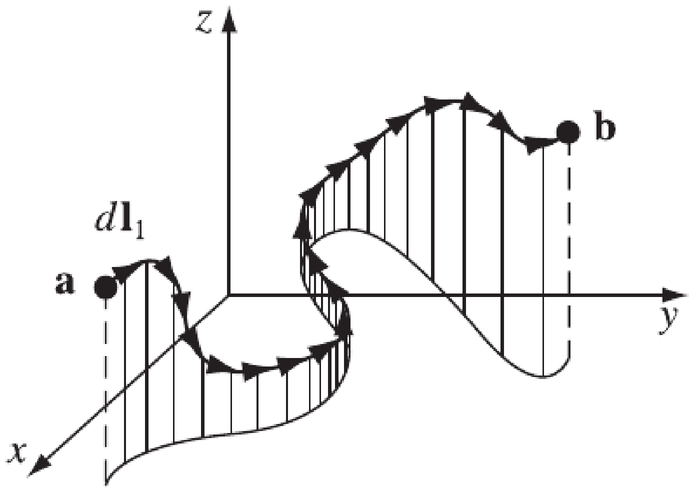
Fundamental Theorem of Gradient

Suppose we have a scalar function of three variables $T(x,y,z)$. Starting from a point **a**, we move a small distance $d\mathbf{l}_1$. The function T changes by an amount

$$dT = (\nabla T) \cdot d\mathbf{l}_1$$

The fundamental theorem of gradient states that the total change in T going from point **a** to point **b** along the selected path is given by

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$$



The right hand side makes no reference to the path, only end points matter. ***Thus the line integral of gradients are path independent.***

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l}$ is independent of the path taken from **a** to **b**.

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b}) - T(\mathbf{a}) = 0$.

A conservative force can be associated with a scalar potential energy function

Fundamental Theorem of Divergence

The fundamental theorem of divergence states that

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}$$

→ The integration of a derivative (in this case, a divergence) over a region (in this case, a volume) is equal to the value of the function at the boundary (in this case the surface that bounds the volume).

→ This theorem is also known as ***Gauss' theorem*** or ***the divergence theorem.***

→ Say, **v** represents the flow of an incompressible fluid and we have a bunch of faucets in the region filled with the fluid. The right-hand side is the flux of **v**. We can measure the total amount of fluid passing per unit time through the surrounding surface in two ways:

1. Count up all the faucets within the volume, recording how much is put out.
2. Go around the boundary, measuring the flow at each point and add it all up

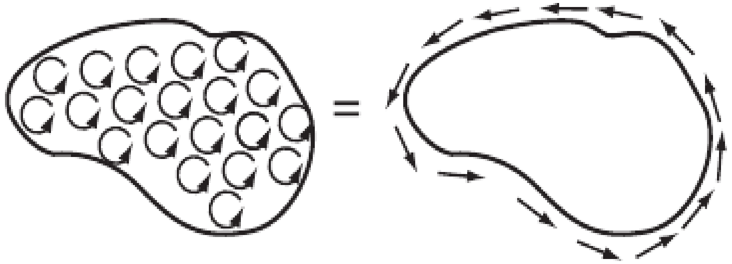
$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface}).$$

Fundamental Theorem of Curl

The fundamental theorem of curl states that

$$\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}$$

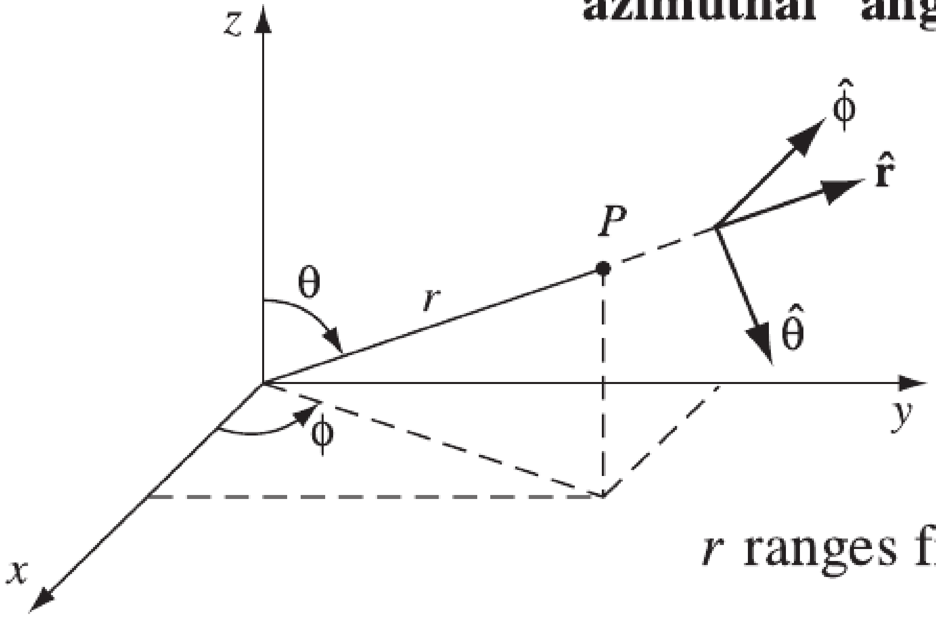
- The integration of a derivative (in this case, the curl) over a region (in this case, a patch of surface) is equal to the value of the function at the boundary (in this case the perimeter of the patch)
- Also known as the ***Stoke's theorem***.
- Curl gives us the swirl of a vector. The integral of the Curl over some surface (more precisely, flux of the curl through that surface) represents the *total amount of swirl.* We can also determine that by going around the edge and finding how much the flow is following the boundary.



Usually, a flux integral depends on the choice of surface you integrate over, but this is not the case with the curls. The line integral makes no reference to the chosen surface.

Spherical Polar Coordinate Systems

You can label a point P by its Cartesian coordinates (x, y, z), but sometimes it is more convenient to use **spherical** coordinates (r, θ, ϕ); r is the distance from the origin (the magnitude of the position vector $\mathbf{r}$), θ (the angle down from the z axis) is called the **polar angle**, and ϕ (the angle around from the x axis) is the **azimuthal angle**. Their relation to Cartesian coordinates can be read from



$$x = r \sin\theta \cos\phi,$$

$$y = r \sin\theta \sin\phi,$$

$$z = r \cos\theta.$$

r ranges from 0 to ∞ θ from 0 to π ϕ from 0 to 2π

Unit Vectors: $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$

A Vector is written as: $\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\phi}};$

A_r, A_θ, and A_ϕ are the radial, polar, and azimuthal components of $\mathbf{A}$.

Spherical Polar Coordinate Systems

To obtain $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$

$$\left.\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}},\\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}},\\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}},
\end{aligned}\right\}$$

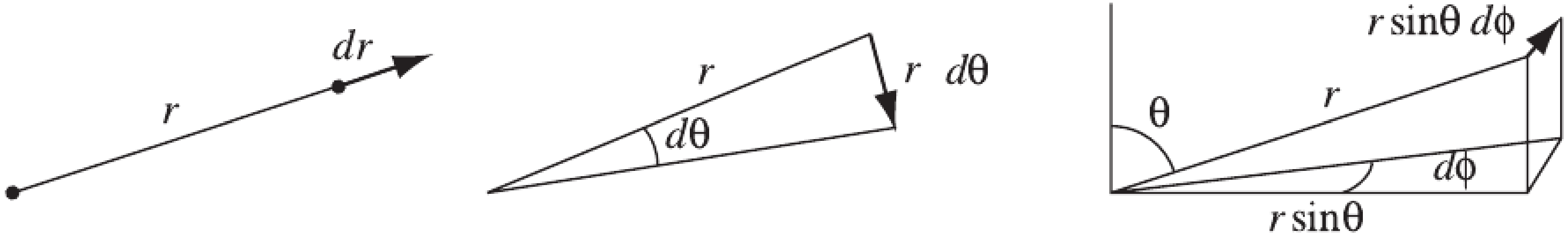
$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r}$$

$$\hat{\phi} = \frac{\hat{z} \times \hat{r}}{\sin\theta}$$

$$\hat{\theta} = \hat{\phi} \times \hat{r}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left(\frac{z}{r}\right)$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

Infinitesimal Displacement:



$$dl_r = dr \qquad dl_\theta = r\,d\theta \qquad dl_\phi = r\sin\theta\,d\phi$$

Thus the general infinitesimal displacement $d\mathbf{l}$ is

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}.$$

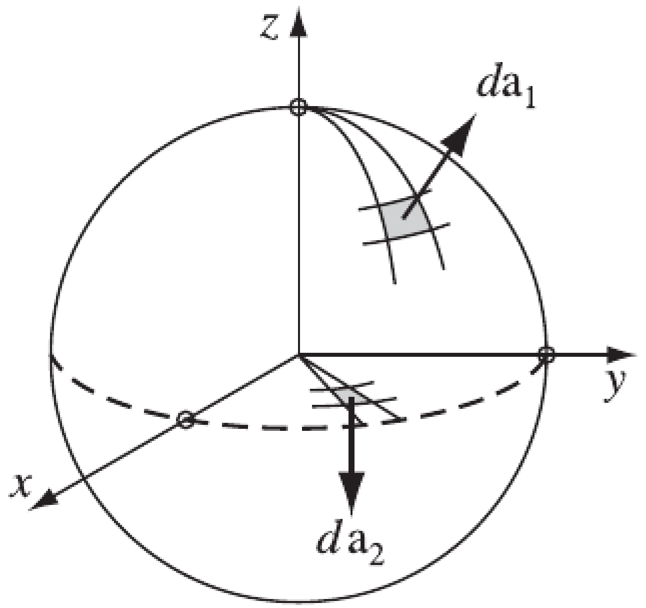
Spherical Polar Coordinate Systems

Infinitesimal Volume:

The infinitesimal volume element $d\tau$, in spherical coordinates, is the product of the three infinitesimal displacements:

$$d\tau = dl_r\, dl_\theta\, dl_\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi .$$

Infinitesimal Surface: *Depends on the choice of surface*



→ If you are integrating on the surface of a sphere,

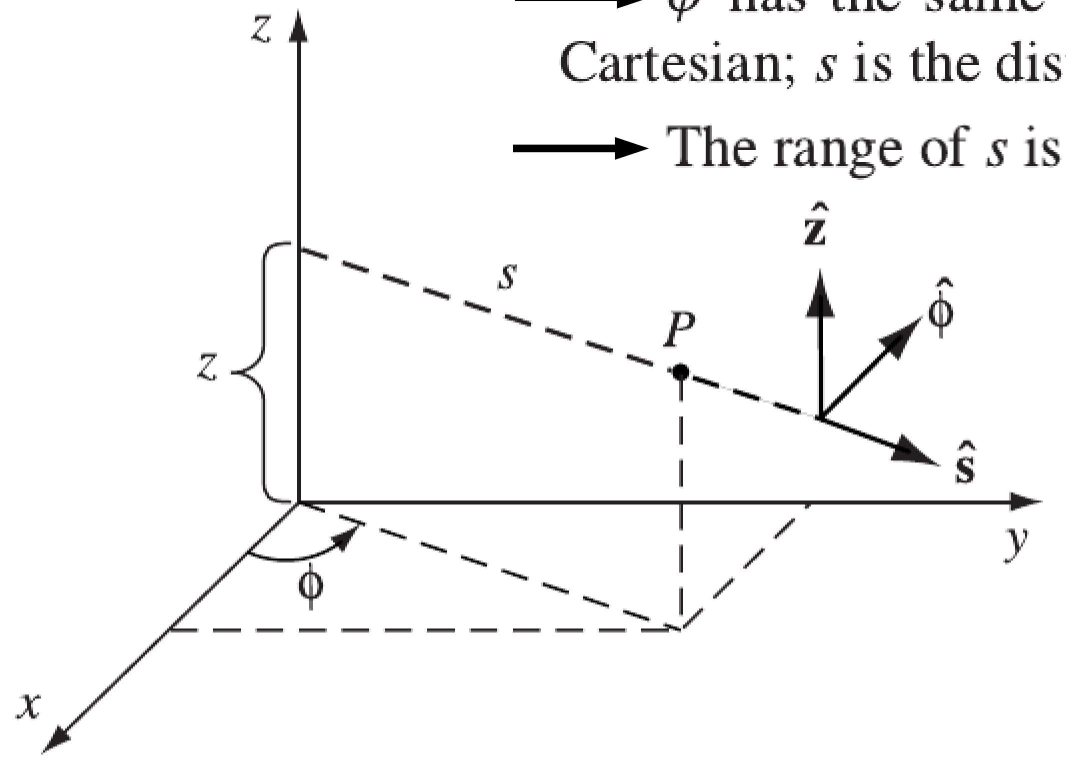
$$d\mathbf{a}_1 = dl_\theta\, dl_\phi\, \hat{\mathbf{r}} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}.$$

→ If the surface lies on the xy plane, so that the polar angle is constant,

$$d\mathbf{a}_2 = dl_r\, dl_\phi\, \hat{\boldsymbol{\theta}} = r\, dr\, d\phi\, \hat{\boldsymbol{\theta}}.$$

Cylindrical coordinate System

The cylindrical coordinates (s, ϕ, z) of a point P



→ ϕ has the same meaning as in spherical coordinates, and z is the same as Cartesian; s is the distance to P *from the z axis*

→ The range of s is $0 \to \infty$, ϕ goes from $0 \to 2\pi$, and z from $-\infty$ to ∞.

$$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z.$$

$$\left.\begin{aligned}
\hat{\mathbf{s}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}, \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \\
\hat{\mathbf{z}} &= \hat{\mathbf{z}}.
\end{aligned}\right\} \qquad \begin{aligned} \hat{s} &= \frac{\vec{s}}{s} \\ \hat{\phi} &= \hat{z}\times\hat{s} \end{aligned}$$

$$\boxed{\begin{aligned} s &= \sqrt{x^2+y^2} \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned}}$$

Unit Vectors: $\hat{\mathbf{s}}, \ \hat{\boldsymbol{\phi}}, \ \hat{\mathbf{z}}$

Vectors written as: $\mathbf{A} = A_s\hat{s} + A_\phi\hat{\phi} + A_z\hat{z}$

Cylindrical coordinate System

Infinitesimal Displacement: $$d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$

Infinitesimal Volume: $$d\tau = s\,ds\,d\phi\,dz$$

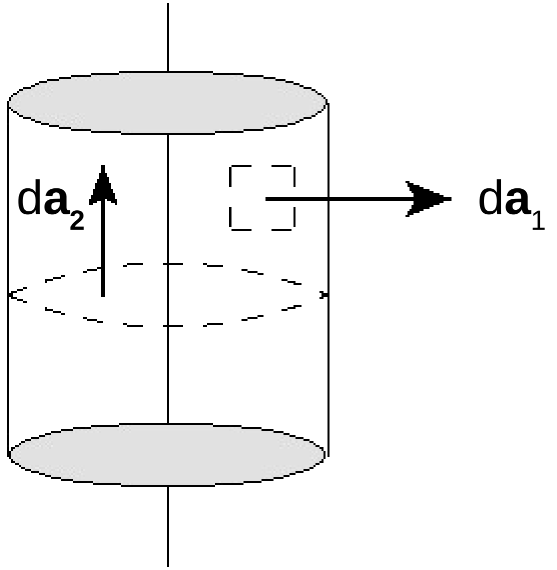
Infinitesimal Surface: *Depends on the choice of surface*

→ If you are integrating on the surface of a cylinder,

$$d\mathbf{a}_1 = s\,d\phi\,dz\ \ \hat{\mathbf{s}}$$

→ If the surface lies on the xy plane,

$$d\mathbf{a}_2 = \ s\,d\phi\,ds\ \ \hat{\mathbf{z}}$$



Vector Derivatives in Curvilinear Coordinate System

In Spherical Polar coordinate system:

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}.$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}.$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}.$$

In Cylindrical coordinate system:

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}.$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}.$$

Curl:

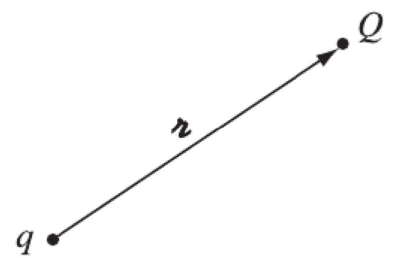
$$\nabla \times \mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}.$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}.$$

Coulomb's Law

What is the force on a test charge Q due to a single point charge q, that is at *rest* a distance $\mathscr{r}$ away?



