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Differential Equations

Generic Elective - 3

Mathematics for Hons. Courses, Semester - III

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CONTENTS

CHAPTER

1

First-Order ODEs

LEARNING OBJECTIVES

After studying the material in this chapter, you should be able to :

- ***Understand the meaning of a differential equation.***
- ***Differentiate between ordinary differential equations and partial differential equations.***
- ***Know basic concepts of ordinary differential equations.***
- ***Differentiate between the general solution and the particular solution of a first-order ordinary differential equation.***
- ***Solve the separable equations.***
- ***Solve the homogeneous differential equations.***
- ***Test the first-order differential equation for exactness and find its solution.***
- ***Find the integrating factor of a non-exact differential equation.***
- ***Identify and solve the first-order linear differential equations.***
- ***Reduce Bernoulli's equation to linear form and then solve it.***
- ***Form a differential equation representing a given family of curves.***
- ***Determine orthogonal trajectories of a given family of curves.***
- ***Discuss the questions of existence and uniqueness of solutions of first-order initial value problem.***

1.1 INTRODUCTION TO DIFFERENTIAL EQUATIONS

Many physical problems are often formulated as mathematical expressions involving variables, functions, equations, and so forth. Any such mathematical expression representing a physical problem is called a **mathematical model** for that problem. Since many physical concepts, such as a velocity and acceleration, are represented mathematically by derivatives, mathematical models often involve equations relating an unknown function and one or more of its derivatives. Such equations are called **differential equations**.

1.2 DIFFERENTIAL EQUATIONS : BASIC CONCEPTS

A **differential equation (DE)** is an equation involving an unknown function and one or more of its derivatives.

The following are a few examples of differential equations :

EXAMPLE 1

(*a*) $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$ (*b*) $\dfrac{d^2y}{dx^2} + 4y = 0$ (*c*) $\left(\dfrac{dy}{dx}\right)^4 + 3y\dfrac{d^2y}{dx^2} = 0$

(*d*) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (*e*) $\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}$ (*f*) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$

Differential equations can be classified as *ordinary* or *partial* differential equations.

1. An **ordinary differential equation (ODE)** is an equation that contains one or more derivatives of an unknown function of a single independent variable.

   For example, equations (*a*), (*b*), and (*c*) of Example 1 are ordinary differential equations – the unknown function *y* depends on a single independent variable *x*.

2. A **partial differential equation (PDE)** is an equation involving one or more partial derivatives of an unknown function that depends on two or more independent variables.

   For example, equations (*d*), (*e*), and (*f*) of Example 1 are partial differential equations – the unknown function *u* depends on two or more independent variables and their partial derivatives appear in the equation.

Typically, partial differential equations (PDEs) are more complicated than ordinary differential equations (ODEs); they will be considered in Chapters 6 and 7. In this chapter we shall be concerned with ordinary differential equations.

Order of an Ordinary Differential Equation

Ordinary differential equations are classified according to their order.

The **order** of an ordinary differential equation is the order of the highest derivative that appears in the equation. For instance, in Example 1, the differential equation (*a*) is of first order, equations (*b*) and (*c*) are of second order, and

$$x^3\frac{d^3y}{dx^3} - 3x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} - 6y = 0$$

is a third-order equation. The most general form of an *n*th-order ordinary differential equation with independent variable *x* and unknown function or dependent variable *y* is

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0, \qquad \left(y^{(k)} = \frac{d^k y}{dx^k}\right)$$

where F is a real-valued function of $n + 2$ variables. In particular, the most general form of a first-order ODE is

$$F(x, y, y') = 0 \qquad \left(y' = \frac{dy}{dx}\right)$$

or in explicit form

$$y' = F(x, y)$$

In this chapter we shall consider first-order ODEs.

Degree of an Ordinary Differential Equation

The **degree** of an ordinary differential equation is the degree(power) of the derivative of the highest order in the equation after it has been expressed in a form free from radicals and fractions so far as derivatives are concerned.

Thus to find the degree of a differential equation, we must express the equation as a polynomial in derivatives.

EXAMPLE 2 Find the order and degree of each of the following ODEs.

(*i*) $\dfrac{dy}{dx} = e^x$ (*ii*) $\dfrac{d^2y}{dx^2} + y = 0$

(*iii*) $\left(\dfrac{d^2y}{dx^2}\right)^2 + x^2\left(\dfrac{dy}{dx}\right)^3 = 0$ (*iv*) $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = \dfrac{d^2y}{dx^2}$

(*v*) $x\dfrac{dy}{dx} + \dfrac{1}{dy/dx} = y$ (*vi*) $\left(\dfrac{dy}{dx}\right)^4 + 3y\dfrac{d^2y}{dx^2} = 0$

SOLUTION (*i*) The given differential equation is $\dfrac{dy}{dx} = e^x$.

In this equation, the order of the highest derivative is 1 and its power is 1. So, it is a differential equation of order 1 and degree 1.

(*ii*) The given differential equation is $\dfrac{d^2y}{dx^2} + y = 0$.

In this equation, the order of the highest derivative is 2 and its power is 1. So, it is a differential equation of order 2 and degree 1.

(*iii*) The given differential equation is $\left(\dfrac{d^2y}{dx^2}\right)^2 + x^2\left(\dfrac{dy}{dx}\right)^3 = 0$.

In this equation, the order of the highest derivative is 2 and its power is 2. So, it is a differential equation of order 2 and degree 2.

(*iv*) The given differential equation is $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = \dfrac{d^2y}{dx^2}$.

In this equation, the order of the highest derivative is 2. So, its order is 2. To find its degree, we express the differential equation as a polynomial in derivatives. To do so, we square both sides and obtain

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

Thus the power of the highest order derivative is 2 and hence its degree is 2.

(*v*) The given differential equation is $x\frac{dy}{dx}+\frac{1}{dy/dx}=y$.

In this equation, the order of the highest derivative is 1. So, its order is 1. To find its degree, we express the equation as a polynomial in derivatives :

$$x\left(\frac{dy}{dx}\right)^2+1 = y\frac{dy}{dx}$$

Thus the power of the highest order derivative is 2 and hence its degree is 2.

(*vi*) The given differential equation is $\left(\frac{dy}{dx}\right)^4+3y\frac{d^2y}{dx^2}=0$.

In this equation, the order of the highest derivative is 2 and its power is 1. So, it is a differential equation of order 2 and degree 1.

Concept of Solution

DEFINITION Solution of a first-order ODE

A function $y=f(x)$ is called a **solution** of a first-order ODE

$$F(x, y, y') = 0, \qquad \ldots(1)$$

on some open interval (a, b) if $f(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity when y and y' are replaced with $f(x)$ and $f'(x)$, respectively. For the sake of brevity, we may say that $y=f(x)$ satisfies the differential equation (1) on (a, b). The graph of $y=f(x)$ is called a **solution curve.**

For example, the function

$$y=e^{-x}$$

is a solution of the differential equation

$$y'+y=0$$

To verify this, we substitute $y=e^{-x}$ and $y'=-e^{-x}$ in the differential equation to get

$$y'+y=-e^{-x}+e^{-x}=0$$

This shows that y and its derivative y' satisfy the differential equation $y'+y=0$. Thus, $y=e^{-x}$ is a solution of this differential equation. Similary, it can be shown that $y=2e^{-x}$, $y=-3e^{-x}$, and

$y = 4e^{-x}$ are also solutions of the differential equation. In fact, for any constant c, the function

$$y = ce^{-x}$$

is a solution of the equation. Such a solution containing an arbitrary constant c is called a **general solution** of the ODE.

Geometrically, the general solution of an ODE represents a family of infinitely many solutions curves, one for each value of the constant c. If we choose a specific value of c, we obtain what is called a **particular solution** of the ODE. Thus, a particular solution of an ODE is a solution that is obtained from the general solution by assigning a suitable value to c.

EXAMPLE 3 Verification of Solution

Verify that $y = \tan(x + c)$ is a solution of the differential equation $y' = 1 + y^2$.

SOLUTION Differentiation of $y = \tan(x + c)$ gives

$$y' = \sec^2(x + c)$$

Substituting the expressions for y and y' into the differential equation, we obtain

$$\sec^2(x + c) = 1 + \tan^2(x + c) = \sec^2(x + c),$$

an identity. Thus, $y = \tan(x + c)$ is a solution.

More generally, we say that a function $y = f(x)$ is a solution of the nth-order ordinary differential equation

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

on some open interval (a, b) provided that the derivatives $f', f'', \ldots, f^{(n)}$ exist on (a, b) and

$$F(x, y, f', f'', \ldots, f^{(n)}) = 0 \qquad \ldots(2)$$

for all $x \in (a, b)$, that is, the differential equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives. A solution of the nth-order ordinary differential equation (2) containing n arbitrary constants is called the **general solution** or the **complete integral** of the ODE.

EXAMPLE 4 Verify that $y = a \cos \pi x + b \sin \pi x$ is a solution of the differential equation

$$y'' + \pi^2 y = 0.$$

SOLUTION Given

$$y = a \cos \pi x + b \sin \pi x$$

$$\therefore \quad y' = -\pi a \sin \pi x + \pi b \cos \pi x$$

$$y'' = -\pi^2 a \cos \pi x - \pi^2 b \sin \pi x$$

$$= -\pi^2(a \cos \pi x + b \sin \pi x)$$

$$= -\pi^2 y$$

$$\Rightarrow \quad y'' + \pi^2 y = 0$$

This shows that $y = a \cos \pi x + b \sin \pi x$ is a solution of the given differential equation.

EXAMPLE 5 Verify that $y = Ax + \frac{B}{x}$ is a solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

SOLUTION Given

$$y = Ax + \frac{B}{x}$$

$$\therefore \quad \frac{dy}{dx} = A - \frac{B}{x^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{2B}{x^3}$$

Hence

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2\left(\frac{2B}{x^3}\right) + x\left(A - \frac{B}{x^2}\right) - \left(Ax + \frac{B}{x}\right)$$

$$= \frac{2B}{x} + Ax - \frac{B}{x} - Ax - \frac{B}{x} = 0$$

This shows that y and the derivatives obtained from it satisfy the given differential equation. Thus $y = Ax + \frac{B}{x}$ is a solution of the given differential equation.

EXAMPLE 6 Verify that $y = e^{-x} + ax + b$ is a solution of the differential equation $e^x y'' = 1$.

SOLUTION Given

$$y = e^{-x} + ax + b$$

$$\therefore \quad y' = -e^{-x} + a \quad \text{and} \quad y'' = e^{-x}$$

$$\Rightarrow \quad e^x y'' = e^x \cdot e^{-x} = 1$$

This shows that y satisfies the given differential equation. Hence $y = e^{-x} + ax + b$ is a solution of the given differential equation.

EXAMPLE 7 Verify that $y = be^x + ce^{2x}$ is a solution of the differential equation

$$y'' - 3y' + 2y = 0$$

SOLUTION Given

$$y = be^x + ce^{2x}$$

$$\therefore \quad y' = be^x + 2ce^{2x} \quad \text{and} \quad y'' = be^x + 4ce^{2x}$$

Hence $\quad y'' - 3y' + 2y = (be^x + 4ce^{2x}) - 3(be^x + 2ce^{2x}) + 2(be^x + ce^{2x}) = 0$

In other words, y and the derivatives obtained from it satisfy the given differential equation. Thus $y = be^x + ce^{2x}$ is a solution of the given differential equation.

EXAMPLE 8 Verify that $y = -\sin x + ax^2 + bx + c$ is a solution of the ODE $y''' = \cos x$.

SOLUTION Differentiating the function $y = -\sin x + ax^2 + bx + c$ thrice with respect x, we get

$$y' = -\cos x + 2ax + b$$

$$y'' = \sin x + 2a$$

$$y''' = \cos x$$

That is, y satisfies the given ODE. Thus, $y = -\sin x + ax^2 + bx + c$ is a solution of the given ODE.

Initial Value Problem (IVP)

A first-order ODE usually has a general solution containing an arbitrary constant, say c. In applications, we usually have to find a unique solution, hence a particular solution, by determining the value of c from a given initial condition, say, the condition that at some point x_0, the solution has the value y_0. This is usually expressed as $y(x_0) = y_0$. Geometrically, this condition means that the solution curve should pass through the pointt (x_0, y_0) in the xy-plane.

A first-order ODE together with an initial condition is called an **initial value problem (IVP)**. Thus, if the first-order ODE is given in explicit form $y' = f(x, y)$, the initial value problem is of the form

$$y' = f(x, y), \qquad y(x_0) = y_0. \qquad \textit{(Initial value problem)}$$

EXAMPLE 9 Verify that $y = ce^x + x + 1$ is a solution of the differential equation $y' = y - x$. Also determine the particular solution satisfying the initial condition $y(0) = 3$.

SOLUTION We have

$$y = ce^x + x + 1 \qquad ...(1)$$

Differentiating both sides with respect to x, we get

$$y' = ce^x + 1 = (ce^x + x + 1) - x = y - x$$

i.e., $$y' = y - x$$

This shows that $y = ce^x + x + 1$ is a solution of the given differential equation. To determine the particular solution that satisfies the initial condition $y(0) = 3$, that is, $y = 3$ when $x = 0$, we substitute $y = 3$ and $x = 0$ into Equation (1). This gives

$$3 = c + 1 \quad \Rightarrow \quad c = 2$$

Hence the particular solution satisfying the initial condition is

$$y = 2e^x + x + 1.$$

EXAMPLE 10 Verify that $y = ce^{0.5x}$ is a solution of the differential equation $y' = 0.5y$. Also determine the particular solution satisfying the initial condition $y(2) = 2$.

SOLUTION We have

$$y = ce^{0.5x} \qquad ...(1)$$

Differentiating both sides with respect to x, we get

$$y' = 0.5ce^{0.5x} = 0.5y$$

This shows that $y = ce^{0.5x}$ is a solution of the ODE $y' = 0.5y$. To determine the particular solution that satisfies the initial condition $y(2) = 2$, that is, $y = 2$ when $x = 2$, we substitute $y = 2$ and $x = 2$ into Equation (1). This gives

$$2 = ce^{0.5(2)} = ce \quad \Rightarrow \quad c = 2/e$$

Hence the particular solution satisfying the initial condition is

$$y = \frac{2}{e} e^{0.5x}.$$

EXERCISE 1.1

Theory Questions :

1. Explain the terms ordinary differential equation (ODE), partial differential equation (PDE), order, general solution, and particular solution. Give examples.

2. What is an initial condition ? How is this condition used in an initial value problem ?

Practical Questions :

1. Verify that $y = -e^{-3x}/3 + c$ is a solution of the differential equation $y' = e^{-3x}$.

2. Verify that $y = 5e^{-2x} + 2x^2 + 2x + 1$ is a solution of the differential equation
$$y' + 2y = 4(x+1)^2.$$

3. Verify that $y = c/x$ is a solution of the differential equation $xy' = -y$.

4. Verify that $y = ae^{2x} + be^{-x}$ is a solution of the differential equation $y'' - y' - 2y = 0$.

5. Verify that $y = e^{3x}(A + Bx)$ is a solution of the differential equation $y'' - 6y' + 9y = 0$.

6. Verify that $y = cx^4$ is a solution of the differential equation $xy' - 4y = 0$.

7. Verify that $y = x^3 + ax^2 + bx + c$ is a solution of the differential equation $y''' = 6$.

8. Verify that $xy = ae^x + bx^{-x} + x^2$ is a solution of the differential equation
$$xy'' + 2y' - xy + x^2 - 2 = 0.$$

9. Verify that $y = (x^3 - x)\log cx$ is a solution of the differential equation
$$(x^3 - x)y' - (3x^2 - 1)y = x(x^2 - 1)^2.$$

10. Verify that $y = Ae^{Bx}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} = \dfrac{1}{y}\left(\dfrac{dy}{dx}\right)^2$.

11. Verify that $y = ce^{3x}$ is a solution of the differential equation $y' = 3y$. Also determine the particular solution satisfying the initial condition $y(0) = 5.7$.

12. Verify that $y = \dfrac{1}{2}\tan(2x + c)$ is a solution of the differential equation $y' = 1 + 4y^2$. Also determine the particular solution satisfying the initial condition $y(0) = 0$.

13. Verify that $y = c\sec x$ is a solution of the differential equation $y' = y\tan x$. Also determine the particular solution satisfying the initial condition $y(0) = \pi/2$.

ANSWERS

11. $y = 5.7e^{3x}$ **12.** $\frac{1}{2}(\tan 2x + n\pi), n = 0, \pm 1, \pm 2, \ldots$ **13.** $y = \frac{\pi}{2}\sec x$

1.3 SOLVING A FIRST-ORDER ORDINARY DIFFERENTIAL EQUATION

In the previous section we discussed the notion of a solution of first-order ODE, the general form of which is given by

$$F(x, y, y') = 0$$

In the remainder of this chapter, we shall restrict ourselves mostly to first-order ODEs which can be written in the so-called **normal form**

$$y' = f(x, y)$$

or in the **standard differential form**

$$M(x, y)dx + N(x, y)dy = 0.$$

In the following, some special types of first-order differential equations are discussed.

Differential Equations of Type $\frac{dy}{dx} = f(x)$

A differential equation of the form

$$\frac{dy}{dx} = f(x) \qquad \ldots(1)$$

can be solved by integrating both sides with respect to x. Thus integrating (1), we get

$$y = \int f(x)dx + c \qquad (c \text{ is a constant})$$

which is a solution of (1).

EXAMPLE 11 Solve the differential equation $\frac{dy}{dx} = 6x^2 + 2x + 3$.

SOLUTION We have

$$\frac{dy}{dx} = 6x^2 + 2x + 3$$

Integrating, we get $\quad y = \int (6x^2 + 2x + 3)dx = 2x^3 + x^2 + 3x + c$

i.e., $\quad y = 2x^3 + x^2 + 3x + c$, which is the required solution.

EXAMPLE 12 Solve the differential equation $(e^x + e^{-x})dy = (e^x - e^{-x})dx$.

SOLUTION We have

$$(e^x + e^{-x})dy = (e^x - e^{-x})dx \quad \Rightarrow \quad \frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Integrating, we get $\qquad y = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \log(e^x + e^{-x}) + c$

i.e., $\qquad y = \log(e^x + e^{-x}) + c$, which is the required solution.

