Hence it leads to a contradiction. ■

1.15 PREDICATE CALCULUS

After getting some idea about statements and statement formulas, now let us pay attention to the concept of a predicate in an atomic statement. The logic associated with the predicates in any statement is called *predicate logic*. In this section, we shall discuss the predicates and predicate calculus. Let us first know what these predicates are?

Consider the two statements:

1. Dog is an animal.
2. Cat is an animal.

Since, both the statements are about two individuals which are animals, so expressing these statements by symbols will require two different symbols to denote them.

But, if a new symbol is introduced to denote the phrase, "is an animal" and if by some method we join it with symbols denoting the name of individuals, we are done. The part, "is an animal" is called a *predicate*. Writing the two statements in this form will reveal the common feature also.

We shall use capital letters to represent predicates and lower case letters to represent the names of individuals or objects in general. So, a statement can be written symbolically in terms of the predicate letter followed by the name(s) of the object(s) to which the predicate is applied. Let us again consider the above two statements. Suppose, the predicate, "is an animal" is represented by the predicate letter A, "Dog' by d and "Cat" by c. Then the statements (1) and (2) can be written as $A(d)$ and $A(c)$. In general, any statement of the type "r is S", where S is a predicate and r is the subject, can be written as $S(r)$. A statement which is expressed by using a predicate letter must have at least one name of an object associated with the predicate.

A predicate may require more than one name to express a statement. A predicate that requires $n (n > 0)$ names is known as an n-place predicate. Let us understand it with some examples:

3. The predicate A in (1) and (2) is a one place predicate
4. Consider the statement: x is greater than y.

Suppose predicate G represents "greater than". Then, G is a 2-place predicate since two names are required to express the statement as $G(x, y)$.

In general, if P is an n-place predicate letter and $x_1, x_2, \ldots, x_n$ are the names of objects, then, $P(x_1, x_2, \ldots, x_n)$ denotes a statement.

1.16 THE STATEMENT FUNCTION, VARIABLE AND QUANTIFIERS

Consider the phrase, "y is a man". Let this be expressed as $M(y)$. It may be noted that $M(y)$ is not a statement. However, if y is replaced by a name an such as a and a, b, s etc. then it will result in a statement. For example, $M(a)$, $M(b)$, and $M(s)$ represent statements.

We define a *simple statement function* of one variable as an expression which consists of a predicate symbol and an individual variable. Here, $M(y)$ is a simple statement function of one variable. The statement function turns to be a statement when the variable is replaced by the name of an object. The statement which is obtained from the statement function by a replacement is called a *substitution instance* of the statement function.

We can form *compound statement functions* by combining one or more simple statement functions and the logical connectives. Suppose $M(x)$ represents "x is a man" and $V(x)$ represents "x is a vegetarian". Then, we can compose many compound statements such as

$$M(x) \wedge V(x),\ {\sim}M(x) \vee V(x),\ {\sim}V(x) \text{ etc.}$$

It is possible to form statement functions of two variables by using statement function of one variable.

For example, let $M(x)$: x is a man

$V(y)$: y is a vegetarian,

Then, $M(x) \wedge V(y)$: x is a man and y is a vegetarian.

But, always it is not possible to express statement functions of two variables using statement functions of one variable.

So far, we have known that statements can be obtained from statement functions by replacing the variables with names of objects. There is another method to obtain the statements. In order to understand this alternative method, let us first consider the following statements:

5. All dogs are animals.
6. Every rose is red.

Clearly, each one is a statement about all individuals or objects belonging to a particular set.

We can also write the above statements in the following way:

(5′) For all x, if x is a dog, then x is an animal.

(6′) For all x, if x is a rose, then x is red.

Now, if a symbol is introduced which denote the phrase "For all x" then the statements (5′) and (6′) can be symbolized.

We use the symbol $(\forall x)$ or (x) to represent the phrase "For all x". Now, using the following:

$D(x)$: x is a dog

$A(x)$: x is an animal

$R(x)$: x is red
$C(x)$: x is rose

(5′) and (6′) can be written as:

5″. $\forall x(D(x) \rightarrow A(x))$

6″. $\forall x(C(x) \rightarrow R(x))$.