Coulomb's Law $\longrightarrow$

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.}$$

Permittivity of free space: $\epsilon_0 = 8.85 \times 10^{-12}\,\dfrac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}.$

C: Coulomb
N: Newton
M: Meter

Separation Vector: $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$

In words, the force is proportional to the product of the charges and inversely proportional to the square of the separation distance.

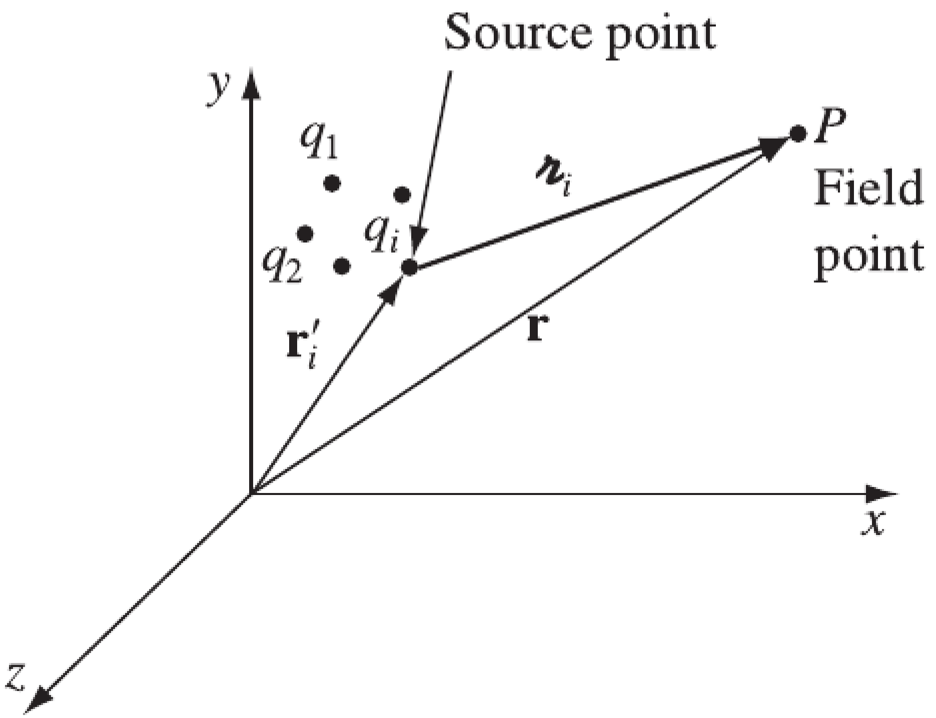
$\longrightarrow$ The force points along the line from q to Q; it is repulsive if q and Q have the same sign, and attractive if their signs are opposite.

Electric Field

If we have *several* point charges $q_1, q_2, \ldots, q_n$, at distances $\mathscr{r}_1, \mathscr{r}_2, \ldots, \mathscr{r}_n$ from Q, the total force on Q is evidently

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \ldots\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \frac{q_3}{\mathscr{r}_3^2}\hat{\boldsymbol{\mathscr{r}}}_3 + \ldots\right),$$



or

$$\boxed{\mathbf{F} = Q\mathbf{E},}$$

where

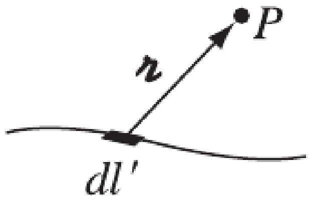
$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i.$$

$\mathbf{E}$ is called the **electric field** of the source charges. Notice that it is a function of position ($\mathbf{r}$), because the separation vectors $\boldsymbol{\mathscr{r}}_i$ depend on the location of the **field point** P

Continuous Charge Distributions

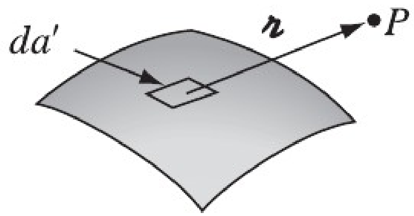
Our definition of electric field assumes that the source of the field is a collection of discrete point charges. If instead the charge is distributed continuously over some region, the sum has to be replaced by an integral.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,dq\,.$$



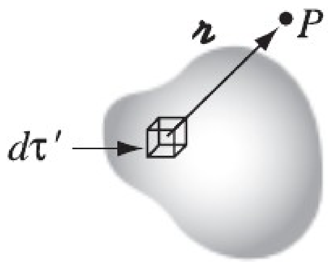
Line charge, λ

$$dq = \lambda\,dl'$$



Surface charge, σ

$$dq = \sigma\,da'$$



Volume charge, ρ

$$dq = \rho\,d\tau'$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,dl'$$

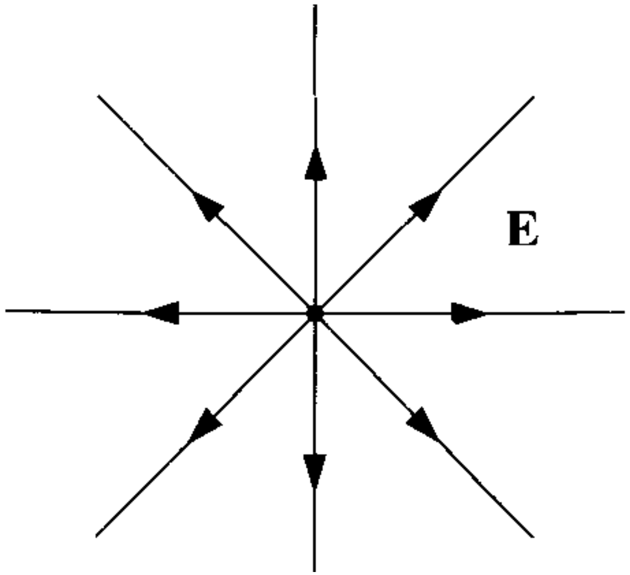
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,da'$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

Electric Field Lines

For a point charge (q):

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}.$$



- The magnitude of the field is indicated by the density of the field lines.
- The density of lines is the total number divided by the area of the sphere.

→ *The field lines are crowded together when the field is strong and spread apart when the field is weaker.*

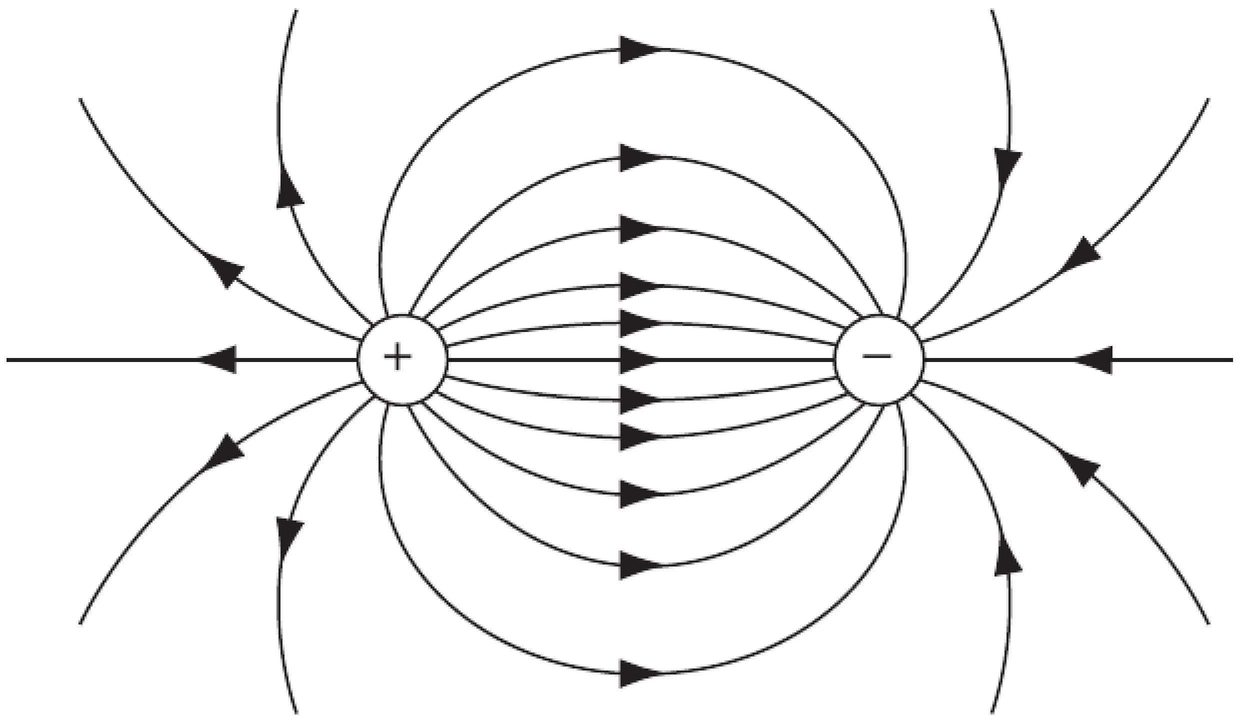
→ *Field lines begin on positive charges and end on negative charges.*

→ *Field lines can never cross.*

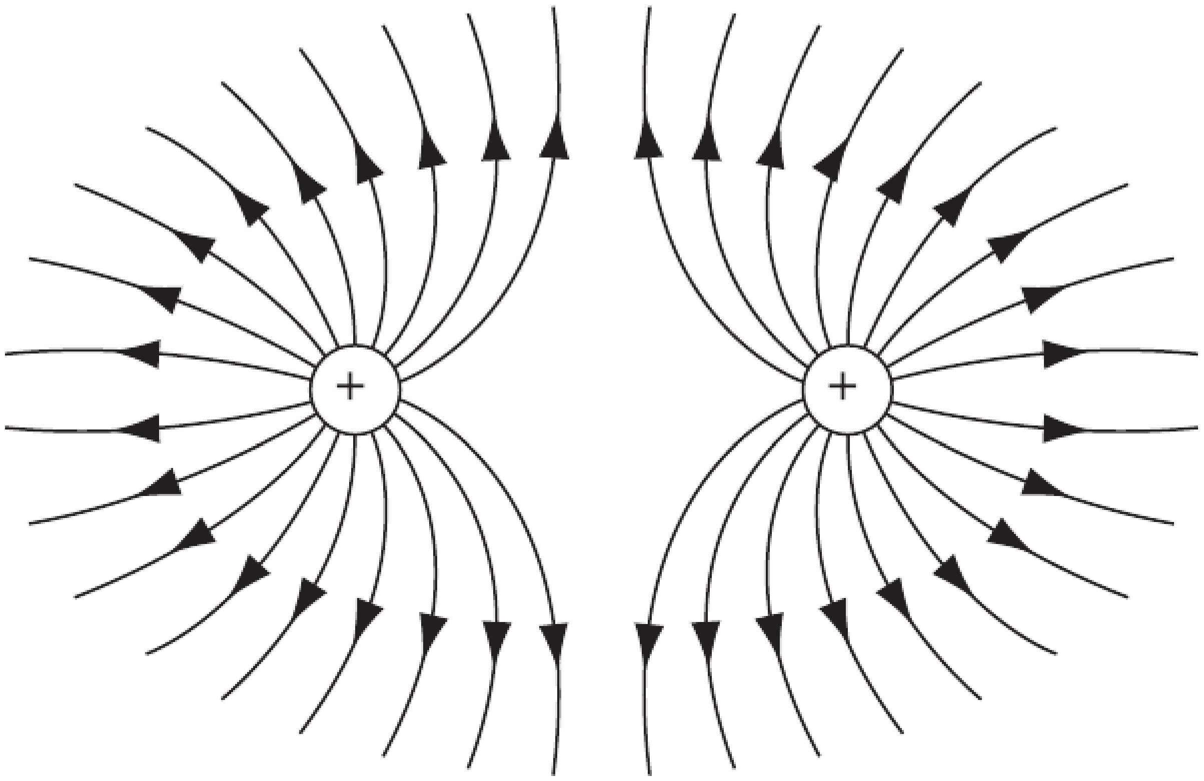
→ *They cannot terminate midair, they may extend out to infinity.*

Electric Field Lines

Sample sketch of field lines of simple configurations:



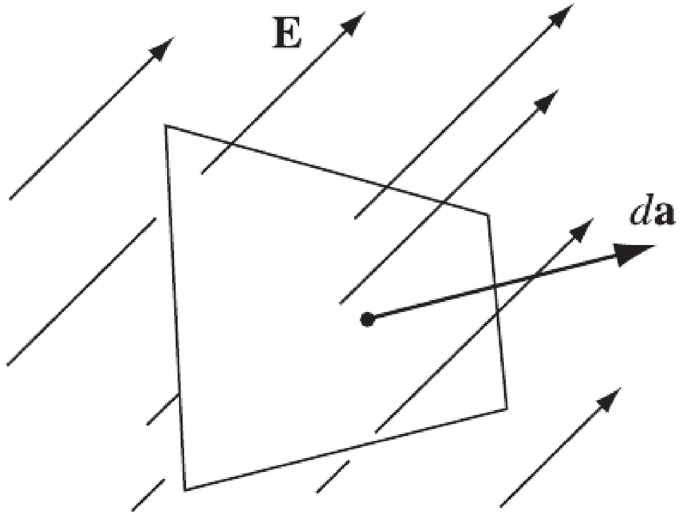
Opposite charges



Equal charges

Electric Flux

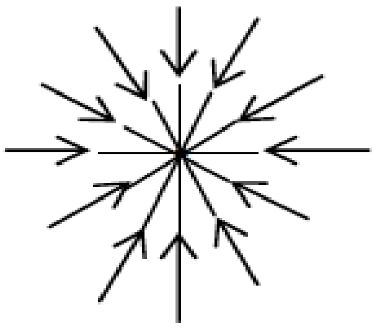
Electric flux through a surface is defined as: $\Phi_E \equiv \int_S \mathbf{E} \cdot d\mathbf{a}$



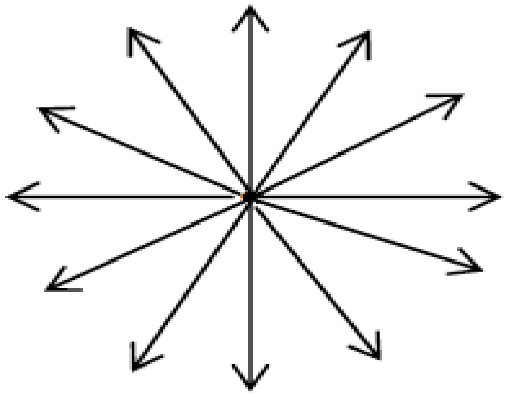
→ For a uniform electric field: $\Phi_E = EA\cos\theta$

$$= \mathbf{E} \cdot \mathbf{A}$$

Angle between the field vector and the area vector.



$q_s = -e$
inward



$q_s = +e$
outward

- Flux leaving a closed surface is positive, whereas flux entering a closed surface is negative.

