7.1 Matrices, Vectors: Addition and Scalar Multiplication

The basic concepts and rules of matrix and vector algebra are introduced in Secs. 7.1 and 7.2 and are followed by **linear systems** (systems of linear equations), a main application, in Sec. 7.3.

Let us first take a leisurely look at matrices before we formalize our discussion. A **matrix** is a rectangular array of numbers or functions which we will enclose in brackets. For example,

$$(1) \qquad \begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$
$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad [a_1 \quad a_2 \quad a_3], \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

are matrices. The numbers (or functions) are called **entries** or, less commonly, *elements* of the matrix. The first matrix in (1) has two **rows**, which are the horizontal lines of entries. Furthermore, it has three **columns**, which are the vertical lines of entries. The second and third matrices are **square matrices**, which means that each has as many rows as columns—3 and 2, respectively. The entries of the second matrix have two indices, signifying their location within the matrix. The first index is the number of the row and the second is the number of the column, so that together the entry's position is uniquely identified. For example, a_{23} (read *a two three*) is in Row 2 and Column 3, etc. The notation is standard and applies to all matrices, including those that are not square.

Matrices having just a single row or column are called **vectors**. Thus, the fourth matrix in (1) has just one row and is called a **row vector**. The last matrix in (1) has just one column and is called a **column vector**. Because the goal of the indexing of entries was to uniquely identify the position of an element within a matrix, one index suffices for vectors, whether they are row or column vectors. Thus, the third entry of the row vector in (1) is denoted by a_3.

Matrices are handy for storing and processing data in applications. Consider the following two common examples.

EXAMPLE 1 Linear Systems, a Major Application of Matrices

We are given a system of linear equations, briefly a **linear system**, such as

$$\begin{aligned} 4x_1 + 6x_2 + 9x_3 &= 6 \\ 6x_1 \qquad\; - 2x_3 &= 20 \\ 5x_1 - 8x_2 + \; x_3 &= 10 \end{aligned}$$

where x_1, x_2, x_3 are the **unknowns**. We form the **coefficient matrix**, call it $\mathbf{A}$, by listing the coefficients of the unknowns in the position in which they appear in the linear equations. In the second equation, there is no unknown x_2, which means that the coefficient of x_2 is 0 and hence in matrix $\mathbf{A}$, $a_{22} = 0$, Thus,

$$\mathbf{A} = \begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix}. \quad \text{We form another matrix} \quad \tilde{\mathbf{A}} = \begin{bmatrix} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{bmatrix}$$

by augmenting **A** with the right sides of the linear system and call it the augmented matrix of the system.

Since we can go back and recapture the system of linear equations directly from the augmented matrix $\tilde{\mathbf{A}}$, $\tilde{\mathbf{A}}$ contains all the information of the system and can thus be used to solve the linear system. This means that we can just use the augmented matrix to do the calculations needed to solve the system. We shall explain this in detail in Sec. 7.3. Meanwhile you may verify by substitution that the solution is $x_1 = 3, x_2 = \frac{1}{2}, x_3 = -1$.

The notation x_1, x_2, x_3 for the unknowns is practical but not essential; we could choose x, y, z or some other letters. ■

EXAMPLE 2 Sales Figures in Matrix Form

Sales figures for three products I, II, III in a store on Monday (Mon), Tuesday (Tues), $\cdots$ may for each week be arranged in a matrix

$$\begin{array}{cc} & \begin{array}{ccccccc} \text{Mon} & \text{Tues} & \text{Wed} & \text{Thur} & \text{Fri} & \text{Sat} & \text{Sun} \end{array} \\ \mathbf{A} = & \begin{bmatrix} 40 & 33 & 81 & 0 & 21 & 47 & 33 \\ 0 & 12 & 78 & 50 & 50 & 96 & 90 \\ 10 & 0 & 0 & 27 & 43 & 78 & 56 \end{bmatrix} \cdot \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \end{array}$$

If the company has 10 stores, we can set up 10 such matrices, one for each store. Then, by adding corresponding entries of these matrices, we can get a matrix showing the total sales of each product on each day. Can you think of other data which can be stored in matrix form? For instance, in transportation or storage problems? Or in listing distances in a network of roads? ■

General Concepts and Notations

Let us formalize what we just have discussed. We shall denote matrices by capital boldface letters **A**, **B**, **C**, $\cdots$, or by writing the general entry in brackets; thus $\mathbf{A} = [a_{jk}]$, and so on. By an ***m* × *n* matrix** (read *m by n matrix*) we mean a matrix with m rows and n columns—rows always come first! $m \times n$ is called the **size** of the matrix. Thus an $m \times n$ matrix is of the form

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \tag{2}$$

The matrices in (1) are of sizes 2×3, 3×3, 2×2, 1×3, and 2×1, respectively.

