CHAPTER 2 DYNAMIC MODELS AND DYNAMIC RESPONSE

2.1 Revisiting Examples 2.4 and 2.5 will be helpful.

(a) $R(s) = A; \quad Y(s) = \dfrac{KA}{\tau s + 1}$

$$y(t) = \frac{KA}{\tau} e^{-t/\tau}; \quad y_{ss} = 0$$

(b) $R(s) = \dfrac{A}{s}; \quad Y(s) = \dfrac{KA}{s(\tau s + 1)}$

$$y(t) = KA(1 - e^{-t/\tau}); \qquad y_{ss} = KA$$

(c) $R(s) = \dfrac{A}{s^2}; \quad Y(s) = \dfrac{KA}{s^2(\tau s + 1)}$

$$y(t) = KA(t - \tau + \tau e^{-t/\tau}); \qquad y_{ss} = KA(t - \tau)$$

(d) $R(s) = \dfrac{A\omega}{s^2 + \omega^2}; \quad \mathrm{Y(s)} = \dfrac{KA\omega}{(\tau s + 1)(s^2 + \omega^2)}$

$$y(t) = \frac{KA\omega\tau}{\tau^2\omega^2 + 1} e^{-t/\tau} + \frac{KA}{\sqrt{\tau^2\omega^2 + 1}} \sin(\omega t + \theta)$$

$$\theta = \tan^{-1}(-\omega\tau)$$

$$y(t)\Big|_{t\to\infty} = \frac{KA}{\sqrt{\tau^2\omega^2 + 1}} \sin(\omega t + \theta)$$

2.2 (a) The following result follows from Eqns (2.57 - 2.58)

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + 2\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

The final value theorem is applicable: the function $sY(s)$ does not have poles on the $j\omega$-axis and right half of s-plane.

$$y_{ss} = \lim_{s\to 0} sY(s) = 1$$

(b) The following result follows from Review Examples 2.2.

$$y(t) = t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{\omega_d} \sin\left(\omega_d t + 2\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

$$y_{ss} = t - \frac{2\zeta}{\omega_n}$$

The final value theorem is not applicable; the function $sY(s)$ has a pole on the $j\omega$-axis.

2.3 Revisiting Review Example 2.1 will be helpful.

$$\frac{E_0(s)}{E_i(s)} = \frac{R/10^4}{RCs+1}\ ;\ E_i(s) = \frac{10}{s^2}$$

$$e_0(t) = 10 \times \frac{R}{10^4}(t - \tau + \tau e^{-t/\tau});\ \tau = RC$$

For the output to track the input with a steady-state delay 100×10^{-6} sec, it is necessary that

$$\frac{R}{10^4} = 1, \text{ and } \tau = RC = 100 \times 10^{-6}$$

This gives $R = 10\ k\Omega$; $C = 0.1\ \mu\text{F}$

Steady-state error = $10\,t$ - $(10\,t - 10\,\tau)$

= 0.001

2.4 Revisiting Review Example 2.2 will be helpful.

$\omega_n = 100$; $\zeta = 3$; $2\zeta / \omega_n = 6/100$ sec = 60 msec

$$\text{Steady-state error} = 25t - \left(25t - 25 \times \frac{2\zeta}{\omega_n}\right)$$

= 1.5

2.5 $M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = F(t) = 1000\mu(t)$

$$Y(s) = \frac{1}{s(s^2 + 10s + 100)}$$

$\omega_n = 10$, $\zeta = 0.5$, $\omega_d = 5\sqrt{3}$

Using Eqns (2.57)-(2.58), we obtain

$$y(t) = 0.01\left[1 - \frac{2}{\sqrt{3}} e^{-5t} \sin(5\sqrt{3}\,t + \tan^{-1}\sqrt{3}\right]$$

2.6 $$\frac{Y(s)}{R(s)} = G(s) = \frac{6}{s^2 + 7s + 6} = \frac{6}{(s+1)(s+6)}$$

$$|G(j\omega)|_{\omega=2} = \frac{3}{5\sqrt{2}};\ \angle G(j2) = -\ 81.87^{o}$$

$$y_{ss} = \frac{3}{5\sqrt{2}} \sin(2t - 81.87^{o})$$

$$= \frac{3}{50} \sin 2t - \frac{21}{50} \cos 2t$$

2.7 $$\frac{Y(s)}{R(s)} = G(s) = \frac{s+3}{s^2 + 7s + 10} = \frac{s+3}{(s+2)(s+5)}$$

(a) $R(s) = \dfrac{1}{s+1}$

$$y(t) = \left(\frac{1}{2} e^{-t} - \frac{1}{3} e^{-2t} - \frac{1}{6} e^{-5t}\right)\mu(t)$$

(b) $(s^2 + 7s + 10)\ Y(s) - sy(0) - \dot{y}(0) - 7y(0)$

$= (s + 3)\ R(s) - sr(0)$

Initial conditions before application of the input are

$$y(0^{-}) = 1,\ \dot{y}(0^{-}) = \frac{1}{2},\ r(0^{-}) = 0$$

$$Y(s) = \frac{s + 15/2}{(s+2)(s+5)} + \frac{s+3}{(s+2)(s+5)(s+1)}$$

$$y(t) = \frac{1}{2} e^{-t} + \frac{3}{2} e^{-2t} - e^{-5t}$$

2.8 $X = \bar{X} + x;\ I = \bar{I} + i$

$$M \frac{d^2(\bar{X} + x)}{dt^2} = F(\bar{X} + x,\ \bar{I} + i) - Mg$$

$$M\ddot{x} = F(\overline{X} + \overline{I}) + \left(\frac{\partial F}{\partial X}\bigg|_{\overline{X},\overline{I}}\right)x + \left(\frac{\partial F}{\partial I}\bigg|_{\overline{X},\overline{I}}\right)i - Mg$$

$$F(\overline{X} + \overline{I}) = Mg = 8.4 \times 10^{-3} \times 9.8 \text{ Newtons}$$

For this value of force, we get from Fig. 2.8b,

$$\overline{X} = 0.27\text{cm};\ \overline{I} = 0.6 \text{ amps}$$

Again from Fig. P 2.8b,

$$K_1 = \frac{\partial F}{\partial X}\bigg|_{\overline{X},\overline{I}} = 0.14 \text{ Newtons/cm}$$

$$K_2 = \frac{\partial F}{\partial I}\bigg|_{\overline{X},\overline{I}} = 0.4 \text{ Newtons/amp}$$

$$\ddot{x} = \frac{K_1}{M}x + \frac{K_2}{M}i;\ \frac{X(s)}{I(s)} = \frac{47.6}{s^2 - 16.67}$$

2.9 $M\ddot{x} + B\dot{x} + Kx = K_1(y - x)$

Gravitational effect has been eliminated by appropriate choice of the zero position.

$$\frac{X(s)}{Y(s)} = G(s) = \frac{0.1667}{(0.0033s + 1)(0.0217s + 1)}$$

$$\omega = 2\pi v / \lambda = 11.63 \text{ rad/sec}$$

$$|G(j\omega)|_{\omega = 11.63} = 0.1615$$

$$x(\text{peak}) = 7.5 \times 0.1615 = 1.2113 \text{ cm}$$

2.10 (a) $M\ddot{x} + B\dot{x} + Kx = F(t)$

Gravitational effect has been eliminated by appropriate choice of the zero position.

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

Force transmitted to the ground

$$= K\,X(s) + Bs\,X(s)$$

$$= \frac{\left(\frac{B}{K}s+1\right)F(s)}{\frac{M}{K}s^2+\frac{B}{K}s+1}$$

$$F(t) = A \sin \omega t$$

Peak amplitude of the force transmitted to the ground at steady-state

$$= \frac{A\sqrt{1+\left(\frac{B}{K}\omega\right)^2}}{\sqrt{\left(1-\frac{M\omega^2}{K}\right)^2+\left(\frac{B\omega}{K}\right)^2}}$$

(b) $M\ddot{x}+B(\dot{x}-\dot{y})+K(x-y)=0$

$$\frac{X(s)}{Y(s)} = \frac{Bs+K}{Ms^2+Bs+K}$$

Peak amplitude of machine vibration

$$= \frac{A\sqrt{1+\left(\frac{B}{K}v\right)^2}}{\sqrt{\left(1-\frac{Mv^2}{K}\right)^2+\left(\frac{Bv}{`K}\right)^2}}$$

2.11 $M_1\ddot{y}_1+K_1(y_1-y_0)+B_1(\dot{y}_1-\dot{y}_0)+K_2(y_1-y_2)+B_2(\dot{y}_1-\dot{y}_2) = 0$

$M_2\ddot{y}_2+K_2(y_2-y_1)+B_2(\dot{y}_2-\dot{y}_1) = 0$

Gravitational effect has been eliminated by appropriate choice of the zero position.

2.12 The following result follows from Section 11.2 (Eqn (11.9)).

(a) $$\frac{E_0(s)}{E_i(s)} = \frac{1+2RC_2s+R^2C_1C_2s^2}{1+R(C_1+2C_2)s+R^2C_1C_2s^2}$$

(b) $$\frac{E_0(s)}{E_i(s)} = \frac{1+2R_1Cs+R_1R_2C^2s^2}{1+(2R_1+R_2)Cs+R_1R_2C^2s^2}$$

2.13 Refer Example 11.6

2.14 Refer Example 11.7

2.15 The following result follows from Example 12.9.

$$x_1 = \phi,\ x_2 = \dot{\phi},\ x_3 = z,\ x_4 = \dot{z},\ r = F(t)$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}r$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4.4537 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0.5809 & 0 & 0 & 0 \end{bmatrix};\ b = \begin{bmatrix} 0 \\ -0.3947 \\ 0 \\ 0.9211 \end{bmatrix}$$

2.16 Mixing valve obeys the following equations:

$$(\overline{Q}_i + q_i)\rho c\theta_H + [Q - (\overline{Q}_i + q_i)]\,\rho c\theta_C = Q\rho c(\overline{Q}_i + \theta_i)$$

$$q_i = k_v x$$

The perturbation equation is

$$K_v(\theta_H - \theta_C)\ x(t) = Q\,\theta_i(t)$$

or,

$$x(t) = K\theta_i(t);\ K = Q / [K_v(\theta_H - \theta_C)]$$

The tank obeys the equations

$$V\rho c\frac{d\theta}{dt} = Q\rho c(\theta_{id} - \theta);\ \theta_{id}(t) = \theta_i(t - \tau_D)$$

This gives

$$\frac{\theta(s)}{\theta_i(s)} = \frac{e^{-1.5s}}{s+1}$$

2.17

$$C_1\frac{d\theta_1}{dt} = q_m(t)\lambda - \frac{\theta_1 - \theta_2}{R_1};\ R_1 = \frac{1}{UA}$$

$$C_2\frac{d\theta_2}{dt} = \frac{\theta_1 - \theta_2}{R_1} + \frac{\theta_i - \theta_2}{R_2};\quad C_2 = V\rho c;\ R_2 = \frac{1}{Q\rho c}$$

From these equations, we get

$$\dot{\theta}_1 = -1.92\ \theta_1 + 1.92\ \theta_2 + 4.46\ q_m$$

$$\dot{\theta}_2 = 0.078\,\theta_1 - 0.2\,\theta_2 + 0.125\,\theta_i$$

2.18 At steady-state

$$0 = \frac{\bar{\theta}_1 - \bar{\theta}_2}{R_1} + \frac{\bar{\theta}_i - \bar{\theta}_2}{R_2}; \text{ this gives } \bar{\theta}_1 = 120°\text{C}$$

$$0 = \bar{Q}_m \lambda - \frac{\bar{\theta}_1 - \bar{\theta}_2}{R_1}; \text{ this gives } \bar{Q}_m = 17.26 \text{ kg / min}$$

2.19 Energy balance on process fluid:

$$V_2 \rho_2 c_2 \frac{d}{dt}(\bar{\theta}_2 + \theta_2) = Q_2 \rho_2 c_2 [\theta_{i2} - (\bar{\theta}_2 + \theta_2)] - UA[\bar{\theta}_2 + \theta_2 - \bar{\theta}_1 - \theta_1]$$

At steady-state

$$0 = Q_2 \rho_2 c_2 (\theta_{i2} - \bar{\theta}_2) - UA(\bar{\theta}_2 - \bar{\theta}_1); \text{ this gives } \bar{\theta}_1 = 40° C$$

The perturbation equation is

$$V_2 \rho_2 c_2 \frac{d\theta_2}{dt} - Q_2 \rho_2 c_2 \theta_2 - UA(\theta_2 - \theta_1)$$

This gives

$$544.5 + \frac{d\theta_2}{dt} + \theta_2 = 0.432\, \theta_1$$

Energy balance on the cooling water:

$$V_1 \rho_1 c_1 \frac{d}{dt}\left(\bar{\theta}_1 + \theta_1\right) = (Q_1 + q_1)\rho_1 c_1 [\theta_{i1} - (\bar{\theta}_1 + \theta_1)]$$

$$+ UA[\bar{\theta}_2 + \theta_2 - \bar{\theta}_1 - \theta_1]$$

At steady-state

$$0 = \bar{Q}_1 \rho_1 c_1 [\theta_{i1} - \bar{\theta}_1] + UA[\bar{\theta}_2 - \bar{\theta}_1]; \text{ this gives}$$

$$\bar{Q}_1 = 5.28 \times 10^{-3} \text{ m}^3\text{/sec}$$

The perturbation equation is

$$V_1 \rho_1 c_1 \frac{d\theta_1}{dt} = (\theta_{i1} - \bar{\theta}_1)\rho_1 c_1 q_1 - \bar{Q}_1 \rho_1 c_1 \theta_1 + UA\,(\theta_2 - \theta_1)$$

This gives

$$184.55\,\frac{d\theta_1}{dt}+\theta_1 = 0.465\;\theta_1 - 1318.2\;q_1$$

Manipulation of the perturbation equation gives

$$\frac{\theta_2(s)}{\theta_1(s)} = -\frac{557.6}{100.5\times 10^3 s^2 + 729s + 0.8}$$

2.20 Tank 1:

$$C_1\frac{dp_1}{dt} = q_1 - q_{10} - q_{11}$$

$$q_{10} = \text{flow through } R_0;\; q_{11} = \text{flow through } R_1$$

$$\frac{A_1}{\rho g}\times\rho g\frac{dh_1}{dt} = q_1 - \frac{\rho g h_1}{R_0} - \frac{\rho g(h_1-h_2)}{R_1}$$

or

$$\frac{dh_1}{dt} = -\frac{\rho g}{A_1}\left(\frac{1}{R_0}+\frac{1}{R_1}\right)h_1 + \frac{\rho g}{A_1R_1}h_2 + \frac{1}{A_1}q_1$$

$$= -3h_1 + 2h_2 + r_1$$

Tank 2:

$$C_2\frac{dp_2}{dt} = q_2 + q_{11} - q_{20}$$

$$q_{20} = \text{flow through } R_2$$

$$\frac{A_2}{\rho g}\times\rho g\frac{dh_2}{dt} = \frac{\rho g(h_1-h_2)}{R_1} - \frac{\rho g h_2}{R_2} + q_2$$

or

$$\frac{dh_2}{dt} = \frac{\rho g}{A_2R_1}h_1 - \frac{\rho g}{A_2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)h_2 + \frac{1}{A_2}q_2$$

$$= 4h_1 - 5h_2 + r_2$$

2.21 $A\dfrac{d}{dt}(\overline{H}+h) = \overline{Q}_1 + q_1 + \overline{Q}_2 + q_2 - \overline{Q} - \overline{q}$

At steady-state

$0 = \overline{Q}_1 + \overline{Q}_2 - \overline{Q}$; this gives $\overline{Q}$ = 30 litres/sec