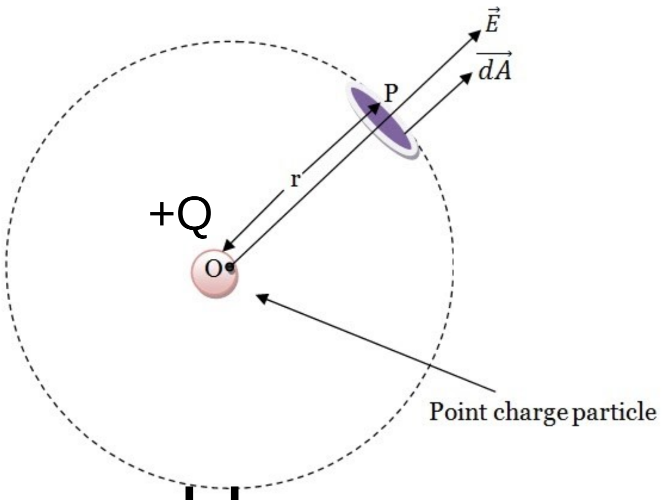
- The net flux through the surface is zero if the number of lines that enter the surface is equal to the number that leave.

Gauss' Law

Elemental area in Spherical-polar $\boxed{d\vec{A} = dA\hat{r} = \left(r^2 \sin\theta d\theta d\varphi\right)\hat{r}}$

The total Flux through the closed Gaussian surface:

$$\Phi_E = \oint_S \vec{E}(\vec{r})\bullet d\vec{A} = \frac{Q}{4\pi\varepsilon_o}\int_S \left(\frac{1}{\cancel{r^2}}\hat{r}\right)\bullet\underbrace{\left(\cancel{r^2}\sin\theta d\theta d\varphi\hat{r}\right)}_{=d\vec{A}}$$



$$\text{Thus:}\quad \Phi_E = \frac{Q}{4\pi\varepsilon_o}\int_{\theta=o}^{\theta=\pi}\int_{\varphi=o}^{\varphi=2\pi}\sin\theta d\theta d\varphi\underbrace{(\hat{r}\bullet\hat{r})}_{=1} = \frac{\cancel{2\pi} Q}{\underbrace{\cancel{4\pi}}_{2}\varepsilon_o}\int_{\theta=o}^{\theta=\pi}\sin\theta d\theta$$

$$= \frac{\cancel{2}Q}{\cancel{2}\varepsilon_o} = \frac{Q}{\varepsilon_o}$$

Imaginary surface of radius R surrounding the charge; ***Gaussian Surface***

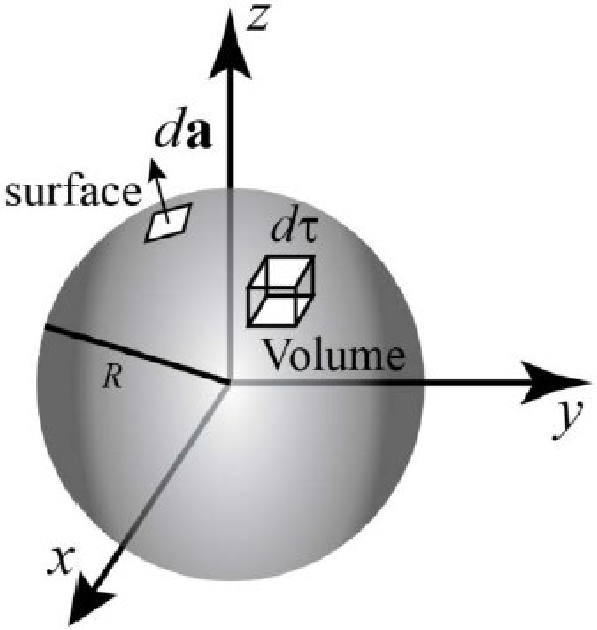
<u>Gauss' Law (in Integral Form)</u>: $\boxed{\Phi_E = \oint_s \vec{E}(\vec{r})\bullet d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_o}}$ $\longrightarrow$ ***For any surface enclosing the charge***

Electric flux through closed surface S = (electric charge enclosed by surface S)/ ε_o

Gauss' Law

As it stands, Gauss's law is an *integral* equation, but we can easily turn it into a *differential* one, by applying the divergence theorem:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau.$$



→ *For a collection of charges enclosed by a Gaussian surface:*

Rewriting Q_{enc} in terms of the charge density ρ, we have

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho\, d\tau.$$

So Gauss's law becomes

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau. \quad \textit{(Holds for any volume)}$$

Therefore,

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.}$$

→ *Gauss' Law in differential form*

Divergence of Electric Field

Let's go back, now, and calculate the divergence of $\mathbf{E}$ directly

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho(\mathbf{r}')\,d\tau'.$$

If, $\mathbf{V} = \frac{\hat{\mathbf{r}}}{r^2}$

$$\nabla\cdot\mathbf{V} = \nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = \frac{0}{r^3} = 0$$

Except at r=0, where it reduces to 0/0; not defined

Then, $\nabla\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)\rho(\mathbf{r}')\,d\tau'$

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}).$$

Dirac delta function, non-zero only at r=0

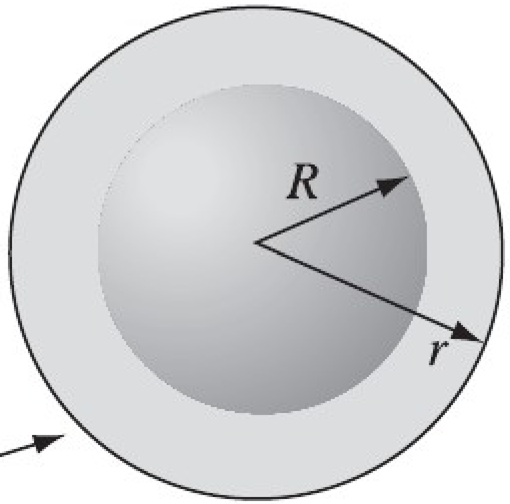
Thus,

$$\nabla\cdot\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int 4\pi\delta^3(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\,d\tau' = \frac{1}{\epsilon_0}\rho(\mathbf{r})$$

Hence,

$$\int_{\mathcal{V}} \nabla\cdot\mathbf{E}\,d\tau = \oint_{\mathcal{S}} \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}\int_{\mathcal{V}} \rho\,d\tau = \frac{1}{\epsilon_0}Q_{\text{enc}}.$$

Find the electric field outside a uniformly charged solid sphere of radius *R* and total charge *q*.



Imagine a spherical surface at radius $r > R$; $\quad Q_{\text{enc}} = q$

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

symmetry allows us to extract $\mathbf{E}$ from under the integral sign: $\mathbf{E}$ certainly points radially outward, as does $d\mathbf{a}$, so we can drop the dot product, and the *magnitude* of $\mathbf{E}$ is constant over the Gaussian surface, so it comes outside the integral:

$$\int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{S}} |\mathbf{E}|\, da = |\mathbf{E}| \int_{\mathcal{S}} da = |\mathbf{E}|\, 4\pi r^2 = \frac{1}{\epsilon_0} q$$

Therefore,

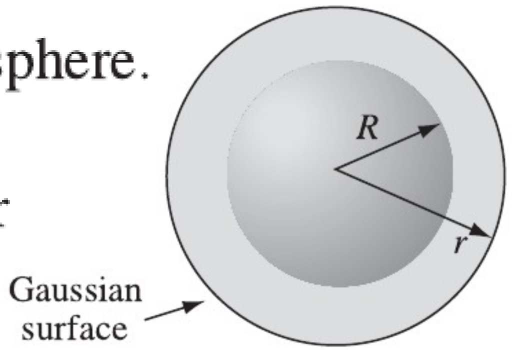
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

Notice a remarkable feature of this result: The field outside the sphere is exactly *the same as it would have been if all the charge had been concentrated at the center.*

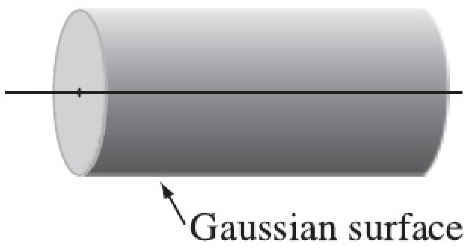
Application of Gauss' Law

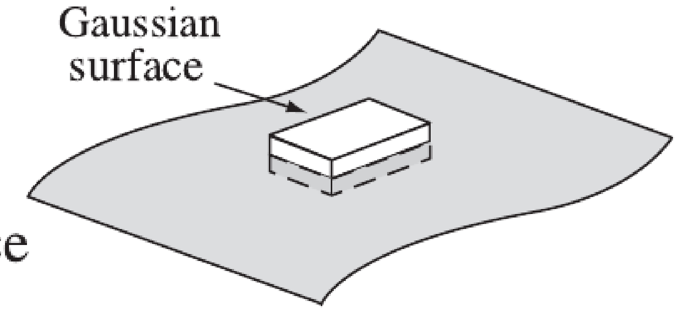
Gauss's law is always *true*, but it is not always *useful*. If ρ had not been uniform (or, at any rate, not spherically symmetrical), or if I had chosen some other shape for my Gaussian surface, it would still have been true that the flux of $\mathbf{E}$ is q/ϵ_0, but $\mathbf{E}$ would not have pointed in the same direction as $d\mathbf{a}$, and its magnitude would not have been constant over the surface, and without that I cannot get $|\mathbf{E}|$ outside of the integral. *Symmetry is crucial* to this application of Gauss's law.

1. *Spherical symmetry.* Make your Gaussian surface a concentric sphere.



2. *Cylindrical symmetry.* Make your Gaussian surface a coaxial cylinder





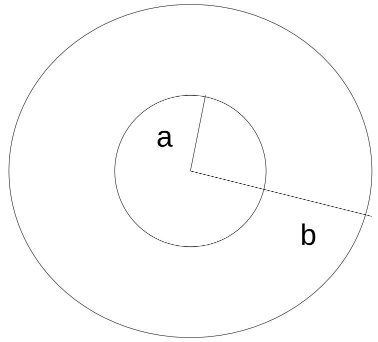
3. *Plane symmetry.* Use a Gaussian "pillbox" that straddles the surface

Application of Gauss' Law

A thick spherical shell carries charge density

$$\rho = \frac{k}{r^2} \quad (a \le r \le b)$$

Find the electric field in the three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $r > b$.



(i) $Q_{\text{enc}} = 0$, so $\boxed{\mathbf{E} = \mathbf{0}.}$

(ii) $\oint \mathbf{E}\cdot d\mathbf{a} = E(4\pi r^2) = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0}\int \rho\, d\tau = \frac{1}{\epsilon_0}\int \frac{k}{\bar{r}^2}\bar{r}^2 \sin\theta\, d\bar{r}\, d\theta\, d\phi$

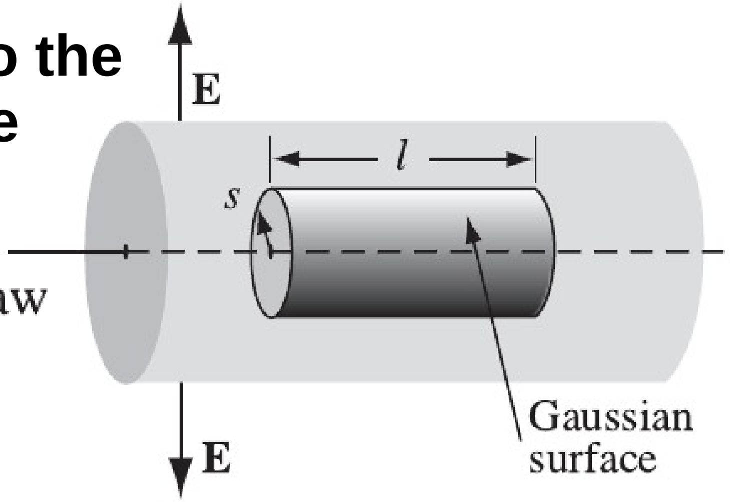
$= \frac{4\pi k}{\epsilon_0}\int_a^r d\bar{r} = \frac{4\pi k}{\epsilon_0}(r-a) \therefore \boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{r-a}{r^2}\right)\hat{\mathbf{r}}.}$

(iii) $E(4\pi r^2) = \frac{4\pi k}{\epsilon_0}\int_a^b d\bar{r} = \frac{4\pi k}{\epsilon_0}(b-a)$, so

$$\boxed{\mathbf{E} = \frac{k}{\epsilon_0}\left(\frac{b-a}{r^2}\right)\hat{\mathbf{r}}.}$$

Application of Gauss' Law

A long cylinder carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant *k*. Find the electric field inside the cylinder.



Draw a Gaussian cylinder of length l and radius s. For this surface, Gauss's law states:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

The enclosed charge is

$$Q_{\text{enc}} = \int \rho \, d\tau = \int (ks')(s' \, ds' \, d\phi \, dz) = 2\pi k l \int_0^s s'^2 \, ds' = \tfrac{2}{3}\pi k l s^3.$$

$\longrightarrow$ integrated ϕ from 0 to 2π, dz from 0 to l.

Now, symmetry dictates that $\mathbf{E}$ must point radially outward, so for the curved portion of the Gaussian cylinder we have:

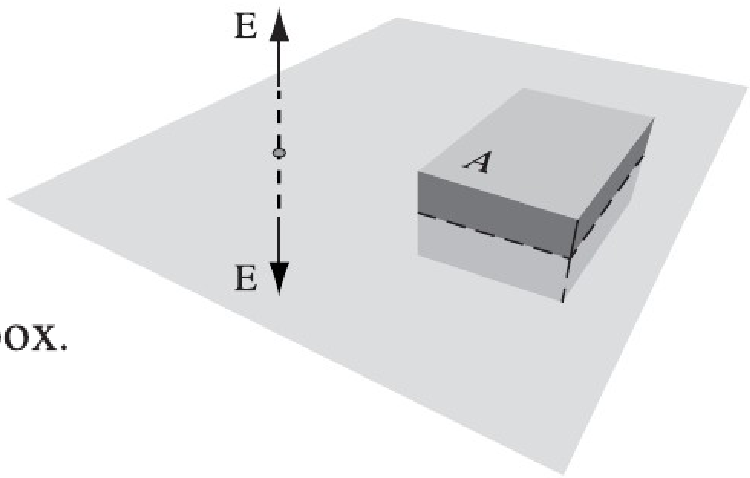
$$\int \mathbf{E} \cdot d\mathbf{a} = \int |\mathbf{E}| \, da = |\mathbf{E}| \int da = |\mathbf{E}| \, 2\pi s l,$$

Hence, $|\mathbf{E}| \, 2\pi s l = \dfrac{1}{\epsilon_0} \dfrac{2}{3} \pi k l s^3 \longrightarrow \mathbf{E} = \dfrac{1}{3\epsilon_0} k s^2 \hat{\mathbf{s}}.$

Note that, the two ends of the surface do not contribute anything since the electric field is perpendicular to the surface.

Application of Gauss' Law

An infinite plane carries a uniform surface charge density σ. Find its electric field.



Apply Gauss's law to this surface:

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

In this case, $Q_{\text{enc}} = \sigma A$, where A is the area of the lid of the pillbox.

$\mathbf{E}$ points away from the plane (upward for points above, downward for points below).