Each entry in (2) has two subscripts. The first is the *row number* and the second is the *column number*. Thus a_{21} is the entry in Row 2 and Column 1.

If $m = n$, we call **A** an $n \times n$ **square matrix**. Then its diagonal containing the entries $a_{11}, a_{22}, \cdots, a_{nn}$ is called the **main diagonal** of **A**. Thus the main diagonals of the two square matrices in (1) are a_{11}, a_{22}, a_{33} and e^{-x}, $4x$, respectively.

Square matrices are particularly important, as we shall see. A matrix of any size $m \times n$ is called a **rectangular matrix**; this includes square matrices as a special case.

Vectors

A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector. We shall denote vectors by ***lowercase*** boldface letters **a**, **b**, $\cdots$ or by its general component in brackets, $\mathbf{a} = [a_j]$, and so on. Our special vectors in (1) suggest that a (general) **row vector** is of the form

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n]. \qquad \text{For instance,} \qquad \mathbf{a} = [-2 \quad 5 \quad 0.8 \quad 0 \quad 1].$$

A **column vector** is of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \qquad \text{For instance,} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

Addition and Scalar Multiplication of Matrices and Vectors

What makes matrices and vectors really useful and particularly suitable for computers is the fact that we can calculate with them almost as easily as with numbers. Indeed, we now introduce rules for addition and for scalar multiplication (multiplication by numbers) that were suggested by practical applications. (Multiplication of matrices by matrices follows in the next section.) We first need the concept of equality.

DEFINITION

Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different**. Thus, matrices of different sizes are always different.

EXAMPLE 3 **Equality of Matrices**

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

Then

$$\mathbf{A} = \mathbf{B} \qquad \text{if and only if} \qquad \begin{aligned} a_{11} &= 4, & a_{12} &= 0, \\ a_{21} &= 3, & a_{22} &= -1. \end{aligned}$$

The following matrices are all different. Explain!

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 0 \\ 4 & 2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 3 \\ 0 & 4 & 2 \end{bmatrix}$$ ■

DEFINITION

Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

As a special case, the **sum** $\mathbf{a} + \mathbf{b}$ of two row vectors or two column vectors, which must have the same number of components, is obtained by adding the corresponding components.

EXAMPLE 4

Addition of Matrices and Vectors

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}, \quad \text{then} \quad \mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}.$$

$\mathbf{A}$ in Example 3 and our present $\mathbf{A}$ cannot be added. If $\mathbf{a} = [5 \quad 7 \quad 2]$ and $\mathbf{b} = [-6 \quad 2 \quad 0]$, then $\mathbf{a} + \mathbf{b} = [-1 \quad 9 \quad 2]$.

An application of matrix addition was suggested in Example 2. Many others will follow. ■

DEFINITION

Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

Here $(-1)\mathbf{A}$ is simply written $-\mathbf{A}$ and is called the **negative** of $\mathbf{A}$. Similarly, $(-k)\mathbf{A}$ is written $-k\mathbf{A}$. Also, $\mathbf{A} + (-\mathbf{B})$ is written $\mathbf{A} - \mathbf{B}$ and is called the **difference** of $\mathbf{A}$ and $\mathbf{B}$ (which must have the same size!).

EXAMPLE 5

Scalar Multiplication

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}, \quad \text{then} \quad -\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}, \quad \frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}, \quad 0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If a matrix $\mathbf{B}$ shows the distances between some cities in miles, $1.609\mathbf{B}$ gives these distances in kilometers. ■

Rules for Matrix Addition and Scalar Multiplication. From the familiar laws for the addition of numbers we obtain similar laws for the addition of matrices of the same size $m \times n$, namely,

$$\textbf{(3)} \qquad \begin{aligned} &\text{(a)} \quad && \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \\ &\text{(b)} && (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \qquad \text{(written } \mathbf{A} + \mathbf{B} + \mathbf{C}\text{)} \\ &\text{(c)} && \mathbf{A} + \mathbf{0} = \mathbf{A} \\ &\text{(d)} && \mathbf{A} + (-\mathbf{A}) = \mathbf{0}. \end{aligned}$$

Here $\mathbf{0}$ denotes the **zero matrix** (of size $m \times n$), that is, the $m \times n$ matrix with all entries zero. If $m = 1$ or $n = 1$, this is a vector, called a **zero vector**.

