5. For each given subset S of $\mathcal{P}_3$, find a subset B of S that is a basis for $\mathcal{V} = \text{span}(S)$.

★**(a)** $S = \{x^3 - 8x^2 + 1, 3x^3 - 2x^2 + x, 4x^3 + 2x - 10, x^3 - 20x^2 - x + 12, x^3 + 24x^2 + 2x - 13, x^3 + 14x^2 - 7x + 18\}$

(b) $S = \{-2x^3 + x + 2, 3x^3 - x^2 + 4x + 6, 8x^3 + x^2 + 6x + 10, -4x^3 - 3x^2 + 3x + 4, -3x^3 - 4x^2 + 8x + 12\}$

★**(c)** S = the set of all polynomials in $\mathcal{P}_3$ with a zero constant term

(d) $S = \mathcal{P}_2$

★**(e)** S = the set of all polynomials in $\mathcal{P}_3$ with the coefficient of the x^2 term equal to the coefficient of the x^3 term

(f) S = the set of all polynomials in $\mathcal{P}_3$ with the coefficient of the x^3 term equal to 8

6. For each given subset S of $\mathcal{M}_{33}$, find a subset B of S that is a basis for $\mathcal{V} = \text{span}(S)$.

★**(a)** $S = \{\mathbf{A} \in \mathcal{M}_{33} \mid \text{each } a_{ij} \text{ is either 0 or 1}\}$

(b) $S = \{\mathbf{A} \in \mathcal{M}_{33} \mid \text{each } a_{ij} \text{ is either 1 or } -1\}$

★**(c)** S = the set of all symmetric 3×3 matrices

(d) S = the set of all nonsingular 3×3 matrices

7. Enlarge each of the following linearly independent subsets T of $\mathbb{R}^5$ to a basis B for $\mathbb{R}^5$ containing T:

★**(a)** $T = \{[1, -3, 0, 1, 4], [2, 2, 1, -3, 1]\}$

(b) $T = \{[1, 1, 1, 1, 1], [0, 1, 1, 1, 1], [0, 0, 1, 1, 1]\}$

★**(c)** $T = \{[1, 0, -1, 0, 0], [0, 1, -1, 1, 0], [2, 3, -8, -1, 0]\}$

8. Enlarge each of the following linearly independent subsets T of $\mathcal{P}_4$ to a basis B for $\mathcal{P}_4$ that contains T:

★**(a)** $T = \{x^3 - x^2, x^4 - 3x^3 + 5x^2 - x\}$

(b) $T = \{3x - 2, x^3 - 6x + 4\}$

★**(c)** $T = \{x^4 - x^3 + x^2 - x + 1, x^3 - x^2 + x - 1, x^2 - x + 1\}$

9. Enlarge each of the following linearly independent subsets T of $\mathcal{M}_{32}$ to a basis B for $\mathcal{M}_{32}$ that contains T:

★**(a)** $T = \left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}$

(b) $T = \left\{ \begin{bmatrix} 0 & -2 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 0 & 1 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -4 & 8 \end{bmatrix} \right\}$

$\star$**(c)** $T = \left\{ \begin{bmatrix} 3 & 0 \\ -1 & 7 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 3 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 6 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$

$\star$**10.** Find a basis for the vector space $\mathcal{U}_4$ consisting of all 4×4 upper triangular matrices.

11. In each case, find the dimension of $\mathcal{V}$ by using an appropriate method to create a basis.

(a) $\mathcal{V} = \text{span}(\{[5,2,1,0,-1], [3,0,1,1,0], [0,0,0,0,0], [-2,4,-2,-4,-2], [0,12,-4,-10,-6], [-6,0,-2,-2,0]\})$, a subspace of $\mathbb{R}^5$

$\star$**(b)** $\mathcal{V} = \{\mathbf{A} \in \mathcal{M}_{33} \mid \text{trace}(\mathbf{A}) = 0\}$, a subspace of $\mathcal{M}_{33}$ (Recall that the trace of a matrix is the sum of the terms on the main diagonal.)

(c) $\mathcal{V} = \text{span}(\{x^4 - x^3 + 2x^2, 2x^4 + x - 5, 2x^3 - 4x^2 + x - 4, 6, x^2 - 1\})$

$\star$**(d)** $\mathcal{V} = \{\mathbf{p} \in \mathcal{P}_6 \mid \mathbf{p} = ax^6 - bx^5 + ax^4 - cx^3 + (a+b+c)x^2 - (a-c)x + (3a - 2b + 16c)$, for real numbers $a, b,$ and $c\}$

12. **(a)** Show that each of these subspaces of $\mathcal{M}_{nn}$ has dimension $(n^2 + n)/2$.

(i) The set of upper triangular $n \times n$ matrices

(ii) The set of lower triangular $n \times n$ matrices

(iii) The set of symmetric $n \times n$ matrices

$\star$**(b)** What is the dimension of the set of skew-symmetric $n \times n$ matrices?