The perturbation equation is

$$A\frac{dh}{dt} = q_1 + q_2 - q$$

The turbulent flow is governed by the relation

$$Q(t) = K\sqrt{H(t)} = f(H)$$

Linearizing about the operating point, we obtain

$$Q(t) = f(\bar{H}) + \left(\frac{\partial f(H)}{\partial H}\bigg|_{H=\bar{H}}\right)(H - \bar{H})$$

$$= \bar{Q} + \frac{K}{2\sqrt{\bar{H}}}h(t)$$

Therefore

$$q(t) = \frac{K}{2\sqrt{\bar{H}}}h(t) = \frac{K\sqrt{\bar{H}}}{2\bar{H}}h(t) = \frac{\bar{Q}}{2\bar{H}}h(t)$$

$$\frac{dh(t)}{dt} = -\frac{1}{A}q(t) + \frac{1}{A}q_1(t) + \frac{1}{A}q_2(t)$$

$$= -0.01\ h(t) + 0.133\ q_1(t) + 0.133\ q_2(t)$$

Mass balance on salt in the tank:

$$A\frac{d}{dt}[(\bar{H} + h(t)\ (\bar{C} + c(t))] = C_1[\bar{Q}_1 + q_1(t)] + C_2[\bar{Q}_2 + q_2(t)]$$

$$- [\bar{C} + c(t)][\bar{Q} + q(t)]$$

At steady state,

$$0 = C_1\bar{Q}_1 + C_2\bar{Q}_2 - \bar{C}\bar{Q}\text{; this gives } \bar{C} = 15$$

The perturbation equation is

$$A\bar{H}\frac{dc(t)}{dt} + A\bar{C}\frac{dh(t)}{dt} = C_1q_1(t) + C_2q_2(t) - \bar{C}q(t) - \bar{Q}c(t)$$

This gives

$$\frac{dc(t)}{dt} = -0.02\ c(t) - 0.004\ q_1(t) + 0.002\ q_2(t)$$

2.22 $e^{-\tau_D s} = 1 - \tau_D s + \dfrac{\tau_D^2 s^2}{2!} - \dfrac{\tau_D^3 s^3}{3!} + \dfrac{\tau_D^4 s^4}{4!} - \dfrac{\tau_D^5 s^5}{5!} + \ldots$

It is easy to calculate with long division that

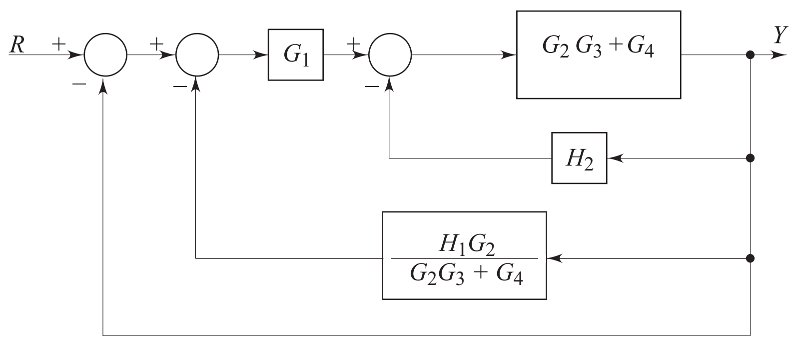
$$\frac{1-\tau_D s/2}{1+\tau_D s/2} = 1 - \tau_D s + \frac{\tau_D^2 s^2}{2} - \frac{\tau_D^3 s^3}{4} + \ldots$$

$$\frac{1-\tau_D s/2+\tau_D^2 s^2/12}{1+\tau_D s/2+\tau_D^2 s^2/12} = 1 - \tau_D s + \frac{\tau_D^2 s^2}{2} - \frac{\tau_D^3 s^3}{6}$$

$$+\frac{\tau_D^4 s^4}{12} + \frac{\tau_D^5 s^5}{144} - \ldots$$

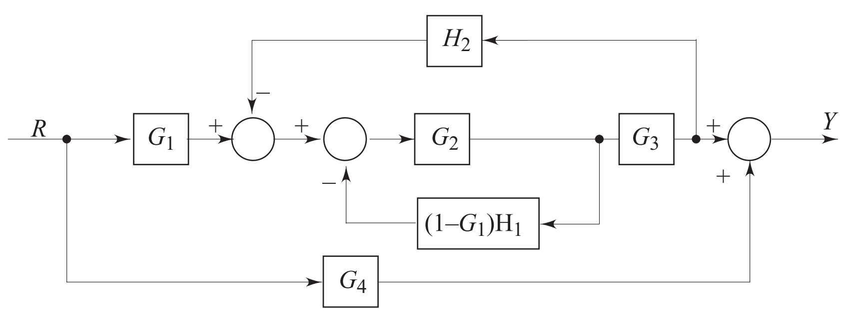
CHAPTER 3 MODELS OF INDUSTRIAL CONTROL DEVICES AND SYSTEMS

3.1



$$\frac{Y(s)}{R(s)} = \frac{G_1(G_2G_3 + G_4)}{1 + (G_2G_3 + G_4)(G_1 + H_2) + G_1H_1G_2}$$

3.2



$$\frac{Y(s)}{R(s)} = G_4 + \frac{G_1G_2G_3}{1 + G_2G_3H_2 + G_2H_1(1 - G_1)}$$

3.3 $$\frac{Y(s)}{R(s)} = (P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4) / \Delta$$

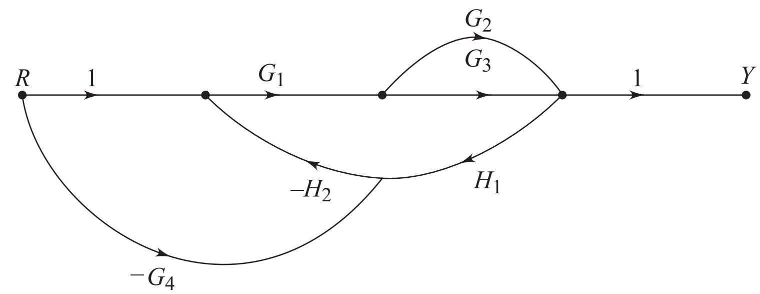
$P_1 = G_1G_2$; $P_2 = G_1G_3$; $P_3 = G_4H_2G_1G_2$;

$P_4 = G_4H_2G_1G_3$

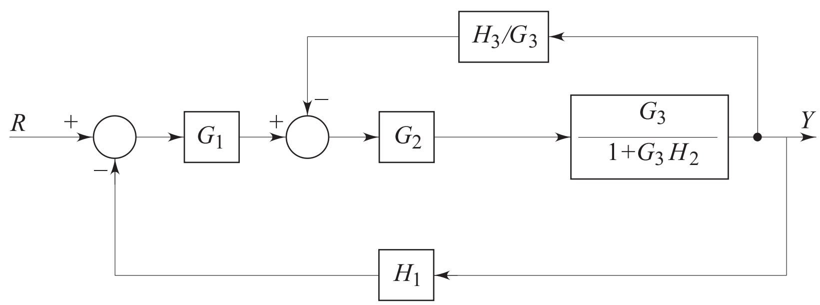
$\Delta = 1 - (-G_1G_3H_1H_2 - G_1G_2H_1H_2)$

$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 1$

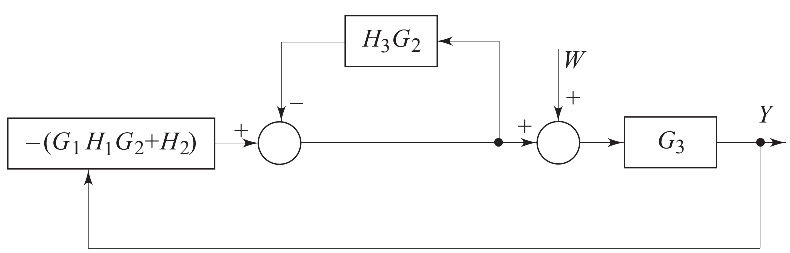
$$\frac{Y(s)}{R(s)} = \frac{(1+G_4H_2)(G_2+G_3)G_1}{1+G_1H_1H_2(G_2+G_3)}$$



3.4

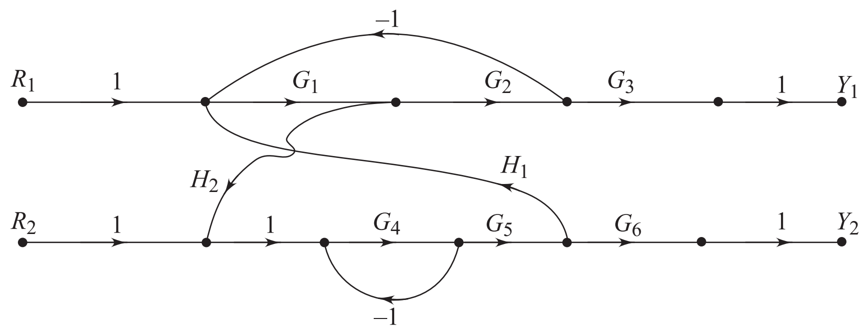


$$\left.\frac{Y(s)}{R(s)}\right|_{W=0} = M(s) = \frac{G_1G_2G_3}{1+G_2H_3+G_3H_2+G_1G_2G_3H_1}$$



$$\left.\frac{Y(s)}{W(s)}\right|_{R=0} = M_W(s) = \frac{G_3(1+H_3G_2)}{1+H_3G_2+G_3(G_1H_1G_2+H_2)}$$

3.5



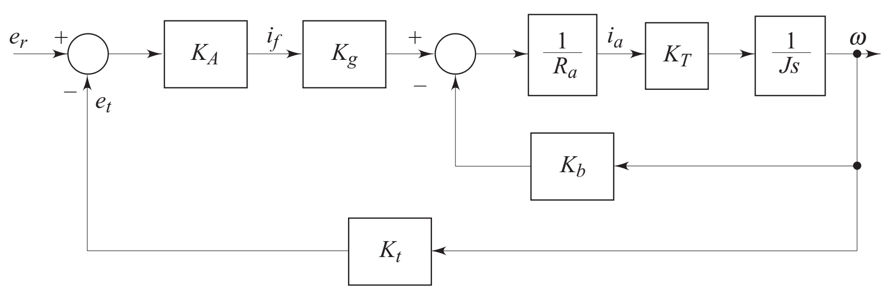
$$\left.\frac{Y_1}{R_1}\right|_{R_2=0} = \frac{P_1\Delta_1}{\Delta}$$

$$= \frac{G_1G_2G_3(1+G_4)}{1-(-G_1G_2-G_4+G_1H_2G_4G_5H_1)+G_1G_2G_4}$$

$$\left.\frac{Y_1}{R_2}\right|_{R_1=0} = \frac{G_4G_5H_1G_1G_2G_3}{\Delta}$$

$$\left.\frac{Y_2}{R_1}\right|_{R_2=0} = \frac{G_1G_4G_5G_6H_2}{\Delta};\ \left.\frac{Y_2}{R_2}\right|_{R_1=0} = \frac{G_4G_5G_6(1+G_1G_2)}{\Delta}$$

3.6



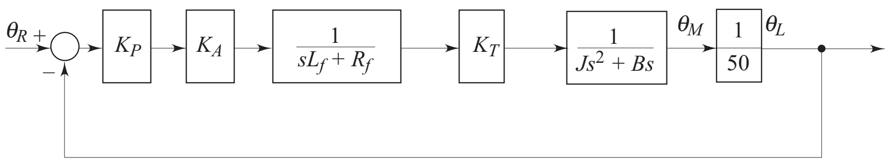
$$\frac{\omega(s)}{E_r(s)} = \frac{50}{s+10.375}$$

For $E_r(s) = 100/s$,

$$\omega(s) = \frac{100 \times 50}{s(s+10.375)}$$

$$\omega(t) = 481.8\,(1 - e^{-10.375\,t})$$

3.7



$$\frac{\theta_L(s)}{\theta_R(s)} = \frac{1}{s(0.1s+1)(0.2s+1)+1}$$

3.8

$$T_M = K_1 e_c + K_2\dot{\theta}_M = \dot{J}_{eq}\,\ddot{\theta}_M + B_{eq}\,\dot{\theta}_M$$

K_2 = slope of the characteristic lines shown in Fig. P3.8b

$$= \frac{3-0}{0-300} = -\,0.01$$

$$K_1 = \left.\frac{\Delta T_M}{\Delta e_c}\right|_{\dot{\theta}_M = 0} = \frac{3-2}{30-20} = 0.1$$

$$J_{eq} = J_M + \left(\frac{N_1 N_3}{N_2 N_4}\right)^2 J_L = 0.0032$$

$$B_{eq} = \left(\frac{N_1 N_3}{N_2 N_4}\right)^2 B_L = 0.00001$$

$$\frac{\theta_M(s)}{E_c(s)} = \frac{10}{s(0.32s+1)}\,;\ \frac{\theta_L(s)}{E_c s} = \frac{1}{s(0.32s+1)}$$

3.9 (i)

$$J_{eq} = J_M + J_L\left(\frac{\dot{\theta}_L}{\dot{\theta}_M}\right)^2 = 0.6$$

$$B_{eq} = B_M + B_L\left(\frac{\dot{\theta}_L}{\dot{\theta}_M}\right)^2 = 0.015$$

$$K_T = K_b$$

$$\frac{\theta_M(s)}{E_a(s)} = \frac{K_T}{s[J_{eq}s + B_{eq})(sL_a + R_a) + K_T K_b]}$$

$$= \frac{16.7}{s(s^2 + 100s + 19.2)}$$

(ii) A sketch of multi-loop configuration may be made on the lines of Example 3.5.

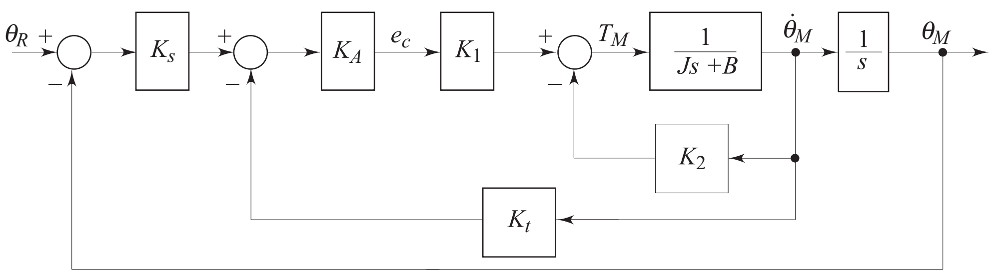
(iii) It will behave as a speed control system.

(iv) Relative stability and speed of response may become unsatisfactory.

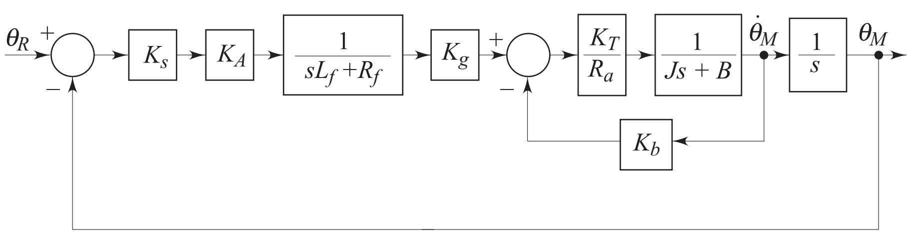
3.10

$$J = J_M = n^2 J_L = 1.5 \times 10^{-5};\ n = 1$$
$$B = B_M + n^2 B_L = 1 \times 10^{-5}$$

$$\frac{\theta_M(s)}{\theta_R(s)} = \frac{13.33}{s^2 + 3.5s + 13.33}$$



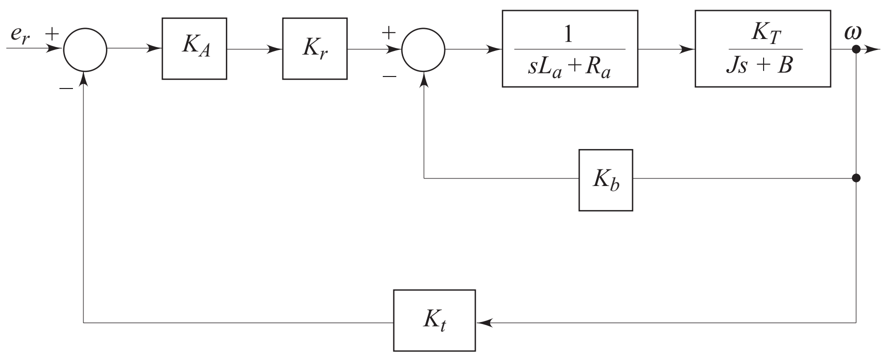
3.11



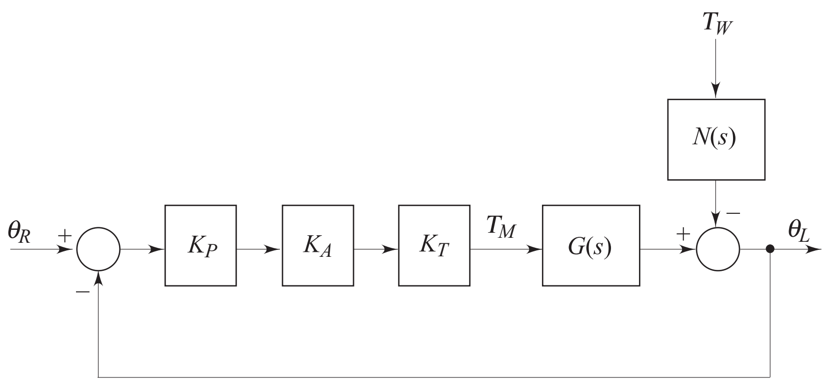
$$\frac{\theta_M(s)}{\theta_R(s)} = \frac{\theta_L(s)}{\theta_R(s)} = \frac{15\times 10^3}{s^3 + 54s^2 + 200s + 15\times 10^3}$$

3.12 (a) $\dfrac{\omega(s)}{E_r(s)} = \dfrac{K}{\dfrac{1}{\omega_n^2}s^2 + \dfrac{2\zeta}{\omega_n}s + 1}$; $K = 17.2$, $\zeta = 1.066$, $\omega_n = 61.2$

(b) Modification in the drive system may be made on the lines of Fig. 3.64.



