5. What happens in Example 3 of the text if you replace $\mathbf{p}$ by $-\mathbf{p}$?

6. What happens in Example 5 if you choose a P at distance $2d$ from the axis of rotation?

7. Rotation. A wheel is rotating about the y-axis with angular speed $\omega = 20 \text{ sec}^{-1}$. The rotation appears clockwise if one looks from the origin in the positive y-direction. Find the velocity and speed at the point $[8, 6, 0]$. Make a sketch.

8. Rotation. What are the velocity and speed in Prob. 7 at the point $(4, 2, -2)$ if the wheel rotates about the line $y = x, z = 0$ with $\omega = 10 \text{ sec}^{-1}$?

9. Scalar triple product. What does $(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = 0$ imply with respect to these vectors?

10. WRITING REPORT. Summarize the most important applications discussed in this section. Give examples. No proofs.

11–23 VECTOR AND SCALAR TRIPLE PRODUCTS

With respect to right-handed Cartesian coordinates, let $\mathbf{a} = [2, 1, 0]$, $\mathbf{b} = [-3, 2, 0]$, $\mathbf{c} = [1, 4, -2]$, and $\mathbf{d} = [5, -1, 3]$. Showing details, find:

11. $\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{a}$, $\mathbf{a} \cdot \mathbf{b}$

12. $3\mathbf{c} \times 5\mathbf{d}$, $15\mathbf{d} \times \mathbf{c}$, $15\mathbf{d} \cdot \mathbf{c}$, $15\mathbf{c} \cdot \mathbf{d}$

13. $\mathbf{c} \times (\mathbf{a} + \mathbf{b})$, $\mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

14. $4\mathbf{b} \times 3\mathbf{c} + 12\mathbf{c} \times \mathbf{b}$

15. $(\mathbf{a} + \mathbf{d}) \times (\mathbf{d} + \mathbf{a})$

16. $(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}$, $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})$

17. $(\mathbf{b} \times \mathbf{c}) \times \mathbf{d}$, $\mathbf{b} \times (\mathbf{c} \times \mathbf{d})$

18. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}$, $\mathbf{a} \times (\mathbf{b} \times \mathbf{a})$

19. $(\mathbf{i}\ \mathbf{j}\ \mathbf{k})$, $(\mathbf{i}\ \mathbf{k}\ \mathbf{j})$

20. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$, $(\mathbf{a}\ \mathbf{b}\ \mathbf{d})\mathbf{c} - (\mathbf{a}\ \mathbf{b}\ \mathbf{c})\mathbf{d}$

21. $4\mathbf{b} \times 3\mathbf{c}$, $12|\mathbf{b} \times \mathbf{c}|$, $12|\mathbf{c} \times \mathbf{b}|$

22. $(\mathbf{a} - \mathbf{b}\ \ \mathbf{c} - \mathbf{b}\ \ \mathbf{d} - \mathbf{b})$, $(\mathbf{a}\ \mathbf{c}\ \mathbf{d})$

23. $\mathbf{b} \times \mathbf{b}$, $(\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{b})$, $\mathbf{b} \cdot \mathbf{b}$

24. TEAM PROJECT. Useful Formulas for Three and Four Vectors. Prove (13)–(16), which are often useful in practical work, and illustrate each formula with two examples. *Hint.* For (13) choose Cartesian coordinates such that $\mathbf{d} = [d_1, 0, 0]$ and $\mathbf{c} = [c_1, c_2, 0]$. Show that each side of (13) then equals $[-b_2c_2d_1, b_1c_2d_1, 0]$, and give reasons why the two sides are then equal in any Cartesian coordinate system. For (14) and (15) use (13).

