Chapter 3

Common families of Distribution

3.1

EX = $\frac{1}{N_1-N_0+1}(\sum_{x=0}^{N_1} i - \sum_{x=0}^{N_0-1} i) = \frac{N_1+N_0}{2}$
EX^2 = $\frac{1}{N_1-N_0+1}(\frac{(N_1)(N_1+1)(2N_1+1)-(N_0-1)(N_0)(2N_0-1)}{6})$
Var = $\frac{(N_1-N_0)(N_1-N_0+2)}{12}$

3.2

(a) P(X = 0 | M = 100,N = 6,K) = $\frac{\binom{6}{0}\binom{94}{K}}{\binom{100}{K}}$ On solving K=32

(b) P(X = 0 | M = 100,N = 6,K) = $\frac{\binom{6}{0}\binom{94}{K}}{\binom{100}{K}} + \frac{\binom{6}{1}\binom{94}{K-1}}{\binom{100}{K}}$

On solving K = 50

3.3

(a) So pedestrian have to pass and that could have happen at any event of time lets say at 10 sec so past 3 secs need to be (1-p)^3. Now for the first 4 seconds car passes at 1st second and rest none would be p(1-p)^{3}We nee to do at any seconds after 4 would be (1 - p(1-p)^3)(1-p)^3

3.4

(a) So EX $= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} i \left(\frac{n-1}{n}\right)^j \left(\frac{1}{n}\right) =$ n
(b) So EX $= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} i \left(\frac{n-i}{n}\right)^j \left(\frac{1}{n}\right) = \frac{n+1}{2}$

3.5

(doubtful)

3.6

(a) binomial distribution
(b) $\sum_{x=0}^{100} \quad {}^{2000}C_x \left(\frac{1}{100}\right)^x \left(\frac{99}{100}\right)^{2000-x}$
(c) $\lambda = 2000 * \frac{1}{100} = 20$
P(X=100) $= \frac{1}{e^{20}} \left[1 + \sum_{i=1}^{i=100} \frac{20^i}{i!}\right] = .99 \Rightarrow 99 percent$

3.7

1 - $\frac{1}{e^{\lambda}}$ - $\frac{\lambda}{e^{\lambda}} = 0.99 \Rightarrow \lambda = 6.638$

3.8

(doubtful)

3.9

(a) X $\sim$ Binomial(60, $\frac{1}{90}$)
P(X $\geq$ 5) = 1 - $\sum_{x=0}^{4} \binom{60}{2x} \left[\frac{1}{90}\right]^{2x} \left[\frac{89}{90}\right]^{60-2x}$ Let say answer is .0006
(b) P $\sim$ Binomial(310, .0006) $\Rightarrow$ P(X $\geq$ 0) = 1 - $\binom{310}{0} .0006^0 (1 - .0006)^{310} =$ .1698
(c) P $\sim$ Binomial(500, .1698) $\Rightarrow$ P(X $\geq$ 0) = 1 - $\binom{310}{0} .0006^0 (1 - .0006)^{310} \sim 1$

3.10

(a) Hyper-geometric distributions

$$\mathrm{P} = \frac{\binom{N}{4}\binom{M}{2}}{\binom{N+M}{4}\binom{N+M-4}{2}}$$

(b) N + M = 496

max (N)(N-1)(N-2)(N-3) * (M)(M-1) on solving N = 331, M = 496 - 331 =

3.11

(a) $\mathrm{P(X = x|N, M, K)} = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}$

P = $\frac{M!}{x!M-x!} * \frac{N-M!}{K-x!(N-M-(K-x))} * \frac{K!}{N!N-K!}$ P = $\binom{K}{x}\left(\frac{M!N-M!}{N!M-x!(N-M-(K-x))}\right)$

P = $\binom{K}{x}\left(\frac{(M*M-1*M-2...M-x+1)*(N-M*N-M-1...N-M-(K-x))}{N*N-1*N-2...N-k+1}\right)$

Applying limits M $\rightarrow \infty$,M $\rightarrow \infty$ N/M M $\rightarrow$ p

