Another consequence of Theorem 4.14 is that any vector space $\mathcal{V}$ having a finite spanning set S must be finite dimensional. This is because a maximal linearly independent subset of S, which must also be finite, is a basis for $\mathcal{V}$ (see Exercise 24).

We also have the following result for spanning sets:

Theorem 4.15 Let $\mathcal{V}$ be a vector space, and let B be a minimal spanning set for $\mathcal{V}$. Then B is a basis for $\mathcal{V}$.

The phrase "B is a **minimal spanning set** for $\mathcal{V}$" means that both of the following are true:

- B is a subset of $\mathcal{V}$ that spans $\mathcal{V}$.
- If $C \subset B$ and $C \neq B$, then C does not span $\mathcal{V}$.

The converse of Theorem 4.15 is true as well (see Exercise 21).

Example 14

Consider the subsets S and B of $\mathbb{R}^3$ given in Example 13. We can use Theorem 4.15 to give another justification that B is a basis for $\mathcal{V} = \text{span}(S)$. Recall from Example 13 that every vector in S is a linear combination of vectors in B, so $S \subseteq \text{span}(B)$. This fact along with $B \subseteq S$ and Corollary 4.6 shows that $\text{span}(B) = \text{span}(S) = \mathcal{V}$. Also, neither vector in B is a scalar multiple of the other, so that neither vector alone can span $\mathcal{V}$ (why?). Hence, B is a minimal spanning set for $\mathcal{V}$, and by Theorem 4.15, B is a basis for $\text{span}(S)$. ■

Dimension of a Subspace

We conclude this section with the result that every subspace of a finite dimensional vector space is also finite dimensional. This is important because it tells us that the theorems we have developed about finite dimension apply to all *subspaces* of our basic examples $\mathbb{R}^n$, $\mathcal{M}_{mn}$, and $\mathcal{P}_n$.

Theorem 4.16 Let $\mathcal{V}$ be a finite dimensional vector space, and let $\mathcal{W}$ be a subspace of $\mathcal{V}$. Then $\mathcal{W}$ is also finite dimensional with $\dim(\mathcal{W}) \leq \dim(\mathcal{V})$. Moreover, $\dim(\mathcal{W}) = \dim(\mathcal{V})$ if and only if $\mathcal{W} = \mathcal{V}$.

The proof of Theorem 4.16 is left for you to do, with hints, in Exercise 22. The only subtle part of this proof involves showing that $\mathcal{W}$ actually has a basis.[4]

[4] Although it is true that *every* vector space has a basis, we must be careful here, because we have not proven this. In fact, Theorem 4.16 establishes that every subspace of a finite dimensional vector space *does* have a basis and that this basis is finite. Although every finite dimensional vector space has a finite basis by definition, the proof that every infinite dimensional vector space has a basis requires advanced set theory and is beyond the scope of this text.

Example 15
Consider the nested sequence of subspaces of $\mathbb{R}^3$ given by $\{\mathbf{0}\} \subset \{$scalar multiples of $[4, -7, 0]\} \subset$ xy-plane $\subset \mathbb{R}^3$. Their respective dimensions are $0, 1, 2$, and 3 (why?). Hence, the dimensions of each successive pair of these subspaces satisfy the inequality given in Theorem 4.16. ∎

Example 16
It can be shown that $B = \{x^3 + 2x^2 - 4x + 18, 3x^2 + 4x - 4, x^3 + 5x^2 - 3, 3x + 2\}$ is a linearly independent subset of $\mathcal{P}_3$. Therefore, by part (2) of Theorem 4.13, B is a basis for $\mathcal{P}_3$. However, we can also reach the same conclusion from Theorem 4.16. For, $\mathcal{W} = \text{span}(B)$ has B as a basis (why?), and hence, $\dim(\mathcal{W}) = 4$. But since $\mathcal{W}$ is a subspace of $\mathcal{P}_3$ and $\dim(\mathcal{P}_3) = 4$, Theorem 4.16 implies that $\mathcal{W} = \mathcal{P}_3$. Hence, B is a basis for $\mathcal{P}_3$. ∎

New Vocabulary

basis
dimension
finite dimensional (vector space)
infinite dimensional (vector space)
maximal linearly independent set
minimal spanning set
standard basis (for $\mathbb{R}^n, \mathcal{M}_{mn}, \mathcal{P}_n$)

Highlights

- A basis is a subset of a vector space that both spans and is linearly independent.
- If a finite basis exists for a vector space, the vector space is said to be finite dimensional.
- For a finite dimensional vector space, all bases have the same number of vectors, and this number is known as the dimension of the vector space.
- The standard basis for $\mathbb{R}^n$ is $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$; $\dim(\mathbb{R}^n) = n$.
- The standard basis for $\mathcal{P}_n$ is $\{1, x, x^2, \ldots, x^n\}$; $\dim(\mathcal{P}_n) = n + 1$.
- The standard basis for $\mathcal{M}_{mn}$ is $\{\Psi_{ij}\}$, where each Ψ_{ij} has a 1 in the (i, j) entry, and zeroes elsewhere; $\dim(\mathcal{M}_{mn}) = m \cdot n$.
- The basis for the trivial vector space $\{\mathbf{0}\}$ is the empty set $\{\ \}$; $\dim(\{\mathbf{0}\}) = 0$.
- If no finite basis exists for a vector space, the vector space is said to be infinite dimensional. $\mathcal{P}$ is an infinite dimensional vector space, as is the set of all real-valued functions (under normal operations).
- In a vector space $\mathcal{V}$ with dimension n, the size of a spanning set S is always $\geq n$. If $|S| = n$, then S is a basis for $\mathcal{V}$.
- In a vector space $\mathcal{V}$ with dimension n, the size of a linearly independent set T is always $\leq n$. If $|T| = n$, then T is a basis for $\mathcal{V}$.

