2024 November.

Q5) a)

I) Stochastic Process.

A stochastic process is a collection $\{x_t : t \in T\}$ of random variables indexed by time or space. Each X_t describes the random state of the system at time t.

II) Queuing theory

Queuing theory models systems where items arrive, wait, get served, and depart. Specify arrival rates, service rates, queue discipline like FIFO, then use mathematical model to compute performance measures such as average waiting time, average queue length.

b)

$$P' = P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

and initial probability of $\pi(0) = p^{\circ} = (0.7 \quad 0.2 \quad 0.1)$

I) $P(x_2 = 3)$

$$p^{(1)} = p^{(0)} \times P$$

$$= (0.7 \quad 0.2 \quad 0.1) \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= (0.22 \quad 0.43 \quad 0.35)$$

$p^{(2)} = p^{(1)} \times P$

$$= \begin{pmatrix} 0.22 & 0.43 & 0.35 \end{pmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= \begin{pmatrix} 0.385 & 0.336 & 0.279 \end{pmatrix}$$

$P(x_2 = 3) = 0.279$

or we can use $\pi(2) = \pi(0) \times P^2$.

II) $P(X_2 = 3 \mid x_1 = 2)$

1 step

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$$

$P(X_2 = 3 \mid x_1 = 2) = 0.2$

III) $P(x_2 = 3 \mid x_0 = 1)$

2 steps.

$$p^{(2)} = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}.$$

$$= \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$P(x_2 = 3 \mid x_0 = 1) = 0.26$

c) I) Let X_n be position of the walker of the n^{th} step,

Each step the walker moves +1 with probability p or -1 with probability $q = 1-p$.

walk starts from the origin, $X_0 = 0$ after n steps,

$$X_n = \sum_{i=1}^{n} W_i \quad ; \quad W_i \text{ is the step at time } i$$

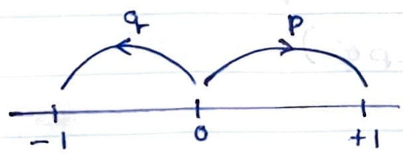
$$W_i = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } q = 1-p \end{cases}$$

when walker not starts from origin,

$$X_n = X_0 + \sum_{i=1}^{n} W_i$$

final position   initial prosition.

II)