Applying limits

P = $\binom{K}{x}\left(\frac{M^x N^{K-x}(1-p)^{K-x}}{N^K}\right)$

P = $\binom{K}{x} p^x(1-p)^{K-x}$

(b) See text

(c) Using some steps from above

P = $\left(\frac{M!N-M!K!N-K!}{x!M-x!N-M-K+x!N!}\right)$

P = $\frac{1}{x!}\left[\frac{M^x K^x}{M^x}\right]\left[1-\frac{K}{N}\right]^{M-x}$

P = $\frac{\lambda^x}{x!}\left[1-\frac{\lambda}{M}\right]^M = \frac{e^{-\lambda}\lambda^x}{x!}$

3.12

$F_X(r-1) = P(X \leq r-1)$

$= P(\text{in n trial total of r}^{\text{th}}\text{ success})$

$= 1 - P(\text{in n trail total for r}^{\text{th}}\text{ failure})$

$= 1 - P(\text{in n trail total for (r+1)}^{\text{th}}\text{ success})$ (doubtful)

3.13

(a) $\mathrm{P(X_T = x)} = \frac{P(X=x)}{P(X>0)}$

$P(X_{\mathrm{T}} = x) = \frac{e^{-\lambda}\lambda^x}{x!(1-e^{-\lambda})}$

EX = $\frac{\lambda}{1-e^{-\lambda}}$ EX^2 = $\frac{\lambda^2+\lambda}{1-e^{-\lambda}}$

VarX = $\frac{\lambda^2+\lambda}{1-e^{-\lambda}}$ - $\left[\frac{\lambda}{1-e^{-\lambda}}\right]^2$

(b) P(X_T = x) = $\frac{P(X=x)}{P(X>0)}$

$= \frac{\binom{r+y-1}{y} p^r (1-p)^y}{1-p^r}$

$EX = \frac{r(1-p)}{p(1-p^r)} EX = \frac{r(1-p)-r^2(1-p)^2}{p^2(1-p^r)} - \left(\frac{r(1-p)}{p(1-p^r)}\right)^2$

3.14

(a) $\sum_{x=1}^{x=\infty} \frac{(1-p)^x}{x} = -log(1-(1-p))$ So P(X = x) = 1

(b) EX = -$\frac{1}{logp}\left[\frac{1-p}{p}\right]$

EX^2 = -$\frac{1}{logp}\left[\frac{1-p}{p^2}\right]$

VarX = $\frac{-(1-p)}{p^2 logp}\left[1+\frac{1-p}{logp}\right]$

3.15

$M_X(t) = \left[\frac{p}{(1-(1-p)e^t)}\right]^r$

$M_X(t) = \left[1+\frac{1r(1-p)(e^t-1)}{r(1-(1-p)e^t)}\right]^r$

$M_X(t) = \lambda(e^t-1)$

3.16

(a) $\Gamma(\alpha) = \int_{t=0}^{t=\infty} t^\alpha e^{-t}dt = [t^\alpha e^{-t}]_0^\infty + \alpha\int_{t=0}^\infty t^{\alpha-1}e^{-t}dt = \alpha\Gamma(\alpha)$

(b) $\left(\int_0^\infty e^{-\frac{z^2}{2}}dz\right)^2 = \left(\int_0^\infty e^{-\frac{u^2}{2}}du\right)\left(\int_0^\infty e^{-\frac{v^2}{2}}dv\right)$

$\int_0^\infty\int_0^\infty e^{\frac{-(u^2+v^2)}{2}}dudv = \frac{\sqrt{\pi}}{2}$ taking u = rcos(θ), v = rsin(θ)

$\Gamma(\frac{1}{2}) = \int_0^\infty w^{-1/2}e^{-w}dw = \sqrt{\pi}$

3.17

$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^a}x^{\alpha-1}e^{\frac{-x}{\beta}}$

EX = $\frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{v+\alpha-1}e^{\frac{-x}{\beta}}dx$ taking y = βx will simplify the expression.