- A maximal linearly independent set in a vector space is a basis.
- A minimal spanning set in a vector space is a basis.
- In a vector space $\mathcal{V}$ with dimension n, the dimension of a subspace $\mathcal{W}$ is always $\leq n$. If $\dim(\mathcal{W}) = n$, then $\mathcal{W} = \mathcal{V}$.

EXERCISES FOR SECTION 4.5

1. Prove that each of the following subsets of $\mathbb{R}^4$ is a basis for $\mathbb{R}^4$ by showing both that it spans $\mathbb{R}^4$ and that is linearly independent:

(a) $\{[2,1,0,0],[0,1,1,-1],[0,-1,2,-2],[3,1,0,-2]\}$

(b) $\{[6,1,1,-1],[1,0,0,9],[-2,3,2,4],[2,2,5,-5]\}$

(c) $\{[1,1,1,1],[1,1,1,-1],[1,1,-1,-1],[1,-1,-1,-1]\}$

(d) $\left\{\left[\frac{15}{2},5,\frac{12}{5},1\right],\left[2,\frac{1}{2},\frac{3}{4},1\right],\left[-\frac{13}{2},1,0,4\right],\left[\frac{18}{5},0,\frac{1}{5},-\frac{1}{5}\right]\right\}$

2. Prove that the following set is a basis for $\mathcal{M}_{22}$ by showing that it spans $\mathcal{M}_{22}$ and is linearly independent:

$$\left\{\begin{bmatrix}1 & 4\\ 2 & 0\end{bmatrix},\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix},\begin{bmatrix}-3 & 1\\ -1 & 0\end{bmatrix},\begin{bmatrix}5 & -2\\ 0 & -3\end{bmatrix}\right\}.$$

3. Show that the subset $\left\{x^4,\ x^4-x^3,\ x^4-x^3+x^2,\ x^4-x^3+x^2-x,\ x^3-1\right\}$ of $\mathcal{P}_4$ is a basis for $\mathcal{P}_4$.

4. Determine which of the following subsets of $\mathbb{R}^4$ form a basis for $\mathbb{R}^4$:

★(a) $S = \{[7,1,2,0],[8,0,1,-1],[1,0,0,-2]\}$

(b) $S = \{[1,3,2,0],[-2,0,6,7],[0,6,10,7]\}$

★(c) $S = \{[7,1,2,0],[8,0,1,-1],[1,0,0,-2],[3,0,1,-1]\}$

(d) $S = \{[1,3,2,0],[-2,0,6,7],[0,6,10,7],[2,10,-3,1]\}$

★(e) $S = \{[1,2,3,2],[1,4,9,3],[6,-2,1,4],[3,1,2,1],[10,-9,-15,6]\}$

5. **(a)** Show that $B = \{[2,3,0,-1],[-1,1,1,-1]\}$ is a maximal linearly independent subset of $S = \{[1,4,1,-2],[-1,1,1,-1],[3,2,-1,0],[2,3,0,-1]\}$.

★(b) Calculate $\dim(\text{span}(S))$.

★(c) Does $\text{span}(S) = \mathbb{R}^4$? Why or why not?

(d) Is B a minimal spanning set for $\text{span}(S)$? Why or why not?

6. **(a)** Show that $B = \{x^3-x^2+2x+1,\ 2x^3+4x-7,\ 3x^3-x^2-6x+6\}$ is a maximal linearly independent subset of $S = \{x^3-x^2+2x+1,\ x-1,\ 2x^3+4x-7, x^3-3x^2-22x+34, 3x^3-x^2-6x+6\}$.

(b) Calculate $\dim(\text{span}(S))$.

(c) Does $\text{span}(S) = \mathcal{P}_3$? Why or why not?

(d) Is B a minimal spanning set for $\text{span}(S)$? Why or why not?

7. Let $\mathcal{W}$ be the solution set to the matrix equation $\mathbf{AX} = \mathbf{O}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ 2 & -1 & 0 & 1 & 3 \\ 1 & -3 & -1 & 1 & 4 \\ 2 & 9 & 4 & -1 & -7 \end{bmatrix}.$$

(a) Show that $\mathcal{W}$ is a subspace of $\mathbb{R}^5$.

(b) Find a basis for $\mathcal{W}$.

(c) Show that $\dim(\mathcal{W})+\text{rank}(\mathbf{A}) = 5$.

8. Prove that every proper nontrivial subspace of $\mathbb{R}^3$ can be thought of, from a geometric point of view, as either a line through the origin or a plane through the origin.

9. Let $\mathbf{f}$ be a polynomial of degree n. Show that the set $\{\mathbf{f}, \mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \ldots, \mathbf{f}^{(n)}\}$ is a basis for $\mathcal{P}_n$ (where $\mathbf{f}^{(i)}$ denotes the ith derivative of $\mathbf{f}$). (Hint: See Exercise 23 in Section 4.4.)

