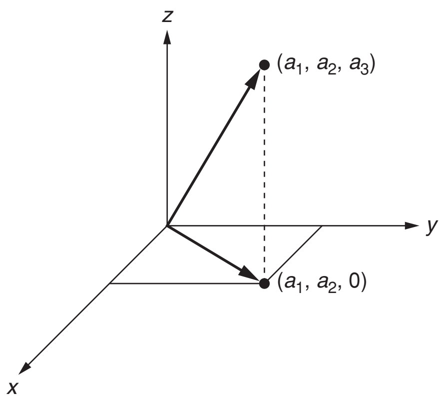

FIGURE 5.3

Projection of $[a_1, a_2, a_3]$ to the xy-plane

consider the mapping $j: \mathbb{R}^4 \to \mathbb{R}^4$ given by $j([a_1, a_2, a_3, a_4]) = [0, a_2, 0, a_4]$. This mapping takes each vector in $\mathbb{R}^4$ to a corresponding vector whose first and third coordinates are zero. The functions h and j are both linear operators (see Exercise 5). Such mappings, where at least one of the coordinates is "zeroed out," are examples of **projection mappings**. You can verify that all such mappings are linear operators. (Other types of projection mappings are illustrated in Exercises 6 and 7.)

Example 9

Rotations: Let θ be a fixed angle in $\mathbb{R}^2$, and let $l: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$l\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

In Exercise 9 you are asked to show that l rotates $[x, y]$ counterclockwise through the angle θ (see Figure 5.4).

Now, let $\mathbf{v}_1 = [x_1, y_1]$ and $\mathbf{v}_2 = [x_2, y_2]$ be two vectors in $\mathbb{R}^2$. Then,

$$\begin{aligned} l(\mathbf{v}_1 + \mathbf{v}_2) &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}(\mathbf{v}_1 + \mathbf{v}_2) \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\mathbf{v}_1 + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\mathbf{v}_2 \\ &= l(\mathbf{v}_1) + l(\mathbf{v}_2). \end{aligned}$$

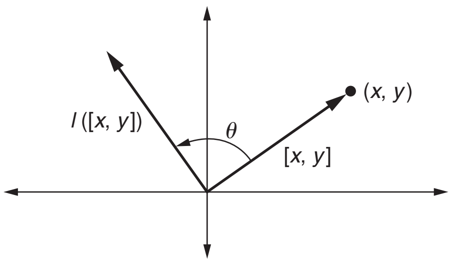

FIGURE 5.4

Counterclockwise rotation of $[x, y]$ through an angle θ in $\mathbb{R}^2$

Similarly, $l(c\mathbf{v}) = cl(\mathbf{v})$, for any $c \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^2$. Hence, l is a linear operator. ∎

Beware! Not all geometric operations are linear operators. Recall that the translation function is not a linear operator!

Multiplication Transformation

The linear operator in Example 9 is actually a special case of the next example, which shows that multiplication by an $m \times n$ matrix is always a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Example 10

Let $\mathbf{A}$ be a given $m \times n$ matrix. We show that the function $f: \mathbb{R}^n \to \mathbb{R}^m$ defined by $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^n$, is a linear transformation. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$. Then $f(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{A}\mathbf{x}_1 + \mathbf{A}\mathbf{x}_2 = f(\mathbf{x}_1) + f(\mathbf{x}_2)$. Also, let $\mathbf{x} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then, $f(c\mathbf{x}) = \mathbf{A}(c\mathbf{x}) = c(\mathbf{A}\mathbf{x}) = cf(\mathbf{x})$. ∎

For a specific example of the multiplication transformation, consider the matrix $\mathbf{A} = \begin{bmatrix} -1 & 4 & 2 \\ 5 & 6 & -3 \end{bmatrix}$. The mapping given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -1 & 4 & 2 \\ 5 & 6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + 4x_2 + 2x_3 \\ 5x_1 + 6x_2 - 3x_3 \end{bmatrix}$$

is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$. In the next section, we will show that the converse of the result in Example 10 also holds; every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is equivalent to multiplication by an appropriate $m \times n$ matrix.

Elementary Properties of Linear Transformations

We now prove some basic properties of linear transformations. From here on, we usually use italicized capital letters, such as "L," to represent linear transformations.

