Chapter 03-Set Theory

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Why set theory is important:

1. Foundation of Mathematics
2. Structures and Relations
3. Logic and Computability
4. Model Theory
5. Applied Mathematics
6. Philosophical and Foundational Questions
7. Interdisciplinary Applications

Interdisciplinary applications of set theory

- Computer Science
- Statistics and Probability
- Economics and Social Sciences
- Engineering and Operations Research
- Biology and Medicine

What is a set?

Think?

Set

A well-defined collection of distinct objects.

- The word **well-defined** refers to a specific property which identify the given object belongs to the set or not.
- Set are usually denoted by upper case letters, A, B, C, etc.
- We typically use the curly bracket notation to refer to a set.
- The **objects** of the sets are called its elements or members and will be denoted by lower case letters, a, b, c, etc.

Examples of Well-Defined Sets

- **Set of hardware components**:
  $A = \{\text{CPU, GPU, RAM}\}$
- **Set of binary digits**:
  $B = \{0, 1\}$
- **Set of Real Numbers Greater Than 5**:

$$R = \{x \in \mathbb{R} \mid x > 5\}$$

Examples of Not Well-Defined Sets

- **Vague Description**:
  $C = \{\text{the best programming language}\}$ — subjective, not well-defined
- **Unclear Boundary**:

$$C = \{x \mid x \text{ is a tall person}\}$$

Can you check whether following examples are a well-defined set or not?

1. The set of tall student in our university.
2. The set of Prime Numbers.
3. Set of Real Numbers Greater Than 5.

- Tabulation: List the elements of the set and enclose them in curly brackets.

  Example: $A = \{a, b, 1, 2, 0\}$ denotes the set A whose elements are $a, b, 1, 2, 0$

  Note:
  The order of the elements in the list does not matter.

- Set-builder form: Starting properties which characterize the elements in the set.

Example:

$$B = \{x | x \text{ is an integers}, x > 0\}$$

$A = \{1, 2, 3\}$

$A = B$

$B = \{3, 2, 1\}$

$B = \{x \mid x$ is an integers, $x > 0\}$

1. Specifying a rule or a verbal description
   Example: "Let A be the set of all odd integers"

2. Enclosing the lists of members within the curly bracket.
   Example:
   1. $C = \{2, 4, 5\}$
   2. $D = \{(2, 4), (-1, 5)\}$

3. Abbreviations can be used if the set is large or infinite.
   Example:
   1. $\{1, 3, 5, ..., 99\}$
   2. $\{4, 8, 12, ...\}$

4. Use set builder notation:
   Example: $F = \{n^3 | 1 \geq n \geq 100\}$

A set is **finite** if it consists of n different elements, where n is some positive integers. Otherwise set is infinite.

Note

- A set which consists of exactly one element is called a **singleton set**.

  Example:

  1. $A = \{5\}$
  2. $B = \{x | x^2 = 1, x > 0\} = \{1\}$

$x^2 = 1$

$x = \pm 1$

$x > 0$

$x = 1$

- A set which contains no elements is called the **empty set** or **null set** and it is denoted by $\varnothing$.

  Example:
  $A = \{x | x^2 = 4, x \text{ is odd}\}$

  $x^2 = 4$
  $x = \pm 2$
  x is odd

- A **universal set** is the set containing all objects or elements and of which all other sets are subsets. It is denoted by $\mathbb{U}$.

$B = \{ \phi \}$

element (not a empty set
singal~~ton~~ set).

Components of a Venn Diagram:

1. Sets
2. Overlap
3. Non-Overlap

Note:

- Single Set Representation: A single circle represents a single set.
- Overlap Representation: Overlapping areas between circles represent elements that belong to both sets simultaneously.

Example: Consider two sets:
$A = \{1, 2, 3\}$.
$B = \{2, 3, 4\}$.

