UNIVERSITY OF RUHUNA

Faculty of Engineering

End-Semester 7 Examination in Engineering: November 2022

Module Number: EE6302 **Module Name: Control System Design**

Model Answers

Q3. a) i) A graphical representation of the closed-loop poles on the S-plane as a system parameter is varied.

[2.0 Marks]

ii) The root locus can be used to describe qualitatively the performance of a system as various parameters are changed. For example, the effect of gain on percent overshoot, settling time, and peak time. Besides transient response, the root locus also gives a graphical representation of a system's stability.

[1.5 Mark]

iii)

$$G_{OL}(s) = \frac{KN(s)}{D(s)}$$

$$G_{CL}(s) = \frac{G_{OL}(s)}{1+G_{OL}(s)} = \frac{\frac{KN(s)}{D(s)}}{1+\frac{KN(s)}{D(s)}} = \frac{KN(s)}{D(s)+KN(s)}$$

The zeros of the closed-loop transfer function, $G_{OL}(s)$:

$$K\ N(s) = 0$$

The zeros of open-loop transfer function $G_{CL}(s)$.

$$K\ N(s) = 0$$

∴ the zeros of the closed-loop transfer function equal to the zeros of open-loop transfer function.

[1.5 Mark]

b) i) Open-loop poles = $-15, -2.5 \pm 0.866j$

There are no open-loop zeros.

$$G(s) = \frac{1}{(s+15)(s+2.5-0.866j)(s+2.5+0.866j)}$$

$$G(s) = \frac{1}{(s+15)((s+2.5)^2-(0.866j)^2)}$$

$$G(s) = \frac{1}{(s+15)(s^2+5s+6.25+0.75)}$$

$$G(s) = \frac{1}{(s+15)(s^2+5s+7)}$$

[1.5 Mark]

ii) Properties of the root locus

$$\sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = 180^o(2h+1); \qquad h = 0, \pm1, \pm2, \ldots$$

Open-loop poles = $-15, -2.5 \pm 0.866j$

There are no open-loop zeros.

$L.H.S = -\angle(s+15) - \angle(s+2.5+0.866j) - \angle(s+2.5+0.866j)$

Substitute $s = -2 + 3.75j$

$$L.H.S = -\angle(-2+3.75j+15) - \angle(-2+3.75j+2.5+0.866j) - \angle(-2 + 3.75j + 2.5 - 0.866j)$$

$L.H.S = -\angle(13+3.75j) - \angle(0.5+4.616j) - \angle(0.5+2.884j)$

$L.H.S = -16.09^o - 83.82^o - 80.16^0$

$L.H.S = \ -180.07^o$

$$\sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = 180^o(2h+1); \qquad h = 0, \pm1, \pm2, \ldots$$

is satisfied. Therefore, $s = -2 + 3.75j$ is on the root locus.

[1.0 Mark]

iii) Root locus: $1 + KG(s) = 0$

$$1 + K\frac{1}{(s+15)(s^2+5s+7)} = 0$$

$(s+15)(s^2+5s+7) + K = 0$

$s^3 + 20s^2 + 82\,s + 105 + K = 0$

On the imaginary axis, $s = j\omega$

$1 + KG(j\omega) = 0$

$(j\omega)^3 + 20(j\omega)^2 + 82\,(j\omega) + 105 + K = 0$

$-j\omega^3 - 20\omega^2 + 82j\omega + 105 + K = 0$

$-20\omega^2 + 105 + K + j\omega(-\omega^2 + 82) = 0$

$-\omega^2 + 82 = 0$

$\omega = \pm 9.05$

[1.5 Marks]

iv)

$$\sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = 180^o(2h+1); \qquad h = 0, \pm1, \pm2, \ldots$$

Consider the angle of departure at open-loop pole $s = -2.5 + 0.866j$ is θ_p.

$$0 - \theta_p - 90^0 - \tan^{-1}\frac{0.866}{15-2.5} = 180^o$$

$\theta_p = -90^0 - 3.96^o - 180^o$

$\theta_p = \ -273.96^o = 86.04^o$

Root locus is symmetrical about real axis. Therefore, angle of departure at open-loop pole $s = -2.5 - 0.866j$ is -86.04^o.

