The Geometry of Linear Equations

Solve system of linear equations

n equation, n unknown

Ex:-> $2x - y = 0$

$-x + 2y = 3$

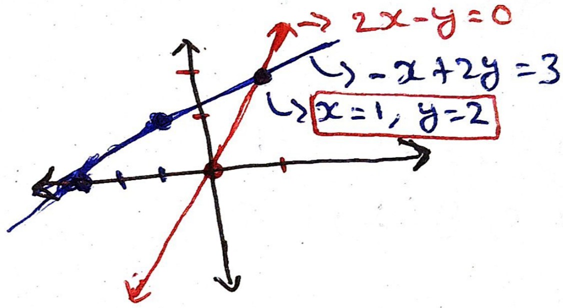
$$\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A \cdot x = b$$

Row Picture (2x2 case)

Geometric view: each equation is a line or plane

$2x - y = 0$	$-x + 2y = 3$
$x = 0, y = 0$	$x = -3; y = 0$
$x = 1, y = 2$	$x = -1, y = 1$



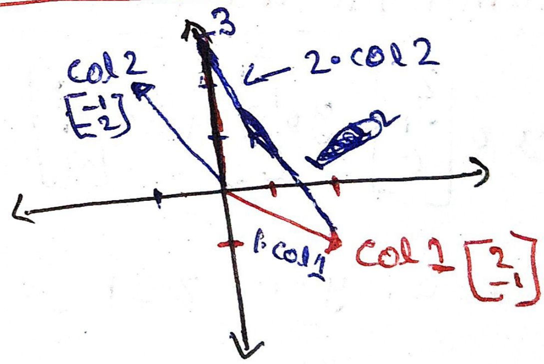
Solution :-> where the two line intersect

The column Picture (Linear combination of vectors)

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Linear combination of columns

$x = 1, y = 2$



* The eqn $Ax = b$ says:

Find numbers $x_1, x_2, \ldots$ so that

$x_1 \cdot (\text{col } 1) + x_2 \cdot (\text{col } 2) + \cdots = b$

↳ This is called linear combination of the columns of A.

Ex → (3×3 case)

$2x - y = 0$

$-x + 2y - z = -1$

$-3y + 4z = 4$

matrix form

$Ax = b$

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}}_{b}$$

A = The coefficient matrix
x = vectors of unknow
b = righ hand side vector

Ax = Take a linear combination of the columns of A weighted by x.

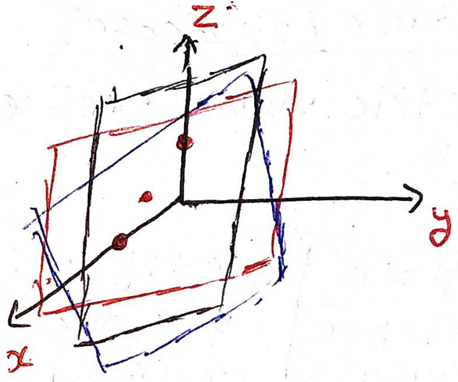
Row pictures (3×3 case)

$-x + 2y - z = -1$

$x = 1,\ y = 0,\ z = 0$

$x = 0,\ y = 0,\ z = 1$

$x = 0,\ y = -\frac{1}{2},\ z = 0$

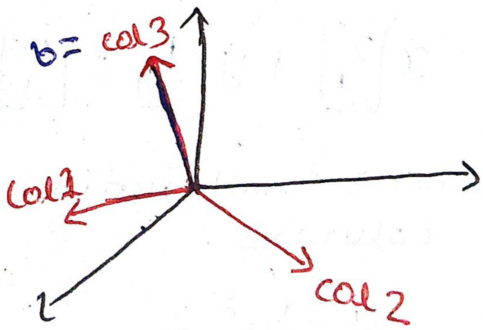


Solution → where three planes intersect.

→ each eqn gives us a plane.