EXAMPLE 13 Solve the differential equation $\sqrt{a+x}\,\frac{dy}{dx} + x = 0$.

SOLUTION The given equation is :

$$\sqrt{a+x}\,\frac{dy}{dx} + x = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{\sqrt{a+x}}$$

Integrating, we get $\qquad y = -\int \frac{x}{\sqrt{a+x}}\,dx = -\int \frac{a+x-a}{\sqrt{a+x}}\,dx$

$$= -\int \sqrt{a+x}\,dx + a\int (a+x)^{-1/2}\,dx$$

$$= -\frac{(a+x)^{3/2}}{3/2} + a\cdot\frac{(a+x)^{1/2}}{1/2} + c$$

i.e., $\qquad y = -\frac{2}{3}(a+x)^{3/2} + 2a\sqrt{a+x} + c$

which is the required solution.

EXAMPLE 14 Solve the initial value problem

$$\frac{dy}{dx} = 2x^3 - x^2 + x - 5; \quad y(0) = 1.$$

SOLUTION The given equation is

$$\frac{dy}{dx} = 2x^3 - x^2 + x - 5$$

Integrating, we get $\qquad y = \frac{2x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - 5x + c \qquad \text{...(1)}$

which is a general solution of the given differential equation. To determine the particular solution that satisfies the initial condition $y(0) = 1$, that is, $y = 1$ when $x = 0$, we substitute $y = 1$ and $x = 0$ into Eq. (1). This gives $c = 1$. Hence the initial value problem has the solution

$$y = \frac{x^4}{2} - \frac{x^3}{3} + \frac{x^2}{2} - 5x + 1.$$

EXERCISE 1.2

Solve each of the following differential equations.

1. $\frac{dy}{dx} = 3x^2 + 2x + 1$ $\qquad$ 2. $\frac{dy}{dx} = x^2 + x - \frac{1}{x}$ $\qquad$ 3. $\frac{dy}{dx} = \frac{x^2 + 4x - 9}{x + 2}$ $\qquad$ 4. $\frac{dy}{dx} = \frac{x}{x^2 + 1}$

5. $\dfrac{dy}{dx} = \log x$ 6. $\dfrac{dy}{dx} = x \log x$ 7. $e^x\left(\dfrac{dy}{dx}\right)+1 = x$ 8. $\sqrt{a-x}dy + xdx = 0$

Solve the following initial value problems.

9. $y' = 6x^2 + 2x + 3, y(0) = 4$ 10. $y' = 2x \log x + x,\ y(2) = 0$

ANSWERS

1. $y = x^3 + x^2 + x + c$
2. $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} - \log x + c$
3. $y = \dfrac{x^2}{2} + 2x - 13\log(x+2) + c$
4. $y = \dfrac{1}{2}\log(x^2 + 1) + c$
5. $y = x \log x - x + c$
6. $y = \dfrac{1}{2}x^2 \log x - \dfrac{1}{4}x^2 + c$
7. $y = -xe^{-x} + c$
8. $y = 2a\sqrt{a-x} - \dfrac{2}{3}(a-x)^{3/2} + c$
9. $y = 2x^3 + x^2 + 3x + 4$
10. $y = x^2(\log x) - 4\log 2$

1.4 DIFFERENTIAL EQUATIONS WITH SEPARABLE VARIABLES

A differential equation of the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad \text{i.e.,} \quad g(y)dy = f(x)dx$$

where $f(x)$ and $g(y)$ are, respectively, functions of x and y, is called a **differential equation with separable variables** or simply a **separable equation**. Thus, a differential equation is said to have separable variables if it is so expressed that the coefficient of dx is only a function of x and that of dy a function of y only.

The solution of such an equation is obtained by integrating each term separately. Hence, its solution is of the form

$$\int g(y)dy = \int f(x)dx + c,$$

where c is an arbitrary constant.

EXAMPLE 15 Solve the differential equation $(1 - y^2)dy - x^2 dx = 0$.

SOLUTION We are given

$$(1 - y^2)dy - x^2 dx = 0,$$

an equation with separable variables. Hence, integrating each term, we get

$$\int(1-y^2)dy - \int x^2 dx = c \quad \text{i.e.,} \quad y - \frac{y^3}{3} - \frac{x^3}{3} = c,$$

which is the required solution.

EXAMPLE 16 Solve the differential equation $y' + (x + 2)y^2 = 0$.

SOLUTION The given ODE can be written as

$$\frac{1}{y^2}dy = -(x + 2)dx,$$

an equation with separable variables. Integrating both sides, we get

$$-\frac{1}{y} = -\frac{x^2}{2} - 2x + k = -\left(\frac{x^2 + 4x - 2k}{2}\right)$$

or, $$y = \frac{2}{x^2 + 4x + c},$$

where $c = -2k$ is an arbitrary constant. This is a general solution.

EXAMPLE 17 Solve the differential equation $y' = 1 + y^2$.

SOLUTION The given ODE is separable because it can be written as

$$\frac{1}{1 + y^2}dy = dx$$

Integrating both sides, we get

$$\tan^{-1}y = x + c \quad \Rightarrow \quad y = \tan(x + c),$$

which is the general solution.

Note It is important to mention that the constant of integration is introduced immediately when the integration is performed. For instance, in the preceding example, if we had written $\tan^{-1}y = x$, then $y = \tan x$, and then introduced c, we would have obtained $y = \tan x + c$, which is not a solution of the given ODE (when $c \neq 0$).

Remark A differential equation of the form

$$f_1(x)\,g_1(y)\,dx + f_2(x)\,g_2(y)\,dy = 0$$

can easily be reduced to having separable variables by dividing both sides by $g_1(y)f_2(x)$, and hence can be solved by the technique just discussed.

EXAMPLE 18 Solve the differential equation $x(1 + y^2)dx + y(1 + x^2)dy = 0$.

SOLUTION The given equation is

$$x(1 + y^2)dx + y(1 + x^2)dy = 0$$

Dividing both sides by $(1 + x^2)(1 + y^2)$, we get

$$\frac{x}{1+x^2}dx + \frac{y}{1+y^2}dy = 0 \qquad \text{(a separable equation)}$$

Integrating each term, we get

$$\int \frac{x}{1+x^2}dx + \int \frac{y}{1+y^2}dy = c \quad i.e., \quad \frac{1}{2}\int \frac{2x}{1+x^2}dx + \frac{1}{2}\int \frac{2y}{1+y^2}dy = c$$

$$\Rightarrow \quad \frac{1}{2}\log(1+x^2) + \frac{1}{2}\log(1+y^2) = c \Rightarrow \log(1+x^2) + \log(1+y^2) = 2c$$

or, $\log(1+x^2)(1+y^2) = 2c \Rightarrow (1+x^2)(1+y^2) = e^{2c} = k$ (say)

i.e., $(1+x^2)(1+y^2) = k,$

which is the required solution.

EXAMPLE 19 Solve the differential equation $3e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0.$

SOLUTION The given equation can be written as

$$3e^x \tan y\, dx = (e^x - 1)\sec^2 y\, dy$$

Dividing both sides by $\tan y(e^x - 1)$, we obtain

$$\frac{3e^x}{e^x-1}dx = \frac{\sec^2 y}{\tan y}dy \qquad \textit{(a separable equation)}$$

Integrating, we get

$$\log \tan y = 3\log(e^x - 1) + c,$$

which is the general solution.

EXAMPLE 20 Solve the differential equation $y - x\dfrac{dy}{dx} = \dfrac{y}{x}.$

SOLUTION The given equation can be written as

$$xy - x^2\frac{dy}{dx} = y \quad \Rightarrow \quad xy - y = x^2\frac{dy}{dx} \quad \text{or} \quad x^2\frac{dy}{dx} = y(x-1)$$

$$\Rightarrow \quad \frac{dy}{y} = \left(\frac{x-1}{x^2}\right)dx \quad i.e., \quad \frac{1}{y}dy = \left(\frac{1}{x} - \frac{1}{x^2}\right)dx \qquad \textit{(a separable equation)}$$

Intergrating both sides, we get

$$\log y = \log x + \frac{1}{x} + c,$$

which is the required solution.

EXAMPLE 21 Solve the differential equation $(x+1)\dfrac{dy}{dx} = 2xy.$

SOLUTION We have

$$(x+1)\frac{dy}{dx} = 2xy \quad \Rightarrow \quad \frac{dy}{y} = \frac{2x}{x+1}dx$$

Integrating both sides, we get

$$\int \frac{1}{y}dy = \int \frac{2x}{x+1}dx + c \quad \text{or} \quad \int \frac{1}{y}dy = 2\int \frac{(x+1)-1}{x+1}dx + c$$

or,
$$\int \frac{1}{y}dy = 2\int dx - 2\int \frac{1}{x+1}dx + c$$

$\Rightarrow$
$$\log y = 2x - 2\log(x+1) + c,$$

which is the required solution.

EXAMPLE 22 Solve the following differential equation $\frac{dy}{dx} = 1 + x + y + xy$.

SOLUTION The given equation is

$$\frac{dy}{dx} = 1 + x + y + xy \quad \text{or} \quad \frac{dy}{dx} = (1+x) + (1+x)y = (1+x)(1+y)$$

$\Rightarrow$
$$\frac{1}{1+y}dy = (1+x)dx \qquad (\textit{a separable equation})$$

Integrating both sides, we get

$$\int \frac{1}{1+y}dy = \int (1+x)dx + c$$

or
$$\log(1+y) = x + \frac{x^2}{2} + c,$$

which is the required solution.

EXAMPLE 23 Solve the initial value problem : $\frac{dy}{dx} = e^{x+y}$, $y(1) = 1$.

SOLUTION We are given

$$\frac{dy}{dx} = e^{x+y} = e^x \cdot e^y \quad \Leftrightarrow \quad e^{-y}dy = e^x dx \quad (\textit{a separable equation})$$

Integrating both sides, we get

$$\int e^{-y}dy = \int e^x dx + c$$

or,
$$-e^{-y} = e^x + c \qquad \ldots(1)$$

From the initial condition $y(1) = 1$ *i.e.*, $y = 1$ when $x = 1$, we obtain

$$-e^{-1} = e + c \quad \Rightarrow \quad c = -\frac{1}{e} - e = -\left(\frac{1+e^2}{e}\right)$$

Substituting the value of c in Eq. (1), we obtain the particular solution

$$-e^{-y} = e^x - \left(\frac{1+e^2}{e}\right).$$

EXAMPLE 24 Solve the differential equation $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

SOLUTION The given differential equation is

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y} \quad \text{or} \quad \frac{dy}{dx} = (e^x + x^2)e^{-y} \quad \text{or} \quad e^y\,dy = (e^x + x^2)dx,$$

an equation with separable variables. Integrating, we obtain

$$\int e^y dy = \int (e^x + x^2)dx + c$$

i.e.,
$$e^y = e^x + \frac{x^3}{3} + c,$$

which is the required solution.

EXAMPLE 25 Solve the initial value problem : $xy\frac{dy}{dx} = y + 2,\ y(2) = 0.$

SOLUTION The given equation can be written as

$$\frac{y}{y+2}dy = \frac{1}{x}dx \quad (\textit{a separable equation})$$

Integrating, we obtain

$$\int \frac{y}{y+2}dy = \int \frac{1}{x}dx + c \quad \text{or} \quad \int \frac{y+2-2}{y+2}dy = \int \frac{1}{x}dx + c$$

or,
$$\int \left(1 - \frac{2}{y+2}\right)dy = \log x + c \quad \Rightarrow \quad y - 2\log(y+2) = \log x + c$$

From the initial condition $y(2) = 0$ *i.e.*, $y = 0$ when $x = 2$, we obtain

$$0 - 2\log 2 = \log 2 + c \quad \Rightarrow \quad c = -3\log 2$$

$$\therefore \quad y - 2\log(y+2) = \log x - 3\log 2 = \log x - \log 2^3 = \log x - \log 8 = \log\left(\frac{x}{8}\right)$$

i.e.,
$$y - 2\log(y+2) = \log\left(\frac{x}{8}\right),$$

which is the required solution.

EXAMPLE 26 Solve the following initial value problem : **[*Delhi Univ. GE-3, 2018*]**

$$y' \tan x = 2y - 8, \quad y(\pi/2) = 0.$$

SOLUTION The given equation can be written as

$$\frac{dy}{dx} = 2(y-4)\cot x \quad \text{or} \quad \frac{dy}{y-4} = 2\cot x\,dx,$$

an equation with separable variables. Integrating, we obtain

$$\int \frac{dy}{y-4} = 2\int \cot x \, dx + \text{ constant}$$

or $$\log(y-4) = 2\log \sin x + \log c = \log \sin^2 x + \log c = \log (c\sin^2 x)$$

$\Rightarrow$ $$y-4 = c\sin^2 x \quad \Rightarrow \quad y = c\sin^2 x + 4,$$

which is the general solution of the given ODE. We now use the initial condition to determine c. From the initial condition $y(\pi/2) = 0$ *i.e.,* $y = 0$ when $x = \pi/2$, we obtain

$$0 = c\sin^2(\pi/2) + 4 = c + 4 \quad \Rightarrow \quad c = -4$$

Thus, the solution to the given initial value problem is

$$y = 4 - 4\sin^2 x = 4(1 - \sin^2 x) = 4\cos^2 x.$$

EXAMPLE 27 Solve the initial value problem : $L\frac{dI}{dt} + RI = 0, \quad I(0) = I_0.$

SOLUTION The given ODE can be written as

$$\frac{1}{I}dI = -\frac{R}{L}dt,$$

an equation with separable variables. Integrating both sides, we obtain

$$\ln I = -\frac{R}{L}t + k \quad \Rightarrow \quad I = e^k e^{-\frac{R}{L}t} = ce^{-\frac{R}{L}t}$$

where $c = e^k$. This is the general solution. We now use the initial condition to determine c. From the initial condition $I(0) = I_0$, *i.e.,* $I = I_0$ when $t = 0$, we obtain

$$I_0 = ce^0 \quad \Rightarrow \quad c = I_0$$

Thus, the given initial value problem has the solution $I = I_0 e^{-\frac{R}{L}t}$.

Differential Equatюns of the type $\frac{dy}{dx} = f(ax + by + c)$

A differential equation of the type $\frac{dy}{dx} = f(ax + by + c)$ can be solved by substituting $ax + by + c = v$ so that $a + b\frac{dy}{dx} = \frac{dv}{dx}$. Upon substitution, the given differential equation reduces to the form in which variables are seperable. The following examples will help illustrate the method.

EXAMPLE 28 Solve the differential equation $\frac{dy}{dx} = 4x + y + 1.$

SOLUTION Put $4x + y + 1 = v$ so that $4 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 4$

Upon substitution, the given differential equation transforms into

$$\frac{dv}{dx} - 4 = v \quad \text{or} \quad \frac{dv}{dx} = 4 + v \quad \text{or} \quad \frac{1}{4+v}dv = dx \quad (\textit{a separable equation})$$

Integrating both sides, we get

$$\log(4 + v) = x + c$$

i.e., $$\log(4x + y + 5) = x + c,$$

which is the required solution.

EXAMPLE 29 Solve the differential equation $y' = (y + 9x)^2$.

SOLUTION Put $v = y + 9x$ so that $v' = y' + 9 \Rightarrow y' = v' - 9$. Upon substitution, the given differential equation becomes

$$v' - 9 = v^2 \quad \text{or} \quad v' = 9 + v^2 \quad \text{or} \quad \frac{dv}{9+v^2} = dx \qquad (\textit{a separable equation})$$

Integrating, we get

$$\frac{1}{3}\tan^{-1}\frac{v}{3} = x + k \Rightarrow \tan^{-1}\frac{v}{3} = 3x + 3k$$

$$\Rightarrow \quad v = 3\tan(3x + c), \text{ where } c = 3k$$

or, $$y + 9x = 3\tan(3x + c) \Rightarrow y = 3\tan(3x + c) - 9x,$$

which is the general solution.

EXAMPLE 30 Solve the differential equation $\frac{dy}{dx} = 1 + e^{x-y}$.

SOLUTION Put $x - y = v$ so that $1 - \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$

Upon substitution, the given differential equation becomes

$$1 - \frac{dv}{dx} = 1 + e^v \quad \text{or} \quad -\frac{dv}{dx} = e^v \quad \text{or} \quad dx + e^{-v}\,dv = 0 \quad (\textit{a separable equation})$$

Integrating, we get

$$\int dx + \int e^{-v}\,dv = c \quad \text{or} \quad x - e^{-v} = c$$

$$\Rightarrow \quad x - e^{-(x-y)} = c \quad \text{or} \quad x = e^{y-x} + c,$$

which is the required solution.