3.13



$$T_M = J_M \ddot{\theta}_M + B(\dot{\theta}_M - \dot{\theta}_L) + K(\theta_M - \theta_L)$$

$$B(\dot{\theta}_M - \dot{\theta}_L) + K(\theta_M - \theta_L) = J_L \ddot{\theta}_L + T_W$$

$$(J_M s^2 + Bs + K)\ \theta_M(s) - (Bs + K)\ \theta_L(s) = T_M(s)$$

$$(Bs + K)\ \theta_M(s) - (J_L s^2 + Bs + K)\ \theta_L(s) = T_W(s)$$

$$\theta_L(s) = G(s)\ T_M(s) - N(s)\ T_W(s)$$

$$\frac{\theta_L(s)}{\theta_R(s)} = \frac{K_P K_A K_T G(s)}{1 + K_P K_A K_T G(s)}$$

$$\frac{\theta_L(s)}{-T_W(s)} = \frac{N(s)}{1 + K_P K_A K_T G(s)}$$

$$G(s) = \frac{Bs + K}{s^2[J_M J_L s^2 + (J_M + J_L)Bs + (J_M + J_L)K]}$$

$$N(s) = \frac{J_M s^2 + Bs + K}{s^2[J_M J_L s^2 + (J_M + J_L)Bs + (J_M + J_L)K]}$$

Revisiting Example 2.7 will be helpful.

3.14 The required block diagram easily follows from Fig. 3.49.

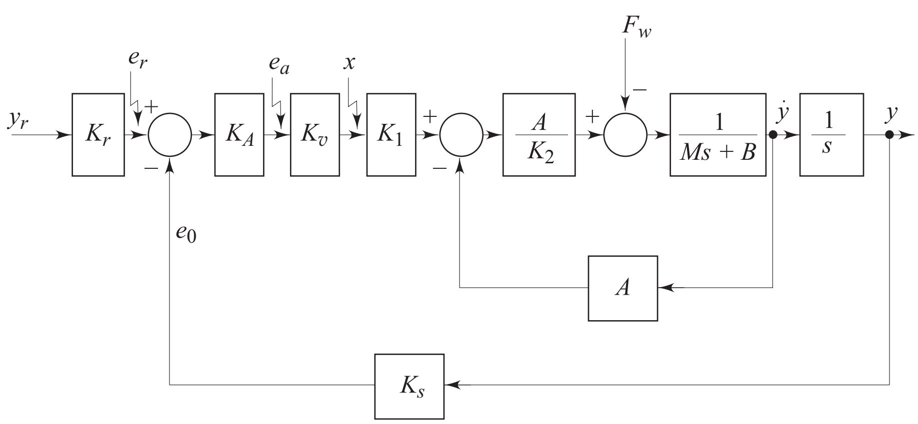
$$K_1 x - A\dot{y} = K_2 \Delta p$$

$$(K_2 \Delta p)\frac{A}{K_2} = M\ddot{y}$$

$$(r - y)\, K_p\, K_A\, K = x$$

$$\frac{Y(s)}{R(s)} = \frac{2}{s^2 + 0.02s + 2}$$

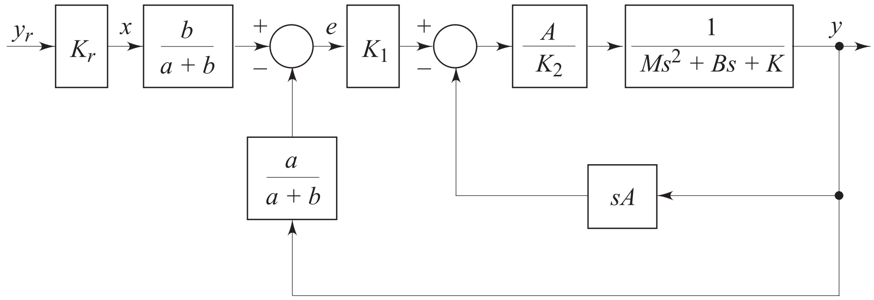
3.15 $$\frac{Y(s)}{Y_r(s)} = \frac{K_A K_v K_1 A K_r / K_2}{Ms^2 + (B + A^2/K_2)s + K_A K_v K_1 K_s A / K_2}$$



3.16 Revisiting Example 3.7 will be helpful. From geometry of the linkage, we get

$$E(s) = \frac{b}{a+b}X(s) - \frac{a}{a+b}Y(s)$$

$$\frac{Y(s)}{Y_r(s)} = \frac{bK_r A K_1 / (a+b)K_2}{Ms^2 + (B + A^2/K_2)s + K + aAK_1/(a+b)K_2}$$

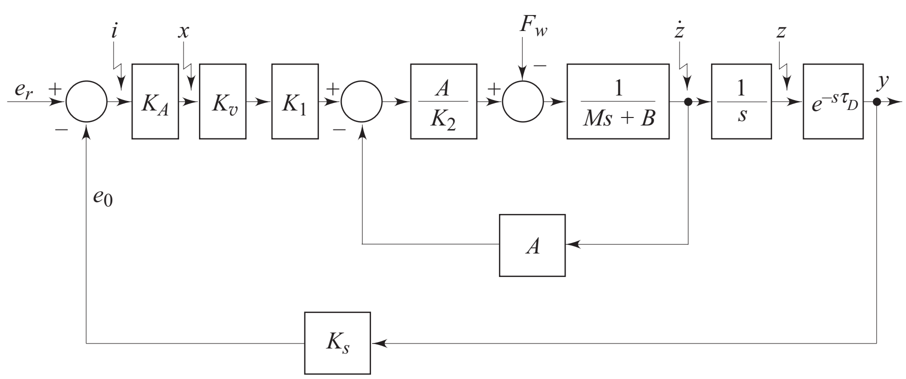


3.17 Let $z(t)$ = displacement of the power piston

$$Y(s) = e^{-s\tau_D} Z(s)\ ;\ \tau_D = \frac{d}{v} \text{ sec}$$

$$\frac{Y(s)}{E_r(s)} = \frac{G(s)}{1+G(s)H(s)}$$

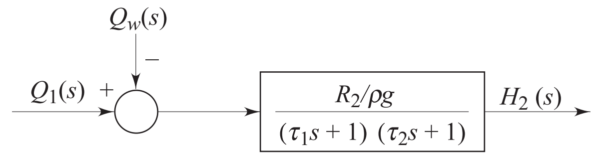
$$G(s) = \frac{(K_A K_v K_1 A / K_2)e^{-s\tau_D}}{s(Ms + B + A^2 / K_2)};\ H(s) = K_s$$



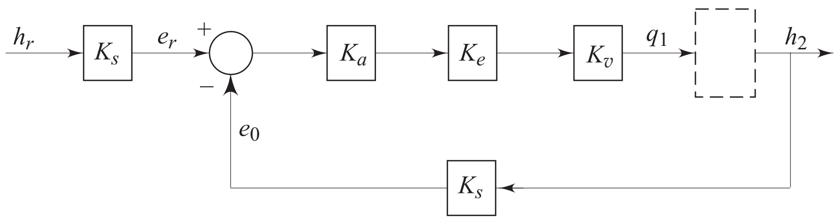
3.18

$$A_1 \frac{dh_1}{dt} = q_1 - q_w - \frac{\rho g h_1}{R_1}$$

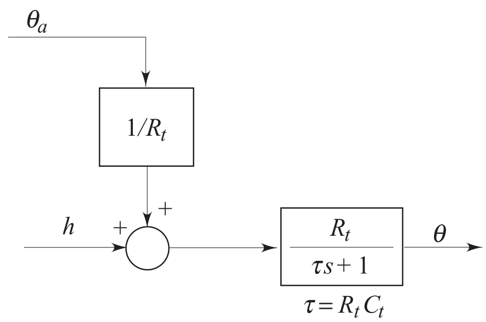
$$A_2 \frac{dh_2}{dt} = \frac{\rho g h_1}{R_1} - \frac{\rho g h_2}{R_2}$$

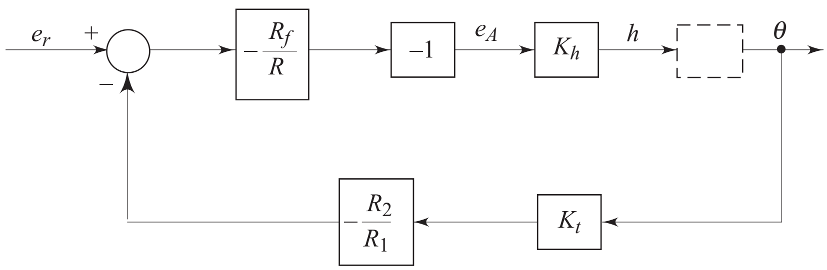


$$\tau_1 = \frac{A_1 R_1}{\rho g}\ ;\ \tau_2 = \frac{A_2 R_2}{\rho g}$$



3.19 $C_t \dfrac{d\theta}{dt} = h - \dfrac{\theta - \theta_a}{R_t}$





3.20 Heater:

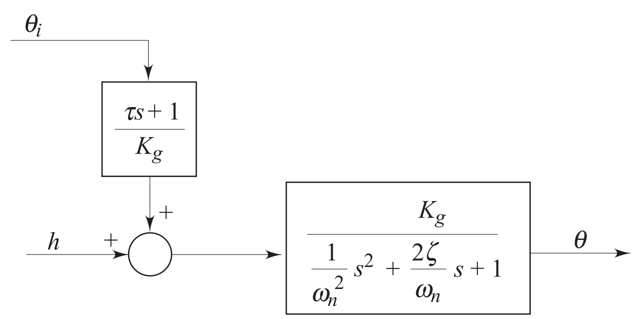
$$C_1 \frac{d\theta_1}{dt} = h - \frac{\theta_1 - \theta}{R_1}$$

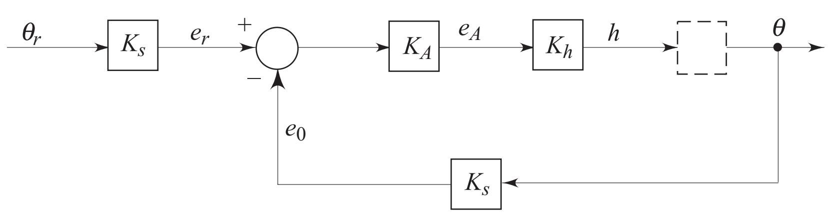
Air: $$V\rho c\frac{d\theta}{dt} = \frac{\theta_1-\theta}{R_1} - Q\rho c(\theta-\theta_i)$$

These equations give us

$$\tau = R_1C_1,\ K_g = \frac{1}{Q\rho c},\ \omega_n = \sqrt{\frac{Q}{R_1C_1V}}$$

$$2\zeta\omega_n = \frac{R_1C_1Q\rho c + V\rho c + C_1}{VR_1C_1\rho c}$$





3.21 Process:

$$V_2\rho_2c_2\frac{d}{dt}(\bar{\theta}_2+\theta_2) = Q_2\rho_2c_2(\bar{\theta}_{i2}+\theta_{i2}) - UA(\bar{\theta}_2+\theta_2-\bar{\theta}_1-\theta_1) - Q_2\rho_2c_2(\bar{\theta}_2+\theta_2)$$

Linearized equation is

$$V_2\rho_2c_2\frac{d\theta_2}{dt} = Q_2\rho_2c_2\theta_{i2} - UA(\theta_2-\theta_1) - Q_2\rho_2c_2\theta_2$$

Rearranging this equation and taking Laplace transform yields

$$(\tau_2 s+1)\theta_2(s) = K_1'\theta_1(s)+K_2'\theta_{i2}(s)$$

$$\tau_2 = 544.5; K_1' = 0.423; K_2' = 0.577$$

Cooling water:

$$V_1\rho_1 c_1 \frac{d}{dt}(\bar{\theta}_1+\theta_1) = (\bar{Q}_1+q_1)\rho_1 c_1(\bar{\theta}_{i1}+\theta_{i1})+UA(\bar{\theta}_2+\theta_2 - \bar{\theta}_1-\theta_1)-(\bar{Q}_1+q_1)\rho_1 c_1(\bar{\theta}_1+\theta_1)$$

Linearized equation is

$$V_1\rho_1 c_1 \frac{d\theta_1}{dt} = \bar{Q}_1\rho_1 c_1\theta_{i1}+\bar{\theta}_{i1}\rho_1 c_1 q_1 + UA(\theta_2-\theta_1) - \bar{Q}_1\rho_1 c_1\theta_1 - \bar{\theta}_1\rho_1 c_1 q_1$$

From this equation, we obtain

$$(\tau_1 s+1)\theta_1(s) = -K_3' Q_1(s)+K_4'\theta_2(s)+K_5'\theta_{i1}(s)$$

$$\tau_1 = \frac{1.82\times 1000\times 4184}{3550\times 5.4+\bar{Q}_1\times 1000\times 4184}$$

$$K_3' = \frac{(-27+\bar{\theta}_1)\times 1000\times 4184}{3550\times 5.4+\bar{Q}_1\times 1000\times 4184}$$

$$K_4' = \frac{3550\times 5.4}{3550\times 5.4+\bar{Q}_1\times 1000\times 4184}$$

$$K_5' = \frac{\bar{Q}_1\times 1000\times 4184}{3550\times 5.4+\bar{Q}_1\times 1000\times 4184}$$

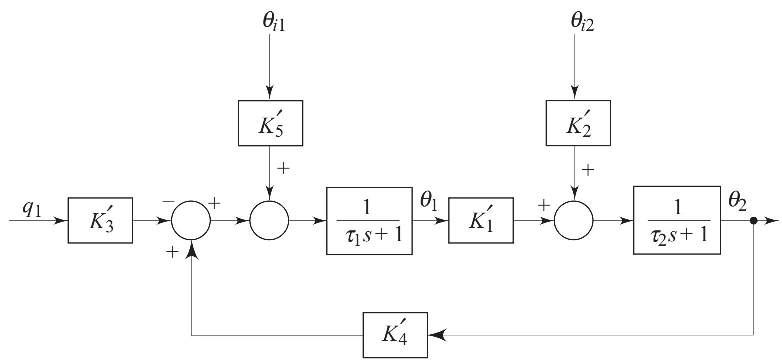
$\bar{\theta}_1$ and $\bar{Q}_1$ are obtained from the following energy-balance equations at steady state:

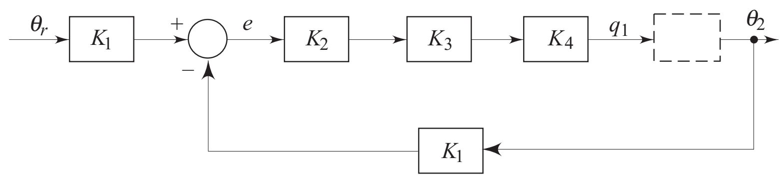
$$\bar{Q}_1\rho_1 c_1(\bar{\theta}_{i1}-\bar{\theta}_1)+UA(\bar{\theta}_2-\bar{\theta}_1) = 0$$

$$\bar{Q}_2\rho_2 c_2(\bar{\theta}_{i2}-\bar{\theta}_2)-UA(\bar{\theta}_2-\bar{\theta}_1) = 0$$

These equations yield

$$\bar{\theta}_1 = 40^\circ\text{C};\quad \bar{Q}_1 = 5.28\times 10^{-3}\ \text{m}^3/\text{sec}$$





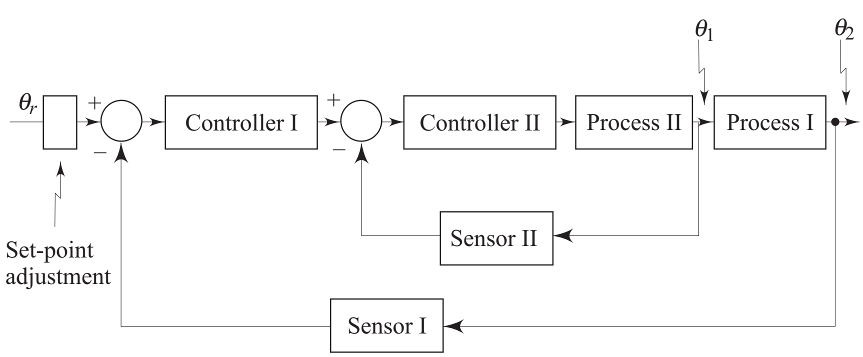
With these parameters,

$$\tau_1 = 184.55 \text{ sec}, \; K_3' = 1318.2, \; K_4' = 0.465, \; K_5' = 0.535$$

$$\frac{\theta_2(s)}{\theta_r(s)} = \frac{G(s)}{1+G(s)}; \; G(s) = \frac{(5.55\times10^{-3})K_1K_2K_3K_4}{s^2+7.26\times10^{-3}s+1.19\times10^{-5}}$$

Refer Fig. 3.57 for suitable hardware required to implement the proposed control scheme.