13. Let $\mathbf{A}$ be an $m \times n$ matrix.

(a) Prove that $S_{\mathbf{A}} = \{\mathbf{X} \in \mathbb{R}^n \mid \mathbf{AX} = \mathbf{0}\}$, the solution set of the homogeneous system $\mathbf{AX} = \mathbf{0}$, is a subspace of $\mathbb{R}^n$.

(b) Prove that $\dim(S_{\mathbf{A}}) + \text{rank}(\mathbf{A}) = n$. (Hint: First consider the case where $\mathbf{A}$ is in reduced row echelon form.)

▶**14.** Prove Theorem 4.17. This proof should be similar to the part of the proof for Theorem 4.16 outlined in parts (a), (b), and (c) of Exercise 22 in Section 4.5. However, change the definition of the set A in that exercise so that each set T is a subset of S rather than of $\mathcal{W}$.

15. Let $\mathcal{W}$ be a subspace of a finite dimensional vector space $\mathcal{V}$.

(a) Show that $\mathcal{V}$ has some basis B with a subset B' that is a basis for $\mathcal{W}$.

$\star$**(b)** If B is any given basis for $\mathcal{V}$, must some subset B' of B be a basis for $\mathcal{W}$? Prove that your answer is correct.

$\star$**(c)** If B is any given basis for $\mathcal{V}$ and $B' \subseteq B$, is there necessarily a subspace $\mathcal{Y}$ of $\mathcal{V}$ such that B' is a basis for $\mathcal{Y}$? Why or why not?

16. Let $\mathcal{V}$ be a finite dimensional vector space, and let $\mathcal{W}$ be a subspace of $\mathcal{V}$.

(a) Prove that $\mathcal{V}$ has a subspace $\mathcal{W}'$ such that every vector in $\mathcal{V}$ can be uniquely expressed as a sum of a vector in $\mathcal{W}$ and a vector in $\mathcal{W}'$. (In other words, show that there is a subspace $\mathcal{W}'$ so that, for every $\mathbf{v}$ in $\mathcal{V}$, there are unique vectors $\mathbf{w} \in \mathcal{W}$ and $\mathbf{w}' \in \mathcal{W}'$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}'$.)

★**(b)** Give an example of a subspace $\mathcal{W}$ of some finite dimensional vector space $\mathcal{V}$ for which the subspace $\mathcal{W}'$ from part (a) is not unique.

17. **(a)** Let S be a finite subset of $\mathbb{R}^n$. Prove that the Simplified Span Method applied to S produces the standard basis for $\mathbb{R}^n$ if and only if $\text{span}(S) = \mathbb{R}^n$.

(b) Let $B \subseteq \mathbb{R}^n$ with $|B| = n$, and let $\mathbf{A}$ be the $n \times n$ matrix whose rows are the vectors in B. Prove that B is a basis for $\mathbb{R}^n$ if and only if $|\mathbf{A}| \neq 0$.

18. Let $\mathbf{A}$ be an $m \times n$ matrix and let S be the set of vectors consisting of the rows of $\mathbf{A}$.

(a) Use the Simplified Span Method to show that $\dim(\text{span}(S)) = \text{rank}(\mathbf{A})$.

(b) Use the Independence Test Method to prove that $\dim(\text{span}(S)) = \text{rank}(\mathbf{A}^T)$.

(c) Use parts (a) and (b) to prove that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$. (We will state this formally as Corollary 5.11 in Section 5.3.)

19. Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be any real numbers, with $n > 2$. Consider the $n \times n$ matrix $\mathbf{A}$ whose (i, j) term is $a_{ij} = \sin(\alpha_i + \beta_j)$. Prove that $|\mathbf{A}| = 0$. (Hint: Consider $\mathbf{x}_1 = [\sin\beta_1, \sin\beta_2, \ldots, \sin\beta_n]$, $\mathbf{x}_2 = [\cos\beta_1, \cos\beta_2, \ldots, \cos\beta_n]$. Show that the row space of $\mathbf{A} \subseteq \text{span}(\{\mathbf{x}_1, \mathbf{x}_2\})$, and hence, dim(row space of $\mathbf{A}$) $< n$.)

★**20.** True or False:

(a) Given any spanning set S for a finite dimensional vector space $\mathcal{V}$, there is some $B \subseteq S$ that is a basis for $\mathcal{V}$.

(b) Given any linearly independent set T in a finite dimensional vector space $\mathcal{V}$, there is a basis B for $\mathcal{V}$ containing T.

(c) If S is a finite spanning set for $\mathbb{R}^n$, then the Simplified Span Method must produce a subset of S that is a basis for $\mathbb{R}^n$.

(d) If S is a finite spanning set for $\mathbb{R}^n$, then the Independence Test Method produces a subset of S that is a basis for $\mathbb{R}^n$.

(e) If S is a finite spanning set for $\mathbb{R}^n$, then the Inspection Method produces a subset of S that is a basis for $\mathbb{R}^n$.

(f) If T is a linearly independent set in $\mathbb{R}^n$, then the Enlarging Method must produce a subset of T that is a basis for $\mathbb{R}^n$.