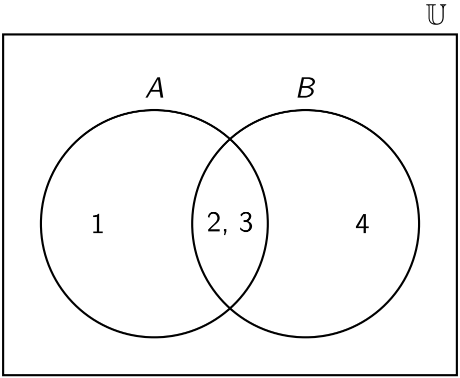


Figure 1: A Venn diagram illustrating two sets A and B within the universal set U.

Usage:

- Comparison
- Logical Relationships
- Problem Solving

Subsets

Let A and B be sets. A is a subset of B if every element of A is also an element of B. This is denoted symbolically as $A \subseteq B$.

$$A \subseteq B \text{ if } \forall x,\ x \in A \Rightarrow x \in B.$$

We also say that A is contained in B or B contains A.

Note:
From the definition it is clear that $A \subseteq A$.

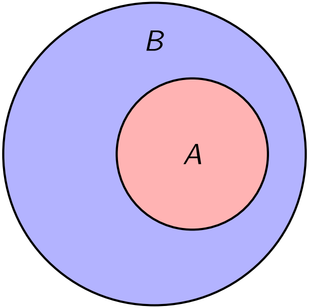


Figure 2: A Venn diagram illustrating that set A is a subset of set B.

Example:

- Universal set:
  $U = \{$all hardware components of a personal computer$\}$
- Subset example:
  $A = \{\text{CPU, RAM}\} \subseteq B = \{\text{CPU, RAM, GPU, SSD}\}$

Exercise: Given the sets

$$A = \{1, 3, 5, \ldots\},$$
$$B = \{5, 10, 15, \ldots\}, \text{ and}$$
$$C = \{x \mid x \text{ is a prime number and } x > 2\},$$

determine whether B and C are subsets of A.

Ans :- $C \subseteq A$, $B \not\subseteq A$

Proper Subset

A set A is a **proper subset** of a set B if every element of A is also an element of B, and A is not equal to B. This is denoted as $A \subsetneq B$.

$$A \subsetneq B \quad \text{if and only if} \quad (A \subseteq B) \land (A \neq B)$$

Example: Consider the sets:

$$A = \{1, 2\}$$

$$B = \{1, 2, 3\}$$

Here, $A \subsetneq B$. → A is a subset of B, but not equal to B

proper subset

Union $\longrightarrow$ මේලය

In set theory, the union of two sets A and B, denoted $A \cup B$, is the set that contains all elements which are in A, in B, or in both A and B.

$$\boxed{A \cup B = \{x \mid x \in A \text{ or } x \in B\}}$$

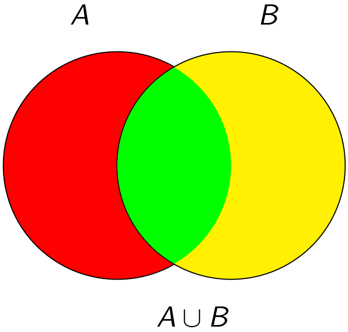
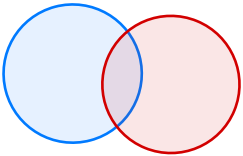


Figure 3: A Venn diagram illustrating the union of sets A and B.

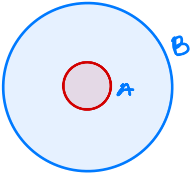
Note:

- $A \cup B = B \cup A$
- $A \subsetneq A \cup B$ and $B \subsetneq A \cup B$



$B, A \subseteq A \cup B$

- If $A \subsetneq B$ then, $A \cup B = B$



$A \cup B = B$

Intersection → ඡේදනය

In set theory, the intersection of two sets A and B, denoted $A \cap B$, is the set that contains all elements which are common to both A and B.