10. (a) Let $\mathbf{A}$ be a 2×2 matrix. Prove that there are real numbers $a_0, a_1, \ldots, a_4$, not all zero, such that $a_4\mathbf{A}^4 + a_3\mathbf{A}^3 + a_2\mathbf{A}^2 + a_1\mathbf{A} + a_0\mathbf{I}_2 = \mathbf{O}_2$. (Hint: You can assume that $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \mathbf{A}^4$, and $\mathbf{I}_2$ are all distinct because if they are not, opposite nonzero coefficients can be chosen for any identical pair to demonstrate that the given statement holds.)

(b) Suppose $\mathbf{B}$ is an $n \times n$ matrix. Show that there must be a nonzero polynomial $\mathbf{p} \in \mathcal{P}_{n^2}$ such that $\mathbf{p}(\mathbf{B}) = \mathbf{O}_n$.

11. (a) Show that $B = \{(x-2), x(x-2), x^2(x-2), x^3(x-2), x^4(x-2)\}$ is a basis for $\mathcal{V} = \{\mathbf{p} \in \mathcal{P}_5 | \mathbf{p}(2) = 0\}$.

⋆(b) What is $\dim(\mathcal{V})$?

⋆(c) Find a basis for $\mathcal{W} = \{\mathbf{p} \in \mathcal{P}_5 \mid \mathbf{p}(2) = \mathbf{p}(3) = 0\}$.

⋆(d) Calculate $\dim(\mathcal{W})$.

⋆12. Let $\mathcal{V}$ be a finite dimensional vector space.

(a) Let S be a subset of $\mathcal{V}$ with $\dim(\mathcal{V}) \leq |S|$. Find an example to show that S need not span $\mathcal{V}$.

(b) Let T be a subset of $\mathcal{V}$ with $|T| \leq \dim(\mathcal{V})$. Find an example to show that T need not be linearly independent.

13. Let S be a subset of a finite dimensional vector space $\mathcal{V}$ such that $|S| = \dim(\mathcal{V})$. If S is not a basis for $\mathcal{V}$, prove that S neither spans $\mathcal{V}$ nor is linearly independent.

14. Let $\mathcal{V}$ be an n-dimensional vector space, and let S be a subset of $\mathcal{V}$ containing exactly n elements. Prove that S spans $\mathcal{V}$ if and only if S is linearly independent.

15. Let $\mathbf{A}$ be a nonsingular $n \times n$ matrix, and let B be a basis for $\mathbb{R}^n$.

(a) Show that $B_1 = \{\mathbf{Av} | \mathbf{v} \in B\}$ is also a basis for $\mathbb{R}^n$. (Treat the vectors in B as column vectors.)

(b) Show that $B_2 = \{\mathbf{vA} | \mathbf{v} \in B\}$ is also a basis for $\mathbb{R}^n$. (Treat the vectors in B as row vectors.)

(c) Letting B be the standard basis for $\mathbb{R}^n$, use the result of part (a) to show that the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.

(d) Prove that the rows of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.

16. Prove that $\mathcal{P}$ is infinite dimensional by showing that no finite subset S of $\mathcal{P}$ can span $\mathcal{P}$, as follows:

(a) Let S be a finite subset of $\mathcal{P}$. Show that $S \subseteq \mathcal{P}_n$, for some n.

(b) Use part (a) to prove that $\text{span}(S) \subseteq \mathcal{P}_n$.

(c) Conclude that S cannot span $\mathcal{P}$.

17. **(a)** Prove that if a vector space $\mathcal{V}$ has an infinite linearly independent subset, then $\mathcal{V}$ is not finite dimensional.

(b) Use part (a) to prove that any vector space having $\mathcal{P}$ as a subspace is not finite dimensional.

18. The purpose of this exercise is to prove Theorem 4.14. Let $\mathcal{V}$, S, and B be as given in the statement of the theorem. Suppose $B \neq S$, and $\mathbf{w} \in S$ with $\mathbf{w} \notin B$.

(a) Explain why it is sufficient to prove that B spans $\mathcal{V}$.

▶**(b)** Prove that if $S \subseteq \text{span}(B)$, then B spans $\mathcal{V}$.

▶**(c)** Let $C = B \cup \{\mathbf{w}\}$. Prove that C is linearly dependent.

(d) Use part (c) to prove that $\mathbf{w} \in \text{span}(B)$. (Also see part (a) of Exercise 26 in Section 4.4.)

(e) Tie together all parts to finish the proof.

19. The purpose of this exercise is to prove Theorem 4.15.

(a) Explain why it is sufficient to prove the following statement: Let S be a spanning set for a vector space $\mathcal{V}$. If S is a minimal spanning set for $\mathcal{V}$, then S is linearly independent.

▶**(b)** State the contrapositive of the statement in part (a).

▶**(c)** Prove the statement from part (b). (Hint: Use Exercise 12 from Section 4.4.)

20. Let B be a basis for a vector space $\mathcal{V}$. Prove that B is a maximal linearly independent subset of $\mathcal{V}$. (Note: You may *not* use $\dim(\mathcal{V})$ in your proof, since $\mathcal{V}$ could be infinite dimensional.)

21. Let B be a basis for a vector space $\mathcal{V}$. Prove that B is a minimal spanning set for $\mathcal{V}$. (Note: You may *not* use $\dim(\mathcal{V})$ in your proof, since $\mathcal{V}$ could be infinite dimensional.)

