IS5311-Discrete Mathematics

Chapter 5-Combinatorics

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Outline

What is Combinatorics?

- Study of counting, arrangement, and combination of objects.
- Importance for Computer Engineering:
    - Algorithm analysis
    - Cryptography and security
    - Network design
    - Resource allocation in operating systems
- Counting the number of possible passwords or routing paths.

Counting Principle

Counting principles are essential tools in combinatorics used to determine the number of possible outcomes in different scenarios.

Basic Counting Principles

- **Product Rule:** sequential independent choices multiply.
- **Sum Rule:** mutually exclusive options add.
- **Division Rule:** account for indistinguishable arrangements by division.

Product Rule

Product Rule

If there are n_1 ways to perform the first task, n_2 ways to perform the second task, and so on, then the total number of ways to perform all tasks is given by the product $n_1 \times n_2 \times \cdots \times n_k$, where k is the number of tasks.

Examples:

Example 1: Build a workstation from 3 CPU types, 4 RAM kits, 2 storage drives:

$3 \times 4 \times 2 = 24$ configurations. Useful for design-space exploration.

Example 2: Choosing an Outfit

Suppose you want to choose an outfit consisting of a shirt and pants. If you have 5 different shirts and 3 different pairs of pants, and each shirt can be paired with each pair of pants, the total number of possible outfits is:

$$\text{Total number of outfits} = 5 \text{ (shirts)} \times 3 \text{ (pants)} = 15$$

Example 3: Creating a Password

Consider creating a 4-digit password where each digit can be any number from 0 to 9. Each digit is chosen independently. Therefore, for each digit, there are 10 possible choices. The total number of possible passwords is:

$$10 \times 10 \times 10 \times 10 \ = 10^4 = 10{,}000$$

Example 4: In a computer system, there are 4 different types of processors, 3 different types of memory modules, and 2 different types of storage drives. How many different combinations of these components can be created if you choose one of each type?

Example 5:
Suppose there are 4 different letters (P, Q, R, and S) which must be placed into 6 different boxes. How many ways are there to place these 4 letters into the 6 different boxes if each letter can go into any of the boxes?

Example 6: How many different number plates can be made if each plate contains a sequence of three uppercase English letters followed by 4 digits?

Example 4 :- $4 \times 3 \times 2 = 24$

Example 5 :- $6 \times 6 \times 6 \times 6 = 6^4$

each letter have 6 choices.

Example 6 :-

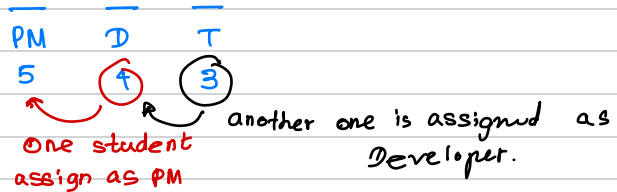___ ___ ___   ___ ___ ___ ___

3 letters   Digits.

$= 26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10$

$=$

Example 6: Suppose there are 5 computer engineering students and 3 distinct project roles (e.g., Project Manager, Developer, Tester) to be assigned. How many ways can you assign these 3 distinct roles to the 5 students if each role must be assigned to exactly one student and any student can take on any role?

Example 7: How many different bit strings of length 8 are there?

Example 6:-

$$\overline{PM} \quad \overline{D} \quad \overline{T}$$

PM   D   T

5   4   3

One student assign as PM

another one is assigned as Developer.

Example. 7:- 2^8

The Sum Rule

can't both occur

The Sum Rule

The *Sum Rule* (or Addition Rule) in combinatorics states that if you have two mutually exclusive events (i.e., events that cannot occur at the same time), then the total number of ways that one of these events can occur is the sum of the number of ways each event can occur.

two types of number plate
→ type 1 → 2 digits, 3 letters
type 2 → 3 digits, 2 letters

add because a plate either Type 1 or Type 2, not both

Examples:

Example 1: Consider a problem where you need to count the number of ways to choose a committee from a set of people, with two options:

1. **Option 1:** Choosing a committee of 3 people from 10 people.
2. **Option 2:** Choosing a committee of 4 people from 15 people.

If these two options are mutually exclusive then the total number of ways to choose a committee under either option is:

$$\text{Total ways} = \text{Ways for Option 1} + \text{Ways for Option 2}$$

$$\text{Total ways} = |A| + |B|$$

$$|A \cup B|$$

Example 2: Suppose there are two types of number plates:

1. Plates that have a sequence of three uppercase English letters followed by 2 digits.
2. Plates that have a sequence of two uppercase English letters followed by 3 digits.

How many different number plates can be made in total?

Solution:

- Type 1 Plates:
  - There are 26 choices for each of the 3 letters.
  - There are 10 choices for each of the 2 digits.
  - The number of possible plates for this type is:
  $$26^3 \times 10^2$$

- Type 2 Plates:
  - There are 26 choices for each of the 2 letters.
  - There are 10 choices for each of the 3 digits.
  - The number of possible plates for this type is:
  $$26^2 \times 10^3$$

  To find the total number of different number plates, add the number of plates for each type:
  $$26^3 \times 10^2 + 26^2 \times 10^3$$
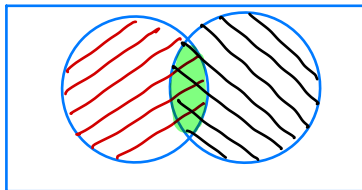
Inclusion-Exclusion Principle

1. **Counting Individual Sets**
2. **Subtracting Overlaps**
3. **Adding Back Overlaps of Triples**
4. **Continuing for Higher Orders**

Inclusion-Exclusion Principle

For Two Sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Here, $|A \cup B|$ is the number of elements in either set A or set B or both. We add $|A|$ and $|B|$ and subtract $|A \cap B|$ to correct for double-counting.

Examples:

Example 1: Consider a computer engineering department with three different student groups:

1. Group A: Students who are taking Programming courses.
2. Group B: Students who are taking Data Structures courses.
3. Group C: Students who are taking Algorithms courses.

We want to find the number of students who are taking at least one of these courses.

Suppose we have the following counts:

- $|A| = 120$ (students taking Programming courses)
- $|B| = 100$ (students taking Data Structures courses)
- $|C| = 90$ (students taking Algorithms courses)
- $|A \cap B| = 50$ (students taking both Programming and Data Structures courses)
- $|A \cap C| = 40$ (students taking both Programming and Algorithms courses)
- $|B \cap C| = 30$ (students taking both Data Structures and

$$|A \cup B \cup C| = |A| + |B| + |C| - \left(|A \cap B| + |B \cap C| + |A \cap C|\right) + |A \cap B \cap C|$$

$$|A \cap B \cap C| = 20 \longrightarrow \text{given}$$

$$\therefore |A \cup B \cup C| = 210$$

Using the Inclusion-Exclusion Principle, we calculate the number of students taking at least one course:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Substituting the given values:

$$|A \cup B \cup C| = 120 + 100 + 90 - 50 - 40 - 30 + 20$$

Performing the arithmetic:

$$|A \cup B \cup C| = 310 - 120 + 20 = 210$$

Thus, the number of students who are taking at least one of the courses is 210.

Example 2: In a secure system, each user must create a password that consists of exactly 10 characters. These characters can be uppercase English letters (26 options), lowercase English letters (26 options), or digits (10 options). Each password must include at least one uppercase letter and at least one digit. How many potential passwords exist under these conditions?

Example 3: How many bit strings of length 8 either start with a 1 bit or end with the two bits 11?

Example 2 :-

$$\overbrace{26, 26, 10}^{52}$$

A - set of passwords that don't contain any uppercase.

B - " " digit.

Total num of pw $= 52^{10}$
(with restriction)

$$= |U|$$

$|A| = 36^{10}$

$|B| = 52^{10}$

$|A \cap B| = 26^{10}$ ← num of PW with only lowercase letter.

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 36^{10} + 52^{10} - 26^{10}$

$=$

$|U| = 62^{10} - (|A \cup B|)$ ← total num PW with atleast one upper letter or atleast one digit.

Example 3:-

$A - 2^7$	1	2	2	2	2	2	2	2		= 128
$B - 2^6$	2	2	2	2	2	2	1	1		= 64
$A \cap B = 2^5$	1	2	2	2	2	2	1	1		= 32

$$(A \cup B) = |A| + |B| - |A \cap B|$$
$$= 128 + 64 - 32$$
$$= 160$$

Example 4: In a computer engineering department, there are 50 students taking Programming, 40 students taking Data Structures, and 30 students taking Algorithms. The following information is given:

- 15 students are taking both Programming and Data Structures.
- 12 students are taking both Programming and Algorithms.
- 8 students are taking both Data Structures and Algorithms.
- 5 students are taking all three subjects (Programming, Data Structures, and Algorithms).

How many students are taking at least one of these subjects?

$|P| = 50, \quad |D| = 40, \quad |A| = 30$

$|P \cap D| = 15$

$|P \cap A| = 12 \qquad, \quad |P \cap A \cap D| = 5$

$|D \cap A| = 8$

$|P \cup D \cup A| = 50 + 40 + 30 + 5 - 15 - 12 - 8$

$\qquad\qquad = \underline{\underline{90}}$

The Division Rule

The Division Rule is used to determine the number of distinct objects when some of them are identical.

The Division Rule

If a set of objects can be arranged in N different ways, but there are k_1 indistinguishable objects of one kind, k_2 indistinguishable objects of another kind, and so on, then the number of distinct arrangements is given by:

$$\frac{N}{k_1! \cdot k_2! \cdot \cdots \cdot k_m!}$$

where $k_1, k_2, \ldots, k_m$ are the numbers of indistinguishable objects of each kind.

Examples:

Example 1: Suppose you have 10 letters, where 4 are 'A's, 3 are 'B's, and 3 are 'C's. To find the number of distinct arrangements of these letters, use the Division Rule:

1. Calculate the total number of arrangements if all letters were distinguishable:

$$10!$$

2. Divide by the factorial of the number of indistinguishable objects for each kind:

$$\frac{10!}{4! \cdot 3! \cdot 3!}$$

Example 2: How many distinct arrangements can be formed using the letters in "COMPUTER ENGINEERING"?

Solution:
For "COMPUTER ENGINEERING":

- Total number of letters, $n = 17$
- Count of each letter:
  1. 'E': 3 (so, $k_E = 3!$)
  2. 'O': 2 (so, $k_O = 2!$)
  3. 'R': 2 (so, $k_R = 2!$)
  4. 'N': 2 (so, $k_N = 2!$)
- Others: 1 each (so, $k_C = 1!$, $k_M = 1!$, $k_P = 1!$, $k_U = 1!$, $k_T = 1!$, $k_G = 1!$, $k_I = 1!$)

Applying the formula:

$$\text{Number of distinct arrangements} = \frac{17!}{3! \cdot 2! \cdot 2! \cdot 2!}$$

Permutations — Order Matters

A permutation is an arrangement of objects in a specific order. Order matters in permutations.

Example: Consider the letters A, B, and C. The permutations of 2 letters out of these 3 are:

- AB
- AC
- BA
- BC
- CA
- CB

There are 6 permutations.

Permutations

If you have a set of *n* distinct objects, the number of ways to arrange all of these objects in order (i.e., the number of permutations) is given by *n*! (n factorial), where:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

Example 1: Suppose you want to find the number of ways to arrange 5 distinct books on a shelf:

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Permutations of a Subset

If you want to arrange r objects out of a set of n distinct objects, the number of permutations is given by:

$$P(n, r) = \frac{n!}{(n - r)!}$$

Example అని అంటారు.

where:

- n is the total number of objects.
- r is the number of objects to arrange.

The number of r-permutations of a set with n elements is denoted by $p(n, r)$.

Example 1: Suppose you have 8 distinct students and want to choose and arrange 3 of them in a line:

$$P(8,3) = \frac{8!}{(8-3)!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5!} = 8 \times 7 \times 6 = 336$$

Permutations of Multisets

If some objects in the set are identical, the formula for the number of distinct permutations changes.

For a multiset with n total objects where there are k_1 identical objects of one type, k_2 identical objects of another type, and so on, the number of distinct permutations is given by:

$$\frac{n!}{k_1! \times k_2! \times \cdots \times k_m!}$$

Example 1: Consider the word "BANANA", where there are 6 letters, but the letter 'A' appears 3 times, and 'N' appears 2 times. The number of distinct permutations is:

$$\frac{6!}{3! \times 2! \times 1!} = \frac{720}{6 \times 2 \times 1} = \frac{720}{12} = 60$$

Example 2: How many different 8-character passwords can be created using the letters from the word "COMPUTER" without repetition of characters?

Solution:
Since all 8 characters in "COMPUTER" are distinct, the number of different passwords is simply:

$$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$$

Combinations — Order Doesn't Matter

A combination is a selection of objects where the order does not matter.

Example :
Consider the letters A, B, and C. The combinations of 2 letters out of these 3 are:

- AB
- AC
- BC

There are 3 combinations.

Combinations

The number of combinations of r objects chosen from n objects can be expressed in several equivalent ways:

$$C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)! \, r!} = nC_r$$

where:

- n is the total number of objects.
- r is the number of objects to choose.
- $n!$ (n factorial) is the product of all positive integers up to n.

Example 1: Suppose you have 5 distinct books, and you want to select 2 of them to take on a trip. The number of ways to choose these 2 books, without considering the order, is:

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \times 4 \times 3!}{2 \times 1 \times 3!} = \frac{5 \times 4}{2 \times 1} = 10$$

Thus, there are 10 different ways to choose 2 books from a set of 5.

Examples 2: Consider a set of 8 different letters. The number of ways to choose 3 letters from this set is:

$$\binom{8}{3} = \frac{8!}{3!(8-3)!} = \frac{8!}{3! \cdot 5!}$$

Example 3:   How many ways are there to choose 5 cards from a deck of 52 cards?

Example 4: How many ways are there to select a subset of 6 objects from a set of 15 objects?

Example 5:   In a Computer Engineering Faculty, there are 120 students in the Software Engineering department and 150 students in the Hardware Engineering department. We need to form a committee of 6 students from the Software Engineering department and 8 students from the Hardware Engineering department. How many ways are there to select this committee?

Example 3 :-

$$\frac{52!}{5!\,(52-5)!}$$

Example 4 :-

$$\frac{15!}{6!\,(15-6)!}$$

Example 5 :-

$$\frac{120!}{6!\,(120-6)!} + \frac{150!}{8!\,(150-8)!}$$

Counting Principles - Quick Reference

Principle	Formula / Rule	Key Word / How to Identify
Sum Rule	m + n	Either / Or, mutually exclusive
Product Rule	m × n	And, sequential tasks
Inclusion-Exclusion	$|A \cup B| = |A| + |B| - |A \cap B|$	Overlapping sets, correct double count
Division Rule	(ways) ÷ (duplicates)	Overcounted arrangements, identical items
Permutation	$P(n, r) = \frac{n!}{(n-r)!}$	Arrange, order matters
Combination	$C(n, r) = \frac{n!}{r!(n-r)!}$	Select, order doesn't matter

Pigeonhole Principle

The Pigeonhole Principle states that if n items are distributed among m containers and $n > m$, then at least one container must contain more than one item.

Example 1:
If there are 10 students and only 9 desks, then at least one desk must have more than one student.

Applications:

- **Proving Existence:** The principle can be used to show that in a group of 13 people, at least two people must have the same number of hairs on their head if the number of hairs ranges from 0 to 12.

- **Scheduling Problems:** In resource allocation problems, if there are more tasks than resources, the principle can show that some resources will be used more than once.

- **Birthday Problem:** In a group of 23 people, there is a better than even chance that at least two people share the same birthday.

Problem: Prove that if 10 pigeons are placed into 9 pigeonholes, then at least one pigeonhole must contain 2 pigeons.

Proof:

- Assume the contrary: each pigeonhole contains at most 1 pigeon.
- Since there are 9 pigeonholes, a maximum of 9 pigeons can be placed into these holes without any pigeonhole containing more than one pigeon.
- However, we have 10 pigeons, which exceeds this maximum. Therefore, at least one pigeonhole must contain more than one pigeon.

The Binomial Theorem

For any positive integer n and any real numbers x and y, the expansion of $(x+y)^n$ is:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0$$

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

where the binomial coefficient $\binom{n}{k}$ is defined as:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad {}^nC_r$$

Example 1: To expand $(2x + 3)^4$ using the binomial theorem, we have:

$$(2x + 3)^4 = \sum_{k=0}^{4} \binom{4}{k} (2x)^{4-k} \cdot 3^k$$

Calculating each term:

$$\text{For } k = 0: \quad \binom{4}{0} (2x)^4 \cdot 3^0 = 1 \cdot 16x^4 = 16x^4$$

$$\text{For } k = 1: \quad \binom{4}{1} (2x)^3 \cdot 3^1 = 4 \cdot 8x^3 \cdot 3 = 96x^3$$

$$\text{For } k = 2: \quad \binom{4}{2} (2x)^2 \cdot 3^2 = 6 \cdot 4x^2 \cdot 9 = 216x^2$$

$$\text{For } k = 3: \quad \binom{4}{3} (2x)^1 \cdot 3^3 = 4 \cdot 2x \cdot 27 = 216x$$

$$\text{For } k = 4: \quad \binom{4}{4} (2x)^0 \cdot 3^4 = 1 \cdot 81 = 81$$

Example 2: To find the <u>coefficient</u> of $x^4 y^5$ in the expansion of $(2x + 3y)^9$, we use the binomial theorem:

$$(2x + 3y)^9 = \sum_{k=0}^{9} \binom{9}{k} (2x)^{9-k} (3y)^k$$

We need the term where $x^{9-k} = x^4$ and $y^k = y^5$. Thus:

$$9 - k = 4 \implies k = 5$$

The term is:

$$\binom{9}{5} (2x)^{9-5} (3y)^5$$

Simplify:

$$\text{Term} = \binom{9}{5} (2x)^4 (3y)^5$$

Compute $(2x)^4$:

$$(2x)^4 = 2^4 \cdot x^4 = \underline{16}x^4$$

Compute $(3y)^5$:

$$(3y)^5 = 3^5 \cdot y^5 = \underline{243}y^5$$

Thus:

$$\text{Term} = \binom{9}{5} \cdot 16x^4 \cdot 243y^5$$

Calculate the binomial coefficient:

$$\binom{9}{5} = \frac{9!}{5! \cdot 4!} = 126$$

The coefficient is:

$$126 \cdot 16 \cdot 243 = 488,088$$

Example 3: Find the coefficient of x^3y^4 in the expansion of $(3x - 2y)^7$.

Example 4: To find the coefficient of x^5 in the expansion of $(-1 + x)^6$, we use the binomial theorem:

Ex:- 3 | $x^3 y^4$

$$(3x - 2y)^7 = \sum_{k=0}^{7} \binom{7}{k} (3x)^{7-k} (-2y)^k$$

$$x^3 = x^{7-k}$$
$$k = 4$$

when $k = 4$

$$\binom{7}{4} (3x)^3 (-2y)^4$$

coef. $= \dfrac{7!}{4! \, 3!} \cdot 3^3 (-2)^4$

$= \dfrac{5 \times 6 \times 7}{6} \cdot 27 \cdot 16$

$= 15120$

Ex: a | $(-1 + x)^6 \Rightarrow$ coef. of x^5

$$(-1+x)^6 = (x-1)^6 = \sum_{k=0}^{6} \binom{6}{k} (x)^{6-k} (-1)^k$$

$$x^{6-k} = x^5 \Rightarrow \begin{array}{l} 6-k = 5 \\ k = 1 \end{array}$$

$$\text{coeff.} = \binom{6}{1}(-1)^1$$
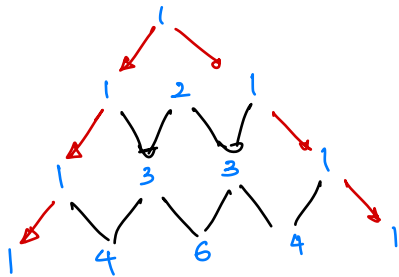
$$= \underline{\underline{-6}}$$

Theorem:
Let n be a positive integer. Then

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0$$

Pascal's Triangle

Pascal's Triangle is a triangular array of numbers where:

- The top row is 1 (row 0).
- Each row corresponds to the coefficients of the binomial expansion $(x + y)^n$ for increasing n.
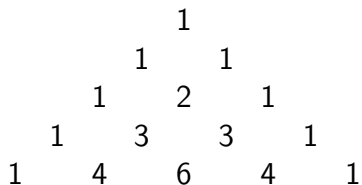- Each number in the triangle is the sum of the two numbers directly above it.

Theorem:
Let n and r be positive integers with $n \geq r$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

The first few rows of Pascal's Triangle are:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\end{array}
$$

Properties:

- **Symmetry:** Each row is symmetric.
- **Binomial Coefficients:** The n-th row corresponds to the coefficients of the binomial expansion $(x + y)^n$.
- **Sum of Elements:** The sum of the elements in the n-th row is 2^n.

Summary — Takeaways

- Combinatorics gives precise counting tools needed across Computer Engineering.
- Use product/sum/division rules appropriately depending on independence and indistinguishability.
- Inclusion–exclusion avoids double counting; pigeonhole proves existence/collisions.
- Binomial identities power both algebraic simplification and DP formulations.

References

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📄 R. P. Grimaldi, *Discrete and Combinatorial Mathematics*.

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Thank you

Questions?

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