$$\boxed{A \cap B = \{x \mid x \in A \text{ and } x \in B\}}$$

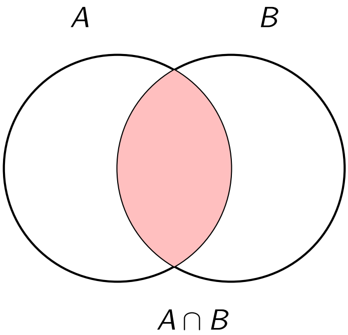
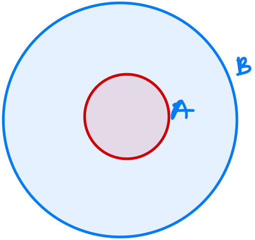


Figure 4: A Venn diagram illustrating the intersection of sets A and B.

Note:

- $A \cap B \subsetneq A$ and $A \cap B \subsetneq B$
- $A \cap B = \emptyset$ that is if A and B do not have any element in common. Then A and B are said to be **disjoint**.
- If $A \subsetneq B$ then, $A \cap B = A$

$A \cap B = A$



all the elements in A

but not in B

Difference

Given two sets A and B, the difference $A \setminus B$ (or $A - B$) is defined as:

$$A \setminus B = \{x \in A \mid x \notin B\}$$

In other words, $A \setminus B$ consists of all elements that are in A and not in B.

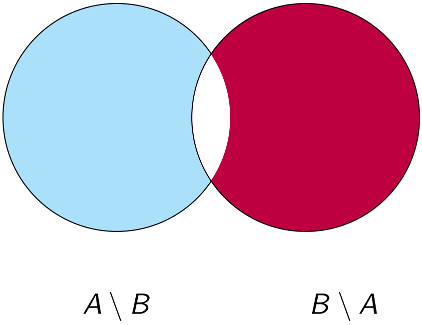


Figure 5: A Venn diagram illustrating the difference .

Note:

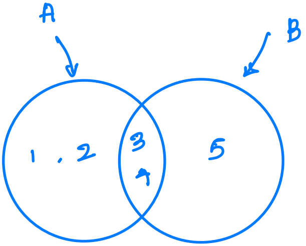
The order of operands matters in set difference. $A \setminus B$ is not the same as $B \setminus A$, except in cases where $A = B$.

Example: Consider two sets:

$$A = \{1, 2, 3, 4\}$$

$$B = \{3, 4, 5\}$$

The difference $A \setminus B$ (or $A - B$) is:



$A / B = \{1, 2\}$

$B / A = \{5\}$

Complement

If $\mathbb{U}$ is the universal set containing all elements under consideration, then the complement of set A, denoted as A' or $\bar{A}$, is defined as:

$$\boxed{A' = \{x \in U \mid x \notin A\}}$$

In other words, A' consists of all elements that belong to $\mathbb{U}$ but do not belong to A.

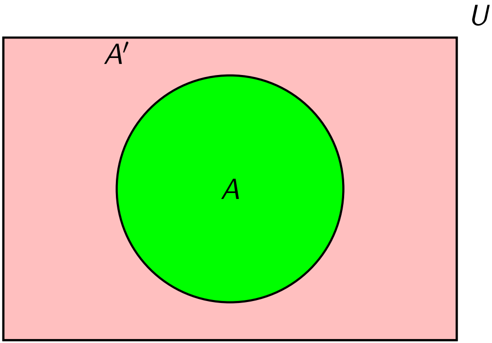


Figure 6: A Venn diagram illustrating set A and its complement A' in the universal set U.

Properties:

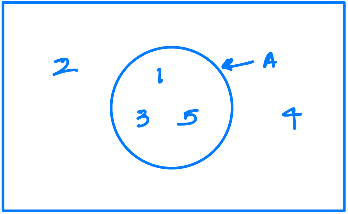
1. Universal Set: The complement A' is defined with respect to a specific universal set $\mathbb{U}$.