22. The purpose of this exercise is to prove Theorem 4.16. Let $\mathcal{V}$ and $\mathcal{W}$ be as given in the theorem. Consider the set A of nonnegative integers defined by $A = \{k \,|\, \text{a set } T \text{ exists with } T \subseteq \mathcal{W}, |T| = k, \text{and } T \text{ linearly independent}\}$.

- **(a)** Prove that $0 \in A$. (Hence, A is nonempty.)
- **(b)** Prove that $k \in A$ implies $k \leq \dim(\mathcal{V})$. (Hint: Use Theorem 4.13.) (Hence, A is finite.)
- ▶**(c)** Let n be the largest element of A. Let T be a linearly independent subset of $\mathcal{W}$ such that $|T| = n$. Prove T is a maximal linearly independent subset of $\mathcal{W}$.
- ▶**(d)** Use part (c) and Theorem 4.14 to prove that T is a basis for $\mathcal{W}$.
- **(e)** Conclude that $\mathcal{W}$ is finite dimensional and use part (b) to show $\dim(\mathcal{W}) \leq \dim(\mathcal{V})$.
- **(f)** Prove that if $\dim(\mathcal{W}) = \dim(\mathcal{V})$, then $\mathcal{W} = \mathcal{V}$. (Hint: Let T be a basis for $\mathcal{W}$ and use part (2) of Theorem 4.13 to show that T is also a basis for $\mathcal{V}$.)
- **(g)** Prove the converse of part (f).

23. Let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$ with $\dim(\mathcal{V}) = n - 1$. (Such a subspace is called a **hyperplane** in $\mathbb{R}^n$.) Prove that there is a nonzero $\mathbf{x} \in \mathbb{R}^n$ such that $\mathcal{V} = \{\mathbf{v} \in \mathbb{R}^n | \mathbf{x} \cdot \mathbf{v} = 0\}$. (Hint: Set up a homogeneous system of equations whose coefficient matrix has a basis for $\mathcal{V}$ as its rows. Then notice that this $(n-1) \times n$ system has at least one nontrivial solution, say $\mathbf{x}$.)

24. Let $\mathcal{V}$ be a vector space and let S be a finite spanning set for $\mathcal{V}$. Prove that $\mathcal{V}$ is finite dimensional.

⋆**25.** True or False:

- **(a)** A set B of vectors in a vector space $\mathcal{V}$ is a basis for $\mathcal{V}$ if B spans $\mathcal{V}$ and B is linearly independent.
- **(b)** All bases for $\mathcal{P}_4$ have four elements.
- **(c)** $\dim(\mathcal{M}_{43}) = 7$.
- **(d)** If S is a spanning set for $\mathcal{W}$ and $\dim(\mathcal{W}) = n$, then $|S| \leq n$.
- **(e)** If T is a linearly independent set in $\mathcal{W}$ and $\dim(\mathcal{W}) = n$, then $|T| = n$.
- **(f)** If T is a linearly independent set in a finite dimensional vector space $\mathcal{W}$ and S is a finite spanning set for $\mathcal{W}$, then $|T| \leq |S|$.
- **(g)** If $\mathcal{W}$ is a subspace of a finite dimensional vector space $\mathcal{V}$, then $\dim(\mathcal{W}) < \dim(\mathcal{V})$.
- **(h)** Every subspace of an infinite dimensional vector space is infinite dimensional.

(i) If T is a maximal linearly independent set for a vector space $\mathcal{V}$ and S is a minimal spanning set for $\mathcal{V}$, then $S = T$.

(j) If $\mathbf{A}$ is a nonsingular 4×4 matrix, then the rows of $\mathbf{A}$ are a basis for $\mathbb{R}^4$.

4.6 CONSTRUCTING SPECIAL BASES

In this section, we present additional methods for finding a basis for a given finite dimensional vector space, starting with either a spanning set or a linearly independent subset.

Using Row Reduction to Construct a Basis

Recall the Simplified Span Method from Section 4.3. Using that method, we were able to simplify the form of span(S) for a subset S of $\mathbb{R}^n$. This was done by creating a matrix $\mathbf{A}$ whose rows are the vectors in S, and then row reducing $\mathbf{A}$ to obtain a reduced row echelon form matrix $\mathbf{C}$. We discovered that a simplified form of span(S) is given by the set of all linear combinations of the nonzero rows of $\mathbf{C}$. Now, each nonzero row of the matrix $\mathbf{C}$ has a (pivot) 1 in a column in which all other rows have zeroes, so the nonzero rows of $\mathbf{C}$ must be linearly independent. Thus, the nonzero rows of $\mathbf{C}$ not only span S but are linearly independent as well, and so they form a basis for span(S). Therefore, whenever we use the Simplified Span Method on a subset S of $\mathbb{R}^n$, we are actually creating a basis for span(S).