(g) Before row reduction, the Simplified Span Method places the vectors of a given spanning set S as columns in a matrix, while the Independence Test Method places the vectors of S as rows.

4.7 COORDINATIZATION

If B is a basis for a vector space $\mathcal{V}$, then we know every vector in $\mathcal{V}$ has a unique expression as a linear combination of the vectors in B. For example, the vector $[a_1,\ldots,a_n]$ in $\mathbb{R}^n$ is written as a linear combination of the standard basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ for $\mathbb{R}^n$ in a natural and unique way as $a_1\mathbf{e}_1+\cdots+a_n\mathbf{e}_n$. Dealing with the standard basis in $\mathbb{R}^n$ is easy because the coefficients in the linear combination are the same as the coordinates of the vector. However, this is not necessarily true for other bases.

In this section, we develop a process, called coordinatization, for representing any vector in a finite dimensional vector space in terms of its coefficients with respect to a given basis. We also determine how the coordinatization changes whenever we switch bases.

Coordinates with Respect to an Ordered Basis

Definition An **ordered basis** for a vector space $\mathcal{V}$ is an ordered n-tuple of vectors $(\mathbf{v}_1,\ldots,\mathbf{v}_n)$ such that the set $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ is a basis for $\mathcal{V}$.

In an ordered basis, the elements are written in a specific order. Thus, $(\mathbf{i},\mathbf{j},\mathbf{k})$ and $(\mathbf{j},\mathbf{i},\mathbf{k})$ are different ordered bases for $\mathbb{R}^3$.

By Theorem 4.9, if $B=(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n)$ is an ordered basis for $\mathcal{V}$, then for every vector $\mathbf{w}\in\mathcal{V}$, there are unique scalars $a_1,a_2,\ldots,a_n$ such that $\mathbf{w}=a_1\mathbf{v}_1+a_2\mathbf{v}_2+\cdots+a_n\mathbf{v}_n$. We use these scalars $a_1,a_2,\ldots,a_n$ to **coordinatize** the vector $\mathbf{w}$ as follows:

Definition Let $B=(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n)$ be an ordered basis for a vector space $\mathcal{V}$. Suppose that $\mathbf{w}=a_1\mathbf{v}_1+a_2\mathbf{v}_2+\cdots+a_n\mathbf{v}_n\in\mathcal{V}$. Then $[\mathbf{w}]_B$, the **coordinatization of w with respect to** B, is the n-vector $[a_1,a_2,\ldots,a_n]$.

The vector $[\mathbf{w}]_B=[a_1,a_2,\ldots,a_n]$ is frequently referred to as "**w expressed in** B**-coordinates**." When useful, we will express $[\mathbf{w}]_B$ as a column vector.

Example 1

Let $B=([4,2],[1,3])$ be an ordered basis for $\mathbb{R}^2$. Notice that $[4,2]=1[4,2]+0[1,3]$, so $[4,2]_B=[1,0]$. Similarly, $[1,3]_B=[0,1]$. From a geometric viewpoint, converting to B-coordinates in $\mathbb{R}^2$ results in a new coordinate system in $\mathbb{R}^2$ with $[4,2]$ and $[1,3]$ as its "unit" vectors. The new coordinate grid consists of parallelograms whose sides are the vectors in B, as shown in Figure 4.6. For example, $[11,13]$ equals $[2,3]$ when expressed in B-coordinates because $[11,13]=2[4,2]+3[1,3]$. In other words, $[11,13]_B=[2,3]$.

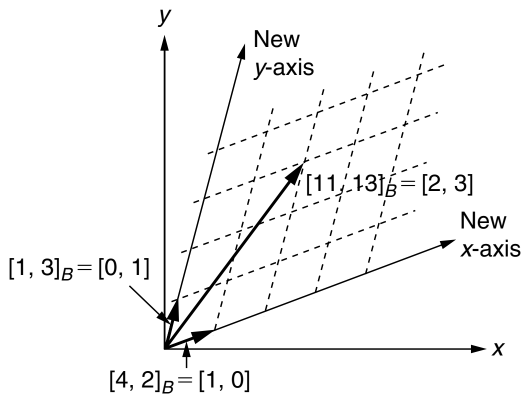

FIGURE 4.6

A B-coordinate grid in $\mathbb{R}^2$: picturing $[11,13]$ in B-coordinates

Example 2

Let $B = (x^3, x^2, x, 1)$, an ordered basis for $\mathcal{P}_3$. Then $[6x^3 - 2x + 18]_B = [6, 0, -2, 18]$, and $[4 - 3x + 9x^2 - 7x^3]_B = [-7, 9, -3, 4]$. Notice also that $[x^3]_B = [1, 0, 0, 0], [x^2]_B = [0, 1, 0, 0]$, $[x]_B = [0, 0, 1, 0]$, and $[1]_B = [0, 0, 0, 1]$.

As part of Example 2, we saw an illustration of the general principle that if $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$, then every vector in B itself has a simple coordinatization. In particular, $[\mathbf{v}_i]_B = \mathbf{e}_i$. You are asked to prove this in Exercise 6.

