CHAPTER

2

Second-Order Linear ODEs

LEARNING OBJECTIVES

After studying the material in this chapter, you should be able to :

- *Know the standard form of second-order linear ODEs.*
- *Differentiate between homogeneous and nonhomogeneous linear ODEs of second order.*
- *Understand the superposition principle or linearity principle.*
- *Know the concepts of general solution, basis and particular solution of homogeneous linear ODEs of second order.*
- *Define an initial value problem for a second-order homogeneous linear ODE.*
- *Solve homogeneous linear ODEs with constant coefficients.*
- *Identify and solve Euler-Cauchy equations.*
- *Understand the concepts of general solution and particular solution of nonhomogeneous linear ODEs of second order.*
- *Use the method of undetermined coefficients to find a particular solution of a nonhomogeneous linear ODE of second order.*
- *Use the method of variation of parameters to find a particular solution of a nonhomogeneous linear ODE of second order.*
- *Discuss the existence and uniqueness of solutions of initial value problems for second-order homogeneous linear ODEs.*

2.1 HOMOGENEOUS LINEAR ODEs OF SECOND ORDER

Recall that a first-order linear differential equation is an equation which can be written in the form

$$y' + P(x)y = Q(x)$$

where P and Q are continuous functions on some interval I. A second-order linear differential equation has an analogous form.

DEFINITION Second-Order Linear Differential Equation

A second-order differential equation in the (unknown) function $y(x)$ is said to be **linear** if it can be written in the form

$$y'' + p(x)y' + q(x)y = r(x) \quad ...(1)$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous on some open interval I(perhaps unbounded). The functions $p(x)$ and $q(x)$ are called the **coefficients** of the equation; the function $r(x)$ on the right-hand side is called the **forcing function** or the **nonhomogeneous term**.

Note Notice that the first term in the second-order linear differential equation (1) is y''. Its coefficient is 1. We call this the **standard form.** However, if the equation begins with, say, $f(x)y''$, then we assume that $f(x)$ is never zero for any $x \in I$. Thus, we can divide by $f(x)$ to get the standard form (1) with y'' as the first term.

An important feature of the second-order linear differential equation is that y'' occurs only to the first power, and if either or both of y and y' occur in the equation, then they do so with power 1 only. Moreover, there are no so-called **cross-product** terms such as yy', yy'', $y'y''$.

A second-order ODE which is not linear is said to be **nonlinear.** For example, the differential equation

$$y'' + xy^2y' - y^3 = e^{xy}$$

is non-linear because it cannot be written in the form (1).

EXAMPLE 1 Second-Order Linear ODEs

(*a*) The differential equation

$$y'' - 5y' + 6y = \cos x$$

is a second-order linear ODE because the dependent variable (or the unknown function) y and its derivatives y' and y'' appear linearly. Here $p(x) = -5$, $q(x) = 6$, and $r(x) = \cos x$ are continuous functions on $(-\infty, \infty)$.

(*b*) The differential equation

$$x^2y'' - 2xy' + 2y = 3x^2 \cos 2x$$

is also a second-order linear ODE because it can be written in the standard form (1)

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 3\cos 2x$$

where $p(x) = -2/x$, $q(x) = 2/x^2$, and $r(x) = 3\cos 2x$ are continuous on any interval that does not contain $x = 0$. For example, we could take $I = (0, \infty)$.

DEFINITION Homogeneous and Nonhomogeneous Linear ODEs

A second-order linear ODE

$$y'' + p(x)y' + q(x)y = r(x) \quad ...(1)$$

is called **homogeneous** if the forcing function $r(x)$ on the right-hand side of (1) is identically zero (denoted by $r(x) \equiv 0$), that is, $r(x) = 0$ for all $x \in I$. Otherwise, it is called

nonhomogeneous. Thus, equation (1) is nonhomogeneous if the forcing function $r(x)$ is not identically zero on I, that is, $r(x) \neq 0$ for some x in I.

For example, the differential equation

$$y'' + \frac{2}{x}y' + y = e^{2x}$$

is a nonhomogeneous linear ODE of second order, whereas the differential equation

$$y'' - 5y' + 6y = 0$$

is a homogeneous linear ODE of second order.

DEFINITION Solution of a Second-Order ODE

A function $y = h(x)$ is called **a solution** of a (linear or nonlinear) second-order ODE on some open interval I if h is defined and twice differentiable throughout that interval and is such that the equation becomes an identity if the unknown function y and its derivatives are replaced by h and its corresponding derivatives.

EXAMPLE 2 Show that the functions $y = e^{2x}$ and $y = e^{3x}$ are solutions of the second-order homogeneous linear ODE

$$y'' - 5y' + 6y = 0$$

for all x.

SOLUTION Differentiating the function $y = e^{2x}$ twice, we obtain

$$y' = 2e^{2x}, \qquad y'' = 4e^{2x}$$

Substituting $y = e^{2x}$, $y' = 2e^{2x}$, and $y'' = 4e^{2x}$ into the given differential equation, we obtain

$$y'' - 5y' + 6y = 4e^{2x} - 10e^{2x} + 6e^{2x} = 0$$

This proves that the function $y = e^{2x}$ is a solution of the given differential equation. Similarly, it can be seen that the function $y = e^{3x}$ is also a solution of the given differential equation. These two functions are not the only solutions to the differential equation. In fact, any of the following are also solutions of the given differential equation :

$$\begin{aligned} y &= -5e^{2x} & \qquad y &= 27e^{2x} \\ y &= 37e^{3x} & y &= 5/7e^{3x} \\ y &= 5e^{2x} - 6e^{3x} & y &= -12e^{2x} - 18e^{3x} \end{aligned}$$

In fact, any linear combination of these solutions, such as

$$y = 3e^{2x} + 2e^{3x}$$

is also a solution. Indeed, differentiation and substitution gives

$$\begin{aligned} y'' - 5y' + 6y &= (3e^{2x} + 2e^{3x})'' - 5(3e^{2x} + 2e^{3x})' + 6(3e^{2x} + 2e^{3x}) \\ &= (12e^{2x} + 18e^{3x}) - 5(6e^{2x} + 6e^{3x}) + 6(3e^{2x} + 2e^{3x}) \\ &= 12e^{2x} + 18e^{3x} - 30e^{2x} - 30e^{3x} + 18e^{2x} + 12e^{3x} \\ &= 0. \end{aligned}$$

Homogeneous Linear ODEs : Superposition Principle

Consider the homogeneous second-order linear differential equation in standard form :

$$y'' + p(x)y' + q(x)y = 0 \qquad ...(1)$$

where the coefficients p and q are continuous on the open interval I.

A particularly useful property of this homogeneous linear equation is the fact that the sum of any two solutions of equation (1) is again a solution, as is any constant multiple of a solution. This is often called the **superposition principle** or **linearity principle**.

> **THEOREM 2.1 Superposition Principle**
>
> If $y_1(x)$ and $y_2(x)$ are two solutions of a homogeneous linear ODE (1) on an open interval I, then for any constants c_1 and c_2, the function $c_1 y_1 + c_2 y_2$ is also a solution of (1) on I.

Proof Since y_1 and y_2 are solutions of equation (1), we have

$$y_1'' + p(x)y_1' + q(x)y_1 = 0, \qquad y_2'' + p(x)y_2' + q(x)y_2 = 0$$

By substituting $y = c_1 y_1 + c_2 y_2$ and its derivatives into (1), and using the basic rules of differentiation, we get

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (c_1 y_1 + c_2 y_2)'' + p(x)(c_1 y_1 + c_2 y_2)' + q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1 y_1'' + c_2 y_2'' + c_1 p(x) y_1' + c_2 p(x) y_2' + c_1 q(x) y_1 + c_2 q(x) y_2 \\ &= c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2) \\ &= 0 \end{aligned}$$

This shows that $y = c_1 y_1 + c_2 y_2$ is a solution of equation (1).

We may state Theorem 2.1 in a very simple form by means of the concept of *linear combination* defined as follows :

A **linear combination** of two functions $y_1(x)$ and $y_2(x)$ is a function of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants.

In terms of this concept, Theorem 2.1 may be stated as follows :

> **THEOREM 2.2 Superposition Principle (Reformulated)**
>
> For a homogeneous linear differential equation (1), any linear combination of two solutions on an open interval I is again a solution of (1).

As an immediate consequence of Theorem 2.1 (or Theorem 2.2), we have the following corollary concerning solutions of the homogeneous linear ODE (1).

> **Corollary 2.3**
>
> 1. A sum $y_1 + y_2$ of two solutions y_1 and y_2 of equation (1) is also a solution (choose $c_1 = c_2 = 1$).
> 2. A constant multiple ky_1 of any solution y_1 of equation (1) is also a solution (choose $c_1 = k$ and $c_2 = 0$).
> 3. The **trivial solution** $y(x) \equiv 0$ is always a solution of the homogeneous linear ODE (1) (choose $c_1 = c_2 = 0$).

EXAMPLE 3 It can be readily verified that the functions $y = \cos x$ and $y = \sin x$ are solutions of the second-order homogeneous linear ODE

$$y'' + y = 0$$

By the superposition principle, any linear combination $c_1 \cos x + c_2 \sin x$ is also a solution of the equation for any constants c_1 and c_2. In particular, the linear combination $2\cos x + 3\sin x$ is a solution.

Remark It is important to emphasize that the superposition principle holds only for homogeneous linear ODEs; it does not hold for nonhomogeneous (linear or nonlinear) ODEs, as is illustrated in the following examples.

EXAMPLE 4 Consider the nonhomogeneous linear ODE

$$y'' + y = 2 \qquad \text{...(1)}$$

It can be easily verified by substitution that the functions $y = 2 + \cos x$ and $y = 2 + \sin x$ are both solutions of equation (1), but their sum is not a solution. Neither is the function $3(2 + \cos x)$ or $2(2 + \sin x)$ a solution of (1).

EXAMPLE 5 Consider the nonlinear differential equation

$$y''y - xy' = 0 \qquad \text{...(1)}$$

It can be easily verified that the functions $y = x^2$ and $y = 1$ are solutions of equation (1), but their sum is not a solution. Neither is the function $-x^2$.

General Solution, Basis and Particular Solution

The fundamental question about solution of a second-order homogeneous linear differential equation is concerned with its general solution – that is, a solution that contains all possible solutions. Note that the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \qquad \text{...(1)}$$

where c_1 and c_2 are arbitrary constants, has the form of a general solution of equation

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{...(2)}$$

So the question is :

If y_1 and y_2 are solutions of (2), *is the expression* (1) *the general solution of* (2) ? *That is, can every solution of equation* (2) *be written as a linear combination of y_1 and y_2 ?*

It turns out that (1) may or may not be the general solution ; it depends on the relation between the solutions y_1 and y_2. The solutions must be *linearly independent*.

DEFINITION Linearly Independent and Linearly Dependent

Two functions $y_1(x)$ and $y_2(x)$ defined on an interval I are called **linearly independent** on I if for any constants c_1 and c_2, the equation

$$c_1 y_1(x) + c_2 y_2(x) \equiv 0 \text{ on } I \quad \Rightarrow \quad c_1 = c_2 = 0$$

The functions y_1 and y_2 are called **linearly dependent** on I if they are not linearly independent. In other words, y_1 and y_2 are linearly dependent on I if there exist constants c_1 and c_2, not both zero, such that

$$c_1 y_1(x) + c_2 y_2(x) = 0 \quad \text{for all } x \text{ in } I.$$

An Important Result *Two functions y_1 and y_2 are linearly dependent on an interval I if and only if one of the functions is a constant multiple of the other. Equivalently, two functions y_1 and y_2 are linearly independent on an interval I if and only if neither of the functions is a constant multiple of the other.*

To prove the above result, let us assume that functions y_1 and y_2 are linearly dependent. Then there exist constants c_1 and c_2, not both 0, such that

$$c_1 y_1(x) + c_2 y_2(x) = 0 \quad \text{for all } x \text{ in } I \qquad ...(1)$$

If $c_1 \neq 0$, then equation (1) implies $y_1 = -\dfrac{c_2}{c_1} y_2$. That is, y_1 is a constant multiple of y_2. Similarly, if $c_2 \neq 0$, then equation (1) implies $y_2 = -\dfrac{c_1}{c_2} y_1$. That is, y_2 is a constant multiple of y_1.

Conversely, suppose that $y_1 = \alpha y_2$ on I for some constant α. Then

$$1\, y_1 + (-\alpha) y_2 \equiv 0 \quad \text{on } I.$$

Thus, by definition, the functions y_1 and y_2 are linearly dependent.

We can reformulate our definition of linearly dependent by using the concept of proportionality. Two functions y_1 and y_2 are called **proportional** on I if for all x on I,

$$(a)\ \ y_1 = k y_2 \qquad \text{or} \qquad (b)\ \ y_2 = l y_2$$

where k and l are numbers, zero or not. Thus, two functions are proportional if their quotient y_1/y_2 (or y_2/y_1) is a constant and hence the above result can be reformulated as follows :

Two functions y_1 and y_2 are linearly dependent on an interval I if and only if they are proportional on I. Equivalently, two functions y_1 and y_2 are linearly independent on an interval I if and only if they are not proportional on I.

For example, the functions $y_1(x) = e^x$ and $y_2(x) = e^{-x}$ are linearly independent on any interval because their quotient $y_1/y_2 = e^{2x}$ is not a constant, whereas $y_1(x) = x^2$ and $y_2(x) = 5x^2$ are dependent because their quotient $y_2/y_1 = 5$ is a constant.

We now introduce the concepts of a general solution of homogeneous linear ODE of second order.

DEFINITION General Solution, Basis, and Particular Solution

Let y_1, y_2 be linearly independent solutions of the second-order homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad ...(1)$$

on an open interval I. Then the **general solution** of (1) on I is a solution of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad ...(2)$$

where c_1, c_2 are arbitrary constants.

Any pair $\{y_1, y_2\}$ of linearly independent solutions of (1) on I is said to form a **basis** (or a **fundamental system**) of solutions of (1) on I.

A **particular solution** of (1) on I is obtained by assigning specific values to c_1, c_2 in (2).

Initial Value Problem

The reader may recall that an initial value problem for a first-order ODE consists of an ODE together with an initial condition $y(x_0) = y_0$. The initial condition is used to determine a value of the arbitrary constant c in the general solution of the ODE. This results in a unique solution, and hence a particular solution of the ODE. We now extend these concepts to second-order ODEs as follows:

An **initial value problem** for a second-order homogeneous linear ODE consists of an ODE

$$y'' + p(x)y' + q(x)y = 0$$

together with two **initial conditions**

$$y(x_0) = k_0 \quad \text{and} \quad y'(x_0) = k_1$$

The initial conditions are used to determine the two arbitrary constants c_1 and c_2 in the general solution $y = c_1 y_1 + c_2 y_2$ of the ODE. This results in a unique solution, and hence a particular solution. Geometrically, the initial conditions mean that the solution curve passes through the point (x_0, k_0) in the xy-plane with slope k_1 at that point.

EXAMPLE 6 Verify by substitution that the functions $y_1 = \cos x$ and $y_2 = \sin x$ are solutions of the ODE $y'' + y = 0$. Then solve the initial value problem

$$y'' + y = 0, \quad y(0) = 2, \quad y'(0) = -0.5.$$

SOLUTION We have

$$y_1' = -\sin x, \qquad y_1'' = -\cos x$$

Replacing y by $y_1 = \cos x$ and y'' by $y_1'' = -\cos x$ in the given ODE, we obtain

$$y'' + y = -\cos x + \cos x = 0$$

This shows that $y_1 = \cos x$ is a solution of the given ODE. Similarly, it can be seen that $y_2 = \sin x$ is a solution of the given ODE. We note that y_1 and y_2 are linearly independent on any interval because their quotient $y_1/y_2 = \cot x$ is not a constant. Hence y_1 and y_2 form a basis of solutions for the given ODE, and the general solution is

$$y = c_1 \cos x + c_2 \sin x,$$

where c_1 and c_2 are arbitrary constants. We now use the initial conditions to find c_1 and c_2. For this we need the derivative

$$y' = -c_1 \sin x + c_2 \cos x$$

From this and the initial conditions $y(0) = 2$, $y'(0) = -0.5$, we have

$$y(0) = c_1 = 2 \qquad \text{and} \qquad y'(0) = c_2 = -0.5$$

Hence the solution of the initial value problem is

$$y = 2\cos x - 0.5\sin x.$$

EXAMPLE 7 Show that e^{3x} and xe^{3x} form a basis of the differential equation $y'' - 6y' + 9y = 0$. Find also the solution that satisfies the conditions $y(0) = -1.4$, $y'(0) = 4.6$.

[*Delhi Univ. GE-3, 2016*]

SOLUTION The given ODE is

$$y'' - 6y' + 9y = 0 \qquad \text{...(1)}$$

We first show by substitution that $y_1 = e^{3x}$ and $y_2 = xe^{3x}$ are solutions of ODE (1). Differentiating the function $y_1 = e^{3x}$ twice, we obtain

$$y_1' = 3e^{3x}, \qquad y_1'' = 9e^{3x}$$

Replacing y by e^{3x}, y' by $3e^{3x}$ and y'' by $9e^{3x}$ in (1), we get

$$y'' - 6y' + 9y = 9e^{3x} - 18e^{3x} + 9e^{3x} = 0$$

This shows that $y_1 = e^{3x}$ is a solution of (1). We next differentiate $y_2 = xe^{3x}$ twice to obtain

$$y_2' = (3x+1)e^{3x}, \qquad y_2'' = (9x+6)e^{3x}$$

Replacing y by xe^{3x}, y' by $(3x+1)e^{3x}$, and y'' by $(9x+6)e^{3x}$ in (1), we get

$$y'' - 6y' + 9y = (9x+6)e^{3x} - 6(3x+1)e^{3x} + 9(xe^{3x})$$
$$= (9x + 6 - 18x - 6 + 9x)e^{3x} = 0$$

Thus, $y_2 = xe^{3x}$ is also a solution of (1).

Hence $y_1 = e^{3x}$ and $y_2 = xe^{3x}$ are both solutions of (1). They are linearly independent since their quotient $y_2/y_1 = xe^{3x}/e^{3x} = x \neq$ constant. So e^{3x}, xe^{3x} form a basis of solutions of (1) and the corresponding general solution is

$$y = c_1e^{3x} + c_2xe^{3x} = (c_1 + c_2x)e^{3x} \qquad \text{...(2)}$$

We now find a particular solution that satisfies the initial conditions $y(0) = -1.4$, $y'(0) = 4.6$. We need the derivative

$$y' = 3(c_1 + c_2x)e^{3x} + c_2e^{3x}$$

From this and the initial conditions, we have

$$y(0) = c_1 = -1.4$$
$$y'(0) = 3c_1 + c_2 = 4.6$$

Solving these equations for c_1 and c_2, we obtain $c_1 = -1.4$ and $c_2 = 8.8$. Substituting these values of c_1 and c_2 in equation (2), we obtain the particular solution of the given initial value problem

$$y = (-1.4 + 8.8x)e^{3x}.$$

EXAMPLE 8 Show that x^2 and x^{-2} form a basis of the differential equation $x^2y''+xy'-4y=0$. Find also the solution that satisfies the conditions $y(1)=11$, $y'(1)=-6$.

[*Delhi Univ. GE-3, 2017*]

SOLUTION The given ODE is

$$x^2y''+xy'-4y=0 \qquad \text{...(1)}$$

We first show that $y_1=x^2$ and $y_2=x^{-2}$ are solutions of ODE (1). Differentiating the function $y_1=x^2$ twice, we obtain

$$y_1'=2x, \qquad y_1''=2$$

Replacing y by x^2, y' by $2x$ and y'' by 2 in (1), we get

$$x^2y''+xy'-4y=x^2(2)+x(2x)-4(x^2)=2x^2+2x^2-4x^2=0$$

This shows that $y_1=x^2$ is a solution of (1). We next differentiate $y_2=x^{-2}$ twice to obtain

$$y_2'=-2x^{-3}, \qquad y_2''=6x^{-4}$$

Replacing y by x^{-2}, y' by $-2x^{-3}$, and y'' by $6x^{-4}$ in (1), we get

$$x^2y''+xy'-4y=x^2(6x^{-4})+x(-2x^{-3})-4(x^{-2})=6x^{-2}-2x^{-2}-4x^{-2}=0$$

This shows $y_2=x^{-2}$ is also a solution of (1).

Hence $y_1=x^2$ and $y_2=x^{-2}$ are both solutions of (1). Since their quotient $y_1/y_2=x^2/x^{-2}=x^4\neq$ constant, they are linearly independent and hence form a basis of solutions of (1). The general solution is

$$y=c_1x^2+c_2x^{-2}, \qquad \text{...(2)}$$

where c_1 and c_2 are arbitrary constants.

We now find the solution that satisfies the initial conditions $y(1)=11$, $y'(1)=-6$. We need the derivative

$$y'=2c_1x-2c_2x^{-3}$$

From this and the initial conditions, we have

$$y(1)=c_1+c_2=11$$
$$y'(1)=2c_1-2c_2=-6$$

Solving these equations for c_1 and c_2, we obtain $c_1=4$, $c_2=7$. Substituting the values of c_1 and c_2, in (2), we obtain the solution of (1) satisfying the initial conditions as

$$y(x)=4x^2+7x^{-2}.$$

EXERCISE 2.1

1. Show that e^x and e^{-x} form a basis of solutions of the ODE $y''-y=0$. Then solve the initial value problem

$$y''-y=0, \ y(0)=6, \ y'(0)=-2.$$

2. Show that $\cos x$ and $\sin x$ form a basis of solutions of the ODE $y''+y=0$. Then solve the initial value problem

$$y''+y=0, \ y(0)=3, \ y'(0)=-0.5.$$

3. Show that e^{x} and e^{-2x} form a basis of solutions of the ODE $y''+y'-2y=0$. Then solve the initial value problem

$$y''+y'-2y=0,\ \ y(0)=4,\ \ y'(0)=-5.$$

4. Show that e^{4x} and e^{-4x} form a basis of solutions of the ODE $y''-16y=0$. Then solve the initial value problem

$$y''-16y=0,\ \ y(0)=3,\ \ y'(0)=8.$$

5. Show that x^3 and x^5 form a basis of solutions of the ODE $x^2y''-7xy'+15y=0$. Then solve the initial value problem

$$x^2y''-7xy'+15y=0,\ \ y(1)=0.4,\ \ y'(1)=1.$$

6. Show that $\cos 5x$ and $\sin 5x$ form a basis of solutions of the ODE $y''+25y=0$. Then solve the initial value problem

$$y''+25y=0,\ \ y(0)=0.8,\ \ y'(0)=-6.5.$$

7. Show that $e^{-0.5x}$ and $xe^{-0.5x}$ form a basis of solutions of the ODE $y''+y'+0.25y=0$. Then solve the initial value problem

$$y''+y'+0.25y=0,\ \ y(0)=3,\ \ y'(0)=-3.5.$$

8. Show that $e^{-x}\cos x$ and $e^{-x}\sin x$ form a basis of solutions of the ODE $y''+2y'+2y=0$. Then solve the initial value problem

$$y''+2y'+2y=0,\ \ y(0)=1,\ \ y'(0)=-1.$$

9. Show that e^{-x} and xe^{-x} form a basis of solutions of the ODE $y''+2y'+y=0$. Then solve the initial value problem

$$y''+2y'+y=0,\ \ y(0)=4,\ \ y'(0)=-6.$$

ANSWERS

1. $y=2e^{x}+4e^{-x}$
2. $y=3\cos x-0.5\sin x$
3. $y=e^{x}+3e^{-2x}$
4. $y=2.5e^{4x}+0.5e^{-4x}$
5. $y=0.5x^3-0.1x^5$
6. $y=0.8\cos 5x-1.3\sin 5x$
7. $y=(3-2x)e^{-0.5x}$
8. $y=e^{-x}\cos x$
9. $y=(4-2x)e^{-x}$

2.2 HOMOGENEOUS LINEAR ODEs WITH CONSTANT COEFFICIENTS

In this section we learn how to solve second-order homogeneous linear ODEs with constant coefficients, that is, equations of the form

$$y''+ay'+by=0 \qquad \text{...(1)}$$

where a and b are constants.

To find the general solution, we first look for a single solution of Eq.(1). We begin by assuming a solution of the form $y = e^{\lambda x}$. Substituting $y = e^{\lambda x}$ and its derivatives

$$y' = \lambda e^{\lambda x} \quad \text{and} \quad y'' = \lambda^2 e^{\lambda x}$$

into the differential equation (1), we obtain

$$\lambda^2 e^{\lambda x} + a(\lambda e^{\lambda x}) + b(e^{\lambda x}) = 0$$

i.e.,

$$e^{\lambda x}(\lambda^2 + a\lambda + b) = 0$$

Because $e^{\lambda x}$ is never zero, we see that $y = e^{\lambda x}$ will be a solution of Eq.(1) precisely when λ is a root of the equation

$$\lambda^2 + a\lambda + b = 0 \qquad \text{...(2)}$$

Equation (2) is called the **characteristic equation** of the differential Eq.(1). Our problem, then, is reduced to the solution of this quadratic equation. Observe that Eq.(2) is formally obtained from Eq.(1) by merely replacing y'' by λ^2, y' by λ, y by 1.

Eq.(2) has two roots, given by the quadratic formula :

$$\lambda_1, \lambda_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

These roots may be real and distinct, real and repeated, or complex conjugate according as the discriminant $a^2 - 4b$ is positive, zero, or negative, respectively.

Case 1. Distinct Real Roots

If the discriminant $a^2 - 4b > 0$, then the characteristic Eq.(2) has 2 distinct real roots λ_1 and λ_2. Moreover, our derivation shows that the functions

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

are solutions of equation (1). These solutions are defined for all x and their quotient $y_1/y_2 = e^{\lambda_1 x}/e^{\lambda_2 x} = e^{(\lambda_1 - \lambda_2)x} \neq$ constant (since $\lambda_1 \neq \lambda_2$). Hence $e^{\lambda_1 x}$, $e^{\lambda_2 x}$ form a basis of solutions of (1) on any interval and the corresponding general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

where c_1 and c_2 are arbitrary constants.

EXAMPLE 9 Find a general solution of the differential equation $y'' + 2y' - 8y = 0$.

SOLUTION The characteristic equation is

$$\lambda^2 + 2\lambda - 8 = 0 \quad \text{or} \quad (\lambda + 4)(\lambda - 2) = 0$$

It has two distinct real roots $\lambda_1 = -4$, $\lambda_2 = 2$. Thus, the general solution of the given ODE is

$$y = c_1 e^{-4x} + c_2 e^{2x}.$$

EXAMPLE 10 Find a general solution of the ODE $10y'' - 7y' + 1.2y = 0$.

SOLUTION Dividing by 10 the given ODE can be put in standard form

$$y'' - 0.7y' + 0.12y = 0$$

The characteristic equation is

$$\lambda^2 - 0.7\lambda + 0{,}12 = 0 \quad \text{or} \quad (\lambda - 0.4)(\lambda - 0.3) = 0$$

It has two distinct roots $\lambda_1 = 0.4$, $\lambda_2 = 0.3$. Thus, the general solution of the given ODE is

$$y = c_1 e^{0.4x} + c_2 e^{0.3x}.$$

Case 2. ***Real Double (or Repeated) Root***

If the discriminant $a^2 - 4b$ is zero, then the characteristic equation has a real double root

$$\lambda = \lambda_1 = \lambda_2 = -a/2$$

and hence we get only one solution, namely, $y = e^{\lambda x}$, However, it can be checked that $y = xe^{\lambda x}$ is also a solution. Since the solutions $e^{\lambda x}$ and $xe^{\lambda x}$ are not proportional, they form a basis. Thus, in the case of a double root λ of (2), a basis of solutions of (1) on any interval is $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$. The corresponding general solution is

$$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x} = (c_1 + c_2 x) e^{\lambda x}$$

where c_1 and c_2 are arbitrary constants.

EXAMPLE 11 Find the general solution of the differential equation $y'' - 6y' + 9y = 0$.

SOLUTION The characteristic equation is

$$\lambda^2 - 6\lambda + 9 = 0 \quad \Leftrightarrow \quad (\lambda - 3)^2 = 0$$

It has the real double root $\lambda = 3$. Hence the general solution is

$$y = c_1 e^{3x} + c_2 x e^{3x} = (c_1 + c_2 x) e^{3x}$$

where c_1 and c_2 are arbitrary constants.

EXAMPLE 12 Find a general solution of the ODE $4y'' - 20y' + 25y = 0$.

SOLUTION Dividing by 4 the given ODE can be put in standard form

$$y'' - 5y' + 6.25y = 0$$

The characteristic equation is

$$\lambda^2 - 5\lambda + 6.25 = 0 \quad \text{or} \quad (\lambda - 2.5)^2 = 0$$

It has the real double root $\lambda = 2.5$. Thus, the general solution of the given ODE is

$$y = (c_1 + c_2 x) e^{2.5x}.$$

Case 3 ***Complex Conjugate Roots***

If the discriminant $a^2 - 4b < 0$, then the characteristic equation (2) has complex conjugate roots :

$$\lambda_1, \lambda_2 = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = -\frac{a}{2} \pm \frac{\sqrt{4b - a^2}}{2} i$$

We set

$$\alpha = -\frac{a}{2}, \qquad \omega = \frac{\sqrt{4b - a^2}}{2}$$

and write the roots as follows:

$$\lambda_1 = \alpha + i\omega, \qquad \lambda_2 = \alpha - i\omega$$

As in the case of real roots, the exponential functions $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are solutions of differential equation (1). Thus, the general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = c_1 e^{(\alpha + i\omega)x} + c_2 e^{(\alpha - i\omega)x}$$

where c_1 and c_2 are arbitrary constants. However, the solutions defined by $e^{(\alpha + i\omega)x}$ and $e^{(\alpha - i\omega)x}$ are complex functions of the real variable x. It is desirable to replace these by two real linearly independent solutions. This can be accomplished by using Eùlar's formula :

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which holds for all real θ. Using this, we have

$$\begin{aligned} c_1 e^{(\alpha + i\omega)x} + c_2 e^{(\alpha - i\omega)x} &= c_1 e^{\alpha x} e^{i\omega x} + c_2 e^{\alpha x} e^{-i\omega x} \\ &= e^{\alpha x}[c_1 e^{i\omega x} + c_2 e^{-i\omega x}] \\ &= e^{\alpha x}[c_1(\cos\omega x + i\sin\omega x) + c_2(\cos\omega x - i\sin\omega x)] \\ &= e^{\alpha x}[(c_1 + c_2)\cos\omega x + i(c_1 - c_2)\sin\omega x] \\ &= e^{\alpha x}[A\cos\omega x + B\sin\omega x] \end{aligned}$$

where $A = c_1 + c_2$, $B = i(c_1 - c_2)$ are two new arbitrary constants. Thus, the general solution corresponding to the conjugate complex roots $\alpha \pm i\omega$ is

$$y(x) = e^{\alpha x}[A\cos\omega x + B\sin\omega x].$$

EXAMPLE 13 Find the general solution of the differential equation $y'' - 4y' + 13y = 0$.

SOLUTION The characteristic equation is

$$\lambda^2 - 4\lambda + 13 = 0 \quad \text{or} \quad (\lambda - 2)^2 + 9 = 0 \quad \Rightarrow \quad \lambda - 2 = \pm 3i \quad \text{or} \quad \lambda = 2 \pm 3i$$

Thus, the characteristic equation has the complex conjugate roots $2 \pm 3i$. Hence the corresponding general solution is

$$y = e^{2x}[A\cos 3x + B\sin 3x]$$

where A and B are arbitrary constants.

EXAMPLE 14 Find the general solution of the ODE $y'' - y' + 2.5y = 0$.

SOLUTION The characteristic equation is

$$\lambda^2 - \lambda + 2.5 = 0 \quad \text{or} \quad (\lambda - 0.5)^2 + 2.25 = 0 \quad \Rightarrow \quad \lambda - 0.5 = \pm 1.5i \quad \text{or} \quad \lambda = 0.5 \pm 1.5i$$

Thus, the characteristic equation has the complex conjugate roots $0.5 \pm 1.5i$. Hence the corresponding general solution is

$$y = e^{0.5x}[A\cos 1.5x + B\sin 1.5x],$$

where A and B are arbitrary constants.

EXAMPLE 15 Find a general solution of the differential equation $y'' + 2.4y' + 4.0y = 0$.

SOLUTION The characteristic equation is

$$\lambda^2 + 2.4\lambda + 4 = 0 \quad \text{or} \quad (\lambda + 1.2)^2 + 2.56 = 0 \quad \Rightarrow \quad \lambda + 1.2 = \pm 1.6i \quad \text{or} \quad \lambda = -1.2 \pm 1.6i$$

Thus, the characteristic equation has the complex conjugate roots $-1.2 \pm 1.6i$. Hence the corresponding general solution is

$$y = e^{-1.2x}[A\cos 1.6x + B\sin 1.6x],$$

where A and B are arbitrary constants.

EXAMPLE 16 Find the general solution of the differential equation $y'' + \omega^2 y = 0$.

SOLUTION The characteristic equation is $\lambda^2 + \omega^2 = 0$. It has the complex conjugare roots $\pm \omega i$. Hence the corresponding general solution is

$$y = A\cos \omega x + B\sin \omega x$$

where A and B are arbitrary constants.

Initial Value Problem : Some Illustrations

EXAMPLE 17 Solve the initial value problem

$$y'' + y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = -5.$$

SOLUTION ***Step 1 General Solution***

The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0 \quad i.e., \quad \lambda^2 + 2\lambda - \lambda - 2 = 0 \quad \text{or,} \quad (\lambda - 1)(\lambda + 2) = 0$$

It has two distinct real roots $\lambda_1 = 1$ and $\lambda_2 = -2$. Thus, the general solution is

$$y = c_1 e^x + c_2 e^{-2x} \qquad \text{...(1)}$$

Step 2 Particular Solution

We now use initial conditions to find constants c_1 and c_2. We need the derivative

$$y' = c_1 e^x - 2c_2 e^{-2x}$$

From this and the initial conditions, we obtain

$$y(0) = c_1 + c_2 = 4 \qquad \text{...(2)}$$

$$y'(0) = c_1 - 2c_2 = -5 \qquad \text{...(3)}$$

Solving Eqs.(2) and (3) for c_1 and c_2, we obtain $c_1 = 1$ and $c_2 = 3$. Thus, the given initial value problem has the solution

$$y = e^x + 3e^{-2x}.$$

EXAMPLE 18 Solve the initial value problem

$$2y'' - 3y' - 2y = 0, \quad y(0) = 3/2, \quad y'(0) = -1/8.$$

SOLUTION ***Step 1 General Solution***

The characteristic equation is

$$2\lambda^2 - 3\lambda - 2 = 0 \quad \Leftrightarrow \quad (2\lambda + 1)(\lambda - 2) = 0$$

It has two distinct real roots $\lambda_1 = -1/2$ and $\lambda_2 = 2$. Thus, the general solution is

$$y = c_1 e^{-x/2} + c_2 e^{2x} \qquad \ldots(1)$$

Step 2 Particular Solution

We now use initial conditions to find constants c_1 and c_2. We need the derivative

$$y' = -\frac{1}{2}c_1 e^{-x/2} + 2c_2 e^{2x}$$

From this and the initial conditions, we obtain

$$y(0) = c_1 + c_2 = 3/2 \qquad \ldots(2)$$

$$y'(0) = -\frac{1}{2}c_1 + 2c_2 = -\frac{1}{8} \qquad \ldots(3)$$

From Eq.(2), $c_2 = \frac{3}{2} - c_1$. Substituting this value of c_2 in Eq.(3), we obtain

$$-\frac{1}{2}c_1 + 2\left(\frac{3}{2} - c_1\right) = -\frac{1}{8}$$

or

$$3 - \frac{5}{2}c_1 = -\frac{1}{8} \quad \Rightarrow \quad c_1 = \frac{2}{5}\left(3 + \frac{1}{8}\right) = \frac{5}{4}$$

$$\therefore \quad c_2 = \frac{3}{2} - c_1 = \frac{3}{2} - \frac{5}{4} = \frac{1}{4}$$

Thus, the given initial value problem has the solution

$$y = \frac{5}{4}e^{-x/2} + \frac{1}{4}e^{2x}.$$

EXAMPLE 19 Solve the initial value problem

$$y'' + y' + 0.25y = 0\,; \quad y(0) = 3,\ y'(0) = -3.5.$$

SOLUTION ***Step 1 General Solution***

The characteristic equation is

$$\lambda^2 + \lambda + 0.25 = 0 \quad \Leftrightarrow \quad (\lambda + 0.5)^2 = 0$$

It has the real double root $\lambda = -0.5$. Hence the general solution is

$$y = (c_1 + c_2 x)e^{-0.5x} \qquad \ldots(1)$$

Step 2 Particular Solution

We now use initial conditions to find constants c_1 and c_2. We need the derivative

$$y' = c_2 e^{-0.5x} - 0.5(c_1 + c_2 x)e^{-0.5x}$$

From this and the initial conditions, we obtain

$$y(0) = c_1 = 3 \quad \ldots(2)$$

$$y'(0) = -0.5c_1 + c_2 = -3.5 \quad \ldots(3)$$

which imply that $c_1 = 3$ and $c_2 = -2$. Thus, the particular solution of the given initial value problem is

$$y = (3 - 2x)e^{-0.5x}.$$

EXAMPLE 20 Solve the initial value problem

$$y'' + 2y' + y = 0, \qquad y(0) = 5, \qquad y'(0) = -3.$$

SOLUTION ***Step 1 General Solution***

The characteristic equation is

$$\lambda^2 + 2\lambda + 1 = 0 \quad \Leftrightarrow \quad (\lambda + 1)^2 = 0$$

It has the real double root $\lambda = -1$. Hence the general solution is

$$y = (c_1 + c_2x)e^{-x} \quad \ldots(1)$$

Step 2 Particular Solution

We now use initial conditions to find constants c_1 and c_2. We need the derivative

$$y' = c_2e^{-x} - (c_1 + c_2x)e^{-x}$$

From this and the initial conditions, we obtain

$$y(0) = c_1 = 5 \quad \ldots(2)$$

$$y'(0) = -c_1 + c_2 = -3 \quad \ldots(3)$$

which imply that $c_1 = 5$ and $c_2 = 2$. Thus, the particular solution of the given initial value problem is

$$y = (5 + 2x)e^{-x}.$$

EXAMPLE 21 Solve the initial value problem

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$ **[*Delhi Univ. GE*-3, 2018]**

SOLUTION ***Step 1 General Solution***

The characteristic equation is

$$\lambda^2 + 0.4\lambda + 9.04 = 0 \quad \text{or} \quad (\lambda + 0.2)^2 + 9 = 0 \quad \Rightarrow \quad \lambda + 0.2 = \pm 3i \quad \text{or} \quad \lambda = -0.2 \pm 3i$$

Thus, the characteristic equation has conjugate complex roots $-0.2 \pm 3i$. Hence, the general solution is

$$y = e^{-0.2x}[A\cos 3x + B\sin 3x],$$

where A and B are arbitrary constants.

Step 2 Particular Solution

We now use initial conditions to find constants A and B. The first initial condition gives

$$y(0) = A = 0$$

Thus, we are left with the expression

$$y = Be^{-0.2x}\sin 3x$$

To apply the second initial condition, we need the derivative

$$\Rightarrow \qquad y' = B(-0.2e^{-0.2x}\sin 3x + 3e^{-0.2x}\cos 3x)$$

The second initial condition gives

$$y'(0) = 3B = 3 \quad \Rightarrow \quad B = 1$$

$\Rightarrow$

Hence the particular solution of the given initial value problem is

$$y = e^{-0.2x}\sin 3x.$$

EXAMPLE 22 Solve the initial value problem

$$y'' - 6y' + 25y = 0, \quad y(0) = -3, \quad y'(0) = -1.$$

SOLUTION ***Step 1 General Solution***

The characteristic equation is

$$\lambda^2 - 6\lambda + 25 = 0 \quad \text{or} \quad (\lambda - 3)^2 + 16 = 0 \quad \Rightarrow \quad \lambda - 3 = \pm 4i \quad \text{or} \quad \lambda = 3 \pm 4i$$

Thus, the characteristic equation has conjugate complex roots $3 \pm 4i$. Hence, the general solution is

$$y = e^{3x}[A\cos 4x + B\sin 4x], \qquad \ldots(1)$$

where A and B are arbitrary constants.

Step 2 Particular Solution

We now use initial conditions to find constants A and B. The first initial condition gives

$$y(0) = A = -3$$

To apply the second initial condition, we need the derivative

$$\Rightarrow \quad y' = e^{3x}(-4A\sin 4x + 4B\cos 4x) + 3e^{3x}(A\cos 4x + B\sin 4x)$$

$$= e^{3x}[(3A + 4B)\cos 4x + (3B - 4A)\sin 4x]$$

The second initial condition gives

$$\Rightarrow \quad y'(0) = 3A + 4B = -1$$

Substituting $A = -3$ in the last equation, we find $B = 2$. Substituting the values of A and B into Eq.(1), we obtain the particular solution of the given initial value problem :

$$y = e^{3x}(-3\cos 4x + 2\sin 4x).$$

Recovering a Differential Equation from Solutions

We can also work backwards using the results above. That is, we can derive a second-order homogeneous linear ODE with constant coefficients that has a given basis of solutions. Let us consider some examples.

EXAMPLE 23 Find a second-order homogeneous linear ODE with constant coefficients that has the basis e^{2x}, e^{-3x} of solutions.

SOLUTION To the given basis e^{2x}, e^{-3x} of solutions, there corresponds the characteristic equation

$$(\lambda - 2)(\lambda + 3) = \lambda^2 + \lambda - 6 = 0$$

Hence, the corresponding ODE is

$$y'' + y' - 6y = 0.$$

EXAMPLE 24 Find a second-order homogeneous linear ODE with constant coefficients that has the basis 1, e^{-2x} of solutions.

SOLUTION To the given basis 1, e^{-2x} (*i.e.*, e^{0x}, e^{-2x}) of solutions, there corresponds the characteristic equation

$$\lambda(\lambda+2)=\lambda^2+2\lambda=0$$

Hence, the corresponding ODE is

$$y''+2y'=0.$$

EXAMPLE 25 Find a second-order homogeneous linear ODE with constant coefficients that has the basis $e^{\sqrt{3}x}$, $xe^{\sqrt{3}x}$ of solutions.

SOLUTION To the given basis $e^{\sqrt{3}x}$, $xe^{\sqrt{3}x}$ of solutions, there corresponds the characteristic equation

$$(\lambda-\sqrt{3})^2=\lambda^2-2\sqrt{3}\,\lambda+3=0$$

Hence, the corresponding ODE is

$$y''-2\sqrt{3}\,y'+3y=0.$$

EXAMPLE 26 Find a second-order homogeneous linear ODE with constant coefficients that has the basis $e^{(-1+2i)x}$, $e^{-(1+2i)x}$ of solutions.

SOLUTION To the given basis $e^{(-1+2i)x}$, $e^{-(1+2i)x}$ of solutions, there corresponds the characteristic equation

$$(\lambda+1-2i)(\lambda+1+2i)=(\lambda+1)^2-(2i)^2=\lambda^2+2\lambda+5=0$$

Hence, the corresponding ODE is

$$y''+2y'+5y=0.$$

EXAMPLE 27 Find a second-order homogeneous linear ODE with constant coefficients that has the function $y(x)=e^x\cos 2x$ as a solution.

SOLUTION Since $e^x\cos 2x$ is a solution, $1+2i$ must be a root of the characteristic equation. We know from algebra that roots of a polynomial equation with real coefficients occur in conjugate pairs. Therefore, $1-2i$ must be the other complex root of the characteristic equation. These roots correspond to the characteristic equation

$$(\lambda-1-2i)(\lambda-1+2i)=(\lambda-1)^2-(2i)^2=\lambda^2-2\lambda+5=0$$

Hence, the corresponding ODE is

$$y''-2y'+5y=0.$$

2.3 DIFFERENTIAL OPERATORS

In calculus, we often use the capital letter D to denote differentiation ; that is, $Dy = \frac{dy}{dx} = y'$. The symbol D is called the **differential operator** because it transforms a given (differentiable) function into its derivative. For example, $D\sin x = (\sin x)' = \cos x$, $D(3x^2 + 5x + 1) = (3x^2 + 5x + 1)' = 6x + 5$. Higher-order derivatives can also be expressed in terms of D in a natural way :

$$D^2y = \frac{d^2y}{dx^2} = y'', \quad D^3y = \frac{d^3y}{dx^3} = y''', \text{ and in general } \quad D^ny = \frac{d^ny}{dx^n} = y^{(n)}$$

Polynomial expressions involving D such as $D - 2I$, $D^2 + 3D + 2I$, $x^2D^2 + 4xD + 5I$, etc., where I is the **identity operator** defined by $Iy = y$, are also differential operators. In particular, the expression

$$L = P(D) = D^2 + aD + bI$$

is called the **second-order differential operator** with constant coefficients.

An important property of second-order differential operator L is that it is **linear** in the sense that if Ly and Lw exist, then for any constants a and b, we have

$$L(ay + bw) = aLy + bLw$$

Using the second-order differential operator L, a second-order homogeneous linear ODE with constant coefficients

$$y'' + ay' + by = 0$$

can be expressed as

$$Ly = P(D)y = (D^2 + aD + bI)y = 0$$

For example, the second-order homogeneous linear ODE

$$y'' - 2y' + 5y = 0$$

can be expressed in terms of differential operator D as

$$(D^2 - 2D + 5I)y = 0.$$

Note Notice that $P(D)$ can be treated just like an algebraic quantity. In particular, $P(D)$ can be factored in the same way as the characteristic polynomial $P(\lambda) = \lambda^2 + a\lambda + b$.

EXERCISE 2.2

Find a general solution of each of the following ODEs.

1. $y'' + 5y' - 6y = 0$
2. $y'' + 8y' + 16y = 0$
3. $y'' - 2y' - 5.25y = 0$
4. $y'' + 2y' + 5y = 0$
5. $y'' + 4\pi y' + 4\pi^2 y = 0$
6. $100y'' + 20y' - 99y = 0$
7. $y'' + 2.6y' + 1.69y = 0$
8. $y'' - 3y = 0$
9. $y'' + y' - 0.96y = 0$
10. $y'' - 225y = 0$
11. $(D^2 + 2D + 4I)y = 0$
12. $(D^2 + 6D + 13I)y = 0$

Solve each of the following initial value problems.

13. $y'' - y' - 6y = 0,\ y(0) = 2,\ y'(0) = 0$ **14.** $y'' - 9y = 0,\ y(0) = -2,\ y'(0) = -12$

15. $y'' - 2y' + y = 0,\ y(0) = 1,\ y'(0) = -1$ **16.** $y'' + 4y' + 5y = 0,\ y(0) = 2,\ y'(0) = -5$

17. $10y'' + 5y' + 0.625y = 0,\ y(0) = 2,\ y'(0) = -4.5$

18. $(D^2 - 2D - 24I)y = 0,\ y(0) = 0,\ y'(0) = 20$

19. $(10D^2 + 18D + 5.6I)y = 0,\ y(0) = 4,\ y'(0) = -3.8$

20. $y'' + 2ky' + (k^2 + \omega^2)y = 0,\ y(0) = 1,\ y'(0) = -k.$

Find an ODE $y'' + ay' + by = 0$ for the given basis.

21. $e^{-x},\ e^{7x}$ **22.** $e^{6x},\ e^{-4x}$

23. $e^{-1.3x},\ xe^{-1.3x}$ **24.** $e^{(-1+i)x},\ e^{-(1+i)x}$

25. $1,\ e^{5x}$ **26.** $e^{3.5x},\ e^{-1.5x}$

ANSWERS

1. $y = c_1e^{-6x} + c_2e^{x}$ **2.** $y = (c_1 + c_2x)e^{-4x}$

3. $y = c_1e^{3.5x} + c_2e^{-1.5x}$ **4.** $y = e^{-x}[A\cos 2x + B\sin 2x]$

5. $y = (c_1 + c_2x)e^{-2\pi x}$ **6.** $y = c_1e^{0.9x} + c_2e^{-1.1x}$

7. $y = (c_1 + c_2x)e^{-1.3x}$ **8.** $y = c_1e^{-\sqrt{3}x} + c_2e^{\sqrt{3}x}$

9. $y = c_1e^{0.6x} + c_2e^{-1.6x}$ **10.** $y = c_1e^{15x} + c_2e^{-15x}$

11. $y = e^{-x}[A\cos\sqrt{3}x + B\sin\sqrt{3}x]$ **12.** $y = e^{-3x}[A\cos 2x + B\sin 2x]$

13. $y = \dfrac{6}{5}e^{-2x} + \dfrac{4}{5}e^{3x}$ **14.** $y = e^{-3x} - 3e^{3x}$

15. $y = (1 - 2x)e^{x}$ **16.** $y = e^{-2x}(2\cos x - \sin x)$

17. $y = (2 - 4x)e^{-0.25x}$ **18.** $y = 2e^{6x} - 2e^{-4x}$

19. $y = 1.8e^{-0.4x} + 2.2e^{-1.4x}$ **20.** $y = e^{-kx}\cos\omega x$

21. $y'' - 6y' - 7y = 0$ **22.** $y'' - 2y' - 24y = 0$

23. $y'' + 2.6y' + 1.69y = 0$ **24.** $y'' + 2y' + 2y = 0$

25. $y'' + 5y' = 0$ **26.** $y'' - 2y' - 5.25y = 0$

2.4 EULER-CAUCHY EQUATIONS

The **Euler-Cauchy equation** is an ordinary differential equation of the form

$$x^2y'' + axy' + by = 0 \qquad \text{...(1)}$$

where the coefficients a and b are constants. The coefficient of the first term x^2y'' is 1. We call this the **standard form**. (If you have kx^2y'', divide by k to get the standard form).

The trick for solving this equation is to try a solution of the form

$$y = x^m$$

where m is to be determined. The idea is similar to that for homogeneous linear ODEs with constant coefficients. Differentiating this function, we obtain

$$y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}$$

Substituting $y = x^m$ and its derivatives $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$ into Eq.(1), we get

$$m(m-1)x^m + amx^m + bx^m = 0$$

or, $$(m(m-1) + am + b)x^m = 0$$

Thus, $y = x^m$ is a solution of the differential equation (1) if and only if m is a root of the **auxiliary equation**

$$m(m-1) + am + b = 0$$

which simplifies to

$$m^2 + (a-1)m + b = 0 \qquad \text{...(2)}$$

There are three different cases to be considered, depending on whether the roots of this quadratic equation are real and distinct, real and equal, or complex. In the last case the roots occur as a conjugate pair.

Case 1. ***Distinct Real Roots***

If the roots m_1 and m_2 of the auxiliary equation (2) are real and distinct, then the solutions

$$y_1 = x^{m_1} \quad \text{and} \quad y_2 = x^{m_2}$$

are linearly independent because their quotient is not constant. Hence x^{m_1}, x^{m_2} form a basis of solutions of (1) for all x for which they are real. The corresponding general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

where c_1 and c_2 are arbitrary constants.

EXAMPLE 28 Solve the Euler-Cauchy equation $x^2y'' - 2xy' - 4y = 0$.

SOLUTION Comparing the given equation with the standard Euler-Cauchy equation

$$x^2y'' + axy' + by = 0,$$

we find $a = -2$, $b = -4$. Hence the given Euler-Cauchy equation has the auxiliary equation

$$m^2 - 3m - 4 = (m+1)(m-4) = 0$$

Its roots are $m_1 = -1$ and $m_2 = 4$. So the general solution is

$$y = c_1x^{-1} + c_2x^4$$

EXAMPLE 29 Find a general solution of the differential equation $(x^2D^2 + xD - 4I)y = 0$, where $D = \dfrac{d}{dx}$. **[*Delhi Univ. GE-3, 2016*]**

SOLUTION The given ODE is an Eular-Cauchy equation

$$x^2y'' + xy' - 4y = 0,$$

with $a = 1$, $b = -4$. The auxiliary equation is

$$m^2 + (a-1)m + b = 0$$

i.e., $$m^2 - 4 = 0 \quad \Rightarrow \quad m = 2, -2$$

Thus, the auxiliary equation has two distinct real roots 2 and -2. Hence the general solution of the given ODE is

$$y = c_1x^2 + c_2x^{-2}.$$

EXAMPLE 30 Find a general solution of the differential equation $(x^2D^2 + 6xD + 6I)y = 0$, where $D = \frac{d}{dx}$. **[Delhi Univ. GE-3, 2017]**

SOLUTION The given ODE is an Eular-Cauchy equation

$$x^2y'' + 6xy' + 6y = 0,$$

with $a = 6$, $b = 6$. The auxiliary equation is

$$m^2 + (a-1)m + b = 0$$

i.e., $$m^2 + 5m + 6 = 0 \quad \text{or} \quad (m+2)(m+3) = 0 \quad \Rightarrow \quad m = -2, -3$$

Thus, the auxiliary equation has two distinct real roots -2 and -3. Hence the general solution of the given ODE is

$$y = c_1x^{-2} + c_2x^{-3}.$$

EXAMPLE 31 Solve the following differential equation $(xD^2 + 4D)y = 0$. Also find the solution satisfying $y(1) = 12$, $y'(1) = -6$. **[Delhi Univ. GE-3, 2018]**

SOLUTION The given ODE can be written as

$$xy'' + 4y' = 0 \quad \Leftrightarrow \quad x^2y'' + 4xy' = 0 \quad (\text{provided } x \neq 0)$$

This is an Eular-Cauchy equation with $a = 4$ and $b = 0$. The auxiliary equation is

$$m^2 + (a-1)m + b = 0$$

i.e., $$m^2 + 3m = 0 \quad \Rightarrow \quad m = 0, -3$$

Thus, the auxiliary equation has two distinct real roots 0 and -3. Hence the general solution of the given ODE is

$$y = c_1 + c_2x^{-3}$$

We now find the particular solution satisfying the conditions $y(1) = 12$, $y'(1) = -6$. We need the derivative

$$y' = -3c_2x^{-4}$$

From this and the initial conditions, we have

$$y(1) = c_1 + c_2 = 12$$
$$y'(1) = \quad -3c_2 = -6$$

Solving these equations for c_1 and c_2, we obtain $c_1 = 10$ and $c_2 = 2$. Substituting these values in the general solution, we get the particular solution of the given initial value problem

$$y = 10 + 2x^{-3}.$$

Case 2. ***Real Double*** **(or** ***Repeated*****)** ***Root***

If the auxiliary equation (2) has a real double root m(that is, $m = m_1 = m_2$), then the two corresponding linearly independent solutions are $y_1 = x^m$ and $y_2 = x^m \ln x$. The corresponding general solution is

$$y = c_1 x^m + c_2 x^m \ln x.$$

EXAMPLE 32 Solve the differential equation : $4x^2y'' + 8xy' + y = 0.$

SOLUTION The given differential equation can be written in standard form of the Euler-Cauchy equation

$$x^2 y'' + 2xy' + \frac{1}{4} y = 0$$

Here $a = 2$, $b = 1/4$. Its auxiliary equation is

$$m^2 + (a-1)m + b = 0$$

i.e., $$m^2 + (2-1)m + \frac{1}{4} = 0 \quad \text{or} \quad 4m^2 + 4m + 1 = 0 \quad \text{or} \quad (2m+1)^2 = 0$$

Thus, it has the real double root $m = -1/2$. Hence the general solution for all positive x is

$$y = c_1 x^{-1/2} + c_2 x^{-1/2} \ln x.$$

EXAMPLE 33 Solve the initial value problem :

$$x^2y'' + 3xy' + y = 0, \ \ y(1) = 4, \ \ y'(1) = -2.$$ **[*Delhi Univ. GE*-3, 2016]**

SOLUTION ***Step* 1** ***General Solution***

The given ODE is an Euler-Cauchy equation with $a = 3$, $b = 1$. The auxiliary equation is

$$m^2 + (a-1)m + b = 0$$

i.e., $$m^2 + 2m + 1 = 0 \quad \text{or,} \quad (m+1) = 0$$

It has the real double root $m = -1$. Hence the general solution for all positive x is

$$y = (c_1 + c_2 \ln x)x^{-1}$$

Step* 2** ***Particular Solution

We now use initial conditions to find constants c_1 and c_2. We need the derivative

$$y' = -(c_1 + c_2 \ln x)x^{-2} + c_2 x^{-2}$$

From this and the initial conditions, we obtain

$$y(1) = c_1 = 4$$
$$y'(1) = -c_1 + c_2 = -2$$

Solving these equations, we find $c_1 = 4$, $c_2 = 2$. Thus, the given initial value problem has the solution

$$y = (4 + 2\ln x)x^{-1}.$$

Case 3. ***Conjugate Complex Roots***

We know that the complex roots of a polynomial equation with real coefficients occur in conjugate pairs. Thus, if $\lambda = \alpha + i\omega$ is a simple root of the auxiliary equation (2), so is the conjugate $\bar{\lambda} = \alpha - i\omega$, and the two corresponding linearly independent solutions are

$$y_1 = x^{\alpha}\cos(\omega\ln x) \quad \text{and} \quad y_2 = x^{\alpha}\sin(\omega\ln x)$$

The corresponding general solution is

$$y = x^{\alpha}[A\cos(\omega\ln x) + B\sin(\omega\ln x)].$$

EXAMPLE 34 Solve the Euler-Cauchy equation : $x^2y'' + 0.6xy' + 16.04y = 0.$

SOLUTION The auxiliary equation of the given ODE is

$$m^2 + (0.6 - 1)m + 16.04 = m^2 - 0.4m + 16.04 = 0$$

Using the quadratic formula, the roots of the auxiliary equation are

$$\frac{0.4 \pm \sqrt{0.16 - 64.16}}{2} = \frac{0.4 \pm 8i}{2} = 0.2 \pm 4i$$

Thus, the general solution of the ODE is

$$y = x^{0.2}[A\cos(4\ln x) + B\sin(4\ln x)].$$

EXAMPLE 35 Solve the initial value problem :

$$2x^2y'' + xy' - y = 0, \quad y(1) = 1, \quad y'(1) = 2.$$

SOLUTION ***Step 1*** ***General Solution***

Dividing by 2, the given ODE can be written in standard form of the Euler-Cauchy equation

$$x^2y'' + \frac{x}{2}y' - \frac{1}{2}y = 0$$

Here $a = 1/2$, $b = -1/2$. Its auxiliary equation is

$$m^2 + (a-1)m + b = 0$$

i.e., $$m^2 + \left(\frac{1}{2} - 1\right)m - \frac{1}{2} = 0 \quad \text{or} \quad 2m^2 - m - 1 = 0 \quad \text{or} \quad (2m+1)(m-1) = 0$$

It has two distinct real roots $m_1 = -1/2$, $m_2 = 1$. Thus, the general solution of the ODE is

$$y = c_1x^{-1/2} + c_2x = c_1\frac{1}{\sqrt{x}} + c_2x$$

where c_1 and c_2 are arbitrary constants.

Step 2 Particular Solution

We now use initial conditions to find constants c_1 and c_2. We need the derivative

$$y' = -\frac{1}{2}c_1 x^{-3/2} + c_2$$

From this and the initial conditions, we obtain

$$y(1) = c_1 + c_2 = 1$$

$$y'(1) = -\frac{1}{2}c_1 + c_2 = 2$$

Solving these equations, we find $c_1 = -2/3$ and $c_2 = 5/3$.

Thus, the particular solution of the initial value problem is

$$y = -\frac{2}{3}x^{-1/2} + \frac{5}{3}x = \frac{5}{3}x - \frac{2}{3}\frac{1}{\sqrt{x}}.$$

EXERCISE 2.3

Find a general solution of each of the following ODEs.

1. $x^2y'' + 2xy' - 6y = 0$
2. $x^2y'' - 7xy' + 16y = 0$
3. $x^2y'' + 0.5xy' + 0.0625y = 0$
4. $x^2y'' + xy' + 4y = 0$
5. $4x^2y'' + 4xy' - y = 0$
6. $x^2y'' + 3xy' + y = 0$
7. $x^2y'' - 2xy' + 2.25y = 0$
8. $x^2y'' + 7xy' + 9y = 0$
9. $x^2y'' + 3xy' + 2y = 0$
10. $x^2y'' - 0.75y = 0$

Solve each of the following initial value problems.

11. $x^2y'' - 6y = 0,\ y(1) = 1,\ y'(1) = 0$
12. $x^2y'' - 4xy' + 6y = 0,\ y(1) = 1,\ y'(1) = 0$
13. $x^2y'' + 3xy' + y = 0,\ y(1) = 4,\ y'(1) = -2$
14. $x^2y'' - 7xy' + 16y = 0,\ y(1) = 4,\ y'(1) = -2$
15. $(x^2D^2 - 2xD + 2.25I)y = 0,\ y(1) = 2.2,\ y'(1) = 2.5$
16. $(x^2D^2 + 1.5xD - 0.5I)y = 0,\ y(1) = 2,\ y'(1) = -11.$

ANSWERS

1. $y = c_1x^2 + c_2x^{-3}$
2. $y = (c_1 + c_2 \ln x)x^4$
3. $y = (c_1 + c_2 \ln x)x^{0.25}$
4. $y = A\cos(2\ln x) + B\sin(2\ln x)$
5. $y = c_1\sqrt{x} + c_2/\sqrt{x}$
6. $y = (c_1 + c_2 \ln x)x^{-1}$
7. $y = (c_1 + c_2 \ln x)x^{1.5}$
8. $y = (c_1 + c_2 \ln x)x^{-3}$
9. $y = x^{-1}[A\cos(\ln x) + B\sin(\ln x)]$
10. $y = c_1x^{1.5} + c_2x^{-0.5}$

11. $y = \frac{2}{5}x^3 + \frac{3}{5}x^{-2}$

12. $y = 3x^2 - 2x^3$

13. $y = (4 + 2\ln x)x^{-1}$

14. $y = (4 - 18\ln x)x^4$

15. $y = (2.2 - 0.8\ln x)x^{1.5}$

16. $y = -6\sqrt{x} + 8x^{-1}$

2.5 EXISTENCE AND UNIQUENESS OF SOLUTIONS. WRONSKIAN

In this section we shall establish the general theory of second-order homogeneous linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \qquad ...(1)$$

where $p(x)$ and $q(x)$ are continuous on some open interval I. Our main concern will be the existence and form of a general solution of (1) as well as the uniqueness of the initial value problem consisting of such an ODE with two initial conditions

$$y(x_0) = k_0, \qquad y'(x_0) = k_1$$

with given values x_0, k and k_1.

The following Theorem (proof omitted) states that such an initial value problem always has a solution that is unique.

> **THEOREM 2.4 Existence and Uniqueness Theorem for Initial Value problems**
> If $p(x)$ and $q(x)$ are continuous on some open interval I containing the point x_0, then the initial value problem
> $$y'' + p(x)y' + q(x)y = 0, \qquad y(x_0) = k_0, \qquad y'(x_0) = k_1$$
> has a unique solution $y(x)$ on I.

Linear Dependence and Independence of Solutions. Wronskian

Recall from Section 2.1 that two functions y_1 and y_2 defined on an interval I are linearly independent if the equation

$$c_1y_1(x) + c_2y_2(x) = 0 \text{ on } I \quad \Rightarrow \quad c_1 = 0, \; c_2 = 0$$

The functions y_1, y_2 are linearly dependent on I if then exist constants c_1 and c_2, not both zero, such that

$$c_1y_1(x) + c_2y_2(x) \equiv 0 \quad \text{on } I$$

It was also observed in Section 2.1 that two functions y_1 and y_2 are linearly dependent on I if and only if they are proportional on I, that is,

$$\text{either} \quad y_1 = ky_2 \quad \text{or} \quad y_2 = ly_1 \quad \text{for all } x \in I,$$

where k and l are numbers, zero or not. In contrast, in the case of linear independence these functions are not proportional.

Linear independence of solutions is crucial for obtaining general solutions. Thus, it would be good to have a criterion for it. For this we introduce the following definition that is relevent for stating the criterion.

DEFINITION Wronskian of Solutions

Let y_1 and y_2 be two solutions of second-order homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = 0$$

The **Wronskian** of y_1, y_2, denoted by $W(y_1, y_2)$ is defined as the determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Note It is important to emphasize that the Wronskian is a function of x that is determined by two solutions y_1, y_2 of ODE (1). For this reason we sometimes use the notation $W(y_1, y_2)(x)$ to emphasize that Wronskian is a function of x. When there is no danger of confusion, we'll shorten the notation to $W(x)$.

The Wronskian can be used to determine whether or not two solutions of (1) are independent as is stated in the following Theorem.

THEOREM 2.5 Linear Dependence and Wronskian

Let y_1 and y_2 be two solutions of the second-order homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{...(1)}$$

where the coefficients $p(x)$ and $q(x)$ are continuous on an open interval I. Then the following statements are equivaluent :

(*a*) y_1, y_2 are linearly dependent on I.

(*b*) $W(y_1, y_2) \equiv 0$ on I.

(*c*) $W(y_1, y_2)(x_0) = 0$ for some x_0 in I.

Proof (*a*) $\Rightarrow$ (*b*) We assume that y_1 and y_2 are linearly dependent on I. Then they must be proportional, that is,

$$\text{either} \quad (a)\ y_1 = ky_2 \quad \text{or} \quad (b)\ y_2 = ly_1 \quad \text{for all } x \in I$$

for some constants k and l.

If $y_1 = ky_2$, then

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = ky_2 y_2' - y_2 ky_2' = 0 \text{ on } I$$

Similarly, if $y_2 = ly_1$, then

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = y_1 l y_1' - l y_1 y_1' = 0 \text{ on } I$$

Thus, in either case, Wronskian is identically zero on I.

(*b*) $\Rightarrow$ (*c*) Obvious.

(*c*) $\Rightarrow$ (*a*) We assume that $W(y_1, y_2) = 0$ for some x_0 in I. We must show that y_1 and y_2 are linearly dependent on I. We consider the homogeneous linear system of equations in the unknowns c_1, c_2

$$\left.\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= 0 \end{aligned}\right\} \qquad \text{...(2)}$$

This system in matrix form is

$$\underbrace{\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}}_{Y}\underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{O} \qquad \ldots(3)$$

From linear algebra the system (2) (or system (3)) has a nontrivial solution if and only if the determinant of the coefficient matrix Y is zero. But this determinant is simply the Wronskian evaluated at $x = x_0$, and, by assumption, $W = 0$. Hence the system (2) (or system (3)) has a nontrivial solution. Thus, there exist c_1 and c_2, not both zero, satisfying linear equations in system (2). With these numbers c_1, c_2, we introduce the function

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

By the superposition principle this function is a solution of (1) on I. From (2), we see that it satisfies the initial conditions

$$y(x_0) = 0, \qquad y'(x_0) = 0,$$

However, the trivial solution $y^* \equiv 0$ of (1) also satisfies the same initial conditions. Since the solution of initial value problems is unique (Theorem 2.4), we have

$$y \equiv y^* \quad \text{i.e.,} \quad c_1 y_1 + c_2 y_2 = 0 \text{ on } I$$

Now since c_1 and c_2 are not both zero, the solutions y_1 and y_2 are linearly dependent on I. This completes the proof.

The following corollary gives a useful characterization of linear independence using the concept of Wronskian.

Corollary 2.6 Linear Independence and Wronskian

Two solutions y_1 and y_2 of (1) on I are linearly independent if and only if their Wronskian $W(y_1, y_2)$ is never zero on I.

EXAMPLE 36 Consider the ODE

$$y'' + \omega^2 y = 0 \qquad \ldots(1)$$

By substitution, it can be shown that $y_1 = \cos\omega x$ and $y_2 = \sin\omega x$ are solutions of (1). Their Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos\omega x & \sin\omega x \\ -\omega\sin\omega x & \omega\cos\omega x \end{vmatrix} = \omega\cos^2\omega x + \omega\sin^2\omega x = \omega$$

Thus, it follows from Corollary 2.6, that y_1 and y_2 are linearly independent if and only if $\omega \neq 0$. For $\omega = 0$, we have the trivial solution $y_2 \equiv 0$, which implies linear dependence because a pair of linearly independent solutions never contains the trivial solution.

EXAMPLE 37 Consider the ODE

$$y'' - 2y' + y = 0 \qquad \ldots(1)$$

By substitution, it can be shown that $y_1 = e^x$ and $y_2 = xe^x$ are solutions of (1) on the interval $(-\infty, \infty)$. By inspection the solutions are linearly independent. This fact can also be seen by evaluating their Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (x+1)e^{2x} - xe^{2x} = e^{2x}, \text{ which is never zero}$$

From Corollary 2.4, these solutions are linearly independent.

EXAMPLE 38 Find a homogeneous linear ODE of second-order for which the functions e^{2x} and e^x are solutions. Also show linear independence by considering their Wronskian.

SOLUTION To the given pair e^{2x}, e^x of solutions, there corresponds the characteristic equation

$$(\lambda - 2)(\lambda - 1) = \lambda^2 - 3\lambda + 2 = 0$$

Hence, the corresponding ODE is

$$y'' - 3y' + 2y = 0$$

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^x \\ 2e^{2x} & e^x \end{vmatrix} = e^{3x} - 2e^{3x} = -e^{3x} \neq 0$$

Thus, the given solutions are linearly independent.

EXAMPLE 39 Find a homogeneous linear ODE of second-order for which the functions e^{kx} and xe^{kx} are solutions. Also show linear independence by considering their Wronskian.

SOLUTION To the given pair e^{kx}, xe^{kx} of solutions, there corresponds the characteristic equation

$$(\lambda - k)^2 = \lambda^2 - 2k\lambda + k^2 = 0$$

Hence, the corresponding ODE is

$$y'' - 2ky' + k^2 y = 0$$

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{kx} & xe^{kx} \\ ke^{kx} & (kx+1)e^{kx} \end{vmatrix} = (kx+1)e^{2kx} - kxe^{2kx} = e^{2kx} \neq 0$$

Thus, the given solutions are linearly independent.

EXAMPLE 40 Find a homogeneous linear ODE of second-order for which the functions $\cos \pi x$ and $\sin \pi x$ are solutions. Also show linear independence by considering their Wronskian.

SOLUTION To the given pair $\cos \pi x$, $\sin \pi x$ of solutions, there corresponds the characteristic equation

$$(\lambda - \pi i)(\lambda + \pi i) = \lambda^2 + \pi^2 = 0$$

Hence, the corresponding ODE is

$$y'' + \pi^2 y = 0$$

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos \pi x & \sin \pi x \\ -\pi \sin \pi x & \pi \cos \pi x \end{vmatrix} = \pi \cos^2 \pi x + \pi \sin^2 \pi x = \pi \neq 0$$

Thus, the given solutions are linearly independent.

EXAMPLE 41 Find a homogeneous linear ODE of second order for which the functions x^3 and x^{-2} are solutions. Also show linear independence by considering their Wronskian.

[*Delhi Univ. GE-3, 2016*]

SOLUTION To the given pair x^3, x^{-2} of solutions, there corresponds the auxiliary equation

$$(m-3)(m+2) = m^2 - m - 6 = m(m-1) - 6 = 0$$

Hence, the ODE (an Eular-Cauchy equation) is

$$x^2y'' - 6y = 0$$

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^3 & x^{-2} \\ 3x^2 & -2x^{-3} \end{vmatrix} = -2 - 3 = -5 \neq 0$$

Thus, the given solutions are linearly independent.

EXAMPLE 42 Find a homogeneous linear ODE of second order for which the functions x^{-3} and $x^{-3}\ln x\,(x > 0)$ are solutions. Also show linear independence by considering their Wronskian.

[*Delhi Univ. GE-3, 2017 2018*]

SOLUTION To the given pair $x^{-3}, x^{-3}\ln x$ of solutions, there corresponds the auxiliarly equation

$$(m+3)^2 = m^2 + 6m + 9 = m^2 - m + 7m + 9 = m(m-1) + 7m + 9 = 0$$

Hence, the ODE (an Eular-Cauchy equation) is

$$x^2y'' + 7xy' + 9y = 0$$

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^{-3} & x^{-3}\ln x \\ -3x^{-4} & -3x^{-4}\ln x + x^{-4} \end{vmatrix} = -3x^{-7}\ln x + x^{-7} + 3x^{-7}\ln x = x^{-7} \neq 0$$

Thus, the given solutions are linearly independent.

EXAMPLE 43 Find a homogeneous linear ODE of second order for which the functions $e^{-2x}\cos\omega x$ and $e^{-2x}\sin\omega x$ are solutions. Also show linear independence by considering their Wronskian.

SOLUTION To the given pair $e^{-2x}\cos\omega x$, $e^{-2x}\sin\omega x$ of solutions, there corresponds the characteristic equation

$$(\lambda + 2 - \omega i)(\lambda + 2 + \omega i) = (\lambda + 2)^2 + \omega^2 = \lambda^2 + 4\lambda + 4 + \omega^2 = 0$$

Hence, the ODE is

$$y'' + 4y' + (4 + \omega^2)y = 0$$

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x}\cos\omega x & e^{-2x}\sin\omega x \\ e^{-2x}(-2\cos\omega x - \omega\sin\omega x) & e^{-2x}(-2\sin\omega x + \omega\cos\omega x) \end{vmatrix}$$

$$= \omega e^{-4x} \neq 0 \qquad (\text{assuming } \omega \neq 0)$$

Thus, the given solutions are linearly independent.

A General Solution of (1) Includes All Solutions

Recall from Section 2.1, that a general solution of the ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad ...(1)$$

on an open interval I is a solution of the form

$$y = c_1 y_1 + c_2 y_2$$

where y_1, y_2 form a basis of solutions of the ODE (1), and c_1 and c_2 are arbitrary constants. The next theorem shows that a general solution to the ODE (1) always exists.

THEOREM 2.7 If $p(x)$ and $q(x)$ are continuous on an open interval I, the ODE (1) has a general solution on I.

Proof Let x_0 be any arbitrary but fixed point in I. By Theorem 2.4, the ODE (1) has a solution $y_1(x)$ on I satisfying the initial conditions

$$y(x_0) = 1, \qquad y'(x_0) = 0$$

and a solution $y_2(x)$ on I satisfying the initial conditions

$$y(x_0) = 0, \qquad y'(x_0) = 1$$

Their Wronskian at $x = x_0$ is

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Hence, by Corollary 2.6, these solutions are linearly independent on I. Consequently, they form a basis of solutions of (1) on I, and $y = c_1 y_1 + c_2 y_2$ is a general solution of (1) on I. This proves the existence of a general solution.

We conclude this section by showing that from a general solution of (1) every solution of (1) can be obtained by choosing suitable values of the arbitrary constants. Hence ODE (1) has no **singular solutions**, that is, solutions that cannot be obtained from a general solution.

THEOREM 2.8 A General Solution of Second-order Homogeneous Linear ODE Includes All Solutions

If the coefficients $p(x)$ and $q(x)$ an continuous on some open interval I, then every solution $y = Y(x)$ of (1) on I is of the form

$$Y(x) = C_1 y_1(x) + C_2 y_2(x)$$

where y_1, y_2 is a basis of solutions of (1) on I and C_1, C_2 are suitable constants.

Proof Let $y = Y(x)$ be any solution of (1) on I. By Theorem 2.7, the ODE (1) has a general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \text{ on } I \qquad ...(1)$$

We have to find suitable values of c_1, c_2 such that $y(x) = Y(x)$ on I. We choose any fixed x_0 in I and show that we can find constants c_1, c_2 such that y and its derivative y' agree with Y and its first derivative Y' at x_0. That is, $y(x_0) = Y(x_0)$ and $y'(x_0) = Y'(x_0)$. Expressed in terms of (1), we have

$$\left.\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= Y(x_0) \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= Y'(x_0) \end{aligned}\right\} \quad \ldots(2)$$

This is a linear system of 2 equations in two unknowns c_1, c_2. This can be expressed in matrix form as

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} Y(x_0) \\ Y'(x_0) \end{bmatrix}$$

The determinant of the coefficient matrix is

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = W(y_1, y_2)(x_0)$$

Since y_1, y_2 form a basis, they are linearly independent and hence their Wronskian is not zero by Corollary 2.6. Thus, the system (2) has a unique solution $c_1 = C_1$, $c_2 = C_2$. Substituting these values in (1), we obtain the particular solution

$$y^*(x) = C_1 y_1(x) + C_2 y_2(x)$$

Now since C_1, C_2 is a solution of (2), we have

$$y^*(x_0) = Y(x_0), \qquad y^{*\prime}(x_0) = Y'(x_0).$$

This means that y^* and Y satisfy the same initial conditions at x_0. The uniqueness theorem (Theorem 2.4) now implies that $y^* \equiv Y$ on I, that is, y^* and Y are equal everywhere on I. This completes the proof.

EXERCISE 2.4

Find a second-order homogeneous linear ODE for which the given functions are solutions. Also show linear independence by considering their Wronskian.

1. $e^{0.5x}$, $e^{-3.5x}$

2. $e^{0.5x}$, $e^{-0.5x}$

3. x^2, x^{-3}

4. $e^{3.4x}$, $e^{-2.5x}$

5. e^{-2x}, xe^{-2x}

6. $x^{1.5}$, $x^{-0.5}$

7. $x^{-1}\cos(\ln x)$, $x^{-1}\sin(\ln x)$

8. $e^{-x}\cos 0.8x$, $e^{-x}\sin 0.8x$,

9. e^x, xe^x

10. $x^{0.25}$, $x^{0.25}\ln x$

ANSWERS

1. $y'' + 3y' - 1.75y = 0$, $W = -4e^{-3x}$

2. $y'' - 0.25y = 0$, $W = -1$

3. $x^2y'' + 2xy' - 6y = 0$, $W = -5x^{-2}$

4. $y'' - 0.9y' - 8.5y = 0$, $W = -5.9e^{0.9x}$

5. $y'' + 4y' + 4y = 0, \quad W = e^{-4x}$

6. $x^2y'' - 0.75y = 0, \quad W = -2$

7. $x^2y'' + 3xy' + 2y = 0, \quad W = x^{-3}$

8. $y'' + 2y' + 1.64y = 0, \quad W = 0.8e^{-2x}$

9. $y'' - 2y' + y = 0, \quad W = e^{2x}$

10. $x^2y'' + 0.5xy' + 0.0625\tilde{y} = 0, \quad W = x^{-0.5}$

2.6 NONHOMOGENEOUS ODEs

In this section we consider the general second-order nonhomogeneous linear ODEs

$$y'' + p(x)y' + q(x)y = r(x) \qquad ...(1)$$

where $p(x)$, $q(x)$, and $r(x)$ are continuous on an open interval I, and $r(x) \neq 0$ on I. We shall see that there is a close connection between solutions of (1) and solutions of the corresponding homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad ...(2)$$

We begin by introducing the concepts of "general solution of (1)" and "particular solution of (1)" which are defined as follows.

DEFINITION General Solution, Particular Solution

A **general solution** of the nonhomogeneous ODE (1) on an open interval I is a solution of the form

$$y(x) = y_h(x) + y_p(x) \qquad ...(3)$$

where $y_h = c_1y_1 + c_2y_2$ is a general solution of the corresponding homogeneous ODE (2) on I and y_p is any solution of (1) on I containing no arbitrary constants.

A **particular solution** of (1) on I is a solution obtained from (3) by assigning specific values to the arbitrary constants c_1 amd c_2 in y_h.

The following result shows that a general solution as defined above actually satisfies (1) and thus justifies the above terminologies.

THEOREM 2.9 Relations of Solutions of (1) to Those of (2)

(*a*) The sum of a solution y of (1) on some open interval I and a solution $\tilde{y}$ of (2) on I is a solution of (1) on I. In particular, (3) is a solution of (1) on I.

(*b*) The difference of two solutions of (1) on I is a solution of (2) on I.

Proof (*a*) Let $L[y]$ denote the left side of (1). That is,

$$L[y] = y'' + p(x)y' + q(x)y$$

Then for any solutions y of (1) and $\tilde{y}$ of (2) on I, we have

$$L[y] = r(x) \quad \text{and} \quad L[\tilde{y}] = 0 \qquad ...(4)$$

Consider

$$\begin{aligned} L[y + \tilde{y}] &= (y + \tilde{y})'' + p(x)(y + \tilde{y})' + q(x)(y + \tilde{y}) \\ &= (y'' + \tilde{y}'') + p(x)(y' + \tilde{y}') + q(x)(y + \tilde{y}) \end{aligned}$$

$$= (y'' + p(x)y' + q(x)y) + (\tilde{y}'' + p(x)\,\tilde{y}' + q(x)\,\tilde{y})$$

$$= L[y] + L[\tilde{y}]$$

$$= r(x) + 0 = r(x) \quad (using\ (4))$$

This proves that $y + \tilde{y}$ is a solution of (1) on I.

(*b*) Let y and y^* be any two solutions of (1) on I. Then

$$L[y] = r(x) \quad \text{and} \quad L[y^*] = r(x) \qquad ...(5)$$

Consider

$$L[y - y^*] = (y - y^*)'' + p(x)(y - y^*)' + q(x)(y - y^*)$$

$$= (y'' - y^{*''}) + p(x)(y' - y^{*'}) + q(x)(y - y^*)$$

$$= (y'' + p(x)y' + q(x)y) - (y^{*''} + p(x)y^{*'} + q(x)y^*)$$

$$= L[y] - L[y^*]$$

$$= r(x) - r(x) = 0 \quad (using\ (5))$$

This proves that $y - y^*$ is a solution of (2) on I.

We saw in Section 2.5 that a general solution of homogeneous ODE (2) includes all solutions. The following theorem (proof omitted) states that the same is true for nonhomogeneous ODEs (1).

THEOREM 2.10 A General Solution of a Nonhomogeneous ODE (1) Includes All Solutions

If the coefficients $p(x)$, $q(x)$, and the function $r(x)$ in (1) are continuous on some open interval I, then every solution of (1) on I is obtained by assigning suitable values to the arbitrary constants c_1 and c_2 in a general solution (3) of (1) on I.

2.7 FINDING A PARTICULAR SOLUTION OF THE NONHOMOGENEOUS ODE

Consider the second-order nonhomogeneous ODE

$$y'' + p(x)y' + q(x)y = r(x) \qquad ...(1)$$

and the corresponding homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0 \qquad ...(2)$$

We know that a general solution of (1) is of the form

$$y = y_h + y_p$$

where $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the homogeneous ODE (2) on I and y_p is a particular solution of (1). Thus, in order to solve the nonhomogeneous ODE (1) or an initial value problem for (1), we need to solve the corresponding homogeneous ODE (2) and find any particular solution y_p of (1). The question is : How can we find a particular solution y_p of (1) ?

There are two methods for finding a particular solution.

(*a*) *Method of undetermined coefficients.*

(*b*) *Method of variation of parameters.*

The method of undetermined coefficients is straightforward but works only for linear ODEs of the form

$$y'' + ay' + by = r(x)$$

where the coefficients a and b are constants and the forcing function $r(x)$ belongs to a restricted class of functions. The method of variation of parameters is more general and works for every function $r(x)$.

2.7.1 Method of Undetermined Coefficients

In this section we shall discuss the method of undetermined coefficients that is used to find a particular solution y_p of nonhomogeneous linear ODEs of the form

$$y'' + ay' + by = r(x) \qquad \text{...(1)}$$

where the coefficients a and b are constants and the forcing function $r(x)$ is an exponential function, a polynomial function, a sine or cosine, or sums or products of such functions. Note that all these functions have derivatives similar to $r(x)$. Thus, it is reasonable to choose a form for y_p similar to $r(x)$, but with unknown coefficients to be determined by substituting that y_p and its derivatives into the ODE. For example, if $r(x)$ is of the form $ae^{\alpha x}$, where a and α are constants, then the choice for y_p would be $y_p = Ae^{\alpha x}$. Similarly, if $r(x)$ is either $a\cos\beta x$ or $a\sin\beta x$, then, because of the rules for differentiating the sine and cosine functions, the choice for y_p would be

$$y_p = A\cos\beta x + B\sin\beta x$$

The choice of y_p for each of the important forms of $r(x)$ is shown in Table 2.1.

TABLE 2.1 : Method of Undetermined Coefficients

Form of $r(x)$	*Choice for $y_p(x)$*
Constant : a	*Constant* : A
$ax + b$	$Ax + B$
ax^n	$A_n x^n + A_{n-1}x^{n-1} + \dots + A_0$
$ae^{\alpha x}$	$Ae^{\alpha x}$
$a\cos\beta x$ or $a\sin\beta x$	$A\cos\beta x + B\sin\beta x$
$ae^{\alpha x}\cos\beta x$ or $ae^{\alpha x}\sin\beta x$	$e^{\alpha x}[A\cos\beta x + B\sin\beta x]$
$ax^n e^{\alpha x}$	$(A_n x^n + A_{n-1}x^{n-1} + \dots + A_0)e^{\alpha x}$
$ax^n\cos\beta x$ or $ax^n\sin\beta x$	$(A_n x^n + A_{n-1}x^{n-1} + \dots + A_0)[A\cos\beta x + B\sin\beta x]$
$ax^n e^{\alpha x}\cos\beta x$ or $ax^n e^{\alpha x}\sin\beta x$	$(A_n x^n + A_{n-1}x^{n-1} + \dots + A_0)e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

The corresponding rules for choosing the form of the particular solution y_p are described as follows.

1. ***Basic Rule*** This rule is applied when $r(x)$ is a single term. If $r(x)$ is one of the functions in the first column in Table 2.1, we choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into the given differential equation.
2. ***Modification Rule*** This rule is applied if a term in our choice for y_p happens to be a solution of the corresponding homogeneous ODE and therefore can't be a solution of the nonhomogeneous ODE. In such a case, we multiply our choice of y_p by x(or by x^2 if this solution corresponds to a double root of the characteristic equation of the corresponding homogeneous ODE) so that no term in y_p is a solution of the corresponding homogeneous ODE.
3. ***Sum Rule*** This rule is applied when $r(x)$ is a sum of functions in the first column of Table 2.1. In this case, the choice for y_p is the sum of the functions in the corresponding lines of the second column of Table 2.1.

Remark In practice it is a good idea to solve the corresponding homogeneous ODE before making the choice for the particular solution y_p of the nonhomogeneous ODE.

The above rules are best illustrated with the help of following example.

EXAMPLE 44 Determine a suitable choice for y_p for each of the following nonhomogeneous ODEs.

(*a*) $y'' + y' - 2y = x^2$ (*b*) $y'' - 2y' + 5y = 6e^{3x}$

(*c*) $y'' - 5y' + 6y = 4e^{2x}$ (*d*) $y'' + y = \sin x$

(*e*) $y'' - 4y' + 13y = e^{2x}\cos 3x$ (*f*) $y'' - 4y' + 4y = e^{2x}$

SOLUTION (*a*) The characteristic equation of the homogeneous ODE $y'' + y' - 2y = 0$ is

$$\lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$$

Thus, the characteristic equation has two distinct real roots 1 and -2. Hence the general solution of the homogeneous ODE is

$$y_h = c_1 e^x + c_2 e^{-2x}$$

Since $r(x) = x^2$, a polynomial of degree 2, we choose a particular solution of the form

$$y_p = Ax^2 + Bx + C.$$

(*b*) The characteristic equation of the homogeneous ODE $y'' - 2y' + 5y = 0$ is

$$\lambda^2 - 2\lambda + 5 = (\lambda - 1)^2 + 4 = 0 \quad \Rightarrow \quad \lambda = 1 \pm 2i$$

Thus, the characteristic equation has complex conjugate roots $1 \pm 2i$. Hence the general solution of the homogeneous ODE is

$$y_h = e^x[c_1 \cos 2x + c_1 \sin 2x]$$

Since $r(x) = 6e^{3x}$, we choose a particular solution of the form

$$y_p = Ae^{3x}.$$

(*c*) The characteristic equation of the homogeneous ODE $y'' - 5y' + 6y = 0$ is

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$$

Thus, the characteristic equation has two distinct real roots 2 and 3. Thus, the general solution of the homogeneous ODE is

$$y_h = c_1 e^{2x} + c_2 e^{3x}$$

Since $r(x) = 4e^{2x}$, the normal choice for y_p would be Ae^{2x}. But this won't work because Ae^{2x} is already a solution of the homogeneous ODE. Thus, we should multiply our choice function by x. That is, we should choose $y_p = Axe^{2x}$.

(*d*) The characteristic equation of the homogeneous ODE $y'' + y = 0$ is

$$\lambda^2 + 1 = 0$$

Thus, the characteristic equation has complex conjugate roots $\pm i$, so the general solution of the homogeneous ODE is

$$y_h = c_1 \cos x + c_2 \sin x$$

Since $r(x) = \sin x$, the normal choice for y_p would be $A \cos x + B \cos x$. But this won't work because it is already a solution of the homogeneous ODE. Thus, we should multiply our choice function by x. That is, we should choose

$$y_p = [A \cos x + B \cos x].$$

(*e*) The characteristic equation of the homogeneous ODE $y'' - 4y' + 13y = 0$ is

$$\lambda^2 - 4\lambda + 13 = (\lambda - 2)^2 + 9 = 0 \quad \Rightarrow \quad \lambda = 2 \pm 3i$$

Thus, the characteristic equation has complex conjugate roots $2 \pm 3i$, so the general solution of the homogeneous ODE is

$$y_h = e^{2x}[c_1 \cos 3x + c_1 \sin 3x]$$

Because $r(x) = e^{2x} \cos 3x$, the normal choice for y_p would be $e^{2x}[c_1 \cos 3x + c_1 \sin 3x]$. But this choice won't work because it is already a solution of the homogeneous ODE. Thus, we should multiply our choice function by x. Thus, instead, we should choose

$$y_p = xe^{2x}[c_1 \cos 3x + c_1 \sin 3x].$$

(*f*) The characteristic equation of the homogeneous ODE $y'' - 4y' + 4y = 0$ is

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$

Thus, the characteristic equation has the real double root 2, so the general solution of the homogeneous ODE is

$$y_h = (c_1 + c_2 x)e^{2x}$$

Because $r(x) = e^{2x}$, the normal choice for y_p should be

$$y_p = Ae^{2x}$$

But this choice won't work because it is already a solution of the homogeneous ODE. Thus, we should multiply our choice function by x. That is, we should choose $y_p = Axe^{2x}$. But this won't work either, $y_p = Axe^{2x}$ is also a solution of the corresponding homogeneous ODE. So, we have to multiply our choice function by x^2. That is, we choose

$$y_p = Ax^2e^{2x}.$$

EXAMPLE 45 Find the general solution of the equation $y'' - 2y' - 3y = 2\sin x$.

SOLUTION ***Step 1. General solution of the homogeneous differential equation.*** The first step is to find the general solution 'y_h' of the corresponding homogeneous ODE :

$$y'' - 2y' - 3y = 0$$

The characteristic equation of the homogeneous ODE is

$$\lambda^2 - 2\lambda - 3 = 0 \qquad \text{or} \qquad (\lambda + 1)(\lambda - 3) = 0$$

Thus, the characteristic equation has two distinct real roots $\lambda_1 = -1$ and $\lambda_2 = 3$. Hence the general solution of the corresponding homogeneous ODE is

$$y_h = c_1 e^{-x} + c_2 e^{3x}.$$

Step 2. Particular solution of the nonhomogeneous equation. The second step is to find the particular solution of the nonhomogeneous ODE :

$$y'' - 2y' - 3y = 2\sin x$$

Because $r(x) = 2\sin x$, we choose a trial solution of the type

$$y_p = A\cos x + B\sin x \quad (A, B \text{ are undetermined coefficients})$$

$$\Rightarrow \qquad y_p' = -A\sin x + B\cos x$$

and $$y_p'' = -A\cos x - B\sin x$$

Substitution into the original differential equation yields

$$(-A\cos x - B\sin x) - 2(-A\sin x + B\cos x) - 3(A\cos x + B\sin x) = 2\sin x$$

$$(-4A - 2B)\cos x + (2A - 4B)\sin x = 2\sin x$$

Equating coefficients of like terms, we obtain

$$-4A - 2B = 0 \qquad \text{and} \qquad 2A - 4B = 2$$

Solving these equations for A and B, we obtain $A = 1/5$ and $B = -2/5$. Therefore,

$$y_p = \frac{1}{5}\cos x - \frac{2}{5}\sin x$$

Step 3 General solution of the nonhomogeneous equation. The general solution of the given nonhomogeneous ODE is

$$y = y_h + y_p = c_1 e^{-x} + c_2 e^{3x} + \frac{1}{5}\cos x - \frac{2}{5}\sin x.$$

EXAMPLE 46 Solve the initial value problem

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \qquad y'(0) = 1.5.$$

SOLUTION ***Step 1. General solution of the homogeneous differential equation.*** The first step is to find the general solution 'y_h' of the corresponding homogeneous ODE :

$$y'' + y = 0$$

The characteristic equation of this homogeneous ODE is $\lambda^2 + 1 = 0$. It has conjugate complex roots $\pm i$. Thus, the homogeneous equation has the general solution

$$y_h(x) = c_1 \cos x + c_2 \sin x$$

Step 2. ***Particular solution of the nonhomogeneous equation.*** The second step is to find the particular solution of the nonhomogeneous ODE :

$$y'' + y = 0.001x^2$$

Since $r(x) = 0.001x^2$, a polynomial of degree 2, we choose a trial solution of the type

$$y_p = Ax^2 + Bx + C$$

$$\Rightarrow \qquad y'_p = 2Ax + B$$

$$\text{and} \qquad y''_p = 2A$$

Substitution into the original differential equation yields

$$2A + (Ax^2 + Bx + C) = 0.001x^2$$

$$\text{or} \qquad Ax^2 + Bx + (2A + C) = 0.001x^2$$

Equating the coefficients of x^2, x and the constant term on both sides, we have

$$A = 0.001, \qquad B = 0 \qquad \text{and} \qquad 2A + C = 0$$

$$\Rightarrow \qquad C = -2A = -0.002$$

Thus, the particular solution of the given nonhomogeneous ODE is

$$y_p = 0.001x^2 - 0.002$$

Step 3 ***General solution of the nonhomogeneous equation.*** The general solution of the given nonhomogeneous ODE is

$$y = y_h + y_p = c_1 \cos x + c_2 \sin x + 0.001x^2 - 0.002 \qquad \ldots(1)$$

We now apply the initial conditions. The first initial condition yields

$$y(0) = c_1 - 0.002 = 0 \quad \Rightarrow \quad c_1 = 0.002$$

$$\Rightarrow \qquad y = 0.002 \cos x + c_2 \sin x + 0.001x^2 - 0.002$$

$$\Rightarrow \qquad y' = -0.002 \sin x + c_2 \cos x + 0.002x$$

The second initial condition yields

$$y'(0) = c_2 = 1.5$$

Thus, the particular solution of the initial value problem is

$$y = 0.002 \sin x + 1.5 \cos x + 0.001x^2 - 0.002.$$

EXAMPLE 47 Solve the initial value problem

$$y'' + 3y' + 2.25y = -10e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

[*Delhi Univ. GE*-3, 2017]

SOLUTION *Step* **1.** ***General solution of the homogeneous differential equation.*** The first step is to find the general solution 'y_h' of the corresponding homogeneous ODE :

$$y'' + 3y' + 2.25y = 0$$

The characteristic equation of this homogeneous ODE is

$$\lambda^2 + 3\lambda + 2.25 = 0 \qquad \text{or} \qquad (\lambda + 1.5)^2 = 0$$

It has the real double root $\lambda = -1.5$. Hence the corresponding homogeneous ODE has the general solution

$$y_h = (c_1 + c_2 x)e^{-1.5x}$$

Step 2. Particular solution of the nonhomogeneous equation. The second step is to find a particular solution of the nonhomogeneous ODE :

$$y'' + 3y' + 2.25y = -10e^{-1.5x}$$

Because $r(x) = -10xe^{-1.5x}$, the normal choice for y_p would be $Ae^{-1.5x}$. But this won't work because $Ae^{-1.5x}$ is a solution of the corresponding homogeneous ODE. Thus, we should multiply our choice function by x. That is, we should choose $y_p = Axe^{-1.5x}$. But this won't work either, $y_p = Axe^{-1.5x}$ is also a solution of the corresponding homogeneous ODE. So, we have to multiply our choice function by x^2. That is, we choose

$$y_p = Ax^2e^{-1.5x}$$

Calculating the derivatives of y_p, we have

$$\Rightarrow \qquad y_p' = A[-1.5x^2e^{-1.5x} + 2xe^{-1.5x}] = A(2x - 1.5x^2)e^{-1.5x}$$

and

$$y_p'' = A[-1.5(2x - 1.5x^2)e^{-1.5x} + (2 - 3x)e^{-1.5x}] = A(2 - 6x + 2.25x^2)e^{-1.5x}$$

Substituting these expressions into the given differential equation yields

$$A(2 - 6x + 2.25x^2)e^{-1.5x} + 3A(2x - 1.5x^2)e^{-1.5x} + 2.25Ax^2e^{-1.5x} = -10e^{-1.5x}$$

Cancelling the factor $e^{-1.5x}$, we get

$$A(2 - 6x + 2.25x^2) + 3A(2x - 1.5x^2) + 2.25Ax^2 = -10$$

Comparing the constants term, we obtain

$$2A = -10 \qquad \Rightarrow \qquad A = -5$$

Thus, a particular solution of the given nonhomogeneous ODE is

$$y_p = -5x^2e^{-1.5x}$$

Step 3 General solution of the nonhomogeneous equation. The general solution of the given nonhomogeneous ODE is

$$y = y_h + y_p = (c_1 + c_2x)e^{-1.5x} - 5x^2e^{-1.5x} \qquad \text{...(1)}$$

Step 4 Solution of the initial value problem. We now use the initial conditions to determine constants c_1 and c_2. The first initial condition yields

$$y(0) = c_1 = 1$$

Differentiation of y yields

$$y' = (c_2 - 1.5c_1 - 1.5c_2x)e^{-1.5x} - 10xe^{-1.5x} + 7.5x^2e^{-1.5x}$$

Setting $x = 0$ and using the second initial condition gives

$$y'(0) = c_2 - 1.5c_1 = 0 \qquad \Rightarrow \qquad c_2 = 1.5c_1 = 1.5$$

Substituting the values of c_1 and c_2 in (1), the particular solution of the initial value problem is

$$y = (1 + 1.5x)e^{-1.5x} - 5x^2e^{-1.5x} = (1 + 1.5x - 5x^2)e^{-1.5x}.$$

EXAMPLE 48 Solve the differential equation : $y'' - 5y' + 6y = 4e^{2x}$.

SOLUTION ***Step 1. General solution of the homogeneous differential equation.***
The first step is to find the general solution 'y_h' of the corresponding homogeneous ODE :

$$y'' - 5y' + 6y = 0$$

The characteristic equation is

$$\lambda^2 - 5\lambda + 6 = 0 \quad \text{or} \quad (\lambda - 2)(\lambda - 3) = 0 \quad \Rightarrow \quad \lambda = 2 \text{ or } \lambda = 3$$

Thus, the characteristic equation has two distinct real roots $\lambda_1 = 3$ and $\lambda_2 = 3$. Hence the homogeneous ODE has the general solution

$$y_h = c_1 e^{2x} + c_2 e^{3x}$$

Step **2.** ***Particular solution of the nonhomogeneous equation.*** The second step is to find a particular solution of the nonhomogeneous ODE :

$$y'' - 5y' + 6y = 4e^{2x}$$

Because $r(x) = 4e^{2x}$, the normal choice for y_p would be Ae^{2x}. But this choice won't work because this function is already a solution of the corresponding homogeneous ODE. So we have to multiply our choice of y_p by x. That is, we choose

$$y_p = Axe^{2x}$$

Calculating the derivatives of y_p, we have

$$y_p' = 2Axe^{2x} + Ae^{2x} \quad \text{and} \quad y_p'' = 4Axe^{2x} + 4Ae^{2x}$$

Substituting these expressions into the given ODE yields

$$(4Axe^{2x} + 4Ae^{2x}) - 5(2Axe^{2x} + Ae^{2x}) + 6(Axe^{2x}) = 4e^{2x}$$

$$\Rightarrow \quad -Ae^{2x} = 4e^{2x} \quad \Rightarrow \quad A = -4$$

Thus, $y_p = -4xe^{2x}$ is a particular solution of the given nonhomogeneous ODE.

Step **3** ***General solution of the nonhomogeneous equation***. The general solution of the given nonhomogeneous differential ODE is

$$y = y_h + y_p = c_1 e^{2x} + c_2 e^{3x} - 4xe^{2x}.$$

EXAMPLE 49 Solve the initial value problem

$$y'' + y' - 12y = 4e^{2x}, \quad y(0) = 7, \quad y'(0) = 0.$$

SOLUTION ***Step 1. General solution of the homogeneous differential equation.***
The first step is to find the general solution 'y_h' of the corresponding homogeneous ODE :

$$y'' + y' - 12y = 0$$

The characteristic equation is

$$\lambda^2 + \lambda - 12 = 0 \quad \text{or} \quad (\lambda - 3)(\lambda + 4) = 0 \quad \Rightarrow \quad \lambda = 3 \text{ or } \lambda = -4$$

Thus, the characteristic equation has two distinct real roots $\lambda_1 = 3$ and $\lambda_2 = -4$. Hence the homogeneous ODE has the general solution

$$y_h = c_1 e^{3x} + c_2 e^{-4x}$$

Step 2. Particular solution of the nonhomogeneous equation. The second step is to find a particular solution of the nonhomogeneous ODE :

$$y'' + y' - 12y = 4e^{2x}$$

Because $r(x) = 4e^{2x}$, we choose a trial solution of the type

$$y_p = Ae^{2x} \qquad (A \text{ is an undetermind coefficient})$$

$$\Rightarrow \qquad y_p' = 2Ae^{2x}, \qquad y_p'' = 4Ae^{2x}$$

Substituting these expressions into the given ODE, we obtain

$$4Ae^{2x} + 2Ae^{2x} - 12Ae^{2x} = 4e^{2x}$$

$$\Rightarrow \qquad 4A + 2A - 12A = 4 \qquad (\text{cancelling } e^{2x})$$

$$\Rightarrow \qquad A = -2/3$$

Thus, the particular solution of the given nonhomogeneous ODE is

$$y_p = -\frac{2}{3}e^{2x}$$

Step 3 General solution of the nonhomogeneous equation. The general solution of the given nonhomogeneous ODE is

$$y = y_h + y_p = c_1e^{3x} + c_2e^{-4x} - \frac{2}{3}e^{2x} \qquad ...(1)$$

Step 4 Solution of the initial value problem. We now use the initial conditions to determine constants c_1 and c_2. Setting $x = 0$ and using the first initial condition gives

$$y(0) = c_1 + c_2 - 2/3 = 7 \qquad ...(i)$$

Differentiation of y yields

$$y' = 3c_1e^{3x} - 4c_2e^{-4x} - \frac{4}{3}e^{2x}$$

From this and the second initial condition, we have

$$y'(0) = 3c_1 - 4c_2 - \frac{4}{3} = 0 \qquad ...(ii)$$

Solving Eq.(i) and (ii) simultaneously, we find that $c_1 = 32/7$ and $c_2 = 65/21$. Substituting these values of c_1 and c_2 in (1), the particular solution of the initial value problem is

$$y = \frac{32}{7}e^{3x} + \frac{65}{21}e^{-4x} - \frac{2}{3}e^{2x}.$$

EXAMPLE 50 Solve the initial value problem

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x, \qquad y(0) = 1, \qquad y'(0) = 2.$$

SOLUTION ***Step 1. General solution of the homogeneous differential equation.*** The first step is to find the general solution 'y_h' of the corresponding homogeneous ODE :

$$y'' - 3y' + 2y = 0$$

The characteristic equation

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$$

has two distinct real roots $\lambda = 1$ and $\lambda = 2$, so the general solution of the homogeneous ODE is

$$y_h = c_1 e^{x} + c_2 e^{2x}$$

Step **2.** ***Particular solution of the nonhomogeneous equation.*** The second step is to find a particular solution of the nonhomogeneous ODE :

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$$

Because $r(x) = 3e^{-x} - 10\cos 3x$ is a sum of two functions in the first column of Table 2.1, we choose for y_p the sum of the functions in the corresponding lines of the second column. That is, we choose

$$y_p = Ae^{-x} + B\cos 3x + C\sin 3x$$

$$\Rightarrow \quad y_p' = -Ae^{-x} - 3B\sin 3x + 3C\cos 3x$$

$$\Rightarrow \quad y_p'' = Ae^{-x} - 9B\cos 3x - 9C\sin 3x$$

Substituting these expressions into the given differential equation, we obtain

$$(Ae^{-x} - 9B\cos 3x - 9C\sin 3x) - 3(-Ae^{-x} - 3B\sin 3x + 3C\cos 3x) + 2(Ae^{-x} + B\cos 3x + C\sin 3x) = 3e^{-x} - 10\cos 3x$$

Regrouping the terms yields

$$6Ae^{-x} + (-7B - 7C)\cos 3x + (9B - 7C)\sin 3x = 3e^{-x} - 10\cos 3x$$

Equating the coefficients of e^{-x}, $\cos 3x$ and $\sin 3x$, we obtain

$$6A = 3 \qquad \ldots(i)$$

$$-7B - 9C = -10 \qquad \ldots(ii)$$

$$9B - 7C = 0 \qquad \ldots(iii)$$

Solving these equations, we get $A = 1/2$, $B = 7/13$ and $C = 9/13$. This gives the particular solution

$$y_p = \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x$$

Step **3** ***General solution of the nonhomogeneous equation.*** The general solution of the given nonhomogeneous ODE is

$$y = y_h + y_p = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x \qquad \ldots(1)$$

Step **4** ***Solution of the initial value problem.*** We now use the initial conditions to determine constants c_1 and c_2. We need to derivative

$$y' = c_1 e^{x} + 2c_2 e^{2x} - \frac{1}{2}e^{-x} - \frac{21}{13}\sin 3x + \frac{27}{13}\cos 3x$$

Using the initial conditions, we have

$$y(0) = c_1 + c_2 + \frac{1}{2} + \frac{7}{13} = 1$$

$$y'(0) = c_1 + 2c_2 - \frac{1}{2} + \frac{27}{13} = 2$$

Solving these equations, we obtain $c_1 = -\frac{1}{2}$, $c_2 = \frac{6}{13}$. Substituting these values of c_1 and c_2 into Eq.(1), we obtain the particular solution of the given initial value problem

$$y = -\frac{1}{2}e^x + \frac{6}{13}c_2 + \frac{1}{2}e^{-x} + \frac{7}{13}\cos 3x + \frac{9}{13}\sin 3x.$$

EXAMPLE 51 Use the method of undetermined coefficients to find the particular solution of the ODE $y'' + 4y' + 5y = 25x^2 + 13\sin 2x$. Also find its general solution. ***[Delhi Univ. GE-3, 2018]***

SOLUTION ***Step 1. General solution of the homogeneous differential equation.***

The first step is to find the general solution 'y_h' of the corresponding homogeneous ODE :

$$y'' + 4y' + 5y = 0$$

The characteristic equation is

$$\lambda^2 + 4\lambda + 5 = 0 \quad \text{or} \quad (\lambda + 2)^2 + 1 = 0 \quad \Rightarrow \quad \lambda = -2 \pm i$$

Thus, the characteristic equation has complex conjugate roots $-2 \pm i$. Hence the corresponding homogeneous ODE has the general solution

$$y_h = e^{-2x}[c_1 \cos x + c_2 \sin x]$$

Step 2. Particular solution of the nonhomogeneous equation. The second step is to find a particular solution of the nonhomogeneous ODE :

$$y'' + 4y' + 5y = 25x^2 + 13\sin 2x$$

Because the forcing function

$$r(x) = 25x^2 + 13\sin 2x$$

is a sum of two functions in the first column of Table 2.1, we choose for y_p the sum of the functions in the corresponding lines of the second column. That is, we choose

$$y_p = Ax^2 + Bx + C + D\cos 2x + E\sin 2x \qquad \text{...(1)}$$

This implies

$$y_p' = 2Ax + B - 2D\sin 2x + 2E\cos 2x$$
$$y_p'' = 2A - 4D\cos 2x - 4E\sin 2x$$

Substituting these expressions into the given differential equation, we obtain

$$(2A - 4D\cos 2x - 4E\sin 2x) + 4(2Ax + B - 2D\sin 2x + 2E\cos 2x)$$
$$+ 5(Ax^2 + Bx + C + D\cos 2x + E\sin 2x) = 25x^2 + 13\sin 2x$$

Regrouping terms, we get

$$(2A + 4B + 5C) + (8A + 5B)x + 5Ax^2 + (8E + D)\cos 2x + (E - 8D)\sin 2x = 25x^2 + 13\sin 2x$$

Equating the coefficients of like powers of x, and the coefficients of $\cos 2x$ and $\sin 2x$, we obtain

$$2A + 4B + 5C = 0, \quad 8A + 5B = 0, \quad 5A = 25, \quad 8E + D = 0, \quad E - 8D = 13,$$

Solving these equations for these 5 unknowns, we obtain

$$A = 5, \quad B = -8, \quad C = 4.4, \quad D = -1.6, \quad E = 0.2$$

Substituting the values of these unknown in Eq. (1), we obtain a particular solution of the given nonhomogeneous ODE

$$y_p = 5x^2 - 8x + 4.4 - 1.6\cos 2x + 0.2\sin 2x$$

Step 3 ***General solution of the nonhomogeneous equation***. The general solution of the given nonhomogeneous ODE is

$$y. = y_h + y_p = e^{-2x}[c_1 \cos x + c_2 \sin x] + 5x^2 - 8x + 4.4 - 1.6\cos 2x + 0.2\sin 2x$$

EXAMPLE 52 Solve the initial value problem

$$y'' + 2y' + 5y = e^{0.5x} + 40\cos 10x - 190\sin 10x, \quad y(0) = 0.16, \quad y'(0) = 40.08.$$

SOLUTION ***Step* 1. *General solution of the homogeneous differential equation.***
The first step is to find the general solution 'y_h' of the corresponding homogeneous ODE :

$$y'' + 2y' + 5y = 0$$

The characteristic equation is

$$\lambda^2 + 2\lambda + 5 = 0$$

or

$$(\lambda + 1)^2 + 4 = 0 \quad \Rightarrow \quad \lambda + 1 = \pm 2i \quad \text{or} \quad \lambda = -1 \pm 2i$$

Hence the corresponding homogeneous ODE has the general solution

$$y_h = e^{-x}[c_1 \cos 2x + c_2 \sin 2x]$$

Step 2. ***Particular solution of the nonhomogeneous equation.*** The second step is to find a particular solution of the nonhomogeneous ODE :

$$y'' + 2y' + 5y = e^{0.5x} + 40\cos 10x - 190\sin 10x$$

Because the forcing function

$$r(x) = e^{0.5x} + 40\cos 10x - 190\sin 10x$$

is a sum of two functions in the first column of Table 2.1, we choose for y_p the sum of the functions in the corresponding lines of the second column. That is, we choose

$$y_p = Ae^{0.5x} + B\cos 10x + C\sin 10x \qquad \ldots(1)$$

Then

$$y_p' = 0.5Ae^{0.5x} - 10B\sin 10x + 10C\cos 10x$$

$\Rightarrow$

$$y_p'' = 0.25Ae^{0.5x} - 100B\cos 10x - 100C\sin 10x$$

Substituting these expressions into the given differential equation, we obtain

$$(0.25Ae^{0.5x} - 100B\cos 10x - 100C\sin 10x) + 2(0.5Ae^{0.5x} - 10B\sin 10x + 10C\cos 10x)$$
$$+ 5(Ae^{0.5x} + B\cos 10x + C\sin 10x) = e^{0.5x} + 40\cos 10x - 190\sin 10x$$

Regrouping terms, we get

$$6.25Ae^{0.5x} + (-95B + 20C)\cos 10x + (-20B - 95C)\sin 10x = e^{0.5x} + 40\cos 10x - 190\sin 10x$$

Equating the coefficients of $e^{0.5x}$, $\cos 10x$ and $\sin 10x$, we obtain

$$6.25A = 1 \quad \text{...}(i)$$
$$-95B + 20C = 40 \quad \text{...}(ii)$$
$$-20B - 95C = -190 \quad \text{...}(iii)$$

Equation (i) gives $A = 1/6.25 = 0.16$. Solving Eqs.(ii) and (iii) simultaneously for B and C, we obtain $B = 0$ and $C = 2$. Substituting the values of A, B and C in Eq.(1), we obtain a particular solution of the given nonhomogeneous ODE

$$y_p = 0.16e^{0.5x} + 2\sin 10x$$

Step 3 General solution of the nonhomogeneous equation. The general solution of the given nonhomogeneous ODE is

$$y = y_h + y_p = e^{-x}(c_1\cos 2x + c_2\sin 2x) + 0.16e^{0.5x} + 2\sin 10x \quad \text{...(2)}$$

Step 4 Solution of the initial value problem. We now use the initial conditions to determine constants c_1 and c_2. We need the derivative

$$y' = e^{-x}(-2c_1\sin 2x + 2c_2\cos 2x - c_1\cos 2x - c_2\sin 2x) + 0.08e^{0.5x} + 20\cos 10x$$

Using the initial conditions, we have

$$y(0) = c_1 + 0.16 = 0.16$$
$$y'(0) = -c_1 + 2c_2 + 0.08 + 20 = 40.08$$

Solving these equations, we obtain $c_1 = 0$, $c_2 = 10$. Substituting these values of c_1 and c_2 into Eq.(2), we obtain the particular solution of the given initial value problem

$$y = 10e^{-x}\sin 2x + 0.16e^{0.5x} + 2\sin 10x.$$

EXAMPLE 53 Solve the initial value problem : $y'' + 3y' + 2y = x^2$.

SOLUTION ***Step 1. General solution of the corresponding homogeneous differential equation.***

The characteristic equation of the corresponding homogeneous differential equation is

$$\lambda^2 + 3\lambda + 2 = 0 \quad i.e., \quad (\lambda + 1)(\lambda + 2) = 0 \quad \Rightarrow \quad \lambda = -1, -2$$

$$\Rightarrow \quad \lambda = -1, \ -2 \quad \textit{Distinct real roots}$$

Thus, the characteristic equation has two distinct real roots -1 and -2. Hence the homogeneous ODE has the general solution

$$y_h = c_1e^{-x} + c_2e^{-2x}$$

Step 2. A particular solution of the nonhomogeneous differential equation.

We now obtain a particular solution, y_p, of the nonhomogeneous ODE

$$y'' + 3y' + 2y = x^2 \quad \text{...(1)}$$

Since $r(x) = x^2$, a polynomial of degree 2, we choose for y_p a trial solution of the form:

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B \quad \text{and} \quad y_p'' = 2A$$

Substituting these expressions into Eq.(1) yields

$$2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$$

or,

$$2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2$$

Equating the coefficients of x^2, x, and the constant term on both sides, we have

$$2A = 1 \qquad ...(i)$$

$$6A + 2B = 0 \qquad ...(ii)$$

$$2A + 3B + 2C = 0 \qquad ...(iii)$$

Eq.(*i*) yields $A = 1/2$. Substituting this value in (*ii*), we get

$$3 + 2B = 0 \quad \Rightarrow \quad B = -3/2$$

Substituting the values of A and B in (*iii*), we get

$$1 - \frac{9}{2} + 2C = 0 \quad \Rightarrow \quad C = \frac{7}{4}$$

Hence a particular solution of the given nonhomogeneous ODE is given by

$$y_p = \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}$$

Step 3 ***General solution of the nonhomogeneous equation***. The general solution of the given nonhomogeneous differential equation is

$$y = y_h + y_p = c_1e^{-x} + c_2e^{-2x} + \frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4}.$$

EXERCISE 2.5

Use the method of undetermined coefficients to find a particular solution of each of the following differential equations. Also give a general solution.

1. $y'' - y' - 6y = e^{2x}$
2. $y'' - 4y' + 3y = 3x^2 - 5$
3. $y'' + 4y = e^{3x}$
4. $y'' + y' - 2y = \sin x$
5. $y'' + y = \sin x$
6. $y'' - 4y' + 3y = e^{3x}$
7. $y'' + 3y' - 10y = 3x^2$
8. $y'' + y' - 2y = x^2$
9. $y'' + 4y' + 3.75y = 109\cos 5x$
10. $y'' - 16y = 19.2e^{4x} + 60e^x$
11. $y'' + 4y' + 4y = e^{-2x}\sin 2x$
12. $y'' + 6y' + 73y = 80e^x\cos 4x$
13. $y'' + 1.44y = 24\cos 1.2x$
14. $y'' + 9y = 18x + 36\sin 3x$
15. $y'' - 5y' + 6y = 4e^{2x}$
16. $y'' - 4y = xe^x + \cos 2x$

Use the method of undetermined coefficients to find the solution of each of the following initial value problems.

17. $y'' - 4y' + 3y = 5e^{-4x}$, $y(0) = 4$, $y'(0) = -1$

18. $y'' + 4y = 16\cos 2x$, $y(0) = 0$, $y'(0) = 0$

19. $y'' - 3y' + 2.25y = 27(x^2 - x)$, $y(0) = 20$, $y'(0) = 30$

20. $y'' + 1.2y' + 0.26y = 1.22e^{0.5x}$, $y(0) = 3.5$, $y'(0) = 0.35$

21. $y'' - 2y' = 12e^{2x} - 8e^{-2x}$, $y(0) = -2$, $y'(0) = 12$.

ANSWERS

1. $y = c_1e^{-2x} + c_2e^{3x} - \frac{1}{4}e^{2x}$

2. $y = c_1e^{x} + c_2e^{3x} + x^2 + \frac{8}{3}x + \frac{11}{9}$

3. $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{13}e^{3x}$

4. $y = c_1e^{x} + c_2e^{-2x} - \frac{1}{10}(\cos x + 3\sin x)$

5. $y = c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x$

6. $y = c_1e^{x} + c_2e^{3x} + \frac{1}{2}xe^{3x}$

7. $y = c_1e^{2x} + c_2e^{-5x} - \frac{3}{10}x^2 - \frac{9}{50}x - \frac{57}{500}$

8. $y = c_1e^{x} + c_2e^{-2x} - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{3}{4}$

9. $y = c_1e^{-1.5x} + c_2e^{-2.5x} - 2.72\cos 5x + 2.56\sin 5x$

10. $y = c_1e^{4x} + c_2e^{-4x} - 2.4xe^{4x} - 4e^{x}$

11. $y = (c_1 + c_2x)e^{-2x} - \frac{1}{4}e^{-2x}\sin 2x$

12. $y = e^{-3x}(A\cos 8x + B\sin 8x) + e^{x}\left(\cos 4x + \frac{1}{2}\sin 4x\right)$

13. $y = c_1\cos 1.2x + c_2\sin 1.2x + 10x\sin 1.2x$

14. $y = A\cos 3x + B\sin 3x + 2x - 6x\cos 3x$

15. $y = c_1e^{2x} + c_2e^{3x} - 4xe^{2x}$

16. $y = c_1e^{2x} + c_2e^{-2x} - \left(\frac{1}{3}x + \frac{2}{9}\right)e^{x} - \frac{1}{8}\cos 2x$

17. $y = 6e^{x} - \frac{15}{7}e^{3x} + \frac{1}{7}e^{-4x}$

18. $y = 4x\sin 2x$

19. $y = 4[(1 + x)e^{1.5x} + 3x^2 + 5x + 4]$

20. $y = e^{-0.1x}[1.5\cos 0.5x - \sin 0.5x] + 2e^{0.5x}$

21. $y = 2e^{2x} - 3 + 6xe^{2x} - e^{-2x}$.

2.7.2 Method of Variation of Parameters

In this section we shall discuss yet another method for finding a particular solution of the second-order nonhomogeneous linear ODE that is given in the standard form

$$y'' + p(x)y' + q(x)y = r(x) \quad \text{...(1)}$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous on some open interval I. We have seen that there is a close connection between solutions of ODE (1) and solutions of the corresponding homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0 \quad ...(2)$$

In fact, a general solution of (1) is of the form

$$y = y_h + y_p$$

where y_h is a general solution of the corresponding homogeneous ODE (2) and y_p is any particular solution of (1). If the forcing function $r(x)$ consists of sums or products of x^n, $e^{\alpha x}$, $\cos\beta x$ or $\sin\beta x$, we can find a particular solution y_p by the method of undetermined coefficients, as we have seen in Section 2.7.1. Since this method is valid only for restricted class of functions $r(x)$, it is desirable to have a method that applies to more general differential equations (1), which we shall now develop. It is called the **method of variation of parameters**. The method of variation of parameters uses a pair of linerarly independent solutions of the corresponding homogeneous ODE (2) to construct a particular solution of ODE (1).

We shall develop this method in connection with the general second-order linear ODE (1). Suppose we know that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the corresponding homogeneous ODE (2). A general solution of the corresponding homogeneous equation is therefore given by

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

The idea in the method of variation of parameters is to replace the arbitrary constants ("the parameters") c_1 and c_2 by functions $u(x)$ and $v(x)$ so that the resulting function

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) \quad ...(3)$$

is a particular solution of the nonhomogeneous ODE (1) (this suggests the name of the method, *variation of parameters*).

The functions $u(x)$ and $v(x)$ will be governed by a system of two equations, one of which is derived by requiring that Eq. 1) is satisfied, and the other of which is chosen to simplify the resulting system. Differentiating (3), we obtain

$$y_p'(x) = u(x)y_1'(x) + v(x)y_2'(x) + u'(x)y_1(x) + v'(x)y_2(x) \quad ...(4)$$

We simplify $y_p'(x)$ by imposing the condition

$$u'(x)y_1(x) + v'(x)y_2(x) = 0 \quad ...(5)$$

With this condition imposed, Eq(4) reduces to

$$y_p'(x) = u(x)y_1'(x) + v(x)y_2'(x) \quad ...(6)$$

Differentiating (6), we obtain

$$y_p''(x) = u(x)y_1''(x) + v(x)y_2''(x) + u'(x)y_1'(x) + v'(x)y_2'(x) \quad ...(7)$$

Substituting y_p and its derivatives according to (3), (6), (7) into (1), we get

$$[u(x)y_1''(x) + v(x)y_2''(x) + u'(x)y_1'(x) + v'(x)y_2'(x)] + p(x)[u(x)y_1'(x) + v(x)y_2'(x)] + q(x)[u(x)y_1(x) + v(x)y_2(x)] = r(x)$$

Regrouping the terms, we get

$$u(x)[y_1''(x) + p(x)y_2'(x) + q(x)y_1(x)] + v(x)[y_2''(x) + p(x)y_2'(x) + q(x)y_2(x)] + u'(x)y_1'(x) + v'(x)y_2'(x) = r(x) \quad \text{...(8)}$$

Since y_1 and y_2 are solutions of the corresponding homogeneous ODE (2), the expressions in the first two brackets in (8) are identically zero. Eq.(8) thus reduces to

$$u'(x)y_1'(x) + v'(x)y_2'(x) = r(x) \quad \text{...(9)}$$

The resulting system that governs $u(x)$ and $v(x)$ is therefore given by (6) and (9) :

$$\left.\begin{aligned} u'(x)y_1(x) + v'(x)y_2(x) &= 0 \\ u'(x)y_1'(x) + v'(x)y_2'(x) &= r(x) \end{aligned}\right\} \quad \text{...(10)}$$

This is a linear system of two algebraic equations for the unknown u' and v'. The determinant of coefficients of this system is

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2'(x)y_1(x)$$

Since y_1 and y_2 are linearly independent solutions of the corresponding homogeneous ODE, we know that

$$W(y_1, y_2) \neq 0$$

Hence the system (10) has a unique solution. Using Cramer's rule, we find that

$$u'(x) = \frac{-y_2(x)r(x)}{W(y_1, y_2)}, \qquad v'(x) = \frac{y_1(x)r(x)}{W(y_1, y_2)}$$

Integrating, we get

$$u(x) = \int \frac{-y_2(x)r(x)}{W(y_1, y_2)}dx, \qquad v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx \quad \text{...(11)}$$

Therefore, a particular solution y_p of Eq.(1) is given by

$$y_p = u(x)y_1(x) + v(x)y_2(x)$$

where $u(x)$ and $v(x)$ are defined by (11).

We now outline the procedure for finding a particular solution of the nonhomogeneous ODE using the method of variation of parameters.

Procedure for Finding a Particular Solution of the nonhomogeneous ODE

$$y'' + p(x)y' + q(x)y = r(x)$$

Step 1 *Find linearly independent solutions of the corresponding homogeneous ODE :*

The first step is to find a pair of linearly independent solutions $\{y_1(x), y_2(x)\}$ of the corresponding homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0$$

Step 2 ***Find a general solution of the corresponding homogeneous ODE :***
The general solution y_h of the corresponding homogeneous ODE is given by

$$y_h = c_1 y_1(x) + c_2 y_2(x) \qquad ...(1)$$

Step 3 ***Find a particular solution of the nonhomogeneous ODE :***
The particular solution y_p of the nonhomogeneous ODE is given by

$$y_p = u(x) y_1(x) + v(x) y_2(x),$$

where $u(x)$ and $v(x)$ are given by

$$u(x) = \int \frac{-y_2(x) r(x)}{W(y_1, y_2)} dx, \qquad v(x) = \int \frac{y_1(x) r(x)}{W(y_1, y_2)} dx.$$

EXAMPLE 54 Use the method of variation of parameters to solve the nonhomogeneous ODE

$$y'' + 3y' + 2y = 30e^{2x}.$$

[Delhi Univ. GE-3, 2016]

SOLUTION **Step 1** ***Find linearly independent solutions of the corresponding homogeneous ODE***
The corresponding homogeneous ODE is

$$y'' + 3y' + 2y = 0 \qquad ...(1)$$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0 \quad \text{or} \quad (\lambda + 1)(\lambda + 2) = 0 \quad \Rightarrow \quad \lambda = -1, -2$$

Thus, the characteristic equation has two distinct real roots -1 and -2. Hence the functions $y_1(x) = e^{-x}$ and $y_2(x) = e^{-2x}$ are two solutions of the homogeneous ODE (1). These solutions are linearly independent because their Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x} \neq 0$$

Step 2 ***Find a general solution of the corresponding homogeneous ODE***
The general solution of the corresponding homogeneous ODE (1) is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

Step 3 ***Find a particular solution of the nonhomogeneous ODE***
By the method of variation of parameters, a particular solution of the given nonhomogeneous ODE is

$$y_p = u(x) e^{-x} + v(x) e^{-2x} \qquad ...(2)$$

where $u(x)$ and $v(x)$ are given by

$$u(x) = \int \frac{-y_2(x) r(x)}{W(y_1, y_2)} dx = \int \frac{-e^{-2x}(30e^{2x})}{-e^{-3x}} dx = 30\int e^{3x} = 10e^{3x}$$

and

$$v(x) = \int \frac{y_1(x) r(x)}{W(y_1, y_2)} dx = \int \frac{e^{-x}(30e^{2x})}{-e^{-3x}} dx = -30\int e^{4x} = -7.5e^{4x}$$

Substituting the expressions for $u(x)$ and $v(x)$ back into equation (2), we get the particular solution of the given nonhomogeneous ODE

$$y_p = u(x)e^{-x} + v(x)e^{-2x} = 10e^{3x} \cdot e^{-x} - 7.5e^{4x}e^{-2x} = 2.5e^{2x}$$

Hence the general solution of the given nonhomogeneous ODE is

$$y = y_h + y_p = c_1e^{-x} + c_2e^{-2x} + 2.5e^{2x}.$$

EXAMPLE 55 Use the method of variation of parameters to solve the nonhomogeneous ODE

$$y'' + y = \tan x.$$

SOLUTION Step 1 ***Find linearly independent solutions of the corresponding homogeneous ODE***

The corresponding homogeneous ODE is

$$y'' + y = 0 \qquad ...(1)$$

The characteristic equation is

$$\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm i$$

Thus, the characteristic equation has two complex conjugate roots $\pm i$. Hence the functions $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are two solutions of the homogeneous ODE (1). These solutions are linearly independent because their Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

Step 2 ***Find a general solution of the corresponding homogeneous ODE***

The general solution of the corresponding homogeneous ODE (1) is

$$y_h = c_1 \cos x + c_2 \sin x$$

Step 3 ***Find a particular solution of the nonhomogeneous ODE***

By the method of variation of parameters, a particular solution of the given nonhomogeneous ODE is

$$y_p = u(x)\cos x + v(x)\sin x, \qquad ...(2)$$

where $u(x)$ and $v(x)$ are given by

$$u(x) = \int \frac{-y_2(x)r(x)}{W(y_1, y_2)}dx = \int -\sin x \tan x \, dx = -\int \frac{\sin^2 x}{\cos x}dx = -\int \frac{1-\cos^2 x}{\cos x}dx$$

$$= \int (\cos x - \sec x)\,dx = \sin x - \ln|\sec x + \tan x|$$

and

$$v(x) = \int \frac{y_1(x)r(x)}{W(y_1, y_2)}dx = \int \cos x \tan x \, dx = \int \sin x \, dx = -\cos x$$

Substituting the expressions for $u(x)$ and $v(x)$ back into equation (2), we get the particular solution of the given nonhomogeneous ODE

$$y_p = \cos x[\sin x - \ln|\sec x + \tan x|] - \sin x \cos x$$

Hence the general solution of the given nonhomogeneous ODE is

$$y = y_h + y_p = c_1 \cos x + c_2 \sin x + \cos x[\sin x - \ln|\sec x + \tan x|] - \sin x \cos x.$$

EXAMPLE 56 Use the method of variation of parameters to find a general solution of the following nonhomogeneous ODE

$$y'' - 2y' + y = e^x \sin x.$$ **[*Delhi Univ. GE-3*, 2018]**

SOLUTION The general solution of the nonhomogeneous ODE

$$y'' - 2y' + y = e^x \sin x \qquad \text{...(1)}$$

is of the form

$$y = y_h + y_p,$$

where y_h is the general solution of the corresponding homogeneous ODE

$$y'' - 2y' + y = 0 \qquad \text{...(2)}$$

and y_p is a particular solution of the ODE (1).

***General solution of the homogeneous ODE* (2)**

The characteristic equation of the homogeneous ODE (2) is

$$\lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad (\lambda - 1)^2 = 0 \quad \Rightarrow \quad \lambda = 1, 1$$

Thus, the characteristic equation has the real double root $\lambda = 1$. Hence the functions $y_1(x) = e^x$ and $y_2(x) = xe^x$ are two solutions of the homogeneous ODE (1). These solutions are linearly independent because their Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (x+1)e^{2x} - xe^{2x} = e^{2x} \neq 0$$

The general solution of the homogeneous ODE (2) is

$$y_h = c_1 e^x + c_2 x e^x$$

***Particular solution of the nonhomogeneous ODE* (1)**

By the method of variation of parameters, a particular solution of the nonhomogeneous ODE (1) is

$$y_p = u(x)\, e^x + v(x) x e^x \qquad \text{...(3)}$$

where $u(x)$ and $v(x)$ are given by

$$u(x) = \int \frac{-y_2(x) r(x)}{W(y_1, y_2)} dx = -\int x \sin x\, dx = -\left[-x\cos x + \int \cos x\, dx\right] = x\cos x - \sin x$$

and

$$v(x) = \int \frac{y_1(x) r(x)}{W(y_1, y_2)} dx = \int \sin x\, dx = -\cos x$$

Substituting the expressions for $u(x)$ and $v(x)$ back into equation (3), we get the particular solution of the nonhomogeneous ODE (1)

$$y_p = xe^x \cos x - e^x \sin x - xe^x \cos x = -e^x \sin x$$

Hence the general solution of the nonhomogeneous ODE (1) is

$$y = y_h + y_p = c_1e^x + c_2xe^x - e^x\sin x.$$

EXERCISE 2.5

Use the method of variation of parameters to find a particular solution of each of the following differential equations. Also give a general solution.

1. $y'' - y = e^x$
2. $y'' + y = \tan x$
3. $y'' + 9y = 3\tan 3x$
4. $y'' - 2y' + y = \dfrac{e^x}{x^2+1}$
5. $y'' + 3y' + 2y = \dfrac{1}{1+e^x}$
6. $y'' + y = \sec x$
7. $y'' + y = \operatorname{cosec} x$
8. $y'' - 4y' + 4y = x^2e^x$
9. $y'' - 2y' + y = e^x\sin x$
10. $y'' - 4y' + 4y = 12e^{2x}/x^4$
11. $(D^2 - 2D + I)y = x^2 + x^{-2}e^x$
12. $(D^2 - 2D + I)y = x^{-3}e^x$

ANSWERS

1. $y = c_1e^x + c_2e^{-x} + \dfrac{1}{2}xe^x$
2. $y = c_1\cos x + c_2\sin x - \cos x\ln|\sec x + \tan x|$
3. $y = c_1\cos 3x + c_2\sin 3x - \dfrac{1}{3}\cos 3x\ln|\sec 3x + \tan 3x|$
4. $y = c_1e^x + c_2xe^x - \dfrac{1}{2}e^x\ln(1+x^2) + xe^x\tan^{-1}x$
5. $y = c_1e^{-x} + c_2e^{-2x} + e^{-x}[\ln(1+e^x)] + e^{-2x}[\ln(1+e^x) - e^x]$
6. $y = c_1\cos x + c_2\sin x + \cos x\ln|\cos x| + x\sin x$
7. $y = c_1\cos x + c_2\sin x - x\cos x + \sin x\ln|\sin x|$
8. $y = (c_1 + c_2x)e^{2x} + (x^2 + 4x + 6)e^x$
9. $y = (c_1 + c_2x)e^x - e^x\sin x$
10. $y = (c_1 + c_2x)e^{2x} + 2x^{-2}e^{2x}$
11. $y = (c_1 + c_2x)e^x + x^2 + 4x - 6 - e^x(\ln|x| +)$
12. $y = (c_1 + c_2x)e^x + \dfrac{e^x}{2x}$