Example 1

Let $S = \{[2,-2,3,5,5],[-1,1,4,14,-8],[4,-4,-2,-14,18],[3,-3,-1,-9,13]\}$, a subset of $\mathbb{R}^5$. We can use the Simplified Span Method to find a basis B for $\mathcal{V} = \text{span}(S)$. We construct the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 3 & 5 & 5 \\ -1 & 1 & 4 & 14 & -8 \\ 4 & -4 & -2 & -14 & 18 \\ 3 & -3 & -1 & -9 & 13 \end{bmatrix},$$

whose rows are the vectors in S. The reduced row echelon form matrix for $\mathbf{A}$ is

$$\mathbf{C} = \begin{bmatrix} 1 & -1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the desired basis for $\mathcal{V}$ is the set $B = \{[1,-1,0,-2,4],[0,0,1,3,-1]\}$ of nonzero rows of $\mathbf{C}$, and $\dim(\mathcal{V}) = 2$.

In general, the Simplified Span Method creates a basis of vectors with a simpler form than the original vectors. This is because a reduced row echelon form matrix has the simplest form of all matrices that are row equivalent to it.

This method can also be adapted to vector spaces other than $\mathbb{R}^n$, as in the next example.

Example 2

Consider the subset $\mathcal{S} = \{x^2 - 3x + 5, 3x^3 + 4x - 8, 6x^3 - x^2 + 11x - 21, 2x^5 - 7x^3 + 5x\}$ of $\mathcal{P}_5$. We use the Simplified Span Method to find a basis for $\mathcal{W} = \text{span}(\mathcal{S})$.

Since $\mathcal{S}$ is a subset of $\mathcal{P}_5$ instead of $\mathbb{R}^n$, we must alter our method slightly. We cannot use the polynomials in $\mathcal{S}$ themselves as rows of a matrix, so we "peel off" their coefficients to create four 6-vectors, which we use as the rows of the following matrix:

$$\mathbf{A} = \begin{array}{c} \begin{array}{cccccc} x^5 & x^4 & x^3 & x^2 & x & 1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 3 & 0 & 4 & -8 \\ 0 & 0 & 6 & -1 & 11 & -21 \\ 2 & 0 & -7 & 0 & 5 & 0 \end{bmatrix} \end{array}.$$

Row reducing this matrix produces

$$\mathbf{C} = \begin{array}{c} \begin{array}{cccccc} x^5 & x^4 & x^3 & x^2 & x & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{43}{6} & -\frac{28}{3} \\ 0 & 0 & 1 & 0 & \frac{4}{3} & -\frac{8}{3} \\ 0 & 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

The nonzero rows of $\mathbf{C}$ yield the following three-element basis for $\mathcal{W}$:

$$D = \left\{ x^5 + \frac{43}{6}x - \frac{28}{3},\ x^3 + \frac{4}{3}x - \frac{8}{3},\ x^2 - 3x + 5 \right\}.$$

Hence, $\dim(\mathcal{W}) = 3$.

Every Spanning Set for a Finite Dimensional Vector Space Contains a Basis

Sometimes, we are interested in reducing a spanning set to a basis by eliminating redundant vectors without changing the form of the original vectors. The next theorem asserts that this is possible; that is, if $\mathcal{V}$ is a finite dimensional vector space, then any spanning set of $\mathcal{V}$, finite or infinite, must contain a basis for $\mathcal{V}$.

Theorem 4.17 If $\mathcal{S}$ is a spanning set for a finite dimensional vector space $\mathcal{V}$, then there is a set $B \subseteq \mathcal{S}$ that is a basis for $\mathcal{V}$.

The proof of this theorem is very similar to the first part of the proof of Theorem 4.16[5] and is left as Exercise 14.

Example 3

Let $\mathcal{S} = \{[1,3,-2],[2,1,4],[0,5,-8],[1,-7,14]\}$, and let $\mathcal{V} = \text{span}(\mathcal{S})$. Theorem 4.17 indicates that some subset of $\mathcal{S}$ is a basis for $\mathcal{V}$. Now, the equations

$$[0,5,-8] = 2[1,3,-2] - [2,1,4] \quad \text{and}$$
$$[1,-7,14] = -3[1,3,-2] + 2[2,1,4]$$

show that the subset $B = \{[1,3,-2],[2,1,4]\}$ is a maximal linearly independent subset of $\mathcal{S}$ (why?). Hence, by Theorem 4.14, B is a basis for $\mathcal{V}$ contained in $\mathcal{S}$. ■

Shrinking a Spanning Set to a Basis Using Row Reduction

As Example 3 illustrates, to find a subset B of a spanning set S that is a basis for span(S), it is necessary to remove enough redundant vectors from S until we are left with a (maximal) linearly independent subset of S. This can be done using the Independence Test Method from Section 4.4. Suppose we row reduce the matrix whose columns are all the vectors in S. Then those vectors of S corresponding to the pivot columns form a linearly independent subset B. This is because if we had row reduced the matrix having just these columns, every column would have had a pivot. Also, no larger subset of S containing B can be linearly independent because reinserting a column corresponding to any of the remaining vectors would result in a nonpivot column after row reduction. Therefore, B is a maximal linearly independent subset of S, and hence is a basis for span(S). This procedure is illustrated in the next two examples.

