}}\sqrt{\mathbf{b} \cdot \mathbf{b}}}.$$

EXAMPLE 1 **Inner Product. Angle Between Vectors**

Find the inner product and the lengths of $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [3, -2, 1]$ as well as the angle between these vectors.

Solution. $\mathbf{a} \cdot \mathbf{b} = 1 \cdot 3 + 2 \cdot (-2) + 0 \cdot 1 = -1$, $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{5}$, $|\mathbf{b}| = \sqrt{\mathbf{b} \cdot \mathbf{b}} = \sqrt{14}$, and (4) gives the angle

$$\gamma = \arccos\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \arccos(-0.11952) = 1.69061 = 96.865°.$$

■

From the definition we see that the inner product has the following properties. For any vectors **a**, **b**, **c** and scalars q_1, q_2,

$$\text{(5)}\quad \begin{aligned} &\text{(a)} && (q_1\mathbf{a} + q_2\mathbf{b}) \cdot \mathbf{c} = q_1\mathbf{a} \cdot \mathbf{c} + q_1\mathbf{b} \cdot \mathbf{c} && \textit{(Linearity)} \\ &\text{(b)} && \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} && \textit{(Symmetry)} \\ &\text{(c)} && \left.\begin{aligned} &\mathbf{a} \cdot \mathbf{a} \geqq 0 \\ &\mathbf{a} \cdot \mathbf{a} = 0 \quad \text{if and only if} \quad \mathbf{a} = \mathbf{0} \end{aligned}\right\} && \textit{(Positive-definiteness).} \end{aligned}$$

Hence *dot multiplication is commutative* as shown by (5b). Furthermore, it is *distributive with respect to vector addition.* This follows from (5a) with $q_1 = 1$ and $q_2 = 1$:

$$\text{(5a*)}\qquad (\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \qquad \textit{(Distributivity).}$$

Furthermore, from (1) and $|\cos \gamma| \leqq 1$ we see that

$$\text{(6)}\qquad |\mathbf{a} \cdot \mathbf{b}| \leqq |\mathbf{a}||\mathbf{b}| \qquad \textit{(Cauchy–Schwarz inequality).}$$

Using this and (3), you may prove (see Prob. 16)

$$\text{(7)}\qquad |\mathbf{a} + \mathbf{b}| \leqq |\mathbf{a}| + |\mathbf{b}| \qquad \textbf{(Triangle inequality).}$$

Geometrically, (7) with $<$ says that one side of a triangle must be shorter than the other two sides together; this motivates the name of (7).

A simple direct calculation with inner products shows that

$$\text{(8)}\qquad |\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2) \qquad \textit{(Parallelogram equality).}$$

Equations (6)–(8) play a basic role in so-called *Hilbert spaces*, which are abstract inner product spaces. Hilbert spaces form the basis of quantum mechanics, for details see [GenRef7] listed in App. 1.

Derivation of (2) from (1). We write $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, as in (8) of Sec. 9.1. If we substitute this into $\mathbf{a} \cdot \mathbf{b}$ and use (5a*), we first have a sum of $3 \times 3 = 9$ products

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1\mathbf{i} \cdot \mathbf{i} + a_1b_2\mathbf{i} \cdot \mathbf{j} + \cdots + a_3b_3\mathbf{k} \cdot \mathbf{k}.$$

Now **i**, **j**, **k** are unit vectors, so that $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ by (3). Since the coordinate axes are perpendicular, so are **i**, **j**, **k**, and Theorem 1 implies that the other six of those nine products are 0, namely, $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = 0$. But this reduces our sum for $\mathbf{a} \cdot \mathbf{b}$ to (2). ■

Applications of Inner Products

Typical applications of inner products are shown in the following examples and in Problem Set 9.2.

EXAMPLE 2 Work Done by a Force Expressed as an Inner Product

This is a major application. It concerns a body on which a *constant* force $\mathbf{p}$ acts. (For a *variable* force, see Sec. 10.1.) Let the body be given a displacement $\mathbf{d}$. Then the work done by $\mathbf{p}$ in the displacement is defined as

$$W = |\mathbf{p}||\mathbf{d}| \cos \alpha = \mathbf{p} \cdot \mathbf{d}, \tag{9}$$

that is, magnitude $|\mathbf{p}|$ of the force times length $|\mathbf{d}|$ of the displacement times the cosine of the angle α between $\mathbf{p}$ and $\mathbf{d}$ (Fig. 179). If $\alpha < 90°$, as in Fig. 179, then $W > 0$. If $\mathbf{p}$ and $\mathbf{d}$ are orthogonal, then the work is zero (why?). If $\alpha > 90°$, then $W < 0$, which means that in the displacement one has to do work against the force. For example, think of swimming across a river at some angle α against the current.

