Chapter 2

Transformations And Expectations

2.1

(a) Y = X^3 , $f_X(x) = 42x^5(1-x)$, 0 < x < 1, $g^{-1}(y) = X^{\frac{1}{3}}$ $f_y = 42x^{\frac{5}{3}}(1-x\frac{1}{3})$ * $\frac{-1}{3}x\frac{-4}{3}$, On integration it comes to 1
(b) $g^{-1}(y) = \frac{Y-3}{4}$, $f_X(x) = 7e^{-7x} \Rightarrow f_y = 7e^{-7\frac{y-3}{4}}$ $Y = (3 <$ y $< \infty)$
(c) Y = X^2, f_X(x) = $30x^2(1-x)^2$, 0 < x < 1 $g^{-1}(y) = X^{\frac{1}{2}}$ $f_y = 20x(1-x\frac{1}{2})$ * $\frac{-1}{2}x\frac{-3}{2}$, On integration it comes to 1

2.2

(a) Y = $X^2 \Rightarrow g^{-1}$(x) = $x^{\frac{1}{2}}$ $f_y = \frac{1}{2}x^{\frac{-1}{2}}$
(b) Y = -log(X) $\Rightarrow g^{-1}$(x) = $e^{-x} \Rightarrow \frac{(n+m+1)!}{n!m!}e^{-xn}(1-e^{-x})^m * (-e^{-x})$
(c) Y = $e^x \Rightarrow f_y = \frac{1}{\sigma^2}x^{\frac{-(logy/\sigma)^2}{2}} * \frac{logy}{y}$

2.3

$f_X(x) = \frac{1}{3}(\frac{2}{3})^x \Rightarrow f_X(y) = \frac{1}{3}(\frac{2}{3})^{\frac{y}{1-y}}$ $Y \rightarrow \{0, \frac{1}{2}, \frac{2}{3} ... \}$

2.4

(a) f(x) = $\begin{cases} \frac{1}{2}\lambda e^{-\lambda x} for x \geq 0 \\ \\ \frac{1}{2}\lambda e^{\lambda x} for x \leq 0 \end{cases}$

$\int_{-\infty}^{0} \frac{1}{2}\lambda e^{-\lambda x} + \int_{0}^{\infty} \frac{1}{2}\lambda e^{\lambda x} = \frac{1}{2} + \frac{1}{2} = 1$

(b) $\int_{-\infty}^{t} \frac{1}{2}\lambda e^{-\lambda x} + \int_{t}^{\infty} \frac{1}{2}\lambda e^{\lambda x} = \frac{t}{2}$

f(x) = $\begin{cases} \frac{1}{2}\lambda e^{\lambda t} for t \geq 0 \\ \\ 1 - \frac{1}{2}\lambda e^{-\lambda t} for t \leq 0 \end{cases}$

(c) P(|X|< t) = 0 for t $\leq$ 0 and 1 - $e^{-\lambda t}$ for t > 0

2.5

answer = $\frac{1}{\pi}\frac{1}{\sqrt{Y(1-Y)^2}}$

2.6

(a) $f_y = \frac{1}{2}e^{-|Y|^{\frac{1}{3}}}\frac{1}{3}Y^{\frac{-2}{3}}$

(b) $f_Y(y) = \frac{3}{8}(1-y)^{\frac{-1}{2}} + \frac{3}{8}(1-y)^{\frac{1}{2}}$, 0 < y < 1

(c) $f_Y(y) = \frac{3}{16}(1-(1-y)^{\frac{1}{2}})^2(1-y)^{\frac{-1}{2}} + \frac{3}{8}(2-y)^2$

2.7

(a),(b) $f_y(y) = \begin{cases} \frac{2}{9}\frac{1}{\sqrt{y}} & \text{if } y < 1 \\ \frac{1}{9} + \frac{1}{9}\frac{1}{\sqrt{y}} & \text{if } y \geq 1 \end{cases}$

2.8

(a) $F_x^{-1}(y) = -ln(1-y)$

(b) $F_x^{-1}(y) = \begin{cases} ln2y \quad 0 < y < \frac{1}{2} \\ \frac{1}{2} \quad y = \frac{1}{2} \\ 1 - ln(2-2y) \quad 0 < y < \frac{1}{2} \end{cases}$

(c) $F_x^{-1}(y) = \begin{cases} ln4y \quad 0 < y \leq \frac{1}{4} \\ -ln(4-4y) \quad \frac{3}{4} < y < 1 \end{cases}$