Example 4

Consider the subset $\mathcal{S} = \{[1,2,-1],[3,6,-3],[4,1,2],[0,0,0],[-1,5,-5]\}$ of $\mathbb{R}^3$. We use the Independence Test Method to find a subset B of $\mathcal{S}$ that is a basis for $\mathcal{V} = \text{span}(\mathcal{S})$. We form the matrix $\mathbf{A}$ whose columns are the vectors in $\mathcal{S}$, and then row reduce

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 & 0 & -1 \\ 2 & 6 & 1 & 0 & 5 \\ -1 & -3 & 2 & 0 & -5 \end{bmatrix} \quad \text{to obtain} \quad \mathbf{C} = \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are nonzero pivots in the first and third columns of $\mathbf{C}$, we choose $B = \{[1,2,-1],[4,1,2]\}$, the first and third vectors in $\mathcal{S}$. Since $|B| = 2, \dim(\mathcal{V}) = 2$. (Hence, $\mathcal{S}$ does not span all of $\mathbb{R}^3$.) ■

[5] Theorem 4.17 is also true for infinite dimensional vector spaces, but the proof requires advanced topics in set theory that are beyond the scope of this book.

This method can also be adapted to vector spaces other than $\mathbb{R}^n$.

Example 5

Let $\mathcal{S} = \{x^3 - 3x^2 + 1, 2x^2 + x, 2x^3 + 3x + 2, 4x - 5\} \subseteq \mathcal{P}_3$. We use the Independence Test Method to find a subset B of $\mathcal{S}$ that is a basis for $\mathcal{V} = \text{span}(\mathcal{S})$. Let $\mathbf{A}$ be the matrix whose columns are the analogous vectors in $\mathbb{R}^4$ for the given vectors in $\mathcal{S}$. Then

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 1 & 3 & 4 \\ 1 & 0 & 2 & -5 \end{bmatrix}, \quad \text{which reduces to} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Because we have nonzero pivots in the first, second, and fourth columns of $\mathbf{C}$, we choose $B = \{x^3 - 3x^2 + 1, 2x^2 + x, 4x - 5\}$. These are the first, second, and fourth vectors in $\mathcal{S}$. Then B is the desired basis for $\mathcal{V}$.

The third vector in $\mathcal{S}$ is a linear combination of previous vectors in $\mathcal{S}$. The first two entries of the third column of $\mathbf{C}$ give the coefficients of that linear combination; that is, $2x^3 + 3x + 2 = 2(x^3 - 3x^2 + 1) + 3(2x^2 + x)$.

The Simplified Span Method and the Independence Test Method for finding a basis are similar enough to cause confusion, so we contrast their various features in Table 4.2.

Shrinking an Infinite Spanning Set to a Basis

The Independence Test Method can sometimes be used successfully when the spanning set $\mathcal{S}$ is infinite.

Table 4.2 Contrasting the Simplified Span Method and Independence Test Method for finding a basis from a given spanning set $\mathcal{S}$

Simplified Span Method	Independence Test Method
The vectors in $\mathcal{S}$ become the *rows* of a matrix.	The vectors in $\mathcal{S}$ become the *columns* of a matrix.
The basis created is *not* a subset of the spanning set $\mathcal{S}$ but contains vectors with a simpler form.	The basis created *is* a subset of the spanning set $\mathcal{S}$.
The nonzero rows of the reduced row echelon form matrix are used as the basis vectors.	The pivot columns of the reduced row echelon form matrix are used to determine which vectors to select from $\mathcal{S}$.

Example 6

Let $\mathcal{V}$ be the subspace of $\mathcal{M}_{22}$ consisting of all 2×2 symmetric matrices. Let S be the set of nonsingular matrices in $\mathcal{V}$, and let $\mathcal{W} = \text{span}(S) = \text{span}(\{\text{nonsingular, symmetric } 2 \times 2 \text{ matrices}\})$. We reduce S to a basis for $\mathcal{W}$ using the Independence Test Method, even though S is infinite. (We prove later that $\mathcal{W} = \mathcal{V}$, and so the basis we construct is actually a basis for $\mathcal{V}$.)

The strategy is to guess a *finite* subset Y of S that spans $\mathcal{W}$. We then use the Independence Test Method on Y to find the desired basis. We try to pick vectors for Y whose forms are as simple as possible to make computation easier. In this case, we choose the set of all nonsingular symmetric 2×2 matrices having only zeroes and ones as entries. That is,

$$Y = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

Now, before continuing, we must ensure that $\text{span}(Y) = \mathcal{W}$. That is, we must show every nonsingular symmetric 2×2 matrix is in $\text{span}(Y)$. In fact, we will show every symmetric 2×2 matrix is in $\text{span}(Y)$ by finding real numbers w, x, y, and z so that

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = w\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + x\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Thus, we must prove that the system

$$\begin{cases} w + x \phantom{{}+y+z} = a \\ \phantom{w+{}} x + y + z = b \\ \phantom{w+{}} x + y + z = b \\ w \phantom{{}+x+y} + z = c \end{cases}$$

has solutions for w, x, y, and z in terms of a, b, and c. But $w = 0$, $x = a$, $y = b - a - c$, $z = c$ certainly satisfies the system. Hence, $\mathcal{V} \subseteq \text{span}(Y)$. Since $\text{span}(Y) \subseteq \mathcal{V}$, we have $\text{span}(Y) = \mathcal{V} = \mathcal{W}$.

We can now use the Independence Test Method on Y. We express the matrices in Y as corresponding vectors in $\mathbb{R}^4$ and create the matrix with these vectors as columns, as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \text{which reduces to} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, the desired basis is

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\},$$

the elements of Y corresponding to the pivot columns of C. ■

The method used in Example 6 is not guaranteed to work when the spanning set $\mathcal{S}$ has infinitely many elements because our choice for the finite set Y might not have the same span as $\mathcal{S}$. When this happens, the choice of a larger set Y may lead to success.