So the top and bottom surfaces yield $\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}|,$

whereas the sides contribute nothing. Thus

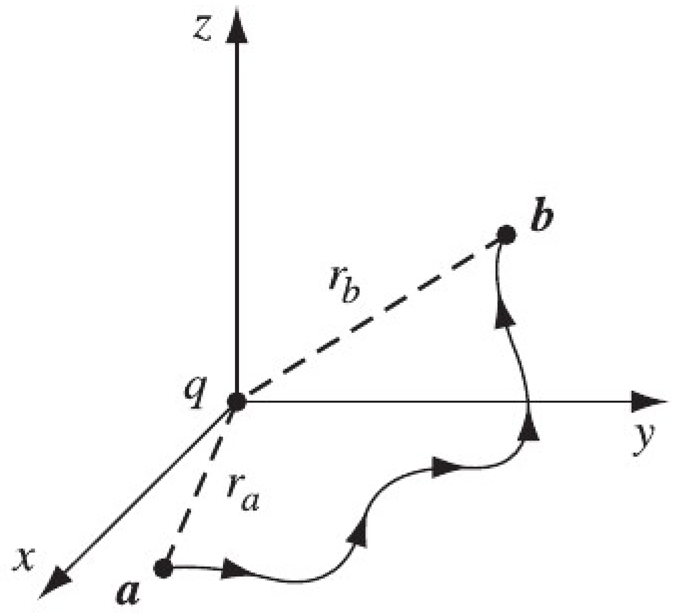
$$2A\,|\mathbf{E}| = \frac{1}{\epsilon_0} \sigma A$$

Hence, $\mathbf{E} = \dfrac{\sigma}{2\epsilon_0} \hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector pointing away from the surface.

Curl of Electric Field

Start with simplest possible configuration: *Electric field due to a single charge, q*

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$



If we calculate the line integral of this field from point **a** to point **b**,

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

In spherical coordinates, $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$, so

$$\mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr.$$

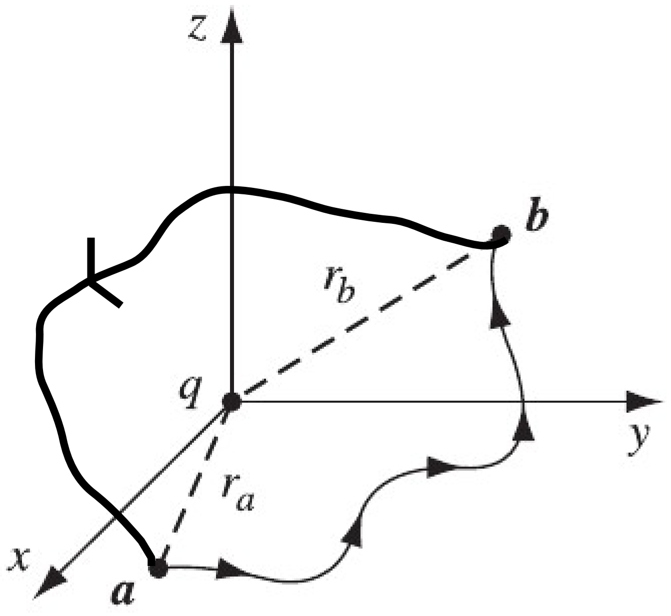
Therefore,

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{a}}^{\mathbf{b}} \frac{q}{r^2}dr = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}\bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right),$$

where r_a is the distance from the origin to the point **a** and r_b is the distance to **b**.

The integral around a *closed* path is evidently zero (for then $r_a = r_b$):

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{l} = 0,}$$



and hence, applying Stokes' theorem,

$$\boxed{\nabla \times \mathbf{E} = \mathbf{0}.}$$

(True for any static charge distribution)

Moreover, if we have many charges, the principle of superposition states that the total field is a vector sum of their individual fields:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots,$$

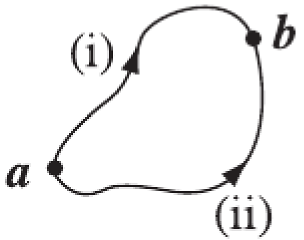
so

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{E}_1 + \mathbf{E}_2 + \ldots) = (\nabla \times \mathbf{E}_1) + (\nabla \times \mathbf{E}_2) + \ldots = \mathbf{0}.$$

Electric Potential

→ ***For a vector to represent an electric field, its curl must be zero.***

→ ***Gradient of a scalar always has zero curl.***



Because $\nabla \times \mathbf{E} = \mathbf{0}$, the line integral of $\mathbf{E}$ around any closed loop is zero

→ Follows from Stoke's theorem.

Because $\oint \mathbf{E} \cdot d\mathbf{l} = 0$, the line integral of $\mathbf{E}$ from point **a** to point **b** is the same for all paths

→ Since the line integral is independent of the path and only depends on the end points, we can define a scalar function,

$$\boxed{V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}.}$$

Here $\mathcal{O}$ is some standard reference point on which we have agreed beforehand; V then depends only on the point $\mathbf{r}$. It is called the **electric potential.**

Electric Potential

→ The potential *difference* between two points **a** and **b** is

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

→ We know from the fundamental theorem of gradients,

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l}$$

Hence,

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

Since, finally, this is true for *any* points **a** and **b**, the integrands must be equal:

$$\boxed{\mathbf{E} = -\nabla V.}$$

Electric Potential

A note on the choice of reference point:

→ The choice is completely arbitrary and the outcome should not be dependent on this choice. Changing the reference points amounts to adding a constant to the potential.

$$V'(\mathbf{r}) = -\int_{\mathcal{O}'}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathcal{O}'}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = K + V(\mathbf{r})$$

where K is the line integral of $\mathbf{E}$ from the old reference point $\mathcal{O}$ to the new one $\mathcal{O}'$.

→ However, adding the constant does not affect the ***potential difference*** *between two points.*

$$V'(\mathbf{b}) - V'(\mathbf{a}) = V(\mathbf{b}) - V(\mathbf{a})$$

→ Gradient of the potential also remains unchanged.

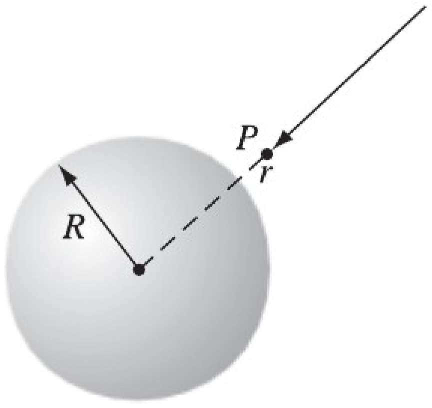
$$\nabla V' = \nabla V$$

Potential as such carries no real physical significance, for at any given point we can adjust its value at will by a suitable relocation of $\mathcal{O}$.

→ *Infinity is a natural choice for the reference point.*

Find the potential inside and outside a spherical shell of radius R carrying a uniform surface charge.

From Gauss's law, the field outside is $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}}$



where q is the total charge on the sphere. The field inside is zero. For points outside the sphere ($r > R$),

$$V(r) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{r} \frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$

To find the potential inside the sphere ($r < R$), we must break the integral into two pieces, using in each region the field that prevails there:

$$V(r) = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{R} \frac{q}{r'^2}\,dr' - \int_{R}^{r}(0)\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{R} + 0 = \frac{1}{4\pi\epsilon_0}\frac{q}{R}.$$

Notice that the potential is *not* zero inside the shell, even though the field is.

Poisson's Equation and Laplace's Equation

Electric field can be written as the gradient of potential: $\mathbf{E} = -\nabla V.$

→ Looking at the divergence and curl of an electric field, $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ and $\nabla \times \mathbf{E} = \mathbf{0}$

In terms of the potential: $\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V$

Then from Gauss' law:

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon_0}.} \longrightarrow$$ ***Poisson's Equation***

→ In regions with no charge, i.e., if $\rho = 0$

$$\nabla^2 V = 0. \longrightarrow$$ ***Laplace's Equation***

The curl of electric field is also consistent since

$$\nabla \times \mathbf{E} = \nabla \times (-\nabla V) = \mathbf{0}.$$

Electric Potential for a charge distribution

Knowing the electric field, one can calculate the potential.

$$\mathbf{E} = (1/4\pi\epsilon_0)(q/r^2)\,\hat{\mathbf{r}}$$

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

Then,

$$\mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr.$$

Setting the reference point at infinity, the potential of a point charge q at the origin is

$$V(r) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E}\cdot d\mathbf{l} = \frac{-1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}\,dr' = \frac{1}{4\pi\epsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$

In general, the potential of a point charge q is

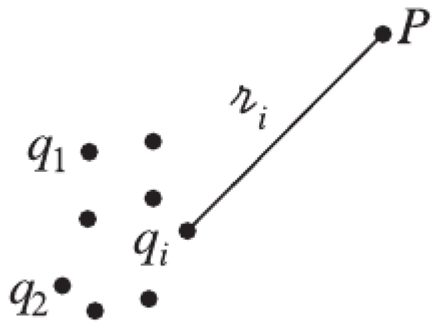
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

where $\mathscr{r}$, as always, is the distance from q to $\mathbf{r}$

Electric Potential for a charge distribution

Following superposition principle, for a collection of charges,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$$



For a continuous charge distribution,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\,dq\,.$$

→ For a line, surface and volume charge distribution respectively,

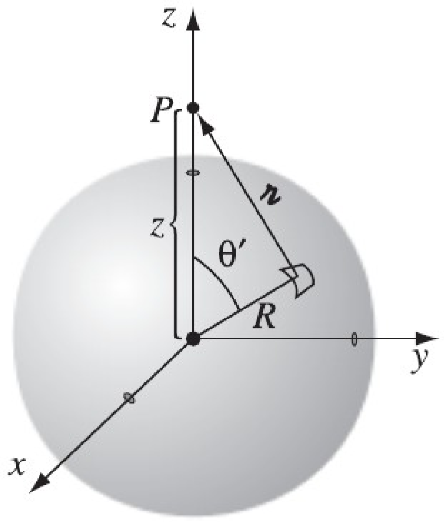
$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}}\,dl' \qquad V = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}}\,da' \qquad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$$

Electric Potential for a charge distribution

Find the electric potential inside and outside a uniformly charged spherical shell of radius R

We'll do it using the charge distribution.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{\mathscr{r}}\,da'$$



We might as well set the point P on the z axis and use the law of cosines to express $\mathscr{r}$:

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta'.$$

An element of surface area on the sphere is $R^2 \sin\theta'\, d\theta'\, d\phi'$, so

$$\begin{aligned}
4\pi\epsilon_0 V(z) &= \sigma \int \frac{R^2 \sin\theta'\, d\theta'\, d\phi'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}} \\
&= 2\pi R^2 \sigma \int_0^{\pi} \frac{\sin\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}\, d\theta' \\
&= 2\pi R^2 \sigma \left(\frac{1}{Rz}\sqrt{R^2 + z^2 - 2Rz\cos\theta'} \right)\Bigg|_0^{\pi} \\
&= \frac{2\pi R\sigma}{z}\left(\sqrt{R^2 + z^2 + 2Rz} - \sqrt{R^2 + z^2 - 2Rz} \right) \\
&= \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2} \right].
\end{aligned}$$

Electric Potential for a charge distribution

At this stage, we must be very careful to take the *positive* root. For points *outside* the sphere, z is greater than R, and hence $\sqrt{(R-z)^2} = z - R$; for points *inside* the sphere, $\sqrt{(R-z)^2} = R - z$. Thus,

$$V(z) = \frac{R\sigma}{2\epsilon_0 z}[(R+z) - (z-R)] = \frac{R^2\sigma}{\epsilon_0 z}, \qquad \text{outside;}$$

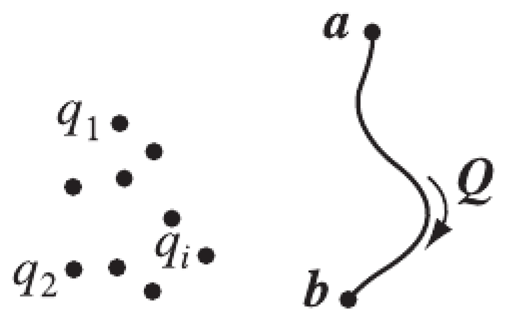
$$V(z) = \frac{R\sigma}{2\epsilon_0 z}[(R+z) - (R-z)] = \frac{R\sigma}{\epsilon_0}, \qquad \text{inside.}$$

In terms of r and the total charge on the shell, $q = 4\pi R^2\sigma$,

$$V(r) = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r} & (r \geq R), \\[2ex] \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{R} & (r \leq R). \end{cases}$$

Work and Energy

We have a stationary charge distribution. A test charge is being moved from point **a** to point **b** in its presence,



- At any point along the path, force acting on the test charge: $\mathbf{F} = Q\mathbf{E}$
- This is the minimum amount of force one has to exert in order to move the test charge against the Coulomb force.
- Work done:

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})]$$

- Work done is independent of the path, only depends on the end points; ***electrostatic force is conservative***. Potential difference between point **a** and **b** is equal to the work done per unit charge.

$$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q}.$$

- If I bring the test charge from infinity to some point with position vector **r**, we can write

$$W = Q[V(\mathbf{r}) - V(\infty)]$$

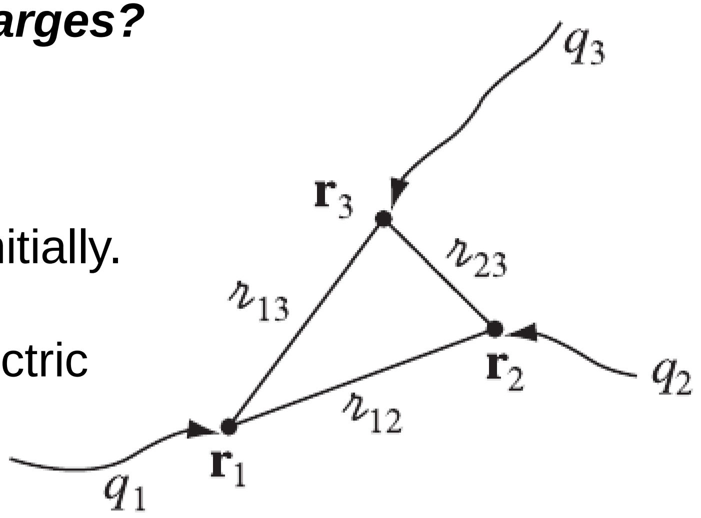
Or,

$$W = QV(\mathbf{r}).$$

Energy of a point charge distribution

How much work would it take to assemble an entire collection of point charges?

Imagine bringing the charges one by one from far away.



- Bringing in the first charge q_1 takes no work since there is no electric field initially.
- In order to bring in the second charge q_2 one needs to work against the electric field of the first charge.

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{\mathscr{r}_{12}} \right)$$

- Similarly, in order to bring in the next charge q_3 work done is

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}} \right)$$

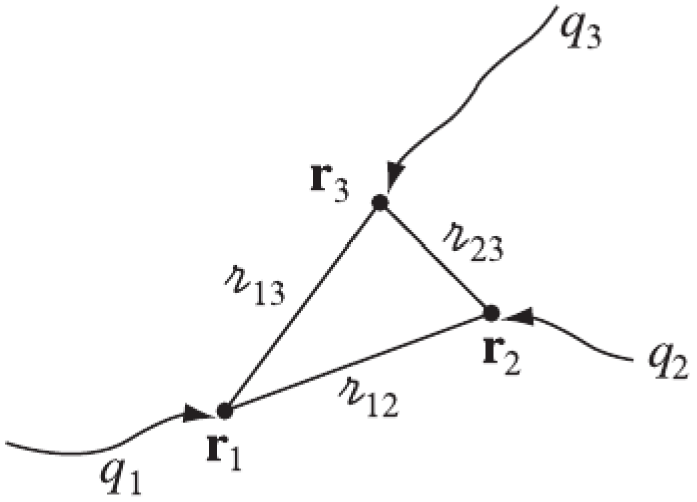
and for the next one,

$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left(\frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}} \right)$$

and so on.....

