Chapter 4

Multiple Random Variables

4.1

(a) $X^2 + Y^2 < 1$
-$\sqrt{1-Y^2} < X < \sqrt{1-Y^2}$
P$(X^2 + Y^2 < 1) = \frac{1}{4}\int_{-1}^{1}\int_{-\sqrt{1-Y^2}}^{\sqrt{1-Y^2}} dxdy = \frac{\pi}{4}$
(b) $\int_{-1}^{1}\int_{Y/2}^{1} dxdy = \frac{1}{2}$
(c) P(|X + Y| < 2) = 1

4.2

(a) $E(ag_1(X,Y)+bg_2(X,Y)+c) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(ag_1(X,Y)+bg_2(X,Y)+c)f_{X,Y}dxdy$
$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ag_1(X,Y)f_{X,Y}dxdy+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} bg_2(X,Y)f_{X,Y}dxdy+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} cg_1(X,Y)f_{X,Y}dxdy$
$= aEg_1(X,Y) + bEg_2(X,Y) + c$
(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x,y)f_{X,Y}(x,y)dxdy$.We know $f_{X,Y} \geq 0$ and $g_1(x,y) \geq 0$
It's enough to prove E$(g_1(X,Y)) \geq 0$
(c) $g_1(x,y) > g_2(x,y)$
$f_{X,Y}(x,y)g_1(x,y) > f_{X,Y}(x,y)g_2(x,y)$
$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)g_1(x,y)dxdy > \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)g_2(x,y)$
(d) same as above

4.3

$\sum_X \sum_Y f_{X,Y}(0,0) = f(0,0) + f(0,1) + f(1,0) + f(1,1) = 1$

4.4

(a) $\int_0^1 \int_0^2 C(x+2y)dxdy = 1$
C = $\frac{1}{4}$
(b)

$$f_X(x) = \left\{ \begin{array}{ll} C(x+2y) & \text{if } 0 < \text{x} < 2 \\ 0 & \text{otherwise} \end{array} \right\} = xy$$

(c) $f_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{y} \frac{1}{4}(x+2y)dxdy = \frac{x^2 y}{8} + \frac{xy^2}{4}$

$$f_X(x) = \left\{ \begin{array}{ll} 0 & x \geq 0, y \geq 0 \\ \frac{x^2 y}{8} + \frac{xy^2}{4} & 0 < x < 2, 0 < y < 1 \\ \frac{x^2}{8} + \frac{x}{4} & 0 < x < 2, y \leq 1 \\ \frac{y}{2} + \frac{y^2}{2} & x \leq 2, 0 < y < 1 \\ 1 & x \leq 2, y \leq 1 \end{array} \right\}$$

(d) $f_z(z) = \frac{9}{8z^2}, 1 < z < 9$

4.5

(a) $\frac{7}{20}$
(b) $\frac{1}{6}$

4.6

Lets create a generic answer let x be the waiting time $\int_2^{1+x} \int_{x_b - x}^{1} dx_a dx_b$
$\int_2^{1+x} (x_b - x - 1) dx_b = \frac{x^2}{2} - x + \frac{1}{2}$

4.7

Let X $\sim$ uniform(0, 30) and Y $\sim$ uniform(0, 60)
$\int_{40}^{50} \int_0^{60-y} dxdy * \frac{1}{300} = .5$

4.8

(a) P(X = M | M = m) = $\frac{1}{2}$
P(X = 2M | M = m) = $\frac{1}{2}$ Logically P(X = x) = $\pi(x) + \pi(x/2)$
P(M = x) = $\pi(x)$ P(M = x|M = x) = $\frac{\pi(x)}{\pi(x)+\pi(x/2)}$

P(M = x|M = x) = $\frac{\pi(x/2)}{\pi(x)+\pi(x/2)}$
(b) $\frac{\pi(x)}{\pi(x)+\pi(x/2)}$2x + $\frac{\pi(x/2)}{\pi(x)+\pi(x/2)}\frac{x}{2}$ > x On solving we would get $\pi(x) < 2\pi(x)$
Substituting $\pi \sim$ exponential(λ) x < 2log2 λ
(c) doubtful

4.9

(a) $F_{X,Y}(x, y) = F_X(x)F_Y(y)$
Differentiation with $\frac{\partial F_{X,Y}(x,y)}{\partial x} = f_X(x)F_Y(y)$
$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = f_X(x)f_Y(y)$
$\int_a^b \int_c^d f_X(x)f_Y(x)dxdy = \int_c^d f_Y(y) \int_a^b f_X(x)dxdy = P(a \leq x \leq b) \int_c^d f_Y(y)dy = P(a \leq x \leq b)P(c \leq y \leq d)$