Using Row Reduction to Coordinatize a Vector

Example 3

Consider the subspace $\mathcal{V}$ of $\mathbb{R}^5$ spanned by the ordered basis

$$C = \big([-4, 5, -1, 0, -1], [1, -3, 2, 2, 5], [1, -2, 1, 1, 3]\big).$$

Notice that the vectors in $\mathcal{V}$ can be put into C-coordinates by solving an appropriate system. For example, to find $[-23, 30, -7, -1, -7]_C$, we solve the equation

$$[-23, 30, -7, -1, -7] = a[-4, 5, -1, 0, -1] + b[1, -3, 2, 2, 5] + c[1, -2, 1, 1, 3].$$

The equivalent system is

$$\begin{cases} -4a + b + c = -23 \\ 5a - 3b - 2c = 30 \\ -a + 2b + c = -7 \\ 2b + c = -1 \\ -a + 5b + 3c = -7 \end{cases}.$$

To solve this system, we row reduce

$$\left[\begin{array}{rrr|r} -4 & 1 & 1 & -23 \\ 5 & -3 & -2 & 30 \\ -1 & 2 & 1 & -7 \\ 0 & 2 & 1 & -1 \\ -1 & 5 & 3 & -7 \end{array}\right] \quad \text{to obtain} \quad \left[\begin{array}{ccc|r} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right].$$

Hence, the (unique) solution for the system is $a = 6, b = -2, c = 3$, and we see that $[-23, 30, -7, -1, -7]_C = [6, -2, 3]$.

On the other hand, vectors in $\mathbb{R}^5$ that are not in span(C) cannot be expressed in C-coordinates. For example, the vector $[1, 2, 3, 4, 5]$ is not in $\mathcal{V} =$ span(C). To see this, consider the system

$$\begin{cases} -4a + b + c = 1 \\ 5a - 3b - 2c = 2 \\ {-a} + 2b + c = 3 \\ \phantom{-4a+{}} 2b + c = 4 \\ {-a} + 5b + 3c = 5 \end{cases}.$$

We solve this system by row reducing

$$\left[\begin{array}{rrr|r} -4 & 1 & 1 & 1 \\ 5 & -3 & -2 & 2 \\ -1 & 2 & 1 & 3 \\ 0 & 2 & 1 & 4 \\ -1 & 5 & 3 & 5 \end{array}\right] \quad \text{to obtain} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right].$$

This result tells us that the system has no solutions, implying that the vector $[1, 2, 3, 4, 5]$ is not in span(S).

■

Notice in Example 3 that the coordinatized vector $[6, -2, 3]$ is more "compact" than the original vector $[-23, 30, -7, -1, -7]$ but still contains the same essential information.

As we saw in Example 3, finding the coordinates of a vector with respect to an ordered basis typically amounts to solving a system of linear equations, which is frequently done using row reduction. The computations we did in Example 3 motivate the following method, which works in general. Although it applies to subspaces of $\mathbb{R}^n$, we can adapt it to other finite dimensional vector spaces, such as $\mathcal{P}_n$ and $\mathcal{M}_{mn}$, as with other techniques we have examined. We handle these other vector spaces "informally" in this chapter, but we will treat them more formally in Section 5.5.

Method for Coordinatizing a Vector with Respect to a Finite Ordered Basis (Coordinatization Method)

Let $\mathcal{V}$ be a nontrivial subspace of $\mathbb{R}^n$, let $B = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ be an ordered basis for $\mathcal{V}$, and let $\mathbf{v} \in \mathbb{R}^n$. To calculate $[\mathbf{v}]_B$, if possible, perform the following steps:

Step 1: Form an augmented matrix $[\mathbf{A} \,|\, \mathbf{v}]$ by using the vectors in B as the *columns* of $\mathbf{A}$, in *order*, and using $\mathbf{v}$ as a column on the right.

Step 2: Row reduce $[\mathbf{A}\,|\,\mathbf{v}]$ to obtain the reduced row echelon form $[\mathbf{C}\,|\,\mathbf{w}]$.

Step 3: If there is a row of $[\mathbf{C}\,|\,\mathbf{w}]$ that contains all zeroes on the left and has a nonzero entry on the right, then $\mathbf{v} \notin \text{span}(B) = \mathcal{V}$, and coordinatization is not possible. Stop.

Step 4: Otherwise, $\mathbf{v} \in \text{span}(B) = \mathcal{V}$. Eliminate all rows consisting entirely of zeroes in $[\mathbf{C}\,|\,\mathbf{w}]$ to obtain $[\mathbf{I}_k\,|\,\mathbf{y}]$. Then, $[\mathbf{v}]_B = \mathbf{y}$, the last column of $[\mathbf{I}_k\,|\,\mathbf{y}]$.