3.22



Refer Fig. 3.57 for suitable hardware required to implement the proposed control scheme.

CHAPTER 4 BASIC PRINCIPLES OF FEEDBACK CONTROL

4.1

$$S_G^M = \frac{1}{1+G(s)H(s)} = \frac{s(s+1)}{s(s+1)+50}$$

$$|S_G^M(j\omega)|_{\omega=1} = 0.0289$$

$$S_H^M = \frac{-G(s)H(s)}{1+G(s)H(s)} = \frac{-50}{s(s+1)+50}$$

$$|S_H^M(j\omega)|_{\omega=1} = 1.02$$

4.2

$$\frac{Y(s)}{R(s)} = M(s) = \frac{P_1\Delta_1}{\Delta}$$

$$P_1 = \frac{K_3K_1}{s^2(s+1)};\ \ \Delta = 1-\left(-\frac{5}{s^2(s+1)}-\frac{K_1K_2}{s+1}-\frac{K_3K_1}{s^2(s+1)}\right);\ \ \Delta_1 = 1$$

$$M(s) = \frac{5K_1}{s^2(s+1+5K_1)+5K_1+5}$$

$$S_{K_1}^M = \frac{\partial M}{\partial K_1}\times\frac{K_1}{M} = \frac{s^2(s+1+5K_1)+5-5K_1s^2}{s^2(s+1+5K_1)+5K_1+5}$$

$$|S_{K_1}^M(j\omega)|_{\omega=0} = \frac{5}{5K_1+5} = 0.5$$

4.3 For $G(s) = 20/(s+1)$, and $R(s) = 1/s$,

$$y(t)|_{\text{open-loop}} = 20\,(1-e^{-t})$$

$$y(t)|_{\text{closed-loop}} = \frac{20}{21}(1-e^{-21t})$$

For $G'(s) = 20/(s+0.4)$, and $R(s) = 1/s$,

$$y(t)|_{\text{open-loop}} = 50\,(1-e^{-0.4t})$$

$$y(t)|_{\text{closed-loop}} = \frac{20}{20.4}(1-e^{-20.4t})$$

The transient response of the closed-loop system is less sensitive to variations in plant parameters

4.4 For $G(s) = 10/(\tau s+1)$, and $R(s) = 1/s$,

$$e_{ss}|_{\text{open-loop}} = 0$$

$$e_{ss}|_{\text{closed-loop}} = \frac{1}{1+10K_p} = 0.0099$$

For $G'(s) = 11/(\tau s+1)$, and $R(s) = 1/s$,

$$e_{ss}|_{\text{open-loop}} = -0.1$$

$$e_{ss}|_{\text{closed-loop}} = 0.009$$

The steady-state response of the closed-loop system is less sensitive to variations in plant parameters.

4.5 $$\tau_f|_{\text{open-loop}} = \frac{L}{R} = \frac{2}{50} = 0.04 \text{ sec}$$

For the feedback system,

$$K_A(e_f - Ki_f) = L\frac{di_f}{dt} + (R+R_s)i_f$$

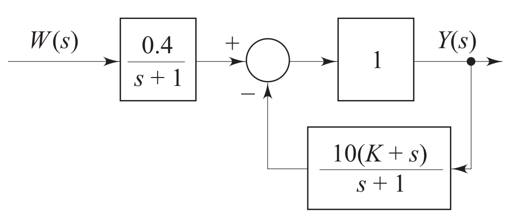
This gives

$$\frac{I_f(s)}{E_f(s)} = \frac{K_A}{sL+R+R_s+K_AK}$$

$$\tau_f|_{\text{closed-loop}} = \frac{L}{R+R_s+K_AK} = \frac{2}{51+90K} = 0.004$$

This gives $K = 4.99$

4.6 The given block diagram is equivalent to a single-loop block diagram given below,



$$\frac{Y(s)}{W(s)} = \frac{0.4}{11s+10K+1}$$

For $\quad W(s) = 1/s,$

$$y_{ss} = \lim_{s\to 0} sY(s) = \frac{0.4}{10K+1} = 0.01$$

This gives $K = 3.9$

4.7 Case I:

$$\frac{Y(s)}{W(s)} = \frac{s(\tau s+1)}{s(\tau s+1)+K}$$

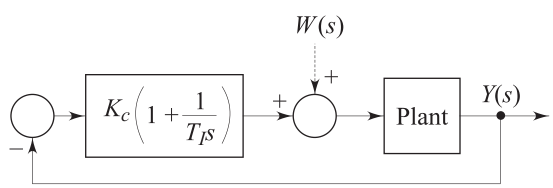
for $W(s) = 1/s$, $y_{ss} = 0$

Case II:

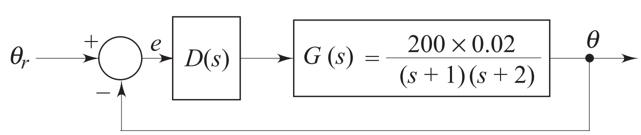
$$\frac{Y(s)}{W(s)} = \frac{K}{s(\tau s+1)+K}$$

For $\quad W(s) = 1/s$, $y_{ss} = 1$

The control scheme shown in the following figure will eliminate the error in Case II.



4.8 A unity-feedback configuration of the given system is shown below.



$$\frac{E(s)}{\theta_r(s)} = \frac{1}{1+D(s)G(s)}; e_{ss} = \lim_{s\to 0} sE(s);\ \theta_r(s) = \frac{1}{s}$$

(i) $e_{ss} = 1/3$

(ii) $e_{ss} = 0$

(iii) $e_{ss} = 1/3$

The integral term improves the steady-state performance and the derivative term has no effect on steady-state error.

4.9 (i) $s^2 + 1 = 0$; oscillatory

(ii) $s^2 + 2s + 1 = 0$; stable

(iii) $s^3 + s + 2 = 0$; unstable

The derivative term improves the relative stability; and the integral term has the opposite effect.

4.10 (a) $e_r = 50 \times 0.03 = 1.5$ volts

(b) $$\frac{V(s)}{E_r(s)} = M(s) = \frac{0.65K_e}{(s+1)(5s+1)+0.0195\,K_e}$$

$$S_{K_e}^M = \frac{\partial M}{\partial K_e} \times \frac{K_e}{M} = \frac{(s+1)(5s+1)}{(s+1)(5s+1)+9.75}$$

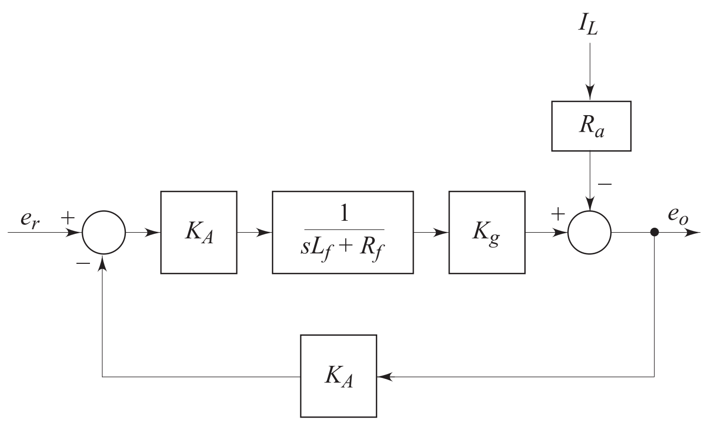
(c) $$V(s) = \frac{325}{(s+1)(5s+1)+9.75}E_r(s) - \frac{19.5(s+1)}{(s+1)(5s+1)+9.75}W(s)$$

Suppose that the engine stalls when constant percent gradient is x%. Then for $E_r(s) = 1.5/s$ and $W(s) = x/s$,

$$v_{ss} = \lim_{s\to 0} sV(s) = \frac{325\times 1.5}{10.75} - \frac{19.5\times x}{10.75} = 0$$

This gives $x = 25$

4.11 (a) The block diagram is shown in the figure below.



(b) $$\left.\frac{E_0(s)}{E_r(s)}\right|_{I_L=0} = \frac{10}{2s+1+10K}$$

For $E_r(s) = 50/s$,

$$e_{oss} = \lim_{s\to 0} sE_0(s) = \frac{500}{1+10K} = 250$$

This gives $K = 0.1$

With $K = 0.1$, we have

$$E_o(s) = \frac{10}{2s+2} E_r(s) - \frac{2s+1}{2s+2} I_L(s)$$

When $E_r(s) = 50/s$ and $I_L(s) = 30/s$, we get

$$e_{oss} = \frac{10 \times 50}{2} - \frac{30}{2} = 235 \text{ V}$$

Let x V be the reference voltage required to restore the terminal voltage of 250 V.

$$\frac{10 \times x}{2} - \frac{30}{2} = 250; \text{ this gives } x = 53\text{V}$$

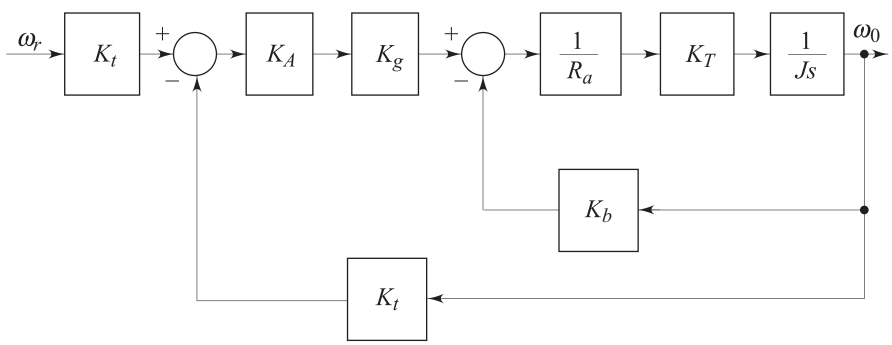
(c) $E_o(s) = \dfrac{10}{2s+1} E_r(s) - I_L(s)$

$e_r = 25$ V gives $e_{oss} = 250$ V when $I_L = 0$

$e_r = 25$ V gives $e_{oss} = 220$ V when $I_L = 30$ amps

(d) The effect of load disturbance on terminal voltage has been reduced due to feedback action.

4.12



(a) $\dfrac{\omega_o(s)}{\omega_r(s)} = M(s) = \dfrac{K_A K_g}{2(s+5) + K_A K_g}$

For $\omega_r(s) = 10/s$,

$$\omega_o(t) = \frac{125}{13}(1 - e^{-130t}); \quad \omega_{oss} = 9.61 \text{ rad/sec}$$

(b) When the feedback loop is opened,

$$\frac{\omega_o(s)}{\omega_r(s)} = \frac{50 K_A}{2(s+5)}$$

K_A = 0.192 gives ω_{oss} = 9.61 for ω_r = 10.

$\omega_o(t) = 9.61\,(1 - e^{-5t})$

Time-constant in the open-loop case is 1/5 sec, and in closed-loop case is 1/130 sec; system dynamics becomes faster with feedback action

(c) $S_{K_A}^M = \dfrac{\partial M}{\partial K_A} \times \dfrac{K_A}{M} = \dfrac{2s+10}{2s+10+K_A K_g}$

$Kg = K\omega_g$, where K is a constant.

$$S_{\omega_g}^M = \frac{\partial M}{\partial K_g} \times \frac{\partial K_g}{\partial \omega_g} \times \frac{\omega_g}{M} = \frac{\partial M}{\partial K_g} \times \frac{K_g}{M} = \frac{2s+10}{2s+10+K_A K_g}$$

The open-loop transfer function of the system is

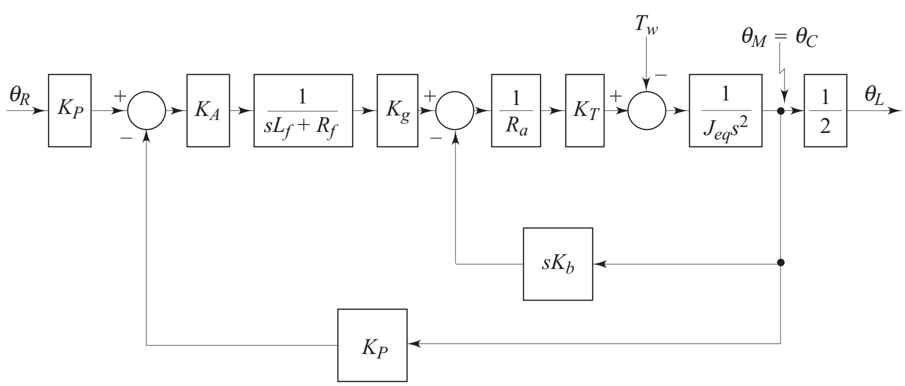
$$G(s) = \frac{K_A K_g}{2\,(s+5)}$$

Therefore

$$S_{K_A}^G = 1 = S_{\omega_g}^G$$

$$\left|S_{K_A}^M\right| < \left|S_{K_A}^G\right|; \left|S_{\omega_g}^M\right| < \left|S_{\omega_g}^G\right|$$

4.13



(b) $\dfrac{\theta_L(s)}{\theta_R(s)} = M(s) = \dfrac{5K_g}{s\,(s+1)\,(s+4)+10\,K_g}$

$$S_{K_g}^M = \frac{\partial M}{\partial K_g} \times \frac{K_g}{M} = \frac{s\,(s+1)\,(s+4)}{s\,(s+1)\,(s+4)+10\times 100}$$

$$\left|S_{K_g}^M(j\omega)\right|_{\omega=0.1} \simeq 4\times 10^{-4}$$

(c) $$\frac{\theta_L(s)}{-T_w(s)} = \frac{(s+4)/2}{s(s+1)(s+4)+10\,K_g}$$

Wind gust torque on the load shaft = 100 N-m.