4.10

(a) take case of $f_Y(3) * f_X(2), f_{Y,X}(3, 2) = 0$
(b) calculate the marginals f_X, f_Y

Y↓X→	1	2	3
2 3 4	$\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$	$\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$	$\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{12}$

4.11

U, V not independent set

4.12

we have y + z > x
z + x > y
y + x > z
constraint on x: y + z > x > y - z
constraint on y, z > 1/2 $\int_0^{l/2} \int_0^{l/2} \int_{y-z}^{y+z} 1dxdydz = 1/4$

4.13

(a) $E[Y - g(X)]^2 = E[(Y - E(Y|X)) + (E(Y|X) - g(X))]^2$
$E[Y - E(Y|X)]^2 + E[E(Y|X) - g(X)]^2 + 2E[(Y - E(Y|X))(E(Y|X) - g(X))]$

$(\because (E(Y|X - g(X)))isaconstantwithrespecttoY = y)$
So E(Y | X) = g(x) to minimize
(b) special part of case a

4.14

(a) $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{-\sqrt{1-y^2}} e^(-x^2+y^2)dxdy = 1 - \frac{1}{\sqrt{e}}$
(b) Using theorem 2.1.8 it can be proven it has a chi-square distribution
0.6826894921370859

4.15

U = X + Y
V = Y
$f_{U,V}(u,v) = \frac{\theta^{u-v}e^{-\theta}}{(u-v)!}\frac{\lambda^v e^{-\lambda}}{v!}$
$f(u|v) = \frac{f_{U,V}(u,v)}{f(u)} = \binom{u}{v}\left(\frac{\theta}{\theta+\lambda}\right)^v\left(\frac{\lambda}{\theta+\lambda}\right)^{u-v}$
$\because$ is other

4.16

(a) The support distribution (U, V) = {u = 1, 2, 3, ..., v = 0, ±1, ±2, ... }
If X $\geq$ Y U = Y, V = X - Y $\implies$ X = U + V $f_{U,V}(u,v) = f_{X,Y}(u+v,u) = (1-p)^u p*(1-p)^{u+v}p$
If Y > X U = Y, V = X - Y $\implies$ X = U - V $f_{U,V}(u,v) = f_{X,Y}(u+v,u) = (1-p)^u p*(1-p)^{u-v}p$
The function can be split independently of u, v
(b) z = $\frac{X}{X+Y}$
X = p(1 - p)$^{\text{x - 1}}$ Y = p(1 - p)$^{\text{y - 1}}$
$f_{U,V}(u,v) = p(1-p)^{\left(\frac{uv}{1-v}-1\right)}p(1-p)^{v-1}$
having summation over will result to answer
(c) U = X, Y = V - U
$p^2(1-p)^{v-2}$

4.17

(a) $f_X(x) = \frac{1}{e^x}$
(b) $\int_i^{i+1} e^x = e^{-i}(1-\frac{1}{e})$

(c) P(X - 4 $\leq$ x | Y $\geq$ 5) = P(X - 4 $\leq$ x | X $\geq$ 4) = e^{-x} as the exponential is memory less

4.18

dudt = rdrdθ
and substituting would result in solving the equations
$\int_0^\infty \int_0^\infty f_{X,Y}(x,y)dxdy = \int_0^{\pi/2} \int_0^\infty \frac{2g(r)}{\pi r} r dr d\theta = 1$

4.19

(b) $f_{X,Y}(x,y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1 - 1} y^{\alpha_2 - 1} e^{-x-y}$
$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left[\frac{uv}{1-u}\right]^{\alpha_1 - 1} v^{\alpha_2 - 1} e^{\left[\frac{uv}{1-u} + v\right]} dv$
$f_U(u) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left[\frac{u}{1-u}\right]^{\alpha_1 - 1} \int_0^\infty v^{\alpha_1 + \alpha_2 - 1} e^{\left[\frac{v}{1-u}\right]} dv$
$f_U(u) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_1)} u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1}$
$\because$ for others
(a) on solving $\frac{1}{2\pi} e^{-u}$