(13) $\mathbf{b} \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{d}$

(14) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a}\ \mathbf{b}\ \mathbf{d})\mathbf{c} - (\mathbf{a}\ \mathbf{b}\ \mathbf{c})\mathbf{d}$

(15) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$

(16) $(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = (\mathbf{b}\ \mathbf{c}\ \mathbf{a}) = (\mathbf{c}\ \mathbf{a}\ \mathbf{b})$
$= -(\mathbf{c}\ \mathbf{b}\ \mathbf{a}) = -(\mathbf{a}\ \mathbf{c}\ \mathbf{b})$

25–35 APPLICATIONS

25. Moment *m* of a force p. Find the moment vector $\mathbf{m}$ and m of $\mathbf{p} = [2, 3, 0]$ about Q: $(2, 1, 0)$ acting on a line through A: $(0, 3, 0)$. Make a sketch.

26. Moment. Solve Prob. 25 if $\mathbf{p} = [1, 0, 3]$, Q: $(2, 0, 3)$, and A: $(4, 3, 5)$.

27. Parallelogram. Find the area if the vertices are $(4, 2, 0)$, $(10, 4, 0)$, $(5, 4, 0)$, and $(11, 6, 0)$. Make a sketch.

28. A remarkable parallelogram. Find the area of the quadrangle Q whose vertices are the midpoints of the sides of the quadrangle P with vertices A: $(2, 1, 0)$, B: $(5, -1. 0)$, C: $(8, 2, 0)$, and D: $(4, 3, 0)$. Verify that Q is a parallelogram.

29. Triangle. Find the area if the vertices are $(0, 0, 1)$, $(2, 0, 5)$, and $(2, 3, 4)$.

30. Plane. Find the plane through the points A: $(1, 2, \frac{1}{4})$, B: $(4, 2, -2)$, and C: $(0, 8, 4)$.

31. Plane. Find the plane through $(1, 3, 4)$, $(1, -2, 6)$, and $(4, 0, 7)$.

32. Parallelepiped. Find the volume if the edge vectors are $\mathbf{i} + \mathbf{j}$, $-2\mathbf{i} + 2\mathbf{k}$, and $-2\mathbf{i} - 3\mathbf{k}$. Make a sketch.

33. Tetrahedron. Find the volume if the vertices are $(1, 1, 1)$, $(5, -7, 3)$, $(7, 4, 8)$, and $(10, 7, 4)$.

34. Tetrahedron. Find the volume if the vertices are $(1, 3, 6)$, $(3, 7, 12)$, $(8, 8, 9)$, and $(2, 2, 8)$.

35. WRITING PROJECT. Applications of Cross Products. Summarize the most important applications we have discussed in this section and give a few simple examples. No proofs.

9.4 Vector and Scalar Functions and Their Fields. Vector Calculus: Derivatives

Our discussion of vector calculus begins with identifying the two types of functions on which it operates. Let P be any point in a domain of definition. Typical domains in applications are three-dimensional, or a surface or a curve in space. Then we define a **vector function** $\mathbf{v}$, whose values are vectors, that is,

$$\mathbf{v} = \mathbf{v}(P) = [v_1(P), v_2(P), v_3(P)]$$

that depends on points P in space. We say that a vector function defines a **vector field** in a domain of definition. Typical domains were just mentioned. Examples of vector fields are the field of tangent vectors of a curve (shown in Fig. 195), normal vectors of a surface (Fig. 196), and velocity field of a rotating body (Fig. 197). Note that vector functions may also depend on time t or on some other parameters.

Similarly, we define a **scalar function** f, whose values are scalars, that is,

$$f = f(P)$$

that depends on P. We say that a scalar function defines a scalar field in that three-dimensional domain or surface or curve in space. Two representative examples of scalar fields are the temperature field of a body and the pressure field of the air in Earth's atmosphere. Note that scalar functions may also depend on some parameter such as time t.