Theorem 5.1 Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces, and let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Let $\mathbf{0}_{\mathcal{V}}$ be the zero vector in $\mathcal{V}$ and $\mathbf{0}_{\mathcal{W}}$ be the zero vector in $\mathcal{W}$. Then

(1) $L(\mathbf{0}_{\mathcal{V}}) = \mathbf{0}_{\mathcal{W}}$

(2) $L(-\mathbf{v}) = -L(\mathbf{v})$, for all $\mathbf{v} \in \mathcal{V}$

(3) $L(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) = a_1L(\mathbf{v}_1) + a_2L(\mathbf{v}_2) + \cdots + a_nL(\mathbf{v}_n)$, for all $a_1, \ldots, a_n \in \mathbb{R}$, and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{V}$, for $n \geq 2$.

Proof.

Part (1):

$$\begin{aligned} L(\mathbf{0}_{\mathcal{V}}) &= L(0\mathbf{0}_{\mathcal{V}}) && \text{part (2) of Theorem 4.1, in } \mathcal{V} \\ &= 0L(\mathbf{0}_{\mathcal{V}}) && \text{property (2) of linear transformation} \\ &= \mathbf{0}_{\mathcal{W}} && \text{part (2) of Theorem 4.1, in } \mathcal{W} \end{aligned}$$

Part (2):

$$\begin{aligned} L(-\mathbf{v}) &= L(-1\mathbf{v}) && \text{part (3) of Theorem 4.1, in } \mathcal{V} \\ &= -1(L(\mathbf{v})) && \text{property (2) of linear transformation} \\ &= -L(\mathbf{v}) && \text{part (3) of Theorem 4.1, in } \mathcal{W} \end{aligned}$$

Part (3): (Abridged) This part is proved by induction. We prove the Base Step $(n = 2)$ here and leave the Inductive Step as Exercise 29. For the Base Step, we must show that $L(a_1\mathbf{v}_1 + a_2\mathbf{v}_2) = a_1L(\mathbf{v}_1) + a_2L(\mathbf{v}_2)$. But,

$$\begin{aligned} L(a_1\mathbf{v}_1 + a_2\mathbf{v}_2) &= L(a_1\mathbf{v}_1) + L(a_2\mathbf{v}_2) && \text{property (1) of linear transformation} \\ &= a_1L(\mathbf{v}_1) + a_2L(\mathbf{v}_2) && \text{property (2) of linear transformation.} \quad \square \end{aligned}$$

The next theorem asserts that the composition $L_2 \circ L_1$ of linear transformations L_1 and L_2 is again a linear transformation (see Appendix B for a review of composition of functions).

Theorem 5.2 Let $\mathcal{V}_1, \mathcal{V}_2$, and $\mathcal{V}_3$ be vector spaces. Let $L_1: \mathcal{V}_1 \to \mathcal{V}_2$ and $L_2: \mathcal{V}_2 \to \mathcal{V}_3$ be linear transformations. Then $L_2 \circ L_1: \mathcal{V}_1 \to \mathcal{V}_3$ given by $(L_2 \circ L_1)(\mathbf{v}) = L_2(L_1(\mathbf{v}))$, for all $\mathbf{v} \in \mathcal{V}_1$, is a linear transformation.

Proof. (Abridged) To show that $L_2 \circ L_1$ is a linear transformation, we must show that for all $c \in \mathbb{R}$ and $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$,

$$(L_2 \circ L_1)(\mathbf{v}_1 + \mathbf{v}_2) = (L_2 \circ L_1)(\mathbf{v}_1) + (L_2 \circ L_1)(\mathbf{v}_2)$$
$$\text{and} \quad (L_2 \circ L_1)(c\mathbf{v}) = c(L_2 \circ L_1)(\mathbf{v}).$$

The first property holds since

$$\begin{aligned}
(L_2 \circ L_1)(\mathbf{v}_1 + \mathbf{v}_2) &= L_2(L_1(\mathbf{v}_1 + \mathbf{v}_2)) \\
&= L_2(L_1(\mathbf{v}_1) + L_1(\mathbf{v}_2)) && \text{because } L_1 \text{ is a linear transformation} \\
&= L_2(L_1(\mathbf{v}_1)) + L_2(L_1(\mathbf{v}_2)) && \text{because } L_2 \text{ is a linear transformation} \\
&= (L_2 \circ L_1)(\mathbf{v}_1) + (L_2 \circ L_1)(\mathbf{v}_2).
\end{aligned}$$