4.20

(a) This transformation is one-to-one
so we can split it into 3 segments
$A_0 = \{-\infty < x_1 < \infty, x_2 = 0\}$
$A_1 = \{-\infty < x_1 < \infty, x_2 > 0\}$
$A_2 = \{-\infty < x_1 < \infty, x_2 < 0\}$
$x_1 = y_2\sqrt{y_1}$
$x_2 = \sqrt{y_1 - y_1 y_2^2}$

$$J_1 = \begin{vmatrix} \frac{1}{2}\frac{y_2}{\sqrt{y_1}} & \sqrt{y_1} \\ \frac{1}{2}\frac{\sqrt{1-y_2^2}}{\sqrt{y_1}} & \frac{y_2\sqrt{y_1}}{\sqrt{1-y_2^2}} \end{vmatrix} = \frac{1}{2\sqrt{1-y_2^2}}$$

$f_{Y_1,Y_2}(y_1,y_2) = 2\left[\frac{1}{2\pi\sigma^2} e^{-\frac{y_1}{2\sigma^2}} \frac{1}{2\sqrt{1-y_2^2}}\right] 0 < y_1 < \infty, -1 < y_2 < 1.$
(b) Y_1 is square distance and Y_2 is the cosine orientation. It says that distance is independent of orientation. (The above can be factorize in y_1 and y_2)

4.21

$f_X(x) = \frac{1}{2}e^{-\frac{x}{2}}$ 0 < x < ∞
$f_Y(y) = \frac{1}{2\pi}$, 0 < y < 2π
$f_{X,Y}(x, y) = \frac{1}{4\pi}e^{-\frac{x}{2}}$
t = X^2 + Y^2
$\theta = tan^{-1}(\frac{y}{x})$
J = $\begin{vmatrix} 2x & 2y \\ \frac{-y}{x^2+y^2} & \frac{-x}{x^2+y^2} \end{vmatrix}$ = 2
$f_{T,\theta}(t, \theta) = \frac{2}{4\pi}e^{-\frac{x^2+y^2}{2}}$, 0 < x^2 + y^2 < ∞, 0 < $\tan^{-1}(\frac{y}{x})$ < 2π
$(\therefore -\infty < x, y < \infty)$

4.22

U = aX + b, V = cY + d
J = $\begin{vmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{c} \end{vmatrix} = \frac{1}{ac}$
$f_{U,V}(u, v) = f_{X,Y}(h_1(u, v), h_2(u, v))|J|$
$f_{U,V}(u, v) = \frac{1}{ac} f\left(\frac{u-b}{a}, \frac{v-d}{c}\right)$

4.23

(a) U = XY, V = Y
J = $\frac{1}{v}$
$f_{U,V}(u, v) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} \left(\frac{u}{v}\right)^{\alpha-1} \left(1 - \frac{u}{v}\right)^{\beta-1} v^{\alpha+\beta-1}(1 - v)^{\gamma-1}\frac{1}{v}, 0 < u <$
$v < 1.$
$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_u^1 v^{\beta-1}(1 - v)^{\gamma-1}(\frac{v-u}{v})^{\beta-1}dv$
$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1 - u)^{\beta+\gamma-1}, 0 < u < 1$
(b) same as a

4.24

$f_{X,Y}(x, y) = \frac{1}{\Gamma(r)\Gamma(s)} x^{r-1}e^{-x}y^{s-1}e^{-y}$
$f_{U,V}(u, v) = f_{X,Y}(x, y)(h_1(u, v), h_2(u, v))|J|$
$|J| = z_1$
$f_{U,V}(u, v) = \frac{1}{\Gamma(r)\Gamma(s)} (z_1z_2)^{r-1}e^{-z_1z_2}(z_1 - z_1z_2)^{s-1}e^{-z_1+z_1z_2}z_1$
$f_{U,V}(u) = \frac{1}{\Gamma(r)\Gamma(s)} \int_0^1 (z_1z_2)^{r-1}e^{-z_1z_2}(z_1 - z_1z_2)^{s-1}e^{-z_1+z_1z_2}z_1dz_2$
$f_U(u) = \frac{1}{\Gamma(r)\Gamma(s)} (z_1)^{r-1+s-1}e^{-z_1} \int_0^1 z_2^{r-1}(1 - z_2)^{s-1}dz_2$

$f_U(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} z_1^{r+s-1} e^{-z_1}$
(b) similarly it can also be derive

4.25

(a) $f_{X,Y}(x, y) = 10$
J = 1
$f_{U,V}(u, v) = 10|J| = 10$
(b) doubtful, its not one to one mapping

4.26

(a) P(X $\geq$ z, Y $\geq$ X)
P(X $\geq$ z, Y $\geq$ X) = $\int_0^z \int_x^\infty \frac{1}{\lambda\mu} e^{-\left[\frac{x}{\lambda}+\frac{y}{\mu}\right]} dydx$
P(Y $\geq$ z, X $\geq$ Y) = $\int_0^z \int_y^\infty \frac{1}{\lambda\mu} e^{-\left[\frac{x}{\lambda}+\frac{y}{\mu}\right]} dxdy$
(b) both are same

4.27

(a) U $\sim$ n(μ + σ, 2σ^2)
U $\sim$ n(μ - σ, 2σ^2)
J = $\frac{1}{2}$
distribution can be broken down buy factorization.