3.18

$M_x(t) = \left[\frac{p}{(1-(1-p)e^{ty})}\right]^r$
Applying limit
$\lim_{p\to 0}\left[\frac{p}{(1-(1-p)e^{ty})}\right]^r$
Using L'Hospital Rule
$M_x(t) = \left[\frac{1}{-[(1-p)te^{tp}-e^{tp}]}\right]^r$
$M_x(t) = \left(\frac{1}{1-t}\right)^r$

3.19

On integration by parts
$\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1}e^{-z} = \sum_{\alpha=\alpha}^{0}\left[\frac{x^{\alpha-1}}{\alpha! e^x}\right]$
On rearranging the terms
$\sum_{y=0}^{\alpha-1}\left[\frac{x^y}{y! e^x}\right]$

3.20

(a) EX = $\sqrt{\frac{\pi}{2}}$, EX^2 = 1
(b) $f_y(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1}e^{\frac{-y}{\beta}}$
Y = g(X) $\Rightarrow$ X = $g^{-1}(Y)$
$f_y(y) = f_x(g^{-1}(Y))\left|\frac{dg^{-1}(Y)}{dx}\right|$
$f_y(y) = \frac{2}{\sqrt{2\pi}} e^{\frac{-}{g^{-1}(x)^2}2}\left|\frac{dg^{-1}(Y)}{dx}\right|$ let $g^{-1}(x)^2 = y$ so $\beta = 2$ on solving α = -$\frac{1}{2}$

3.21

Integral is infinite f(x) = $\int_{-\infty}^{\infty} \frac{1}{\pi} * \frac{1}{1+x^2} * e^{tx} dx$

3.22

(a), (b), (c), (d), (e) Integration and parts, Do rough work

3.23

(a) $\int_{x=\alpha}^{\infty} \frac{\beta\alpha^{\beta}}{x^{\beta+1}} dx = -\left[\frac{\beta\alpha^{\beta}}{\beta x^{\beta}}\right]_{\alpha}^{\infty} = 1$
(b)EX $= \int_{x=\alpha}^{\infty} x \frac{\beta\alpha^{\beta}}{x^{\beta+1}} dx = \frac{\alpha\beta}{1-\beta}$
EX^2 $= \int_{x=\alpha}^{\infty} x^2 \frac{\beta\alpha^{\beta}}{x^{\beta+1}} dx = \frac{\alpha\beta}{1-\beta} = \frac{\alpha\beta^2}{2-\beta}$
Var $= \frac{\alpha\beta^2}{2-\beta}$ - $\left[\frac{\alpha\beta}{1-\beta}\right]^2$
(c) If $\beta < 2$ the var is infinite

3.24

$h_T(t) = \lim_{\delta \to 0} \frac{P(t \leq T < t+\delta)}{P(T \geq t)}$
As $\delta \to 0$, t $\to$ T
$h_T(t) = \frac{f_T(t)}{1-F_T(t)}$

3.25

(a), (b), (c) simple calculation

3.26

differentiate the function and solve it for zero

3.27

for (a), (b), (c), (d) Equate with all the function and get the desired result.
$f(x|\theta) = h(x)c(\theta)exp\left(\sum_{i=1}^{k} w_i(\theta)t_i(x)\right)$ Equate term by term

3.28

Equate terms and get the result!

3.29

(a) $f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}$
h(x) = 1 for all x;

c(θ) = c(μ, σ) = $\frac{1}{\sqrt{2\pi\sigma}}e^{\left(\frac{-\mu^2}{2\sigma^2}\right)}$, -$\infty$ < μ < ∞, σ > 0
$w_1(\mu,\sigma) = \frac{1}{\sigma^2}, \sigma > 0; w_2(\mu,\sigma) = \frac{\mu}{\sigma^2}\sigma > 0;$
$t_1(x) = -\frac{x^2}{2}; and t_2(x) = x$
Natural Parameter = (η_1, η_2) = $(-\frac{1}{2\sigma^2}, \frac{\mu}{\sigma})$, $\eta_1 < 0$ & $\infty < \eta_2 < \infty$
(b) For the gamma(α, β),
f(x) = $\left(\frac{1}{\Gamma\beta^\alpha}\right)\left(e^{\alpha-1}log\ x - \frac{x}{\beta}\right)$
natural parameter is (η_1, η_2) = (α - 1, -$\frac{1}{\beta}$)
Varying the α, β $\{(\eta_1, \eta_2) : \eta_1 > -1, \eta_2 < 0\}$