We leave the proof of the second property as Exercise 33. □

Example 11

Let L_1 represent the rotation of vectors in $\mathbb{R}^2$ through a fixed angle θ (as in Example 9), and let L_2 represent the reflection of vectors in $\mathbb{R}^2$ through the x-axis. That is, if $\mathbf{v} = [v_1, v_2]$, then

$$L_1(\mathbf{v}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad L_2(\mathbf{v}) = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}.$$

Because L_1 and L_2 are both linear transformations, Theorem 5.2 asserts that

$$L_2(L_1(\mathbf{v})) = L_2\left(\begin{bmatrix} v_1\cos\theta - v_2\sin\theta \\ v_1\sin\theta + v_2\cos\theta \end{bmatrix}\right) = \begin{bmatrix} v_1\cos\theta - v_2\sin\theta \\ -v_1\sin\theta - v_2\cos\theta \end{bmatrix}$$

is also a linear transformation. $L_2 \circ L_1$ represents a rotation of $\mathbf{v}$ through θ followed by a reflection through the x-axis.

Theorem 5.2 generalizes naturally to more than two linear transformations. That is, if $L_1, L_2, \ldots, L_k$ are linear transformations and the composition $L_k \circ \cdots \circ L_2 \circ L_1$ makes sense, then $L_k \circ \cdots \circ L_2 \circ L_1$ is also a linear transformation.

Linear Transformations and Subspaces

The final theorem of this section assures us that, under a linear transformation $L: \mathcal{V} \to \mathcal{W}$, subspaces of $\mathcal{V}$ "correspond" to subspaces of $\mathcal{W}$, and vice versa.

Theorem 5.3 Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation.

(1) If $\mathcal{V}'$ is a subspace of $\mathcal{V}$, then $L(\mathcal{V}') = \{L(\mathbf{v}) \mid \mathbf{v} \in \mathcal{V}'\}$, the image of $\mathcal{V}'$ in $\mathcal{W}$, is a subspace of $\mathcal{W}$. In particular, the range of L is a subspace of $\mathcal{W}$.

(2) If $\mathcal{W}'$ is a subspace of $\mathcal{W}$, then $L^{-1}(\mathcal{W}') = \{\mathbf{v} \mid L(\mathbf{v}) \in \mathcal{W}'\}$, the pre-image of $\mathcal{W}'$ in $\mathcal{V}$, is a subspace of $\mathcal{V}$.

We prove part (1) and leave part (2) as Exercise 31.

Proof. Part (1): Suppose that $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and that $\mathcal{V}'$ is a subspace of $\mathcal{V}$. Now, $L(\mathcal{V}')$, the image of $\mathcal{V}'$ in $\mathcal{W}$ (see Figure 5.5), is certainly nonempty (why?). Hence, to show that $L(\mathcal{V}')$ is a subspace of $\mathcal{W}$, we must prove that $L(\mathcal{V}')$ is closed under addition and scalar multiplication.

First, suppose that $\mathbf{w}_1, \mathbf{w}_2 \in L(\mathcal{V}')$. Then, by definition of $L(\mathcal{V}')$, we have $\mathbf{w}_1 = L(\mathbf{v}_1)$ and $\mathbf{w}_2 = L(\mathbf{v}_2)$, for some $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}'$. Then, $\mathbf{w}_1 + \mathbf{w}_2 = L(\mathbf{v}_1) + L(\mathbf{v}_2) = L(\mathbf{v}_1 + \mathbf{v}_2)$ because L is a linear transformation. However, since $\mathcal{V}'$ is a subspace of $\mathcal{V}$, $(\mathbf{v}_1 + \mathbf{v}_2) \in \mathcal{V}'$. Thus, $(\mathbf{w}_1 + \mathbf{w}_2)$ is the image of $(\mathbf{v}_1 + \mathbf{v}_2) \in \mathcal{V}'$, and so $(\mathbf{w}_1 + \mathbf{w}_2) \in L(\mathcal{V}')$. Hence, $L(\mathcal{V}')$ is closed under addition.