Energy of a point charge distribution

Hence, total work necessary to assemble the first four charges:



$$W = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}}\right)$$

→ In general, for *n* number of charges one can write,

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

The restriction on the limit of the second sum, $j > i$ is to avoid double counting.

One can also write it as:

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

(also represents the energy stored in the charge configuration)

Hence,

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right)$$

Or,

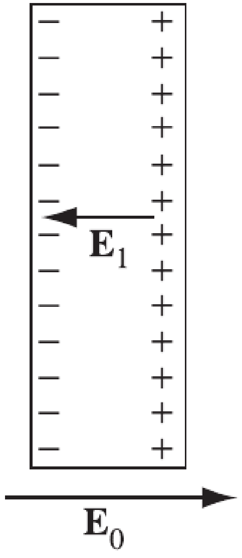
$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i).$$

Potential at $\boldsymbol{r}_i$ due to all other charges.

Conductors

A perfect conductor has an unlimited supply of free electrons.

→ In reality, there are no ideal conductors, but metals come pretty close.



Electric field inside a conductor is zero.

→ If a conductor is put in an external electric field, the induced charges produce an electric field inside the conductor such that the field inside is oriented opposite to the direction of the external electric field. The two fields cancel each other inside. the charges flow inside until this equilibrium is reached.

Hence, inside the conductor the charge density must be zero since

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$$

→ If electric field inside is zero, so is the charge density, which means there are equal number of positive and negative charges.

Any net charge it may have, can only reside on the surface. Conductors have equipotential surfaces.

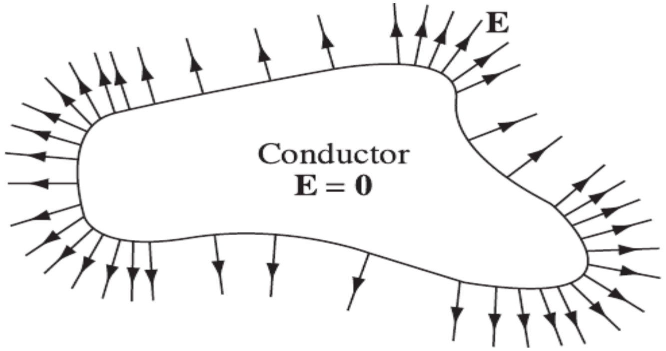
→ If **a** and **b** are two points on the surface of a conductor,

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = 0$$

Hence, $V(\mathbf{a}) = V(\mathbf{b})$.

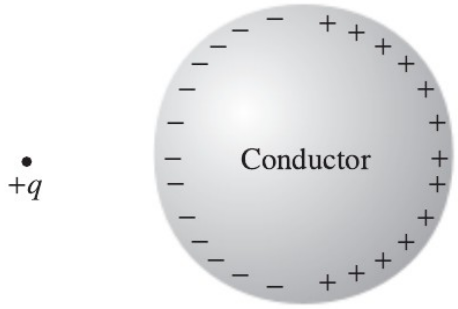
Conductors

Electric field just outside the conductor is perpendicular to the surface.



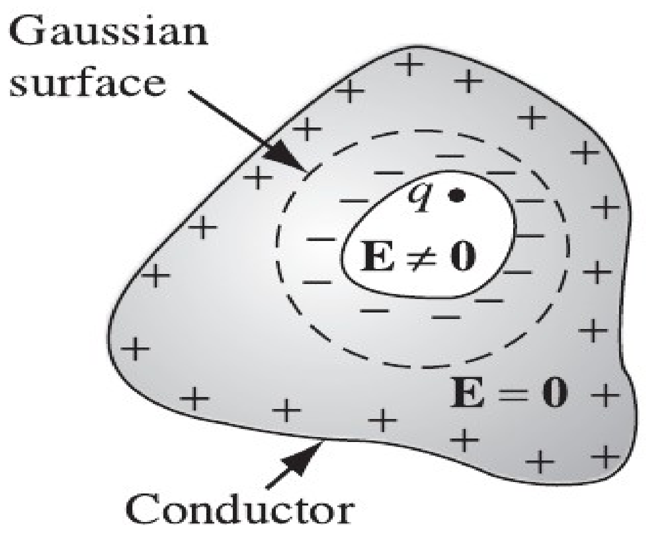
→ Otherwise, the charges would flow around until they kill off the tangential component.

Induced Charges



If you hold a charge $+q$ near an uncharged conductor, the two will attract one another. The reason for this is that q will pull minus charges over to the near side and repel plus charges to the far side. The charges move around in such a way as to kill off the electric field inside the conductor. Since the negative induced charge is closer to q, there is a net force of attraction.

Cavity inside a Conductor



- Outside the conductor electric field is non-zero
- Outside the cavity but inside the conductor electric field is zero
- Inside the cavity electric field is non-zero

- The total charge induced on the cavity wall is equal and opposite to the charge inside the cavity.

For the Gaussian surface shown in the figure, $\oint \mathbf{E} \cdot d\mathbf{a} = 0$ since, we are inside the conductor but outside cavity

Here,

$$Q_{\text{enc}} = q + q_{\text{induced}}$$

Hence,

$$q_{\text{induced}} = -q.$$

- Field outside the conductor: $\mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}}$

Capacitors

Say, we have two conductors carrying +*Q* and -*Q* charges respectively.

Since the potential is uniform over any conductor, the potential difference between them

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}$$

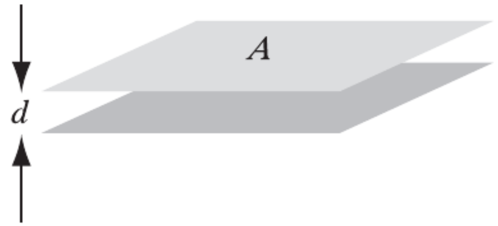
V is proportional to the electric charge, since the electric field is proportial to the charge.

→ The proportionality constant is called the ***capacitance(C)*** of the system.

$$C \equiv \frac{Q}{V}$$

Capacitance depends upon the sizes, shapes and distance between the two conductors.

For a parallel plate capacitor with +*Q* and -*Q* charges uniformly distributed over the plates,,



The uniform charge density, $\sigma = Q/A$

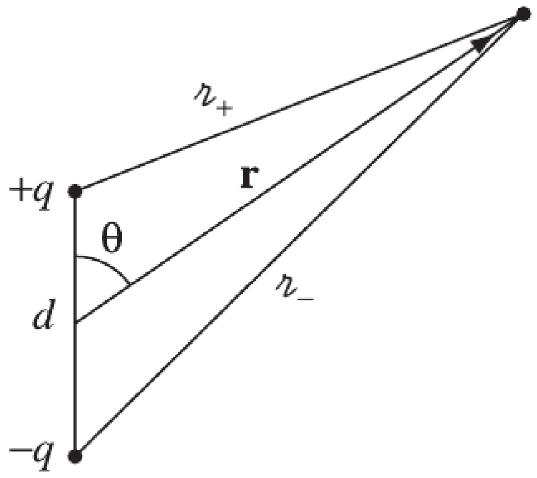
Electric field in between the plates is $(1/\epsilon_0)Q/A$

Potential difference: $V = \frac{Q}{A\epsilon_0}d$

Capacitance: $C = \frac{A\epsilon_0}{d}.$

Electric Dipole

An electric dipole with equal and opposite charges separated by a distance *d*. Calculate the potential at a distance ***r***.



Let $\mathscr{r}_-$ be the distance from $-q$ and $\mathscr{r}_+$ the distance from $+q$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

We have,

$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

For $r \gg d$, binomial expansion yields

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

Then

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta$$

Hence,

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}.$$

Electric Dipole

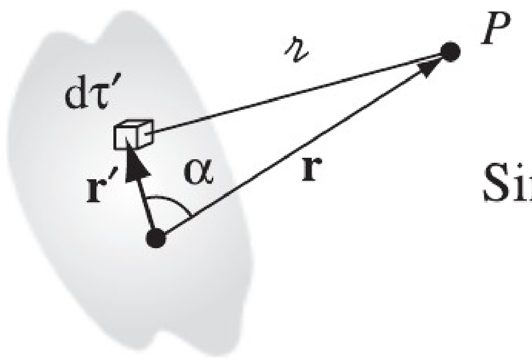
For a given charge distribution, the most dominant contribution to potential comes from the monopole term.

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

where, $Q = \int \rho \, d\tau$

For a point charge at origin, this is the exact contribution. All higher order contribution vanish.

→ If the total charge happens to be zero, then most dominant contribution comes from the nno-zero dipole term.



$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r' \cos\alpha \, \rho(\mathbf{r}') \, d\tau'$$

Since α is the angle between $\mathbf{r}'$ and $\mathbf{r}$

$$r' \cos\alpha = \hat{\mathbf{r}} \cdot \mathbf{r}'$$

Then

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}} \cdot \int \mathbf{r}' \rho(\mathbf{r}') \, d\tau'$$

Electric Dipole

Define dipole moment:

$$\mathbf{p} \equiv \int \mathbf{r}' \rho(\mathbf{r}')\, d\tau'$$

Then we can write,

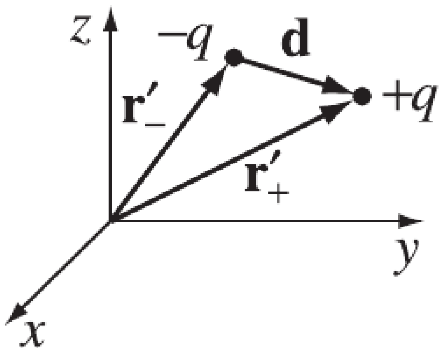
$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

For a collection of point charges,

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i$$

For a physical dipole it reduces to:

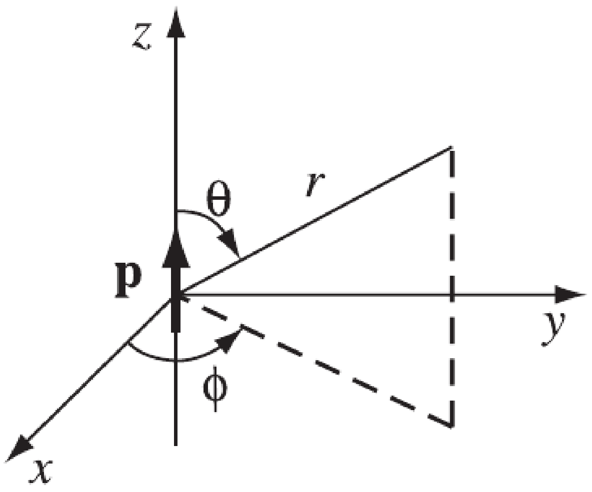
$$\mathbf{p} = q\mathbf{r}'_+ - q\mathbf{r}'_- = q(\mathbf{r}'_+ - \mathbf{r}'_-) = q\mathbf{d}.$$



→ valid only for $r \gg d$

Electric Field due to a Dipole

We have found out the potential in terms of dipole moment:



$$V_{\text{dip}}(r,\theta) = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

To get the field, we take the negative gradient of V:

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3},$$

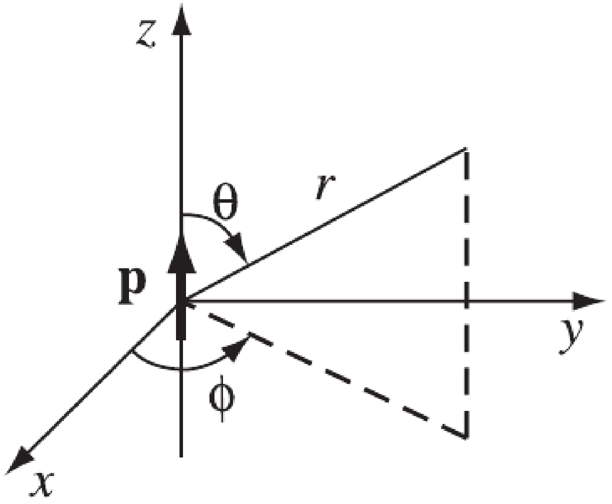
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3},$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0.$$

Thus,

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$

The electric field can be rewritten as



$$\mathbf{E}_{\rm dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3}[3(\mathbf{p}.\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]$$

Since,

$$\begin{aligned}\mathbf{p} &= (\mathbf{p}.\hat{\mathbf{r}})\ \hat{\mathbf{r}} + (\mathbf{p}.\hat{\theta})\ \hat{\theta} \\ &= p\cos\theta\ \hat{\mathbf{r}} - p\sin\theta\ \hat{\theta}\end{aligned}$$

Then,

$$\begin{aligned}&3(\mathbf{p}.\hat{\mathbf{r}})\ \hat{\mathbf{r}} - \mathbf{p} \\ &= 3p\cos\theta\ \hat{\mathbf{r}} - p\cos\theta\ \hat{\mathbf{r}} + p\sin\theta\ \hat{\theta} \\ &= 2p\cos\theta\ \hat{\mathbf{r}} + p\sin\theta\ \hat{\theta}\end{aligned}$$

Dielectrics

Almost all everyday objects upto a good approximation, belong to one of the two large categories: **conductors** and **insulators** or **dielectrics**.

→ Conductors have an unlimited supply of free electrons that move through the material.

→ In dielectrics all charges are attached to specific atoms or molecules. They can move a little bit within the atom or molecule but the microscopic displacements are not as significant as inside a conductor. The cumulative effect of these microscopic movements accounts for the characteristics of a dielectric.

→ Depending on the amount of restriction on the movement of the charges inside the material we can differentiate between insulators and dielectrics. Insulators have the smallest dielectric constant.

Atomic Dipole

When neutral atom is placed in an electric field:

→ The positive and negative charge cores are separated; the nucleus pushed in the direction of the field and the electrons the opposite way.

→ The positive and negative core attract each other; it holds the atom together.

→ The atom is polarised and now has a tiny dipole moment pointing in the same direction as the external electric field.

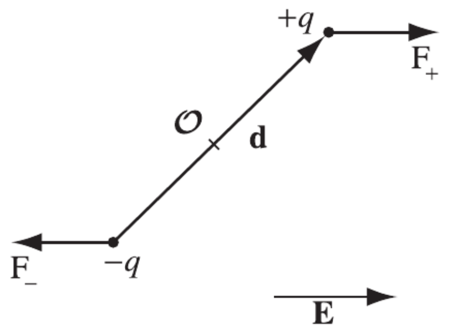
$$\mathbf{p} = \alpha \mathbf{E}.$$

The constant of proportionality α is called **atomic polarizability.**

Some molecules have built-in permanent dipole moment. When such polar molecules are placed in an electric field:

→ If the field is uniform, the force on the positive end exactly cancels the force on the negative end.

→ There is a torque.



$$\mathbf{N} = (\mathbf{r}_+ \times \mathbf{F}_+) + (\mathbf{r}_- \times \mathbf{F}_-)$$
$$= \left[(\mathbf{d}/2) \times (q\mathbf{E})\right] + \left[(-\mathbf{d}/2) \times (-q\mathbf{E})\right] = q\mathbf{d} \times \mathbf{E}.$$

Hence, we can write,

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}.$$

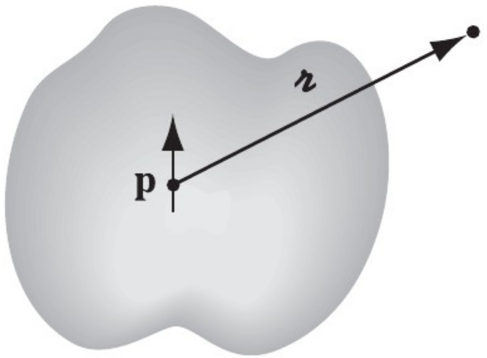
Polarization

Notice that these two mechanisms produce the same basic result: *a lot of little dipoles pointing along the direction of the field*—the material becomes **polarized**. A convenient measure of this effect is

$$\mathbf{P} \equiv \textit{ dipole moment per unit volume,}$$

which is called the **polarization**.

