CHAPTER

2

Transcendental and Polynomial Equations

2.1 INTRODUCTION

A problem of great importance in applied mathematics and engineering is that of determining the roots of an equation of the form

$$f(x) = 0. \tag{2.1}$$

The function $f(x)$ may be given explicitly, for example

$$f(x) = P_n(x)$$
$$= a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n,\ a_0 \neq 0$$

a polynomial of degree n in x, or $f(x)$ may be known only implicitly as a transcendental function.

Definition 2.1 A number ξ is a solution of $f(x) = 0$ if $f(\xi) \equiv 0$. Such a solution ξ is a **root** or a **zero** of $f(x) = 0$.

Geometrically, a root of the equation (2.1) is the value of x at which the graph of $y = f(x)$ intersects the x-axis.

Definition 2.2 If we can write (2.1) as

$$f(x) = (x - \xi)^m\, g(x) = 0$$

where $g(x)$ is bounded and $g(\xi) \neq 0$, then ξ is called a **multiple root** of multiplicity m.

In this case, $f(\xi) = f'(\xi) = \ldots = f^{(m-1)}(\xi) = 0,\ f^{(m)}(\xi) \neq 0$.

For $m = 1$, the number ξ is said to be a **simple root**.

There are generally two types of methods used to find the roots of the equation (2.1).

Direct Methods

These methods give the exact value of the roots in a finite number of steps. Further, the methods give all the roots at the same time. For example, a direct method gives the root of a linear or first degree equation

$$a_0 x + a_1 = 0,\ a_0 \neq 0 \tag{2.2}$$

as
$$x = -a_1/a_0.$$
Similarly, the roots of the quadratic equation
$$a_0 x^2 + a_1 x + a_2 = 0,\ a_0 \neq 0 \tag{2.3}$$
are given by
$$x = \frac{-a_1 \pm \sqrt{(a_1^2 - 4a_0 a_2)}}{2a_0}$$

Iterative Methods

These methods are based on the idea of successive approximations, i.e., starting with one or more initial approximations to the root, we obtain a sequence of approximations or iterates $\{x_k\}$, which in the limit converges to the root. The methods give only one root at a time. For example, to solve the quadratic equation (2.3) we may choose any one of the following iteration methods:

$$\text{(a)}\ x_{k+1} = -\frac{a_2 + a_0 x_k^2}{a_1},\ k = 0, 1, 2, \cdots$$
$$\text{(b)}\ x_{k+1} = -\frac{a_2}{a_0 x_k + a_1},\ k = 0, 1, 2, \cdots$$
$$\text{(c)}\ x_{k+1} = -\frac{a_2 + a_1 x_k}{a_0 x_k},\ k = 0, 1, 2, \cdots \tag{2.4}$$

The convergence of the sequence $\{x_k\}$ to the number ξ, the root of the equation (2.3) depends on the rearrangement (2.4) and the choice of the starting approximation x_0.

Definition 2.3 A sequence of iterates $\{x_k\}$ is said to converge to the root ξ, if
$$\lim_{k\to\infty} |x_k - \xi| = 0 \text{ or } \lim_{k\to\infty} x_k = \xi.$$

If $x_k, x_{k-1}, \cdots, x_{k-m+1}$ are m approximations to the root, then a multipoint iteration method is defined as
$$x_{k+1} = \phi(x_k, x_{k-1}, \cdots, x_{k-m+1}). \tag{2.5}$$
The function ϕ is called the multipoint iteration function.

For $m = 1$, we get the one point iteration method
$$x_{k+1} = \phi(x_k). \tag{2.6}$$

Thus, given one or more initial approximations to the root, we require a suitable iteration function ϕ for a given function $f(x)$, such that the sequence of iterates obtained from (2.5) or (2.6) converges to the root ξ. In practice, except in rare cases, it is not possible to find ξ which satisfies the given equation exactly. We, therefore, attempt to find an approximate root ξ^* such that either
$$|f(\xi^*)| < \varepsilon$$
or
$$|x_{k+1} - x_k| < \varepsilon \tag{2.7}$$

where x_k and x_{k+1} are two consecutive iterates and ε is the prescribed **error tolerance**.

Initial Approximations

Initial approximations to the root are often known from the physical considerations of the problem. Otherwise, graphical methods are generally used to obtain initial approximations to the root. Since the value of x, at which the graph of $y = f(x)$ intersects the x-axis, gives the root of $f(x) = 0$, any value in the neighbourhood of this point may be taken as an initial approximation to the root (see Fig. 2.1 a, b). If the equation $f(x) = 0$ can be conveniently written in the form $f_1(x) = f_2(x)$, then the point of intersection of the graphs of $y = f_1(x)$ and $y = f_2(x)$ gives the root of $f(x) = 0$ and therefore any value in the neighbourhood of this point can be taken as an initial approximation to the root (see Fig. 2.1c).

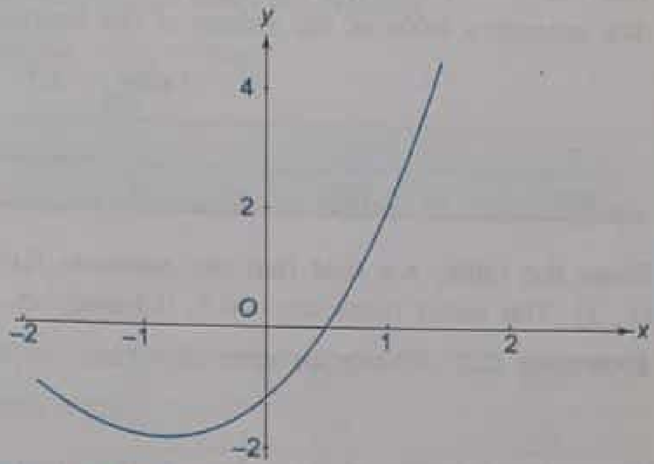

Fig. 2.1 (a). Graph of $y = x^2 + 2x - 1$.

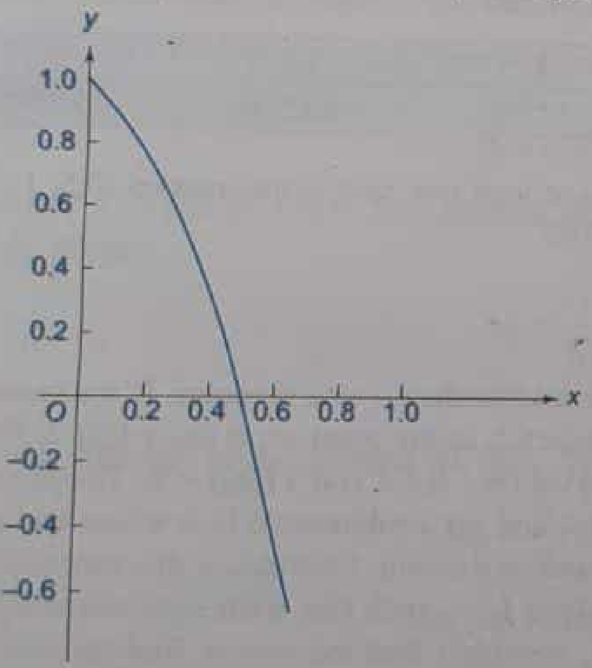

Fig. 2.1 (b). Graph of $y = \cos x - xe^x$.

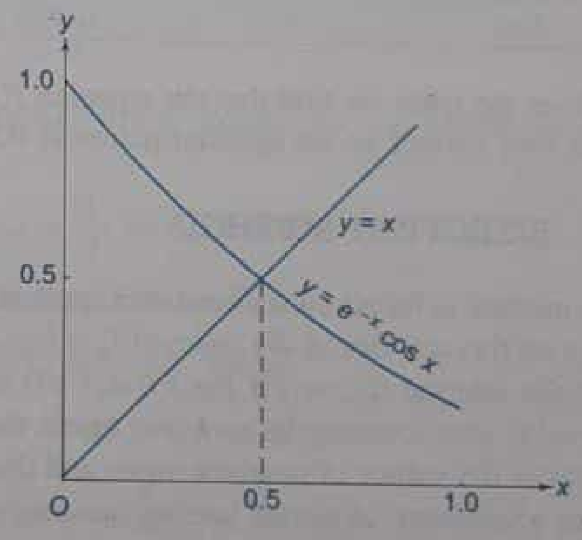

Fig. 2.1 (c). Graph of $y = x$ and $y = e^{-x} \cos x$.

Another commonly used method to obtain the initial approximation to the root is based upon the **Intermediate Value Theorem**, which states:

Theorem 2.1 *If $f(x)$ is a continuous function on some interval $[a, b]$ and $f(a)\, f(b) < 0$, then the equation $f(x) = 0$ has at least one real root or an odd number of real roots in the interval (a, b).*

We can set up a table of values of $f(x)$ for various values of x and obtain a suitable initial approximation to the root.

Example 2.1 The equation

$$8x^3 - 12x^2 - 2x + 3 = 0$$

has three real roots. Find the intervals each of unit length containing each one of these roots.

We prepare a table of the values of the function $f(x)$ for various values of x (Table 2.1).

Table 2.1 Values of f(x)

x	-2	-1	0	1	2	3
$f(x)$	-105	-15	3	-3	15	105

From the table, we find that the equation $f(x) = 0$ has roots in the intervals $(-1, 0)$, $(0, 1)$ and $(1, 2)$. The exact roots are -0.5, 0.5 and 1.5.

Example 2.2 Obtain an interval which contains a root of the equation

$$f(x) = \cos x - x\, e^x = 0.$$

We prepare a table of the values of the function $f(x)$ for various values of x (Table 2.2).

Table 2.2 Values of $f(x)$

x	0	0.5	1	1.5	2
$f(x)$	1	0.0532	−2.1780	−6.6518	−15.1942

From the table we find that the equation $f(x) = 0$ has at least one root in the interval $(0.5, 1)$. The exact root correct to ten decimal places is 0.5177573637.

2.2 BISECTION METHOD

This method is based on the repeated application of the intermediate value theorem. If we know that a root of $f(x) = 0$ lies in the interval $I_0 = (a_0, b_0)$, we bisect I_0 at the point $m_1 = (a_0 + b_0)/2$. Denote by I_1 the interval (a_0, m_1) if $f(a_0)\, f(m_1) < 0$ or the interval (m_1, b_0) if $f(m_1)\, f(b_0) < 0$. Therefore, the interval I_1 also contains the root. We bisect the interval I_1 and get a subinterval I_2 at whose end points $f(x)$ takes the values of opposite signs and therefore contains the root. Continuing this procedure, we obtain a sequence of nested sets of sub-intervals $I_0 \supset I_1 \supset I_2 \cdots$ such that each subinterval contains the root. After repeating the bisection process q times, we either find the root or find the interval I_q of length $(b_0 - a_0)/2^q$ which contains the root. We take the midpoint of the last subinterval as the desired approximation to the root. This root has error not greater than one-half of the length of the interval of which it is the midpoint. Thus, we have

$$m_{k+1} = a_k + \frac{1}{2}(b_k - a_k), \quad k = 0, 1, 2, \ldots$$

where

$$(a_{k+1}, b_{k+1}) = \begin{cases} (a_k, m_{k+1}), \text{ if } f(a_k)\, f(m_{k+1}) < 0 \\ (m_{k+1}, b_k), \text{ if } f(m_{k+1})\, f(b_k) < 0. \end{cases}$$

We notice that this method uses only the end points of the interval $[a_k, b_k]$ for which $f(a_k)\, f(b_k) < 0$ and not the values of $f(x)$ at these end points, to obtain the next approximation to the root. The method is simple to use and the sequence of approximations always converges to the root for any $f(x)$ which is continuous in the interval that contains the root. If the permissible error is ε, then the approximate number of iterations required may be determined from the relation

$$\frac{b_0 - a_0}{2^n} \le \varepsilon \quad \text{or} \quad n \ge \frac{\log (b_0 - a_0) - \log \varepsilon}{\log 2}$$

Since n is an integer, we take n as the next nearest integer.

The minimum number of iterations required for converging to a root in the interval (0, 1) for a given ε are listed in Table 2.3.

Table 2.3 Number of Iterations

ε	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
n	7	10	14	17	20	24

Thus, the bisection method requires a large number of iterations to achieve a reasonable degree of accuracy for the root. It requires one function evaluation for each iteration.

Example 2.3 Perform five iterations of the bisection method to obtain the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0.$$

Since $f(0) > 0$ and $f(1) < 0$, the smallest positive root lies in the interval (0, 1). Taking $a_0 = 0$, $b_0 = 1$, we get

$$m_1 = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(0 + 1) = 0.5$$

$$f(m_1) = -\ 1.375 \text{ and } f(a_0)\, f(m_1) < 0.$$

Thus, the root lies in the interval (0, 0.5). Taking $a_1 = 0$, $b_1 = 0.5$, we get

$$m_2 = \frac{1}{2}(a_1 + b_1) = \frac{1}{2}(0 + 0.5) = 0.25$$

$$f(m_2) = f(0.25) = -\ 0.234375 \text{ and } f(a_1)\, f(m_2) < 0.$$

Thus the root lies in the interval (0, 0.25). The sequence of intervals is given in Table 2.4.

Table 2.4 Sequence of Intervals for the Bisection Method

k	a_{k-1}	b_{k-1}	m_k	$f(m_k)\, f(a_{k-1})$
1	0	1	0.5	< 0
2	0	0.5	0.25	< 0
3	0	0.25	0.125	> 0
4	0.125	0.25	0.1875	> 0
5	0.1875	0.25	0.21875	< 0

Hence, the root lies in (0.1875, 0.21875). The approximate root is taken as the midpoint of this interval, that is 0.203125.

Example 2.4 Perform five iterations of the bisection method to obtain a root of the equation

$$f(x) = \cos x - xe^x = 0$$

Since $f(0) = 1 > 0$ and $f(1) = -2.1780 < 0$, the root lies in the interval (0, 1). Taking the initial approximations as $a_0 = 0$, $b_0 = 1$, we get

$$m_1 = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(0 + 1) = 0.5$$

$$f(m_1) = f(0.5) = 0.0532 \text{ and } f(a_0)\, f(m_1) > 0.$$

Therefore, the root lies in the interval (0.5, 1.0).

Taking $a_1 = 0.5$, $b_1 = 1.0$, we get

$$m_2 = \frac{1}{2}(a_1 + b_1) = \frac{1}{2}(0.5 + 1.0) = 0.75$$

$$f(m_2) = -0.8561 \text{ and } f(a_1)\, f(m_2) < 0.$$

Therefore, the root lies in the interval (0.5, 0.75). The sequence of intervals is given in Table 2.5.

Table 2.5 Sequence of Intervals for the Bisection Method

k	a_{k-1}	b_{k-1}	m_k	$f(m_k)\, f(a_{k-1})$
1	0	1	0.5	> 0
2	0.5	1	0.75	< 0
3	0.5	0.75	0.625	< 0
4	0.5	0.625	0.5625	< 0
5	0.5	0.5625	0.53125	< 0

Hence, the root lies in the interval (0.5, 0.53125). The approximate root is taken as the midpoint of this interval, that is, 0.515625.

2.3 ITERATION METHODS BASED ON FIRST DEGREE EQUATION

We have already seen that if $f(x) = 0$ is a first degree equation in x then it can be readily solved. We now study the iteration methods which will produce exact results whenever $f(x) = 0$ is a first degree equation. Thus, if we approximate $f(x)$ by a first degree equation in the neighbourhood of the root, then we may write

$$f(x) = a_0\, x + a_1 = 0. \tag{2.8}$$

The solution of (2.8) is given by

$$x = -\frac{a_1}{a_0} \tag{2.9}$$

where $a_0 \neq 0$ and a_1 are arbitrary parameters to be determined by prescribing two appropriate conditions on $f(x)$ and/ or its derivatives.

Secant and Regula-Falsi Methods

If x_{k-1} and x_k are two approximations to the root, then we determine a_0 and a_1 in (2.8) by using the conditions

$$f_{k-1} = a_0 x_{k-1} + a_1$$
$$f_k = a_0 x_k + a_1$$

where $f_{k-1} = f(x_{k-1})$ and $f_k = f(x_k)$.

On solving, we obtain

$$a_0 = (f_k - f_{k-1})/(x_k - x_{k-1})$$
$$a_1 = (x_k f_{k-1} - x_{k-1} f_k)/(x_k - x_{k-1}). \tag{2.10}$$

From the equations (2.9) and (2.10), the next approximation x_{k+1} to the root is given by

$$x_{k+1} = \frac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}}. \tag{2.11}$$

which may also be written as

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k, \; k = 1, 2, \cdots \tag{2.12}$$

This is called the **secant** or the **chord method**.

Geometrically, in this method we replace the function $f(x)$ by a straight line or a chord passing through the points (x_k, f_k) and (x_{k-1}, f_{k-1}) and take the point of intersection of the straight line with the x-axis as the next approximation to the root (Fig. 2.2a). If the approximations are such that $f_k f_{k-1} < 0$, then the method (2.11) or (2.12) is known as the **Regula-Falsi method**. The method is shown graphically in Fig. 2.2b. Since (x_{k-1}, f_{k-1}), (x_k, f_k) are known before the start of the iteration, the secant and the Regular-Falsi methods require one function evaluation per iteration.

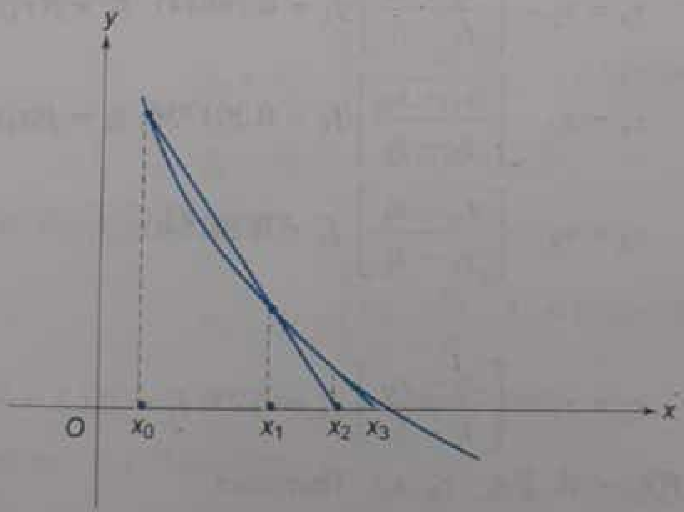

Fig. 2.2 (a). Secant method.

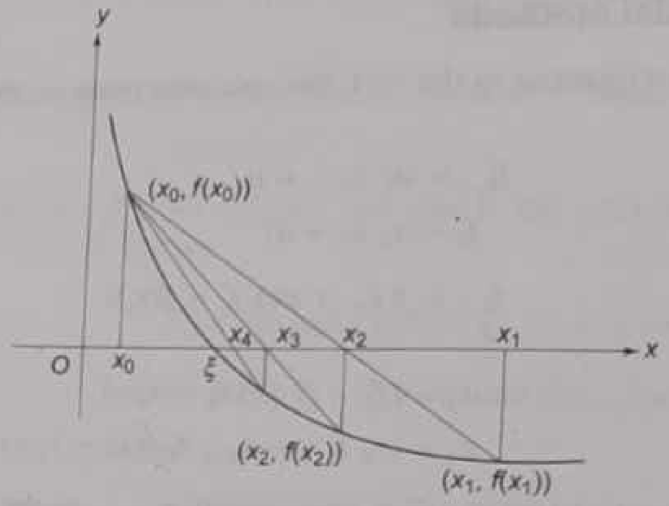

Fig. 2.2 (b). The Regula-Falsi method.

Example 2.5 A real root of the equation

$$f(x) = x^3 - 5x + 1 = 0$$

lies in the interval (0, 1). Perform four iterations of the secant method and the Regula-Falsi method to obtain this root.

We have

$$x_0 = 0,\ x_1 = 1,\ f_0 = f(x_0) = 1,\ f_1 = f(x_1) = -3.$$

Secant method

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f_1 - f_0}\right] f_1 = 0.25,\ f_2 = f(x_2) = -0.234375.$$

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f_2 - f_1}\right] f_2 = 0.186441,\ f_3 = f(x_3) = 0.074276.$$

$$x_4 = x_3 - \left[\frac{x_3 - x_2}{f_3 - f_2}\right] f_3 = 0.201736,\ f_4 = f(x_4) = -0.000470.$$

$$x_5 = x_4 - \left[\frac{x_4 - x_3}{f_4 - f_3}\right] f_4 = 0.201640.$$

Regula-Falsi method

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f_1 - f_0}\right] f_1 = 0.25,\ f_2 = f(x_2) = -0.234375.$$

Since $f(x_0)\, f(x_2) < 0$, $\xi \in (x_0, x_2)$. Therefore,

$$x_3 = x_2 - \left[\frac{x_2 - x_0}{f_2 - f_0}\right] f_2 = 0.202532,\ f_3 = f(x_3) = -0.004352.$$

Since $f(x_0)\, f(x_3) < 0$, $\xi \in (x_0, x_3)$. Therefore,

$$x_4 = x_3 - \left[\frac{x_3 - x_0}{f_3 - f_0}\right] f_3 = 0.201654,\ f_4 = f(x_4) = -\ 0.000070.$$

Since $f(x_0)\, f(x_4) < 0$, $\xi \in (x_0, x_4)$. Therefore,

$$x_5 = x_4 - \left[\frac{x_4 - x_0}{f_4 - f_0}\right] f_4 = 0.201640.$$

Example 2.6 Use the secant and Regula-Falsi methods to determine the root of the equation

$$\cos x - x\, e^x = 0.$$

Taking the initial approximations as $x_0 = 0$, $x_1 = 1$, we obtain for the secant method

$$f(0) = 1,$$

$$f(1) = \cos 1 - e = -\ 2.177979523$$

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f_1 - f_0}\right] f_1 = 0.3146653378$$

$$f_2 = f(x_2) = 0.519871175$$

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f_2 - f_1}\right] f_2 = 0.4467281466$$

$$f_3 = f(x_3) = 0.203544710$$

$$x_4 = x_3 - \left[\frac{x_3 - x_2}{f_3 - f_2}\right] f_3 = 0.5317058606.$$

Now, for the Regula-Falsi method, we get

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f_1 - f_0}\right] f_1 = 0.3146653378,$$

$$f_2 = f(x_2) = 0.519871175.$$

Since $f(x_1)\, f(x_2) < 0$, $\xi \in (x_1, x_2)$. Therefore,

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f_2 - f_1}\right] f_2 = 0.4467281466$$

$$f_3 = f(x_3) = 0.203544710.$$

Since $f(x_1)\, f(x_3) < 0$, $\xi \in (x_1, x_3)$. Therefore,

$$x_4 = x_3 - \left[\frac{x_3 - x_1}{f_3 - f_1}\right] f_3 = 0.4940153366.$$

The computed results are tabulated in Table 2.6.

Table 2.6 Approximations to the Root by the Secant and the Regula-Falsi Methods

k	Secant Method		Regula-Falsi Method	
	x_{k+1}	$f(x_{k+1})$	x_{k+1}	$f(x_{k+1})$
1	0.3146653378	0.519871	0.3146653378	0.519871
2	0.4467281466	0.203545	0.4467281446	0.203545
3	0.5317058606	−0.429311(−01)	0.4940153366	0.708023(−01)
4	0.5169044676	0.259276(−02)	0.5099461404	0.236077(−01)
5	0.5177474653	0.301119(−04)	0.5152010099	0.776011(−02)
6	0.5177573708	−0.215132(−07)	0.5169222100	0.253886(−02)
7	0.5177573637	0.178663(−12)	0.5174846768	0.829358(−03)
8	0.5177573637	0.222045(−15)	0.5176683450	0.270786(−03)
10	—	—	0.5177478783	0.288554(−04)
20	—	—	0.5177573636	0.396288(−09)

The numbers within the parentheses denote exponentiation.

Newton-Raphson Method

We determine a_0 and a_1 in (2.8) using the conditions

$$f_k = a_0 x_k + a_1$$
$$f'_k = a_0 \tag{2.13}$$

where a prime denotes differentiation with respect to x.

On substituting a_0 and a_1 from (2.13) in (2.9) and representing the approximate value of x by x_{k+1}, we obtain

$$x_{k+1} = x_k - \frac{f_k}{f'_k},\ k = 0, 1, \ldots \tag{2.14}$$

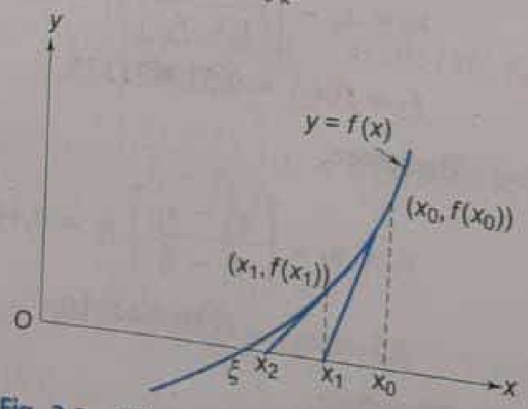

Fig. 2.3. The Newton-Raphson method.

This method is called the **Newton-Raphson** method. The method (2.14) may also be obtained directly from (2.12) by taking the limit $x_{k-1} \to x_k$. In the limit when $x_{k-1} \to x_k$, the chord passing through the points (x_k, f_k) and (x_{k-1}, f_{k-1}) becomes the tangent at the point (x_k, f_k). Thus, in this case the problem of finding the root of the equation (2.1) is equivalent to finding the point of intersection of the tangent to the curve $y = f(x)$ at the point (x_k, f_k) with the x-axis. The method is shown graphically in Fig. 2.3. The Newton-Raphson method requires two evaluations f_k, f'_k for each iteration.

Alternative

Let x_k be an approximation to the root of the equation $f(x) = 0$. Let Δx be an increment in x such that $x_k + \Delta x$ is an exact root. Therefore,

$$f(x_k + \Delta x) \equiv 0.$$

Expanding in Taylor series about the point x_k, we get

$$f(x_k) + \Delta x\, f'(x_k) + \frac{1}{2!}(\Delta x)^2 f''(x_k) + \cdots = 0.$$

Neglecting the second and higher powers of Δx, we obtain

$$f(x_k) + \Delta x\, f'(x_k) = 0$$

or

$$\Delta x = -\frac{f(x_k)}{f'(x_k)}.$$

Hence, we obtain the iteration method

$$x_{k+1} = x_k + \Delta x = x_k - \frac{f(x_k)}{f'(x_k)},\ k = 0, 1, \cdots$$

which is same as (2.14).

Example 2.7 Perform four iterations of the Newton-Raphson method to find the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0.$$

The smallest positive root lies in the interval (0, 1). Take the initial approximation as $x_0 = 0.5$. We have

$$f(x) = x^3 - 5x + 1,\ f'(x) = 3x^2 - 5.$$

Using the Newton-Raphson method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

we get

$$x_{k+1} = x_k - \frac{x_k^3 - 5x_k + 1}{3x_k^2 - 5} = \frac{2x_k^3 - 1}{3x_k^2 - 5},\ k = 0, 1, \cdots$$

Starting with $x_0 = 0.5$, we obtain

$$x_1 = 0.176471,\ x_2 = 0.201568.$$

$$x_3 = 0.201640,\ x_4 = 0.201640.$$

The exact value correct to six decimal places is 0.201640.

Chapter 2

Example 2.8 Perform four iterations of the Newton-Raphson method to obtain the approximate value of $(17)^{1/3}$ starting with the initial approximation $x_0 = 2$.

Let $x = (17)^{1/3}$. We obtain $x^3 = 17$ and $f(x) = x^3 - 17 = 0$. Using the Newton-Raphson method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \; k = 0, 1, \cdots$$

we get

$$x_{k+1} = x_k - \frac{x_k^3 - 17}{3x_k^2} = \frac{2x_k^3 + 17}{3x_k^2}, \; k = 0, 1, \cdots$$

Starting with $x_0 = 2$, we obtain

$$x_1 = \frac{2x_0^3 + 17}{3x_0^2} = 2.75, \qquad x_2 = \frac{2x_1^3 + 17}{3x_1^2} = 2.582645$$

$$x_3 = \frac{2x_2^3 + 17}{3x_2^2} = 2.571332, \quad x_4 = \frac{2x_3^3 + 17}{3x_3^2} = 2.571282.$$

The exact value correct to six decimal places is 2.571282.

Example 2.9 Apply Newton-Raphson's method to determine a root of the equation

$$f(x) = \cos x - xe^x = 0$$

such that $|f(x^*)| < 10^{-8}$, where x^* is the approximation to the root. Take the initial approximation as $x_0 = 1$.

We write (2.14) in the form

$$x_{k+1} = x_k - \Delta x_k, \; k = 0, 1, 2, \cdots$$

where $$\Delta x_k = \frac{f(x_k)}{f'(x_k)} = \frac{(\cos x_k - x_k e^{x_k})}{(-\sin x_k - x_k e^{x_k} - e^{x_k})}$$

Starting with $x_0 = 1$, we get

$$\Delta x_0 = \frac{\cos x_0 - x_0 e^{x_0}}{-\sin x_0 - x_0 e^{x_0} - e^{x_0}} = \frac{-2.17797952}{-6.27803464} = 0.34692060$$

$$x_1 = x_0 - \Delta x_0 = 1 - 0.34692060 = 0.65307940$$

$$\Delta x_1 = \frac{\cos x_1 - x_1 e^{x_1}}{-\sin x_1 - x_1 e^{x_1} - e^{x_1}} = \frac{-0.46064211}{-3.78394215} = 0.12173603$$

$$x_2 = x_1 - \Delta x_1 = 0.53134337.$$

The results obtained are given in Table 2.7.

Table 2.7 Approximations to the Root by the Newton-Raphson Method

k	x_k	Δx_k	x_{k+1}	$f(x_{k+1})$
0	1.0	0.3469	0.65307940	– 0.4606
1	0.65307940	0.1217	0.53134337	– 0.4180(– 1)
2	0.53134337	0.1343(– 1)	0.51790991	– 0.4641(– 3)
3	0.51790991	0.1525(– 3)	0.51775738	– 0.5926(– 7)
4	0.51775738	0.1948(– 7)	0.51775736	– 0.2910(– 10)

Example 2.10 Show that the initial approximation x_0 for finding $1/N$, where N is a positive integer, by the Newton-Raphson method must satisfy $0 < x_0 < 2/N$, for convergence.

We write
$$f(x) = \frac{1}{x} - N = 0.$$

The Newton-Raphson method becomes
$$x_{n+1} = 2x_n - N x_n^2.$$

Let us now draw the graphs of $y = x$ and $y = 2x - Nx^2$. The second curve is the parabola
$$\left(x - \frac{1}{N}\right)^2 = -\frac{1}{N}\left(y - \frac{1}{N}\right).$$

The graphs are given in Fig. 2.4. The point of intersection of these two curves is the required value $1/N$. From Fig. 2.4, we find that for any initial approximation outside the range $0 < x_0 < 2/N$, the method diverges. If $x_0 = 0$, the iterations do not converge to $1/N$ but remain zero always. This shows the importance of choosing a suitable initial approximation.

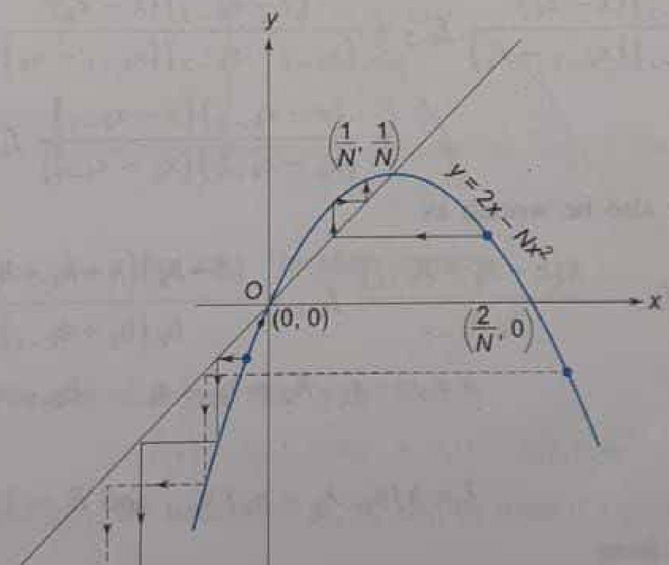

Fig. 2.4. Choice of the initial approximation.

2.4 ITERATION METHODS BASED ON SECOND DEGREE EQUATION

We assume for the function $f(x)$ a polynomial of degree two and write as
$$f(x) = a_0 x^2 + a_1 x + a_2 = 0, \; a_0 \neq 0 \tag{2.15}$$

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$$x_1^* = x_0 - \frac{f_0}{f_0'} = 0.65307940,\ f_1^* = -0.46064211$$

$$x_1 = x_1^* - \frac{f_1^*}{f_0'} = 0.57970578$$

$$f_1 = -0.19844837,\ f_1' = -3.36836305$$

$$x_2^* = x_1 - \frac{f_1}{f_1'} = 0.52079041,\ f_2^* = -0.00925036$$

$$x_2 = x_2^* - \frac{f_2^*}{f_1'} = 0.51804416$$

$$f_2 = -0.00087268,\ f_2' = -3.04358498$$

$$x_3^* = x_2 - \frac{f_2}{f_2'} = 0.51775743,\ f_3^* = -0.00000002$$

$$x_3 = x_3^* - \frac{f_3^*}{f_2'} = 0.51775736.$$

2.5 RATE OF CONVERGENCE

We now study the rate at which the iteration method converges if the initial approximation to the root is sufficiently close to the desired root.

Definition 2.4 An iterative method is said to be of **order** p or has the rate of **convergence** p, if p is the largest positive real number for which there exists a finite constant $C \neq 0$ such that

$$|\varepsilon_{k+1}| \leq C|\varepsilon_k|^p \tag{2.34}$$

where $\varepsilon_k = x_k - \xi$ is the error in the kth iterate.

The constant C is called the **asymptotic error constant** and usually depends on derivatives of $f(x)$ at $x = \xi$.

Secant Method

We assume that ξ is a simple root of $f(x) = 0$. Substituting $x_k = \xi + \varepsilon_k$ in (2.12) we obtain

$$\varepsilon_{k+1} = \varepsilon_k - \frac{(\varepsilon_k - \varepsilon_{k-1}) f(\xi + \varepsilon_k)}{f(\xi + \varepsilon_k) - f(\xi + \varepsilon_{k-1})} \tag{2.35}$$

Expanding $f(\xi + \varepsilon_k)$ and $f(\xi + \varepsilon_{k-1})$ in Taylor's series about the point ξ and noting that $f(\xi) = 0$, we get

$$\varepsilon_{k+1} = \varepsilon_k - \frac{(\varepsilon_k - \varepsilon_{k-1})\left[\varepsilon_k f'(\xi) + \frac{1}{2}\varepsilon_k^2 f''(\xi) + \cdots\right]}{(\varepsilon_k - \varepsilon_{k-1}) f'(\xi) + \frac{1}{2}(\varepsilon_k^2 - \varepsilon_{k-1}^2) f''(\xi) + \cdots}$$

$$= \varepsilon_k - \left[\varepsilon_k + \frac{1}{2}\varepsilon_k^2 \frac{f''(\xi)}{f'(\xi)} + \cdots\right]\left[1 + \frac{1}{2}(\varepsilon_{k-1} + \varepsilon_k)\frac{f''(\xi)}{f'(\xi)} + \cdots\right]^{-1}$$

or

$$\varepsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k \varepsilon_{k-1} + O\left(\varepsilon_k^2 \varepsilon_{k-1} + \varepsilon_k \varepsilon_{k-1}^2\right)$$

or

$$\varepsilon_{k+1} = C\varepsilon_k \varepsilon_{k-1} \tag{2.36}$$

where $C = \dfrac{1}{2}\dfrac{f''(\xi)}{f'(\xi)}$ and higher powers of ε_k are neglected.

The relation of the form (2.36) is called the **error equation**. Keeping in view the definition of the rate of convergence, we seek a relation of the form

$$\varepsilon_{k+1} = A\varepsilon_k^p \tag{2.37}$$

where A and p are to be determined.

From (2.37) we have

$$\varepsilon_k = A\varepsilon_{k-1}^p \text{ or } \varepsilon_{k-1} = A^{-1/p}\,\varepsilon_k^{1/p}$$

Substituting the values of ε_{k+1} and ε_{k-1} in (2.36) we obtain

$$\varepsilon_k^p = CA^{-(1+1/p)}\,\varepsilon_k^{1+1/p}. \tag{2.38}$$

Comparing the powers of ε_k on both sides, we get

$$p = 1 + \frac{1}{p}$$

which gives

$$p = \frac{1}{2}\left(1 \pm \sqrt{5}\right).$$

Neglecting the minus sign, we find that the rate of convergence for the secant method (2.12) is $p = 1.618$.

From (2.38), we also obtain $A = C^{p/(p+1)}$.

Regula-Falsi Method

If the function $f(x)$ in the equation $f(x) = 0$ is convex in the interval (x_0, x_1) that contains the root, then one of the points x_0 or x_1 is always fixed and the other point varies with k. If the point x_0 is fixed, then the function $f(x)$ is approximated by the straight line passing through the points (x_0, f_0) and (x_k, f_k), $k = 1, 2, \ldots$. The error equation (2.36) becomes

$$\varepsilon_{k+1} = C\,\varepsilon_0\,\varepsilon_k$$

where $C = \dfrac{1}{2}\dfrac{f''(\xi)}{f'(\xi)}$ and $\varepsilon_0 = x_0 - \xi$ is independent of k. Therefore, we can write

$$\varepsilon_{k+1} = C^* \varepsilon_k \tag{2.39}$$

where $C^* = C\varepsilon_0$ is the asymptotic error constant. Hence, the Regula-Falsi method has linear rate of convergence.

Newton-Raphson Method

On substituting $x_k = \xi + \varepsilon_k$ in (2.14) and expanding $f(\xi + \varepsilon_k), f'(\xi + \varepsilon_k)$ in Taylor's series about the point ξ, we obtain

$$\varepsilon_{k+1} = \varepsilon_k - \frac{\left[\varepsilon_k f'(\xi) + \frac{1}{2}\varepsilon_k^2 f''(\xi) + \cdots\right]}{f'(\xi) + \varepsilon_k f''(\xi) + \cdots}$$

$$= \varepsilon_k - \left[\varepsilon_k + \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 + \cdots\right]\left[1 + \frac{f''(\xi)}{f'(\xi)}\varepsilon_k + \cdots\right]^{-1}$$

$$\varepsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 + O(\varepsilon_k^3).$$

On neglecting ε_k^3 and higher powers of ε_k, we get

$$\varepsilon_{k+1} = C\,\varepsilon_k^2 \tag{2.40}$$

where $$C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}.$$

Thus, the Newton-Raphson method has second order convergence.

Muller Method

On substituting $x_j = \xi + \varepsilon_j$, $j = k-2, k-1, k$ and expanding $f(\xi + \varepsilon_j)$ in Taylor's series about the point ξ in (2.25a) and using $f(\xi) = 0$, we get

$$D = (\varepsilon_k - \varepsilon_{k-2})(\varepsilon_k - \varepsilon_{k-1})(\varepsilon_{k-1} - \varepsilon_{k-2})$$

$$a_2 = \varepsilon_k f'(\xi) + \frac{1}{2}\varepsilon_k^2 f''(\xi) + \frac{1}{6}\varepsilon_k^3 f'''(\xi) + \cdots$$

$$\begin{aligned} a_1 = \frac{1}{D}\Big[& (\varepsilon_k - \varepsilon_{k-2})^2 \Big\{(\varepsilon_k - \varepsilon_{k-1}) f'(\xi) + \frac{1}{2}(\varepsilon_k^2 - \varepsilon_{k-1}^2) f''(\xi) \\ & + \frac{1}{6}(\varepsilon_k^3 - \varepsilon_{k-1}^3) f'''(\xi) + \cdots\Big\} \\ & - (\varepsilon_k - \varepsilon_{k-1})^2 \Big\{(\varepsilon_k - \varepsilon_{k-2}) f'(\xi) + \frac{1}{2}(\varepsilon_k^2 - \varepsilon_{k-2}^2) f''(\xi) \\ & + \frac{1}{6}(\varepsilon_k^3 - \varepsilon_{k-2}^3) f'''(\xi) + \cdots\Big\}\Big] \end{aligned}$$

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