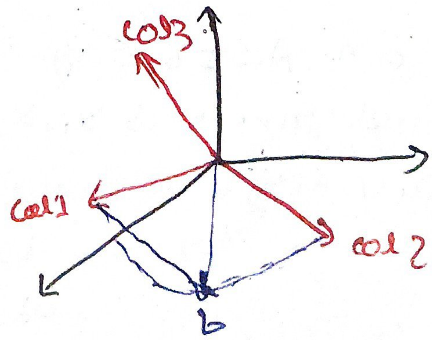
column picture (3×3 case)

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$



$x = 0,\ y = 0,\ z = 1$

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$



$x = 1,\ y = 1,\ z = 0$

Q-> Can I solve $Ax = b$ for every b?

or

Q-> Do the linear combs of the columns fill 3d space?

Ans-> For this A, answer is yes

When it works vs when it fails

-> If the columns ~~are~~ of A are independent (not in the same line / plane), their combinations fills the whole space. Then we can solve $Ax = b$ for any b. This is a non singular (invertible) matrix.

-> If some columns is a combination of others, you can only reach a plane / line inside the space: many b's are impossible. This is singular (non invertible) matrix.

Two ways to multiply a matrix by a vector

$Ax = b$

Column view

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

$\boxed{Ax \text{ is a comb of columns of } A}$

Row view

Dot each row of A with x

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\times1 + 5\times2 \\ 1\times1 + 3\times2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Q-> Solve $\begin{cases} 2x+y=3 \\ x-2y=-1 \end{cases}$ and find out its row picture and column picture.

Sol:->

$x-2y=-1$

$x=-1+2y$

$x=-1+2\times 1$

$\boxed{x=1}$

$2x+y=3$

$2(-1+2y)+y=3$

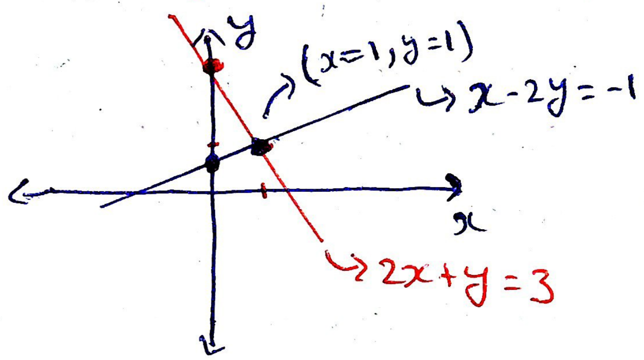
$-2+4y+y=3$

$5y=5$

$\boxed{y=1}$

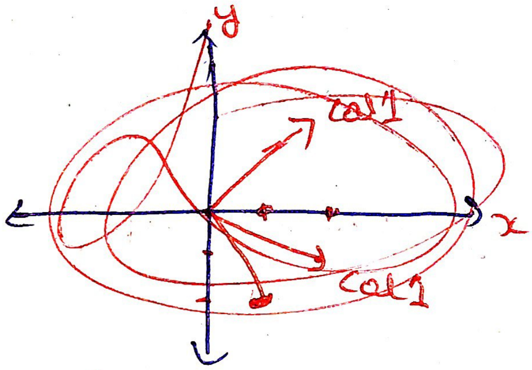
Row Picture

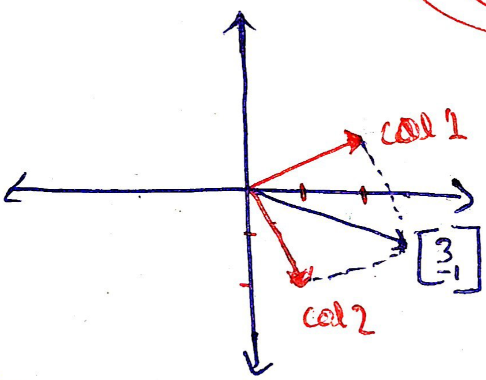
$2x+y=3$	$x-2y=-1$
$x=0, y=3$	$x=0, y=\frac{1}{2}$
$x=1, y=1$	$x=1, y=1$



Column Picture

$$x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}3\\-1\end{bmatrix}$$





matrix form

$$A=\begin{bmatrix}v_1 & v_2\end{bmatrix}=\begin{bmatrix}2 & 1\\1 & -2\end{bmatrix}$$

$$\begin{bmatrix}2 & 1\\1 & -2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\-1\end{bmatrix}$$

$ax=b \quad x=\frac{b}{a}=a^{-1}b$

$$A^{-1}A=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=A^{-1}\begin{bmatrix}3\\-1\end{bmatrix}$$