Next, suppose that $c \in \mathbb{R}$ and $\mathbf{w} \in L(\mathcal{V}')$. By definition of $L(\mathcal{V}')$, $\mathbf{w} = L(\mathbf{v})$, for some $\mathbf{v} \in \mathcal{V}'$. Then, $c\mathbf{w} = cL(\mathbf{v}) = L(c\mathbf{v})$ since L is a linear transformation. Now, $c\mathbf{v} \in \mathcal{V}'$, because $\mathcal{V}'$ is a subspace of $\mathcal{V}$. Thus, $c\mathbf{w}$ is the image of $c\mathbf{v} \in \mathcal{V}'$, and so $c\mathbf{w} \in L(\mathcal{V}')$. Hence, $L(\mathcal{V}')$ is closed under scalar multiplication. □

Example 12

Let $L: \mathcal{M}_{22} \to \mathbb{R}^3$, where $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [b, 0, c]$. L is a linear transformation (verify!). By Theorem 5.3, the range of any linear transformation is a subspace of the codomain. Hence, the range of $L = \{[b, 0, c] \mid b, c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Also, consider the subspace $\mathcal{U}_2 = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| a, b, d \in \mathbb{R} \right\}$ of $\mathcal{M}_{22}$. Then the image of $\mathcal{U}_2$ under L is $\{[b, 0, 0] \mid b \in \mathbb{R}\}$. This image is a subspace of $\mathbb{R}^3$, as Theorem 5.3 asserts. Finally, consider the subspace $\mathcal{W} = \{[b, e, 2b] \mid b, e \in \mathbb{R}\}$ of $\mathbb{R}^3$. The pre-image of $\mathcal{W}$ consists of all

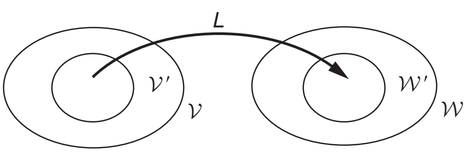

FIGURE 5.5

Subspaces of $\mathcal{V}$ correspond to subspaces of $\mathcal{W}$ under a linear transformation $L: \mathcal{V} \to \mathcal{W}$

matrices in $\mathcal{M}_{22}$ of the form $\begin{bmatrix} a & b \\ 2b & d \end{bmatrix}$. Notice that this pre-image is a subspace of $\mathcal{M}_{22}$, as claimed by Theorem 5.3. ■

New Vocabulary

codomain (of a linear transformation)
composition of linear transformations
contraction (mapping)
dilation (mapping)
domain (of a linear transformation)
identity linear operator
image (of a vector in the domain)
linear operator
linear transformation
pre-image (of a vector in the codomain)
projection (mapping)
range (of a linear transformation)
reflection (mapping)
rotation (mapping)
shear (mapping)
translation (mapping)
zero linear operator

Highlights

- A linear transformation is a function from one vector space to another that preserves the operations of addition and scalar multiplication. That is, under a linear transformation, the image of a linear combination of vectors is the linear combination of the images of the vectors having the same coefficients.
- A linear operator is a linear transformation from a vector space to itself.
- A nontrivial translation of the plane ($\mathbb{R}^2$) or of space ($\mathbb{R}^3$) is never a linear operator, but all of the following are linear operators: contraction (of $\mathbb{R}^n$), dilation (of $\mathbb{R}^n$), reflection of space through the xy-plane (or xz-plane or yz-plane), rotation of the plane about the origin through a given angle θ, projection (of $\mathbb{R}^n$) in which one or more of the coordinates are zeroed out.
- Multiplication of vectors in $\mathbb{R}^n$ on the left by a fixed $m \times n$ matrix $\mathbf{A}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Multiplying a vector on the left by the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is equivalent to rotating the vector counterclockwise about the origin through the angle θ.
- Linear transformations always map the zero vector of the domain to the zero vector of the codomain.
- A composition of linear transformations is a linear transformation.
- Under a linear transformation, subspaces of the domain map to subspaces of the codomain, and the pre-image of a subspace of the codomain is a subspace of the domain.

EXERCISES FOR SECTION 5.1

1. Determine which of the following functions are linear transformations. Prove that your answers are correct. Which are linear operators?

★**(a)** $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f([x,y]) = [3x - 4y, -x + 2y]$

★**(b)** $h: \mathbb{R}^4 \to \mathbb{R}^4$ given by $h([x_1,x_2,x_3,x_4]) = [x_1 + 2, x_2 - 1, x_3, -3]$

(c) $k: \mathbb{R}^3 \to \mathbb{R}^3$ given by $k([x_1,x_2,x_3]) = [x_2, x_3, x_1]$

★**(d)** $l: \mathcal{M}_{22} \to \mathcal{M}_{22}$ given by $l\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a - 2c + d & 3b - c \\ -4a & b + c - 3d \end{bmatrix}$

(e) $n: \mathcal{M}_{22} \to \mathbb{R}$ given by $n\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

