Recurrence Relations for decreasing functions

1) Substitution method

2) Master's theorem method

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ex-1

$$T(n) = \begin{cases} 1 & n = 0 \\ T(n-1) + 1 & n > 0 \end{cases}$$

① Solving By Substitution Method

$T(n) = T(n-1) + 1$ — ①

$T(n-1) = T(n-1-1) + 1$ — ②

$T(n-2) = T(n-2-1) + 1$ — ③

Substituting 2 in 1

$T(n) = T(n-2) + 1 + 1 = T(n-2) + 2$ — ④

Substituting 3 in 4

$T(n) = T(n-3) + 1 + 2$

$= T(n-3) + 3$

⋮

$= T(\underline{n-k}) + k$

$n - k = 0$

$n = k$

$= T(0) + n$

$= n+1$ Sol^n $O(n)$

ex-2

$$T(n) = \begin{cases} 1 & n=0 \\ T(n-1)+n & n>0 \end{cases}$$

Using Substitution method

$T(n) = T(n-1) + n$ — ①

$T(n-1) = T(n-1-1) + n$ — ②

$T(n-2) = T(n-3) + n$ — ③

Substituting 2 in 1

$T(n) = T(n-2) + n + n$

$T(n) = T(n-2) + 2n$ — ④

Substituting 3 in 4

$T(n) = T(n-3) + n + 2n$

$= T(n-3) + 3n$

⋮ k steps.

$= T(n-k) + kn$

$n-k=0$

$n=k$

$= T(0) + n \times n$

$= \underline{\underline{1 + n^2}}$ $\underline{\underline{\text{Sol}^n}} = O(n^2)$

<u>ex-3</u> $T(n) = \begin{cases} 1 & n = 0 \\ T(n-1) + \log n & n > 0 \end{cases}$

using <u>substitution</u> <u>method</u>

$T(n) = T(n-1) + \log n \quad -(1)$

$T(n-1) = T(n-2) + \log(n-1) \quad -(2)$

$T(n-2) = T(n-3) + \log(n-2) \quad -(3)$

Substituting 2 in 1

$T(n) = T(n-2) + \log(n-1) + \log n \quad -(4)$

Substituting 3 in 4

$T(n) = T(n-3) + \log(n-2) + \log(n-1) + \log n$

$= T(n-k) + \log 1 + \log 2 + \cdots \log(n-1) + \log n$

$\vdots$

$n - k = 0$

$n = k$

$= 1 + \log 1 + \log 2 + \cdots \log(n-1) + \log n$

$= 1 + \log(n)(n-1)\cdots(2)(1)$

$= 1 + \log n! \text{ ——— } O(n\log n)$

* Now, see the pattern

$T(n) = T(n-1) + 1 \text{ —— } O(n)$

$T(n) = T(n-1) + n \text{ —— } O(n^2)$

$T(n) = T(n-1) + \log n \text{ —— } O(n\log n).$

Q) $T(n) = T(n-1) + n^2$?

$= O(n^3)$

This analogy is Master's theorem

Master's Theorem for Decreasing functions

More patterns

1) $2T(n-1) + 1 \text{ —— } O(2^n)$

2) $T(n) = 3T(n-1) + 1 \text{ —— } O(3^n)$

3) $T(n) = 4T(n-1) + n \text{ —— } O(n4^n)$

Master's Theorem Rule

Let's consider the General form for recurrence relation of decreasing functions as :-

$$\boxed{T(n) = aT(n-b) + f(n)}$$

where we assume that

1) $a > 0$
2) $b > 0$
3) $f(n)$ is in form of n^k where $k \geq 0$

Rule 1

1) if $a = 1$

$soln = O(n * f(n))$

2) if $a > 1$

$soln = O(f(n) * a^n)$

if $b > 1$ then

$$\text{sol}^n = O\left(f(n) + a^{n/b}\right)$$

3) $a < 1$

$$\text{soln} = O(f(n))$$

This is Master's theorem for recurrence relation with decreasing functions.