Notation. If we introduce Cartesian coordinates x, y, z, then, instead of writing $\mathbf{v}(P)$ for the vector function, we can write

$$\mathbf{v}(x, y, z) = [v_1(x, y, z), \quad v_2(x, y, z), \quad v_3(x, y, z)].$$

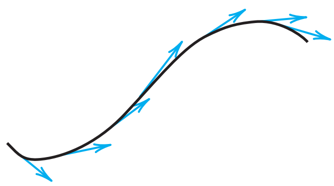

Fig. 195. Field of tangent vectors of a curve

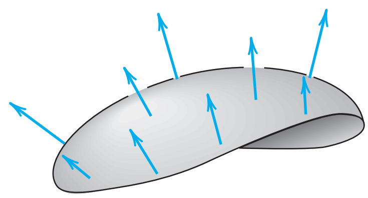

Fig. 196. Field of normal vectors of a surface

We have to keep in mind that the components depend on our choice of coordinate system, whereas a vector field that has a physical or geometric meaning should have magnitude and direction depending only on P, not on the choice of coordinate system.

Similarly, for a scalar function, we write

$$f(P) = f(x, y, z).$$

We illustrate our discussion of vector functions, scalar functions, vector fields, and scalar fields by the following three examples.

EXAMPLE 1 Scalar Function (Euclidean Distance in Space)

The distance $f(P)$ of any point P from a fixed point P_0 in space is a scalar function whose domain of definition is the whole space. $f(P)$ defines a scalar field in space. If we introduce a Cartesian coordinate system and P_0 has the coordinates x_0, y_0, z_0, then f is given by the well-known formula

$$f(P) = f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

where x, y, z are the coordinates of P. If we replace the given Cartesian coordinate system with another such system by translating and rotating the given system, then the values of the coordinates of P and P_0 will in general change, but $f(P)$ will have the same value as before. Hence $f(P)$ is a scalar function. The direction cosines of the straight line through P and P_0 are not scalars because their values depend on the choice of the coordinate system. ■

EXAMPLE 2 **Vector Field (Velocity Field)**

At any instant the velocity vectors $\mathbf{v}(P)$ of a rotating body B constitute a vector field, called the **velocity field** of the rotation. If we introduce a Cartesian coordinate system having the origin on the axis of rotation, then (see Example 5 in Sec. 9.3)

$$\mathbf{v}(x, y, z) = \mathbf{w} \times \mathbf{r} = \mathbf{w} \times [x, y, z] = \mathbf{w} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \tag{1}$$

where x, y, z are the coordinates of any point P of B at the instant under consideration. If the coordinates are such that the z-axis is the axis of rotation and $\mathbf{w}$ points in the positive z-direction, then $\mathbf{w} = \omega\mathbf{k}$ and

$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \omega[-y, x, 0] = \omega(-y\mathbf{i} + x\mathbf{j}).$$

An example of a rotating body and the corresponding velocity field are shown in Fig. 197. ■

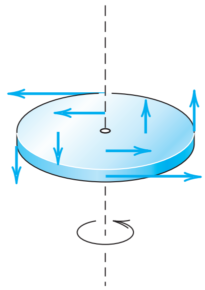

Fig. 197. Velocity field of a rotating body

EXAMPLE 3 **Vector Field (Field of Force, Gravitational Field)**

Let a particle A of mass M be fixed at a point P_0 and let a particle B of mass m be free to take up various positions P in space. Then A attracts B. According to **Newton's law of gravitation** the corresponding gravitational force $\mathbf{p}$ is directed from P to P_0, and its magnitude is proportional to $1/r^2$, where r is the distance between P and P_0, say,

$$|\mathbf{p}| = \frac{c}{r^2}, \qquad c = GMm. \tag{2}$$

Here $G = 6.67 \cdot 10^{-8}\ \mathrm{cm}^3/(\mathrm{g} \cdot \mathrm{sec}^2)$ is the gravitational constant. Hence $\mathbf{p}$ defines a vector field in space. If we introduce Cartesian coordinates such that P_0 has the coordinates x_0, y_0, z_0 and P has the coordinates x, y, z, then by the Pythagorean theorem,