[1.0 Mark]

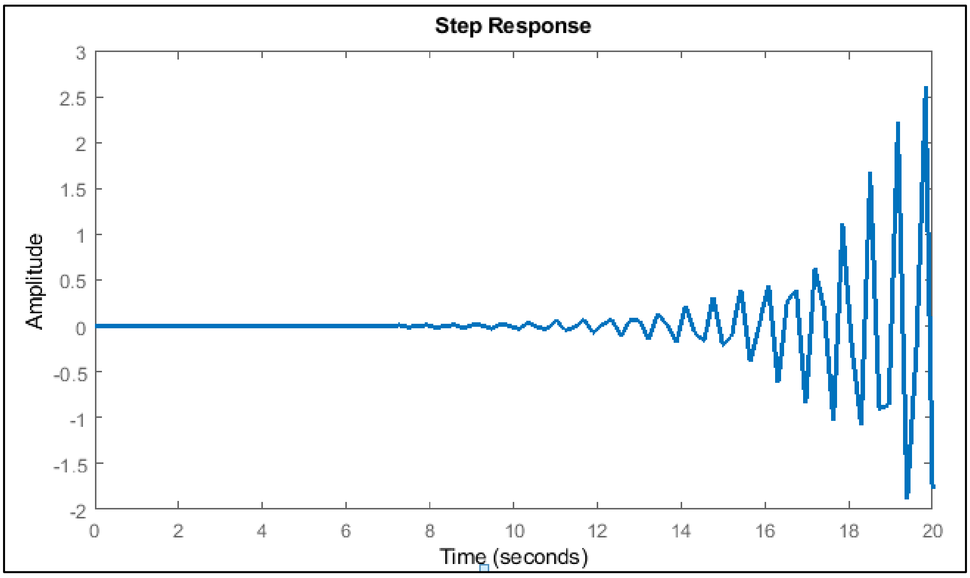
v) Gain K on the imaginary axis crossings of the root locus

$$-20\omega^2 + 105 + K = 0$$
$$K = 20\omega^2 - 105$$
$$K = 20 \times 9.05^2 - 105 = 1533.05$$

The closed-loop system is stable for $K < 1533.05$

[1.0 Mark]

vi)



[1.0 Mark]

Q4. a) i) Proportional

$$y_c(t) = K_p e(t)$$

Integral

$$y_c(t) = K_i \int_0^t e(t)\,dt$$

Derivative

$$y_c(t) = K_d \frac{de(t)}{dt}$$

[1.5 Marks]

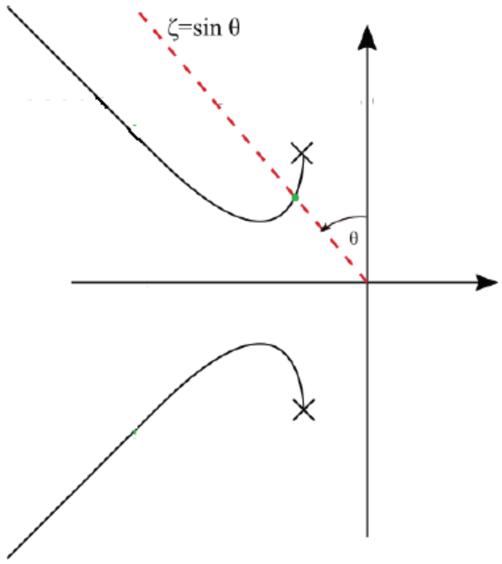
ii) The relationship between the damping ratio (ζ) and the percent overshoot is given by the following equation.

$$M_p\% = 100\, e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

From this equation damping ratio corresponding to certain overshoot can be calculated.

$$\sin\theta = \zeta$$

In the s-plane, root locus and the $\sin\theta = \zeta$ line be drawn as follows. The gain corresponding to the crossing point of the root locus and $\sin\theta = \zeta$ line results the desired percent overshoot.



[1.5 Marks]

iii) When the gain is increased, the steady state error is reduced.
The slow response of an overdamped system can be made faster by increasing the gain.
However, the maximum overshoot will increase with increasing gain.

[1.0 Mark]

b) i)

$$M_p\% = 100\, e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$
$$M_p\% = 20$$

$$20 = 100\, e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$
$$e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.2$$
$$\frac{-\zeta\pi}{\sqrt{1-\zeta^2}} = \ln(0.2)$$
$$-\zeta\pi = -1.609\sqrt{1-\zeta^2}$$
$$(\zeta\pi)^2 = 1.609^2(1-\zeta^2)$$
$$\zeta = 0.456$$

[1.0 Mark]

Peak time of the compensated system; $T_p = 0.5$ s.
Suppose the second order dominant poles of the compensated system are $s = -\sigma + j\omega$

$$T_p = \frac{\pi}{\omega}$$
$$\omega = \frac{\pi}{T_p} = 6.283$$
$$\sin\theta = 0.456$$
$$\tan\theta = \frac{\sigma}{\omega}$$

$$\sigma = \omega\ \tan\theta = 6.283\tan(\sin^{-1} 0.456) = 3.219$$
$$s = -3.219 + j6.283$$

[1.0 Mark]

Root locus of the compensated system;
$$1 + KG_{PD}(s)G(s) = 0$$

$$G_{PD}(s) = s + z_c$$

$$1 + K(s + z_c)\frac{1}{(s+1)(s+2)(s+12)} = 0$$
Using the second property of the root locus
$$\sum \angle\theta_z - \sum \angle\theta_p = 180(2h \pm 1)$$
$$\angle(s + z_c) - \big(\angle(s+1) + \angle(s+2) + (s+12)\big) = 180$$

$s = -3.219 + j6.283$ should be a point on the root locus of the compensated system.
$$\theta_{zc} - \big(\angle(-3.219 + j6.283 + 1) + \angle(-3.219 + j6.283 + 2) + (-3.219 + j6.283 + 12)\big) = 180$$
$$\theta_{zc} - \tan^{-1}\frac{6.283}{-3.219+1} - \tan^{-1}\frac{6.283}{-3.219+2} - \tan^{-1}\frac{6.283}{-3.219+12} = 180$$
$$\theta_{zc} = 66.02$$

$$\theta_{zc} = 66.02 = \tan^{-1}\frac{6.283}{-3.219 + z_c}$$
$$z_c = 6.015$$

[2.0 Marks]

Now we need to design the compensator gain.
$s = -3.219 + j6.283$ is a point on the root locus of the compensated system.

$$1 + K(s + z_c)\frac{1}{(s+1)(s+2)(s+12)} = 0$$

$$1 + K(s + 6.015)\frac{1}{(s+1)(s+2)(s+12)} = 0$$

$$1 + K(-3.219 + j6.283 + 6.015)\frac{1}{(-3.219 + j6.283 + 1)(-3.219 + j6.283 + 2)(-3.219 + j6.283 + 12)} = 0$$

$$K = -\frac{(-3.219 + j6.283 + 1)(-3.219 + j6.283 + 2)(-3.219 + j6.283 + 12)}{(-3.219 + j6.283 + 6.015)}$$

$$K = 66.9$$

PD compensator = 66.9 (s+6.015)

[1.0 Mark]

Justify the second order approximation.

Closed-loop poles of the compensated system

$$1 + 66.9(s + 6.015)\frac{1}{(s+1)(s+2)(s+12)} = 0$$

$$s^3 + 15s^2 + 104.9s + 426.404 = 0$$

$$s = -8.565, -3.217 \pm j6.279$$

Closed-loop zeros of the compensated system

$$s = -6.015$$

Closed-loop pole -8.565 and closed-loop zero -6.015 located close by. Therefore, those can be canceled out. Hence second order approximation is valid.

[1.0 Mark]

ii)

$$e_{ss} = \lim_{s\to 0}\frac{1}{1 + G_{OL}(s)}$$

$$e_{ss} = \lim_{s\to 0}\frac{1}{1 + 66.9(s + 6.015)\frac{1}{(s+1)(s+2)(s+12)}}$$

$$e_{ss} = \frac{1}{1 + 66.9 \times (0 + 6.015)\frac{1}{(0+1)(0+2)(0+12)}}$$

$$e_{ss} = 0.056$$

[1.0 Mark]

iii) Add a PI controller in addition to the PD controller designed above.

[1.0 Mark]

Q5. a)

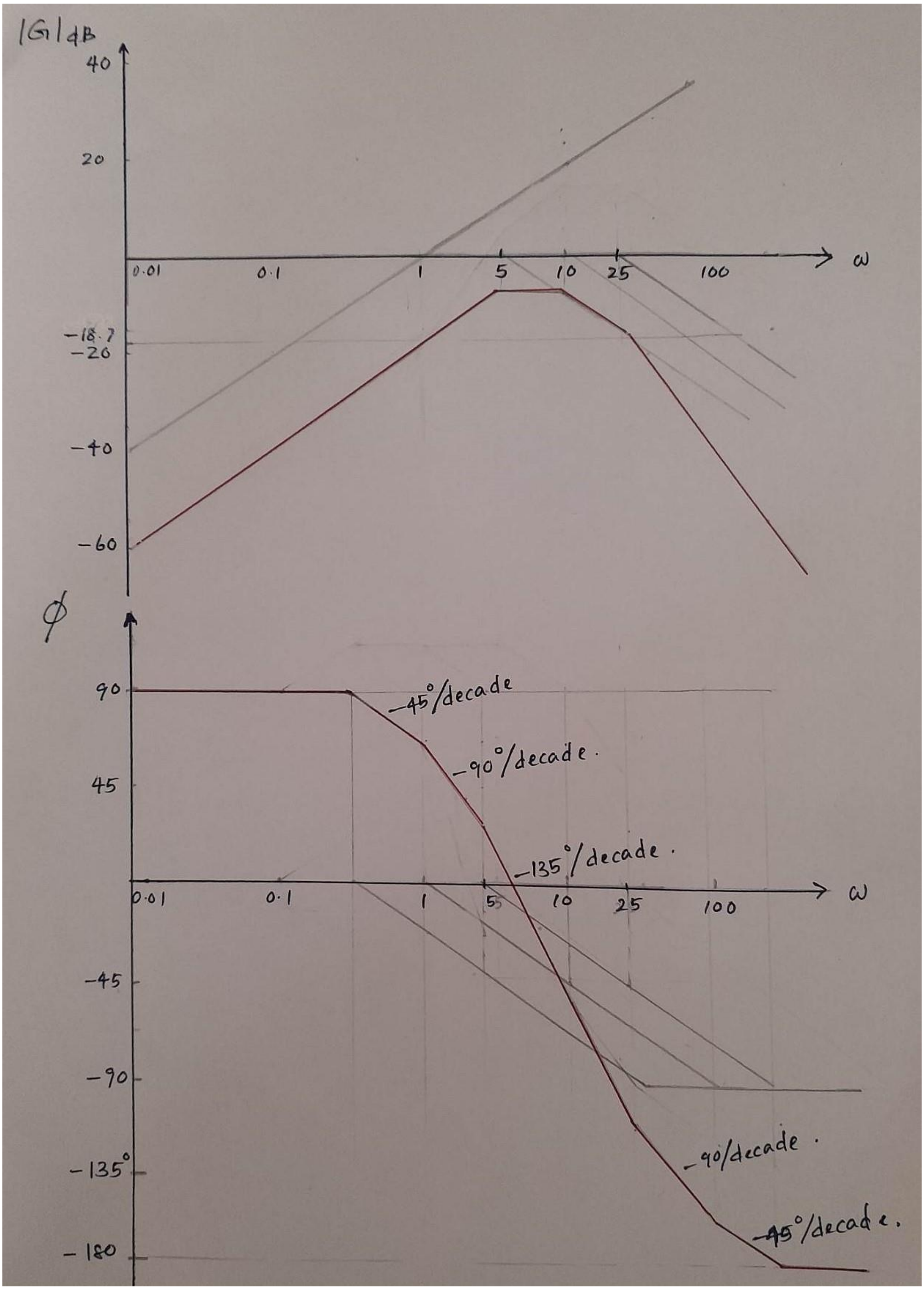
$$G(j\omega) = \frac{145j\omega}{(j\omega+5)(j\omega+10)(j\omega+25)}$$

$$G(j\omega) = \frac{20j\omega}{5(j\frac{\omega}{5}+1)10(j\frac{\omega}{10}+1)25(j\frac{\omega}{25}+1)}$$

$$G(j\omega) = \frac{0.116j\omega}{(j\frac{\omega}{5}+1)(j\frac{\omega}{10}+1)(j\frac{\omega}{25}+1)}$$

$$|G(j\omega)|\ dB = 20\ \log 0.116 + 20\ \log|j\omega| + 20\ \log\left|\frac{1}{j\frac{\omega}{5}+1}\right| + 20\ \log\left|\frac{1}{j\frac{\omega}{10}+1}\right| + 20\ \log\left|\frac{1}{j\frac{\omega}{25}+1}\right|$$

$$\angle G(j\omega) = \angle j\omega + \angle\frac{1}{j\frac{\omega}{5}+1} + \angle\frac{1}{j\frac{\omega}{10}+1} + \angle\frac{1}{j\frac{\omega}{25}+1}$$


|G|dB
40
20
-18.7
-20
-40
-60
0.01
0.1
1
5
10
25
100
ω
ϕ
90
45
-45
-90
-135°
-180
-45°/decade
-90°/decade.
-135°/decade.
-90/decade.
-45°/decade.

[3.0 Marks]

b) i) $$e_{ss} = \lim_{s\to 0}\frac{1}{1+G_{OL}(s)} = \frac{1}{1+G_{OL}(0)}$$
$$G_{OL}(s) = \frac{K}{(s^2+3s+7)}$$
$$G_{OL}(0) = \frac{K}{7}$$
$$0.01 = \frac{1}{1+\frac{K}{7}}$$
$$K = 693$$

[1.0 Mark]

ii)
$$M_p\% = 100\, e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$
$$M_p\% = 20$$
$$\zeta = 0.456$$

$$\phi_{PM,req} = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}} = 48.2^o$$

[1.0 Mark]

iii) Calculate the phase margin of the gain adjusted system
$$G(s) = \frac{693}{(s^2+3s+7)}$$
$$G(j\omega) = \frac{693}{((j\omega)^2+3j\omega+7)} = \frac{693}{7-\omega^2+3j\omega}\times\frac{7-\omega^2+3j\omega}{7-\omega^2-3j\omega}$$
$$G(j\omega) = \frac{693(7-\omega^2+3j\omega)}{(7-\omega^2)^2+9\omega^2}$$
$$|G(j\omega)| = 693\left\{\left(\frac{7-\omega^2}{(7-\omega^2)^2+9\omega^2}\right)^2+\left(\frac{3\omega}{(7-\omega^2)^2+9\omega^2}\right)^2\right\}^{1/2}$$
At phase margin frequency $|G(j\omega)| = 1$
$$693\left\{\left(\frac{7-\omega^2}{(7-\omega^2)^2+9\omega^2}\right)^2+\left(\frac{3\omega}{(7-\omega^2)^2+9\omega^2}\right)^2\right\}^{1/2} = 1$$
$$693^2[(7-\omega^2)^2+9\omega^2] = ((7-\omega^2)^2+9\omega^2)^2$$
$$\omega^2 = x$$
$$693^2 = (7-x)^2+9x$$
$$x = 695.47, -690.47$$
$$\omega = 26.37$$

Phase margin of the gain adjusted system
$$G(j\omega) = \left.\frac{693}{7-\omega^2+3j\omega}\right|_{\omega=26.37} = 1\angle-173.4$$
$$\phi_{PM,org} = 180-173.4 = 6.6$$

[2.0 Marks]

Phase contribution required from the lead compensator
$\phi_{lead} = \phi_{PM,req} - \phi_{PM,org} + 10 = 48.2 - 6.6 + 10 = 51.6$

$\phi_{lead} = \phi_{lead,max} = \sin^{-1}\frac{1-\beta}{1+\beta}$
$51.6 = \sin^{-1}\frac{1-\beta}{1+\beta}$
$\beta = 0.124$

[1.5 Marks]

Compensator's magnitude at the peak of the phase curve
$G_{lead}(j\omega_{max}) = \frac{1}{\sqrt{\beta}} = \frac{1}{\sqrt{0.124}} = 2.839 = 9.06 \text{ dB}$

[1.0 Mark]

Frequency at which the uncompensated system's magnitude equals to the negative of the lead compensator's magnitude at the peak of the compensator's phase curve.
$|G(j\omega)| = -9.06$ dB = 0.353

$693^2 = 0.353^2[(7-x)^2 + 9x]$
$3854047 = (7-x)^2 + 9x$
$x = 1965.7, -1960.7$
$\omega = 44.3$

[1.0 Mark]

$\omega = \omega_{max} = 44.3$
$\omega_{max} = \frac{1}{T\sqrt{\beta}}$
$T = \frac{1}{\omega_{max}\sqrt{\beta}} = \frac{1}{44.3\sqrt{0.124}} = 0.064$

[1.0 Mark]

$$G_{lead}(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} = \frac{1}{0.124}\frac{s+\frac{1}{0.064}}{s+\frac{1}{0.124\times 0.064}} = 8.06\ \frac{s+15.625}{s+126}$$

[0.5 Mark]