EXERCISE 1.3

Solve each of the following differential equations (1 – 16) :

1. $\frac{dy}{dx} = 2\sec 2y$

2. $\sin \pi x \frac{dy}{dx} = y \cos \pi x$

3. $\frac{dy}{dx} = (e^x + 1)y$

4. $(x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0$

5. $\frac{dy}{dx} = e^{x+y} + x^2e^y$

6. $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$

7. $\frac{dy}{dx} = 1 - x + y - xy$

8. $(e^x + 1)y\,dy = (y+1)e^x dx$

9. $(x-1)\frac{dy}{dx} = 2xy$

10. $x \log x\, dy - y\, dx = 0$

11. $(1 + x^2)dy = xy\, dx$

12. $y\,dx - x\,dy = xy\,dx$

13. $x^2(y-1)dx + y^2(x-1)dy = 0$

14. $\frac{dr}{dt} = -2tr$

15. $(x + y + 1)\frac{dy}{dx} = 1$

16. $\frac{dy}{dx} + 1 = e^{x+y}$

Solve each of the following initial value problems (17 – 22) :

17. $x(1 + y^2)dx - y(1 + x^2)dy = 0,\ y(1) = 0$

18. $(x-1)\frac{dy}{dx} = 2xy, \quad y(0) = 1$

19. $e^x\frac{dy}{dx} = 3y^3, \quad y(0) = \frac{1}{2}$

20. $\frac{dy}{dx} = x^2e^{-3y}, \quad y(0) = 0$

21. $e^{2x}y' = 2(x+2)y^3, \quad y(0) = 1/\sqrt{5}$

22. $\frac{dr}{d\theta} = b\left[\left(\frac{dr}{d\theta}\right)\cos\theta + r\sin\theta\right], \ r(\pi/2) = \pi.$

ANSWERS

1. $y = \frac{1}{2}\sin^{-1}(4x + c)$

2. $y = c(\sin \pi x)^{1/\pi}$

3. $\log y = e^x + x + c$

4. $\log y + \frac{1}{y} + \frac{1}{x} - x = c$

5. $-e^{-y} = e^x + \frac{x^3}{3} + c$

6. $xe^x - e^x\sqrt{1-y^2} = c$

7. $\log(1 + y) = x - \frac{x^2}{2} + c$

8. $y = \log(1 + y)(1 + e^x) + c$

9. $y = c(x-1)^2 e^{2x}$

10. $y = c \log x$

11. $y = c\sqrt{1+x^2}$

12. $y = cxe^{-x}$

13. $x^2 + y^2 + 2(x+y) + 2\log(x-1)(y-1) = c$

14. $r = ce^{-t^2}$

15. $y - \log(x+y+2) = c$

16. $(x+c)e^{x+y} + 1 = 0$

17. $(1+x^2) = 2(1+y^2)$

18. $y = (x-1)^2 e^{2(x-2)}$

19. $\dfrac{1}{6y^2} = \dfrac{1}{e^x} - \dfrac{1}{3}$

20. $e^{3y} = x^3 + 1$

21. $y = \dfrac{e^x}{\sqrt{2x+5}}$

22. $r = \pi(1 - b\cos\theta)$

1.5 HOMOGENEOUS DIFFERENTIAL EQUATIONS

We begin by defining the notion of a homogeneous function.

DEFINITION A function $z = f(x, y)$ is said to be **homogeneous of degree n** (n being a constant), if

$$z = x^n \cdot F\left(\frac{y}{x}\right)$$

That is, a function $z = f(x, y)$ is homogeneous if it can be expressed as a product of x^n and a function of $\frac{y}{x}$. For example, consider the function z defined by

$$z = 2x^2y + xy^2 - y^3$$

Then z can be expressed as

$$z = x^3\left\{2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 - \left(\frac{y}{x}\right)^3\right\}$$

and hence z is homogeneous of degree 3.

Notice that in the above example the function z is a polynomial in x and y such that the degree of each term is 3, which is the degree of homogeneity of the function. In general, we have the following remark for such functions.

Remark 1 A polynomial function in two variables x and y is homogeneous if the degree of each term in the polynomial is same. In that case, the degree of homogeneity is the common degree of each term. In particular, a polynomial function containing only one term is always homogeneous. For example, the function $z = x^2y^3$ is homogeneous of degree 5.

We next give an example of a homogeneous function which is not a polynomial. For instance, consider the function z defined by

$$z = \frac{x^3 + y^3}{x - y}$$

Then z can be expressed as

$$z = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-\frac{y}{x}\right]} = x^2\left\{\frac{1+\left(\frac{y}{x}\right)^3}{1-\frac{y}{x}}\right\}$$

This shows that z is homogeneous of degree 2.

Remark 2 A rational function (*i.e.*, quotient of two polynomials) in two variables is homogeneous if both numerator and denominator are separately homogeneous. In that case, the degree of homogeneity of the given function is degree of numerator minus degree of denominator. To quote some examples, each of the following functions is homogeneous :

(*i*) $z = \dfrac{ax^2 + 2hxy + by^2}{cx + dy}$: *homogeneous of degree* 1

(*ii*) $z = \dfrac{x+y}{x-y}$: *homogeneous of degree* 0

(*iii*) $z = \dfrac{x^2+y^2}{x^3-y^3}$: *homogeneous of degree* –1

(*iv*) $z = \dfrac{x^2+y^2}{xy}$: *homogeneous of degree* 0

Homogeneous Differential Equations

A differential equation of the form $\dfrac{dy}{dx} = f(x,y)$ is said to be **homogeneous** if $f(x,y)$ is a homogeneous function of degree 0. Thus, homogeneous differential equations are of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \qquad \text{...(1)}$$

For example, the following differential equations :

(*i*) $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ (*ii*) $\dfrac{dy}{dx} = \dfrac{x^2+xy+y^2}{x^2}$

(*iii*) $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{x^2+xy}$ (*iv*) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\left(\dfrac{y}{x}\right)$

are all homogeneous.

The procedure for solving a homogeneous differential equation of the form (1) is as follows :

Put $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Upon substitution, equation (1) becomes

$$v + x\frac{dv}{dx} = F(v) \quad \text{or} \quad x\frac{dv}{dx} = F(v) - v \quad \text{or} \quad \frac{dv}{F(v)-v} = \frac{dx}{x},$$

which is a differential equation having separable variables and can be easily solved.

EXAMPLE 31 Solve the differential equation $x\dfrac{dv}{dx} = x + y$.

SOLUTION The given equation can be written as

$$\frac{dy}{dx} = \frac{x+y}{x} = 1 + \frac{y}{x} \quad \text{(a homogeneous differential equation)} \quad ...(1)$$

Put $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substituting in (1), we get

$$v + x\frac{dv}{dx} = 1 + v \quad \Rightarrow \quad x\frac{dv}{dx} = 1 \quad \text{or} \quad dv = \frac{dx}{x}$$

Integrating, we get $\quad v = \log x + c$

Substituting the value of v, we get

$$\frac{y}{x} = \log x + c \quad \text{or} \quad y = x \log x + cx,$$

which is the required solution.

EXAMPLE 32 Solve the differential equation $(x^2 + xy)dy = (x^2 + y^2)dx$.

SOLUTION The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} \quad \text{(a homogeneous differential equation)} \quad ...(1)$$

Put $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substituting in (1), we get

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{x^2 + vx^2} = \frac{1+v^2}{1+v} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v = \frac{1-v}{1+v}$$

$$\Rightarrow \quad \frac{1+v}{1-v}dv = \frac{dx}{x} \quad \text{or} \quad \left(-1 + \frac{2}{1-v}\right)dv = \frac{dx}{x}$$

Integrating, we get $-v - 2\log(1 - v) = \log x + c$

Substituting the value of v, we get

$$-\frac{y}{x} - 2\log\left(1 - \frac{y}{x}\right) = \log x + c \quad \Rightarrow \quad -2\log\left(\frac{x-y}{x}\right) - \log x = \frac{y}{x} + c$$

or, $\quad -2\log(x-y) + 2\log x - \log x = \dfrac{y}{x} + c \qquad$ *i.e.*, $\quad \log x - \log(x-y)^2 = \dfrac{y}{x} + c$

or, $$\log\left\{\frac{x}{(x-y)^2}\right\} = \frac{y}{x} + c$$

which is the required solution.

EXAMPLE 33 Solve the differential equation $\dfrac{dy}{dx} = \dfrac{2y-x}{2x-y}$.

SOLUTION The given equation is clearly a homogeneous differential equation.

Put $\quad y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{2v-1}{2-v} \Rightarrow x\frac{dv}{dx} = \frac{2v-1}{2-v} - v \quad \text{or} \quad x\frac{dv}{dx} = \frac{v^2-1}{2-v}$$

$$\Rightarrow \quad \frac{2-v}{v^2-1}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{2}{v^2-1}dv - \int\frac{v}{v^2-1}dv = \int\frac{1}{x}dx + \log c \quad \text{or} \quad \log\left(\frac{v-1}{v+1}\right) - \frac{1}{2}\log(v^2-1) = \log x + \log c$$

$$\text{or,} \quad \log\left(\frac{v-1}{v+1}\right) - \log\sqrt{v^2-1} = \log(xc) \Rightarrow \log\left(\frac{v-1}{v+1}\right)\frac{1}{\sqrt{v^2-1}} = \log(xc)$$

$$\Rightarrow \quad \left(\frac{v-1}{v+1}\right)\left(\frac{1}{\sqrt{v^2-1}}\right) = xc \Rightarrow \left(\frac{y-x}{y+x}\right)\left(\frac{x}{\sqrt{y^2-x^2}}\right) = xc$$

$$\text{or,} \quad \frac{\sqrt{y-x}}{(y+x)^{3/2}} = c \quad \text{or} \quad \sqrt{y-x} = c(y+x)^{3/2},$$

which is the required solution.

EXAMPLE 34 Solve the differential equation $(x+y)+(y-x)\dfrac{dy}{dx} = 0$.

SOLUTION The given differential equation is

$$(x+y)+(y-x)\frac{dy}{dx} = 0$$

i.e., $\quad (x-y)\dfrac{dy}{dx} = x+y \Rightarrow \dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ (*a homogeneous differential equation*)

Put $\quad y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{1+v}{1-v} \Rightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v}$$

$$\Rightarrow \quad \frac{1-v}{1+v^2}dv = \frac{dx}{x} \quad \text{or} \quad \left[\frac{1}{1+v^2} - \frac{1}{2}\cdot\frac{2v}{1+v^2}\right]dv = \frac{dx}{x}$$

Integrating, we get

$$\tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + c$$

$$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right) = \log x + c,$$

which is the required solution.

EXAMPLE 35 Solve the differential equation $x^2 y\,dx - (x^3 + y^3)dy = 0$.

SOLUTION The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$$

which is clearly a homogeneous differential equation.

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{vx^3}{x^3 + v^3x^3} = \frac{v}{1+v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{v - v - v^4}{1+v^3} = -\frac{v^4}{1+v^3}$$

$$\Rightarrow \quad \frac{1+v^3}{v^4}dv + \frac{dx}{x} = 0 \quad \text{or} \quad \left(v^{-4} + \frac{1}{v}\right)dv + \frac{dx}{x} = 0$$

Integrating, we get

$$\frac{v^{-3}}{-3} + \log v + \log x = c \quad \text{or} \quad -\frac{1}{3v^3} + \log vx = c$$

$$\text{or,} \quad -\frac{x^3}{3y^3} + \log y = c \quad \Rightarrow \quad \log y = c + \frac{x^3}{3y^3}$$

which is the required solution.

EXAMPLE 36 Solve the differential equation $(3xy + y^2)dx + (x^2 + xy)\,dy = 0$.

SOLUTION The given equation can be written as

$$\frac{dy}{dx} = -\left(\frac{3xy + y^2}{x^2 + xy}\right),$$

which is clearly a homogeneous differential equation.

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \quad v + x\frac{dv}{dx} = -\left\{\frac{3vx^2 + v^2x^2}{x^2 + vx^2}\right\} = -\left\{\frac{3v + v^2}{1+v}\right\}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left\{\frac{3v + v^2}{1+v} + v\right\} = -\left\{\frac{3v + v^2 + v + v^2}{1+v}\right\} = -2\left\{\frac{v^2 + 2v}{v+1}\right\}$$

$$\Rightarrow \quad \frac{v+1}{v^2 + 2v}dv + 2\frac{dx}{x} = 0$$

or, $$\frac{2v+2}{v^2+2v}dv + \frac{4}{x}dx = 0 \qquad \textit{(Multiplying by 2)}$$

Integrating, we get

$$\log(v^2 + 2v) + 4\log x = \log c \quad \text{or} \quad \log\left(\frac{y^2}{x^2} + \frac{2y}{x}\right) + \log x^4 = \log c$$

or, $$\log\left(\frac{y^2 + 2xy}{x^2}\right) + \log x^4 = \log c \quad \text{or} \quad \log\left(\frac{y^2 + 2xy}{x^2} \cdot x^4\right) = \log c$$

$$\Rightarrow \quad \log x^2(y^2 + 2xy) = \log c \quad \Rightarrow \quad x^2(y^2 + 2xy) = c$$

which is the required solution.

EXAMPLE 37 Solve the differential equation $x\frac{dy}{dx} = y - \sqrt{x^2 + y^2}$.

SOLUTION The given equation can be written as

$$\frac{dy}{dx} = \frac{y}{x} - \frac{\sqrt{x^2+y^2}}{x} = \frac{y}{x} - \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

Put $$y = vx \text{ so that } \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = v - \sqrt{1+v^2} \quad \Rightarrow \quad x\frac{dv}{dx} = -\sqrt{1+v^2} \quad \text{or} \quad \frac{dv}{\sqrt{v^2+1}} + \frac{dx}{x} = 0$$

Integrating, we get

$$\log\left(v + \sqrt{v^2+1}\right) + \log x = \log c \quad \text{or} \quad \log\left[x\left(v + \sqrt{v^2+1}\right)\right] = \log c$$

or, $$\left[vx + \sqrt{(vx)^2 + x^2}\right] = c \quad \text{or} \quad y + \sqrt{y^2 + x^2} = c$$

which is the required solution.

Equations Reducible to Homogeneous Form

A differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}, \quad \text{where} \quad \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \qquad ...(1)$$

can be reduced to homogeneous form by substituting $x = X + h$ and $y = Y + k$, where h and k are constants to be determined suitably.

We have $\quad \frac{dx}{dX} = 1, \quad \frac{dy}{dY} = 1$ so that

$$\frac{dy}{dx} = \frac{dy}{dY}\cdot\frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{dY}{dX}$$

Upon substitution, Eq. (1) becomes

$$\frac{dY}{dX} = \frac{(a_1X+b_1Y)+(a_1h+b_1k+c_1)}{(a_2X+b_2Y)+(a_2h+b_2k+c_2)} \qquad ...(2)$$

Choose h and k so that

$$a_1h + b_1k + c_1 = 0$$
$$a_2h + b_2k + c_2 = 0$$

Solving these equations for h and k, we get

$$h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \qquad k = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

Eq. (2), now, becomes

$$\frac{dY}{dX} = \frac{a_1X+b_1Y}{a_2X+b_2Y}$$

which being a homogeneous equation can be solved by means of the substitution $Y = VX$.

A Particular Case : A differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}, \quad \text{where} \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = k(\text{say}) \qquad ...(3)$$

can be put in the form

$$\frac{dy}{dx} = \frac{k(a_2x+b_2y)+c_1}{a_2x+b_2y+c_2} \qquad ...(4)$$

Put $a_2x + b_2y = z$ so that $a_2 + b_2\frac{dy}{dx} = \frac{dz}{dx}$. Equation (4), then, becomes

$$\frac{dz}{dx} = b_2\frac{kz+c_1}{z+c_2} + a_2$$

which is a differential equation having separable variables.

EXAMPLE 38 Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x+2y-1}{2x+y+1}$.

SOLUTION Substituting $x = X + h,\ y = Y + k$, the given equation becomes

$$\frac{dY}{dX} = \frac{X+2Y+(h+2k-1)}{2X+Y+(2h+k+1)}$$

Choose h and k so that $h + 2k - 1 = 0,\ 2h + k + 1 = 0$.

Solving these equations for h and k, we get $h = -1$ and $k = 1$

$$\therefore \qquad \frac{dY}{dX} = \frac{X+2Y}{2X+Y}$$

Put $Y = VX$ so that $\dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

$$\therefore \qquad V + X\frac{dV}{dX} = \frac{1+2V}{2+V} \quad \Rightarrow \quad X\frac{dV}{dX} = \frac{1-V^2}{2+V}$$

$$\text{or,} \qquad \frac{2+V}{1-V^2}dV = \frac{dX}{X} \qquad \text{or} \qquad \left(\frac{2}{V^2-1} + \frac{V}{V^2-1}\right)dV + \frac{dX}{X} = 0$$

Integrating, we get

$$\log\left(\frac{V-1}{V+1}\right) + \frac{1}{2}\log(V^2-1) + \log X = \log c$$

$$\Rightarrow \quad \left(\frac{V-1}{V+1}\right)\cdot\sqrt{V^2-1}\cdot X = c \qquad \text{or} \qquad \frac{Y-X}{Y+X}\sqrt{Y^2-X^2} = c \qquad \left(\because V = \frac{Y}{X}\right)$$

i.e., $\qquad (Y-X)^{3/2} = c(Y+X)^{1/2}$, where $Y = y - 1,\ X = x + 1$.

EXAMPLE 39 Solve the differential equation $(2x + y + 1)dx + (4x + 2y - 1)dy = 0$.

SOLUTION The given equation can be put in the form

$$\frac{dy}{dx} = -\frac{2x+y+1}{4x+2y-1} = -\frac{2x+y+1}{2(2x+y)-1}$$

Put $\qquad 2x + y = z$ so that $2 + \dfrac{dy}{dx} = \dfrac{dz}{dx} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dz}{dx} - 2$

$$\therefore \qquad \frac{dz}{dx} - 2 = -\frac{z+1}{2z-1} \quad \Rightarrow \quad \frac{dz}{dx} = \frac{3z-3}{2z-1} \qquad \text{or} \qquad \frac{2z-1}{3(z-1)}dz = dx$$

Integrating, we get

$$x + k = \frac{1}{3}\int\frac{2z-1}{z-1}dz = \frac{2}{3}\int\left[1+\frac{1}{2(z-1)}\right]dz = \frac{2}{3}\left[z+\frac{1}{2}\log(z-1)\right]$$

i.e., $\qquad 3x + c = 2z + \log(z-1)$, where $c = 3k$

$\therefore \quad \log(2x + y - 1) + x + 2y = c$, which is the required solution.