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \qquad (\geqq 0).$$

Assuming that $r > 0$ and introducing the vector

$$\mathbf{r} = [x - x_0,\quad y - y_0,\quad z - z_0] = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k},$$

we have $|\mathbf{r}| = r$, and $(-1/r)\mathbf{r}$ is a unit vector in the direction of $\mathbf{p}$; the minus sign indicates that $\mathbf{p}$ is directed from P to P_0 (Fig. 198). From this and (2) we obtain

$$\begin{aligned} \mathbf{p} = |\mathbf{p}|\left(-\frac{1}{r}\mathbf{r}\right) &= -\frac{c}{r^3}\mathbf{r} = \left[-c\,\frac{x - x_0}{r^3},\quad -c\,\frac{y - y_0}{r^3},\quad -c\,\frac{z - z_0}{r^3}\right] \\ &= -c\,\frac{x - x_0}{r^3}\,\mathbf{i} - c\,\frac{y - y_0}{r^3}\,\mathbf{j} - c\,\frac{z - z_0}{r^3}\,\mathbf{k}. \end{aligned} \tag{3}$$

This vector function describes the gravitational force acting on B. ■

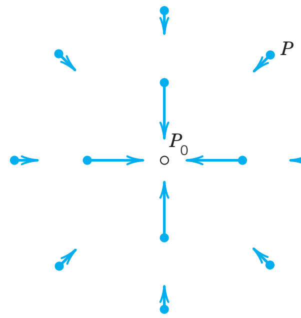

Fig. 198. Gravitational field in Example 3

Vector Calculus

The student may be pleased to learn that many of the concepts covered in (regular) calculus carry over to vector calculus. Indeed, we show how the basic concepts of convergence, continuity, and differentiability from calculus can be defined for vector functions in a simple and natural way. Most important of these is the derivative of a vector function.

Convergence. An infinite sequence of vectors $\mathbf{a}_{(n)}$, $n = 1, 2, \cdots,$ is said to **converge** if there is a vector $\mathbf{a}$ such that

$$\lim_{n\to\infty} |\mathbf{a}_{(n)} - \mathbf{a}| = 0. \tag{4}$$

$\mathbf{a}$ is called the **limit vector** of that sequence, and we write

$$\lim_{n\to\infty} \mathbf{a}_{(n)} = \mathbf{a}. \tag{5}$$

If the vectors are given in Cartesian coordinates, then this sequence of vectors converges to $\mathbf{a}$ if and only if the three sequences of components of the vectors converge to the corresponding components of $\mathbf{a}$. We leave the simple proof to the student.

Similarly, a vector function $\mathbf{v}(t)$ of a real variable t is said to have the **limit** $\boldsymbol{l}$ as t approaches t_0, if $\mathbf{v}(t)$ is defined in some neighborhood of t_0 (possibly except at t_0) and

$$\lim_{t\to t_0} |\mathbf{v}(t) - \boldsymbol{l}| = 0. \tag{6}$$

Then we write

$$\lim_{t\to t_0} \mathbf{v}(t) = \boldsymbol{l}. \tag{7}$$

Here, a *neighborhood* of t_0 is an interval (segment) on the t-axis containing t_0 as an interior point (not as an endpoint).

Continuity. A vector function $\mathbf{v}(t)$ is said to be **continuous** at $t = t_0$ if it is defined in some neighborhood of t_0 (including at t_0 itself!) and

$$\lim_{t\to t_0} \mathbf{v}(t) = \mathbf{v}(t_0). \tag{8}$$

If we introduce a Cartesian coordinate system, we may write

$$\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)] = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}.$$

Then $\mathbf{v}(t)$ is continuous at t_0 if and only if its three components are continuous at t_0.

We now state the most important of these definitions.

DEFINITION

Derivative of a Vector Function

A vector function $\mathbf{v}(t)$ is said to be **differentiable** at a point t if the following limit exists:

$$\mathbf{v}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}. \tag{9}$$

This vector $\mathbf{v}'(t)$ is called the **derivative** of $\mathbf{v}(t)$. See Fig. 199.