Say, we have a polarized material with all the dipoles pointing in the same diection.



→ For a single dipole, the electrostatic potential can be written as

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

where, $\boldsymbol{\mathscr{r}}$ is the vector from the dipole to the point at which we are trying to find the potential. We have a dipole moment $\mathbf{p} = \mathbf{P}\,d\tau'$ in each volume element $d\tau'$. Hence the total potential is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}}\frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$

Field due to a polarized object

The total potential can be written as,

$$V = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \mathbf{P} \cdot \nabla' \left(\frac{1}{\eta}\right) d\tau' \qquad \text{since,} \qquad \nabla' \left(\frac{1}{\eta}\right) = \frac{\hat{\boldsymbol{\eta}}}{\eta^2}$$

Using the identity $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

$$V = \frac{1}{4\pi\epsilon_0} \left[\int_{\mathcal{V}} \nabla' \cdot \left(\frac{\mathbf{P}}{\eta}\right) d\tau' - \int_{\mathcal{V}} \frac{1}{\eta} (\nabla' \cdot \mathbf{P}) \, d\tau' \right]$$

Using divergence theorem,

$$V = \underbrace{\frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{1}{\eta} \mathbf{P} \cdot d\mathbf{a}'}_{\text{Potential due to a surface charge distribution}} - \underbrace{\frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{1}{\eta} (\nabla' \cdot \mathbf{P}) \, d\tau'}_{\text{Potential due to a volume charge distribution}}$$

Potential due to a surface charge distribution

Potential due to a volume charge distribution

$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}} \qquad \rho_b \equiv -\nabla \cdot \mathbf{P}$$

Bound Charges

Potential of a polarized object is the same as that produced by a volume and a surface charge distribution.

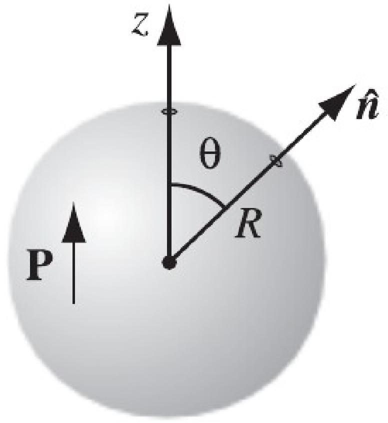
Bound charges

$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$\rho_b \equiv -\nabla \cdot \mathbf{P}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{\sigma_b}{\mathscr{r}} \, da' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}} \, d\tau'$$

Ex: Find the electric field produced by a uniformly polarized sphere of radius R



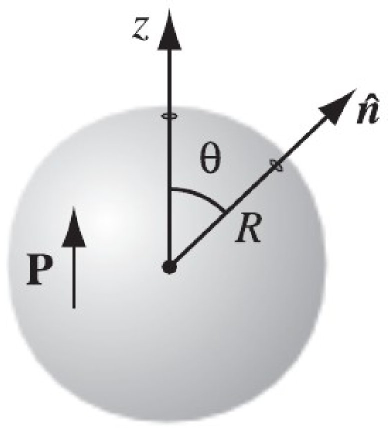
→ The volume bound charge density ρ_b is zero, since $\mathbf{P}$ is uniform

→ choose the z axis to coincide with the direction of polarization

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = P \cos\theta$$

→ charge density $P \cos\theta$ plastered over the surface of a sphere

We know the expression for electric potential for such a system



$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta, & \text{for} \quad r \le R, \\[2ex] \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta, & \text{for} \quad r \ge R. \end{cases}$$

Since $r\cos\theta = z$, the *field* inside the sphere is *uniform*:

$$\mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{1}{3\epsilon_0}\mathbf{P}, \quad \text{for} \quad r < R.$$

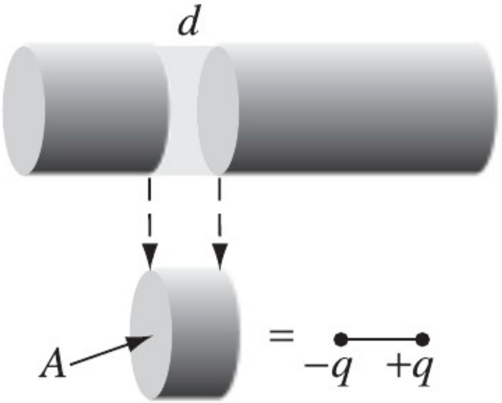
Physical interpretation of Bound Charges:

Say, we have a string of dipoles made up of same amount of +ve and -ve charges.

→ Bound Charges

To calculate the amount of bound charges resulting from a polarization

→ A tube of dielectric parallel to the polarization vector



The dipole moment of the tiny chunk shown

→ $P(Ad)$, where A is the cross-sectional area of the tube and d is the length of the chunk.

→ In terms of the charge (q) at the end, this same dipole moment can be written qd.

The bound charge that piles up at the right end of the tube is $q = PA$

Bound Charges

If the ends have been sliced off perpendicularly, the surface charge density is $\sigma_b = \dfrac{q}{A} = P$

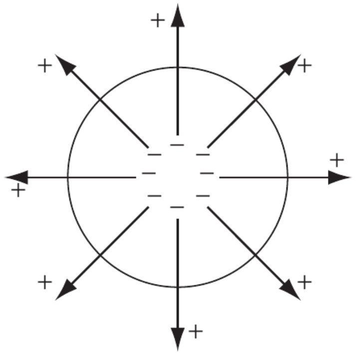
For an oblique cut the *charge* is still the same, but $A = A_{\text{end}} \cos\theta$, so

$$A = A_{\text{end}} \cos\theta$$

$$\sigma_b = \frac{q}{A_{\text{end}}} = P\cos\theta = \mathbf{P}\cdot\hat{\mathbf{n}}$$

The effect of the polarization, then, is to paint a bound charge $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$ over the surface of the material.

→ If the polarization is nonuniform, we get accumulations of bound charge *within* the material, as well as on the surface.



Bound Charges

Ex: Prove that the total bound charge is zero.

Total charge $$Q = \oint\limits_{surf} \sigma_b \, da' + \int\limits_{vol} \rho_b \, d\tau'$$

$$= \oint\limits_{surf} \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} da' - \int\limits_{vol} \mathbf{\nabla}' \cdot \mathbf{P}(\mathbf{r}') d\boldsymbol{\tau}'$$

$$= \oint\limits_{surf} \mathbf{P}(\mathbf{r}') \cdot d\mathbf{a}' - \int\limits_{vol} \mathbf{\nabla}' \cdot \mathbf{P}(\mathbf{r}') d\boldsymbol{\tau}'$$

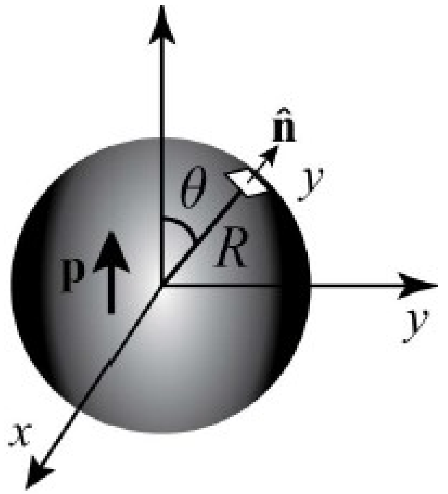
$$= \int\limits_{vol} \mathbf{\nabla}' \cdot \mathbf{P}(\mathbf{r}') d\boldsymbol{\tau}' - \int\limits_{vol} \mathbf{\nabla}' \cdot \mathbf{P}(\mathbf{r}') d\tau' = 0$$

Bound Charges

Ex: Find the electric field of a sphere of radius R when it is uniformly polarized $\mathbf{P}(\mathbf{r}') = P\,\hat{\mathbf{z}}$.

$$\rho_b = -\nabla' \cdot \mathbf{P}(\mathbf{r}') = -\nabla' \cdot (P\,\hat{\mathbf{z}}) = 0$$

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} = P\cos\theta$$



The potential due to a uniformly polarized sphere is equal to the potential due to a spherical surface charge density $\sigma_b = P\cos\theta$

For $r \geq R$
$$V(r,\theta) = \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta = \frac{R^3}{3\epsilon_0}\frac{\mathbf{P}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{3\epsilon_0}\frac{\left(\frac{4\pi}{3}\right)R^3}{\left(\frac{4\pi}{3}\right)}\frac{\mathbf{P}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$\boxed{\mathbf{p} = \left(\frac{4\pi}{3}\right)R^3\mathbf{P}}$$

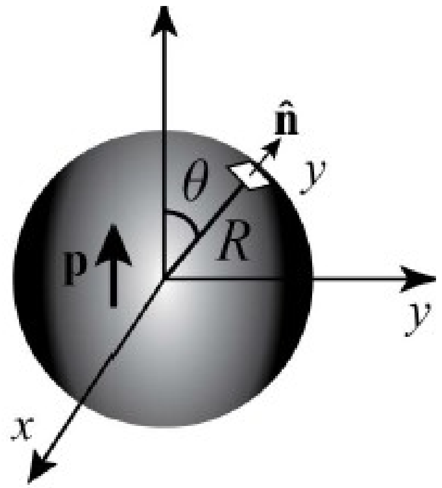
total dipole moment of the sphere

$$\Rightarrow \quad \mathbf{E} = -\nabla V = \mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

For $r \leq R$
$$V(r,\theta) = \frac{P}{3\epsilon_0}r\cos\theta \Rightarrow \mathbf{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{\mathbf{z}} = -\frac{p}{4\pi\epsilon_0 R^3}\hat{\mathbf{z}}$$

The Field of a Polarized Object:

Find the bound charges of a sphere of radius R when it is non-uniformly polarized, $\mathbf{P}(\mathbf{r}') = k\mathbf{r}$.



Volume charge

$$\rho_b = -\nabla' \cdot \mathbf{P}(\mathbf{r}')$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) \quad = -\frac{1}{r^2}3kr^2 \quad = -3k$$

Surface charge

$$\sigma_b = \mathbf{P}(\mathbf{r}') \cdot \widehat{\mathbf{n}'} \qquad = kR$$

Q: What is the electric field outside the sphere?

Volume and surface charge distributions are both symmetric with respect to the center of the sphere. So, the total charge can be thought of as being concentrated at the center

$$\text{Total charge } Q = \oint_{surf} \sigma_b \, da' + \int_{vol} \rho_b \, d\tau'$$

$$= kR \times 4\pi R^2 \qquad +(-3k) \times \frac{4\pi}{3} R^3$$

$$= 0$$

So the electric field outside the sphere is zero.

Gauss' Law in Presence of Dielectrics

The net effect of Polarization is the accumulation of bound charges.

$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}} \qquad\qquad \rho_b \equiv -\nabla \cdot \mathbf{P}$$

These bound charges give rise to electric field.

→ Hence with in a dielectric, the total electric field = *Field due to the free charges*

\+

Field due to bound charges

→ The total charge density, $\rho = \rho_b + \rho_f$

→ Using Gauss' law, $\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$

where $\mathbf{E}$ is now the *total* field, not just that portion generated by polarization.

Hence,

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

Gauss' Law in Presence of Dielectrics

We write, $\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$

Electric Displacement

In terms of $\mathbf{D}$, Gauss's law reads

$$\nabla \cdot \mathbf{D} = \rho_f$$

Hence, the integral form:

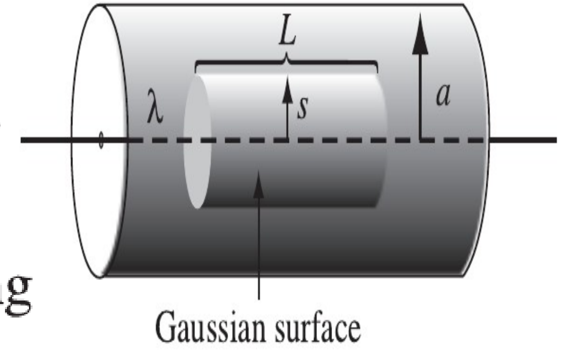
$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

$Q_{f_{\text{enc}}}$ denotes the total free charge enclosed in the volume.

This is particularly useful since we only need to know the free charge in the system, which is something we can control. The bound charges appear only after polarization. In a typical problem, initially we just start with the free charge densities.

Electric Displacement

A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a. Find the electric displacement.



Drawing a cylindrical Gaussian surface, of radius s and length L, and applying

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}} \longrightarrow D(2\pi s L) = \lambda L$$

Hence,

$$\mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}$$

Notice that this formula holds both within the insulation and outside it. In the latter region, $\mathbf{P} = 0$, so

$$\mathbf{E} = \frac{1}{\epsilon_0}\mathbf{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}, \qquad \text{for } s > a.$$

Inside the rubber, the electric field cannot be determined, since we do not know $\mathbf{P}$.

Linear Dielectrics

Linear dielectrics is a class of dielectrics for which the induced polarization is proportional to the applied electric field.

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}.$$

$\chi_e \longrightarrow$ **Electric susceptibility** of the medium; depends on microscopic structure of the substance in question and also on external conditions such as temperature.

In linear media we have

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E},$$

so $\mathbf{D}$ is *also* proportional to $\mathbf{E}$:

$$\mathbf{D} = \epsilon \mathbf{E},$$

where

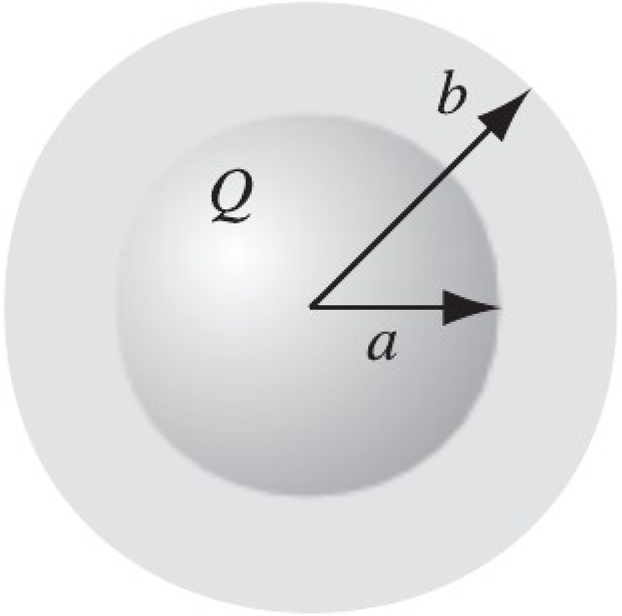
Permittivity of free space

$$\epsilon \equiv \epsilon_0 (1 + \chi_e).$$

Permittivity of a material

(Dielectric Constant) $\epsilon_r \equiv 1 + \chi_e = \dfrac{\epsilon}{\epsilon_0}$

A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).