Example 4

Let $\mathcal{V}$ be the subspace of $\mathbb{R}^3$ spanned by the ordered basis

$$B = ([2,-1,3],[3,2,1]).$$

We use the Coordinatization Method to find $[\mathbf{v}]_B$, where $\mathbf{v} = [5,-6,11]$. To do this, we set up the augmented matrix

$$\left[\begin{array}{rr|r} 2 & 3 & 5 \\ -1 & 2 & -6 \\ 3 & 1 & 11 \end{array}\right], \quad \text{which row reduces to} \quad \left[\begin{array}{rr|r} 1 & 0 & 4 \\ 0 & 1 & -1 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right].$$

Ignoring the bottom row of zeroes, we discover $[\mathbf{v}]_B = [4,-1]$.

Similarly, applying the Coordinatization Method to the vector $[1,2,3]$, we see that

$$\left[\begin{array}{rr|r} 2 & 3 & 1 \\ -1 & 2 & 2 \\ 3 & 1 & 3 \end{array}\right] \quad \text{row reduces to} \quad \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

From the third row, we see that coordinatization of $[1,2,3]$ with respect to B is not possible by Step 3 of the Coordinatization Method. ■

Fundamental Properties of Coordinatization

The following theorem shows that the coordinatization of a vector behaves in a manner similar to the original vector with respect to addition and scalar multiplication:

Theorem 4.19 Let $B = (\mathbf{v}_1,\ldots,\mathbf{v}_n)$ be an ordered basis for a vector space $\mathcal{V}$. Suppose $\mathbf{w}_1,\ldots,\mathbf{w}_k \in \mathcal{V}$ and $a_1,\ldots,a_k$ are scalars. Then

(1) $[\mathbf{w}_1 + \mathbf{w}_2]_B = [\mathbf{w}_1]_B + [\mathbf{w}_2]_B$

(2) $[a_1\mathbf{w}_1]_B = a_1[\mathbf{w}_1]_B$

(3) $[a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_k\mathbf{w}_k]_B = a_1[\mathbf{w}_1]_B + a_2[\mathbf{w}_2]_B + \cdots + a_k[\mathbf{w}_k]_B$

Figure 4.7 illustrates part (1) of this theorem. Moving along either path from the upper left to the lower right in the diagram produces the same answer. (Such a picture is called a **commutative diagram**.)

Part (3) asserts that to put a linear combination of vectors in $\mathcal{V}$ into B-coordinates, we can first find the B-coordinates of each vector individually and then calculate the analogous linear combination in $\mathbb{R}^n$. The proof of Theorem 4.19 is left for you to do in Exercise 13.

Example 5

Recall the subspace $\mathcal{V}$ of $\mathbb{R}^5$ from Example 3 spanned by the ordered basis

$$C = \big([-4,5,-1,0,-1],[1,-3,2,2,5],[1,-2,1,1,3]\big).$$

Consider the vectors $\mathbf{x} = [1,0,-1,0,4], \mathbf{y} = [0,1,-1,0,3], \mathbf{z} = [0,0,0,1,5]$. Applying the Coordinatization Method to $\mathbf{x}$, we find that the augmented matrix

$$\left[\begin{array}{rrr|r} -4 & 1 & 1 & 1 \\ 5 & -3 & -2 & 0 \\ -1 & 2 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ -1 & 5 & 3 & 4 \end{array}\right] \quad \text{row reduces to} \quad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 10 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right].$$

Ignoring the last two rows of zeroes, we obtain $[\mathbf{x}]_C = [1,-5,10]$. In a similar manner we can calculate $[\mathbf{y}]_C = [1,-4,8]$ and $[\mathbf{z}]_C = [1,-3,7]$.

Using Theorem 4.19, it is now a simple matter to find the coordinatization of any linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. For example, consider the vector $2\mathbf{x} - 7\mathbf{y} + 3\mathbf{z}$, which is easily computed to be $[2,-7,5,3,2]$. Theorem 4.19 asserts that $[2\mathbf{x} - 7\mathbf{y} + 3\mathbf{z}]_C = 2[\mathbf{x}]_C - 7[\mathbf{y}]_C + 3[\mathbf{z}]_C = 2[1,-5,10] - 7[1,-4,8] + 3[1,-3,7] = [-2,9,-15]$. This result is easily checked by noting that $-2[-4,5,-1,0,-1] + 9[1,-3,2,2,5] - 15[1,-2,1,1,3]$ really does equal $[2,-7,5,3,2]$. ■

$$\begin{tikzcd}
\overset{\mathbf{w}_1,\,\mathbf{w}_2}{\mathcal{V}} \arrow[r, "+"] \arrow[d, "\text{Coordinatization}"'] & \overset{\mathbf{w}_1+\mathbf{w}_2}{\mathcal{V}} \arrow[d, "\text{Coordinatization}"] \\
\underset{[\mathbf{w}_1]_B,\,[\mathbf{w}_2]_B}{\mathbb{R}^n} \arrow[r, "+"] & \underset{[\mathbf{w}_1]_B+[\mathbf{w}_2]_B}{\mathbb{R}^n}
\end{tikzcd}$$

FIGURE 4.7

Commutative diagram involving addition and coordinatization of vectors

The Transition Matrix for Change of Coordinates

Our next goal is to determine how the coordinates of a vector change when we convert from one ordered basis to another.