Therefore, on the motor shaft

$$T_w(s) = n \times 100/s = 50/s$$

$$\theta_{Lss} = \lim_{s\to 0} s\theta_L(s) = 0.1 \text{ rad}$$

(d) Replace K_A by $D(s) = K_c\left(1+\frac{1}{T_I s}\right)$ to reduce the steady-state error to zero value.

4.14 (a) $$\left.\frac{\theta(s)}{\theta_r(s)}\right|_{\text{open-loop}} = G(s) = \frac{K_P K_A K_e K(25s+1)}{s(0.25s^2+0.02s+1)}$$

$$\left.\frac{\theta(s)}{\theta_r(s)}\right|_{\text{closed-loop}} = M(s) = \frac{G(s)}{1+G(s)}$$

$$S_K^G = 1;\; S_K^M = \frac{s(0.25s^2+0.02s+1)}{s(0.25s^2+0.02s+1)+K_P K_A K_e K(25s+1)}$$

$$\left|S_K^M\right| < \left|S_K^G\right|$$

(b) A PD control scheme with proportional controller in the forward path and derivative action realized by feeding back $\dot{\theta}$, will increase aircraft damping.

4.15 (a) $$\frac{\theta(s)}{T_w(s)} = \frac{1/B}{s\left(\frac{J}{B}s+1\right)+\frac{sK_bK_T}{BR_a}+\frac{K_TK_A}{BR_a}}$$

For $T_w(s) = K_w/s$,

$$\theta_{ss} = \lim_{s\to 0} s\theta(s) = \frac{K_w R_a}{K_T K_A}$$

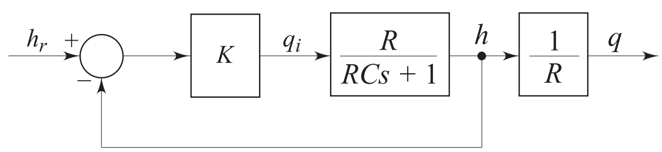
(b) $$\frac{\theta(s)}{\theta_r(s)} = M(s) = \frac{K_T K_A / BR_a}{s\left(\frac{J}{B}s+1\right) + \frac{sK_b K_T}{BR_a} + \frac{K_T K_A}{BR_a}}$$

$$S_J^M = \frac{\partial M}{\partial J} \times \frac{J}{M} = \frac{-\frac{J}{B}s^2}{s\left(\frac{J}{B}s+1\right) + \frac{sK_b K_T}{BR_a} + \frac{K_T K_A}{BR_a}}$$

(c) Excessive large magnitudes of signals at various levels in a control system can drive the devices into nonlinear region of operation. The requirement of linear operation of devices under various operating conditions imposes a constraint on the use of large values of K_A.

4.16 (a) $\frac{dV}{dt} = q_i - q$; $V = C\,h(t)$, $q = h(t)/R$

(b)



(c) $$\frac{H(s)}{H_r(s)} = M(s) = \frac{KR}{RCs + 1 + KR}$$

$$S_R^M = \frac{\partial M}{\partial R} \times \frac{R}{M} = \frac{1}{RCs + 1 + KR}$$

$$S_R^M\Big|_{s=0} = \frac{1}{1 + KR}$$

(d) For $H_r(s) = 1/s$, $h_{ss} = \frac{KR}{1 + KR}$; $e_{ss} = \frac{1}{1 + KR}$

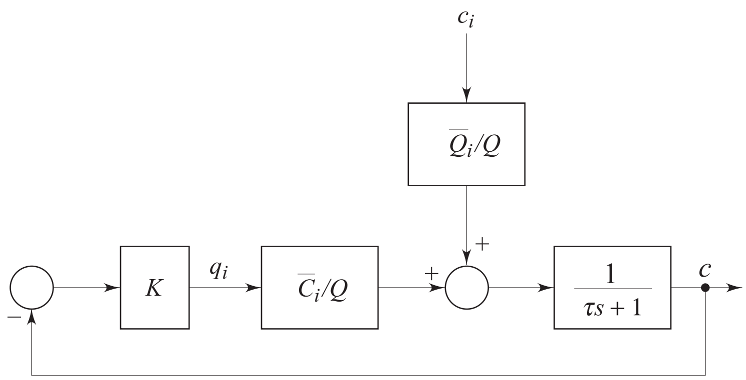
Replacing K by $K_c\left(1 + \frac{1}{T_I s}\right)$ reduces the steady-state error to zero; and the system becomes insensitive to changes in R under steady dc conditions.

4.17 $V\frac{d}{dt}(\overline{C} + c) = (\overline{Q}_i + q_i)(\overline{C}_i + c_i) - Q(\overline{C} + c)$

The perturbation dynamics is given by

$$\tau\frac{dc(t)}{dt} + c(t) = \frac{\overline{C}_i}{Q}q_i(t) + \frac{\overline{Q}_i}{Q}c_i(t); \tau = \frac{V}{Q}$$

Block diagram of the closed-loop system is shown below.



$$\frac{C(s)}{C_i(s)} = M(s) = \frac{\overline{Q}_i / Q}{\tau s + 1 + K\overline{C}_i / Q}$$

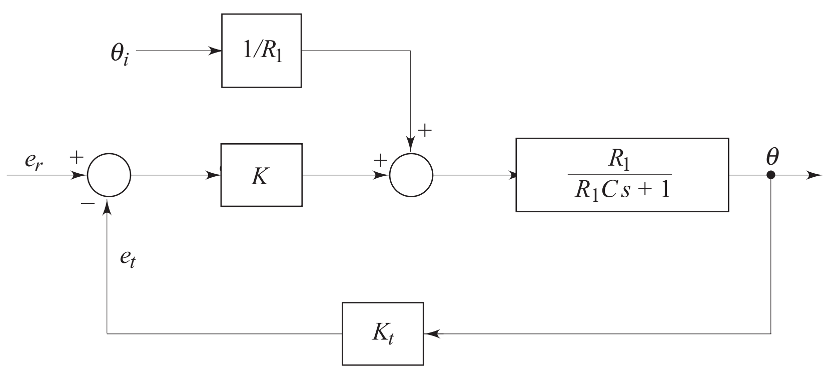
For $C_i(s) = A/s$,

$$c_{ss} = \frac{A\overline{Q}_i}{Q + K\overline{C}_i}$$

With integral control, c_{ss} becomes zero.

4.18 (b) Open-loop case:

(i) $$\frac{\theta(s)}{E_r(s)} = G(s) = \frac{KR_1}{R_1Cs+1}; S_K^G = 1$$



(ii) $\dfrac{\theta(s)}{\theta_i(s)} = \dfrac{1}{R_i Cs}$. For $\theta_i(s) = 1/s$, $\theta_{ss} = 1$

(iii) $\tau = R_1 C$

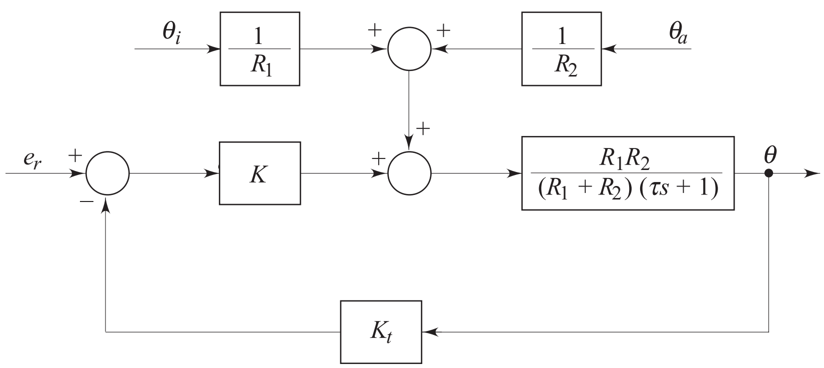
Closed-loop case:

(i) $\dfrac{\theta(s)}{E_r(s)} = M(s) = \dfrac{KR_1}{R_1Cs + 1 + KK_tR_1}$; $S_K^M = \dfrac{R_1Cs + 1}{R_1Cs + 1 + KK_tR_1}$

(ii) $\dfrac{\theta(s)}{\theta_i(s)} = \dfrac{1}{R_1Cs + 1 + KK_tR_1}$. For $\theta_i(s) = 1/s$, $\theta_{ss} = \dfrac{1}{1 + KK_tR_1}$

(iii) $\tau = \dfrac{R_1C}{1 + KK_tR_1}$

(c) $K(e_r - e_t) = C\dfrac{d\theta}{dt} + \dfrac{\theta - \theta_i}{R_1} + \dfrac{\theta - \theta_a}{R_2}$; $R_2 = \dfrac{1}{UA}$, $\tau = \dfrac{R_1R_2C}{R_1 + R_2}$



$$\left.\frac{\theta(s)}{\theta_a(s)}\right|_{\text{open-loop}} = \frac{R_1}{(R_1 + R_2)(\tau s + 1)};\ \theta_{ss}\big|_{\text{open-loop}} = R_1\,(R_1 + R_2)$$

$$\left.\frac{\theta(s)}{\theta_a(s)}\right|_{\text{closed-loop}} = \frac{R_1}{(R_1 + R_2)(\tau s + 1) + KK_tR_1R_2}$$

$$\theta_{ss}\big|_{\text{closed-loop}} = \frac{R_1}{R_1 + R_2 + KK_tR_1R_2}$$

4.19 (a) $\dfrac{Y(s)}{W(s)} = \dfrac{0.1\,(s+1)}{2s^2 + 3s + 1.125}$. For $W(s) = 1/s$, $y_{ss} = 0.089$

(b) Replace K_c by $K_c\left(1+\dfrac{1}{T_I s}\right)$

(c) $$\frac{Y(s)}{W(s)} = \frac{0.1\,(s+1)}{2s^2+3s+1.125}\left[1-\frac{K_f K_c}{s+1}\right]$$

For $W(s) = 1/s$, $y_{ss} = 0.089\,(1 - K_f K_c)$

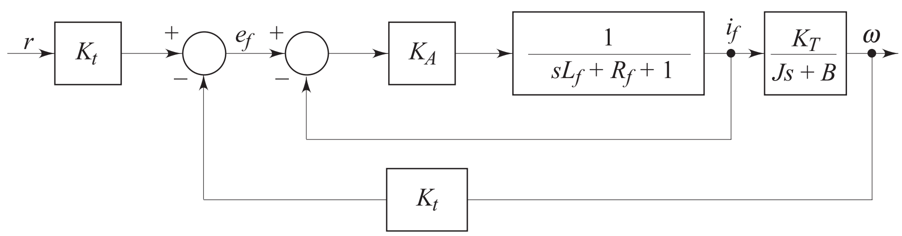
$y_{ss} = 0$ when $K_f = 1/K_c = 0.8$

The PI control scheme increases the order of the system; this makes it less stable.

The feedforward scheme does not affect the characteristic roots of the system. A difficulty with feedforward compensation is that it is an open-loop technique; it contains no self-correcting action. If the value of K_c is not accurately known, the gain K_f will not cancel the disturbance completely.

The usefulness of a method must be determined in the light of the specific performance requirements. If the uncertainty is large, self-correcting action can be obtained by using PI control law in conjunction with the feedforward compensation.

4.20

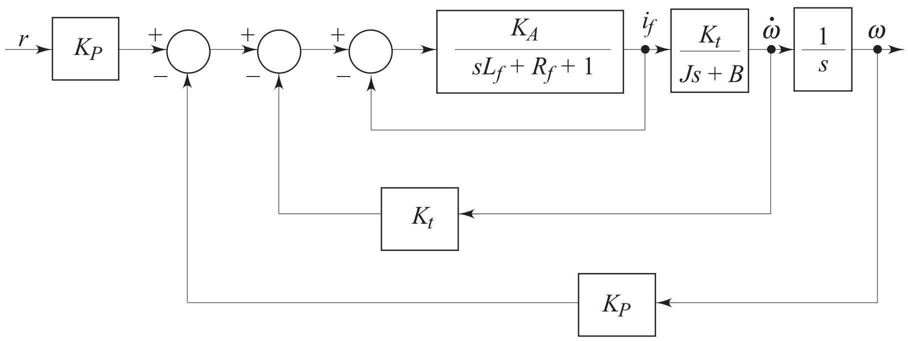


Time-constant (without current feedback) = $L_f/(R_f + 1)$

$$\frac{I_f(s)}{E_f(s)} = \frac{K_A}{sL_f + R_f + K_A + 1}$$

Time-constant (with current feedback) = $\dfrac{L_f}{R_f + K_A + 1}$

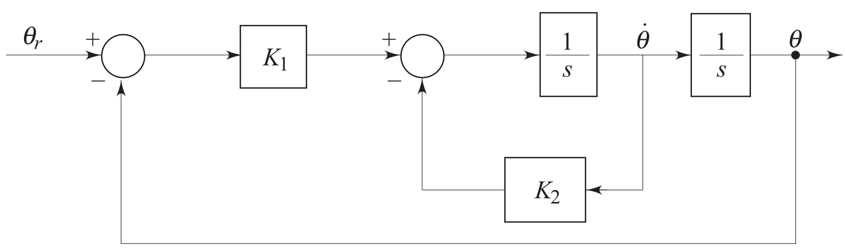
Position control system is shown below.



4.21 (a) $s^2 + K_c T_D s + K_c = (s + 1 + j1.732)(s + 1 - j\,1.732)$

This gives $K_c = 4$, $T_D = 0.5$

(b) Alternative control scheme is shown below.



$$\frac{\theta(s)}{\theta_r(s)} = \frac{K_1}{s^2 + K_2 s + K_1}; \; K_1 = 4, \; K_2 = 2$$

4.22 Fig. P4.22a:

$$\frac{\theta(s)}{\theta_r(s)} = M(s) = \frac{25K}{s^2 + 5s + 25K}$$

$$S_K^M = \frac{\partial M}{\partial K} \times \frac{K}{M} = \frac{s(s+5)}{s^2 + 5s + 25}$$

$$\left|S_K^M(j\omega)\right|_{\omega = 5} = 1.41$$

Fig. P4.22b:

$$\frac{\theta(s)}{\theta_r(s)} = M(s) = \frac{KK_2}{s^2 + (1 + KK_1)_s + KK_2} = \frac{25\,K}{s^2 + 5s + 25K}$$

This gives $K_2 = 25$ and $K_1 = 4$

$$M(s) = \frac{25K}{s^2 + (1+4K)s + 25K}$$

$$S_K^M = \frac{s(s+1)}{s^2 + 5s + 25}$$

$$|S_K^M(j\omega)|_{\omega=5} \approx 1$$

This shows the superiority of the two-loop system over a single-loop system.

CHAPTER 5 CONCEPTS OF STABILITY AND THE ROUTH STABILITY CRITERION

5.1 (a) All the elements in the first column of the Routh array are +ve. Therefore, all the roots are in the left-half plane.

(b) Two sign changes are found in the first column of the Routh array. Therefore, two roots are in the right-half plane and the rest in the left-half plane.

(c) s^3-row of the Routh array has zero pivot element, but the entire row is not all zeros. We replace the pivot element by ε and then proceed with the construction of the Routh array. As $\varepsilon \to 0$, two sign changes are found in the first column of the Routh array. Therefore, two roots are in right-half plane and the rest in the left-half plane.

(d) s^1-row of the Routh array is an all-zero row. Auxiliary polynomial formed using the elements of s^2-row is given by

$$A(s) = s^2 + 1$$

We replace the elements of s^1-row with the coefficients of

$$\frac{dA(s)}{ds} = 2s + 0$$

and proceed with the construction of the Routh array. There are no sign changes in the resulting Routh array; the characteristic polynomial does not have any root in the right-half plane. The roots of the 2nd-order auxiliary polynomial are therefore purely imaginary.

The given characteristic equation has two roots on the imaginary axis and the rest in the left-half plane.

(e) Since all the coefficients of the given characteristic polynomial are not of the same sign, the system is unstable. The Routh array formation is required only if the number of roots in the right-half plane are to be determined.

Only one sign change (the s^0-row has – ve pivot element) is found in the Routh array. Therefore one root is in the right-half plane and the rest in the left-half plane.