★**(f)** $r: \mathcal{P}_3 \to \mathcal{P}_2$ given by $r(ax^3 + bx^2 + cx + d) = (\sqrt[3]{a})x^2 - b^2x + c$

(g) $s: \mathbb{R}^3 \to \mathbb{R}^3$ given by $s([x_1,x_2,x_3]) = [\cos x_1, \sin x_2, e^{x_3}]$

★**(h)** $t: \mathcal{P}_3 \to \mathbb{R}$ given by $t(a_3x^3 + a_2x^2 + a_1x + a_0) = a_3 + a_2 + a_1 + a_0$

(i) $u: \mathbb{R}^4 \to \mathbb{R}$ given by $u\left([x_1,x_2,x_3,x_4]\right) = |x_2|$

★**(j)** $v: \mathcal{P}_2 \to \mathbb{R}$ given by $v\left(ax^2 + bx + c\right) = abc$

★**(k)** $g: \mathcal{M}_{32} \to \mathcal{P}_4$ given by $g\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) = a_{11}x^4 - a_{21}x^2 + a_{31}$

★**(l)** $e: \mathbb{R}^2 \to \mathbb{R}$ given by $e([x,y]) = \sqrt{x^2 + y^2}$

2. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces.

(a) Show that the identity mapping $i: \mathcal{V} \to \mathcal{V}$ given by $i(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v} \in \mathcal{V}$, is a linear operator.

(b) Show that the zero mapping $z: \mathcal{V} \to \mathcal{W}$ given by $z(\mathbf{v}) = \mathbf{0}_{\mathcal{W}}$, for all $\mathbf{v} \in \mathcal{V}$, is a linear transformation.

3. Let k be a fixed scalar in $\mathbb{R}$. Show that the mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ given by $f([x_1,x_2,\ldots,x_n]) = k[x_1,x_2,\ldots,x_n]$ is a linear operator.

4. **(a)** Show that $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by $f([x,y,z]) = [-x,y,z]$ (reflection of a vector through the yz-plane) is a linear operator.

(b) What mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ would reflect a vector through the xz-plane? Is it a linear operator? Why or why not?

(c) What mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ would reflect a vector through the y-axis? through the x-axis? Are these linear operators? Why or why not?

5. Show that the projection mappings $h: \mathbb{R}^3 \to \mathbb{R}^3$ given by $h([a_1,a_2,a_3]) = [a_1,a_2,0]$ and $j: \mathbb{R}^4 \to \mathbb{R}^4$ given by $j([a_1,a_2,a_3,a_4]) = [0,a_2,0,a_4]$ are linear operators.

6. The mapping $f: \mathbb{R}^n \to \mathbb{R}$ given by $f([x_1, x_2, \ldots, x_i, \ldots, x_n]) = x_i$ is another type of projection mapping. Show that f is a linear transformation.

7. Let $\mathbf{x}$ be a fixed nonzero vector in $\mathbb{R}^3$. Show that the mapping $g: \mathbb{R}^3 \to \mathbb{R}^3$ given by $g(\mathbf{y}) = \mathbf{proj_x y}$ is a linear operator.

8. Let $\mathbf{x}$ be a fixed vector in $\mathbb{R}^n$. Prove that $L: \mathbb{R}^n \to \mathbb{R}$ given by $L(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ is a linear transformation.

9. Let θ be a fixed angle in the xy-plane. Show that the linear operator $L: \mathbb{R}^2 \to \mathbb{R}^2$ given by $L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ rotates the vector $[x, y]$ counterclockwise through the angle θ in the plane. (Hint: Consider the vector $[x', y']$, obtained by rotating $[x, y]$ counterclockwise through the angle θ. Let $r = \sqrt{x^2 + y^2}$. Then $x = r\cos\alpha$ and $y = r\sin\alpha$, where α is the angle shown in Figure 5.6. Notice that $x' = r(\cos(\theta + \alpha))$ and $y' = r(\sin(\theta + \alpha))$. Then show that $L([x, y]) = [x', y']$.)

10. (a) Explain why the mapping $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is a linear operator.

(b) Show that the mapping L in part (a) rotates every vector in $\mathbb{R}^3$ about the z-axis through an angle of θ (as measured relative to the xy-plane).