4.28

(a) U = $\frac{X}{X+Y}$
V = X
X = V, Y = $\frac{V}{U}(1-U)$
|J| = $\frac{v}{u^2}$
$f_U(v) = \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}\left[v^2+v^2\frac{[1-u]^2}{u^2}\right]} dv$
$f_U(v) = \frac{1}{\pi} \frac{1}{u^2+(1-u)^2}$
(b) $f_U(u) = \frac{1}{\pi} \frac{1}{1+u^2}$
(c) If two distribution have std normal distribution, then X/|Y| has Cauchy distribution

4.29

(a) let X/Y = cot(z)
$f_(R,\theta)(r,\theta) = \frac{1}{2\pi}$
also
$f_(\theta)(\theta) = \frac{1}{2\pi}$
$f_Z(z) = \frac{1}{2\pi}\left|\frac{1}{1+z^2}\right|(0 < \theta < \pi) + \frac{1}{2\pi}\left|\frac{1}{1+z^2}\right|(\pi < \theta < 2\pi)$
$f_Z(z) = \frac{1}{\pi}\frac{1}{1+z^2}$
(b) $\frac{2XY}{\sqrt{X^2+Y^2}} = Rsin(2\theta)$
$restisdoubtful$

4.30

(a) EY = E(E(Y|X))
E(Y|X) = x EY = EX = $\frac{1}{2}$
VarY = E(Var(Y|X)) + Var(E(Y|X))
VarY = EX^2 + VarX
VarY = $\frac{1}{3}$ + $\frac{1}{3} - \frac{1}{4}$ = $\frac{5}{12}$
Cov(X, Y) = EXY - $\mu_\alpha\mu_\beta$
EXY = E(E(XY|X)) = E(XE(Y|X)) = E(X^2) = $\frac{1}{3}$ EXY = $\frac{1}{3}$ - EXEY = $\frac{1}{3} - (\frac{1}{2}\frac{1}{2})$
= $\frac{1}{12}$
(b) X is uniform distribution hace independent

4.31

(a) EY = $\frac{n}{2}$
VarY = $\frac{n^2}{16} + \frac{n}{6}$
(b) P(Y = y, X $\leq$ x) = $\binom{n}{y} x^y(1-x)^{n-y}$
(c) = P(Y = y) = $\binom{n}{y} \int_0^\infty x^y(1-x)^{n-y}$
P(y=y) = $\binom{n}{y} \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}$

4.32

(a) $f_Y(y) = \int_0^\infty \frac{e^{-\Lambda}\Lambda^y}{y!}\frac{1}{\Gamma(\alpha)\beta^\alpha}\Lambda^{\alpha-1}e^{-\frac{\Lambda}{\beta}}d\Lambda$
$f_Y(y) = \frac{1}{y!\Gamma(\alpha)\beta^\alpha}\int_0^\infty e^{-\Lambda-\frac{\Lambda}{\beta}}\Lambda^{y+\alpha-1}d\Lambda$
$f_Y(y) = \frac{1}{y!\Gamma(\alpha)\beta^\alpha}\Gamma(\alpha+y)(\frac{\beta}{1+\beta})^{\alpha+y}$
$f_Y(y) = \frac{\Gamma(\alpha+y)}{y!\Gamma(\alpha)}(\frac{1}{1+\beta})^\alpha(\frac{\beta}{1+\beta})^\alpha$
$f_Y(y) = \binom{y+\alpha-1}{y}(\frac{1}{1+\beta})^\alpha(\frac{\beta}{1+\beta})^\alpha$

EY = $\alpha\beta$
VarY = $\alpha\beta + \alpha\beta^2$

4.33

$Y|N \sim binomial(N, p)$
$N|A \sim Poisson(\Lambda)$
$Y|\Lambda = \sum_{n=y}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} e^{-\Lambda} \frac{\Lambda^n}{n!}$
$\frac{(p\Lambda)^y e^{-p\Lambda}}{y!}$
Combine with other result to form your final answer