(f) s^3-row of the Routh array is an all-zero row. The auxillary polynomial is

$$A(s) = 9s^4 + 0s^2 + 36$$

We replace elements of s^3-row with the coefficients of

$$\frac{dA(s)}{ds} = 36s^3 + 0s$$

The resulting Routh array has s^2-row with zero pivot element but the entire row is not all zeros. We replace the pivot element with ε and proceed with the construction of the Routh array. Two sign changes are found in the first column; the characteristic polynomial $\Delta(s)$ has two roots in the right-half plane. Three possibilities exist: (i) the 4th-order auxiliary polynomial has all the four purely imaginary roots and the two right-half plane roots are contributed by the factor $\Delta(s)/A(s)$, (ii) the 4th-order auxiliary polynomial has complex roots with quadrantal symmetry (two roots in the right-half plane) and the factor $\Delta(s)/A(s)$ has all the roots is the left-half plane, and (iii) the 4th-order auxiliary polynomial has two real-axis roots and two imaginary-axis roots with quadrantal symmetry and the factor $\Delta(s)/A(s)$ has one root in the right-half plane. Examining $A(s)$ we find that auxiliary-polynomial roots are $s = -1 \pm j1, 1 \pm j1$.

The given system, therefore, has two roots in the right-half plane and the rest in the left-hand plane.

5.2 (a) s^1-row of the Routh array is an all-zero row. The auxiliary polynomial is

$$A(s) = s^2 + 9$$

By long division

$$\Delta(s)/A(s) = (s^4 + 2s^3 + 11s^2 + 18s + 18)/A(s) = s^2 + 2s + 2$$

Therefore

$$\begin{aligned} \Delta(s) &= (s^2 + 9)(s^2 + 2s + 2) \\ &= (s + j3)(s - j3)(s + 1 + j1)(s + 1 - j1) \end{aligned}$$

(b) s^3-row of the Routh array is an all-zero row. The auxiliary polynomial is

$$\begin{aligned} A(s) &= s^4 + 24s^2 - 25 \\ \Delta(s)/A(s) &= (s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50)/A(s) \\ &= s + 2 \\ \Delta(s) &= (s + 2)(s^4 + 24s^2 - 25) \\ &= (s + 2)(s^2 - 1)(s^2 + 25) \\ &= (s + 2)(s + 1)(s - 1)(s + j5)(s - j5) \end{aligned}$$

(c) s^3-row of the Routh array is an all-zero row. The auxiliary polynomial is

$$\begin{aligned} A(s) &= s^4 + 3s^2 + 2 \\ \Delta(s)/A(s) &= (s^6 + 3s^5 + 5s^4 + 9s^3 + 8s^2 + 6s + 4)/A(s) \\ &= s^2 + 3s + 2 \\ \Delta(s) &= (s^2 + 3s + 2)(s^4 + 3s^2 + 2) \\ &= (s^2 + 3s + 2)(s^2 + 1)(s^2 + 2) \\ &= (s + 1)(s + 2)(s + j1)(s - j1)(s + j\sqrt{2})(s - j\sqrt{2}) \end{aligned}$$

5.3 With $s = \hat{s} - 1$, the characteristic equation becomes

$$\hat{s}^3 + \hat{s}^2 + \hat{s} + 1 = 0$$

$\hat{s}^1$-row in the Routh array is an all-zero row. The auxiliary polynomial is

$$A(\hat{s}) = \hat{s}^2 + 1 = (\hat{s} + j1)(\hat{s} - j1)$$

We replace the elements of $\hat{s}^2$-row with the coefficients of

$$\frac{dA(\hat{s})}{d\hat{s}} = 2\hat{s} + 0$$

and proceed with the construction of the Routh array. There are no sign changes in the resulting Routh array. The s-polynomial does not have roots to the right of the line at $s = -1$, and there are two roots at $s = -1 \pm j1$. The largest time-constant is therefore 1 sec.

5.4 (a) Unstable for all values of K

(b) Unstable for all values of K

(c) $K < 14/9$

(d) $K > 0.528$

5.5 With $s = \hat{s} - 1$, the characteristic equation becomes

$\hat{s}^3 + 3K\hat{s}^2 + (K + 2)\hat{s} + 4 = 0$

From the Routh array, we find that for $K > 0.528$, all the s-plane roots lie to the left of the line at $s = -1$.

5.6 $$G(s) = \frac{4K}{2s+1}\left[\frac{20/(s+1)}{1+20\times 0.2/(s+1)}\right]$$

$$H(s) = \frac{0.05}{4s+1}$$

$1 + G(s)H(s) = 0$ gives

$8s^3 + 46s^2 + 31s + 5 + 4K = 0$

From the Routh array, we find that the closed-loop system is stable for $K < 43.3$.

5.7 (i) $s^3 + 5s^2 + 9s + K = 0$; $K < 45$

(ii) $s^3 + 5s^2 + (9 - K)s + K = 0$; $K < 7.5$

(iii) $s^4 + 7s^3 + 19s^2 + (18 - K)s + 2K = 0$; $K < 10.1$

5.8 (a) $\Delta(s) = s^3 + 10s^2 + (21 + K)s + 13K = 0$

(i) $K < 70$ (ii) $K = 70$

(iii) for $K = 70$, $A(s) = s^2 + 91 = 0$

$\Delta(s) = (s^2 + 91)(s + 10) = (s + j9.54)(s - j9.54)(s + 10)$

(b) $\Delta(s) = s^3 + 5s^2 + (K - 6)s + K = 0$

(i) $K > 7.5$ (ii) $K = 7.5$

(iii) For $K = 7.5$, $A(s) = s^2 + 1.5 = 0$

$\Delta(s) = (s^2 + 1.5)(s + 5) = (s + j1.223)(s - j1.223)(s + 5)$

(c) $\Delta(s) = s^4 + 7s^3 + 15s^2 + (25 + K)s + 2K = 0$

(i) $K < 28.1$ (ii) $K = 28.1$

(iii) For $K = 28.1$, $A(s) = s^2 + 7.58 = 0$

$$\Delta(s) = (s^2 + 7.58)(s^2 + 7s + 7.414)$$
$$= (s + j2.75)(s - j2.75)(s + 5.7)(s + 1.3)$$

5.9 (a) $\Delta(s) = s^4 + 12s^3 + 69s^2 + 198s + 200 + K = 0$

From the Routh array we find that the system is stable for $K < 666.25$.

For $K = 666.25$,

$$A(s) = 52.5s^2 + 866.25 = 0$$

This gives

$$s = \pm\, j\, 4.06$$

Frequency of oscillations is 4.06 rad/sec when $K = 666.25$.

(b) $\Delta(s) = s^4 + 3s^3 + 12s^2 + (K - 16)s + K = 0$

From the Routh array, we find that the system is stable for $23.3 < K < 35.7$.

For $K = 23.3$,

$$A(s) = 9.57s^2 + 23.3 = 0;\ s = \pm\, j1.56$$

For $K = 35.7$,

$$A(s) = 5.43s^2 + 35.7 = 0;\ s = \pm\, j2.56$$

Frequency of oscillation is 1.56 rad/sec when $K = 23.3$; and 2.56 rad/sec when $K = 35.7$.

5.10 $\Delta(s) = s^3 + as^2 + (2 + K)s + 1 + K = 0$

From the Routh array we find that for the system to oscillate,

$(2 + K)a = 1 + K$

Oscillation frequency $= \sqrt{\dfrac{1+K}{a}} = 2$

These equations give $a = 0.75$, $K = 2$

5.11 $\Delta(s) = 0.02s^3 + 0.3s^2 + s + K = 0$

(a) $K < 15$

(b) For $K = 15$, the auxiliary equation is $A(s) = s^2 + 50 = 0$. The oscillation frequency is 7.07 rad/sec at $K = 15$.

(c) With $K = 7.5$ and $s = \hat{s} - 1$,

$\Delta(\hat{s}) = 0.2\,\hat{s}^3 + 2.4\,\hat{s}^2 + 4.6\,\hat{s} + 67.8 = 0$

Two sign changes are there in the first column of the Routh array; therefore two roots have time-constant larger than 1 sec.

5.12 (a) Unstable for all K_c

(b) For stability, $T_D > \tau$.

5.13 (a) $\Delta(s) = s^3 + 15s^2 + 50s + 25K = 0$; $K < 30$

(b) $\Delta(s) = \tau s^3 + (1 + 5\tau)s^2 + 5s + 50 = 0;\ \tau < 0.2$

(c) $\Delta(s) = \tau s^3 + (1 + 5\tau)s^2 + 5s + 2.5K = 0;\ K < \left(\dfrac{2}{\tau} + 10\right)$

5.14 $\Delta(s) = s^4 + 5s^3 + 6s^2 + K_c s + K_c\alpha = 0;\ \alpha = \dfrac{1}{T_I}$

From the Routh array, we find that $K < (30 - 25\alpha)$ results in system stability. Larger the α (smaller the T_I), lower is the limit on gain for stability.

5.15 $\Delta(s) = s^3 + 15s^2 + (50 + 100K_t)s + 100K = 0$

From the Routh array, we find that for stability $K < (7.5 + 15K_t)$. Limit on K for stability increases with increasing value of K_t.

CHAPTER 6 THE PERFORMANCE OF FEEDBACK SYSTEMS

6.1 (a) Refer Section 6.3 for the derivation

(b) The characteristic equation is

$$s^2 + 10s + 100 = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

(i) $\omega_n = 10$ rad/sec; $\zeta = 0.5$; $\omega_d = \omega_n\sqrt{1-\zeta^2} = 8.66$ rad/sec

(ii) $t_r = \dfrac{\pi - \cos^{-1}\zeta}{\omega_d} = 0.242$ sec; $t_p = \dfrac{\pi}{\omega_d} = 0.363$ sec

$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 16.32\%$; $t_s = \dfrac{4}{\zeta\omega_n} = 0.8$ sec

(iii) $K_p = \lim_{s\to 0} G(s) = \infty$; $K_v = \lim_{s\to 0} sG(s) = 10$;

$K_a = \lim_{s\to 0} s^2G(s) = 0$

(iv) $e_{ss} = \dfrac{1}{1+K_p} = 0$; $e_{ss} = \dfrac{1}{K_v} = 0.1$; $e_{ss} = \dfrac{1}{K_a} = \infty$

6.2 Characteristic equation is

$$s^2 + 10s + 10K_A = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

(a) $K_A = 2.5$ gives $\zeta = 1$ which meets the response requirements.

(b) $K_p = \infty$, $K_v = 2.5$

$$e_{ss} = \frac{5}{1+K_p} + \frac{1}{K_v} = 0.4$$

6.3 (a) Using Routh criterion, it can be checked that the close-loop system is stable.

(i) $e_{ss} = \dfrac{10}{K_v} = \dfrac{10}{\infty} = 0$; (ii) $e_{ss} = \dfrac{10}{K_v} + \dfrac{0.2}{K_a} = \dfrac{10}{\infty} + \dfrac{0.2}{0.1} = 2$

(b) The closed-loop system is unstable.

6.4 The closed-loop system is stable. This can easily be verified by Routh criterion.

$$G(s) = \frac{5}{(0.1s+1)(s+1)(0.2s+1)}$$

$$e_{ss}|_{r\,=\,10} = \frac{10}{1+K_p} = \frac{10}{1+5}$$

$$\frac{Y(s)}{W(s)} = \frac{0.1s+1}{(s+1)(0.2s+1)(0.1s+1)+5}$$

For $W(s) = 4/s$, $y_{ss} = 4/6$

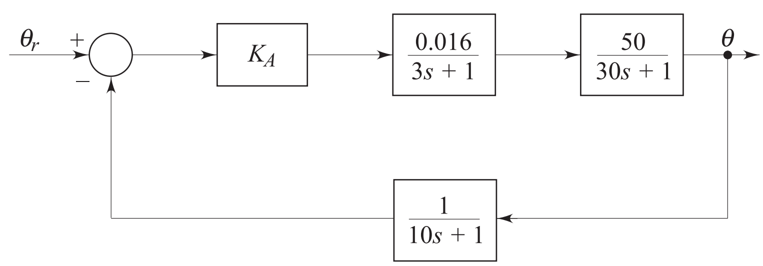
$$e_{ss}|_{\text{total}} = 7/3$$

6.5 (a) The characteristic equation of the system is

$900s^3 + 420s^2 + 43s + 1 + 0.8\,K_A = 0$

Using Routh criterion, we find that the system is stable for $K_A < 23.84$.

(b) Rearrangement of the block diagram of Fig. P6.5 is shown below.



$$G(s) = \frac{0.016 \times 50K_A}{(3s+1)(30s+1)}$$

$$H(s) = \frac{1}{10s+1}\text{; the dc gain of } H(s) \text{ is unity.}$$

$$K_p = \lim_{s\to 0} G(s) = 0.8\,K_A\,;\; e_{ss} = \frac{10}{1+K_p} = 1$$

This gives $K_A = 11.25$

6.6 (a) The system is stable for $K_c < 9$.

$K_p = \lim_{s\to 0} D(s)\,G(s) = K_c$; $e_{ss} = \dfrac{1}{1+K_c}$

e_{ss}= 0.1 (10%) is the minimum possible value for steady-state error. Therefore e_{ss} less than 2% is not possible with proportional compensator.

(b) Replace K_c by $D(s) = 3 + \dfrac{K_I}{s}$. The closed-loop system is stable for $0 < K_1 < 3$. Any value in this range satisfies the static accuracy requirements.

6.7 (a) $K_v = \dfrac{1000K_c}{10} = 1000$

$$\zeta = \frac{10 + 1000K_D}{2 \times 100} = 0.5$$

These equations give $K_c = 10$ and $K_D = 0.09$.

(b) The closed-loop poles have real part $= -\zeta\omega_n = -50$. The zero is present at $s = -K_c/K_D = -111.11$. The zero will result in pronounced early peak. ζ is not an accurate estimate of M_p.

6.8 (a) $K_v = K_I = 10$

(b) With $K_I = 10$, the characteristic equation becomes

$$\Delta(s) = s^3 + 10s^2 + 100(1 + K_c)s + 1000 = 0$$

$$\text{With } s = \hat{s} - 1,$$

$$\Delta(\hat{s}) = \hat{s}^3 + 7\hat{s}^2 + [100(1 + K_c) - 17]\hat{s} + 1009 - 100\,(1 + K_c) = 0$$

In Routh array formation, $K_c = 0.41$ results in auxiliary equation with purely imaginary roots. Therefore $K_c = 0.41$ results in dominant s-plane poles with real part $= -1$.

(c) Routh array formation for $\Delta(\hat{s})$ gives the following auxiliary equation.

$$A(\hat{s}) = 7\hat{s}^2 + 868 = 0;\ \hat{s} = \pm j11.14$$

The complex-conjugate s-plane roots are $-1 \pm j11.14$.

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 11.14;\ \zeta\omega_n = 1$$

These equations give $\zeta = 0.09$.

(d) $\Delta(\hat{s}) = (\hat{s} + 7)(\hat{s} + j11.14)(\hat{s} - j11.14) = 0$

The third s-plane closed-loop pole is at $s = -8$. The dominance condition in terms of third closed-loop pole is reasonably satisfied. The zero is at $s = -K_I/K_c = -24.39$. The zero will not give pronounced early peak. Therefore ζ approximately represents M_p.

6.9 (a) With $T_I = \infty$,

$$D(s)G(s) = \frac{80(1 + T_D s)}{s^2 + 8s + 80}$$

The characteristic polynomial becomes

$$\Delta(s) = s^2 + (8 + 80T_D)s + 160$$

$$T_D = 0.216 \text{ gives } \zeta = 1$$

(b) The zero is at $s = -1/0.216 = -4.63$. Real part of the closed-loop pole $= -\zeta\omega_n = -12.65$. Therefore, small overshoot will be observed.

(c) The characteristic equation becomes

$$s^3 + 25.28s^2 + 160s + \frac{80}{T_I} = 0$$

$T_I > 0.0198$ for stability.