EXERCISE 1.4

Solve each of the following differential equations (1 – 18) :

1. $2xy\dfrac{dy}{dx} = x^2 + y^2$

2. $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$

3. $x^2\dfrac{dy}{dx} = x^2 - 2y^2 + xy$

4. $\dfrac{dy}{dx} = \dfrac{4x^2 + y^2}{xy}$

5. $\dfrac{dy}{dx} = \dfrac{x - y}{x + y}$

6. $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$

7. $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$

8. $(x^2 + 3xy + y^2)dx - x^2\,dy = 0$

9. $2xy\dfrac{dy}{dx} = x^2 + 3y^2$

10. $(x - y)\dfrac{dy}{dx} = x + 3y$

11. $x^2\,dy + y(x + y)dx = 0$

12. $2\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}$

13. $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$

14. $x\dfrac{dy}{dx} - y = 2\sqrt{y^2 - x^2}$

15. $\dfrac{dy}{dx} = \dfrac{2x - y + 1}{x - 2y + 1}$

16. $\dfrac{dy}{dx} = \dfrac{2x - y + 3}{x + 2y + 4}$

17. $(4x + 6y + 5)dx = (2x + 3y + 4)dy$

18. $(2x + 2y + 3)dy = (x + y + 1)dx$

19. Solve the initial value problem : $(x^2 - y^2)dx + 2xy\,dy = 0,\quad y(1) = 1.$

20. Solve the initial value problem : $2xyy' = 3y^2 + x^2,\quad y(1) = 2.$

ANSWERS

1. $x = c(x^2 - y^2)$

2. $x^2 + y^2 - cx = 0$

3. $\dfrac{1}{2\sqrt{2}}\log\left(\dfrac{x + \sqrt{2}y}{x - \sqrt{2}y}\right) - \log x = c$

4. $y^2 = x^2(8\log x + c)$

5. $y^2 + 2xy - x^2 = c$

6. $\dfrac{y}{x} - \log y = c$

7. $x^2 - y^2 = c(x^2 + y^2)$

8. $\dfrac{x}{x + y} + \log x = c$

9. $x^2 + y^2 = cx^3$

10. $-\dfrac{2x}{x + y} = \log\{c(x + y)\}.$

11. $x^2y = c(y + 2x)$

12. $y - x = c\sqrt{xy}$

13. $y = xe^{cx}$

14. $y + \sqrt{y^2 - x^2} = cx^3$

15. $x^2 + y^2 - xy + x - y = c$

16. $y^2 - x^2 + xy + 4y - 3x = c$

17. $\frac{1}{8}(2x + 3y) + \frac{9}{64}\log(16x + 24y + 23) = x + c$

18. $x + y + \frac{4}{3} = ce^{3(x-2y)}$

19. $x^2 + y^2 = 2x$

20. $y = x\sqrt{5x - 1}$

1.6 EXACT DIFFERENTIAL EQUATIONS

Recall from calculus that if $u(x, y)$ is a function of two variables possessing continuous partial derivatives, then the **total differential** of u, denoted by du, is given by

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Note that if $u(x, y) = c$, where c is a constant, then $du = 0$. For example, if $u = 2x^2y + y^3 = c$, then

$$du = 4xydx + (2x^2 + 3y^2)dy = 0 \qquad ...(1)$$

This gives

$$\frac{dy}{dx} = -\frac{4xy}{2x^2 + 3y^2},$$

which is an ordinary differential equation that can be solved moving backward. Note that equation (1) is of the form

$$M(x, y)dx + N(x, y)dy = 0 \qquad \left(M(x, y) = \frac{\partial u}{\partial x},\ N(x, y) = \frac{\partial u}{\partial y}\right)$$

whose general solution is $u = 2x^2y + y^3 = c$. Such an equation is called an *exact differential equation.*

DEFINITION Exact Differential Equation

A first-order ordinary differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0 \qquad ...(1)$$

is said to be an **exact differential equation** if its differential form $M(x, y)dx + N(x, y)dy$ is the total differential

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

of some function $u(x, y)$. In other words, equation (1) is an exact differential equation if there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial u}{\partial y} = N(x, y)$$

The general solution of the equation is $u(x, y) = c$.

The following theorem (proof omitted) enables us to examine whether equation (1) is exact or not.

THEOREM 1.1 Test for Exactness

Let M and N be continuous and have continuous partial derivatives on some open disc in the xy-plane. The differential equation $M(x, y)dx + N(x, y)dy = 0$ is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

EXAMPLE 40 Test the following differential equations for exactness.

(*a*) $(xy^2 + x)dx + yx^2dy = 0$ (*b*) $\cos y\,dx + (y^2 - x\sin y)dy = 0$

SOLUTION (*a*) The differential equation $(xy^2 + x)dx + yx^2dy = 0$ is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

with $$M = xy^2 + x \quad \text{and} \quad N = yx^2$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 2xy$$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

(*b*) The differential equation $\cos y\,dx + (y^2 - x\sin y)dy = 0$ is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

with $$M = \cos y \quad \text{and} \quad N = y^2 - x\sin y$$

$$\therefore \quad \frac{\partial M}{\partial y} = -\sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = -\sin y$$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact.

Procedure for Solving an Exact Differential Equation

We now outline the procedure for solving an exact differential equation.

Step 1. ***Test for exactness*** **:** The first step is to verify that the given differential equation is exact using the test for exactness :

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Step 2. ***Write the system of two differential equations that define the function $u(x, y)$*** **:** The system of two differential equations that define the function $u(x, y)$ are :

$$\frac{\partial u}{\partial x} = M(x, y) \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial y} = N(x, y) \qquad \text{...(2)}$$

Step 3. ***Determine the function $u(x, y)$*** **:** To determine the function $u(x,y)$, we integrate equation (1) with respect to x to get

$$u(x, y) = \int M(x,y)\,dx + h(y) \qquad ...(3)$$

In this integration, y is to be regarded as constant, and $h(y)$ plays the role of a constant of integration.

Step 4. ***Determine the function $h(y)$*** **:** To determine the function $h(y)$, we proceed as follows :

- Differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)dx + h(y)\right]$$

- Use equation (2) of step 2 to get

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)dx + h(y)\right] = N(x,y)$$

- Solve the above equation to get dh/dy.
- Integrate dh/dy to get $h(y)$.

Step 5. ***Find the general solution*** **:** The general solution of the exact differential equation is then given by $u(x, y) = c$.

Note Notice that the formula in Step 3 was obtained by using the first equation in Step 2. Instead of the first equation, we may equally well use the second equation. Then instead of equation (3), we first have

$$u(x, y) = \int N(x,y) \; + k(x), \qquad ...(3^*)$$

obtained by integrating the equation (2) with respect to y. To determine $k(x)$, we derive $\partial u/\partial x$ from (3*), use equation (1) to get dk/dx, and integrate to get $k(x)$.

Alternative Procedure for Solving an Exact Differential Equation

We now give an alternative procedure for solving an exact differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

Step 1. Integrate M with respect to x keeping y as a constant.

Step 2. Integrate those terms in N which do not involve x with respect to y.

Step 3. The sum of the two expressions thus obtained equated to a constant is the required solution.

The following examples help illustrate the procedure.

EXAMPLE 41 Solve the equation $(2xy - \sin x)dx + (x^2 - \cos y)dy = 0$.

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $$M = 2xy - \sin x \quad \text{and} \quad N = x^2 - \cos y$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x$$

$\because \ \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact. Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x, y) = 2xy - \sin x \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = N(x, y) = x^2 - \cos y \qquad ...(2)$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int (2xy - \sin x)dx = x^2 y + \cos x + h(y), \qquad ...(3)$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = x^2 + \frac{dh}{dy} \qquad ...(4)$$

From (2) and (4), it follows that

$$x^2 + \frac{dh}{dy} = x^2 - \cos y \quad \Rightarrow \quad \frac{dh}{dy} = -\cos y \quad \Rightarrow \quad h(y) = -\sin y + c_1$$

Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = x^2 y + \cos x - \sin y + c_1$$

Thus, the general solution of the given equation is

$$x^2 y + \cos x - \sin y = c,$$

where c_1 is merged with c.

Alternative Method

Here

$$M(x, y) = 2xy - \sin x, \qquad N = x^2 - \cos y$$

It has already been proved that the given equation is exact. Thus, to find its solution, we proceed as follows.

Step 1. Integrate $M(x, y)$ with respect to x keeping y as a constant

$$\int M(x, y)dx = \int (2xy - \sin x)dx = x^2 y + \cos x$$

Step 2. Integrate these terms in $N(x, y)$ which do not involve x with respect to y. The only term in $N(x, y)$ not involving x is $-\cos y$ and its integral is $-\sin y$.

Step 3. Obtain the general solution : The sum of the two expressions obtained in Step 1 and Step 2 equated to a constant is the general solution. Thus, the general solution is

$$x^2y + \cos x - \sin y = c, \quad \text{same as before.}$$

EXAMPLE 42 Using exactness, solve the following differential equation

$$(3x^2 + 2x + \sin(x+y))\,dx + \sin(x+y)\,dy = 0$$ **[Delhi Univ. GE-3, 2017]**

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $$M = 3x^2 + 2x + \sin(x+y) \quad \text{and} \quad N = \sin(x+y)$$

$$\therefore \qquad \frac{\partial M}{\partial y} = \cos(x+y) \quad \text{and} \quad \frac{\partial N}{\partial x} = \cos(x+y)$$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact. Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x, y) = 3x^2 + 2x + \sin(x+y) \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial y} = N(x, y) = \sin(x+y) \qquad \text{...(2)}$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int (3x^2 + 2x + \sin(x+y))\,dx = x^3 + x^2 - \cos(x+y) + h(y), \qquad \text{...(3)}$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = \sin(x+y) + \frac{dh}{dy} \qquad \text{...(4)}$$

From (2) and (4), it follows that

$$\sin(x+y) + \frac{dh}{dy} = \sin(x+y) \;\Rightarrow\; \frac{dh}{dy} = 0 \;\Rightarrow\; h(y) = 0,$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting this value of $h(y)$ into equation (3), we get

$$u(x, y) = x^3 + x^2 - \cos(x+y)$$

Thus, the general solution of the given equation is

$$x^3 + x^2 - \cos(x+y) = c.$$

EXAMPLE 43 Using exactness, solve the following differential equation

$$\cos(x+y)\,dx + (3y^2 + 2y + \cos(x+y))\,dy = 0$$ **[Delhi Univ. GE-3, 2018]**

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $$M = \cos(x + y) \quad \text{and} \quad N = 3y^2 + 2y + \cos(x + y)$$

$$\therefore \quad \frac{\partial M}{\partial y} = -\sin(x + y) \quad \text{and} \quad \frac{\partial N}{\partial x} = -\sin(x + y)$$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact. Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x, y) = \cos(x + y) \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = N(x, y) = 3y^2 + 2y + \cos(x + y) \qquad ...(2)$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int \cos(x+y)\,dx = \sin(x+y) + h(y), \qquad ...(3)$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{dh}{dy} \qquad ...(4)$$

From (2) and (4), it follows that

$$\cos(x+y) + \frac{dh}{dy} = 3y^2 + 2y + \cos(x+y) \quad \Rightarrow \quad \frac{dh}{dy} = 3y^2 + 2y \quad \Rightarrow \quad h(y) = y^3 + y^2,$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = \sin(x + y) + y^3 + y^2$$

Thus, the general solution of the given equation is

$$\sin(x + y) + y^3 + y^2 = c.$$

EXAMPLE 44 Find the general solution of $2xe^y dx + (x^2e^y + \cos y)dy = 0$.

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $$M = 2xe^y \quad \text{and} \quad N = x^2e^y + \cos y$$

$$\therefore \quad \frac{\partial M}{\partial y} = 2xe^y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2xe^y$$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact. Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x, y) = 2xe^{y} \quad ...(1)$$

$$\frac{\partial u}{\partial y} = N(x, y) = x^2e^{y} + \cos y \quad ...(2)$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int 2xe^{y}\, dx = x^2e^{y} + h(y), \quad ...(3)$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = x^2e^{y} + \frac{dh}{dy} \quad ...(4)$$

From (2) and (4), it follows that

$$x^2e^{y} + \frac{dh}{dy} = x^2e^{y} + \cos y \quad \Rightarrow \quad \frac{dh}{dy} = \cos y \quad \Rightarrow \quad h(y) = \sin y,$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = x^2e^{y} + \sin y$$

Thus, the general solution of the given equation is

$$x^2e^{y} + \sin y = c.$$

EXAMPLE 45 Find the general solution of $(\cos x - x\sin x + y^2)dx + 2xy\, dy = 0,\ y(\pi) = 1.$

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $\qquad M = \cos x - x\sin x + y^2 \quad \text{and} \quad N = 2xy$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2y$$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact. Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x, y) = \cos x - x\sin x + y^2 \quad ...(1)$$

$$\frac{\partial u}{\partial y} = N(x, y) = 2xy \quad ...(2)$$

Because $N(x, y)$ is simpler than $M(x, y)$, it is better to begin by integrating $N(x, y)$ with respect to y, keeping x as a constant

$$u(x, y) = \int N(x, y)\, dy = \int 2xy\, dy = xy^2 + g(x), \quad ...(3)$$

where $g(x)$ plays the role of constant of integration. To find $g(x)$, we differentiate equation (3) partially with respect to x to get

$$\frac{\partial u}{\partial x} = y^2 + g'(x) \qquad ...(4)$$

From (2) and (4), it follows that

$$y^2 + g'(x) = \cos x - x\sin x + y^2 \quad \Rightarrow \quad g'(x) = \cos x - x\sin x$$

$$\Rightarrow \quad g(x) = \int(\cos x - x\sin x)\,dx + c_1$$

$$= \sin x - \left[-x\cos x + \int \cos x\right] + c_1$$

$$= \sin x + x\cos x - \sin x + c_1 = x\cos x + c_1$$

Substituting the expression for $g(x)$ into equation (3), we get

$$u(x, y) = xy^2 + x\cos x + c_1$$

Thus, the general solution of the given equation is

$$xy^2 + x\cos x = c.$$

We now use the initial condition to determine c. From the initial condition $y(\pi) = 1$, *i.e.*, $y = 1$ when $x = \pi$, we get

$$\pi(1)^2 + \pi\cos\pi = c \quad \Rightarrow \quad c = 0$$

Hence, the initial value problem has the solution

$$xy^2 + x\cos x = 0.$$

EXAMPLE 46 Solve the initial value problem :

$$(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0, \quad y(0) = 2.$$

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $\quad M = 2x\cos y + 3x^2y \quad$ and $\quad N = x^3 - x^2\sin y - y$

$$\therefore \quad \frac{\partial M}{\partial y} = -2x\sin y + 3x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = -2x\sin y + 3x^2$$

$\because \ \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact. Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x, y) = 2x\cos y + 3x^2y \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = N(x, y) = x^3 - x^2\sin y - y \qquad ...(2)$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int(2x\cos y + 3x^2y)\,dx = x^2\cos y + x^3y + h(y), \qquad ...(3)$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = -x^2 \sin y + x^3 + \frac{dh}{dy} \qquad ...(4)$$

From (2) and (4), it follows that

$$-x^2 \sin y + x^3 + \frac{dh}{dy} = x^3 - x^2 \sin y - y \quad \Rightarrow \quad \frac{dh}{dy} = -y \quad \Rightarrow \quad h(y) = -\frac{y^2}{2},$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = x^2 \cos y + x^3 y - \frac{y^2}{2}$$

Thus, the general solution of the given equation is

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = c.$$

We now use the initial condition to determine c. The initial condition $y(0) = 2$ gives

$$0 + 0 - 2 = c \quad \Rightarrow \quad c = -2$$

Hence, the particular solution to the given initial value problem is

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = -2.$$

EXAMPLE 47 Solve the differential equation : $(2xy - 3x^2)dx + (x^2 - 2y)dy = 0$.

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $\quad M = 2xy - 3x^2 \quad$ and $\quad N = x^2 - 2y$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x$$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact. Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x, y) = 2xy - 3x^2 \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = N(x, y) = x^2 - 2y \qquad ...(2)$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int (2xy - 3x^2)\,dx = x^2 y - x^3 + h(y), \qquad ...(3)$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = x^2 + \frac{dh}{dy} \quad ...(4)$$

From (2) and (4), it follows that

$$x^2 + \frac{dh}{dy} = x^2 - 2y \quad \Rightarrow \quad \frac{dh}{dy} = -2y \quad \Rightarrow \quad h(y) = -y^2,$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = x^2y - x^3 - y^2$$

Thus, the general solution of the given equation is

$$x^2y - x^3 - y^2 = c.$$

EXAMPLE 48 Solve the differential equation : $(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0.$

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $\quad M = 1 + ye^{xy}$ and $N = 2y + xe^{xy}$

$$\therefore \quad \frac{\partial M}{\partial y} = (1 + xy)e^{xy} \quad \text{and} \quad \frac{\partial N}{\partial x} = (1 + xy)e^{xy}$$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the given equation is exact. Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x, y) = 1 + ye^{xy} \quad ...(1)$$

$$\frac{\partial u}{\partial y} = N(x, y) = 2y + xe^{xy} \quad ...(2)$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int (1 + ye^{xy})\, dx = x + e^{xy} + h(y), \quad ...(3)$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = xe^{xy} + \frac{dh}{dy} \quad ...(4)$$

From (2) and (4), it follows that

$$xe^{xy} + \frac{dh}{dy} = 2y + xe^{xy} \quad \Rightarrow \quad \frac{dh}{dy} = 2y \quad \Rightarrow \quad h(y) = y^2,$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = x + e^{xy} + y^2$$

Thus, the general solution of the given equation is

$$x + e^{xy} + y^2 = c.$$

EXAMPLE 49 Under what conditions for the constants A, B, C, D is the following equation exact?

$$(Ax + By)dx + (Cx + Dy)dy = 0$$

Solve the exact equation.

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $\qquad M = Ax + By \quad \text{and} \quad N = Cx + Dy$

$$\therefore \qquad \frac{\partial M}{\partial y} = B \quad \text{and} \quad \frac{\partial N}{\partial x} = C$$

The given equation will be exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, that is, if $B = C$. Assuming this condition, we have an exact equation and as such there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M(x, y) = Ax + By \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = N(x, y) = Cx + Dy \qquad ...(2)$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int (Ax + By)\,dx = \frac{Ax^2}{2} + Bxy + h(y), \qquad ...(3)$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = Bx + \frac{dh}{dy} \qquad ...(4)$$

From (2) and (4), it follows that

$$Bx + \frac{dh}{dy} = Cx + Dy \;\Rightarrow\; \frac{dh}{dy} = Dy \;\; (\because\; B = C \text{ by assumption}) \;\Rightarrow\; h(y) = \frac{Dy^2}{2},$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = \frac{Ax^2}{2} + Bxy + \frac{Dy^2}{2}$$

Thus, the general solution of the given equation is $\dfrac{Ax^2}{2} + Bxy + \dfrac{Dy^2}{2} = c.$

Reduction to Exact Form : Integrating Factors

Consider the differential equation

$$M(x, y)dx + N(x, y)dy = 0 \qquad ...(1)$$

Even if this equation is not exact, there may be a function $F(x, y)$ such that if we multiply both sides of (1) by F, the resulting equation

$$FMdx + FNdy = 0$$

becomes exact and hence we can solve it by the method discussed earlier. The function $F(x, y)$ is called an **integrating factor** of (1). For example, consider the first-order differential equation

$$-y\,dx + x\,dy = 0 \qquad ...(2)$$

This equation is not exact, because $M = -y$ and $N = x$ so that $\frac{\partial M}{\partial y} = -1$, but $\frac{\partial N}{\partial x} = 1$. However, if we multiply this equation by $1/x^2$, then the resulting equation

$$-\frac{y}{x^2}dx + \frac{1}{x}dy = 0 \qquad ...(3)$$

is exact. In fact,

$$-\frac{y}{x^2}dx + \frac{1}{x}dy = d\left(\frac{y}{x}\right)$$

Thus, equation (3) can be written as

$$d\left(\frac{y}{x}\right) = 0 \qquad ...(4)$$

Integration of equation (4) gives the general solution

$$y/x = c = \text{constant}$$

The function $F(x) = 1/x^2$ on multiplying by which equation (2) becomes exact is an integrating factor of equation (2).