Hence matrix addition is *commutative* and *associative* [by (3a) and (3b)].
Similarly, for scalar multiplication we obtain the rules

$$
\textbf{(4)}\qquad
\begin{aligned}
&\text{(a)} && c(\mathbf{A}+\mathbf{B}) = c\mathbf{A}+c\mathbf{B} \\
&\text{(b)} && (c+k)\mathbf{A} = c\mathbf{A}+k\mathbf{A} \\
&\text{(c)} && c(k\mathbf{A}) = (ck)\mathbf{A} \qquad \text{(written } ck\mathbf{A}\text{)} \\
&\text{(d)} && 1\mathbf{A} = \mathbf{A}.
\end{aligned}
$$

PROBLEM SET 7.1

1–7 GENERAL QUESTIONS

1. **Equality.** Give reasons why the five matrices in Example 3 are all different.
2. **Double subscript notation.** If you write the matrix in Example 2 in the form $\mathbf{A} = [a_{jk}]$, what is a_{31}? a_{13}? a_{26}? a_{33}?
3. **Sizes.** What sizes do the matrices in Examples 1, 2, 3, and 5 have?
4. **Main diagonal.** What is the main diagonal of **A** in Example 1? Of **A** and **B** in Example 3?
5. **Scalar multiplication.** If **A** in Example 2 shows the number of items sold, what is the matrix **B** of units sold if a unit consists of **(a)** 5 items and **(b)** 10 items?
6. If a 12×12 matrix **A** shows the distances between 12 cities in kilometers, how can you obtain from **A** the matrix **B** showing these distances in miles?
7. **Addition of vectors.** Can you add: A row and a column vector with different numbers of components? With the same number of components? Two row vectors with the same number of components but different numbers of zeros? A vector and a scalar? A vector with four components and a 2×2 matrix?

8–16 ADDITION AND SCALAR MULTIPLICATION OF MATRICES AND VECTORS

Let

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 4 \\ 6 & 5 & 5 \\ 1 & 0 & -3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 5 & 2 \\ 5 & 3 & 4 \\ -2 & 4 & -2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 5 & 2 \\ -2 & 4 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} -4 & 1 \\ 5 & 0 \\ 2 & -1 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} 0 & 2 \\ 3 & 4 \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 1.5 \\ 0 \\ -3.0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} -5 \\ -30 \\ 10 \end{bmatrix}.$$

Find the following expressions, indicating which of the rules in (3) or (4) they illustrate, or give reasons why they are not defined.

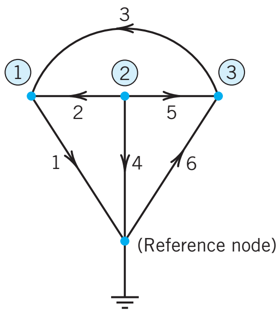
8. $2\mathbf{A} + 4\mathbf{B}$, $4\mathbf{B} + 2\mathbf{A}$, $0\mathbf{A} + \mathbf{B}$, $0.4\mathbf{B} - 4.2\mathbf{A}$
9. $3\mathbf{A}$, $0.5\mathbf{B}$, $3\mathbf{A} + 0.5\mathbf{B}$, $3\mathbf{A} + 0.5\mathbf{B} + \mathbf{C}$
10. $(4 \cdot 3)\mathbf{A}$, $4(3\mathbf{A})$, $14\mathbf{B} - 3\mathbf{B}$, $11\mathbf{B}$
11. $8\mathbf{C} + 10\mathbf{D}$, $2(5\mathbf{D} + 4\mathbf{C})$, $0.6\mathbf{C} - 0.6\mathbf{D}$, $0.6(\mathbf{C} - \mathbf{D})$
12. $(\mathbf{C} + \mathbf{D}) + \mathbf{E}$, $(\mathbf{D} + \mathbf{E}) + \mathbf{C}$, $0(\mathbf{C} - \mathbf{E}) + 4\mathbf{D}$, $\mathbf{A} - 0\mathbf{C}$
13. $(2 \cdot 7)\mathbf{C}$, $2(7\mathbf{C})$, $-\mathbf{D} + 0\mathbf{E}$, $\mathbf{E} - \mathbf{D} + \mathbf{C} + \mathbf{u}$
14. $(5\mathbf{u} + 5\mathbf{v}) - \frac{1}{2}\mathbf{w}$, $-20(\mathbf{u} + \mathbf{v}) + 2\mathbf{w}$, $\mathbf{E} - (\mathbf{u} + \mathbf{v})$, $10(\mathbf{u} + \mathbf{v}) + \mathbf{w}$
15. $(\mathbf{u} + \mathbf{v}) - \mathbf{w}$, $\mathbf{u} + (\mathbf{v} - \mathbf{w})$, $\mathbf{C} + 0\mathbf{w}$, $0\mathbf{E} + \mathbf{u} - \mathbf{v}$
16. $15\mathbf{v} - 3\mathbf{w} - 0\mathbf{u}$, $-3\mathbf{w} + 15\mathbf{v}$, $\mathbf{D} - \mathbf{u} + 3\mathbf{C}$, $8.5\mathbf{w} - 11.1\mathbf{u} + 0.4\mathbf{v}$
17. **Resultant of forces.** If the above vectors **u**, **v**, **w** represent forces in space, their sum is called their *resultant*. Calculate it.
18. **Equilibrium.** By definition, forces are *in equilibrium* if their resultant is the zero vector. Find a force **p** such that the above **u**, **v**, **w**, and **p** are in equilibrium.
19. **General rules.** Prove (3) and (4) for general 2×3 matrices and scalars c and k.