Finding a Basis from a Spanning Set by Inspection

When a spanning set $\mathcal{S}$ for a vector space $\mathcal{V}$ is given, it is sometimes easier to select a maximal linearly independent subset of $\mathcal{S}$ (and hence, a basis for $\mathcal{V}$) by process of elimination rather than row reduction. The idea behind the following method is to inspect each of the vectors in the given spanning set $\mathcal{S}$ in turn and eliminate any that are redundant; that is, any vectors in $\mathcal{S}$ that are linear combinations of previous vectors.

The formal technique presented in the following method resembles a proof by induction in that there is a "Base" Step followed by an "Inductive" Step that is repeated until the desired basis is found. The method stops when we run out of vectors to choose in the Inductive Step that are linearly independent of those previously chosen.[6]

Method for Finding a Basis from a Spanning Set by Inspection (Inspection Method)

Let $\mathcal{S}$ be a finite set of vectors spanning a vector space $\mathcal{V}$.

(1) **Base Step:** Choose $\mathbf{v}_1 \neq \mathbf{0}$ in $\mathcal{S}$.

Repeat the following step as many times as possible:

(2) **Inductive Step:** Assuming $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ have already been chosen from $\mathcal{S}$, choose $\mathbf{v}_k \in \mathcal{S}$ such that $\mathbf{v}_k \notin \text{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\})$.

The final set constructed is a basis for $\mathcal{V}$.

The Inspection Method is useful when you can determine easily (without tedious computations) which vectors to choose next in the Inductive Step. Otherwise, you should apply the Independence Test Method.

Example 7

Let $\mathcal{S} = \{[0,0,0], [2,-8,12], [-1,4,-6], [7,2,2]\}$, a subset of $\mathbb{R}^3$. Let $\mathcal{V} = \text{span}(\mathcal{S})$, a subspace of $\mathbb{R}^3$. We use the Inspection Method to find a subset B of $\mathcal{S}$ that is a basis for $\mathcal{V}$.

The Base Step is to choose $\mathbf{v}_1$, a nonzero vector in $\mathcal{S}$. So, we skip over the first vector listed in $\mathcal{S}$, $[0,0,0]$ and let $\mathbf{v}_1 = [2,-8,12]$.

[6] We assume that $\mathcal{S}$ has at least one nonzero vector. Otherwise, $\mathcal{V}$ would be the trivial vector space. In this case, the desired basis for $\mathcal{V}$ is the empty set, { }.

Moving on to the Inductive Step, we look for $\mathbf{v}_2$ in S so that $\mathbf{v}_2 \notin \text{span}(\{\mathbf{v}_1\})$. Hence, $\mathbf{v}_2$ may not be a scalar multiple of $\mathbf{v}_1$. Therefore, we may not choose $[-1,4,-6]$ because $[-1,4,-6] = -\frac{1}{2}[2,-8,12]$. Instead, we choose $\mathbf{v}_2 = [7,2,2]$.

At this point, there are no more vectors in S for us to try, so the induction process must terminate here. Therefore, $B = \{\mathbf{v}_1, \mathbf{v}_2\} = \{[2,-8,12],[7,2,2]\}$ is the desired basis for $\mathcal{V}$. Notice that $\mathcal{V} = \text{span}(B)$ is not all of $\mathbb{R}^3$ because $\dim(\mathcal{V}) = 2 \neq \dim\left(\mathbb{R}^3\right)$. (You can verify, for example, that the vector $[1,0,0] \in \mathbb{R}^3$ cannot be expressed as a linear combination of the vectors in B and hence is not in $\mathcal{V} = \text{span}(B)$.) ■

Every Linearly Independent Set in a Finite Dimensional Vector Space Is Contained in Some Basis

Suppose that $T = \{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$ is a linearly independent set of vectors in a finite dimensional vector space $\mathcal{V}$. Because $\mathcal{V}$ is finite dimensional, it has a finite basis, say $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Consider the set $T \cup A$. Now, $T \cup A$ certainly spans $\mathcal{V}$ (since A alone spans $\mathcal{V}$). We can therefore apply the Independence Test Method to $T \cup A$ to produce a basis B for $\mathcal{V}$. If we order the vectors in $T \cup A$ so that all the vectors in T are listed first, then none of these vectors will be eliminated, since no vector in T is a linear combination of vectors listed earlier in T. In this manner we construct a basis B for $\mathcal{V}$ that contains T. We have just proved the following:

Theorem 4.18 Let T be a linearly independent subset of a finite dimensional vector space $\mathcal{V}$. Then $\mathcal{V}$ has a basis B with $T \subseteq B$.

Compare this result with Theorem 4.17.

We modify slightly the method outlined just before Theorem 4.18 to find a basis for a finite dimensional vector space containing a given linearly independent subset T.

Method for Finding a Basis by Enlarging a Linearly Independent Subset (Enlarging Method)

Suppose that $T = \{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$ is a linearly independent subset of a finite dimensional vector space $\mathcal{V}$.

Step 1: Find a finite spanning set $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ for $\mathcal{V}$.

Step 2: Form the ordered spanning set $S = \{\mathbf{t}_1, \ldots, \mathbf{t}_k, \mathbf{a}_1, \ldots, \mathbf{a}_n\}$ for $\mathcal{V}$.