6.10 (a) $G(s) = \dfrac{K}{s(s+1)}$; $K_v = \lim\limits_{s\to 0} sG(s) = K$

$K = 10$ will give steady-state unit-ramp following error of 0.1.

$$\Delta(s) = s^2 + s + K = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$K = 10 \text{ gives } \zeta = 0.158.$$

(b) $G(s) = \dfrac{10}{s(s+1+10K_t)}$

$$\Delta(s) = s^2 + (1 + 10K_t)s + 10 = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$K_t = 0.216$ gives $\zeta = 0.5$.

$$K_v = \lim_{s\to 0} sG(s) = \frac{10}{1+10K_t} = \frac{10}{3.16}$$

$e_{ss} = 0.316$

(c) $G(s) = \dfrac{10K_A}{s(s+1+10K_t)}$

$$K_v = \frac{10K_A}{1+10K_t}$$

$$\Delta(s) = s^2 + (1 + 10K_t)s + 10\,K_A = s^2 + 2\zeta\omega_n s + \omega_n^2$$

From these equations, we find that

$K_A = 10$ and $K_t = 0.9$ give $e_{ss} = 0.1$ and $\zeta = 0.5$.

6.11 (a) $\zeta^2 = \dfrac{(\text{In } M_p)^2}{(\text{In } M_p)^2 + \pi^2}$

$M_p = 0.1$ gives $\zeta = 0.59$

$t_s = \dfrac{4}{\zeta\omega_n} = 0.05$. Therefore, $\omega_n = 135.59$

$G(s) = \dfrac{5K_1}{s(0.2s + 1 + 100K_2)}$

$\Delta(s) = 0.2s^2 + s(1 + 100K_2) + 5K_1 = 0$

Therefore $s^2 + 5(1 + 100K_2)s + 25K_1 = s^2 + 2\zeta\omega_n s + \omega_n^2$

This equation gives $K_1 = 735.39$ and $K_2 = 0.31$

(b) $K_p = \infty$, $K_v = \dfrac{5K_1}{1 + 100K_2} = 114.9$, $K_a = 0$

6.12 $\dfrac{\theta(s)}{T_w(s)} = \dfrac{4}{s(s+1) + 10K + 10K_t s}$

For $T_w(s) = 1/s$, $\theta_{ss} = \dfrac{4}{10K} = 0.05$

This gives $K = 8$

$\Delta(s) = s^2 + (1 + 10K_t)s + 10K = s^2 + 2\zeta\omega_n s + \omega_n^2$

$K_t = 0.79$ gives $\zeta = 0.5$

6.13 $\dfrac{Y(s)}{F(s)} = \dfrac{1}{Ms^2 + Bs + K}$. For $F(s) = 9/s$, $y_{ss} = \dfrac{9}{K} = 0.03$

This gives $K = 300$ Newtons/m

$e^{-\pi\zeta/\sqrt{1-\zeta^2}} = \dfrac{0.003}{0.03} = 0.1$

This gives $\zeta = 0.59$

$t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{0.81\omega_n} = 2$

This gives $\omega_n = 1.94$ rad/sec

From the equation

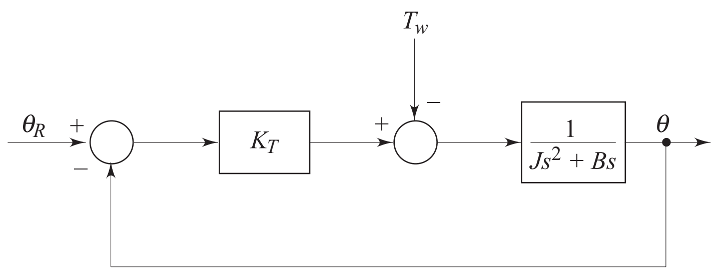
$$s^2 + \frac{B}{M}s + \frac{K}{M} = s^2 + 2\zeta\omega_n s + \omega_n^2$$

we obtain

$$M = 79.71 \text{ kg}, B = 182.47 \text{ Newtons per m/sec}$$

6.14

$$\frac{\theta(s)}{\theta_R(s)} = \frac{K_T / J}{s^2 + \frac{B}{J}s + \frac{K_T}{J}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



$$\zeta^2 = \frac{(\ln M_p)^2}{(\ln M_p)^2 + \pi^2}$$

For $M_p = 0.25$, $\zeta = 0.4$

Steady-state error to unit-ramp input = $\dfrac{2\zeta}{\omega_n} = 0.04$

This gives $\omega_n = 20$

$$\frac{\theta(s)}{-T_w(s)} = \frac{1}{Js^2 + Bs + K_T}$$

$$\theta_{ss} = \frac{10}{K_T} = 0.01$$

From these equations, we obtain the following values of system parameters.

$K_T = 1000$ N-m/rad

$B = 40$ N-m/(rad/sec)

$J = 2.5$ kg-m^2

6.15 $J\dot{\omega} + B\omega = K_T(\omega_r - \omega)$

$$\frac{\omega(s)}{\omega_r(s)} = \frac{45}{100s+50}$$

$$\omega_r = 50 \times \frac{2\pi}{60} \text{ rad/sec = step input}$$

$$\omega(t) = 1.5\pi\,(1 - e^{-0.5t})$$

$$\omega_{ss} = 1.5\pi \text{ rad/sec = 45 rpm; } e_{ss} = 5 \text{ rpm}$$

A control scheme employing gain adjustment with integral error compensation will remove the offset.

6.16 $J\ddot{\theta} + B\dot{\theta} = K_T(\theta_r - \theta)$

$$\frac{\theta(s)}{\theta_r(s)} = \frac{2400}{150s^2 + 600s + 2400}$$

$$\theta_r(s) = \frac{\pi}{3s}\,;\ \theta(\text{s}) = \frac{\pi}{3}\left[\frac{16}{s(s^2+4s+16)}\right]$$

$$\theta(t) = \frac{\pi}{3}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \cos^{-1}\zeta)\right]$$

$$= \frac{\pi}{3}\left[1 - 1.1547e^{-2t}\sin\left(3.46t + \frac{\pi}{3}\right)\right]$$

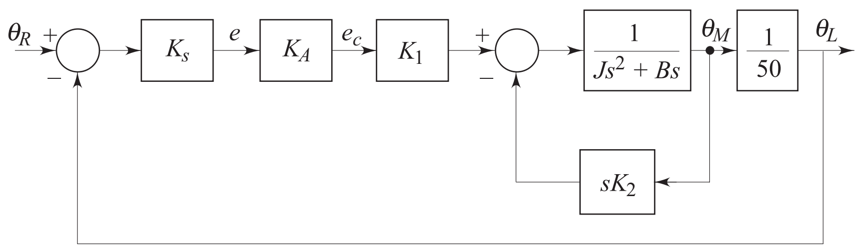
$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 16.3\%$$

The following control schemes have the potential of eliminating overshoot:

(i) gain adjustment with derivative error compensation; and

(ii) gain adjustment with derivative output compensation.

6.17



(b) $G(s) = \dfrac{4K_A}{s(s+100.5)}$

$K_v = 4K_A/100.5$

For a ramp input $\theta_R(t) = \pi t$, the steady-state error $= \dfrac{\pi}{K_v}$

The specification is 5 deg or $\dfrac{5\pi}{180}$ rad.

Therefore

$$\frac{100.5\pi}{4K_A} = \frac{\pi}{180}\text{ ; this gives } K_A = 904.5$$

The characteristic equation is

$$s^2 + 100.5s + 4 \times 904.5 = 0$$

$$\omega_n = 60.15;\ \zeta = 0.835;\ M_p = 0.85\%;\ t_s = 0.0796 \text{ sec}$$

(c) With PI control the system becomes type-2; and the steady-state error to ramp inputs is zero provided the closed-loop system is stable.

$$G(s) = \frac{4K_A\left(1+\dfrac{1}{s}\right)}{s(s+100.5)}\text{ ; } K_A = 904.5$$

The characteristic equation is

$$s^3 + 100.5s^2 + 4 \times 904.5s + 4 \times 904.5 = 0$$

It can easily be verified that the closed-loop system is stable.

(d) The characteristic equation can be expressed as

$$(s + 1.0291)\ (s^2 + 99.4508s + 3514.6332) = 0$$

or

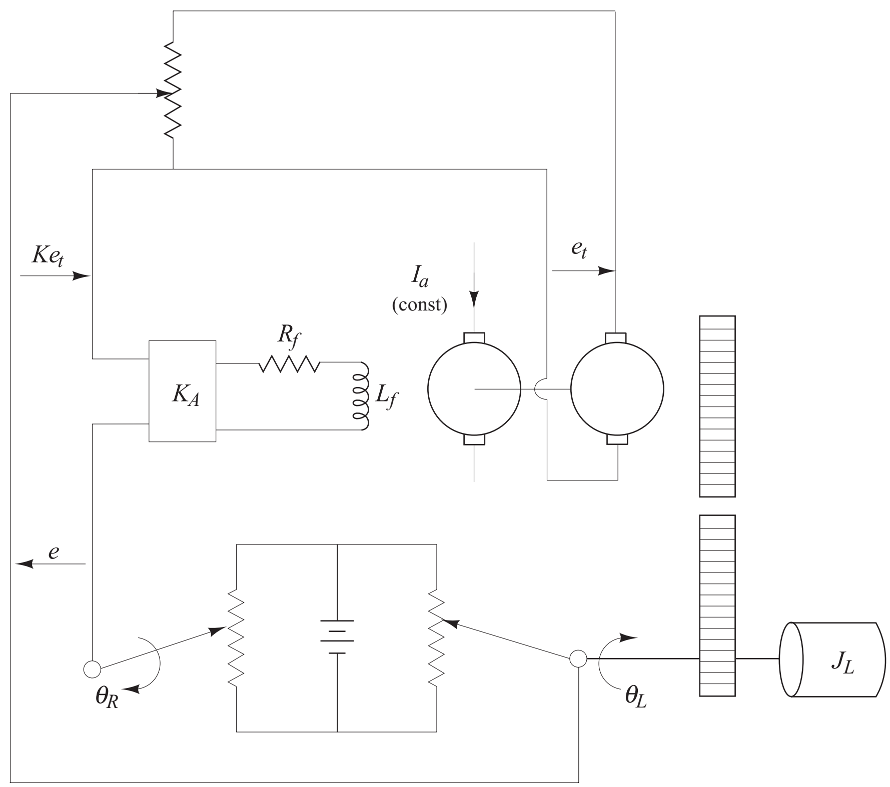
$$(s + 1.0291)\ (s + 49.7254 + j32.2803)\ (s + 49.7254 - j32.2803) = 0$$

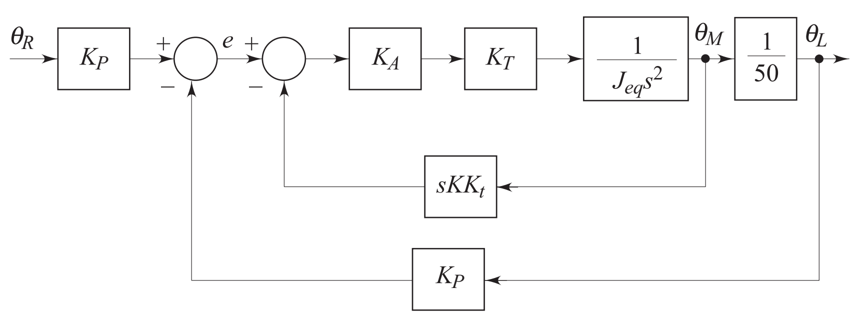
The dominance condition is satisfied because the real pole is very close to the zero. Therefore, the transient response resembles that of a second-order system with characteristic equation

$$s^2 + 99.4508s + 3514.6332 = 0$$

$$\omega_n = 59.28;\ \zeta = 0.84;\ M_p = 0.772\%;\ t_s = 0.08 \text{ sec}$$

6.18



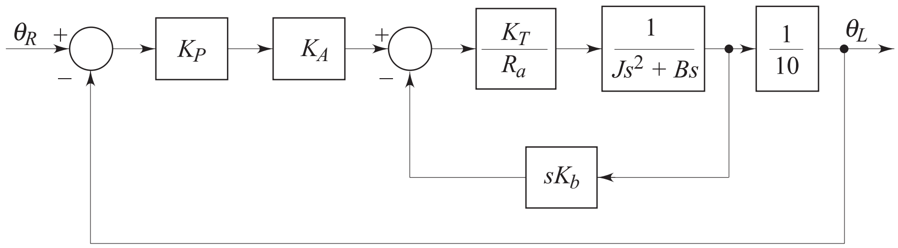


(c) The characteristic equation of the system is

$$s^2 + 100\ KK_A s + 6K_A = s^2 + 2\zeta\omega_n\ s + \omega_n^2 = 0$$

This gives $K_A = 2.67$; $K = 0.024$

6.19



(b) Open-loop transfer function $G(s) = \dfrac{20K_A}{s(4s+202)}$

$$K_v = \frac{20K_A}{202} = \frac{1}{0.01}\text{; this gives } K_A = 1010.$$

The characteristic equation becomes

$$s^2 + 50.5s + 5050 = 0 = s^2 + 2\zeta\omega_n s + \omega_n^2$$

This gives $\zeta = 0.355$; $M_p = 30.33\%$

(c) $G(s) = \dfrac{20(K_A + sK_D)}{s(4s+202)}$

$$K_v = \frac{20K_A}{202} = 100\text{; no effect on } e_{ss}.$$

Characteristic equation of the system is

$$s^2 + (50.5 + 5K_D)s + 5050 = 0 = s^2 + 2\zeta\omega_n s + \omega_n^2$$

This gives $K_D = 6.95$.

For $\zeta = 0.6$, $M_p = 9.5\%$

(d) System zero: $s = \dfrac{-K_A}{K_D} = -145.32$

Real part of closed-loop poles $= -\zeta\omega_n = -42.636$.

The zero will affect the overshoot. The peak overshoot will be slightly more than 9.5%.

6.20 (a) $G(s) = \dfrac{3K_A}{s(0.3s+1)(1+2s)}$

$$K_v = 3K_A;\ e_{ss} = \frac{0.03}{3K_A} = \frac{0.01}{K_A}$$

(b) $$\frac{Y(s)}{F_w(s)} = \frac{0.3(1+2s)}{s(0.3s+1)(1+2s)+3K_A}$$

$$y_{ss} = \frac{0.1}{K_A}$$

(c) Characteristic equation is

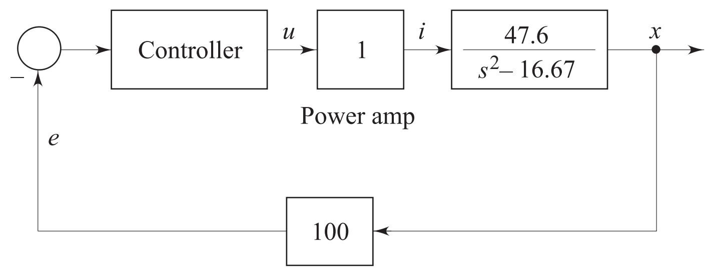
$$0.6s^3 + 2.3s^2 + s + 3K_A = 0$$

For stability, $K_A < 1.28$.

Minimum value of error in part (a) = 7.81×10^{-3}

Minimum value of error in part (b) = 0.078

6.21



(a) Characteristic equation of the system is

$$s^2 - 16.67s + 4760K_c = 0$$

Unstable or oscillatory for all $K_c > 0$.

(b) $U(s) = -K_c(1 + sT_D)\,E(s)$

Characteristic equation becomes

$$s^2 + 4760\,K_cT_Ds + 4760K_c - 16.67 = 0$$

For stability, $K_c > 3.5 \times 10^{-3}$ and $T_D > 0$.

(c) $$\zeta^2 = \frac{[\ln(0.2)]^2}{[\ln(0.2)]^2 + \pi^2}\ ;\ \zeta = 0.456$$

$$t_s = \frac{4}{\zeta\omega_n} = 0.4;\ \omega_n = 21.93$$

$s^2 + 4760\ K_cT_Ds + 4760\ K_c - 16.67 = s^2 + 2\zeta\omega_n s + \omega_n^2$

This gives $K_c = 0.1045$, $T_D = 0.04$.