2. Identity: $A \cup A' = \mathbb{U}$ and $A \cap A' = \emptyset$, where $\emptyset$ denotes the empty set.

3. Notation: The complement of set A can be denoted as A', $\bar{A}$, or sometimes $U \setminus A$.

Example: Consider a universal set $U = \{1, 2, 3, 4, 5\}$ and a set $A = \{1, 3, 5\}$. The complement of A in U, denoted A' or $\bar{A}$, is:

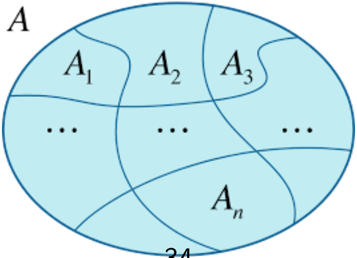
A' or $\bar{A} = \{2, 4\}$



Partition

Let S be a set. A partition of S is a collection $\{S_1, S_2, \ldots, S_n\}$ of subsets of S such that:

1. $S_1, S_2, \ldots, S_n$ are non-empty sets.
2. Every element of S is in exactly one of the sets S_i (i.e., $S = \bigcup_{i=1}^{n} S_i$).
3. The sets S_i are pairwise disjoint (i.e., $S_i \cap S_j = \emptyset$ for all $i \neq j$).



Example: Consider the set $S = \{1, 2, 3, 4, 5, 6\}$. A partition of S could be $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$.

1. Each subset $\{1, 2\}, \{3, 4\}, \{5, 6\}$ is non-empty.
2. Every element of S (which are $\{1, 2, 3, 4, 5, 6\}$) is in exactly one of the subsets.
3. The subsets $\{1, 2\}, \{3, 4\}, \{5, 6\}$ are pairwise disjoint.

$\{1,2\} \cap \{3,4\} = \phi$, $\{3,4\} \cap \{5,6\} = \phi$,

$\{1,2\} \cap \{5,6\} = \phi$ $\longrightarrow$ not common element.

- Union: The union of all subsets in a partition equals the original set S.

  In the example, $\{1,2\} \cup \{3,4\} \cup \{5,6\} = \{1,2,3,4,5,6\}$.

- Disjointness: The subsets in a partition are pairwise disjoint, ensuring that no element appears in more than one subset within the partition.

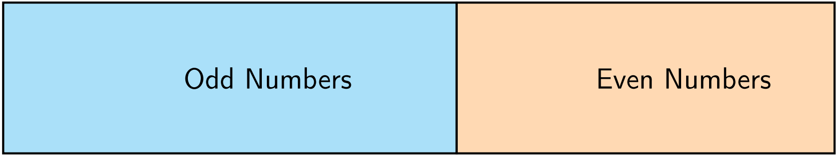


Figure 7: A partition of the natural numbers into odd and even numbers on a number line.

Ordered Pairs

An ordered pair is denoted as (a, b), where a is the first element and b is the second element.

Note:

Two ordered pairs (a, b) and (c, d) are equal if and only if their corresponding elements are equal.

Cartesian Product

Let A and B be two sets. The **Cartesian product** $A \times B$ is defined as the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$:

$$\boxed{A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}}$$

The number of elements in the Cartesian product $A \times B$ is determined by multiplying the number of elements in set A by the number of elements in set B.

$$\boxed{|A \times B| = |A| \times |B|}$$

Exercise:

Let $A = \{1, 2\}$ and $B = \{x, y, z\}$.

1. Find the number of elements in the Cartesian product $A \times B$.
2. List all the elements of $A \times B$.

$|A| = 2$ elements

$|B| = 3$ elements

$$|A \times B| = |A| \times |B| = 2 \times 3 = 6$$

$$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z)\}$$

Note:
If $|A \times B| = k$, then there are 2^k subsets in $A \times B$. Each subset of $A \times B$ represents a relation from A to B.
Given that the number of elements in $A \times B$ is k:

$$|A \times B| = k$$

The number of subsets (relations) of $A \times B$ is 2^k:

$$\text{Number of subsets in } A \times B = 2^k$$

Example: Let $A = \{1, 2\}$ and $B = \{x, y\}$. The Cartesian product $A \times B$ is:

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

In this case, $k = |A \times B| = 4$. Therefore, the number of subsets (relations) in $A \times B$ is:

$$2^k = 2^4 = 16$$

there are 16 subsets.