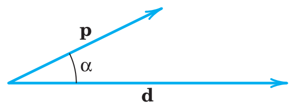

Fig. 179. Work done by a force

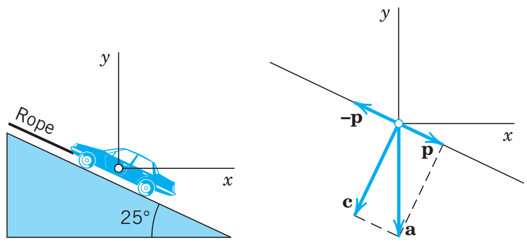

Fig. 180. Example 3

EXAMPLE 3 Component of a Force in a Given Direction

What force in the rope in Fig. 180 will hold a car of 5000 lb in equilibrium if the ramp makes an angle of 25° with the horizontal?

Solution. Introducing coordinates as shown, the weight is $\mathbf{a} = [0, -5000]$ because this force points downward, in the negative y-direction. We have to represent $\mathbf{a}$ as a sum (resultant) of two forces, $\mathbf{a} = \mathbf{c} + \mathbf{p}$, where $\mathbf{c}$ is the force the car exerts on the ramp, which is of no interest to us, and $\mathbf{p}$ is parallel to the rope. A vector in the direction of the rope is (see Fig. 180)

$$\mathbf{b} = [-1, \tan 25°] = [-1, 0.46631], \qquad \text{thus} \qquad |\mathbf{b}| = 1.10338,$$

The direction of the unit vector $\mathbf{u}$ is opposite to the direction of the rope so that

$$\mathbf{u} = -\frac{1}{|\mathbf{b}|}\mathbf{b} = [0.90631, -0.42262].$$

Since $|\mathbf{u}| = 1$ and $\cos \gamma > 0$, we see that we can write our result as

$$|\mathbf{p}| = (|\mathbf{a}| \cos \gamma)|\mathbf{u}| = \mathbf{a} \cdot \mathbf{u} = -\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{5000 \cdot 0.46631}{1.10338} = 2113 \text{ [lb]}.$$

We can also note that $\gamma = 90° - 25° = 65°$ is the angle between $\mathbf{a}$ and $\mathbf{p}$ so that

$$|\mathbf{p}| = |\mathbf{a}| \cos \gamma = 5000 \cos 65° = 2113 \text{ [lb]}.$$

Answer: About 2100 lb. ∎

Example 3 is typical of applications that deal with the **component** or **projection** *of a vector* **a** *in the direction of a vector* **b** $(\neq \mathbf{0})$. If we denote by p the length of the orthogonal projection of **a** on a straight line l parallel to **b** as shown in Fig. 181, then

$$p = |\mathbf{a}| \cos \gamma. \tag{10}$$

Here p is taken with the plus sign if $p\mathbf{b}$ has the direction of **b** and with the minus sign if $p\mathbf{b}$ has the direction opposite to **b**.

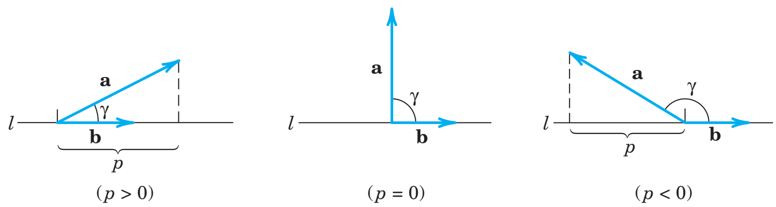

Fig. 181. Component of a vector **a** in the direction of a vector **b**

Multiplying (10) by $|\mathbf{b}|/|\mathbf{b}| = 1$, we have $\mathbf{a} \cdot \mathbf{b}$ in the numerator and thus

$$p = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \qquad (\mathbf{b} \neq \mathbf{0}). \tag{11}$$

If **b** is a unit vector, as it is often used for fixing a direction, then (11) simply gives

$$p = \mathbf{a} \cdot \mathbf{b} \qquad (|\mathbf{b}| = 1). \tag{12}$$

Figure 182 shows the projection p of **a** in the direction of **b** (as in Fig. 181) and the projection $q = |\mathbf{b}| \cos \gamma$ of **b** in the direction of **a**.

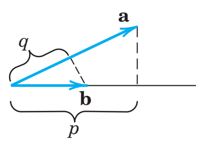

Fig. 182. Projections p of **a** on **b** and q of **b** on **a**

EXAMPLE 4 **Orthonormal Basis**

By definition, an *orthonormal basis* for 3-space is a basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ consisting of orthogonal unit vectors. It has the great advantage that the determination of the coefficients in representations $\mathbf{v} = l_1\mathbf{a} + l_2\mathbf{b} + l_3\mathbf{c}$ of a given vector **v** is very simple. We claim that $l_1 = \mathbf{a} \cdot \mathbf{v}$, $l_2 = \mathbf{b} \cdot \mathbf{v}$, $l_3 = \mathbf{c} \cdot \mathbf{v}$. Indeed, this follows simply by taking the inner products of the representation with **a**, **b**, **c**, respectively, and using the orthonormality of the basis, $\mathbf{a} \cdot \mathbf{v} = l_1\mathbf{a} \cdot \mathbf{a} + l_2\mathbf{a} \cdot \mathbf{b} + l_3\mathbf{a} \cdot \mathbf{c} = l_1$, etc.