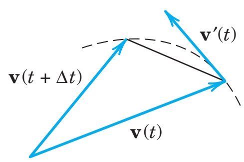

Fig. 199. Derivative of a vector function

In components with respect to a given Cartesian coordinate system,

$$\mathbf{v}'(t) = [v_1'(t), \quad v_2'(t), \quad v_3'(t)]. \tag{10}$$

Hence the derivative $\mathbf{v}'(t)$ ***is obtained by differentiating each component separately.*** For instance, if $\mathbf{v} = [t, t^2, 0]$, then $\mathbf{v}' = [1, 2t, 0]$.

Equation (10) follows from (9) and conversely because (9) is a "vector form" of the usual formula of calculus by which the derivative of a function of a single variable is defined. [The curve in Fig. 199 is the locus of the terminal points representing $\mathbf{v}(t)$ for values of the independent variable in some interval containing t and $t + \Delta t$ in (9)]. It follows that the familiar differentiation rules continue to hold for differentiating vector functions, for instance,

$$(c\mathbf{v})' = c\mathbf{v}' \qquad (c \text{ constant}),$$

$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

and in particular

$$(\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' \tag{11}$$

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}' \tag{12}$$

$$(\mathbf{u}\quad \mathbf{v}\quad \mathbf{w})' = (\mathbf{u}'\quad \mathbf{v}\quad \mathbf{w}) + (\mathbf{u}\quad \mathbf{v}'\quad \mathbf{w}) + (\mathbf{u}\quad \mathbf{v}\quad \mathbf{w}'). \tag{13}$$

The simple proofs are left to the student. In (12), note the order of the vectors carefully because cross multiplication is not commutative.

EXAMPLE 4 Derivative of a Vector Function of Constant Length

Let $\mathbf{v}(t)$ be a vector function whose length is constant, say, $|\mathbf{v}(t)| = c$. Then $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v} = c^2$, and $(\mathbf{v} \cdot \mathbf{v})' = 2\mathbf{v} \cdot \mathbf{v}' = 0$, by differentiation [see (11)]. This yields the following result. *The derivative of a vector function* $\mathbf{v}(t)$ *of constant length is either the zero vector or is perpendicular to* $\mathbf{v}(t)$. ■

Partial Derivatives of a Vector Function

Our present discussion shows that partial differentiation of vector functions of two or more variables can be introduced as follows. Suppose that the components of a vector function

$$\mathbf{v} = [v_1, \quad v_2, \quad v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

are differentiable functions of n variables $t_1, \cdots, t_n$. Then the **partial derivative** of $\mathbf{v}$ with respect to t_m is denoted by $\partial\mathbf{v}/\partial t_m$ and is defined as the vector function

$$\frac{\partial \mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k}.$$

Similarly, second partial derivatives are

$$\frac{\partial^2 \mathbf{v}}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m}\mathbf{i} + \frac{\partial^2 v_2}{\partial t_l \partial t_m}\mathbf{j} + \frac{\partial^2 v_3}{\partial t_l \partial t_m}\mathbf{k},$$

and so on.

EXAMPLE 5 Partial Derivatives

Let $\mathbf{r}(t_1, t_2) = a \cos t_1 \,\mathbf{i} + a \sin t_1 \,\mathbf{j} + t_2 \,\mathbf{k}$. Then $\dfrac{\partial \mathbf{r}}{\partial t_1} = -a \sin t_1 \,\mathbf{i} + a \cos t_1 \,\mathbf{j}$ and $\dfrac{\partial \mathbf{r}}{\partial t_2} = \mathbf{k}$. ■

Various physical and geometric applications of derivatives of vector functions will be discussed in the next sections as well as in Chap. 10.

PROBLEM SET 9.4

1–8 SCALAR FIELDS IN THE PLANE

Let the temperature T in a body be independent of z so that it is given by a scalar function $T = T(x, t)$. Identify the isotherms $T(x, y) = \text{const}$. Sketch some of them.