Definition Suppose that $\mathcal{V}$ is a nontrivial n-dimensional vector space with ordered bases B and C. Let $\mathbf{P}$ be the $n \times n$ matrix whose ith column, for $1 \leq i \leq n$, equals $[\mathbf{b}_i]_C$, where $\mathbf{b}_i$ is the ith basis vector in B. Then $\mathbf{P}$ is called the **transition matrix from B-coordinates to C-coordinates**.

We often refer to the matrix $\mathbf{P}$ in this definition as the "**transition matrix from B to C**."

Example 6

Recall from Example 5 the subspace $\mathcal{V}$ of $\mathbb{R}^5$ that is spanned by the ordered basis $C = ([-4,5,-1,0,-1],[1,-3,2,2,5],[1,-2,1,1,3])$. Using the Simplified Span Method on the vectors in C produces the vectors $\mathbf{x} = [1,0,-1,0,4]$, $\mathbf{y} = [0,1,-1,0,3]$, and $\mathbf{z} = [0,0,0,1,5]$ from Example 5. Thus $B = (\mathbf{x},\mathbf{y},\mathbf{z})$ is also an ordered basis for $\mathcal{V}$. To find the transition matrix from B to C we must solve for the C-coordinates of each vector in B. In Example 5, we used the Coordinatization Method on each of $\mathbf{x},\mathbf{y}$, and $\mathbf{z}$ in turn. However, we could have obtained the same result by applying the Coordinatization Method to $\mathbf{x},\mathbf{y}$, and $\mathbf{z}$ simultaneously — that is, by row reducing the augmented matrix

$$\left[\begin{array}{rrr|rrr} -4 & 1 & 1 & 1 & 0 & 0 \\ 5 & -3 & -2 & 0 & 1 & 0 \\ -1 & 2 & 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ -1 & 5 & 3 & 4 & 3 & 5 \end{array}\right] \quad \text{to obtain} \quad \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -5 & -4 & -3 \\ 0 & 0 & 1 & 10 & 8 & 7 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right].$$

This gives $[\mathbf{x}]_C = [1,-5,10]$, $[\mathbf{y}]_C = [1,-4,8]$, and $[\mathbf{z}]_C = [1,-3,7]$ (as we saw earlier). These vectors form the columns of the transition matrix from B to C, namely,

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ -5 & -4 & -3 \\ 10 & 8 & 7 \end{bmatrix}.$$

■

Example 6 illustrates that solving for the columns of the transition matrix can be accomplished efficiently by performing a single row reduction using an augmented matrix with several columns to the right of the augmentation bar. Hence, we have the following:

Method for Calculating a Transition Matrix (Transition Matrix Method)
To find the transition matrix $\mathbf{P}$ from B to C where B and C are ordered bases for a nontrivial k-dimensional subspace of $\mathbb{R}^n$, use row reduction on

$$\left[\begin{array}{cccc|cccc} \text{1st} & \text{2nd} & & k\text{th} & \text{1st} & \text{2nd} & & k\text{th} \\ \text{vector} & \text{vector} & \cdots & \text{vector} & \text{vector} & \text{vector} & \cdots & \text{vector} \\ \text{in} & \text{in} & & \text{in} & \text{in} & \text{in} & & \text{in} \\ C & C & & C & B & B & & B \end{array}\right]$$

to produce $\left[\begin{array}{c|c} \mathbf{I}_k & \mathbf{P} \\ \hline \text{rows of} & \text{zeroes} \end{array}\right].$

In Exercise 8, you are asked to show that, in the special cases where either B or C is the standard basis in $\mathbb{R}^n$, there are simple expressions for the transition matrix from B to C.

Example 7

Consider the following ordered bases for $\mathcal{U}_2$:

$$B = \left(\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right) \quad \text{and} \quad C = \left(\begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix} \right).$$

Expressing the matrices in B and C as column vectors, we use the Transition Matrix Method to find the transition matrix from B to C by row reducing

$$\left[\begin{array}{ccc|ccc} 22 & 12 & 33 & 7 & 1 & 1 \\ 7 & 4 & 12 & 3 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & -1 & 1 \end{array}\right] \quad \text{to obtain} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -4 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right].$$

Ignoring the final row of zeroes, we see that the transition matrix from B to C is given by

$$\mathbf{P} = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Change of Coordinates Using the Transition Matrix

The next theorem shows that the transition matrix can be used to change the coordinatization of a vector $\mathbf{v}$ from one ordered basis B to another ordered basis C. That is, if $[\mathbf{v}]_B$ is known, then $[\mathbf{v}]_C$ can be found by using the transition matrix from B to C.

Theorem 4.20 Suppose that B and C are ordered bases for a nontrivial n-dimensional vector space $\mathcal{V}$, and let $\mathbf{P}$ be an $n \times n$ matrix. Then $\mathbf{P}$ is the transition matrix from B to C if and only if for every $\mathbf{v} \in \mathcal{V}$, $\mathbf{P}[\mathbf{v}]_B = [\mathbf{v}]_C$.