(c) f(x) = $\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)e^{(\alpha-1)log\ x+(\beta-1)log\ (1-x)}$
Natural Parameter: (η_1, η_2) = (α - 1), (β - 1), Natural Parameter Space: {(η_1, η_2): η_1 > -1, η_2 > -1}
(d) h(x) = $\frac{1}{x!}]I_{\{0,1,2,...\}}(x)$, c($\theta$) = $e^{-\theta}$, w$_1$(θ) = log θ, $t_1(x) = x$
So Natural Parameter is η = log θ
Natural Parameter Space is {η: -∞ < η < ∞}
(e) h(x) = $\binom{x-1}{r-1}$ $I_{\{r,r+1,r+2,...\}}$ c(p) = $\frac{p}{1-p}^r$ w$_1$(p) = log(1 - p)
t$_1$(x) = x
Natural Space is η = log(1 - p)
Natural Parameter Space = {η: η < 0}

3.30

(a), (b) put values and get it

3.31

(a) $\int \left(h(x)c(\theta)exp(\sum_{i=0}^{i=k} w_i(\theta)t_i(x))\right) = 1$
Differentiating both side with $\frac{d}{d\theta_i}$
$c^{'}(\theta)\left(\int h(x)exp(\sum_{i=0}^{i=k} w_i(\theta)t_i(x))\right)+c(\theta)\left(\int h(x)exp(\sum_{i=0}^{i=k} w_i(\theta)t_i(x)(\sum_{i=0}^{i=k}\frac{dw_i}{d\theta_j}(\theta)t_i(x)))\right)$
Rearranging terms
E$\left[(\sum_{i=0}^{i=k}\frac{dw_i}{d\theta_j}(\theta)t_i(x)\right] = -\frac{d}{d\theta_j}logc(\theta)$
(b) $c^{''}(\theta)\left(\int h(x)exp(\sum_{i=0}^{i=k} w_i(\theta)t_i(x))\right) + 2c^{'}(\theta)\left(\int h(x)exp(\sum_{i=0}^{i=k} w_i(\theta)t_i(x)(\sum_{i=0}^{i=k}\frac{dw_i}{d\theta_j}(\theta)t_i(x)))\right)+$
$c(\theta)\left(\int h(x)exp(\sum_{i=0}^{i=k} w_i(\theta)t_i(x)(\sum_{i=0}^{i=k}\frac{d^2w_i}{d\theta_j^2}(\theta)t_i(x))) + \int h(x)exp(\sum_{i=0}^{i=k} w_i(\theta)t_i(x)(\sum_{i=0}^{i=k}\frac{dw_i}{d\theta_j}(\theta)t_i(x))^2)\right)$
$\Rightarrow \frac{c^{''}(\theta)}{c(\theta)}+2\frac{c^{'}(\theta)}{c(\theta)}E\left[\sum_{i=0}^{i=k}\frac{dw_i}{d\theta_j}(\theta)t_i(x)\right]+E\left[\sum_{i=0}^{i=k}\frac{d^2w_i}{d\theta_j^2}(\theta)t_i(x)\right]+E\left[(\sum_{i=0}^{i=k}\frac{dw_i}{d\theta_j}(\theta)t_i(x))^2\right]$
$\Rightarrow \frac{d^2}{dx^2}log\ (x)+(\frac{d}{dx}\ log\ x)^2-2\left(\frac{d}{dx}\ log\ x\right)^2+E\left[\sum_{i=0}^{i=k}\frac{d^2w_i}{d\theta_j^2}(\theta)t_i(x)\right]+E\left[(\sum_{i=0}^{i=k}\frac{dw_i}{d\theta_j}(\theta)t_i(x))^2\right]$