1. $T = x^2 - y^2$
2. $T = xy$
3. $T = 3x - 4y$
4. $T = \arctan(y/x)$
5. $T = y/(x^2 + y^2)$
6. $T = x/(x^2 + y^2)$
7. $T = 9x^2 + 4y^2$
8. **CAS PROJECT. Scalar Fields in the Plane.** Sketch or graph isotherms of the following fields and describe what they look like.

(a) $x^2 - 4x - y^2$ **(b)** $x^2y - y^3/3$
(c) $\cos x \sinh y$ **(d)** $\sin x \sinh y$
(e) $e^x \sin y$ **(f)** $e^{2x} \cos 2y$
(g) $x^4 - 6x^2y^2 + y^4$ **(h)** $x^2 - 2x - y^2$

9–14 SCALAR FIELDS IN SPACE

What kind of surfaces are the **level surfaces** $f(x, y, z) = \text{const}$?

9. $f = 4x - 3y + 2z$
10. $f = 9(x^2 + y^2) + z^2$
11. $f = 5x^2 + 2y^2$
12. $f = z - \sqrt{x^2 + y^2}$
13. $f = z - (x^2 + y^2)$
14. $f = x - y^2$

15–20 **VECTOR FIELDS**

Sketch figures similar to Fig. 198. Try to interpet the field of **v** as a velocity field.

15. $\mathbf{v} = \mathbf{i} + \mathbf{j}$

16. $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$

17. $\mathbf{v} = x\mathbf{j}$

18. $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$

19. $\mathbf{v} = x\mathbf{i} - y\mathbf{j}$

20. $\mathbf{v} = y\mathbf{i} - x\mathbf{j}$

21. CAS PROJECT. Vector Fields. Plot by arrows:

(a) $\mathbf{v} = [x, x^2]$ **(b)** $\mathbf{v} = [1/y, 1/x]$

(c) $\mathbf{v} = [\cos x, \sin x]$ **(d)** $\mathbf{v} = e^{-(x^2+y^2)}[x, -y]$

22–25 **DIFFERENTIATION**

22. Find the first and second derivatives of $\mathbf{r} = [3 \cos 2t, 3 \sin 2t, 4t]$.

23. Prove (11)–(13). Give two typical examples for each formula.

24. Find the first partial derivatives of $\mathbf{v}_1 = [e^x \cos y, e^x \sin y]$ and $\mathbf{v}_2 = [\cos x \cosh y, -\sin x \sinh y]$.

25. WRITING PROJECT. Differentiation of Vector Functions. Summarize the essential ideas and facts and give examples of your own.

9.5 Curves. Arc Length. Curvature. Torsion

Vector calculus has important applications to curves (Sec. 9.5) and surfaces (to be covered in Sec. 10.5) in physics and geometry. The application of vector calculus to geometry is a field known as **differential geometry**. Differential geometric methods are applied to problems in mechanics, computer-aided as well as traditional engineering design, geodesy, geography, space travel, and relativity theory. For details, see [GenRef8] and [GenRef9] in App. 1.

Bodies that move in space form paths that may be represented by curves C. This and other applications show the need for **parametric representations** of C with **parameter** t, which may denote time or something else (see Fig. 200). A typical parametric representation is given by

(1) $$\mathbf{r}(t) = [x(t), \quad y(t), \quad z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

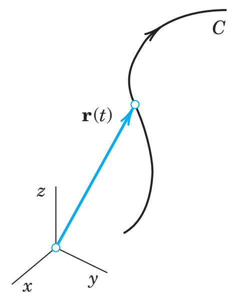

Fig. 200. Parametric representation of a curve

Here t is the parameter and x, y, z are Cartesian coordinates, that is, the usual rectangular coordinates as shown in Sec. 9.1. To each value $t = t_0$, there corresponds a point of C with position vector $\mathbf{r}(t_0)$ whose coordinates are $x(t_0), y(t_0), z(t_0)$. This is illustrated in Figs. 201 and 202.

The use of parametric representations has key advantages over other representations that involve projections into the xy-plane and xz-plane or involve a pair of equations with y or with z as independent variable. The projections look like this:

(2) $$y = f(x), \qquad z = g(x).$$