if $X_0 = 0$,

$$E(X_n) = E\left(X_0 + \sum_{i=1}^{n} W_i\right)$$

$$= 0 + \sum_{i=1}^{n} W_i$$

$$= \sum_{i=1}^{n} \left(p \times 1 + q \times (-1)\right)$$

$$= \sum_{i=1}^{n} p - q$$

$$E(X_n) = n(p-q)$$

$$Var(X_n) = \sum_{i=1}^{n} \{ E(X_n^2) - [E(X_n)]^2 \}$$

$$= \sum_{i=1}^{n} E(X_n^2) - \sum_{i=1}^{n} [E(X_n)]^2$$

$$= \sum_{i=1}^{n} [p \times 1^2 + q \times (-1)^2] - n(p-q)^2$$

$$= \sum_{i=1}^{n} (p+q) - n(p-q)^2$$

$$= n(p+q - p^2 - q^2 + 2pq)$$

$$p + q = 1$$

$$(p+q)^2 = 1$$

$$p^2 + q^2 + 2pq = 1$$

$$2pq = 1 - p^2 - q^2$$

$$-p^2 - q^2 = 2pq - 1$$

$$Var(X_n) = n(\underbrace{p+q}_{1} \underbrace{-p^2-q^2}_{2pq-1} + 2pq)$$

$$= n(1 + 2pq - 1 + 2pq)$$

$$= 4pqn$$

$$Var(X_n) = 4npq$$

III) $\mu = n(p-q)$

$\sigma = \sqrt{4npq}$

we have to find, $p(35 \le x \le 45)$

$n = 100$,

$p = 0.7$,

$q = 0.3$

$\mu = 100(0.7 - 0.3)$

$= 40$

$\sigma = \sqrt{4npq}$

$= \sqrt{4 \times 100 \times 0.7 \times 0.3}$

$= 9.165$

$X_n \approx N(40, 9.165)$

$P(35 - 0.5 \le X \le 45 + 0.5)$

$P(34.5 \le X \le 45.5)$

$P\left(\frac{34.5 - 40}{9.165} \le Z_n \le \frac{45.5 - 40}{9.165}\right)$

$P(-0.6 \le Z_n \le 0.6)$

$P(Z < 0.6) = 0.7257 \rightarrow$ from Z table

$\therefore P(35 \le X_{100} \le 45) = 0.7257 - (1 - 0.7257)$

$= \underline{\underline{0.4514}}$

2024 January

Q5) a) I) Stochastic process.

A stochastic process is a collection $\{X_t : t \in T\}$ of random variable indexed by time or space. Each X_t describes the random state of the system at time t.

II) Markov property

A process has the Markov property if the future state depends only on the present state and not on the past state. This means the process is memory less.

b) I) State space : The set of four transaction types.

$S_x = \{\text{type } 1, \text{ type } 2, \text{ type } 3, \text{ type } 4\}$

Parameter space : $n = 0, 1, 2 \ldots$ counting number of transaction submitted.

$T_x = \{0, 1, 2, 3 \ldots\}$

II) state space $= \{0, 1, 2 \ldots N\}$

N is the maximum capacity of the garage.

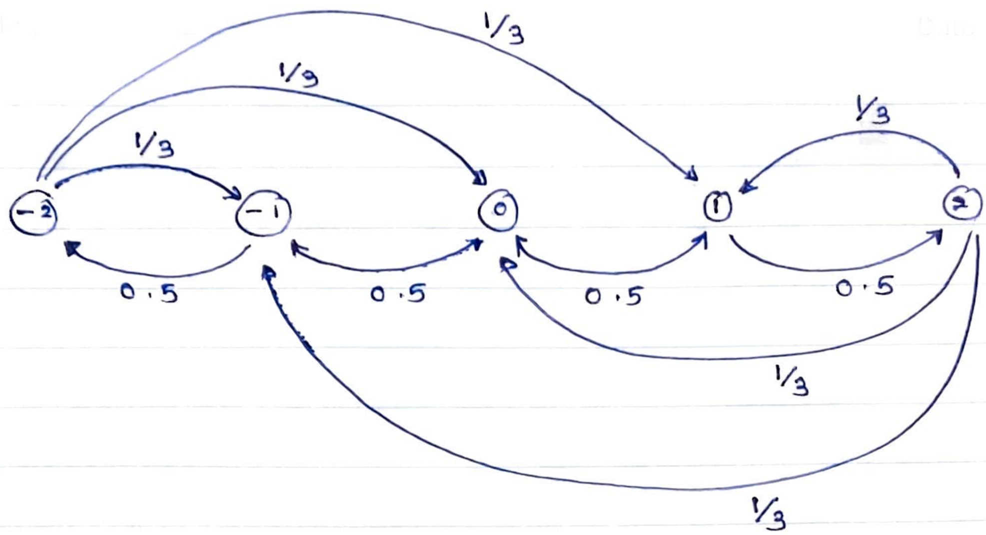
parameter space $= t$ ($t > 0$ parking time in hours).

c) I) Finite states of random walk : $\{-2, -1, 0, 1, 2\}$

if in state $i = -1, 0, 1$ it moves to $i-1$ or $i+1$ with probability 0.5 each

if in state -2, it can move to ~~only -1~~ $-1, 0, 1$ with probability $1/3$

if in state 2, it can move to ~~only 1~~ $-1, 0, 1$ with probability $1/3$



II) $P = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$

III) $P^2 = P \cdot P$

$$= \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & 1/6 & 1/3 & 1/6 & 1/6 \\ 0 & 5/12 & 1/6 & 5/12 & 0 \\ 1/4 & \cancel{1/4}\,0 & 1/2 & 0 & 1/4 \\ 0 & 5/12 & 1/6 & 5/12 & \cancel{1/4}\,0 \\ 1/6 & 1/6 & 1/3 & 1/6 & 1/6 \end{bmatrix}$$