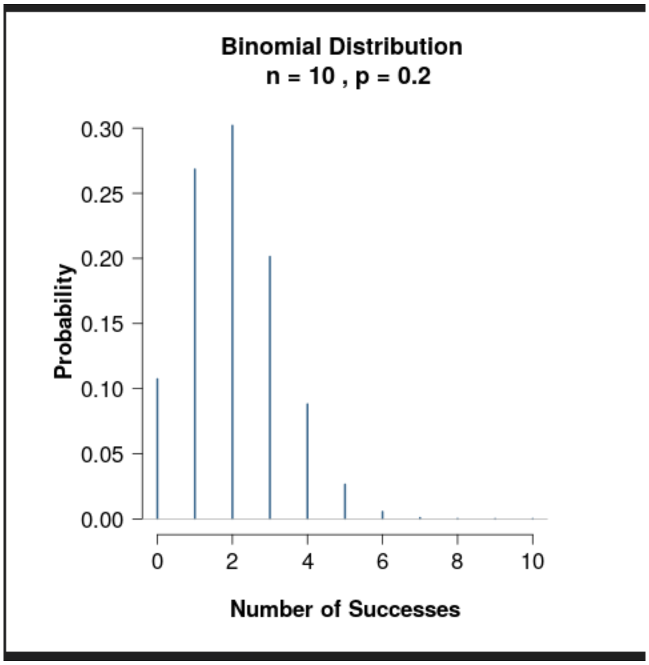

Figure 2.1:

2.9

Logically cdf $F_x(x) \sim$ Uniform(0,1) $F_x^{-1}(x) = \begin{cases} 0 & -\infty < x \leq 1 \\ \frac{(x-1)^2}{4} & 1 < x < 3 \\ 1 & 3 \leq \infty \end{cases}$

2.10

Refer to figure 2.1
logically P(Y $>$ y) $\geq$ P([U $>$ y])
P(Y $<$ y) $\leq$ P([U $<$ y])
for some x = x_0 P(Y $>$ y) = P([U $>$ y]) but at next point ϵ, P(Y $>$ y) will be stagnant since its a discrete probability. But P([U $>$ y]) will eventually be lifted some points up. So the two cases are justified

2.11

(a) f_X(x) = $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$
$\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$

$\Rightarrow$ Y = X^2

$f_y(y) = 2(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} | \frac{1}{\sqrt{2\pi y}} |)$

Integrating gives 1

(b)f_X(x) = $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ EY = $\frac{2}{\pi}$ EY_2 = 1

(EY) = 1 - $\frac{2}{\pi}$

2.12

skipping

2.13

$f_X(X) = =$ p(1-p)^x + (1-p)p^x

EX = p(1-p)$[\frac{1}{p^2} + \frac{1}{(1-p)^2}]$

2.14

(doubt) EX = $\int_0^\infty x f_X(x)$ Integration by parts

EX = $[xF_X(x)]_0^\infty \quad - \quad \int_0^\infty F_X(x)dx$ Now let say the identity is true

$\int_0^\infty [1 - F_X(x)]dx = \int_0^\infty dx \quad - \quad \int_0^\infty F_x(x)dx$

So 1^{st} term doesn't make any sense

2.15

$\int_{-\infty}^\infty x f_1(x)dx + \int_{-\infty}^\infty x f_2(x)dx = \int_{-\infty}^\infty x(f_1(x)+f_2(x))dx = \int_{-\infty}^\infty x(min(f_1(x), f_2(x)) + max(f_1(x), f_2(x)))dx$ = (X $\vee$ Y) + (X $\wedge$ Y) = X + Y

Hence by rearranging we get the equation

2.16

$\int_0^\infty ae^{-\lambda t} + (1-a)e^{-\mu t} = \frac{a}{\lambda} + \frac{1-a}{\mu}$

2.17

(a) m = $2^{\frac{1}{3}}$

(b) m = 0

2.18

E|X-a| $= \int_{-\infty}^{\infty} |x-a| f(x) = \int_{-\infty}^{a} (-x+a) f(x) + \int_{a}^{\infty} (x-a) f(x)$. On differentiating,
$\frac{dE|x-a|}{da} = \int_{-\infty}^{a} f(x) - \int_{a}^{\infty} f(x) = 0 \Rightarrow \int_{-\infty}^{a} f(x) = \int_{a}^{\infty} f(x)$