The symbols (x) or $\forall x$ are called *universal quantifiers* and represent "for all x", "Every x", "for any x". It may be noted that statements remain unchanged if the quantifying variable, say x, is changed or replaced by another variable, say y, throughout. For example, the statements $(x)\ (D(x) \rightarrow A(x))$ and $(y)\ (D(y) \rightarrow A(y))$ are equivalent.

Now let us introduce another quantifier which symbolizes the expression such as "there exists some" or "there is at least one". This quantifier is called *existential quantifier* and symbolized as $(\exists x)$. The area of logic that deals with predicates and quantifiers is called *predicate calculus*.

Example 1.31

(a) Symbolize the expression "Some roses are red".

Solution: $(\exists x)\ (C(x) \wedge R(x))$

(b) Symbolize the expression "Some numbers are irrational".

Solution:

Let I: x is irrational.
N: x is a number.

Then, the above statement can be symbolized as

$$(\exists x)\ (N(x) \wedge I(x))$$

It may be observed that a conjunction is used with the existential quantifier and an implication is used with the universal quantifier. ■

1.16.1 Predicate Formulas

From the earlier discussions, we know that $P(x_1, x_2, \ldots, x_n)$ denotes an n-place predicate formula in which P is the predicate and $(x_1, x_2, \ldots, x_n)$ are n individual variables or objects. Normally, the expression $P(x_1, x_2, \ldots, x_n)$ is called an *atomic formula* of predicate calculus. Now, we present some examples of atomic formulas.

Example 1.32 $S \quad M(y), \quad P(b, a, x), \quad M(x, y, z)$

Some rules should be kept in mind while obtaining a well-formed formula (wff) of a predicate calculus. The rules are:

1. An atomic formula is a well formed formula.
2. If A and B are well formed formulas (well-formed formulas), then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \equiv B)$ are also well-formed formulas.

3. If M is a well formed formula, then $\overline{M}$ is also a well-formed formula.
4. If M is a well formed formula, then $(x)M$ and $(\exists x)M$ are also well-formed formulas, where x is any variable.
5. Only the formulas which are obtained by applying rules (1) to (4) are well formed formulas.

Note: As mentioned earlier, we shall use the term "formula" for "well-formed formula" in rest of the chapter.

1.17 FREE AND BOUND VARIABLE

If any formula contains a part like $(\exists x)\ M(x)$ or $(\forall x)\ M(x)$, then that part is called *x-bound* part of the formula. Any occurrence of x in an x-bound part, is known as a *bound occurrence* of x. Any occurrence of x which is not a bound occurrence is known as *free occurrence.* The formula that immediately follows the quantifier is known as the *scope of the quantifier.* In other words, the *scope of a quantifier* is the part of a logical expression to which the quantifier is applied. So, a variable is free if it lies outside the scope of all quantifiers in the formula which specifies this variable. For example, the formula $M(x)$ in $(\exists x)\ M(x)$ or $(\forall x)\ M(x)$ is known as the *scope of the quantifier.*

Example 1.33

(i) $(x)\ M(x)$

Here, $M(x)$ is the scope of the quantifier

(ii) $(\exists x)\ (M(x) \wedge N(x))$

Here, $[M(x) \wedge N(x)]$ is the scope of quantifier and all occurrences of x are bound.

(iii) $M(x) \wedge \exists(x)\ N(x)$

Here, scope of $(\exists x)$ is $N(x)$ and it contains bound occurrence of x while occurrence of x in M is free. ■

It should be kept in mind that bound variable can be replaced by any other variable but not by a constant. Hence, the formulas $(x)\ M(y, x)$ and $(z)\ M(y, z)$ are same.

It may be observed that, if there is a free variable in the formula, then we have a statement function and in the case when every occurrence of a variable is bound and no variable has free occurrence, then we get a statement.

Example 1.34 Write the following predicate in symbolic form: "Someone in your school has visited Agra".

Solution: Let $S(x)$: x is in your school.

$A(x)$: x has visited Agra.

We symbolize the predicate as

$$(\exists x)\ (S(x) \wedge A(x)).$$

■