⋆(c) What matrix should be multiplied times $[x, y, z]$ to create the linear operator that rotates $\mathbb{R}^3$ about the y-axis through an angle θ (relative to the xz-plane)? (Hint: When looking down from the positive y-axis toward

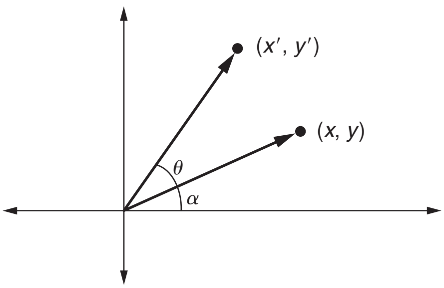

FIGURE 5.6

The vectors $[x, y]$ and $[x', y']$

the xz-plane in a right-handed system, the positive z-axis rotates 90° counterclockwise into the positive x-axis.)

11. **Shears**: Let $f_1, f_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

and

$$f_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}.$$

The mapping f_1 is called a **shear in the x-direction with factor k**; f_2 is called a **shear in the y-direction with factor k**. The effect of these functions (for $k > 1$) on the vector $[1, 1]$ is shown in Figure 5.7. Show that f_1 and f_2 are linear operators directly, without using Example 10.

12. Let $f \colon \mathcal{M}_{nn} \to \mathbb{R}$ be given by $f(\mathbf{A}) = \operatorname{trace}(\mathbf{A})$. (The trace is defined in Exercise 14 of Section 1.4.) Prove that f is a linear transformation.

13. Show that the mappings $g, h \colon \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $g(\mathbf{A}) = \mathbf{A} + \mathbf{A}^T$ and $h(\mathbf{A}) = \mathbf{A} - \mathbf{A}^T$ are linear operators on $\mathcal{M}_{nn}$.

14. **(a)** Show that if $\mathbf{p} \in \mathcal{P}_n$, then the (indefinite integral) function $f \colon \mathcal{P}_n \to \mathcal{P}_{n+1}$, where $f(\mathbf{p})$ is the vector $\int \mathbf{p}(x)\, dx$ with zero constant term, is a linear transformation.

(b) Show that if $\mathbf{p} \in \mathcal{P}_n$, then the (definite integral) function $g \colon \mathcal{P}_n \to \mathbb{R}$ given by $g(\mathbf{p}) = \int_a^b \mathbf{p}\, dx$ is a linear transformation, for any fixed $a, b \in \mathbb{R}$.

15. Let $\mathcal{V}$ be the vector space of all functions f from $\mathbb{R}$ to $\mathbb{R}$ that are infinitely differentiable (that is, for which $f^{(n)}$, the nth derivative of f, exists for every

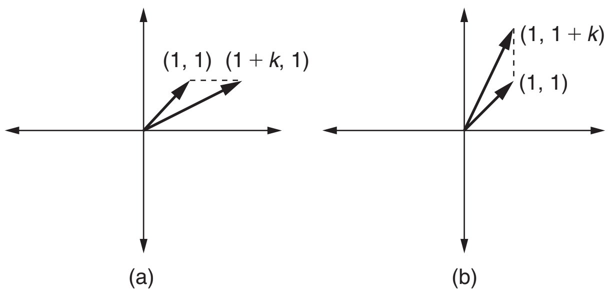

FIGURE 5.7

(a) Shear in the x-direction; (b) shear in the y-direction (both for $k > 0$)

integer $n \geq 1$). Use induction and Theorem 5.2 to show that for any given integer $k \geq 1$, $L: \mathcal{V} \to \mathcal{V}$ given by $L(f) = f^{(k)}$ is a linear operator.

16. Consider the function $f: \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $f(\mathbf{A}) = \mathbf{BA}$, where $\mathbf{B}$ is some fixed $n \times n$ matrix. Show that f is a linear operator.

17. Let $\mathbf{B}$ be a fixed nonsingular matrix in $\mathcal{M}_{nn}$. Show that the mapping $f: \mathcal{M}_{nn} \to \mathcal{M}_{nn}$ given by $f(\mathbf{A}) = \mathbf{B}^{-1}\mathbf{AB}$ is a linear operator.

18. Let a be a fixed real number.

(a) Let $L: \mathcal{P}_n \to \mathbb{R}$ be given by $L(\mathbf{p}(x)) = \mathbf{p}(a)$. (That is, L evaluates polynomials in $\mathcal{P}_n$ at $x = a$.) Show that L is a linear transformation.

(b) Let $L: \mathcal{P}_n \to \mathcal{P}_n$ be given by $L(\mathbf{p}(x)) = \mathbf{p}(x + a)$. (For example, when a is positive, L shifts the graph of $\mathbf{p}(x)$ to the *left* by a units.) Prove that L is a linear operator.