Rearranging terms $E\left[\left(\left(\sum_{i=0}^{i=k}\frac{dw_i}{d\theta_j}(\theta)t_i(x)\right)\right)^2\right] - \left(\frac{d}{dx}\ log\ x\right)^2 = -\frac{d^2}{d\theta_j^2}log\ c(\theta) - E\left[\sum_{i=0}^{i=k}\frac{d^2w_i}{d\theta_j^2}(\theta)t_i(x)\right]$
Right Hand Site is Var function

3.32

(a) E$\left[\sum_{i=0}^{i=k}\frac{\partial\eta_i}{\partial\eta_j}\right] = t_j(x)$ double differentiation leads to 0
E$\left[\sum_{i=0}^{i=k}\frac{\partial^2\eta_i}{\partial\eta_j^2}\right] = 0$ Substitute it we will get the terms
(b) Substitute an solve

3.33

(a), (b), (c), (d) solve yourself

3.34

(a) $\bar{X} \sim n(\lambda, \lambda/n)$ You split the term according to exponential family
$\mu = \lambda, \sigma^2 = \lambda/n$
$n\sigma^2 = \mu$
The parameters of theta lies on parabola
(b) $\bar{X} \sim n(p, (1-p)/n)$ You split the term according to exponential family
$n^{'} = p, p^{'} = p(1-p)/n$
On rearranging the equation becomes parabola $p' = n'(1-n')/n$
(c) $\bar{X} \sim n(r(1-p)/p, r(1-p)/np^2$
$p' = r'/(np)$
The parameter lies in 1st quadrant

3.35

as previous questions

3.36

(a), (b), (c)

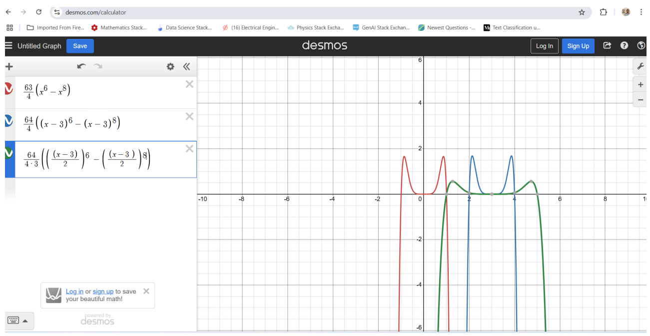

Figure 3.1:

3.37

$\left(\int_{-\infty}^{m}(\frac{1}{\sigma})f(\frac{x-\mu}{\sigma})dx\right) = (\int_{m}^{\infty}(\frac{1}{\sigma})f(\frac{x-\mu}{\sigma})dx)$
t $= \frac{x-\mu}{\sigma}$ substituting and differentiating
σdt = dx
$\left(\int_{-\infty}^{\frac{m-\mu}{\sigma}} f(t)dt\right) = \frac{1}{2}$
$\frac{m-\mu}{\sigma} = 0 \Rightarrow$ m = μ

3.38

$(\frac{1}{\sigma})f(\frac{x-\mu}{\sigma})$
$x_\alpha = \sigma z_\alpha + \mu$
$\alpha = P(Z > z_\alpha) = \int_{z_\alpha}^{\infty} f(z)dz$
$z_\alpha = \frac{x_\alpha - \mu}{\sigma}$ $\alpha = \int_{z_\alpha}^{\infty} f(z)dz$
Generalizing it $z = \frac{x-\mu}{\sigma}$ $\int_{\frac{x-\mu}{\sigma}}^{\infty}(\frac{1}{\sigma})f(\frac{x-\mu}{\sigma})dx = P(\frac{X-\mu}{\sigma} > \frac{x_\alpha-\mu}{\sigma}) = P(X > x_\alpha) = \alpha$