How to find Integrating Factors

For differential equations of the simpler types, integrating factor may be obtained by inspection or perhaps after some trials keeping in mind the following exact forms :

(*i*) $x\,dy + y\,dx = d(xy)$

(*ii*) $\frac{-y\,dx + x\,dy}{x^2} = d\left(\frac{y}{x}\right)$

(*iii*) $\frac{-y\,dx + x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

(*iv*) $\frac{-y\,dx + x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$

(*v*) $\frac{y\,dx - x\,dy}{xy} = d\left(\ln\frac{x}{y}\right)$

However, in general, to find integrating factors, we make use of the following theorems.

THEOREM 1.2 Integrating Factor $F(x)$

If the first-order ordinary differential equation

$$M(x, y)dx + N(x, y)dy = 0 \quad ...(1)$$

is such that

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = R(x), \text{ a function of } x \text{ only}$$

then the integrating factor $F(x)$ of equation (1) is given by

$$F(x) = e^{\int R(x)dx}.$$

THEOREM 1.3 Integrating Factor $F^*(y)$

If the first-order ordinary differential equation

$$M(x, y)dx + N(x, y)dy = 0 \quad ...(1)$$

is such that

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = R^*(y), \text{ a function of } y \text{ only}$$

then the integrating factor $F^*(y)$ of equation (1) is given by

$$F^*(y) = e^{\int R^*(y)dy}.$$

EXAMPLE 50 Find an integrating factor and solve the differential equation

$$(x^2 + y^2 + 1)dx - 2xydy = 0.$$

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $\quad M = x^2 + y^2 + 1 \quad$ and $\quad N = -2xy$

Now, $$\frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = -2y$$

$\because \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the given equation is *not* exact. However,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{2xy}[2y + 2y] = -\frac{4y}{2xy} = -\frac{2}{x}, \text{ a function of } x \text{ alone}$$

Therefore, an integrating factor is given by

$$F(x) = e^{\int -2/x\,dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}.$$

Multiplying both sides of the given equation by x^{-2}, we obtain the exact differential equation

$$\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx - \frac{2y}{x}dy = 0$$

Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = 1+\frac{y^2}{x^2}+\frac{1}{x^2} \quad \text{...(1)}$$

$$\frac{\partial u}{\partial y} = -\frac{2y}{x} \quad \text{...(2)}$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx = x-\frac{y^2}{x}-\frac{1}{x}+h(y), \quad \text{...(3)}$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = -\frac{2y}{x}+\frac{dh}{dy} \quad \text{...(4)}$$

From (2) and (4), it follows that

$$-\frac{2y}{x}+\frac{dh}{dy} = -\frac{2y}{x} \quad \Rightarrow \quad \frac{dh}{dy} = 0 \quad \Rightarrow \quad h(y) = 0,$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = x-\frac{y^2}{x}-\frac{1}{x}.$$

Thus, the general solution of the given equation is

$$x-\frac{y^2}{x}-\frac{1}{x} = c.$$

EXAMPLE 51 Find an integrating factor and solve the differential equation

$$(e^{x+y}-y)dx+(xe^{x+y}+1)dy = 0.$$ **[*Delhi Univ. GE-3, 2016*]**

SOLUTION The given differential equation is of the form

$$M(x, y)dx+N(x, y)dy = 0$$

where $M = e^{x+y}-y$ and $N = xe^{x+y}+1$

Now, $\dfrac{\partial M}{\partial y} = e^{x+y}-1$ and $\dfrac{\partial N}{\partial x} = xe^{x+y}+e^{x+y}$

$\because \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the given equation is *not* exact. However,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{1}{(xe^{x+y}+1)}\left[e^{x+y}-1-xe^{x+y}-e^{x+y}\right] = -\frac{xe^{x+y}+1}{xe^{x+y}+1} = -1$$

Therefore, an integrating factor is given by

$$F(x) = e^{\int -1 dx} = e^{-x}$$

Multiplying both sides of the given equation by e^{-x}, we obtain the exact differential equation

$$(e^y - ye^{-x})dx + (xe^y + e^{-x})dy = 0$$

Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = e^y - ye^{-x} \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = xe^y + e^{-x} \qquad ...(2)$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int (e^y - ye^{-x})dx = xe^y + ye^{-x} + h(y), \qquad ...(3)$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = xe^y + e^{-x} + \frac{dh}{dy} \qquad ...(4)$$

From (2) and (4), it follows that

$$xe^y + e^{-x} + \frac{dh}{dy} = xe^y + e^{-x} \quad \Rightarrow \quad \frac{dh}{dy} = 0 \quad \Rightarrow \quad h(y) = 0,$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = xe^y + ye^{-x}$$

Thus, the general solution of the given equation is

$$xe^y + ye^{-x} = c.$$

EXAMPLE 52 Find an integrating factor and solve the differential equation

$$(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0.$$

[*Delhi Univ.* GE-3, 2017]

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $\quad M = e^{x+y} + ye^y \quad$ and $\quad N = xe^y - 1$

Now, $\quad \frac{\partial M}{\partial y} = e^{x+y} + e^y + ye^y \quad$ and $\quad \frac{\partial N}{\partial x} = e^y$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given equation is *not* exact. However,

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{(e^{x+y} + ye^y)}\left[e^y - e^{x+y} - e^y - ye^y\right] = -\frac{e^{x+y} + ye^y}{e^{x+y} + ye^y} = -1$$

Therefore, an integrating factor is given by

$$F^*(y) = e^{\int -1 dy} = e^{-y}$$

Multiplying both sides of the given equation by e^{-y}, we obtain the exact differential equation

$$(e^x + y)dx + (x - e^{-y})dy = 0$$

Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = e^x + y \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial y} = x - e^{-y} \qquad \text{...(2)}$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int (e^x + y)dx = e^x + xy + h(y), \qquad \text{...(3)}$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = x + \frac{dh}{dy} \qquad \text{...(4)}$$

From (2) and (4), it follows that

$$x + \frac{dh}{dy} = x - e^{-y} \quad \Rightarrow \quad \frac{dh}{dy} = -e^{-y} \quad \Rightarrow \quad h(y) = e^{-y},$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = e^x + xy + e^{-y}$$

Thus, the general solution of the given equation is

$$e^x + xy + e^{-y} = c.$$

EXAMPLE 53 Find an integrating factor and solve the differential equation

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$$

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $\quad M = y^4 + 2y \quad$ and $\quad N = xy^3 + 2y^4 - 4x$

Now, $\quad \dfrac{\partial M}{\partial y} = 4y^3 + 2 \quad$ and $\quad \dfrac{\partial N}{\partial x} = y^3 - 4$

$\because \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the given equation is *not* exact. However,

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y^4 + 2y}\left[y^3 - 4 - 4y^3 - 2\right] = \frac{-3(y^3 + 2)}{y(y^3 + 2)} = -\frac{3}{y}, \quad \text{a function of } y \text{ alone}$$

Therefore, an integrating factor is given by

$$F^*(y) = e^{\int -3/y\,dy} = e^{-3\ln y} = e^{\ln y^{-3}} = y^{-3}$$

Multiplying both sides of the given equation by y^{-3}, we obtain the exact differential equation

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$$

Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = y + \frac{2}{y^2} \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = x + 2y - \frac{4x}{y^3} \qquad ...(2)$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int\left(y + \frac{2}{y^2}\right)dx = xy + \frac{2x}{y^2} + h(y), \qquad ...(3)$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = x - \frac{4x}{y^3} + \frac{dh}{dy} \qquad ...(4)$$

From (2) and (4), it follows that

$$x - \frac{4x}{y^3} + \frac{dh}{dy} = x + 2y - \frac{4x}{y^3} \quad \Rightarrow \quad \frac{dh}{dy} = 2y \quad \Rightarrow \quad h(y) = y^2,$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = xy + \frac{2x}{y^2} + y^2$$

Thus, the general solution of the given equation is

$$xy + \frac{2x}{y^2} + y^2 = c.$$

EXAMPLE 54 Find an integrating factor and solve the differential equation

$$(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0.$$

SOLUTION The given differential equation is of the form

$$M(x, y)dx + N(x, y)dy = 0$$

where $\quad M = 2xy^4e^y + 2xy^3 + y \quad$ and $\quad N = x^2y^4e^y - x^2y^2 - 3x$

Now, $$\frac{\partial M}{\partial y} = 2xy^4e^y + 8xy^3e^y + 6xy^2 + 1 \quad \text{and} \quad \frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$\because \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the given equation is *not* exact. However,

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{2xy^4e^y + 2xy^3 + y}\left[2xy^4e^y - 2xy^2 - 3 - 2xy^4e^y - 8xy^3e^y - 6xy^2 - 1\right]$$

$$= \frac{-4(2xy^3e^y + 2xy^2 + 1)}{y(2xy^3e^y + 2xy^2 + 1)} = -\frac{4}{y}, \quad \text{a function of } y \text{ alone}$$

Therefore, an integrating factor is given by

$$F^*(y) = e^{\int -4/y\,dy} = e^{-4\ln y} = e^{\ln y^{-4}} = y^{-4}$$

Multiplying both sides of the given equation by y^{-4}, we obtain the exact differential equation

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0$$

Thus, there exists a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = 2xe^y + \frac{2x}{y} + \frac{1}{y^3} \qquad \text{...(1)}$$

$$\frac{\partial u}{\partial y} = x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4} \qquad \text{...(2)}$$

Integrating equation (1) with respect to x, treating y as a constant, we obtain

$$u(x, y) = \int\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx = x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} + h(y), \qquad \text{...(3)}$$

where $h(y)$ plays the role of constant of integration. To find $h(y)$, we differentiate equation (3) partially with respect to y to get

$$\frac{\partial u}{\partial y} = x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4} + \frac{dh}{dy}, \qquad \text{...(4)}$$

From (2) and (4), it follows that

$$x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4} + \frac{dh}{dy} = x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4} \quad \Rightarrow \quad \frac{dh}{dy} = 0 \quad \Rightarrow \quad h(y) = 0,$$

where we have taken the constant of integration equal to zero without loss of generality. Substituting the expression for $h(y)$ into equation (3), we get

$$u(x, y) = x^2e^y + \frac{x^2}{y} + \frac{x}{y^3}$$

Thus, the general solution of the given equation is

$$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c.$$

EXERCISE 1.5

Test for exactness each of the following differential equations. If exact, solve. If not, find an integrating factor and then solve. Also, if an initial condition is given, determine the corresponding particular solution.

1. $(e^y - ye^x)dx + (xe^y - e^x)dy = 0$
2. $e^x \cos y\, dx - e^x \sin y\, dy = 0$
3. $\left(2x+\frac{1}{y}-\frac{y}{x^2}\right)dx+\left(2y+\frac{1}{x}-\frac{x}{y^2}\right)dy=0$
4. $-2xy\sin(x^2)dx + \cos(x^2)dy = 0$
5. $\left(-\frac{y}{x^2}+2\cos 2x\right)dx + \left(\frac{1}{x}-2\sin 2y\right)dy=0$
6. $e^{2x}(2\cos y\, dx - \sin y\, dy) = 0,\ y(0) = 0$
7. $-ydx + xdy = 0$ **Hint :** $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{2}{x}$
8. $\left(\cos xy+\frac{x}{y}\right)dx+\left(1+\frac{x}{y}\cos xy\right)dy=0$ **Hint :** $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y}$
9. $(\sin y\cos y + x\cos 2y)dx + xdy = 0$ **Hint :** $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 2\tan y$
10. $(x^4 + y^2)dx - xydy = 0, y(2) = 1$ **Hint :** $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{3}{x}$

ANSWERS

1. Exact, $xe^y - ye^x = c$
2. Exact, $e^x \cos y = c$
3. Exact, $x^2 + \frac{x}{y} + \frac{y}{x} + y^2 = c$
4. Exact, $y\cos(x^2) = c$
5. Exact, $\frac{y}{x} + \sin 2x + \cos 2y = c$
6. Exact, $e^{2x}\cos y = 1$
7. Non-exact, $\frac{y}{x} = c$
8. Non-exact, $\sin xy + \frac{1}{2}x^2 + \frac{1}{2}y^2 = c$
9. Non-exact, $x\tan y + \frac{1}{2}x^2 = c.$
10. Non-exact, $x^2 - \frac{y^2}{x^2} = 3.75$

1.7 LINEAR DIFFERENTIAL EQUATIONS

A first-order ordinary differential equation is said to be **linear** if the dependent variable and its derivatives occur in the first degree only and are not multiplied together.

Thus the most general first-order linear differential equation is of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{or} \qquad y' + Py = Q \quad \left(y' = \frac{dy}{dx}\right) \qquad \ldots(1)$$

where P and Q are functions of x. It is understood that, in Eq. (1), y is the dependent variable. If the equation begins with, say, $f(x)y'$, then divide by $f(x)$ to have the **standard form** (1) with y' as the first term, which is practical.

To solve Eq. (1), we multiply both sides by $e^{\int P\,dx}$ to get

$$e^{\int P\,dx}\frac{dy}{dx} + Py\,e^{\int P\,dx} = Q\,e^{\int P\,dx} \qquad \ldots(2)$$

But L.H.S. of Eq. (2) equals $\frac{d}{dx}\left(y\,e^{\int P\,dx}\right)$. Thus, we get

$$\frac{d}{dx}\left(y\,e^{\int P\,dx}\right) = Q\,e^{\int P\,dx}$$

Integrating, we get

$$y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c \qquad \text{or} \qquad y = e^{-\int P\,dx}\left[\int Q\,e^{\int P\,dx}\,dx + c\right]$$

This gives the general solution of Equation (1).

The factor $e^{\int P\,dx}$, on multiplying by which the L.H.S. of Eq. (1) becomes the derivative of a function of x and y, is called the **integrating factor**. We shall denote it by *I.F.* Thus the solution of Equation (1) can also be expressed as

$$y(I.F.) = \int Q(I.F.)\,dx + c$$

Note The reader may recall that $e^{\ln t} = t$ for any t. This fact is frequently used in this section.

EXAMPLE 55 Solve the differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2$.

SOLUTION The given equation is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \quad \text{where } P = \frac{1}{x} \quad \text{and} \quad Q = x^2$$

$$\therefore \qquad I.F. = e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Thus, the general solution is given by

$$y(I.F.) = \int Q(I.F.)\,dx + c \qquad i.e., \qquad yx = \int x^2 \cdot x\,dx + c$$

or,
$$xy = \int x^3\,dx + c = \frac{x^4}{4} + c$$

$$\Rightarrow \qquad y = \frac{x^3}{4} + cx^{-1}.$$

EXAMPLE 56 Solve the differential equation $\frac{dy}{dx} - \frac{y}{x} = 2x^2$.

SOLUTION The given equation is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \quad \text{where } P = -\frac{1}{x} \text{ and } Q = 2x^2$$

$$\therefore \qquad I.F. = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = x^{-1} = \frac{1}{x}$$

Thus, the general solution is given by

$$y(I.F.) = \int Q(I.F.)dx + c$$

i.e.,
$$y\left(\frac{1}{x}\right) = \int 2x^2 \cdot \frac{1}{x} dx + c = \int 2x\,dx + c = x^2 + c$$

$$\Rightarrow \qquad y = x^3 + cx.$$

EXAMPLE 57 Solve the initial value problem

$$x^2 y' + 3xy = \frac{1}{x}, \quad y(1) = -1.$$

SOLUTION Dividing by x^2, the given equation can be expressed as

$$y' + \frac{3}{x} y = \frac{1}{x^3},$$

a standard linear differential equation of the form

$$y' + Py = Q, \quad \text{where} \quad P = \frac{3}{x}, \quad Q = \frac{1}{x^3}$$

$$\therefore \qquad I.F. = e^{\int P\,dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

Thus, the general solution is given by

$$y(I.F.) = \int Q(I.F.)dx + c$$

i.e.,
$$yx^3 = \int \frac{1}{x^3} \cdot x^3\,dx + c = x + c$$

$$\Rightarrow \qquad y = x^{-2} + cx^{-3} \qquad \text{...(1)}$$

From the initial condition $y(1) = -1$, *i.e.,* $y = -1$ when $x = 1$, we have

$$-1 = 1 + c \quad \Rightarrow \quad c = -2$$

Substituting this value of c in equation (1), we get the particular solution of the given initial value problem

$$y = x^{-2} - 2x^{-3}.$$

EXAMPLE 58 Solve the initial value problem

$$y' \tan x = 2y - 8, \quad y(\pi/2) = 0.$$ *[Delhi Univ. GE-3, 2018]*

SOLUTION ***First Method*** **:** *Separation of variables method*

The given equation can be expressed as

$$\frac{dy}{y-4} = 2\frac{\cos x}{\sin x}\,dx,$$

an equation with separable variables. Integrating both sides, we get

$$\ln(y-4) = 2\ln(\sin x) + \ln c$$

$$\Rightarrow \quad \ln(y-4) = \ln(c\sin^2 x)$$

$$\Rightarrow \quad y - 4 = c\sin^2 x$$

or
$$y = c\sin^2 x + 4 \qquad \ldots(1)$$

From the initial condition $y(\pi/2) = 0$, *i.e.*, $y = 0$ when $x = \pi/2$, we get

$$0 = c + 4 \quad \Rightarrow \quad c = -4$$

Substituting this value of c in equation (1), we get the particular solution of the given initial value problem

$$y = 4 - 4\sin^2 x.$$

Second Method **:** *Method of linearization*

The given equation can be expressed as

$$y' - (2\cot x)y = -8\cot x,$$

a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \quad \text{where } P = -2\cot x \text{ and } Q = -8\cot x$$

$$\therefore \quad I.F. = e^{\int P\,dx} = e^{-2\int \cot x\,dx} = e^{-2\ln\sin x} = \frac{1}{\sin^2 x}$$

Thus, the general solution is given by

$$y(I.F.) = \int Q(I.F.)dx + c^*$$

i.e.,
$$y\left(\frac{1}{\sin^2 x}\right) = -8\int \frac{\cot x}{\sin^2 x}dx + c^*$$

or,
$$y\left(\frac{1}{\sin^2 x}\right) = -8\int \cot x \operatorname{cosec}^2 x\,dx + c^*$$

$$= 4\cot^2 x + c^* \quad \left(\text{using } \int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1},\ n \neq -1\right)$$

$$\Rightarrow \quad y = 4\cos^2 x + c^*\sin^2 x$$

$$y = c\sin^2 x + 4, \text{ where } c = c^* - 4$$

We now continue as before.