19. Let $\mathbf{A}$ be a fixed matrix in $\mathcal{M}_{nn}$. Define $f: \mathcal{P}_n \to \mathcal{M}_{nn}$ by

$$f(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$$
$$= a_n \mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_1 \mathbf{A} + a_0 \mathbf{I}_n.$$

Show that f is a linear transformation.

20. Let $\mathcal{V}$ be the unusual vector space from Example 7 in Section 4.1. Show that $L: \mathcal{V} \to \mathbb{R}$ given by $L(x) = \ln(x)$ is a linear transformation.

21. Let $\mathcal{V}$ be a vector space, and let $\mathbf{x} \neq \mathbf{0}$ be a fixed vector in $\mathcal{V}$. Prove that the translation function $f: \mathcal{V} \to \mathcal{V}$ given by $f(\mathbf{v}) = \mathbf{v} + \mathbf{x}$ is not a linear transformation.

22. Show that if $\mathbf{A}$ is a fixed matrix in $\mathcal{M}_{mn}$ and $\mathbf{y} \neq \mathbf{0}$ is a fixed vector in $\mathbb{R}^m$, then the mapping $f: \mathbb{R}^n \to \mathbb{R}^m$ given by $f(\mathbf{x}) = \mathbf{Ax} + \mathbf{y}$ is not a linear transformation by showing that part (1) of Theorem 5.1 fails for f.

23. Prove that $f: \mathcal{M}_{33} \to \mathbb{R}$ given by $f(\mathbf{A}) = |\mathbf{A}|$ is not a linear transformation. (A similar result is true for $\mathcal{M}_{nn}$, for $n > 1$.)

24. Suppose $L_1: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $L_2: \mathcal{V} \to \mathcal{W}$ is defined by $L_2(\mathbf{v}) = L_1(2\mathbf{v})$. Show that L_2 is a linear transformation.

25. Suppose $L: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear operator and $L([1,0,0]) = [-2,1,0]$, $L([0,1,0]) = [3,-2,1]$, and $L([0,0,1]) = [0,-1,3]$. Find $L([-3,2,4])$. Give a formula for $L([x,y,z])$, for any $[x,y,z] \in \mathbb{R}^3$.

★26. Suppose $L: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator and $L(\mathbf{i} + \mathbf{j}) = \mathbf{i} - 3\mathbf{j}$ and $L(-2\mathbf{i} + 3\mathbf{j}) = -4\mathbf{i} + 2\mathbf{j}$. Express $L(\mathbf{i})$ and $L(\mathbf{j})$ as linear combinations of $\mathbf{i}$ and $\mathbf{j}$.

27. Let $L: \mathcal{V} \to \mathcal{W}$ be a linear transformation. Show that $L(\mathbf{x} - \mathbf{y}) = L(\mathbf{x}) - L(\mathbf{y})$, for all vectors $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

28. Part (3) of Theorem 5.1 assures us that if $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation, then $L(a\mathbf{v}_1 + b\mathbf{v}_2) = aL(\mathbf{v}_1) + bL(\mathbf{v}_2)$, for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$ and all $a, b \in \mathbb{R}$. Prove that the converse of this statement is true. (Hint: Consider two cases: first $a = b = 1$ and then $b = 0$.)

►**29.** Finish the proof of part (3) of Theorem 5.1 by doing the Inductive Step.

30. **(a)** Suppose that $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation. Show that if $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)\}$ is a linearly independent set of n distinct vectors in $\mathcal{W}$, for some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{V}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set in $\mathcal{V}$.

⋆**(b)** Find a counterexample to the converse of part (a).

►**31.** Finish the proof of Theorem 5.3 by showing that if $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation and $\mathcal{W}'$ is a subspace of $\mathcal{W}$ with pre-image $L^{-1}(\mathcal{W}')$, then $L^{-1}(\mathcal{W}')$ is a subspace of $\mathcal{V}$.

32. Show that every linear operator $L: \mathbb{R} \to \mathbb{R}$ has the form $L(\mathbf{x}) = c\mathbf{x}$, for some $c \in \mathbb{R}$.