Proof. Let B and C be ordered bases for a vector space $\mathcal{V}$, with $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$.

First, suppose $\mathbf{P}$ is the transition matrix from B to C. Let $\mathbf{v} \in \mathcal{V}$. We want to show $\mathbf{P}[\mathbf{v}]_B = [\mathbf{v}]_C$. Suppose $[\mathbf{v}]_B = [a_1, \ldots, a_n]$. Then $\mathbf{v} = a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n$. Hence,

$$\begin{aligned}\mathbf{P}[\mathbf{v}]_B &= \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \\ &= a_1 \begin{bmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{n1} \end{bmatrix} + a_2 \begin{bmatrix} p_{12} \\ p_{22} \\ \vdots \\ p_{n2} \end{bmatrix} + \cdots + a_n \begin{bmatrix} p_{1n} \\ p_{2n} \\ \vdots \\ p_{nn} \end{bmatrix}.\end{aligned}$$

However, $\mathbf{P}$ is the transition matrix from B to C, so the ith column of $\mathbf{P}$ equals $[\mathbf{b}_i]_C$. Therefore,

$$\begin{aligned}\mathbf{P}[\mathbf{v}]_B &= a_1[\mathbf{b}_1]_C + a_2[\mathbf{b}_2]_C + \cdots + a_n[\mathbf{b}_n]_C \\ &= [a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n]_C && \text{by Theorem 4.19} \\ &= [\mathbf{v}]_C.\end{aligned}$$

Conversely, suppose that $\mathbf{P}$ is an $n \times n$ matrix and that $\mathbf{P}[\mathbf{v}]_B = [\mathbf{v}]_C$ for every $\mathbf{v} \in \mathcal{V}$. We show that $\mathbf{P}$ is the transition matrix from B to C. By definition, it is enough to show that the ith column of $\mathbf{P}$ is equal to $[\mathbf{b}_i]_C$. Since $\mathbf{P}[\mathbf{v}]_B = [\mathbf{v}]_C$, for all $\mathbf{v} \in \mathcal{V}$, let $\mathbf{v} = \mathbf{b}_i$. Then since $[\mathbf{v}]_B = \mathbf{e}_i$, we have $\mathbf{P}[\mathbf{v}]_B = \mathbf{P}\mathbf{e}_i = [\mathbf{b}_i]_C$. But $\mathbf{P}\mathbf{e}_i = i$th column of $\mathbf{P}$, which completes the proof. □

Example 8

Recall the ordered bases for $\mathcal{U}_2$ from Example 7:

$$B = \left(\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right) \quad \text{and} \quad C = \left(\begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix} \right).$$

In that example, we found that the transition matrix $\mathbf{P}$ from B to C is

$$\mathbf{P} = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

This gives a quick way of changing the coordinatization of any vector in $\mathcal{U}_2$ from B-coordinates to C-coordinates. For example, let $\mathbf{v} = \begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix}$. Since

$$\begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix} = 4\begin{bmatrix} 7 & 3 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} - 6\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

we know that

$$[\mathbf{v}]_B = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}. \quad \text{But then,} \quad \mathbf{P}[\mathbf{v}]_B = \begin{bmatrix} -8 \\ -19 \\ 13 \end{bmatrix},$$

and so $[\mathbf{v}]_C = [-8, -19, 13]$ by Theorem 4.20. We can easily verify this by checking that

$$\begin{bmatrix} 25 & 24 \\ 0 & -9 \end{bmatrix} = -8\begin{bmatrix} 22 & 7 \\ 0 & 2 \end{bmatrix} - 19\begin{bmatrix} 12 & 4 \\ 0 & 1 \end{bmatrix} + 13\begin{bmatrix} 33 & 12 \\ 0 & 2 \end{bmatrix}.$$

■

Algebra of the Transition Matrix

The next theorem shows that the cumulative effect of two transitions between bases is represented by the product of the transition matrices in *reverse* order.

> **Theorem 4.21** Suppose that B, C, and D are ordered bases for a nontrivial finite dimensional vector space $\mathcal{V}$. Let $\mathbf{P}$ be the transition matrix from B to C, and let $\mathbf{Q}$ be the transition matrix from C to D. Then $\mathbf{QP}$ is the transition matrix from B to D.

The proof of this theorem is left as Exercise 14.

Example 9

Consider the ordered bases B and C for $\mathcal{P}_2$ given by

$$B = (-x^2 + 4x + 2,\, 2x^2 - x - 1,\, -x^2 + 2x + 1) \quad \text{and}$$

$$C = (x^2 - 2x - 3,\, 2x^2 - 1,\, x^2 + x + 1).$$

Also consider the standard basis $S = (x^2, x, 1)$ for $\mathcal{P}_2$.