→ let's begin by calculating $\mathbf{D}$,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for all points } r > a.$$

Then,

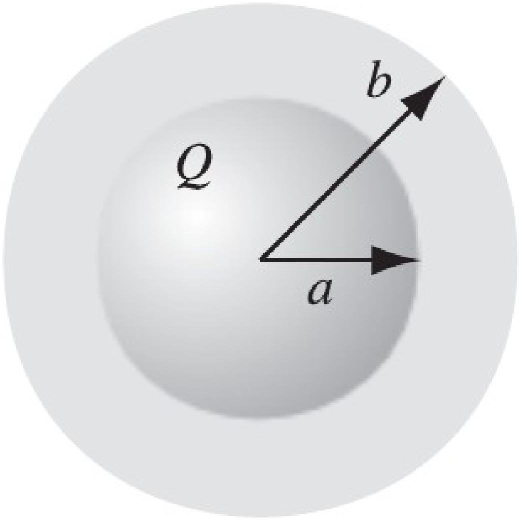
$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\[2ex] \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

The potential at the center is therefore

$$V = -\int_{\infty}^{0} \mathbf{E} \cdot d\mathbf{l} = -\int_{\infty}^{b} \left(\frac{Q}{4\pi\epsilon_0 r^2}\right) dr - \int_{b}^{a} \left(\frac{Q}{4\pi\epsilon r^2}\right) dr - \int_{a}^{0} (0)\, dr$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right).$$

→ *To find the bound charges:*



$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = \frac{\epsilon_0 \chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}},$$

in the dielectric, and hence

$$\rho_b = -\nabla \cdot \mathbf{P} = 0,$$

while

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0 \chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface,} \\[2ex] \dfrac{-\epsilon_0 \chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface.} \end{cases}$$

(unit vector points outward with respect to the dielectrics)

Linear Dielectrics

For electric displacement vector,

$$\nabla \cdot \mathbf{D} = \rho_f \quad \text{and} \quad \nabla \times \mathbf{D} = \mathbf{0}$$

→ $\mathbf{D}$ can be found from the free charge just as though the dielectric were not there:

$$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{vac}}$$

where $\mathbf{E}_{\text{vac}}$ is the field the same free charge distribution would produce in the absence of any dielectric.

→ Inside linear dielectric,

$$\mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \frac{1}{\epsilon_r}\mathbf{E}_{\text{vac}}$$

Conclusion: When all space is filled with a homogeneous linear dielectric, the field everywhere is simply reduced by a factor of one over the dielectric constant.

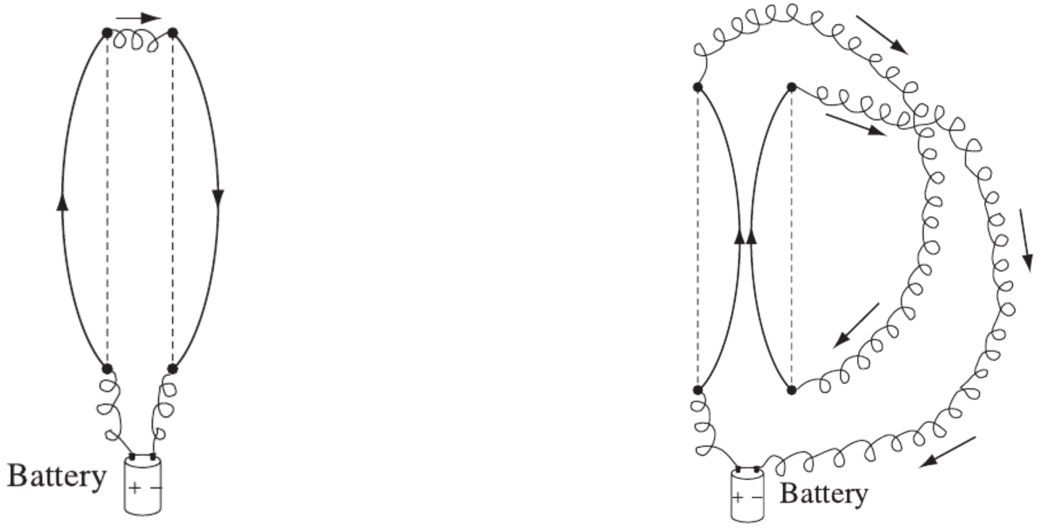
For example, if a free charge q is embedded in a large dielectric, the field it produces is

$$\mathbf{E} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{\mathbf{r}} \quad (\text{that's } \epsilon, \text{ not } \epsilon_0)$$

Magnetic Field

Up to this point, we only have been considering static electric charges. Now we will study the force between charges in motion.

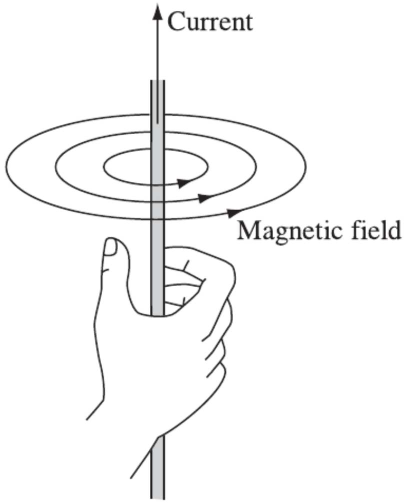
→ Imagine two wires hanging from the ceiling. Current passes up on one wire, and back down on the other. we see that the wires repel each other. If the current passes in the same direction in both wires, they are pulled closer.



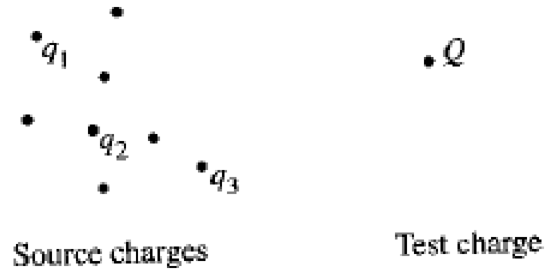
(a) Currents in opposite directions repel.

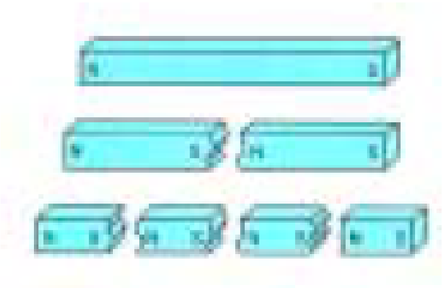
(b) Currents in same directions attract.

Whatever force accounts for the attraction of parallel currents and the repulsion of antiparallel ones is *not* electrostatic in nature. It is our first encounter with a *magnetic* force. Whereas a *stationary* charge produces only an electric field **E** in the space around it, a *moving* charge generates, in addition, a magnetic field **B**.



Now, if you hold up a tiny compass in the vicinity of a current-carrying wire, you quickly discover a very peculiar thing: The field does not point *toward* the wire, nor *away* from it, but rather it *circles around the wire*. In fact, if you grab the wire with your right hand—thumb in the direction of the current—your fingers curl around in the direction of the magnetic field





If one try to isolate the poles by cutting the magnet, a curious thing happens: One obtains two magnets. No matter how thinly the magnet is sliced, each fragment always have two poles. Even down to the atomic level, no one has found an isolated magnetic pole, called a monopole. Thus magnetic field lines form closed loops.

Magnetostatics

Stationary charges $\longrightarrow$ Constant Electric field $\longrightarrow$ Electrostatics

Steady currents $\longrightarrow$ Constant Magnetic field $\longrightarrow$ Magnetostatics

Force on a point charge *Q*:

Electric Force: $\mathbf{F}_{\mathrm{elec}} = Q\mathbf{E}$

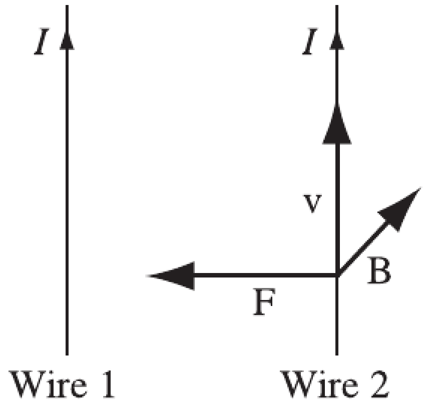
Magnetic Force: $\mathbf{F}_{\mathrm{mag}} = Q(\mathbf{v} \times \mathbf{B})$ **Lorentz Force Law**

Total Force: $\boxed{\mathbf{F} = \mathbf{F}_{\mathrm{elec}} + \mathbf{F}_{\mathrm{mag}} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$

Magnetic Force

Lorentz force explains the observation with two parallel current carrying wires

→ Current in same direction; magnetic field into the page as shown; Attractive force between the wires.



Work done by Magnetic Force

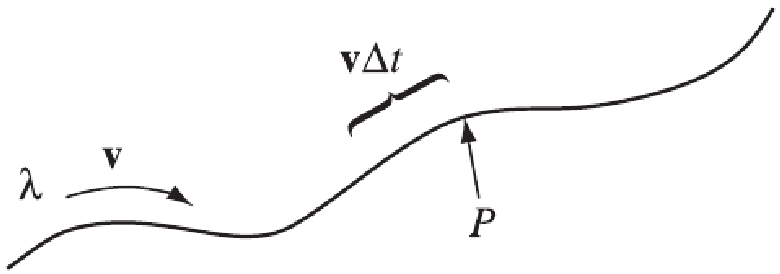
$$\mathbf{W}_{\mathrm{mag}} = \int \mathbf{F}_{\mathrm{mag}} \cdot d\mathbf{l} = \int Q(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = \int Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt$$

$$= 0$$

- Magnetic forces do no work
- Magnetic forces can change the direction in which a particle moves.
- Magnetic forces do not change the speed with which a particle moves.

Current

- Current is charge flow per unit time $I = \frac{dq}{dt}$
- It is measured in Coulombs per second, or Amperes (A).



Charge flowing in a wire is described by **Current**

$$\mathbf{I} = \frac{dq}{dt} = \frac{\lambda\, d\mathbf{l}}{dt} = \lambda \mathbf{v}$$

- The direction of current is in the direction of charge-flow.

- Conventionally, this is the direction opposite to the flow of electrons.

$$\boxed{\mathbf{F}_{\text{mag}} = \int (\mathbf{I} \times \mathbf{B})\, dl = I \int (d\mathbf{l} \times \mathbf{B})}$$

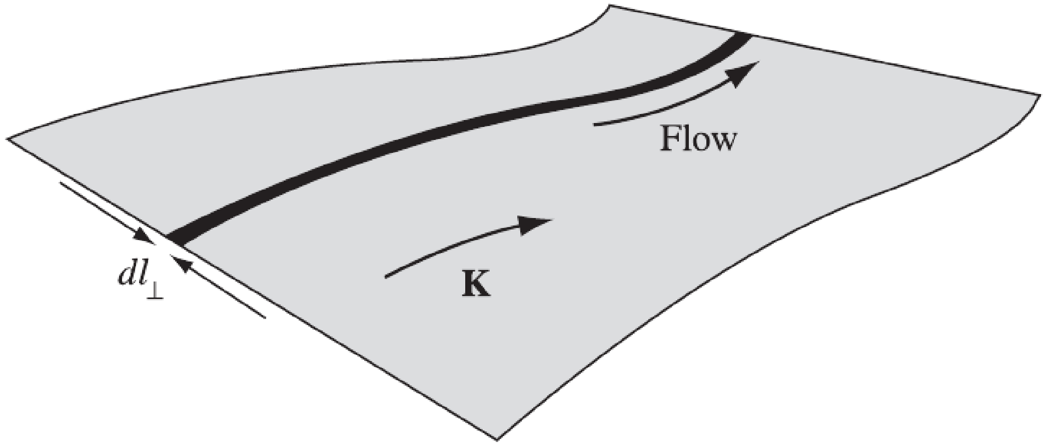
Magnetic force on a current carrying wire:

$$\mathbf{F}_{\text{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{v} \times \mathbf{B})\, dq = \int (\mathbf{v} \times \mathbf{B})\, \lambda dl = \int (\mathbf{I} \times \mathbf{B})\, dl$$

Current

Charge flowing on a surface is described by **surface current density**

$$\mathbf{K} = \frac{d\mathbf{I}}{dl_\perp} = \sigma \mathbf{v}$$



- Current density is a vector quantity.

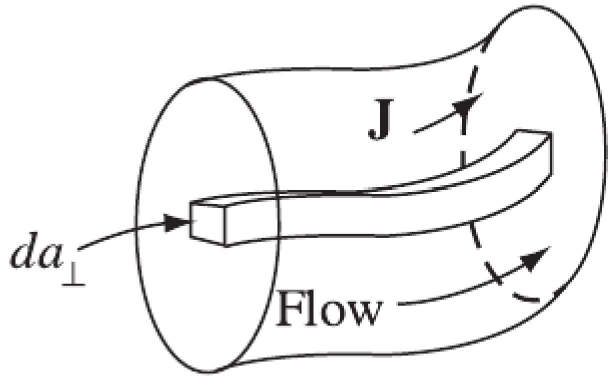
Magnetic force on the surface current:

$$\mathbf{F}_{\text{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{v} \times \mathbf{B})\, \sigma da = \int (\mathbf{K} \times \mathbf{B})\, da$$

Current

Charge flowing in a volume is described by **volume current density**

$$\mathbf{J} = \frac{d\mathbf{I}}{da_\perp} = \rho\mathbf{v}$$



- Current density is a vector quantity.

Magnetic force on the volume current:

$$\mathbf{F}_{\mathbf{mag}} = (\mathbf{v} \times \mathbf{B})Q = \int (\mathbf{v} \times \mathbf{B})\, \rho\, d\tau = \int (\mathbf{J} \times \mathbf{B})\, d\tau$$

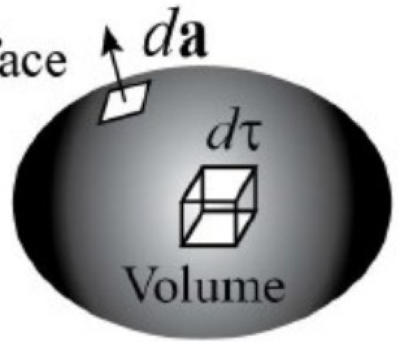
Steady Current

→ A continuous flow of charge that has been going on forever without change or charges piling up anywhere

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial \mathbf{J}}{\partial t} = \mathbf{0}$$

(Results in constant magnetic field)

Continuity Equation



Starting from the volume current density definition, $\mathbf{J} \equiv \dfrac{d\mathbf{I}}{da_\perp}$

→ the total current crossing a surface $\mathcal{S}$ can be written as

$$I = \int_{\mathcal{S}} J\, da_\perp = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a}.$$

(The dot product serves neatly to pick out the appropriate component of $d\mathbf{a}$.) In particular, the charge per unit time leaving a volume $\mathcal{V}$ is

$$\oint_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau.$$

Because charge is conserved, whatever flows out through the surface must come at the expense of what remains inside:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{J})\, d\tau = -\frac{d}{dt}\int_{\mathcal{V}} \rho\, d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}\right) d\tau.$$

(The minus sign reflects the fact that an *outward* flow *decreases* the charge left in $\mathcal{V}$.) Since this applies to *any* volume, we conclude that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

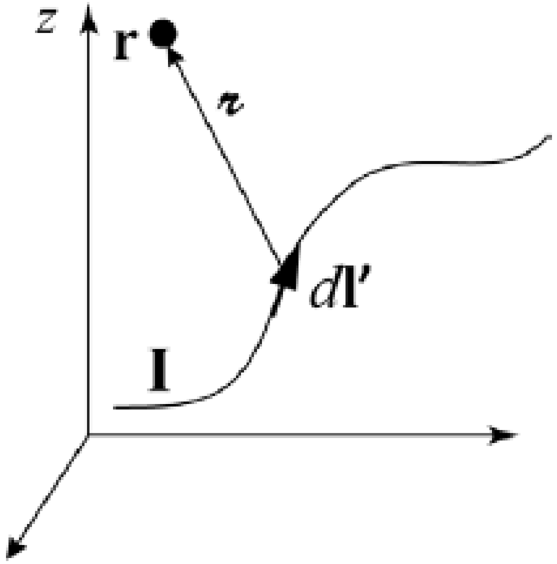
This is the precise mathematical statement of local charge conservation; it is called the **continuity equation**