Recurrence Relations :

Dividing functions

1) Substitution Method

ex-1

$$T(n) = \begin{cases} 1 & n = 1 \\ T(n/2) + 1 & n > 1 \end{cases}$$

$$T(n) = T(n/2) + 1 \quad \text{—} \ 1$$

$$T\left(\frac{n}{2}\right) = T\left(n/2^2\right) + 1 \quad \text{—} \ 2$$

$$T\left(n/2^2\right) = T\left(n/2^3\right) + 1 \quad \text{—} \ 3$$

Substituting 2 in 1

$$T(n) = T\left(n/2^2\right) + 2 \quad \text{—} \ 4$$

Substituting 3 in 4

$$T(n) = T\left(\frac{n}{2^3}\right) + 3$$

$$\vdots$$

$$= T\left(\frac{n}{2^k}\right) + K$$

$\Downarrow$

for this to be 1

$n/2^k = 1 \quad , \; n = 2^k$

$\log n = \log 2^k$

$k \log 2 = \log n$

$k = \log n$

$$T(n) = 1 + \log n$$

$$= O(\log n)$$

Answer.

ex-2

$$T(n) = \begin{cases} 1 & n = 1 \\ T(n/2) + n & n > 0 \end{cases}$$

$T(n) = T(n/2) + n$ —— 1

$T(n/2) = T(n/2^2) + n/2$ —— 2

$T(n/2^2) = T(n/2^3) + n/2^2$ —— 3

Substituting 2 in 1

$T(n) = T(n/2^2) + \frac{n}{2} + n$ —— 4

Substituting 3 in 4

$$T(n) = T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} + \frac{n}{2} + n \text{ ---}$$

$$\vdots$$

$$= \underbrace{T\left(\frac{n}{2^k}\right)}_{\text{This has to be 1}} + \frac{n}{2^{k-1}} + \frac{n}{2^{k-2}} + \text{---} \frac{n}{2} + n$$

$\therefore k = \log n$.

$$= 1 + n\left[\underbrace{\frac{1}{2^{k-1}} + \frac{1}{2^{k-2}} + \text{---} + \frac{1}{2}}_{\text{This part is 1}} + 1\right]$$

$$= 1 + n[1 + 1]$$

$$= 1 + 2n = O(2n) \text{ Ans.}$$

ex-3

$$T(n) = \begin{cases} 1 & n = 1 \\ 2T(n/2) + n & n > 0 \end{cases}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n \quad \text{—— } 1$$

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{2^2}\right) + \frac{n}{2} \quad \text{—— } 2$$

$$T\left(\frac{n}{2^2}\right) = 2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2} \quad \text{—— } 3$$

Substituting 2 in 1

$$T(n) = 2\left[2T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right] + n$$

$$T(n) = 4T\left(\frac{n}{2^2}\right) + n + n \quad \text{—— } 4$$

Substituting 3 in 4

$$T(n) = 4\left[2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right] + n + n$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + n + n + n$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 3n$$

⋮ k times

$n =$

$$= 2^k T\left(\frac{n}{2^k}\right) + kn$$

$\Rightarrow$ This will be 1 when $k = \log n$

$\frac{n}{2^k} = 1$

$$= n \times 1 + \underbrace{n \log n}_{\text{Bigger}}$$

$n = 2^k$

$k = \log n$

$$= \underline{O(n \log n)} \text{ Answer}$$

Master's Theorem for Dividing functions

General form of recurrence relations for dividing functions.

$$\boxed{T(n) = aT\left(\frac{n}{b}\right) + f(n)}$$

where we assume that

1) $a \geq 1$
2) $b > 1$
3) $f(n)$ is in form $(n^k \log^p n)$

For finding solution using Master's Theorem, we need to find 1) $\log_b^a$ & 2) K.

Based on these values we have many cases & many solutions as well which we will see one by one.

Case 1 : if $\log_b^a > k$ then

soln $\Theta(n^{\log_b^a})$

Case 2 : if $\log_b^a = k$, then

a) if $p > -1$, soln $= \Theta(n^k \log^{p+1} n)$

b) if $p = -1$, soln $= \Theta(n^k \log \log n)$

c) if $p < -1$, soln $= \Theta(n^k)$

Case 3 :- if $\log_b^a < k$, then.