For example, the unit vectors **i**, **j**, **k** in (8), Sec. 9.1, associated with a Cartesian coordinate system form an orthonormal basis, called the **standard basis** with respect to the given coordinate system. ■

EXAMPLE 5 **Orthogonal Straight Lines in the Plane**

Find the straight line L_1 through the point P: (1, 3) in the xy-plane and perpendicular to the straight line $L_2: x - 2y + 2 = 0$; see Fig. 183.

Solution. The idea is to write a general straight line $L_1: a_1x + a_2y = c$ as $\mathbf{a} \bullet \mathbf{r} = c$ with $\mathbf{a} = [a_1, a_2] \neq \mathbf{0}$ and $\mathbf{r} = [x, y]$, according to (2). Now the line L_1^* through the origin and parallel to L_1 is $\mathbf{a} \bullet \mathbf{r} = 0$. Hence, by Theorem 1, the vector $\mathbf{a}$ is perpendicular to $\mathbf{r}$. Hence it is perpendicular to L_1^* and also to L_1 because L_1 and L_1^* are parallel. $\mathbf{a}$ is called a **normal vector** of L_1 (and of L_1^*).

Now a normal vector of the given line $x - 2y + 2 = 0$ is $\mathbf{b} = [1, -2]$. Thus L_1 is perpendicular to L_2 if $\mathbf{b} \bullet \mathbf{a} = a_1 - 2a_2 = 0$, for instance, if $\mathbf{a} = [2, 1]$. Hence L_1 is given by $2x + y = c$. It passes through P: (1, 3) when $2 \cdot 1 + 3 = c = 5$. *Answer:* $y = -2x + 5$. Show that the point of intersection is $(x, y) = (1.6, 1.8)$. ■

EXAMPLE 6 **Normal Vector to a Plane**

Find a unit vector perpendicular to the plane $4x + 2y + 4z = -7$.

Solution. Using (2), we may write any plane in space as

$$\mathbf{a} \bullet \mathbf{r} = a_1x + a_2y + a_3z = c \tag{13}$$

where $\mathbf{a} = [a_1, a_2, a_3] \neq \mathbf{0}$ and $\mathbf{r} = [x, y, z]$. The unit vector in the direction of $\mathbf{a}$ is (Fig. 184)

$$\mathbf{n} = \frac{1}{|\mathbf{a}|}\mathbf{a}.$$

Dividing by $|\mathbf{a}|$, we obtain from (13)

$$\mathbf{n} \bullet \mathbf{r} = p \qquad \text{where} \qquad p = \frac{c}{|\mathbf{a}|}. \tag{14}$$

From (12) we see that p is the projection of $\mathbf{r}$ in the direction of $\mathbf{n}$. This projection has the same constant value $c/|\mathbf{a}|$ for the position vector $\mathbf{r}$ of any point in the plane. Clearly this holds if and only if $\mathbf{n}$ is perpendicular to the plane. $\mathbf{n}$ is called a **unit normal vector** of the plane (the other being $-\mathbf{n}$).

Furthermore, from this and the definition of projection, it follows that $|p|$ is the distance of the plane from the origin. Representation (14) is called **Hesse's[2] normal form** of a plane. In our case, $\mathbf{a} = [4, 2, 4]$, $c = -7$, $|\mathbf{a}| = 6$, $\mathbf{n} = \frac{1}{6}\mathbf{a} = [\frac{2}{3}, \frac{1}{3}, \frac{2}{3}]$, and the plane has the distance $\frac{7}{6}$ from the origin. ■

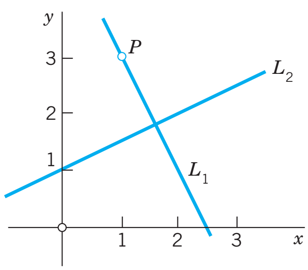

Fig. 183. Example 5

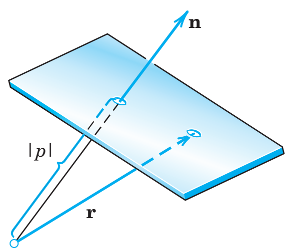

Fig. 184. Normal vector to a plane

[2]LUDWIG OTTO HESSE (1811–1874), German mathematician who contributed to the theory of curves and surfaces.