There are 16 possible relations in $A \times B$.

subset = $\{1\}, \{2\}, \{x\}, \{y\}, \{1, x\}, \{1, y\}, \{2, x\},$
$\{2, y\}, \{1, x, y\}, \{2, x, y\}, \{1, 2, x\},$
$\{1, 2, y\}, \{\}, \{1, 2, x, y\}, \{1, 2\}, \{x, y\}$

Relations

Let A and B be sets. A relation R from A to B is a subset of $A \times B$, i.e., $R \subseteq A \times B$.

Suppose R is a relation from set A to set B. For $a \in A$ and $b \in B$, one of the following statements is true:

1. If $(a, b) \in R$, we say that *"a is R-related to b"* and we write it as aRb.
2. If $(a, b) \notin R$, we say that *"a is not R-related to b"* and we write it as $a \not R b$.

Examples

1. Let $A = \{1, 2\}$ and $B = \{x, y\}$. Consider a relation R such that:

$$R = \{(1, x), (2, y)\}$$

For this relation R:

- $(1, x) \in R$ implies that $1Rx$
- $(1, y) \notin R$ implies that $1\not{R}y$
- $(2, x) \notin R$ implies that $2\not{R}x$
- $(2, y) \in R$ implies that $2Ry$

$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$

$R = \{(1, x), (2, y)\}$

$R \subseteq A \times B$

$(1, x) \in R$, 1 is R- related to x $\Rightarrow$ $1Rx$

$(1, y) \notin R$, 1 is not R- related to y $\Rightarrow$ $1\not{R}y$

- Consider the sets:

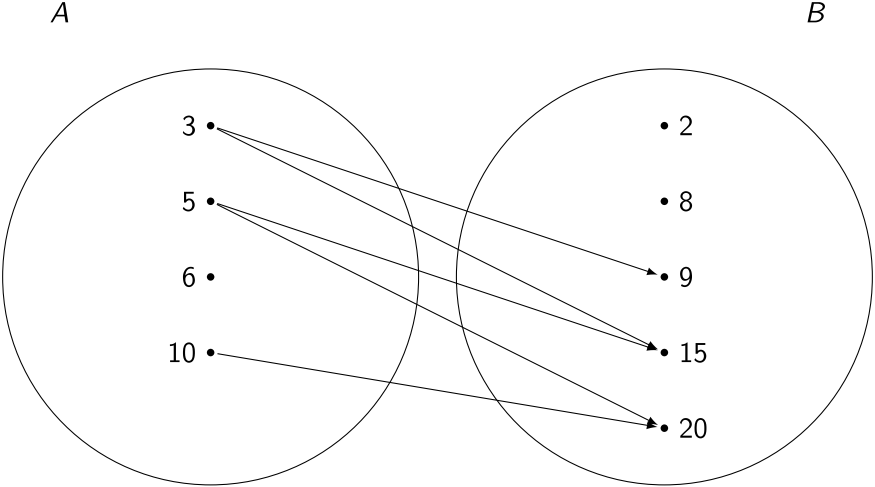
$$A = \{3, 5, 6, 10\}$$

$$B = \{2, 8, 9, 15, 20\}$$

The relation R is defined by m divides n, where $m \in A$ and $n \in B$.