2.19

$E|X-a|^2 = \int_{-\infty}^{\infty} (x-a)^2 f(x)dx = \frac{dE}{dx} = \int_{-\infty}^{\infty} 2(x-a)f(x)dx \Rightarrow \int_{-\infty}^{\infty} xf(x) = a\int_{-\infty}^{\infty} f(x)dx = a(1) \Rightarrow EX = a$

2.20

$EX = \sum_{k=0}^{k=\infty} k(1-p)^k p = \frac{1}{p^2}$

2.21

$Eg(X) = \int_{-\infty}^{\infty} g(X) f_X(x)dx = \int_{-\infty}^{\infty} y f_X(g^{-1}y)|\frac{dg^{-1}(y)}{dy}|dy = \int_{-\infty}^{\infty} y f_y(y)dy = EY$

2.22

(a) Integration leads to 1.
(b) EX $= \frac{2\beta}{\sqrt{\pi}}$, Var(X) $= \beta^2(\frac{3}{2} - \frac{4}{\pi})$

2.23

$f_y(y) = \frac{1}{2}y^{\frac{-1}{2}}$
EY $= \frac{1}{3}$
EY^2 $= \frac{1}{5}$
Var(y) $= \frac{4}{45}$

2.24

(a), (b), (c) same as above

2.25

Let A $= \int_0^X f_X(x)dx$ Now $\int_0^{-X} f_X(x)dx \Rightarrow Y = -X \Rightarrow dY = -dX \Rightarrow \int_{-X}^0 f_X(x)dx \Rightarrow -\int_Y^0 f_Y(y)dy = A$
Hence symmetrical

2.26

$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$
$M_X(-t) = \int_{-\infty}^{\infty} e^{-tx} f_X(x)dx \Rightarrow \int_{-\infty}^{\infty} e^{t(-x)} f_X(-x)dx. Let j = -x \Rightarrow dj = -dx - \int_{\infty}^{-\infty} e^{tj} f_X(j)dj = M_x(t)$

2.27

(a) gaussian, |x|, x^2
(b) $\int_{-\infty}^{a} f_X(x) = \int_a^{\infty} f_X(x) \Rightarrow x = a - y \Rightarrow \int_0^{\infty} f_X(a-y)dy = \int_{-\infty}^{a} f_X(a-y)dy \Rightarrow$a is the median
(c) Wrong sol $\int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{a} x f_X(x)dx + \int_a^{\infty} x f_X(x)dx$
let for first part x $\rightarrow$ a - x and second part x $\rightarrow x - a$
$-\int_{\infty}^{0}((a-x)f_X(a-x))dx + \int_0^{\infty}((x-a)f_X(x-a)dx) = \int_0^{\infty}((a-x)f_X(a-x))dx + \int_0^{\infty}((x-a)f_X(x-a)dx) = \int_0^{\infty}((a-x)f_X(a-x))dx + \int_0^{\infty}((x-a)f_X(a-x)dx) = 0$
symmetric function
(d) It has a monotonic slope. Hence it cant have a symmetric pdf
(e) EX = 1, m median = ln2

2.28

(a) Gaussian distribution
(b) uniform distribution
(c) Let assume that the point of symmetry is not the modial point. Since the function is symmetric there will be another x for which the mode value will be same. This is in contradiction as our function is unimodal. Hence its symmetric about the mode.
(d) for a $\geq$ x $\geq$ y where a = 0 and f(0) $\geq$ x $\geq$ y.

2.29

(a) Let think logically x-a will shift the a^{th} to zero. Hence symmetrical.
(b) $\alpha_3 = 2$

(c) (i) $\alpha_4 = 3$ (ii) $\alpha_4 = \frac{9}{5}$ (iii) $\alpha_4 = 6$

2.30

(a) EX^2 = n(n-1)p^2 + np. EX = np So E(X(X-1)) = n(n-1)p^2 (binomial)
EX^2 = λ^2 + λ EX = λ E(x(x-1)) = λ^2
(b) Var(binomial) = n(n-1)p^2 - np^2 Var(binomial) = λ^2 + λ - λ^2 = λ
(c) I wont be able to solve

2.31

(a) M$_X$(t) = $\frac{e^(tx)-1}{ct}$
(b) M$_X$(t) = $\frac{2}{c^2}((c-1)e^c - c + 1)$
(c) M$_X$(t) = $\frac{4e^{\alpha t}}{4-\beta^2 t^2}$
(d) P(X = x) = $\sum_{x=0}^{x=\infty}\binom{\text{r + x - 1}}{\text{x}} p^r(1-p)^x dx \Rightarrow$
$M^X(t) = \sum_{x=0}^{x=\infty} e^{tx}\binom{\text{r + x - 1}}{\text{x}} p^r(1-p)^x dx \Rightarrow \sum_{x=0}^{x=\infty} e^{tx}\binom{\text{r + x - 1}}{\text{x}} p^r((1-p)e^t)^x(1-((1-p)e^t))^r dx = 1 \Rightarrow M_X(t) = (\frac{p}{(1-((1-p)e^t))})^r$