Step 3: Use either the Independence Test Method or the Inspection Method on S to produce a subset B of S.

Then B is a basis for $\mathcal{V}$ containing T.

The basis produced by this method is easier to use if the additional vectors in the set A have a simple form. Ideally, we choose A to be the standard basis for $\mathcal{V}$.

Example 8

Consider the linearly independent subset $T = \{[2,0,4,-12],[0,-1,-3,9]\}$ of $\mathcal{V} = \mathbb{R}^4$. We use the Enlarging Method to find a basis for $\mathbb{R}^4$ that contains T.

Step 1: We choose A to be the standard basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ for $\mathbb{R}^4$.

Step 2: We create

$$S = \{[2,0,4,-12],[0,-1,-3,9],[1,0,0,0],[0,1,0,0],[0,0,1,0],[0,0,0,1]\}.$$

Step 3: The matrix

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 4 & -3 & 0 & 0 & 1 & 0 \\ -12 & 9 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{reduces to} \quad \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{4} & 0 & -\frac{1}{12} \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}.$$

Since columns $1, 2, 3$, and 5 have nonzero pivots, the Independence Test Method indicates that the set $B = \{[2,0,4,-12],[0,-1,-3,9],[1,0,0,0],[0,0,1,0]\}$ is a basis for $\mathbb{R}^4$ containing T.

In general, we can use the Enlarging Method only when we already know a finite spanning set to use for A. Otherwise, we can make an intelligent guess, just as we did when using the Independence Test Method on an infinite spanning set. However, we must then take care to verify that the resulting set actually spans the vector space.

New Vocabulary

Enlarging Method

Inspection Method

Highlights

- Every spanning set of a finite dimensional vector space $\mathcal{V}$ has a subset that is a basis for $\mathcal{V}$.
- Every linearly independent set of a finite dimensional vector space $\mathcal{V}$ can be enlarged to a basis for $\mathcal{V}$.
- The Simplified Span Method is useful for finding a basis (in simplified form) for the span of a given set of vectors (by row reducing the matrix whose rows are the given vectors).
- The Independence Test Method is useful for finding a *subset* of a given set of vectors that is a basis for the span of the vectors.

- The Inspection Method is useful for finding a subset of a spanning set that is a basis (by eliminating those vectors that are linear combinations of earlier vectors).
- The Enlarging Method is useful for enlarging a linearly independent set to a basis (for a finite dimensional vector space).

EXERCISES FOR SECTION 4.6

1. For each of the given subsets $\mathcal{S}$ of $\mathbb{R}^5$, find a basis for $\mathcal{V} = \text{span}(\mathcal{S})$ using the Simplified Span Method:

⋆(a) $\mathcal{S} = \{[1,2,3,-1,0],[3,6,8,-2,0],[-1,-1,-3,1,1],[-2,-3,-5,1,1]\}$

(b) $\mathcal{S} = \{[3,2,-1,0,1],[1,-1,0,3,1],[4,1,-1,3,2],[3,7,-2,-9,-1],$
$[-1,-4,1,6,1]\}$

(c) $\mathcal{S} = \{[0,1,1,0,6],[2,-1,0,-2,1],[-1,2,1,1,2],[3,-2,0,-2,-3],$
$[1,1,1,-1,4],[2,-1,-1,1,3]\}$

⋆(d) $\mathcal{S} = \{[1,1,1,1,1],[1,2,3,4,5],[0,1,2,3,4],[0,0,4,0,-1]\}$

⋆2. Adapt the Simplified Span Method to find a basis for the subspace of $\mathcal{P}_3$ spanned by $\mathcal{S} = \{x^3 - 3x^2 + 2, 2x^3 - 7x^2 + x - 3, 4x^3 - 13x^2 + x + 5\}$.

⋆3. Adapt the Simplified Span Method to find a basis for the subspace of $\mathcal{M}_{32}$ spanned by

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 & 4 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 1 & -1 \\ 4 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 7 \\ -1 & -2 \\ 2 & -3 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 2 & -1 \\ 6 & 12 \end{bmatrix} \right\}.$$

4. For each given subset $\mathcal{S}$ of $\mathbb{R}^3$, find a subset B of $\mathcal{S}$ that is a basis for $\mathcal{V} = \text{span}(\mathcal{S})$.

⋆(a) $\mathcal{S} = \{[3,1,-2],[0,0,0],[6,2,-3]\}$

(b) $\mathcal{S} = \{[4,7,1],[1,0,0],[6,7,1],[-4,0,0]\}$

⋆(c) $\mathcal{S} = \{[1,3,-2],[2,1,4],[3,-6,18],[0,1,-1],[-2,1,-6]\}$

(d) $\mathcal{S} = \{[1,4,-2],[-2,-8,4],[2,-8,5],[0,-7,2]\}$

⋆(e) $\mathcal{S} = \{[3,-2,2],[1,2,-1],[3,-2,7],[-1,-10,6]\}$

(f) $\mathcal{S} = \{[3,1,0],[2,-1,7],[0,0,0],[0,5,-21],[6,2,0],[1,5,7]\}$

(g) $\mathcal{S}$ = the set of all 3-vectors whose second coordinate is zero

⋆(h) $\mathcal{S}$ = the set of all 3-vectors whose second coordinate is -3 times its first coordinate plus its third coordinate