4.34

(a) $Ee^{tx} = EE(e^{Ht}|N) = EE(E^{(X_1+X_2+X_3\ldots)t}|N) = E\left(\frac{log 1-e^t(1-p)}{logp}\right)^N$
$E\left(\frac{log 1-e^t(1-p)}{logp}\right)^N = \sum_{n=0}^{\infty}\left(\frac{log\{1-e^t(1-p)\}}{logp}\right)^n \frac{e^{-\lambda}\lambda^n}{n!}$
$E(e^{tx}) = \left(\frac{p}{1-e^t(1-p)}\right)^{\frac{-\lambda}{logp}}$
moment of negative binomial distribution

4.35

(a) P(X = x | P = p) = $\binom{n}{x} p^x (1-p)^{n-x}$
P(X = x) = $\int_0^1 \binom{n}{x} p^x (1-p)^(n-x)p^{\alpha-1}(1-p)^{\beta-1}$
Later can be derive
(b) $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \binom{r+x-1}{x} \frac{\Gamma(\alpha+r)\Gamma(x+\beta)}{\Gamma(\alpha+r+x+\beta)}$

4.36

(a) $VarX = E(Var(X|Y)) + Var(E(X|Y))$
$E[np(1-p)] + Var(np)$
$nEP - nEP^2 + n^2[EP^2 - (EP)^2]$
$n^2EP^2 - n^2(EP)^2 + nEP - nEP^2$
$n(n-1)EP^2 - n^2(EP)^2 + nEP - n(n-1)(EP)^2 + n(n-1)(EP)^2$
$n(n-1)VarP - n^2(EP)^2 + nEP + n^2(EP)^2 - n(EP)^2$
$n(n-1)VarP + nEP - n(EP)^2$
$n(n-1)VarP + nEP(1-EP)$
(b) simple formulae

4.37

(a) EY = $\sum_{i=1}^{n} EX_i = \sum_{i=1}^{n} EEX_i|P_i = \frac{n\alpha}{\alpha+\beta}$
(b) if all X_i are independent
Var(Y) = $\sum_{i=1}^{n} Var(X_i)$
$= nVar(X_i)$
$= n\frac{\alpha\beta}{\{\alpha+\beta\}^2}$
(c) EY = $\frac{\alpha}{\alpha+\beta}\sum_{i=1}^{k} n_i$
Var(Y) = $E(\sum_{i=1}^{k} X_i)^2 - (E(\sum_{i=1}^{k} X_i))^2 \quad = EX_1^2 + EX_2^2 + EX_3^2 ... 2EX_1X_2 + 2EX_1X_3 ... - (EX_1)^2 - (EX_2)^2 - (EX_3)^2 - 2EX_1EX_2 - 2EX_1EX_3 ...$ ($\because$ sample are i.i.d $2EX_iEX_j = 2EX_iX_j$)
(c) same approach solve

4.38

(a, b) same method as above

4.39

(a) $f(x) = \int_0^{\lambda} \frac{1}{v} e^{-\frac{x}{v}} p_{\lambda}(v) dv = \frac{x^{r-1} e^{-\frac{x}{\lambda}}}{\Gamma(r)\lambda^r}$
(b) simple integration
(c) $\frac{dlog(f(x))}{dx} = \frac{r-1}{x} - \frac{1}{\lambda}$
$\frac{d}{dx} log \int_0^{\infty} (e^{-\frac{x}{v}}/v) q_{\lambda}(v) dx < 0$ contradiction

4.40

(a) $f(x_1, x_2, ..., x_n) = \frac{m!}{x_1! x_2! ... x_n!} p_1^{x_1} p_2^{x_2} ... p_n^{x_n}$
$f(x_j) = \frac{m!}{x_j!(m-x_j)!} p_j^{x_j} (1-p_j)^{m-x_j} \sum_{x_i \neq x_j} \frac{(m-x_j)!}{x_1 x_2 ... x_{j-1} x_{j+1} ... x_m} \frac{p_1}{1-p_j}^{x_1} ... \frac{p_n}{1-p_j}^{x_n}$ the side term is another multinomial
$f(x_j) = \frac{m!}{x_j!(m-x_j)!} p_j^{x_j} (1-p_j)^{m-x_j}$
(b) Expand on the same idea to get
$f(x_i, x_j) = \frac{m!}{x_j!(m-x_j)!} p_j^{x_j} (1-p_j)^{m-x_j} \frac{m-x_j!}{x_i!(m-x_i-x_j)!} \left[\frac{p_i}{1-p_j}\right]^{x_i} \left[1 - \frac{p_i}{1-p_j}\right]^{(m-x_i-x_j)}$
proceed further

