Riddle: n people that are connected to each other using 2 ways of communication; phone and mail, prove that they can decode only one of these two ways and still remain connected

Solution: Let's denote the people as vertices, the communication relations are the edges between them, and the type of communication (wither phone/email) resembles the way of communication.

Now, we shall prove by contradiction that there are two people, u and v, that after cancelling one way of communication (without loss of generality), phone, are inaccessible to each other.

After removing the phone edges, there is no path between u and v, therefore they are in different connectivity components after removing the phone edges between them, and the edge between them is a phone edge.

If we were to remove the mail edges, u and v were in the same connectivity component. Otherwise, if they were on different connectivity components, we shall deduce that they were not connected in the original graph, but it is a fully-connected graph – therefore the contradiction.

Therefore, we conclude that before removing the phone edges, all the rest of the vertices were connected to either u or v in a phone edge.

If that were not true, there has been a vertex connected to both of them in an email edge, and it was possible to reach through him from u to v in contradiction to the initial assumption.

To sum up, all vertices, besides u and v, are connected to either of them in a phone edge, and both u and v are connected to each other in a phone edge.

The subgraph of phone is a connected graph.

Riddle: 20 people, all of them are connected to each other, 18 connections are removed, prove that the graph is still connected

Solution I: We shall prove that it is possible to remove 18 edges from the graph and the graph would remain a connected graph. Let's assign red color to k vertices and blue color to the rest. The number of edges between red and blue vertices is $k \cdot (20 - k)$. It is easy to see that the minimal value of the number of edges is when $k = 1$, therefore a cut of 19 edges between one vertex to the rest of 19 vertices. For any value of k between 1 and 19, we shall produce two full graphs of k and (20-k) vertices with the same color, and $k \cdot (20 - k)$ edges with different colors in each side. We can still remove 18 of the edges with different colors, and remain with one edge at least with different colors on each side that will connect the two full subgraphs, therefore the graph remains a connected graph.

Solution II: Corollary: If G is a full graph with n vertices, then G remains a connected graph after removing n-2 edges at most. Proof with induction on n: *base:* for $n \leq 2$ it is trivial, no edge is removed and graph remains a connected graph. *step:* we shall assume that the claim is correct for n (therefore the graph yielded from the fully graph with n vertices, after removing n-2 edges, is a connected graph) – and we shall prove that claims is correct for $n + 1$, therefore the graph G yielded from the fully graph (with $n + 1$ vertices) after removing up to n-1 edges is a connected graph. Let denote v a vertex in graph G; if $d(v) = n$, then the vertex id connected to the other vertices and graph is a connected graph. Let's assume that there is a vertex v, to which $d(v) < n$, therefore there is at least one edge that has been removed and v is in one of it's ends.

Let H be the graph after removing vertex v and it's edges from graph G. Therefore H is produced from the full graph G (with n vertices) after removing $n - 2$ edges. According to the induction claim, H is a connected graph. To show that G is a connected graph, it's enough to prove that there is at least one edge, vw, when w is vertex in graph H, and it is true since $d(v) \geq n - (n - 1) = 1$. Therefore G is a connected graph.