Magnetic Field due to Steady Current

The Biot-Savart Law

The magnetic field produced by a steady line current

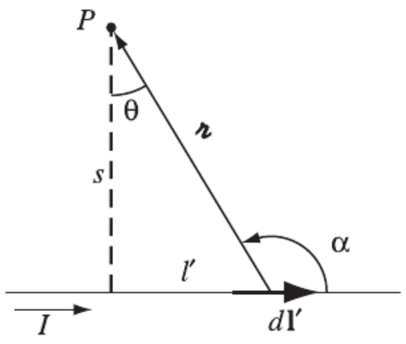
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

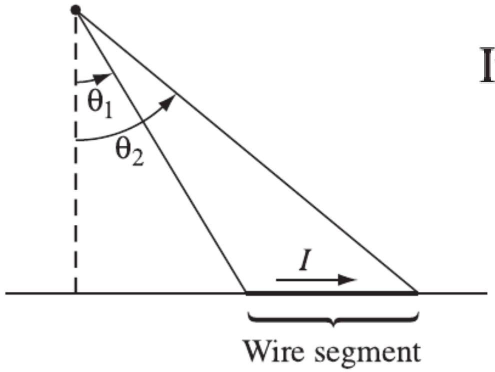


- μ_0 is the permeability of free space
- $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$
- The unit of magnetic field is Newton per Ampere-meter, or Tesla
- 1 Tesla is a very strong magnetic field. Earth's magnetic field is about 10^{-4} times smaller
- Biot-Savart law for magnetic field is analogous to Coulomb's law for electric field

Application of Biot-Savart Law

Find the magnetic field a distance s from a long straight wire carrying a steady current I





In the diagram, $(d\mathbf{l}' \times \hat{\mathscr{r}})$ points *out* of the page, and has the magnitude

$$dl' \sin\alpha = dl' \cos\theta .$$

Also, $l' = s\tan\theta$, so $\quad dl' = \dfrac{s}{\cos^2\theta}\,d\theta,$

and $s = \mathscr{r}\cos\theta$, so $\quad \dfrac{1}{\mathscr{r}^2} = \dfrac{\cos^2\theta}{s^2}.$

Thus

$$B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\,d\theta$$

$$= \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1).$$

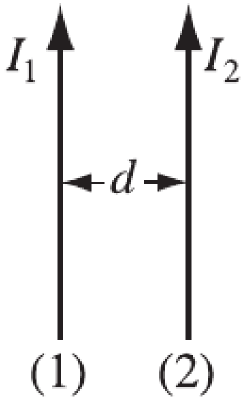
In the case of an *infinite* wire,

→ $\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$, so we obtain

$$B = \frac{\mu_0 I}{2\pi s}.$$

Application of Biot-Savart Law

For an infinitely long current carrying wire the resultant magnetic field is $B = \dfrac{\mu_0 I}{2\pi s}$.



As an application, let's find the force of attraction between two long, parallel wires a distance d apart, carrying currents I_1 and I_2

The field at (2) due to (1) is $B = \dfrac{\mu_0 I_1}{2\pi d}$,

and it points into the page. The Lorentz force law predicts a force directed towards (1), of magnitude

$$F = I_2 \left(\frac{\mu_0 I_1}{2\pi d} \right) \int dl.$$

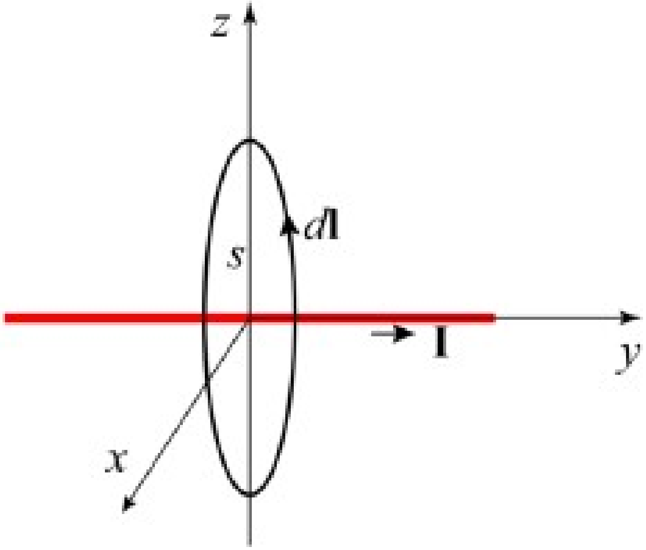
The *total* force, not surprisingly, is infinite, but the force per unit length is $f = \dfrac{\mu_0}{2\pi} \dfrac{I_1 I_2}{d}$.

→ If the currents are antiparallel (one up, one down), the force is repulsive

Divergence and Curl of Magnetic Field

What is the divergence of **B** ? $\nabla \cdot \mathbf{B} = 0$

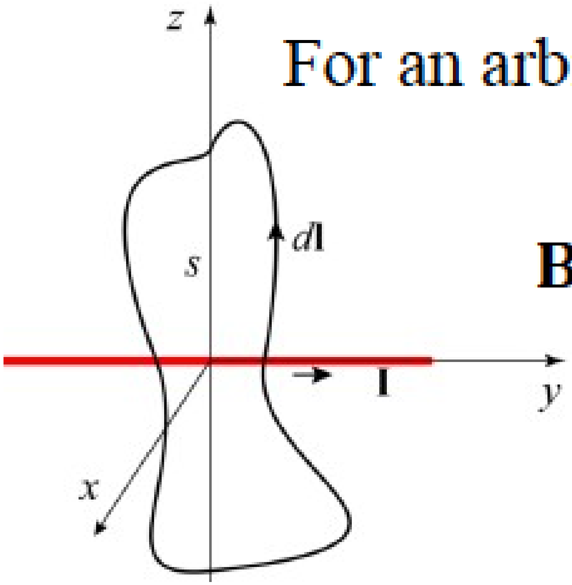
What is the curl of **B** ? Should be $\nabla \times \mathbf{B} \neq \mathbf{0}$



$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s} \oint dl = \frac{\mu_0 I}{2\pi s} 2\pi s = \mu_0 I$$

The line integral is independent of *s*

$$\boxed{B(s) = \frac{\mu_0 I}{2\pi s}}$$



For an arbitrary path enclosing the current carrying wire

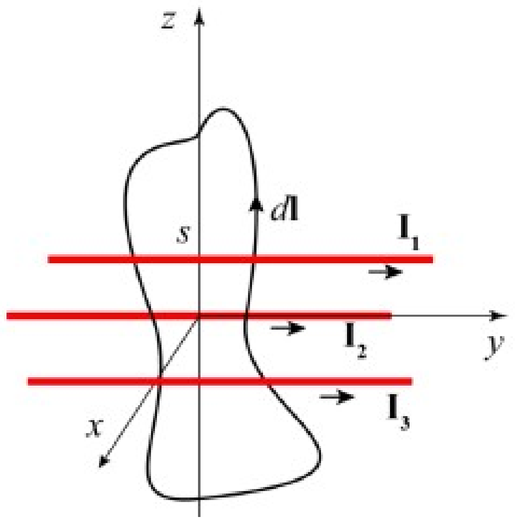
cylindrical coordinate

$$\mathbf{B}(s) = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \qquad d\mathbf{l} = ds\,\hat{\mathbf{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}} \cdot (ds\,\hat{\mathbf{s}} + s d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}) = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I$$

Divergence and Curl of Magnetic Field

If the path encloses more than one current carrying wire



$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_1 + \mu_0 I_2 + \mu_0 I_3 = \mu_0 I_{\text{enc}}$$

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}$$

Hence,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} \Rightarrow \boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

It is valid in general

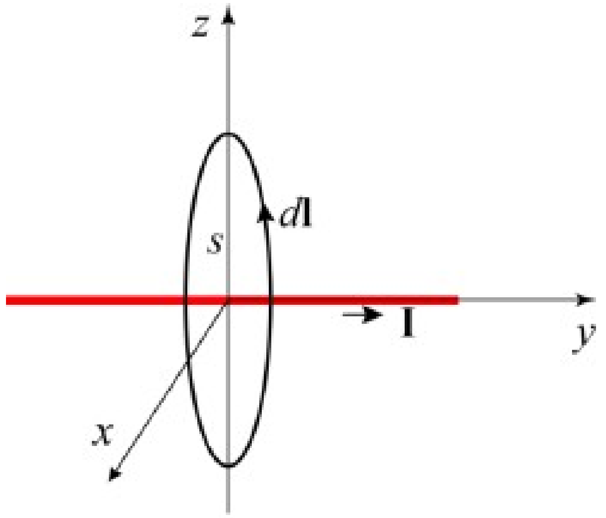
The Ampere's Law

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ Ampere's law in differential form

$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$ Ampere's law in integral form

Ampere's Law

- Ampere's law is analogous to Gauss's law
- Ampere's law makes the calculation of magnetic field very easy if there is symmetry.
- If there is no symmetry, one has to use Biot-Savart law to calculate the magnetic field.



→ *Application of Ampere's Law:* Calculate the magnetic field due to an infinitely long straight wire carrying a steady current I .

Make an Amperian loop of radius s enclosing the current

Since it is an infinite wire, the magnetic field must be circularly symmetric. Therefore,

$$\oint \mathbf{B} \cdot d\mathbf{l} \quad = B \oint dl \quad = B 2\pi s \quad = \mu_0 I_{\text{enc}} \quad = \mu_0 I$$

Hence, $$\boxed{B = \frac{\mu_0 I}{2\pi s}}$$

Magnetostatics and Electrostatics

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,\rho(\mathbf{r}')d\tau$$

Coulomb's Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau'$$

Biot-Savart Law

$$\mathbf{F}_{\mathrm{elec}} = Q\mathbf{E}$$

Electric Force

$$\mathbf{F}_{\mathrm{mag}} = Q(\mathbf{v} \times \mathbf{B})$$

Magnetic Force

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Gauss's Law

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

No Name

$$\boldsymbol{\nabla} \times \mathbf{E} = 0$$

No Name

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0\mathbf{J}$$

Amperes's Law

Vector Potential

If the divergence of a vector field $\mathbf{F}$ is zero everywhere, $(\nabla \cdot \mathbf{F} = 0)$, then:

(1) $\int \mathbf{F} \cdot d\mathbf{a}$ is independent of surface.

(2) $\oint \mathbf{F} \cdot d\mathbf{a} = 0$ for any closed surface.

This is because of the divergence theorem

$$\int_{Vol} (\nabla \cdot \mathbf{F}) d\tau = \oint_{Surf} \mathbf{F} \cdot d\mathbf{a}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$\mathbf{F}$ is the curl of a vector function: $\mathbf{F} = \nabla \times \mathbf{A}$

Magnetic Vector Potential

The vector potential is not unique. A gradient ∇V of a scalar function can be added to $\mathbf{A}$ without affecting the curl, since the curl of a gradient is zero.

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Vector Potential

What happens to the Ampere's Law ?

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

$$\Rightarrow \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

- This is not in a very nice form.
- Ampere's law in terms of **B** seems better

- However, if we can ensure that $\nabla \cdot \mathbf{A} = 0$, we can have it in a nice form.
- This can be done since we know that a $\nabla\lambda$ can be added to **A** without changing **B**

Suppose we start with $\mathbf{A}_0$, such that, $\mathbf{B} = \nabla \times \mathbf{A}_0$ but, $\nabla \cdot \mathbf{A}_0 \neq \mathbf{0}$.

Then, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \nabla(\nabla \cdot \mathbf{A}_0) - \nabla^2 \mathbf{A}_0 = \mu_0 \mathbf{J}$

Re-define by adding $\nabla\lambda$: $\mathbf{A}_0 + \nabla\lambda \equiv \mathbf{A}$ such that $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2 \lambda = 0$

Then $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Rightarrow \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \Rightarrow -\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$

Thus, one can always redefine the vector potential such that $\nabla \cdot \mathbf{A} = 0$

Vector Potential

Recall: $\nabla^2 V = -\dfrac{\rho}{\epsilon_0}$ (Poisson's Equation)

The solution is: $V(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau'$

So, $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau'$ This is simpler than Biot-Savart Law.

For surface current: $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}}\,da'$

For line current: $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} I \displaystyle\int \frac{d\mathbf{l}'}{\mathscr{r}}$

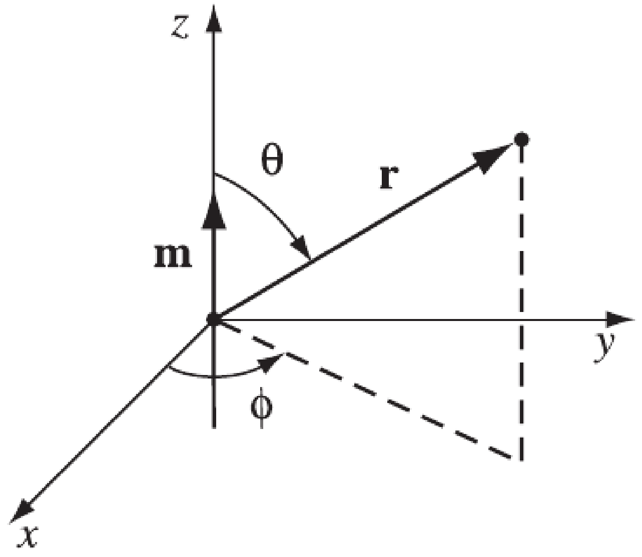
Magnetic Dipole

Magnetic field due to a magnetic dipole

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

$$\boxed{\mathbf{m} \equiv I\int d\mathbf{a}'}$$

Magnetic dipole moment



Take $\mathbf{m} = m\,\hat{\mathbf{z}}$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}}$$

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla\times\mathbf{A}_{\text{dip}}(\mathbf{r})$$

$$= \frac{\mu_0}{4\pi}\frac{m}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Recall

$\mathbf{p} = p\hat{\mathbf{z}}$

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

Magnetization

What is Magnetization? - magnetic dipole moment per unit volume

- The magnetic dipole moment is caused by electric charges in motion:
  (i) electrons orbiting around nuclei & (ii) electrons spinning about their own axes.
- In some material, magnetization is in the direction parallel to **B** (Paramagnets).
- In some other material, magnetization is opposite to **B** (Diamagnets).
- In other, there can be magnetization even in the absence of **B** (Ferromagnets).

Ampere's Law in a magnetised material

$$\mathbf{J}_b(\mathbf{r}') = \nabla' \times \mathbf{M}(\mathbf{r}')$$

Bound Volume Current Density

$$\mathbf{K}_b(\mathbf{r}') = \mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}$$

Bound Surface Current Density

Total volume current is

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$$

Free Volume Current Density

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad = \mu_0(\mathbf{J}_f + \mathbf{J}_b) \quad = \mu_0(\mathbf{J}_f + \nabla \times \mathbf{M})$$

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}_f$$

Define: $\mathbf{H} \equiv \dfrac{\mathbf{B}}{\mu_0} - \mathbf{M}$ $\qquad \mathbf{B} \equiv \mu_0(\mathbf{H} + \mathbf{M})$

Auxiliary Field

$$\nabla \times \mathbf{H} = \mathbf{J}_f$$

Ampere's law in magnetized material (differential form)

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}}$$

Ampere's law in magnetized material (integral form)

Magnetic Susceptibility and Permeability

In a Linear media,

$$\mathbf{M} = \chi_m \mathbf{H}$$

χ_m is called the magnetic susceptibility

χ_m is a dimensionless quantity

χ_m is positive for paramagnetic and negative for diamagnets

The magnetic field thus becomes

$$\mathbf{B} \equiv \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(\mathbf{H} + \chi_m \mathbf{H}) = \mu_0(1 + \chi_m)\mathbf{H}$$

So, $\mathbf{B} \equiv \mu \mathbf{H}$

$\mu \equiv \mu_0(1 + \chi_m)$
is called the permeability of the material