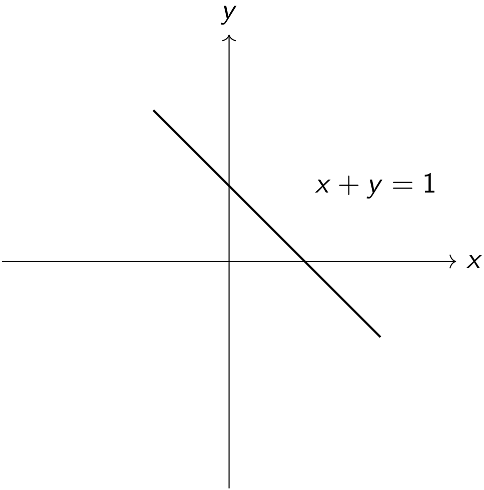
The relation R is:

$$R = \{(3, 9), (3, 15), (5, 15), (5, 20), (10, 20)\}$$


A
B
3
5
6
10
2
8
9
15
20

- The relation R is defined by:

$$R: \quad x + y = 1, \quad x, y \in \mathbb{R}$$



Empty Relation

The empty relation $\emptyset$ is defined as:

$$\emptyset = \{\}$$

There are no pairs in the empty relation.

For any sets A and B:

$$\emptyset \subseteq R \subseteq A \times B$$

Example: Consider a set $A = \{1, 2\}$ and a set $B = \{3, 4\}$. The empty relation between A and B is:

$$\emptyset = \{\}$$

Universal Relation

The universal relation $A \times B$ is defined as:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

It contains every possible pair where a is related to b.

For any sets A and B:

$$A \times B \subseteq A \times B$$

Example: Let $A = \{1, 2\}$ and $B = \{3, 4\}$. The universal relation $A \times B$ is:

Inverse Relation

If R is a relation containing pairs (a, b), then the inverse relation R^{-1} is:

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

It swaps the elements of each pair in R.

Example:

Suppose $R = \{(1, 2), (3, 4)\}$. The inverse relation R^{-1} is:

$R^{-1} = \{(2,1),(4,3) \mid (1,2),(3,4) \in R\}$

Reflexive

R is reflexive on A if $(a, a) \in R$ for all $a \in A$.

Example: Suppose $A = \{1, 2, 3\}$ and
$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$.

Here, R is reflexive because $(1, 1), (2, 2)$, and $(3, 3)$ are included.

Symmetric

R is symmetric on A if $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.

Example: Let $A = \{1, 2, 3\}$ and
$R = \{(1, 2), (2, 1), (1, 1), (2, 2), (3, 3)\}$.

R is symmetric because for every pair (a, b) in R, (b, a) is also in R.

Transitive

R is transitive on A if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Example: - Consider $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3), (1, 3), (2, 2), (3, 3)\}$.

R is transitive because for every pair (a, b) and (b, c) in R, (a, c) is also in R.

Antisymmetric

R is antisymmetric on A if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$.

Note:

- If two distinct elements are related in both directions, that cannot happen.
- Only the same element can relate to itself in both directions.

Example: Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3), (3, 3), (1, 1)\}$.

R is antisymmetric because if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

Equivalence Relations

An equivalence relation R on a set A is a binary relation such that the relation is

- Reflexive
- Symmetric
- Transitive

$A = \{1, 2, 3, 4\}$

1 → $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\}$

2 → $R = \{(1,1), (1,3), (3,1)\}$

3 → $R = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\}$

A → Reflexive $(a,a) \in R \quad \forall a \in A$

B → Symmetric $(a,b) \in R \Rightarrow (b,a) \in R \quad \forall a \in A$

C → Transitive $(a,b) \in R \ \& \ (b,c) \in R \Rightarrow (a,c) \in R$

$\forall a, b, c \in A$

D → Antisymmetric $(a,b) \in R \ \& \ (b,c) \in R \Rightarrow a = b \quad \forall a, b \in A$

1 ⇒ not reflective → $3, 3$ → නැ

not symmetric → $3, 4$ → $4, 3$ නැ

not transitive → $(3,4), (4,1)$ තිබ්බම

$(3,1)$ නැ

not antisymmetric → $(3,4), (4,1)$ තිබ්බම

$3 \neq 4$

Examples:

- **Equivalence Relation: $=$ on a set of numbers A**
  - Reflexive: $a = a$ for all $a \in A$.
  - Symmetric: $a = b \Rightarrow b = a$ for all $a, b \in A$.
  - Transitive: $a = b$ and $b = c \Rightarrow a = c$ for all $a, b, c \in A$.