2.32

$M_X(0) = 0, But M_X(0) = 1 as the distribution of pmf should be 1. So no$

2.33

$\frac{d}{dt}S(t) = \frac{M_X'(t)}{M_X(t)} Putting t = 0 we get M_X(0) = 1, M_X' = EX,$ Same go by differentiating divide rule and you will
$(M_X^t)^2 \Rightarrow EX^2 - (EX)^2$

2.34

(a) $M_X(x)\sum_{x=0}^{x=\infty}\frac{e^{tx}e^{-\lambda}\lambda^x}{x!}$ Using Taylor series $e^{-\lambda} * e^{\lambda e^t}$ Hence we derive the $M_X(t)$. (Rest is Maths) EX = λ EX^2 = $\lambda^2 + \lambda$ Var(x) = λ
(b) $M_X(X)\sum_{x=0}^{x=\infty} p(1-p)_x$ (Think in terms of Geometric Mean.) $M_X(t) = \frac{p}{1-(1-p)e^t}$
EX = $\frac{1-p}{p}$, EX^2 = $\frac{p(1-p)+2(1-p)^2}{p^2}$ Var(x) = $\frac{1-p}{p^2}$
(c) Yes its an mgf (Some complicated mathematics) EX = μ EX^2 = μ^2 + σ^2.

Var(x) = σ^2

2.35

(a) $\int_{\infty}^{\infty} x^r * \frac{1}{\sqrt{2\pi x}} * e^{-(logx)^2/2} = e^{r^2/2}$
(b) $e^{r^2/2-2\pi^2}$

2.36

$\int_0^{\infty} \frac{e^{tx}}{\sqrt{2\pi x}} e^{-(log)^2/2} dx$
$= \lim_{x\to\infty} e^{tx-log^2(x)}$
$\lim_{x\to\infty} \frac{e^{tx}}{e^{log^2(x)}}$Taking log
$= \lim_{x\to\infty} \frac{tx}{log^2(x)}$ On solving $\lim_{x\to\infty} tx/2 \to \infty$

2.37

(a)
(b)
(c) (i) $M_X(t) = e^{K_1}$
ii $M_X(t) = e^{K_2}$

(d) make transformation e^x

2.38

(a) $M_X(t) = [p/(1-(1-p)e^t)]^r$
(b) $M_X(t) = [p/(1-(1-p)e^2pt)]^r$ On solving $[\frac{1}{1-2t}]^r$

2.39

(a) $e^{-\lambda x}$
(b) $\frac{-1}{\lambda^2}$
(c) $\frac{-1}{t^2}$
(d) $\frac{1}{(1-t)^2}$

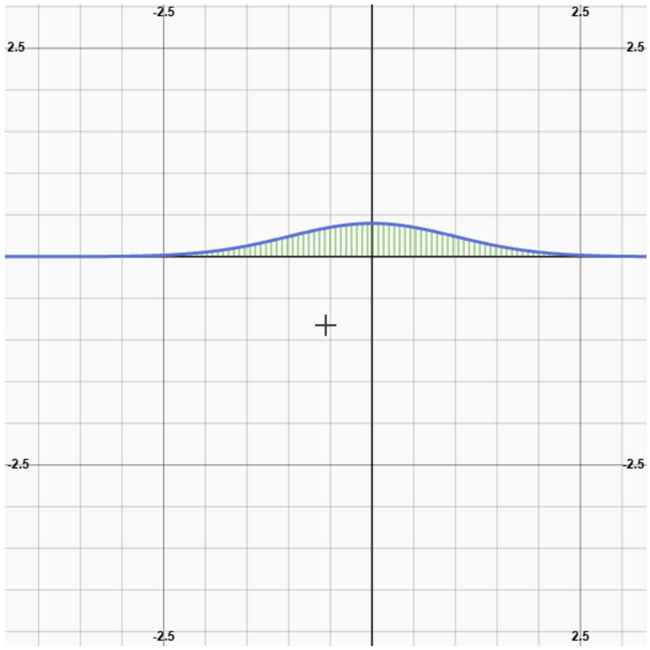

Figure 2.2:

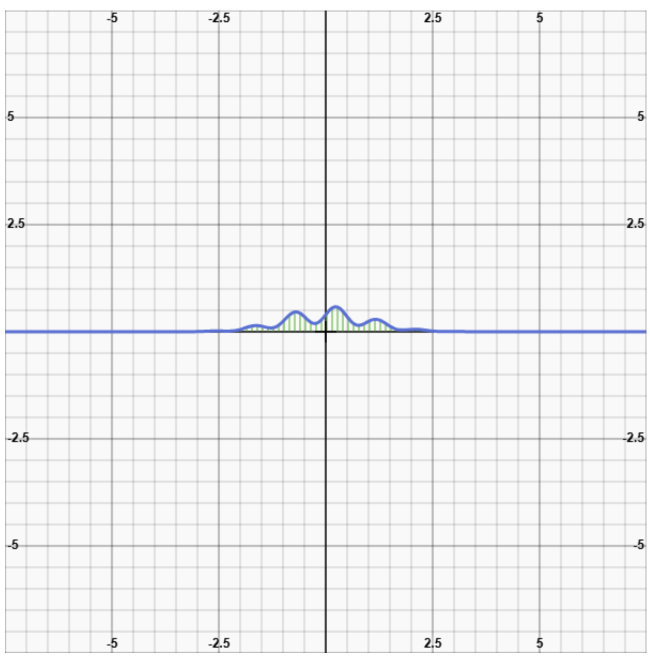

Figure 2.3:

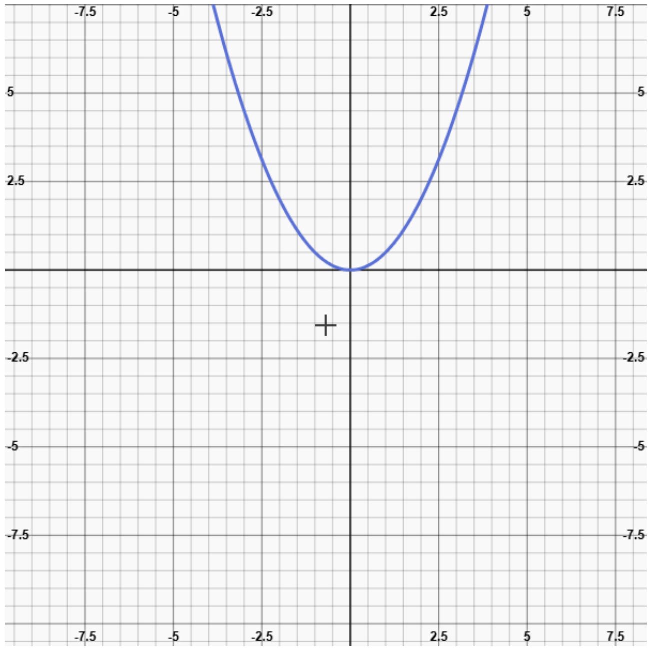

Figure 2.4:

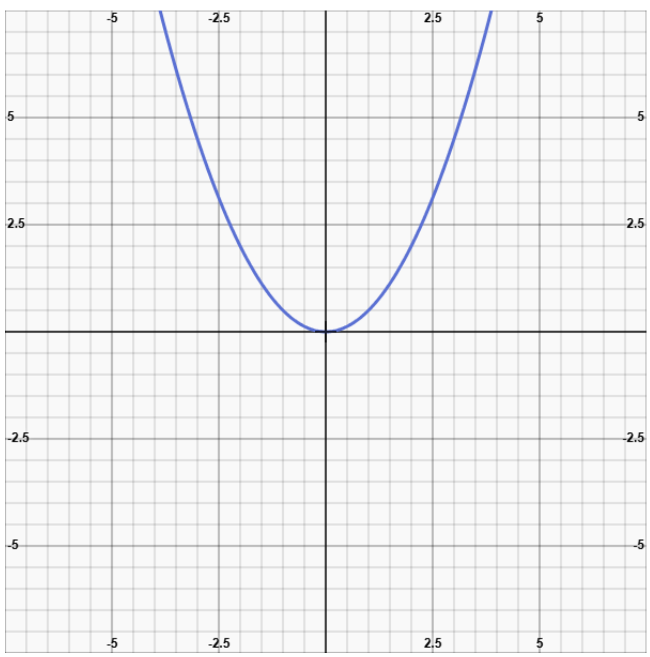

Figure 2.5: