Linear Algebra Review

ORIE 4741

September 15, 2019

Outline

Linear Independence and Dependence

Definition

The sequence of vectors $v_1, v_2, ..., v_n$ is **linearly independent** if the only combination that gives the zero vector is $0v_1 + 0v_2 + ... + 0v_n$.

Example

The vectors [1, 0, 0], [0, 1, 0], and [0, 0, 1] are linearly independent

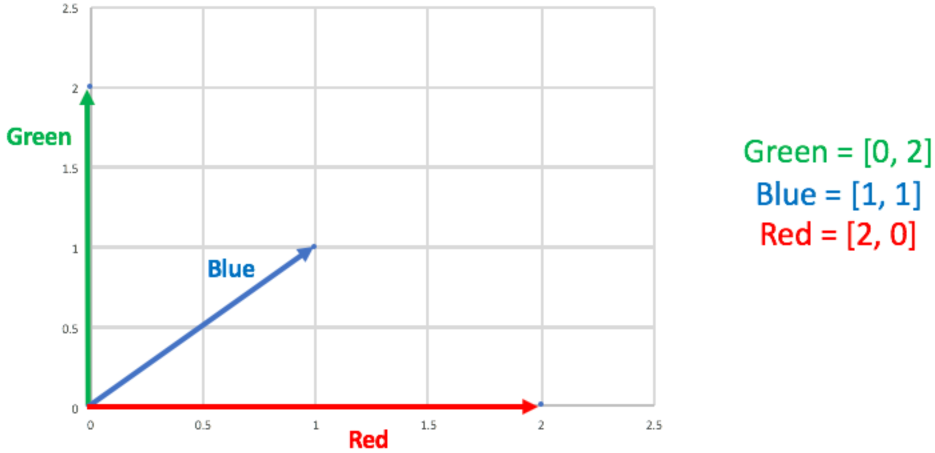
Definition

The sequence of vectors $v_1, v_2, ..., v_n$ is **linearly dependent** if there exists a combination that gives the zero vector other than $0v_1 + 0v_2 + ... + 0v_n$.

Example

The vectors [1, 2, 0], [2, 4, 0], and [0, 0, 1] are linearly dependent because $-2[1, 2, 0] + 1[2, 4, 0] + 0[0, 0, 1] = [0, 0, 0]$.

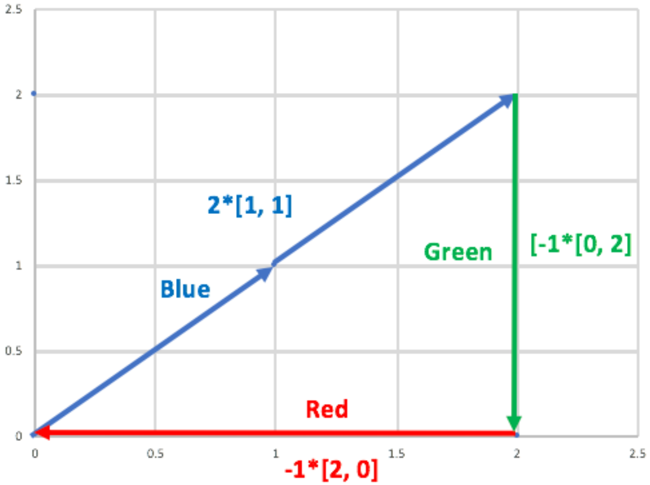
Can you find a linear combination of these vectors that equals 0?

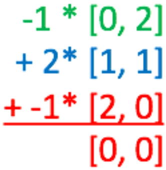


Linear Dependence

Geometric Definition

Any multiple of $-1 * \mathit{Green} + 2 * \mathit{Blue} + -2 * \mathit{Red} = 0$





Matrix Rank

Row Rank

Definition

The **row rank** of a matrix is the number of linearly independent rows in a matrix.

Example

Each row in matrix A is linearly independent, so row rank(A) $= 3$. Two rows in matrix B are linearly independent, so row rank(B) $= 2$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

Column Rank

Definition

The **column rank** of a matrix is the number of linearly independent columns in a matrix.

Example

Each column in matrix A is linearly independent, so row rank(A) = 3. The first two columns in matrix B are linearly independent, so row rank(B) = 2.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

Did you notice the row ranks and the column ranks for the matrices were the same?

Definition

The **rank** of a matrix is the number of linearly independent rows (columns) in a matrix.

Full-Rank

Definition

A matrix B is **full-rank** if:

$$rank(B) = min\{\#\ columns\ in\ B,\ \#\ of\ rows\ in\ B\}$$

Example

Both A and B are full-rank.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Invertible Matrices

Matrix Multiplication

Multiply A*B where:

$$A= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \; B= \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Matrix Multiplication

Multiply A*B where:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1*5)+(2*7) & (1*6)+2*8 \\ (3*5)+(4*7) & (3*6)+4*8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Identity Matrix

Definition

The **identity matrix**, I, is an nxn matrix with 1s on the diagonal and 0s everywhere else with the property that $\mathrm{I}x = x\mathrm{I} = x$ for any vector x ($\mathrm{I}A = A\mathrm{I} = A$ for any matrix A).

Example

An example of the 3x3 identity matrix is:

$$\mathrm{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition of Invertible Matrix

Definition

The matrix A is **invertible** if there exists a matrix A^{-1} such that

- $A^{-1}A = \mathrm{I}$
- $AA^{-1} = \mathrm{I}$

Note: An invertible matrix should be square (same number of rows and columns) and have full-rank.

Invertible Matrix Example

Let $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$.

Then, the inverse of A is: $A^{-1} = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$.

Check that A^{-1} satisfies the definition of an inverse:

$$AA^{-1} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ Therefore, A^{-1} is the inverse of A.

Norms

Vector Norm

Definition

The **norm** of a vector x, denoted $||x||$ or $||x||_2$, is the length of x:

$$||x|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

Example

$$\text{Let x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \text{ then}$$

$$||x|| = \sqrt{(1^2 + 2^2 + 3^2 + 4^2)} = \sqrt{1 + 4 + 9 + 16} = \sqrt{30}$$

Projection Matrix

Span

Definition

Let A be a matrix. The *A***span** of the columns of A is the set of all linear combinations of the columns of A.

Example

The rows of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ spans all of $\mathbb{R}^2$.

Projection Matrix

Definition

A square matrix P is the projection matrix onto the *span*(*columns of* A) if

$$P^2 = P.$$

Example

$P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is a projection matrix because $P^2 = P$.

Projection Matrix Example Continued

Example

$P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is a projection matrix that projects onto $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$P \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$P \begin{bmatrix} 2 \\ 27 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$P \begin{bmatrix} 254 \\ -1000 \end{bmatrix} = \begin{bmatrix} 627 \\ -627 \end{bmatrix} = 627 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Helpful Matrix Hints

Helpful Matrix Properties and Identities

- $AB \neq BA$
- $A^T B^T = (BA)^T$
- $A(B + C) = AB + AC$
- $(AB)C = A(BC)$
- $\mathrm{I}A = A\mathrm{I} = \mathrm{A}$
- $AA^{-1} = A^{-1}A = \mathrm{I}$

Gradients

Multivariate Derivatives

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ $\frac{\partial f}{\partial z}$, where

$$f(x, y, z) = 5x + 2x^2 + xy + 3xy^2z - 4y^3 + 2yz + 4z^2 - x^4yz^2.$$

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$\frac{\partial f}{\partial x} = 5 + 4x + y + 3y^2z - 4x^3yz^2$

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$$\frac{\partial f}{\partial x} = 5 + 4x + y + 3y^2z - 4x^3yz^2$$

$$\frac{\partial f}{\partial y} = x + 6xyz - 12y^2 + 2z - x^4z^2$$

Multivariate Derivatives

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$\frac{\partial f}{\partial x} = 5 + 4x + y + 3y^2z - 4x^3yz^2$

$\frac{\partial f}{\partial y} = x + 6xyz - 12y^2 + 2z - x^4z^2$

$\frac{\partial f}{\partial z} = 3xy^2 + 2y + 8z - 2x^4yz4$

Gradient

Definition

Let $f(x_1, x_2, ..., x_n)$ be a multivariable function. The **gradient of f**, ∇f, is the multivariable generalization of the derivative of f. The gradient is a vector, where each row corresponds to a partial derivative with respect to a variable of the function.

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Example Gradient

Let $f(x, y, z) = x^2 + 3xy + 4xyz + z$.

Then ∇f is:

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x + 3y + 4yz \\ 3x + 4xz \\ 4xy + 1 \end{bmatrix}$$

Computational Complexity Notation

Big O Notation

Big O notation is used to describe the run-time (computational complexity) of algorithms.

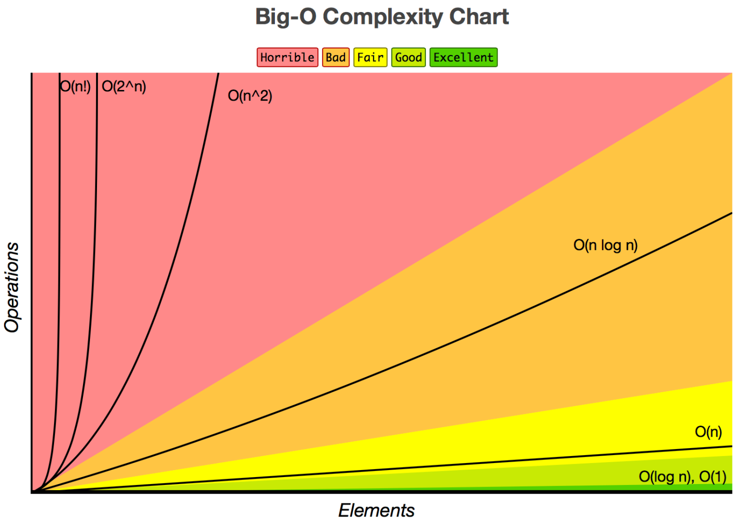
Definition

Let f(n) be the run-time of some algorithm. If f(n) = O(g(n)), then there exists a constant C and a constant N such that:
$|f(x)| \leq C * |g(x)|$ for all $x > N$.

Example

When an algorithm runs in time O(n). That means it runs in linear time. So, where n is the amount of data, if an algorithm runs in time 5n, 1000n+50000, n, or n-45, the algorithm runs in time O(n).

Big O Notation



http://bigocheatsheet.com

References

Strang, Gilbert. (2009, 4th ed.). Introduction to Linear Algebra.

Ling-Hsiao Ly's 2012 Lecture 3 Projection and Projection Matrices Notes
http://www.ss.ncu.edu.tw/ lyu/lecture$_f iles_e n/lyu_L A_N otes/Lyu_L A_2 012/Lyu_L A_{3}{}_{2}012.pdf$