- **Modular Congruence (Modulo Operation):**

$$a \equiv b \pmod{m}$$

  - Reflexive: $a \equiv a \pmod{m}$
  - Symmetric: If $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$
  - Transitive: If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$

$a \equiv b \pmod{m}$

$m \mid a-b \Rightarrow m \mid -(a-b) \rightarrow m \mid b-a$

$\downarrow$

$b \equiv a \pmod{m}$

$a \equiv b \pmod{m}$

$m \mid a-b \Rightarrow a-b = mk$

$b \equiv c \pmod{m}$

$b-c = mk'$

$a-b+b-c = m(k+k')$

$a-c = m(k+k')$

$m \mid a-c \Rightarrow a \equiv c \pmod{m}$

Exercise:

1. Consider the relation R on the set of all people, where two people a and b are related, denoted aRb, if and only if a and b have the same birthday. Determine whether this relation is an equivalence relation or not.

2. Consider the relation R on the set of real numbers, where two numbers a and b are related, denoted aRb, if and only if a and b have the same absolute value. Determine whether this relation is an equivalence relation or not.

Partial Ordering Relation

A partial order $\leq$ on a set A satisfies the following properties for all elements $a, b, c \in A$:

- Reflexive
- Antisymmetric
- Transitive

Examples:

- **Subset Ordering:**
  - Set: $\mathcal{P}(X)$, the power set of a set X.
  - Relation: $A \subseteq B$ (subset relation).
  - Properties:
    - Reflexive: $A \subseteq A$ for all subsets $A \subseteq X$.
    - Antisymmetric: If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
    - Transitive: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

- **Divisibility Ordering:**
  - Set: $\mathbb{Z}^+$, the set of positive integers.
  - Relation: $a \mid b$ (a divides b).
  - Properties:
    - Reflexive: $a \mid a$ for all positive integers a.
    - Antisymmetric: If $a \mid b$ and $b \mid a$, then $a = b$.
    - Transitive: If $a \mid b$ and $b \mid c$, then $a \mid c$.

Matrix of Relations

Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be finite sets with cardinalities m and n, respectively. A relation R from A to B can be represented by an $m \times n$ matrix R defined as follows:

$$R_{ij} = \begin{cases} 1 & \text{if } a_i \, R \, b_j \\ 0 & \text{otherwise} \end{cases}$$

The matrix R is called the adjacency matrix (or the relation matrix) of the relation R.

Examples:

- Consider example where we define a relation R between sets $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$, with the relation $R = \{(1, b), (2, a), (2, c), (3, a), (4, b), (4, c)\}$.

  Matrix representation of relation R:

	a	b	c
1	0	1	0
2	1	0	1
3	1	0	0
4	0	1	1

  - Row 1 (1): $1Rb$
  - Row 2 (2): $2Ra, 2Rc$
  - Row 3 (3): $3Ra$
  - Row 4 (4): $4Rb, 4Rc$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$A = \{a_1, a_2, a_3\}$ $B = \{b_1, b_2, b_3, b_4\}$

3 4

$R = \{(a_1, b_2), (a_2, b_3), (a_3, b_1), (a_3, b_4)\}$

$$R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{3 \times 4} \end{array}$$

Exercise: Consider the sets $A = \{8, 9, 14, 21\}$ and $B = \{10, 15, 20, 30, 35\}$. Define a relation R such that two elements $a \in A$ and $b \in B$ are related, denoted aRb, if and only if a and b have the same parity (both even or both odd).

1. Determine whether this relation is an equivalence relation.

2. Construct the relation matrix for the sets A and B based on this parity relation.