4.41

(a), (b) Take either (1-y) or (1 - x) take that out and integrate
(c), (d) solve for the values

4.42

Cov(X, a) = E(Xa) - E(a)EX
Cov(X, a) = aEX - aEX = 0
Corr = 0, not correlated

4.43

(a) $Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$
$Corr(X,Y) = \frac{\mu_X \sigma_Y}{\sqrt{(\mu_X^2 \sigma_Y^2 + \mu_Y^2 \sigma_X^2 + \sigma_X^2 \sigma_Y^2)}}$

4.44

(a) σ^2
(b) 0

4.45

(a) simple algebric identity

4.46

(a) Integrate by keeping other fix
(b) divide the bivariate by f(x)
(c) take U = X+Y, V = Y

4.47

(a) $EX = c_x, EY = c_y, VarX = EX^2 - (EX)^2 = E(a_x Z_1 + b_x Z_2 + c_x)^2 - (E(a_x Z_1 + b_x Z_2 + c_x))^2 = a_x^2 + b_x^2, similarly for both of 2$
(b) substitute

(c) use jacobian transformation
(d) there are infinite number of solutions

4.48

(a) P(Z $\leq$ z) = P(Z $\leq$ z and XY $>$ 0) + P(-Z $\leq$ z and XY $<$ 0)
P(Z $\leq$ z and Y $<$ 0) + P(-Z $\leq$ z and Y $<$ 0)
P(Z $\leq$ z)P(Y $<$ 0) + P(Z $\geq$ -z)P(Y $<$ 0)
P(Z $\leq$ z)(P(Y $<$ 0) + P(Y $>$ 0))
P(Z $\leq$ z)
(b)
Z $>$ 0, X $>$ 0, Y $>$ 0
Z $>$ 0, X $<$ 0, Y $>$ 0
Z $<$ 0, X $<$ 0, Y $<$ 0
Z $<$ 0, X $>$ 0, Y $<$ 0
Z and Y have same sign

4.49

(a), (b) solve

4.50

(a) $f_X(x) = \int_{-\infty}^{\infty}(af_1(x)g_1(y) + (1-a)f_2(x)g_2(y)) = af_1(x) + (1-a)f_2(x)$, similarly other
(b) f(x, y) = $f_X(x)f_Y(y)$, solve it and get the conditions
(c) Cov(X, Y) = EXY - EXEY = $a(1-a)[\mu_1 - \mu_2][\epsilon_1 - \epsilon_2] = 0$ for uncorrelated variable
(d) Take any random binomial say $f_1 \sim binomial(n, p_1), f_1 \sim binomial(n, p_2), g_1 \sim binomial(n, p_1), g_2 \sim binomial(n, p_2)$

4.51

solve, simple

4.52

(a) $f(X/Y \leq t)$
$U = X/Y, V = Y$

$Y = V, X = UV, |J| = v$
$f_{U,V}(u, v) = 1 \times v$
$f_U(u) = \int_0^1 vdv = \frac{1}{2}$
$P(u \leq t) = \begin{cases} 0 & t \leq 0 \\ \frac{t}{2} & 0 \leq t \leq 2 \\ 1 & 2 \leq t \end{cases}$ (b) $f(XY \leq t)$
$Y = V, X = \frac{u}{v}, |J| = \frac{1}{v}$
$f_{U,V}(u, v) = \frac{1}{v}$
$f_U(u) = \infty$
(c) same for z

4.53

(a) $f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-\frac{1}{2}[x^2+y^2]}$
Transformation $Z^2 = X^2 + Y^2, W = Y$ Take jacobain and solve
$f_Z(z) = \frac{z}{2} e^{-\frac{z^2}{2}}$

4.54

(DOUBTFUL)

4.55

Example 4.6.8 will be used for getting gamma distribution
$f_y(y) = \frac{(-logy)^{n-1}}{\Gamma(n)}$

4.56

P(x, y, z) = P(x)P(y)P(z)
P(X $\geq$ z) = $\frac{1}{\lambda} e^{-\frac{x}{\lambda}}$
$P(x, y, z) = 3(1 - e^{-\frac{y}{\lambda}})^2 e^{-\frac{y}{\lambda}}$

4.57