a) if $p \geq 0$, soln $= \Theta(n^k \log^p n)$

b) if $p < 0$, soln $= O(n^k)$

ex-1 $T(n) = 2T(n/2) + 1$

$a = 2,\ b = 2$

$f(n) = 1 = n^0 \log^0 n$

$k = 0,\ \log_2^2 = 1$

Case 1 $\log_2^2 > k$

then $\Theta(n^{\log_2^2}) = \Theta(n)$

ex-2 $T(n) = 4T(n/2) + n$

$a = 4,\ b = 2\ \ f(n) = n = (n^1 \log^0 n)$

$\log_2^4 = 2,\ \ k = 1$

Case 1 $\log_b^a > k$

then soln $= \Theta(n^{\log_2^4}) = \underline{\underline{\Theta(n^2)}}$

ex-3 $T(n) = 8T(n/2) + n$

$a = 8,\ b = 2$

$\log_2^8 = 3\ \ ,f(n) = n = n^1 \log^0 n$

$k = 1$

$\log_b^a > K$

Case 1

$sol^n = \Theta(n^{\log_b^a}) = \Theta(n^3)$

ex-4 $T(n) = 9T\left(\frac{n}{3}\right) + 1$

$a = 9, \; b = 3, \; \log_3^9 = 2$

$f(n) = 1, \; n^0 \log^0 n =$

$= K = 0, \; P = 0$

$\log_b^a > K$ Case 1

Ans $= \Theta(n^{\log_b^a}) = \Theta(n^2)$

ex-5 $T(n) = 4T(n/_2) + n$

$a = 4, \; b = 2 \quad f(n) = n^1 \log^0 n$

$K = 1$

Case 1 $\log_2^4 = 2 > K$

Ans $= \Theta(n^2)$

ex-6 $T(n) = 8T(n/_2) + n\log n$

$a = 8, \; b = 2 \quad f(n) = n^1 \log^1 n$

$\log_2^8 = 3 \quad |$

$K = 1, \; p = 1$

$\log_b^a > K$

$\therefore$ Ans $= \Theta(n^3)$

ex-7 $\quad T(n) = 2T(n/2) + n$

$a = 2, \quad b = 2$

$\log_2^2 = 1 \quad , f(n) = n^1 \log^0 n \; , K = 1$

$\log_b^a = K \quad$, Case 2

Checking P, $\quad P = 0 \quad P > -1$

Case 2a $\quad$ Ans. $= \Theta(n^k \log^{p+1} n)$

$= \Theta(n \log n)$

ex-8 $\quad T(n) = 4T(n/2) + n^2$

$\log_b^a = \log_2^4 = 2, \quad f(n) = n^2 \log^0 n$

$\Rightarrow K = 2, P = 0.$

$\log_b^a = K$

Case 2a $\quad$ Ans $= (n^k \log^{p+1} n)$

$= \Theta(n^2 \log n)$

ex-9 $\quad T(n) = 4T(n/2) + n^2 (\log n)$

$a = 4,\ b = 2$ $\qquad f(n) = n^2 \log n$

$\log_2^4 = 2 \quad |$ $\qquad = n^k \log^p n$

$\Rightarrow k = 2,\ p = 1$

$\log_b^a = k$

Case 2 $\quad p = 1 > -1$

Ans = $\Theta(n^2 \log \log n) =$

$= \Theta(n^2 \log^2 n)$

ex-10 $\quad T(n) = 8T(n/2) + n^3$

$\log_2^8 = 3, \quad k = 3 \quad , p = 0$

Ans. by Case 2a $= \Theta(n^3 \log n)$

ex-11 $\quad T(n) = 2T(n/2) + \frac{n}{\log n}$

$= 2T(n/2) + n \log^{-1} n$

$a = 2,\ b = 2$

$\log_2^2 = 1,\ k = 1 \quad p = -1$

Case 2 b $\quad$ Ans = $(n \log \log n)$

ex-12 $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log^2 n}$

$T(n) = 2T\left(\frac{n}{2}\right) + n\log^{-2} n$

$a=2, b=2, \log_2^2 = 1, k=1, p=-2$

Case 2c, Ans $= \Theta(n^k) = \Theta(n)$

ex-13 $T(n) = T\left(\frac{n}{2}\right) + n^2$

$a=1, b=2$ $f(n) = n^2 \log^0 n$

$\log_2^1 = 0$

$K=2, P=0$

Case 3 $\log_b^a < K$

$P \geq 0$ Ans $\Theta(n^2 \log^0 n)$

$= \Theta(n^2)$

ex-14 $T(n) = 2T\left(\frac{n}{2}\right) + n^2$

$a=2, b=2$

$\log_2^2 = 1,$ $K=2, P=0$

Case 3 Ans. $\Theta(n^k \log^0 n) = \Theta(n^2)$

ex-15 $T(n) = 2T(n/2) + n^2\log^2 n$

$a = 2,\ b = 2$

$\log_2^2 = 1,\quad k = 2 \quad p = 2$

Case 3a $\Rightarrow$ Ans $\Theta(n^2\log^2 n)$

ex-16 $T(n) = 4T(n/2) + \dfrac{n^3}{\log n}$

$T(n) = 4T(n/2) + n^3\log^{-1} n$

$a = 4,\ b = 2 \quad k = 3,\ p = -1$

$\log_2^4 = 2$

$\log_b^a < k$, Case 3

$p < 0$, Case 3b

Ans: $\Theta(n^k) = \Theta(n^3)$