EXAMPLE 59 Solve the differential equation $y' + 4x^2y = (4x^2 - x)e^{-x^2/2}$.

SOLUTION The given equation is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \quad \text{where } P = 4x^2 \text{ and } Q = (4x^2 - x)e^{-x^2/2}$$

$$\therefore \qquad I.F. = e^{\int P\,dx} = e^{\int 4x^2\,dx} = e^{4x^3/3}$$

Thus, the general solution is given by

$$y(I.F.) = \int Q(I.F.)dx + c$$

i.e.,

$$y(e^{4x^3/3}) = \int (4x^2 - x)e^{-x^2/2} \cdot e^{4x^3/3}\,dx + c = \int (4x^2 - x)e^{(4x^3/3)-(x^2/2)}\,dx + c$$

To evaluate $\int (4x^2 - x)e^{(4x^3/3)-(x^2/2)}\,dx$, we let $\frac{4x^3}{3} - \frac{x^2}{2} = t$ so that $(4x^2 - x)dx = dt$

$$\therefore \qquad \int (4x^2 - x)e^{(4x^3/3)-(x^2/2)}\,dx = \int e^t\,dt = e^t = e^{(4x^3/3)-(x^2/2)}$$

Thus, we have

$$y(e^{4x^3/3}) = e^{(4x^3/3)-(x^2/2)} + c$$

$$\Rightarrow \qquad y = e^{-x^2/2} + ce^{-4x^3/3}.$$

EXAMPLE 60 Solve the differential equation $x\frac{dy}{dx} + y = x\log x$.

SOLUTION Dividing by x, the given equation can be expressed as

$$\frac{dy}{dx} + \frac{1}{x}y = \log x,$$

a linear differential of the form

$$\frac{dy}{dx} + Py = Q, \quad \text{where } P = \frac{1}{x} \text{ and } Q = \log x$$

Thus, the general solution is given by

$$I.F. = e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\therefore \qquad y(I.F.) = \int Q(I.F.)dx + c$$

i.e.,

$$yx = \int \log x \cdot x\,dx + c$$

or,

$$xy = \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2}dx + c \qquad \textit{(Integrating by parts)}$$

$$= \frac{x^2 \log x}{2} - \frac{1}{2}\int x\,dx + c = \frac{x^2 \log x}{2} - \frac{x^2}{4} + c$$

$$\Rightarrow \qquad 4xy = 2x^2 \log x - x^2 + k, \text{ where } k = 4c.$$

EXAMPLE 61 Solve the differential equation $\frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x}$.

SOLUTION The given equation is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \quad \text{where } P = \frac{1}{x\log x} \text{ and } Q = \frac{2}{x}$$

$$\therefore \qquad I.F. = e^{\int \frac{1}{x\log x}dx}$$

To evaluate $\int \frac{1}{x\log x}dx$, we let $\log x = t$ so that $\frac{1}{x}dx = dt$.

$$\therefore \qquad \int \frac{1}{x\log x}dx = \int \frac{dt}{t} = \log t = \log(\log x)$$

$$\Rightarrow \qquad I.F. = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$$

Thus, the general solution is given by

$$y(I.F.) = \int Q(I.F.)dx + c$$

i.e.,

$$y(\log x) = \int \frac{2}{x}\cdot \log x\,dx + c$$

or,

$$y(\log x) = (\log x)^2 + c \quad \Rightarrow \quad y = \log x + \frac{c}{\log x}.$$

EXAMPLE 62 Solve the differential equation $(y + \log x)dx - x\,dy = 0$.

SOLUTION The given equation can be written as

$$\frac{dy}{dx} = \frac{y + \log x}{x} \quad \text{or} \quad \frac{dy}{dx} - \frac{1}{x}y = \frac{\log x}{x}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \quad \text{where } P = -\frac{1}{x} \text{ and } Q = \frac{\log x}{x}$$

$$I.F. = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = x^{-1} = \frac{1}{x}$$

Thus, the general solution is given by

$$y(I.F.) = \int Q(I.F.)dx + c$$

i.e.,
$$y\left(\frac{1}{x}\right) = \int \log x \cdot \frac{1}{x^2}\,dx + c$$

or,
$$\frac{y}{x} = -\frac{\log x}{x} + \int \frac{1}{x^2}\,dx + c \qquad \textit{(Integration by parts)}$$

i.e.,
$$\frac{y}{x} = -\frac{\log x}{x} - \frac{1}{x} + c$$

$\Rightarrow$
$$y = -(\log x + 1) + cx.$$

EXAMPLE 63 Solve the initial value problem

$$\frac{dy}{dx} + 2y\sin 2x = 2e^{\cos 2x}, \qquad y(0) = 0.$$

SOLUTION The given equation is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \quad \text{where } P = 2\sin 2x \text{ and } Q = 2e^{\cos 2x}$$

$\therefore$
$$I.F. = e^{\int P\,dx} = e^{\int 2\sin 2x\,dx} = e^{-\cos 2x}$$

Thus, the general solution is given by

$$y(I.F.) = \int Q(I.F.)\,dx + c$$

i.e.,
$$y(e^{-\cos 2x}) = \int 2e^{\cos 2x} \cdot e^{-\cos 2x}\,dx + c = \int 2\,dx + c = 2x + c$$

$\Rightarrow$
$$y = 2xe^{\cos 2x} + ce^{\cos 2x}$$

From the initial condition

$$y(0) = 0 \text{ i.e., } y = 0 \text{ when } x = 0$$

we get
$$0 = 0 + ce^0 \quad \Rightarrow \quad c = 0$$

Thus, the particular solution of the given initial value problem is $y = 2xe^{\cos 2x}$.

EXAMPLE 64 Solve the following differential equation $dy/dx = (x + y + 1)/(x + 1)$.

SOLUTION The given equation can be written as

$$\frac{dy}{dx} = 1 + \frac{y}{x+1} \quad \text{or} \quad \frac{dy}{dx} - \frac{1}{x+1}y = 1$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \quad \text{where } P = -\frac{1}{x+1} \text{ and } Q = 1$$

$$I.F. = e^{\int P\,dx} = e^{\int -\frac{1}{x+1}dx} = e^{-\log(x+1)} = \frac{1}{x+1}$$

Thus, the general solution is given by

$$y(I.F.) = \int Q(I.F.)dx + c$$

i.e.,

$$\frac{y}{x+1} = \int \frac{1}{x+1}\,dx + c$$

or,

$$\frac{y}{x+1} = \log(x+1) + c \quad \text{or} \quad y = (x+1)\log(x+1) + c(x+1).$$

EXAMPLE 65 Solve the differential equation $y' \sin 2y + x\cos 2y = 2x$.

SOLUTION We first note that the given equation is not linear. However, it can be transformed to linear form by substituting

$$\cos 2y = z \quad \text{so that} \quad -2\sin 2y\frac{dy}{dx} = \frac{dz}{dx} \quad \Rightarrow \quad \sin 2y\frac{dy}{dx} = -\frac{1}{2}\frac{dz}{dx}$$

This substitution transforms the given equation to

$$-\frac{1}{2}\frac{dz}{dx} + xz = 2x \qquad \text{or} \qquad \frac{dz}{dx} - 2xz = -4x$$

This is a linear differential equation with z as the dependent variable. Here, $P = -2x$ and $Q = -4x$.

$$I.F. = e^{\int P dx} = e^{\int -2x dx} = e^{-x^2}$$

Thus, the general solution is given by

$$z(I.F.) = \int Q(I.F.)dx + c$$

i.e.,

$$z(e^{-x^2}) = \int (-4x)e^{-x^2}\,dx + c = 2e^{-x^2} + c$$

$$\Rightarrow \qquad z = 2 + ce^{x^2} \qquad \text{or} \qquad \cos 2y = 2 + ce^{x^2}$$

$$\Rightarrow \qquad y = \frac{1}{2}\cos^{-1}(2 + ce^{x^2}).$$

Remark Sometimes, a differential equation can be put in the form

$$\frac{dx}{dy} + Px = Q \qquad \text{...(1)}$$

where P and Q are functions of y. Obviously, Eq. (1) can be thought of as a linear differential equation with x as the dependent variable. Hence, its solution is given by

$$x(I.F.) = \int Q(I.F.)\,dy + c \qquad \text{where} \quad I.F. = e^{\int P dy}$$

EXAMPLE 66 Solve the differential equation $(x + 2y^3)\dfrac{dy}{dx} = y$.

SOLUTION The given equation is obviously not linear, if we take y as the dependent variable.

However, the equation can be written as

$$y\frac{dx}{dy} = x + 2y^3 \qquad \text{or} \qquad \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

which is a linear differential equation with x as the dependent variable. Here $P = -\frac{1}{y}$ and $Q = 2y^2$.

$$\therefore \qquad I.F. = e^{\int P\,dy} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$$

Thus, the general solution is given by

$$x(I.F.) = \int Q(I.F.)\,dy + c$$

i.e.,
$$x\left(\frac{1}{y}\right) = \int 2y^2 \cdot \frac{1}{y}dy + c = \int 2y\,dy + c = y^2 + c$$

or,
$$\frac{x}{y} = y^2 + c \qquad \text{or} \qquad x = y(c + y^2).$$

EXAMPLE 67 Solve the differential equation $(x + y + 1)\frac{dy}{dx} = 1$.

SOLUTION The given equation can be written as

$$\frac{dx}{dy} = x + y + 1 \qquad \text{or} \qquad \frac{dx}{dy} - x = y + 1,$$

which is a linear differential equation with x as the dependent variable. Here $P = -1$ and $Q = y + 1$.

$$\therefore \qquad I.F. = e^{\int P\,dy} = e^{\int -dy} = e^{-y}$$

Thus, the general solution is given by

$$x(I.F.) = \int Q(I.F.)\,dy + c$$

i.e.,
$$xe^{-y} = \int (y+1)e^{-y}dy + c = -(y+1)e^{-y} + \int e^{-y}dy + c \quad (\textit{Integrating by parts})$$
$$= -(y+1)e^{-y} - e^{-y} + c = -(y+2)e^{-y} + c$$

or,
$$xe^{-y} = -(y+2)e^{-y} + c \quad \text{or} \quad x = -(y+2) + ce^{y}.$$

EXAMPLE 68 Solve the differential equation $\frac{dx}{dy} = \frac{y^3 + 2xe^y}{ye^y}$.

SOLUTION The given equation is

$$\frac{dx}{dy} = \frac{y^3 + 2xe^y}{ye^y} \qquad \text{or} \qquad \frac{dx}{dy} - \frac{2}{y}x = y^2e^{-y}$$

which is a linear differential equation with x as the dependent variable. Here $P = -\frac{2}{y}$ and $Q = y^2 e^{-y}$.

$$\therefore \quad I.F. = e^{\int P\,dy} = e^{-\int \frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$$

Thus, the general solution is given by

$$x(I.F.) = \int Q(I.F.)\,dy + c$$

i.e.,
$$x\left(\frac{1}{y^2}\right) = \int y^2 e^{-y} \cdot \frac{1}{y^2} dy + c$$

or,
$$\frac{x}{y^2} = \int e^{-y} dy + c = -e^{-y} + c$$

$$\Rightarrow \quad x = cy^2 - e^{-y} y^2.$$

Equations Reducible to Linear Form (Bernoulli's Equation)

A differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)\,y^n, \qquad \ldots(1)$$

where n is a real number, is known as **Bernoulli's** equation. This equation looks a lot like a linear equation except for the y^n. If $n = 0$ or $n = 1$, equation (1) is linear and we can solve it. Otherwise, it is non-linear but can be reduced to linear form by making the substitution $y^{1-n} = z$. Let us consider few examples.

EXAMPLE 69 Solve the differential equation $\frac{dy}{dx} + \frac{y}{x} = y^2 \log x$.

SOLUTION The given equation is a Bernoulli's equation. Dividing both sides of the given equation by y^2, we get

$$y^{-2}\frac{dy}{dx} + y^{-1} \cdot \frac{1}{x} = \log x \qquad \ldots(1)$$

Put $\quad y^{-1} = z$ so that $-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$

Equation (1) thus reduces to

$$-\frac{dz}{dx} + \frac{1}{x}z = \log x \quad \text{or} \quad \frac{dz}{dx} - \frac{1}{x}z = -\log x$$

which is a linear differential equation with z as the dependent variable. Here $P = -\frac{1}{x}$ and $Q = -\log x$.

$$\therefore \quad I.F. = e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

Thus, its general solution is given by

$$z(I.F.) = \int Q(I.F.)\,dx + c$$

i.e.,
$$z\left(\frac{1}{x}\right) = \int (-\log x)\cdot\frac{1}{x}\,dx + c = -\frac{1}{2}(\log x)^2 + c$$

or,
$$\frac{z}{x} = -\frac{1}{2}(\log x)^2 + c \quad \text{or} \quad \frac{1}{xy} = -\frac{(\log x)^2}{2} + c,$$

which is the required solution.

EXAMPLE 70 Solve the differential equation $y' = 5.7y - 6.5y^2$. **[Delhi Univ. GE-3, 2017]**

SOLUTION The given equation can be written as

$$y' - 5.7y = -6.5y^2 : \textit{ Bernoulli's equation}$$

Division by y^2 yields

$$y^{-2}y' - 5.7y = -6.5 \quad \text{or} \quad -y^{-2}y' + 5.7y^{-1} = 6.5 \qquad ...(1)$$

Set $\quad y^{-1} = z$ so that $-y^{-2}y' = z'$.

Equation (1) thus reduces to

$$z' + 5.7z = 6.5$$

which is a linear differential equation with z as the dependent variable.

$$I.F. = e^{\int 5.7\,dx} = e^{5.7x}$$

Thus, the general solution is given by

$$z(I.F.) = \int Q(I.F.)\,dx + c$$

i.e.,
$$ze^{5.7x} = \int 6.5e^{5.7x}\,dx + c = \frac{6.5}{5.7}e^{5.7x} + c \quad \Rightarrow \quad z = \frac{6.5}{5.7} + ce^{-5.7x}$$

or,
$$\frac{1}{y} = \frac{6.5}{5.7} + ce^{-5.7x} \quad \text{or,} \quad y = \frac{1}{\frac{6.5}{5.7} + ce^{-5.7x}}.$$

EXAMPLE 71 Solve the differential equation $y' = Ay - By^2$. **[Delhi Univ. GE-3, 2018]**

SOLUTION The given equation can be written as

$$y' - Ay = -By^2 \quad : \textit{ Bernoulli's equation}$$

Division by y^2 yields

$$y^{-2}y' - Ay = -B \quad \text{or} \quad -y^{-2}y' + Ay^{-1} = B \qquad ...(1)$$

Set $\quad y^{-1} = z$ so that $-y^{-2}y' = z'$.

Equation (1) thus reduces to

$$z' + Az = B$$

which is a linear differential equation with z as the dependent variable.

$$I.F. = e^{\int A dx} = e^{Ax}$$

Thus, the general solution is given by

$$z(I.F.) = \int Q(I.F.) dx + c$$

i.e.,
$$ze^{Ax} = \int Be^{Ax} dx + c = \frac{B}{A}e^{Ax} + c \quad \Rightarrow \quad z = \frac{B}{A} + ce^{-Ax}$$

or,
$$\frac{1}{y} = \frac{B}{A} + ce^{-Ax} \quad \text{or,} \quad y = \frac{1}{\frac{B}{A} + ce^{-Ax}}.$$

Note Notice that problem in Example 70 is a particular case of problem in Example 71.

EXAMPLE 72 Solve the differential equation

$$y' + (x+1)y = e^{x^2} y^3, \quad y(0) = 0.5.$$ **[*Delhi Univ. GE-3, 2016*]**

SOLUTION The given ODE is a Bernoulli's equation. Dividing both sides of the given equation by y^3, we obtain

$$y^{-3}y' + (x+1)y^{-2} = e^{x^2} \qquad ...(1)$$

Put
$$y^{-2} = z \text{ so that } -2y^{-3}\frac{dy}{dx} = \frac{dz}{dx}$$

Equation (1) thus reduces to

$$-\frac{1}{2}\frac{dz}{dx} + (x+1)z = e^{x^2} \quad \text{or} \quad \frac{dz}{dx} - 2(x+1)dz = -2e^{x^2}$$

which is a linear differential equation with z as the dependent variable. Here $P = -2(x+1)$ and $Q = -2e^{x^2}$.

$$\therefore \quad I.F. = e^{\int P dx} = e^{-2\int (x+1)dx} = e^{-x^2 - 2x}$$

Thus its solution is given by

$$z(I.F.) = \int Q(I.F.) dx + c$$

i.e.,
$$z \cdot e^{-x^2 - 2x} = -2\int e^{x^2} \cdot e^{-x^2 - 2x} dx + c$$

or,
$$z \cdot e^{-x^2 - 2x} = -2\int e^{-2x} dx + c$$

or, $$z \cdot e^{-x^2-2x} = e^{-2x} + c$$

$\Rightarrow$ $$z = e^{x^2} + ce^{x^2+2x}$$

i.e., $$y^{-2} = e^{x^2} + ce^{x^2+2x} \qquad ...(2)$$

We now use the initial condition : $y(0) = 0.5$ *i.e.,* $y = 0.5$ when $x = 0$. Using this condition, from (2), we obtain

$$(0.5)^{-2} = e^0 + ce^0 \quad \text{or} \quad 4 = 1 + c \quad \Rightarrow \quad c = 3$$

Substituting the value of c into equation (2), we obtain the solution of the given initial value problem

$$y^{-2} = e^{x^2} + 3e^{x^2+2x} \quad \text{or} \quad y^{-2} = e^{x^2}(1 + 3e^{2x}).$$

EXAMPLE 73 Solve the differential equation $\dfrac{dy}{dx} + xy = y^2 e^{x^2/2} \log x$.

SOLUTION The given ODE is a Bernoulli's equation. Dividing both sides of the given equation by y^2, we obtain

$$y^{-2}\frac{dy}{dx} + y^{-1}x = e^{x^2/2}\log x \qquad ...(1)$$

Put $$y^{-1} = z \text{ so that } -y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

Equation (1) thus reduces to

$$-\frac{dz}{dx} + xz = e^{x^2/2}\log x \quad \text{or} \quad \frac{dz}{dx} - xz = -e^{x^2/2}\log x,$$

which is a linear differential equation with z as the dependent variable. Here $P = -x$ and $Q = -e^{x^2/2}\log x$.

$$I.F. = e^{\int P dx} = e^{-\int x dx} = e^{-x^2/2}$$

Thus, its general solution is given by

$$z(I.F.) = \int Q(I.F.)\, dx + c$$

i.e., $$z \cdot e^{-x^2/2} = -\int \log x \cdot e^{x^2/2} \cdot e^{-x^2/2} dx + c$$

or, $$ze^{-x^2/2} = -\int \log x\, dx + c = -(x\log x - x) + c$$

or, $$y^{-1}e^{-x^2/2} = c + x - x\log x$$

$\Rightarrow$ $$ye^{x^2/2}(c + x - x\log x) = 1.$$

EXERCISE 1.6

Solve each of the following differential equations :

1. $\dfrac{dy}{dx} + 2y = 6e^x$
2. $\dfrac{dy}{dx} + \dfrac{y}{x} = e^{-x}$
3. $\dfrac{dy}{dx} - y = e^{2x}$
4. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^3$
5. $x\dfrac{dy}{dx} + 2y = x^2$
6. $(x^2-1)\dfrac{dy}{dx} + 2xy = 1$
7. $\dfrac{dy}{dx} + \dfrac{y}{x^2} = 2xe^{1/x}$
8. $\dfrac{dy}{dx} = \dfrac{y}{x} + \log x$
9. $x\log x\dfrac{dy}{dx} + y = \dfrac{2}{x}\log x$
10. $x\dfrac{dy}{dx} - y = (x-1)e^x$
11. $\dfrac{dy}{dx} + 2xy = xe^{-x^2/2}$
12. $x\dfrac{dy}{dx} + y = x^2 + 1$
13. $x\log x\dfrac{dy}{dx} + y = 2\log x$
14. $x\,dx + (x-y^2)dy = 0$
15. $y\,dx - (x + 2y^3) = 0$
16. $\dfrac{dy}{dx} + \dfrac{(2x+1)}{x}y = e^{-2x}$
17. $\dfrac{dy}{dx} + xy = x^3y^3$
18. $x\dfrac{dy}{dx} + y = xy^3$

Solve the following initial value problems :

19. $x^2y' + 3xy = 1/x, \quad y(1) = -1$
20. $y' + y = y^2, \quad y(0) = -1$
21. $y' + 4y\cot 2x = 6\cos 2x, \quad y(\pi/4) = 2$
22. $y' + 2y\sin 2x = 2e^{\cos 2x}, \quad y(0) = 0$
23. $2yy' + y^2\sin x = \sin x, \quad y(0) = \sqrt{2}$
24. $(x^2+1)y' + 4xy = x, \quad y(2) = 1$

ANSWERS

1. $y = 2e^x + ce^{-2x}$
2. $y = -\left(\dfrac{x+1}{x}\right)e^{-x} + \dfrac{c}{x}$
3. $y = e^{2x} + ce^x$
4. $5xy = x^5c$
5. $y = \dfrac{x^2}{4} + cx^{-2}$
6. $y(1-x^2) = -x + c$
7. $y = e^{1/x}(x^2 + c)$
8. $y = \dfrac{1}{2}x(\log x)^2 + cx$
9. $y\log x = -\dfrac{2(1+\log x)}{x} + c$
10. $y = e^x + cx$
11. $y = e^{-x^2}\left(c + \dfrac{1}{2}x^2\right)$
12. $xy = \dfrac{x^3}{3} + x + c$

13. $y \log x = c + (\log x)^2$

14. $3xy = y^3 + c$

15. $x = y^3 + cy$

16. $xye^{2x} = \frac{x^2}{2} + c$

17. $y^{-1} = ce^{x^2} + 1 + x^2$

18. $(2 + cx)\, xy^2 = 1$

19. $y = x^{-2} - 2x^{-3}$

20. $y = \frac{1}{1-2e^x}$

21. $y = \sin 2x + \frac{1}{\sin^2 2x}$

21. $y = 2xe^{\cos 2x}$

23. $y^2 = 1 - ce^{\cos x}$, $c = -1/e$

24. $y(x^2+1)^2 = \frac{(x^2+1)}{4} + \frac{75}{4}$

1.8 FORMATION OF AN ORDINARY DIFFERENTIAL EQUATION

Consider an equation

$$f(x, y, c_1, c_2, \ldots, c_n) = 0 \quad \ldots(1)$$

representing a family of curves depending on n constants $c_1, c_2, \ldots, c_n$. If we differentiate Eq. (1) successively n times with respect to x, we get n equations of the form

$$\left.\begin{array}{r} f_1(x, y, y', c_1, c_2, \ldots, c_n) = 0 \\ f_2(x, y, y', y'', c_1, c_2, \ldots, c_n) = 0 \\ \vdots \\ f_n\left(x, y, y', y'', \ldots, y^{(n)}, c_1, c_2, \ldots, c_n\right) = 0 \end{array}\right\} \quad \ldots(2)$$

By eliminating n constants $c_1, c_2, \ldots, c_n$ from $(n + 1)$ equations given in (1) and (2), we can form a differential equation of order n, say

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

Such an equation is called the differential equation representing the family of curves given by Eq. (1).

EXAMPLE 74 Form the differential equation representing the family of curves $y = mx$, where m is a constant.

SOLUTION The given equation is

$$y = mx \quad \ldots(1)$$

where m is a constant. Differentiating Eq. (1) with respect to x, we get

$$\frac{dy}{dx} = m$$

Substituting the value of m in Eq. (1), we get

$$y = x\frac{dy}{dx}$$

which is the required differential equation.

EXAMPLE 75 Form the differential equation of the family of curves represented by the equation $(2x + a)^2 + y^2 = a^2$, where a is a constant.

SOLUTION The given equation is

$$(2x+a)^2 + y^2 = a^2 \quad \Leftrightarrow \quad 4x^2 + a^2 + 4ax + y^2 = a^2$$

$$\Rightarrow \quad 4x^2 + 4ax + y^2 = 0 \qquad \text{...(1)}$$

Differentiating Eq. (1) with respect to x, we get

$$8x + 4a + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad 8x^2 + 4ax + 2xy\frac{dy}{dx} = 0$$

(Multiplying both sides by x)

$$i.e., \quad 4x^2 + (4x^2 + 4ax) + 2xy\frac{dy}{dx} = 0 \quad \text{or} \quad 4x^2 - y^2 + 2xy\frac{dy}{dx} = 0 \qquad \text{(using (1))}$$

which is the required differential equation.

EXAMPLE 76 Form the differential equation corresponding to $Ax^2 + By^2 = 1$ by eliminating a and b.

SOLUTION The given equation is

$$Ax^2 + By^2 = 1 \qquad \text{...(1)}$$

Differentiating Eq. (1) with respect to x, we get

$$2Ax + 2By\frac{dy}{dx} = 0 \quad \Rightarrow \quad Ax + By\frac{dy}{dx} = 0 \qquad \text{...(2)}$$

Differentiating Eq. (2) with respect to x, we get

$$A + B\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0$$

$$\Rightarrow \quad Ax + Bx\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0 \qquad \text{(Multiplying both sides by } x\text{)}$$

$$\Rightarrow \quad -By\frac{dy}{dx} + Bx\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0 \qquad \text{(using (2))}$$

$$\Rightarrow \quad -y\frac{dy}{dx} + xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 0, \qquad \text{(Dividing both sides by } B\text{)}$$

which is the required differential equation.

EXAMPLE 77 Form the differential equation corresponding to $y^2 = a(b - x^2)$ by eliminating a and b.

SOLUTION The given equation is

$$y^2 = a(b - x^2) \qquad \text{...(1)}$$

Differentiating Eq. (1) with respect to x, we get

$$2y\frac{dy}{dx} = -2ax \quad \Rightarrow \quad y\frac{dy}{dx} = -ax \qquad ...(2)$$

Differentiating Eq. (2) with respect to x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a$$

$$\Rightarrow \quad x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = -ax \qquad \textit{(Multiplying both sides by x)}$$

$$\Rightarrow \quad x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y\frac{dy}{dx} \qquad \textit{(using (2))}$$

which is the required differential equation.

EXAMPLE 78 Form the differential equation corresponding to $(x-a)^2 + (y-b)^2 = r^2$ by eliminating the constants a and b.

SOLUTION The given equation is

$$(x-a)^2 + (y-b)^2 = r^2 \qquad ...(1)$$

Differentiating Eq. (1) with respect to x, we get

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0 \quad \Rightarrow \quad (x-a) + (y-b)\frac{dy}{dx} = 0 \qquad ...(2)$$

Differentiating Eq. (2) with respect to x, we get

$$1 + (y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad ...(3)$$

To get the required differential equation, we must eliminate a and b from Eqs. (1), (2) and (3). From Eq. (3), we get

$$y - b = -\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

Substituting the value of $(y-b)$ in Eq. (2), we get

$$x - a = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$$

Substituting the values of $(x-a)$ and $(y-b)$ in Eq. (1), we get

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2}+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}=r^2 \quad\Rightarrow\quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^3=r^2\left(\frac{d^2y}{dx^2}\right)^2,$$

which is the required differential equation.

EXAMPLE 79 Find the differential equation of all circles which pass through the origin and whose centres lie on the y-axis.

SOLUTION The family of circles with the desired properties can be represented by

$$x^2+(y-a)^2 = a^2 \quad \text{or} \quad x^2+y^2-2ay=0 \qquad ...(1)$$

where a is an arbitrary constant. Differentiating Eq. (1) with respect to x, we get

$$2x+2(y-a)\frac{dy}{dx} = 0 \quad \text{or} \quad x+(y-a)\frac{dy}{dx}=0 \;\Rightarrow\; a=\frac{x+y\frac{dy}{dx}}{\frac{dy}{dx}} \qquad ...(2)$$

Substituting the value of a in Eq. (1), we obtain

$$x^2+y^2-2y\left[\frac{x+y\frac{dy}{dx}}{\frac{dy}{dx}}\right]=0 \quad\Rightarrow\quad (x^2+y^2)\frac{dy}{dx}-2xy-2y^2\frac{dy}{dx}=0$$

or

$$(x^2-y^2)\frac{dy}{dx}=2xy,$$

which is the required differential equation.

EXAMPLE 80 Find the differential equation of all circles in the first quadrant which touch the coordinate axes.

SOLUTION The family of circles with the desired properties can be represented by

$$(x-a)^2+(y-a)^2 = a^2 \qquad ...(1)$$

where a is an arbitrary constant. Differentiating Eq. (1) with respect to x, we get

$$2(x-a)+2(y-a)\frac{dy}{dx} = 0 \quad \text{or} \quad (x-a)+(y-a)\frac{dy}{dx}=0$$

$$\Rightarrow \qquad a=\frac{x+y\frac{dy}{dx}}{1+\frac{dy}{dx}}=\frac{x+yp}{1+p}, \quad \text{where } p=\frac{dy}{dx}.$$

Substituting the value of a in Eq. (1), we obtain

$$\left[x-\frac{x+yp}{1+p}\right]^2+\left[y-\frac{x+yp}{1+p}\right]^2=\left[\frac{x+yp}{1+p}\right]^2 \Rightarrow (px-py)^2+(y-x)^2=(x+py)^2$$

$$\Rightarrow \quad p^2(x-y)^2+(y-x)^2=(x+py)^2 \quad i.e., \quad (x-y)^2(p^2+1)=(x+py)^2,$$

which is the required differential equation.

EXAMPLE 81 Find the differential equation corresponding to $y=Ae^{2x}+Be^{-2x}$ by eliminating A and B.

SOLUTION Given

$$y = Ae^{2x}+Be^{-2x} \qquad \text{...(1)}$$

Differentiation of Eq. (1) with respect to x gives

$$\frac{dy}{dx} = 2Ae^{2x}-2Be^{-2x}$$

Differentiating the above equation with respect to x, we get

$$\frac{d^2y}{dx^2}=4Ae^{2x}+4Be^{-2x}=4(Ae^{2x}+Be^{-2x})=4y \quad i.e., \quad \frac{d^2y}{dx^2}=4y$$

which is the required differential equation.

EXAMPLE 82 Form the differential equation of the family of curves represented by $c(y+c)^2=x^3$.

SOLUTION Given

$$c(y+c)^2 = x^3 \qquad \text{...(1)}$$

Differentiating both sides of Eq. (1) with respect to x, we get

$$2c(y+c)\frac{dy}{dx} = 3x^2 \qquad \text{...(2)}$$

To find the required differential equation, we must eliminate c from (1) and (2). Dividing (1) by (2), we get

$$\frac{y+c}{2\frac{dy}{dx}}=\frac{x}{3} \Rightarrow y+c=\frac{2}{3}x\frac{dy}{dx} \Rightarrow c=\frac{2}{3}x\frac{dy}{dx}-y$$

Substituting the value of c in (1), we get

$$\left[\frac{2}{3}x\frac{dy}{dx}-y\right]\left[\frac{2}{3}x\frac{dy}{dx}\right]^2=x^3 \Rightarrow \frac{8}{27}x^3\left(\frac{dy}{dx}\right)^3-\frac{4}{9}x^2y\left(\frac{dy}{dx}\right)^2=x^3$$

$$\Rightarrow \quad 8x\left(\frac{dy}{dx}\right)^3-12y\left(\frac{dy}{dx}\right)^2=27x \Rightarrow x\left\{8\left(\frac{dy}{dx}\right)^3-27\right\}=12y\left(\frac{dy}{dx}\right)^2$$

which is the required differential equation.

EXAMPLE 83 Find the differential equation of all parabolas having their axis of symmetry coincident with the x-axis.

SOLUTION The equation of the family of parabolas having the desired property is

$$y^2 = 4a(x - h) \qquad \ldots(1)$$

where a and h are arbitrary constants. Differentiating Eq. (1) with respect to x, we get

$$2y\frac{dy}{dx} = 4a \quad \text{or} \quad y\frac{dy}{dx} = 2a \qquad \ldots(2)$$

Differentiating Eq. (2) with respect to x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

which is the required differential equation.

EXERCISE 1.7

1. Form the differential equation corresponding to $y^2 = 4a(x + a)$ by eliminating a.
2. Form the differential equation corresponding to $y^2 - 2ay + x^2 = a^2$ by eliminating a.
3. Form the differential equation of the family of curves represented by $y = c(x - c)^2$, where c is a constant.
4. Form the differential equation of the family of curves represented by $(2x - a)^2 - y^2 = a^2$, where a is a constant.
5. Form the differential equation of the family of curves represented by $y = ae^{bx}$, where a and b are constants.
6. Form the differential equation of the family of curves represented by $(x + a)^2 - 2y^2 = a^2$, where a is a constant.
7. Find the differential equation of the family of curves $= Ae^x + Be^{-x}$, where A and B are arbitrary constants.
8. Find the differential equation corresponding to $y^2 = (x - c)^3$ by eliminating c.
9. Find the differential equation corresponding to the equation $y = ax^2 + bx + c$ by eliminating a, band c.
10. Find the differential equation of all the circles which pass through the origin and whose centres lie on the x-axis.

ANSWERS

1. $y\left[1 - \left(\frac{dy}{dx}\right)^2\right] = 2x\frac{dy}{dx}$
2. $(x^2 - 2y^2)\,p^2 - 4pxy - x^2 = 0,\ p = \frac{dy}{dx}$
3. $\left(\frac{dy}{dx}\right)^3 = 4\left[x\frac{dy}{dx} - 2y\right]$
4. $2xy\frac{dy}{dx} - 4x^2 - y^2 = 0$
5. $y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

6. $4xy\frac{dy}{dx} = x^2 + 2y^2$ 7. $\frac{d^2y}{dx^2} = y$ 8. $8\left(\frac{dy}{dx}\right)^3 = 27y$

9. $\frac{d^3y}{dx^3} = 0$ 10. $(x^2 - y^2) + 2xy\frac{dy}{dx} = 0$

1.9 ORTHOGONAL TRAJECTORIES

An important application of first-order differential equations is to find a family of curves that intersect a given family of curves perpendicularly. The new curves are called **orthogonal trajectories** of the given curves and vice-versa. Orthogonal trajectories, therefore, are two families of curves that always intersect at right angles.

EXAMPLE 84 The family $y = mx$ of straight lines through the origin and the family $x^2 + y^2 = r^2$ of concentric circles with center at the origin, are orthogonal trajectories of each other, as shown in Figure 1.1.

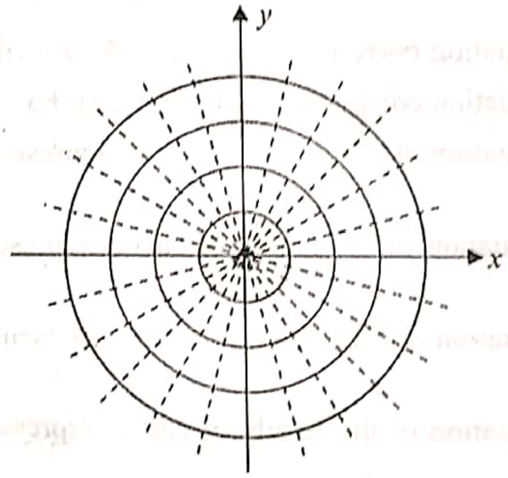

FIGURE 1.1

It is important to mention here that the **angle of intersection** between two intersecting curves is defined to be the angle between the tangents of the curves at the intersection point. The term **orthogonal** means *perpendicular* and *trajectory* means path or curve. From calculus it is known that two straight lines will be perpendicular if the product of their slopes is -1, that is, if slope of one is the negative reciprocal of the slope of the other. Since the slope of curve is given by the derivative, two families of curves

$$F(x, y, c_1) = 0 \quad \text{and} \quad G(x, y, c_2) = 0$$

will be orthogonal if the product of their derivatives is -1.

Procedure for Determining the Orthogonal Trajectories

We now outline the procedure for determining the orthogonal trajectories for a given family

$$F(x, y, c) = 0 \qquad ...(1)$$

of curves in the xy-plane. Here each curve in the family is specified by some value of c. The Eq.(1) is said to represent a **one-parameter family of curves**, and c is called the **parameter** of the family.

Step 1. Find an ODE for which the given family is a general solution. This is obtained by differentiating the given family of curves with respect to x and eliminating the constant c appearing in $F(x, y, c) = 0$.

Step 2. Write down the ODE of the orthogonal trajectories, that is, the ODE whose general solution gives the orthogonal trajectories of the given curves. This ODE is obtained by replacing $\frac{dy}{dx}$ by $\frac{-1}{dy/dx}$ in the differential equation obtained in Step 1.

Step 3. Solve the ODE obtained in Step 2 to get the equation for the orthogonal trajectories of the given family.

EXAMPLE 85 Find the orthogonal trajectories of the family of curves $y = cx^2$, where c is a constant.

SOLUTION **Step 1.** Find an ODE for which the given family is a general solution. The family of curves can be written as

$$\frac{y}{x^2} = c$$

Differentiating this equation with respect to x yields

$$\frac{x^2y' - 2xy}{x^4} = 0 \quad \Rightarrow \quad x^2y' - 2xy = 0 \quad \text{or} \quad y' = \frac{2y}{x}$$

This is the ordinary differential equation of the given family of curves.

Step 2. The second step is to write down the ordinary differential equation of the orthogonal trajectories, that is, the ODE whose general solution gives the orthogonal trajectories of the given curves. This ordinary differential equation is

$$\frac{dy}{dx} = -\frac{x}{2y}$$

or,
$$xdx + 2ydy = 0 \qquad \text{...(1)}$$

Step 3. Solve the ODE (1) to get the equation for the orthogonal trajectories of the given family. Integrating equation (1) gives

$$\frac{x^2}{2} + y^2 = a, \qquad \text{...(2)}$$

where a is an arbitrary constant. Thus, the orthogonal trajectories of the given family of curves is given by equation (2).

EXAMPLE 86 Find the orthogonal trajectories of the family of curves $y^2 = 2x^2 + c$, where c is a real parameter. **[*Delhi Univ. GE-3, 2018*]**

SOLUTION Differentiating the equation

$$y^2 = 2x^2 + c$$

with respect to x, we obtain

$$2y\frac{dy}{dx} = 4x$$

or
$$\frac{dy}{dx} = \frac{2x}{y}$$

The last equation is the ODE of the given family of curves. The ODE of the orthogonal trajectories, therefore, is

$$\frac{dy}{dx} = -\frac{y}{2x}$$

or,
$$\frac{2}{y}dy + \frac{1}{x}dx = 0$$

Integrating, we obtain

$$2\ln y + \ln x = \ln a \qquad i.e., \qquad \ln y^2 x = \ln a \qquad \Rightarrow \qquad y^2 x = a,$$

where a is a arbitrary constant. Thus, the orthogonal trajectories of the given family of curves is $y^2 x = a$.

EXAMPLE 87 Find the orthogonal trajectories of the family of curves given by $x = c\sqrt{y}$.

[*Delhi Univ. GE-3, 2017*]

SOLUTION The given family of curves can be written as

$$\frac{x}{\sqrt{y}} = c$$

Differentiating this equation yields

$$\frac{\sqrt{y} - \dfrac{x}{2\sqrt{y}}\dfrac{dy}{dx}}{y} = 0 \qquad \Rightarrow \qquad \sqrt{y} = \frac{x}{2\sqrt{y}}\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{2y}{x}$$

The last equation is the ODE of the given family of curves. The ODE of the orthogonal trajectories, therefore, is

$$\frac{dy}{dx} = -\frac{x}{2y} \qquad \Leftrightarrow \qquad x\,dx + 2y\,dy = 0$$

Integrating the last equation, we obtain

$$\frac{x^2}{2} + y^2 = a, \qquad ...(1)$$

where a is an arbitrary constant. Thus, the orthogonal trajectories are the family of curves given by Eq.(1).

EXAMPLE 88 Find the orthogonal trajectories of the family of curves $y = 4x + c$, where c is an arbitrary constant.

SOLUTION Differentiating the equation

$$y = 4x + c \qquad ...(1)$$

with respect ot x, we obtain

$$\frac{dy}{dx} = 4 \qquad ...(2)$$

Eq.(2) represents the ODE of the given family of curves. The ODE of the orthogonal trajectories, therefore, is

$$\frac{dy}{dx} = -\frac{1}{4}$$

Integrating, we obtain

$$y = -\frac{1}{4}x + a$$

where a is an arbitrary constant. Thus, the orthogonal trajectories of the given family of curves is $y = -\frac{1}{4}x + a$.

EXAMPLE 89 Find the orthogonal trajectories of the family of curves $y = ce^{-3x}$, where c is a constant. **[*Delhi Univ. GE-3*, 2016]**

SOLUTION The given family of curves can be written as

$$e^{3x}y = c$$

Differentiating this equation with respect to x, we obtain

$$3e^{3x}y + e^{3x}\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad 3y + \frac{dy}{dx} = 0 \qquad (\because \; e^{3x} \text{ is never zero})$$

$$\Rightarrow \qquad \frac{dy}{dx} = -3y$$

The last equation is the ODE of the given family of curves. The ODE of the orthogonal trajectories, therefore, is

$$\frac{dy}{dx} = \frac{1}{3y} \quad \Leftrightarrow \quad 3y\,dy = dx$$

Integrating the last equation, we obtain

$$\frac{3y^2}{2} = x + a,$$

which is the orthogonal trajectories of the given family of curves, where a is a constant.

EXAMPLE 90 Find the orthogonal trajectories of the family of curves $x^2 + (y-c)^2 = c^2$, where c is a constant.

SOLUTION Differentiating the equation

$$x^2 + (y-c)^2 = c^2, \quad \text{or} \quad x^2 + y^2 - 2cy = 0 \qquad ...(1)$$

with respect to x, we get

$$2x + 2(y-c)\frac{dy}{dx} = 0 \quad \text{or} \quad x + (y-c)\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad c = \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}} \qquad ...(2)$$

Substituting the value of c in Eq.(1), we obtain

$$x^2 + y^2 - 2y\left[\frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}\right] = 0 \quad \Rightarrow \quad (x^2+y^2)\frac{dy}{dx} - 2xy - 2y^2\frac{dy}{dx} = 0$$

or, $$(x^2 - y^2)\frac{dy}{dx} = 2xy \quad \text{or} \quad \frac{dy}{dx} = \frac{2xy}{x^2-y^2}$$

which is the ODE of the given family of curves. The ODE of the orthogonal trajectories, therefore, is

$$\frac{dy}{dx} = \frac{y^2-x^2}{2xy} \quad : \text{a homogeneoud differential equation} \qquad ...(2)$$

The general solution of this differential equation gives the orthogonal trajectories of the given family of curves. To find the general solution of (2), we put

$$y = vx \quad \text{so that} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in (2), we get

$$v + x\frac{dv}{dx} = \frac{v^2-1}{2v} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{-1-v^2}{2v}$$

$$\Rightarrow \qquad \frac{2v}{1+v^2}dv + \frac{dx}{x} = 0$$

Integrating, we get

$$\ln(1+v^2) + \ln x = \ln a \quad \Leftrightarrow \quad \ln(1+v^2)x = \ln a \quad \Leftrightarrow \quad (1+v^2)x = a$$

$$\Rightarrow \qquad \left(1 + \frac{y^2}{x^2}\right)x = a \quad \Rightarrow \quad x^2 + y^2 = ax$$

Thus, the orthogonal trajectories of the given family of curves is : $x^2 + y^2 = ax$, where a is an arbitrary constant.

EXERCISE 1.8

1. Find the orthogonal trajectories of the family of curves $y = 2x + c$, where c is an arbitrary constant.
2. Find the orthogonal trajectories of the family of curves $y = c/x$, where c is an arbitrary constant.
3. Find the orthogonal trajectories of the family of curves $x^2 + y^2 = c^2$, where c is an arbitrary constant.
4. Find the orthogonal trajectories of the family of curves $x^2y = c$, where c is an arbitrary constant.
5. Find the orthogonal trajectories of the family of curves $y = ce^{x^2/2}$, where c is an arbitrary constant.
6. Find the orthogonal trajectories of the family of curves $x^2 - y^2 = c$, where c is an arbitrary constant.
7. Find the orthogonal trajectories of the family of curves $4x^2 + y^2 = c$, where c is an arbitrary constant.
8. Find the orthogonal trajectories of the family of curves $x = ce^{y/4}$, where c is an arbitrary constant.
9. Find the orthogonal trajectories of the family of curves $y^2 = cx^3$, where c is an arbitrary constant.

ANSWERS

1. $y = -\frac{1}{2}x + a$
2. $\frac{y^2}{2} - \frac{x^2}{2} = a$
3. $y = ax$
4. $y^2 - \frac{x^2}{2} = a$
5. $x = ae^{-y^2/2}$
6. $xy = a$
7. $x = ay^4$
8. $y + \frac{x^2}{8} = a$
9. $\frac{3}{2}y^2 + x^2 = a$

1.10 EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this section we shall discuss the questions of existence and uniqueness of solutions of the first-order initial value problems of the form

$$y' = f(x, y), \quad y(x_0) = y_0 \qquad \text{...(1)}$$

To solve an initial value problem of the form (1), we must find a function $y(x)$ that not only satisfies the differential equation $y' = f(x, y)$ but also satisfies the initial condition that it has the value y_0 when x has value x_0. Stated geometrically, solving the initial value problem is to find an integral curve of the differential equation $y' = f(x, y)$ that passes through the point (x_0, y_0).

EXAMPLE 91 Solve the initial value problem

$$y' = 3x^2, \qquad y(0) = 1.$$

SOLUTION To solve this problem, we must find a function $y(x)$ that not only satisfies the differential equation $y' = 3x^2$ but also satisfies the initial condition that it has the value 1 when x has value 0. The differential equation $y' = 3x^2$ has a one-parameter family of solutions given by

$$y = x^3 + c \qquad \text{...(1)}$$

From the initial condition $y(0) = 1$ *i.e.*, $y = 1$ when $x = 0$, we find

$$1 = 0 + c \quad \Rightarrow \quad c = 1$$

Substituting this value of c into equation (1), we have

$$y = x^3 + 1$$

which is the solution of the given initial value problem.

Note Note that in the above problem we were able to find a solution of the initial value problem. But do all initial problems have solutions ? To answer this question, let us consider the following example.

EXAMPLE 92 Solve, if possible, the initial value problem

$$y' = \frac{1}{x}, \qquad y(0) = 0.$$

SOLUTION The given initial value problem has no solution, because no solution

$$y(x) = \int \frac{1}{x} dx = \ln|x| + c$$

of the differential equation $y' = \frac{1}{x}$ is defined at $x = 0$.

The next example shows that an initial value problem may have even infinitely many solutions.

EXAMPLE 93 Solve the initial value problem

$$xy' = y - 2, \qquad y(0) = 2.$$

SOLUTION The given initial value problem has infinitely many solutions, namely, $y = 2 + cx$, where c is an arbitrary constant because $y(0) = 2$ for all c.

From these examples, we see that an initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

may have no solution, a unique solution or infinitely many solutions. This fact leads to the following two fundamental questions.

1. **Problem of Existence :** Under what conditions does an initial value problem actually have at least one solution ?
2. **Problem of Uniqueness :** Under what conditions does such a problem have at most one solution ?

We now state the existence theorem which says that if a function $f(x, y)$ is continuous on some rectangle R in the xy-plane containing the point (x_0, y_0), then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

has at least one solution that is defined on some open interval containing the point x_0.

THEOREM 1.4 Existence Theorem

Let the function $f(x, y)$ be continuous at all points in some rectangle R (Figure 1.2)

$$R : |x - x_0| < a, \quad |y - y_0| < b$$

and bounded in R, that is, there is a number K such that

$$|f(x, y)| \leq K \quad \text{for all } (x, y) \text{ in } R$$

Then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

has at least one solution $y(x)$ that is defined for all x in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, where $\alpha = \min\{a, b/K\}$.

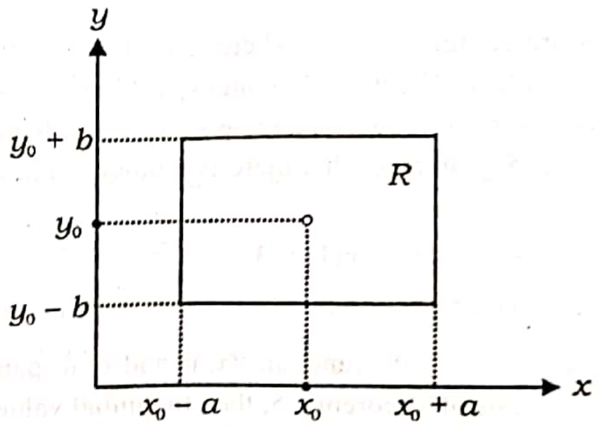

FIGURE 1.2 *Rectangle R in the existence and uniqueness theorems*

The next theorem, called the **uniqueness theorem**, states that the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

has one and only solution that is defined on some open interval containing the point x_0, provided that both the function f and its partial derivative f_y are continuous on some rectangle R in the xy-plane that contains the point (x_0, y_0) in its interior.

THEOREM 1.5 Uniqueness Theorem

Let the function $f(x, y)$ and its partial derivative $f_y = \partial f/\partial y$ be continuous at all points in some rectangle

$$R : |x - x_0| \leq a, \quad |y - y_0| \leq b$$

and bounded in R, that is, there are numbers K and M such that

$$(a) \ |f(x, y)| \leq K \qquad (b) \ |f_y(x, y)| \leq M \qquad \text{for all } (x, y) \text{ in } R$$

Then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

has at most one solution $y(x)$. Thus, by Theorem 1.4, the problem has one and only one solution that is defined for all x in the subinterval $|x - x_0| \leq \alpha$ of the interval $|x - x_0| < a$, where $\alpha = \min\{a, b/K\}$.

Let us consider the following example to illustrate Theorem 1.5.

EXAMPLE 94 Show that the following initial value problem has a unique solution defined on some sufficiently small interval I containing the point $x_0 = 1$:

$$y' = x^2 + y^2, \quad y(1) = 3.$$

SOLUTION We will use Theorem 1.5. For this we first check the hypotheses of Theorem 1.5. We have

$$f(x, y) = x^2 + y^2 \quad \text{and} \quad f_y(x, y) = 2y$$

Both of these functions are continuous everywhere, and, in particular, they are continuous on every rectangle R in the xy-plane. The initial condition $y(1) = 3$ means that $x_0 = 1$ and $y_0 = 3$, and the point $(1, 3)$ certainly lies in some such rectangle R. Thus, all hypotheses of Theorem 1.5 are satisfied. Hence Theorem 1.5 guarantees that there is a unique solution $y(x)$ of the initial value problem

$$y' = x^2 + y^2, \quad y(1) = 3$$

on some open x-interval containing the point $x_0 = 1$.

The following example shows that, if the function $f(x, y)$ and/or its partial derivative $f_y(x, y)$ fail to satisfy the continuity hypothesis of Theorem 1.5, then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

may have either no solution or infinitely many solutions.

EXAMPLE 95 Does the initial value problem $xy' = 2y$, $y(0) = 1$ have a solution ?

SOLUTION The given differential equation is

$$xy' = 2y \qquad \ldots(1)$$

Here,

$$f(x, y) = \frac{2y}{x} \quad \text{and} \quad f_y(x, y) = \frac{2}{x}$$

These functions are both continuous everywhere *except* for $x = 0$ (that is, along the y-axis). Thus, by Theorem 1.5, we can conclude that the given initial value problem must have a unique solution near any point in the xy-plane where $x \neq 0$. Indeed, we see immediately (by separating the variables) that

$$y(x) = cx^2$$

satisfies equation (1) for any value of the constant c and for all values of the variable x. All these solution curves are parabolas which pass through the origin $(0, 0)$ but none of these passes through any other point on the y-axis. Thus, the initial value problem

$$xy' = 2y, \qquad y(0) = 1$$

has no solution. However, if we replace the initial condition $y(0) = 1$ by $y(0) = 0$, then the initial value problem

$$xy' = 2y, \qquad y(0) = 0$$

has infinitely many solutions.

EXERCISE 1.9

Theory Questions :

1. Does every first-order ODE have a solution ? A general solution ? What do you know about uniqueness of solutions ?

Practical Questions :

Show that each of the following initial value problems has a unique solution defined on some sufficiently small interval about $x_0 = 1$

1. $y' = 2x^2y^2$; $y(1) = -1$
2. $y' = x \ln y$; $y(1) = 1$
3. $y' = x^2 \sin y$; $y(1) = -2$
4. $y' = \dfrac{y^2}{x-2}$; $y(1) = 0$.