33. Finish the proof of Theorem 5.2 by proving property (2) of a linear transformation for $L_2 \circ L_1$.

34. Let $L_1, L_2: \mathcal{V} \to \mathcal{W}$ be linear transformations. Define $(L_1 \oplus L_2): \mathcal{V} \to \mathcal{W}$ by $(L_1 \oplus L_2)(\mathbf{v}) = L_1(\mathbf{v}) + L_2(\mathbf{v})$ (where the latter addition takes place in $\mathcal{W}$). Also define $(c \odot L_1): \mathcal{V} \to \mathcal{W}$ by $(c \odot L_1)(\mathbf{v}) = c\,(L_1(\mathbf{v}))$ (where the latter scalar multiplication takes place in $\mathcal{W}$).

(a) Show that $(L_1 \oplus L_2)$ and $(c \odot L_1)$ are linear transformations.

(b) Use the results in part (a) above and part (b) of Exercise 2 to show that the set of all linear transformations from $\mathcal{V}$ to $\mathcal{W}$ is a vector space under the operations $\oplus$ and $\odot$.

35. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a nonzero linear operator. Show that L maps a line to either a line or a point.

⋆**36.** True or False:

(a) If $L: \mathcal{V} \to \mathcal{W}$ is a function between vector spaces for which $L(c\mathbf{v}) = cL(\mathbf{v})$, then L is a linear transformation.

(b) If $\mathcal{V}$ is an n-dimensional vector space with ordered basis B, then $L: \mathcal{V} \to \mathbb{R}^n$ given by $L(\mathbf{v}) = [\mathbf{v}]_B$ is a linear transformation.

(c) The function $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L([x, y, z]) = [x + 1, y - 2, z + 3]$ is a linear operator.

(d) If $\mathbf{A}$ is a 4×3 matrix, then $L(\mathbf{v}) = \mathbf{A}\mathbf{v}$ is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$.

(e) A linear transformation from $\mathcal{V}$ to $\mathcal{W}$ always maps $\mathbf{0}_{\mathcal{V}}$ to $\mathbf{0}_{\mathcal{W}}$.

(f) If $M_1: \mathcal{V} \to \mathcal{W}$ and $M_2: \mathcal{W} \to \mathcal{X}$ are linear transformations, then $M_1 \circ M_2$ is a well-defined linear transformation.

(g) If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation, then the image of any subspace of $\mathcal{V}$ is a subspace of $\mathcal{W}$.

(h) If $L: \mathcal{V} \to \mathcal{W}$ is a linear transformation, then the pre-image of $\{\mathbf{0}_{\mathcal{W}}\}$ is a subspace of $\mathcal{V}$.

5.2 THE MATRIX OF A LINEAR TRANSFORMATION

In this section, we show that the behavior of any linear transformation $L: \mathcal{V} \to \mathcal{W}$ is determined by its effect on a basis for $\mathcal{V}$. In particular, when $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional and ordered bases for $\mathcal{V}$ and $\mathcal{W}$ are chosen, we can obtain a matrix corresponding to L that is useful in computing images under L. Finally, we investigate how the matrix for L changes as the bases for $\mathcal{V}$ and $\mathcal{W}$ change.

A Linear Transformation Is Determined by Its Action on a Basis

If the action of a linear transformation $L: \mathcal{V} \to \mathcal{W}$ on a basis for $\mathcal{V}$ is known, then the action of L can be computed for all elements of $\mathcal{V}$, as we see in the next example.

Example 1

You can quickly verify that

$$B = \big([0,4,0,1], [-2,5,0,2], [-3,5,1,1], [-1,2,0,1]\big)$$

is an ordered basis for $\mathbb{R}^4$. Now suppose that $L: \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation for which

$$L\big([0,4,0,1]\big) = [3,1,2], \qquad L([-2,5,0,2]) = [2,-1,1],$$

$$L([-3,5,1,1]) = [-4,3,0], \quad \text{and} \quad L([-1,2,0,1]) = [6,1,-1].$$

We can use the values of L on B to compute L for other vectors in $\mathbb{R}^4$. For example, let $\mathbf{v} = [-4,14,1,4]$. By using row reduction, we see that $[\mathbf{v}]_B = [2,-1,1,3]$ (verify!). So,

$$\begin{aligned} L(\mathbf{v}) &= L\big(2[0,4,0,1] - 1[-2,5,0,2] + 1[-3,5,1,1] + 3[-1,2,0,1]\big) \\ &= 2L\big([0,4,0,1]\big) - 1L([-2,5,0,2]) + 1L([-3,5,1,1]) \\ &\quad + 3L([-1,2,0,1]) \\ &= 2[3,1,2] - [2,-1,1] + [-4,3,0] + 3[6,1,-1] \\ &= [18,9,0]. \end{aligned}$$