Now, row reducing

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 1 & -1 & 2 & -1 \\ -2 & 0 & 1 & 4 & -1 & 2 \\ -3 & -1 & 1 & 2 & -1 & 1 \end{array}\right] \quad \text{to obtain} \quad \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -9 & 3 & -5 \\ 0 & 1 & 0 & 11 & -3 & 6 \\ 0 & 0 & 1 & -14 & 5 & -8 \end{array}\right],$$

we see that the transition matrix from B to C is

$$\mathbf{P} = \begin{bmatrix} -9 & 3 & -5 \\ 11 & -3 & 6 \\ -14 & 5 & -8 \end{bmatrix}.$$

Because it is simple to express each vector in C in S-coordinates, we can quickly calculate that the transition matrix from C to S is

$$\mathbf{Q} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \\ -3 & -1 & 1 \end{bmatrix}.$$

Then, by Theorem 4.21, the product

$$\mathbf{QP} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} -9 & 3 & -5 \\ 11 & -3 & 6 \\ -14 & 5 & -8 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 4 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$$

is the transition matrix from B to S. This matrix is correct, since the columns of $\mathbf{QP}$ are, in fact, the vectors of B expressed in S-coordinates. ■

The next theorem shows how to reverse a transition from one basis to another. The proof of this theorem is left as Exercise 15.

Theorem 4.22 Let B and C be ordered bases for a nontrivial finite dimensional vector space $\mathcal{V}$, and let $\mathbf{P}$ be the transition matrix from B to C. Then $\mathbf{P}$ is nonsingular, and $\mathbf{P}^{-1}$ is the transition matrix from C to B.

Let us return to the situation in Example 9 and use the inverses of the transition matrices to find the B-coordinates of a polynomial in $\mathcal{P}_2$.

Example 10

Consider again the bases B, C, and S in Example 9 and the transition matrices $\mathbf{P}$ from B to C and $\mathbf{Q}$ from C to S. From Theorem 4.22, the transition matrices from C to B and from S to C, respectively, are

$$\mathbf{P}^{-1} = \begin{bmatrix} -6 & -1 & 3 \\ 4 & 2 & -1 \\ 13 & 3 & -6 \end{bmatrix} \quad \text{and} \quad \mathbf{Q}^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 4 & -3 \\ 2 & -5 & 4 \end{bmatrix}.$$

Now,

$$[\mathbf{v}]_B = \mathbf{P}^{-1}[\mathbf{v}]_C = \mathbf{P}^{-1}\left(\mathbf{Q}^{-1}[\mathbf{v}]_S\right) = \left(\mathbf{P}^{-1}\mathbf{Q}^{-1}\right)[\mathbf{v}]_S,$$

and so $\mathbf{P}^{-1}\mathbf{Q}^{-1}$ acts as the transition matrix from S to B (see Figure 4.8). For example, if $\mathbf{v} = x^2 + 7x + 3$, then

$$[\mathbf{v}]_B = \left(\mathbf{P}^{-1}\mathbf{Q}^{-1}\right)[\mathbf{v}]_S$$

$$= \begin{bmatrix} -6 & -1 & 3 \\ 4 & 2 & -1 \\ 13 & 3 & -6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 2 \\ -1 & 4 & -3 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}.$$

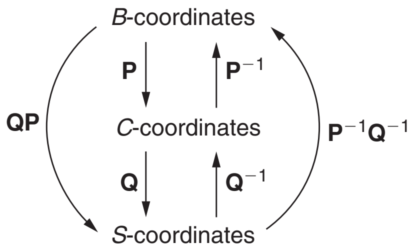

FIGURE 4.8

Transition matrices used to convert among B-, C-, and S-coordinates in $\mathcal{P}_2$

Diagonalization and the Transition Matrix

The matrix $\mathbf{P}$ obtained in the process of diagonalizing an $n \times n$ matrix turns out to be a transition matrix between two different bases for $\mathbb{R}^n$, as we see in our final example.

Example 11

Consider

$$\mathbf{A} = \begin{bmatrix} 14 & -15 & -30 \\ 6 & -7 & -12 \\ 3 & -3 & -7 \end{bmatrix}.$$

A quick calculation produces $p_{\mathbf{A}}(x) = x^3 - 3x - 2 = (x-2)(x+1)^2$. Row reducing $(2\mathbf{I}_3 - \mathbf{A})$ yields a fundamental eigenvector $\mathbf{v}_1 = [5, 2, 1]$. The set $\{\mathbf{v}_1\}$ is a basis for the eigenspace E_2. Similarly, we row reduce $(-1\mathbf{I}_3 - \mathbf{A})$ to obtain fundamental eigenvectors $\mathbf{v}_2 = [1, 1, 0]$ and $\mathbf{v}_3 = [2, 0, 1]$. The set $\{\mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for the eigenspace E_{-1}.

Let $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. These vectors are linearly independent (see the remarks before Example 13 in Section 4.4), and thus B is a basis for $\mathbb{R}^3$ by Theorem 4.13. Let S be the standard basis. Then, the transition matrix $\mathbf{P}$ from B to S is given by the matrix whose columns are the