20. TEAM PROJECT. Matrices for Networks. Matrices have various engineering applications, as we shall see. For instance, they can be used to characterize connections in electrical networks, in nets of roads, in production processes, etc., as follows.

(a) Nodal Incidence Matrix. The network in Fig. 155 consists of six *branches* (connections) and four *nodes* (points where two or more branches come together). One node is the *reference node* (grounded node, whose voltage is zero). We number the other nodes and number and direct the branches. This we do arbitrarily. The network can now be described by a matrix $\mathbf{A} = [a_{jk}]$, where

$$a_{jk} = \begin{cases} +1 & \text{if branch } k \text{ leaves node } (j) \\ -1 & \text{if branch } k \text{ enters node } (j) \\ 0 & \text{if branch } k \text{ does not touch node } (j). \end{cases}$$

$\mathbf{A}$ is called the *nodal incidence matrix* of the network. Show that for the network in Fig. 155 the matrix $\mathbf{A}$ has the given form.



$$\begin{array}{c} \text{Branch} \\ \text{Node } (1) \\ \text{Node } (2) \\ \text{Node } (3) \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Fig. 155. Network and nodal incidence matrix in Team Project 20(a)

(b) Find the nodal incidence matrices of the networks in Fig. 156.

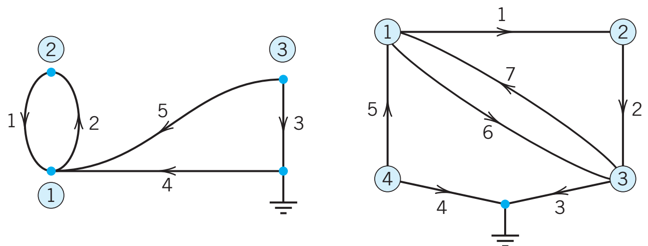


Fig. 156. Electrical networks in Team Project 20(b)

(c) Sketch the three networks corresponding to the nodal incidence matrices

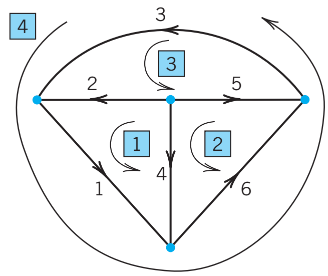
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix}.$$

(d) Mesh Incidence Matrix. A network can also be characterized by the *mesh incidence matrix* $\mathbf{M} = [m_{jk}]$, where

$$m_{jk} = \begin{cases} +1 & \text{if branch } k \text{ is in mesh } \boxed{j} \\ & \text{and has the same orientation} \\ -1 & \text{if branch } k \text{ is in mesh } \boxed{j} \\ & \text{and has the opposite orientation} \\ 0 & \text{if branch } k \text{ is not in mesh } \boxed{j} \end{cases}$$

and a mesh is a loop with no branch in its interior (or in its exterior). Here, the meshes are numbered and directed (oriented) in an arbitrary fashion. Show that for the network in Fig. 157, the matrix $\mathbf{M}$ has the given form, where Row 1 corresponds to mesh 1, etc.



$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Fig. 157. Network and matrix **M** in Team Project 20(d)