3.39

(a) $\int_{-\infty}^{\mu}\frac{1}{\sigma\pi(1+(\frac{x-\mu}{\sigma}))}dx = \int_{\mu}^{\infty}\frac{1}{\sigma\pi(1+(\frac{x-\mu}{\sigma}))}dx$ substituting z = $\frac{x-\mu}{\sigma}$
we will get standard Cauchy with median 0. So x = μ
(b) $\int_{-1}^{1}\frac{dx}{\sigma\pi\left(1+(\frac{x-\mu}{\sigma})^2\right)}$, putting μ = 0, σ = 1. We get $\frac{1}{2}$ With symmetry we get $\frac{1}{4}$ adding a general location (μ) and scale family (σ)

3.40

$f^*(x) = \frac{1}{\sigma} f(\frac{x-\mu}{\sigma})$

3.41

Scholastically greater $F_X(t) \leq F_Y(t)$ for all t
Say let X, Y = $X + \mu'$ are two random variables for same family distribution where μ' is fixed differences between the their mean. P(t $\geq$ x) > P(t $\geq$ y) as for same σ the area as the distribution is same P(X) will be more than P(Y) $F_X(x) > F_Y(y)$
Y is scholastically greater than X also Y is greater than X hence Provved
(b) Refer next section

3.42

(a) For location family μ changes keeping the scale parameter constant, Refer to above section a logic you will get for the same.
(b) Let $\sigma_1 > \sigma_2$
$F(x|\sigma_1) = P(X_1 \leq x) = P(\sigma_1 Z \leq x) = P(Z \leq x\sigma_1) = F(x/\sigma_1) \leq F(x/\sigma_2) = P(Z \leq x/\sigma_2) = P(\sigma_2 Z \leq x) = P(X_2 \leq x) = F(x|\sigma_2)$.

3.43

(a) $P(X \leq x|\theta_1) \leq P(X \leq x|\theta_2)$
$\Rightarrow P(\frac{1}{Y} \leq \frac{1}{y}|\theta_1) \leq P(\frac{1}{Y} \leq \frac{1}{y}|\theta_2)$
$\Rightarrow P(y \leq Y|\theta_1) \leq P(y \leq Y|\theta_2)$
$\Rightarrow P(Y \leq y|\theta_1) \geq P(Y \leq y|\theta_2)$
(b) P(X $\leq$ x | θ) $\leq$ P(X $\leq$ x | θ + $d\theta$)
P(X le x | $\frac{1}{\theta}$), P(X $\leq$ x | $\frac{1}{\theta}$ - $\frac{d\theta}{\theta^2}$)
Clearly P(X le x | $\frac{1}{\theta}$)$\geq$ P(X $\leq$ x | $\frac{1}{\theta}$ - $\frac{d\theta}{\theta^2}$)

3.44

Using Chebychev Inequality's
P(|X| $\geq$ b) $\leq$ $\frac{E|X|}{b}$
P(|X| $\geq$ b) = P(X^2 $\geq$ b) $\leq$ $\frac{EX^2}{b^2}$
E|X|/b = 1/3 > 2/9 = EX^2/b^2

E|X|/b = 1/ $\sqrt{2} < 1 = EX^2/b^2$.

3.45

(a) $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx \ \geq$
$\int_a^{\infty} e^{tx} f_X(x)dx \ \geq e^{ta} \int_a^{\infty} f_X(x)dx = e^{ta} P(X \geq a)$
(b) same as above
(c) It must be a non negative

3.46

Draw graph

3.47

(a) P(Z $\geq$ t) $\leq \int_t^{\infty} \frac{x^2+1}{x^2+1} e^{-\frac{x^2}{2}} dx = \int_t^{\infty} \frac{x^2}{x^2+1} e^{-\frac{x^2}{2}} dx + \int_t^{\infty} \frac{1}{x^2+1} e^{-\frac{x^2}{2}} dx$
On solving first you will get the equality. Second term must be discarded

3.48

(a),(b),(c) Divide P(X = x+1)/P(X = x) to derive it for all

3.49

(a)(b) solve mathematically

3.50

(a)(b) solve mathematically