6.22 (a) System-type number = 2

(b) $\dfrac{H(s)}{-Q_w(s)} = \dfrac{0.5s(0.1s+1)}{s^2(0.1s+1)+0.5(K_c+sK_D)}$

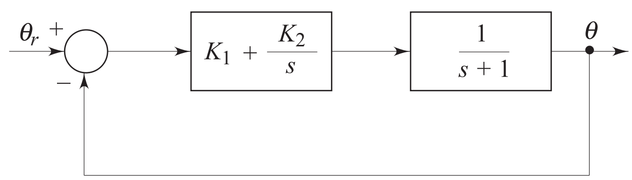
$h_{ss} = -\dfrac{0.5}{0.5K_c}$; $K_c > 10$

The characteristic equation is

$$0.1s^3 + s^2 + 0.5K_Ds + 0.5K_c = 0$$

For stability, $K_D > 0.1K_c$

6.23



Characteristic equation is

$s^2 + (1 + K_1)s + K_2 = 0 = s^2 + 2\zeta\omega_n s + \omega_n^2$

$M_p = 20\% \rightarrow \zeta = 0.456$; $t_s = 2 \rightarrow \omega_n = 4.386$

$K_2 = 19.237$; $K_1 = 3$

CHAPTER 7 COMPENSATOR DESIGN USING ROOT LOCUS PLOTS

7.1 (a) $-\sigma_A = -2$; $\phi_A = 60^\circ, 180^\circ, 300^\circ$; intersection with $j\omega$-axis at $\pm j5.042$; angle of departure ϕ_p from $-1 + j4 = 40^\circ$.

(b) $-\sigma_A = -1.25$; $\phi_A = 45^\circ, 135^\circ, 225^\circ, 315^\circ$; intersection with $j\omega$-axis at $\pm j1.1$; angle of departure ϕ_p from $-1 + j1 = -71.6^\circ$; multiple roots at $s = -2.3$.

(c) Angle of arrival ϕ_z at $-3 + j4 = 77.5^\circ$; multiple roots at $s = -0.45$.

(d) $-\sigma_A = -1.33$; $\phi_A = 60^\circ$; $180^\circ, 300^\circ$; intersection with $j\omega$-axis at $\pm j\sqrt{5}$; angle of departure ϕ_p from $-2 + j1 = -63.43^\circ$; multiple roots at $s = -1, -1.67$.

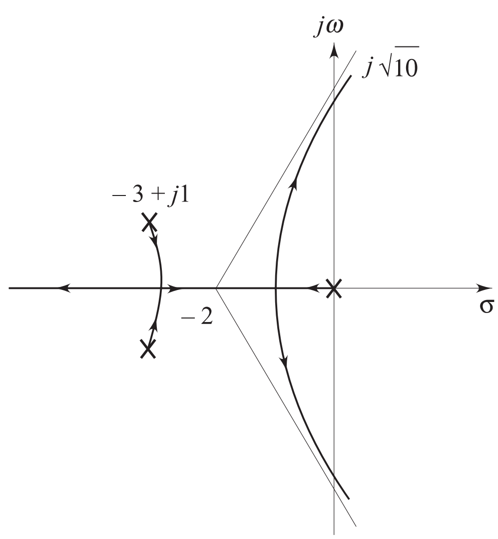
(e) $-\sigma_A = -1.5$; $\phi_A = 90^\circ, 270^\circ$

(f) $-\sigma_A = -5.5$; $\phi_A = 90^\circ, 270^\circ$; multiple roots at $s = -2.31, -5.18$.

(g) $-\sigma_A = 0.5$; $\phi_A = 90^\circ, 270^\circ$; intersection with $j\omega$-axis at $\pm j2.92$.

7.2 (i) $-\sigma_A = -2$; $\phi_A = 60^\circ, 180^\circ, 300^\circ$; intersection with $j\omega$-axis at $\pm j\sqrt{10}$; angle of departure ϕ_p from $-3 + j1 = -71.5^\circ$.

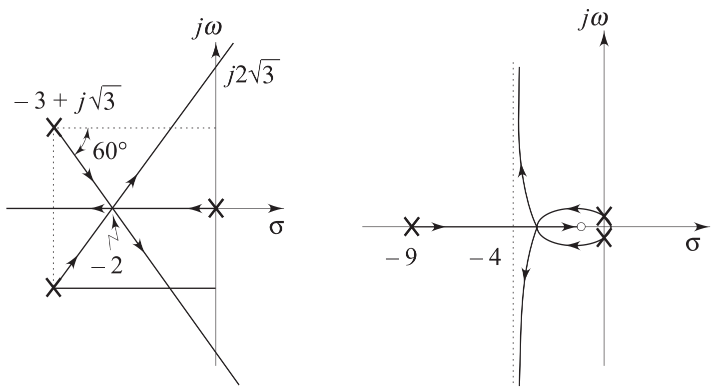
Two root loci break away from $s = -1.1835$ at $\pm 90^\circ$. Two root loci approach $s = -2.8165$ at $\pm 90^\circ$



(i)

(ii) Intersection with $j\omega$-axis at $\pm\ j2\sqrt{3}$; angle of departure ϕ_p from $-3 + j\sqrt{3} = -60^\circ$.

Three root loci approach the point $s = -2$, and then break away in directions 120º apart.



(ii) (iii)

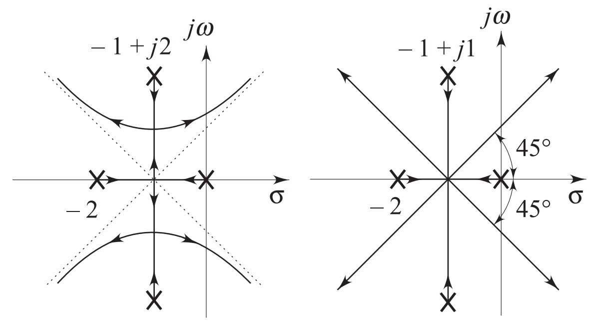
(iii) $-\sigma_A = -4$; $\phi_A = 90^\circ, 270^\circ$

The characteristic equation has three roots at $s = -3$; the three root loci originating from open-loop poles approach this point and then breakaway. Tangents to the three loci breaking away from $s = -3$ are 120º apart.

(iv) $-\sigma_A = -1$; $\phi_A = 45^\circ, 135^\circ, -45^\circ, -135^\circ$; intersection with the $j\omega$ axis at $\pm\ j$ 1.58; angle of departure ϕ_p from $-1 + j2 = -90^\circ$

There are two roots at $s = -1$, two roots at $s = -1 + j1.22$, and two roots at $s = -1 - j1.22$. The break away directions are shown in the figure.

(v) There are four roots at $s = -1$. The break away directions are shown in the figure.



(iv) (v)

7.3 $-\sigma_A = -1$; ϕ_A = 60º, 180º, 300º; intersection with the $j\omega$-axis at $\pm\ j\sqrt{2}$; multiple roots at $s = -0.42$.
The $\zeta = 0.5$ loci passes through the origin and makes an angle of $\theta = \cos^{-1}\zeta$ = 60º with the negative real axis. The point $s_d = -0.33 + j0.58$ on this line satisfies the angle criterion. By magnitude criterion, the value of K at this point is found to be 1.04. Using this value of K, the third pole is found at $s = -2.33$. Therefore,

$$M(s) = \frac{1.04}{(s+0.33+j0.58)(s+0.33-j0.58)(s+2.33)}$$

7.4 $-\sigma_A = -3$; ϕ_A = 45º, 135º, 225º, 315º; intersection with the $j\omega$-axis at $\pm j3.25$; departure angle ϕ_p from $-4 + j4$ = 225º; multiple roots at $s = -1.5$.
At the intersection of the $\zeta = 0.707$ line with the root locus, the value of K, by magnitude criterion is 130. The remaining pair of complex roots for $K = 130$ can be approximately located graphically. It turns out that real part of complex pair away from $j\omega$-axis is approximately four times as large as that of the pair near $j\omega$-axis. Therefore, the transient response term due to the pair away from the $j\omega$-axis will decay much more rapidly than the transient response term due to the pair near $j\omega$-axis.

7.5 $-\sigma_A = -2.67$; ϕ_A = 60º, 180º, 300º ; intersection with $j\omega$-axis at $\pm\ j\sqrt{32}$; departure angle ϕ_p from $-4 + j4 = -45$º.
The point $s_d = -2 + j3.4$ on the $\zeta = 0.5$ line satisfies the angle criterion. By magnitude criterion, the value of K at this point is found to be 65. Using this value of K, the third pole is found at $s = -4$; the dominance condition is not satisfied.

7.6 $-\sigma_A = -2$; ϕ_A = 60º, 180º, 300º; intersection with the $j\omega$-axis at $\pm j\sqrt{5}$; multiple roots at $s = -0.473$.

The least value of K to give an oscillatory response is the value of K at the multiple roots (K = 1.128).

The greatest value of K that can be used before continuous oscillations occur is the value of K at the point of intersection with the $j\omega$-axis. From the Routh array formulation it is found that the greatest value of K is 30 and at this gain continuous oscillations of frequency $\sqrt{5}$ rad/sec occur.

7.7 The proof given in Example 7.1.

7.8 The proof follows from Example 7.1.
Draw a line from the origin tangential to the circular part of the locus. This line corresponds to damping ratio for maximum oscillatory response. The tangential line corresponds to ζ = 0.82 and the value of K at the point of tangency is 2.1.

7.9 The proof given in Example 7.2

7.10 $s = \sigma + j\omega$

$$\tan^{-1}\frac{\omega+1}{\sigma} + \tan^{-1}\frac{\omega-1}{\sigma} = \tan^{-1}\frac{\omega}{\sigma} + \tan^{-1}\frac{\omega}{\sigma+2} \pm 180°(2q+1)$$

Taking tangents on both sides of this equation, and noting that

$$\tan\left[\tan^{-1}\frac{\omega}{\sigma+2} \pm 180°\right] = \frac{\omega}{\sigma+2},$$

we obtain

$$\frac{\dfrac{\omega+1}{\sigma} + \dfrac{\omega-1}{\sigma}}{1-\left(\dfrac{\omega+1}{\sigma}\right)\left(\dfrac{\omega-1}{\sigma}\right)} = \frac{\dfrac{\omega}{\sigma} + \dfrac{\omega}{\sigma+2}}{1-\left(\dfrac{\omega}{\sigma}\right)\left(\dfrac{\omega}{\sigma+2}\right)}$$

Manipulation of this equation gives

$$\left[\left(\sigma-\frac{1}{2}\right)^2 + \omega^2 - \frac{5}{4}\right] = 0$$

There exists a circular root locus with centre at σ = ½, ω = 0 and the radius equal to $\sqrt{5}/2$.

7.11 Use the result in Problem 7.7 for plotting the root locus. Complex-root branches form a circle.
The ζ = 0.707 line intersects the root locus at two points; the values of K at these points are 1 and 5. The point that corresponds to K = 5 results in lower value of t_s; therefore we choose $K = 5(s_d = -3 + j3)$.

$$\text{For } \zeta = 0.707,\ M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 4.3\%$$

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{3} \text{ sec}$$

$$G(s) = \frac{K(s+4)}{(s+2)(s-1)}$$

$$E(s) = \frac{1}{1+G(s)} R(s) = \frac{(s+2)(s+1)}{(s+2)(s-1)+K(s+4)} R(s)$$

For $R(s) = 1/s$ (assuming selection of K that results in stable closed-loop poles)

$$e_{ss} = \lim_{s\to 0} sE(s) = \frac{-2}{-2+4K}$$

For $K = 5$, $e_{ss} = -1/9$

Therefore, steady-state error is 11.11%.

7.12 Breakaway points obtained from the solution of the equation $dK/ds = 0$, are $s = -0.634, -2.366$. Two root loci break away from the real axis at $s = -0.634$, and break into the real axis at $s = -2.366$. By determining a sufficient number of points that satisfy the angle criterion, it can be found that the root locus is a circle with centre at -1.5 that passes through the breakaway points.

Both the breakaway points correspond to $\zeta = 1$. For minimum steady-state error we should select larger value of K.

$s = -0.634$ corresponds to $K = 0.0718$

$s = -2.366$ corresponds to $K = 14$

7.13 The characteristic equation is

$$1 + \frac{0.8K}{(10s+1)(30s+1)(3s+1)} = 0$$

or
$$1 + \frac{K'}{\left(s+\frac{1}{10}\right)\left(s+\frac{1}{30}\right)\left(s+\frac{1}{3}\right)} = 0 \ ;\ K' = 0.000888\, K$$

$-\sigma_A = -0.155$; $\phi_A = 60°, 180°, 300°$; intersection with the $j\omega$-axis at $\pm j\,0.22$; multiple roots at $s = -0.063$

The point on the $\zeta = 0.707$ line that satisfies the angle criterion is $s_d = -0.06 \pm j0.06$; the value of K' at this point, obtained by magnitude criterion, is 0.00131. Therefore $K = 1.475$.

7.14 (a) Ideal feedback sensor:

$$1 + \frac{K}{s(0.1s+1)} = 1 + \frac{10K}{s(s+10)} = 0$$

$K = 10$ gives $\zeta = 0.5$.

(b) Sensor with appreciable time-constant:

$$1 + \frac{K}{s(0.1s+1)(0.02s+1)} = 0$$

or
$$1 + \frac{K'}{s(s+10)(s+20)} = 0 \; ; \; K' = 500 \text{ K}$$

Locate a root locus point on the real axis that gives $K' = 5000$ (i.e., $K = 10$). Complex-conjugate roots can then be found by long division. The process gives $\zeta = 0.4$.

7.15 Characteristic equation is given by

$$\Delta(s) = s^3 + 6s^2 + 8s + 0.1K(s + 10) = 0$$

which can be arranged as

$$1 + \frac{K'(s+10)}{s(s+2)(s+4)} = 0 \; ; \; K' = 0.1K$$

$-\sigma_A = 2$; $\phi_A = 90°, -90°$; intersection with the $j\omega$-axis at $\pm\, j\sqrt{20}$; multiple roots at $s = -0.8951$.

The point on $\zeta = 0.5$ line that satisfies the angle criterion is $s_d = -0.75 + j1.3$. The value of K' at this point, obtained by magnitude criterion, is 1.0154. Hence $K = 10.154$.

The third closed-loop pole is found at $s = -4.5$

7.16 Characteristic equation of the system is

$$s^3 + s + 10K_t s + 10 = 0$$

which can be rearranged as

$$1 + \frac{Ks}{s^2 + s + 10} = 0 \; ; \; K = 10K_t$$

Notice that a zero is located at the origin and open-loop poles are located at $s = -0.5 \pm j3.1225$. As per the result of Problem 7.9, a circular root locus exists with the centre at zero and radius equal to $\sqrt{10}$.

The point on the $\zeta = 0.7$ line that satisfies the angle criterion is $s_d = -2.214 + j2.258$.

The gain K corresponding to this point is 3.427. Hence the desired value of velocity feedback gain K_t is 0.3427.

7.17 Characteristic equation of the system is

$$\Delta(s) = s(s + 1)(s + 4) + 20K_t s + 20 = 0$$

which may be rearranged as

$$1 + \frac{Ks}{(s + j2)(s - j2)(s + 5)} = 0 \; ; K = 20K_t$$

$-\sigma_A = -2.5$; $\phi_A = 90°, -90°$; departure angle ϕ_p from $s = j2$ is 158.2°.

The following two points on $\zeta = 0.4$ line satisfy the angle criterion:

$$s_1 = -1.05 + j2.41,\; s_2 = -2.16 + j4.97$$

The value of K at s_1 is 0.449, and at s_2 is 1.4130.

The third pole corresponding to $K = 0.449$ is found at $s = -2.9$, and for $K = 1.413$ at -0.68.
From the root locus plot, one may infer that the closed-loop pole at $s = -0.68$ is close to system zero at the origin. This is not the case. In fact the closed-loop system does not have a zero at the origin:

$$\frac{Y(s)}{R(s)} = \frac{20}{s(s+1)(s+4) + 20(1 + K_t s)}$$

In the root locus plot, the zero at the origin was introduced because of the process of modifying the characteristic equation so that the adjustable variable $K = 20K_t$ appears as a multiplying factor.

$K_t = 0.449$ satisfies the dominance condition to a reasonable extent. For $K_t = 1.413$, complex-conjugate poles are not dominant.

7.18 The characteristic equation of the system is

$$s(s + 1)(s + 3) + 2s + 2\alpha = 0$$

which may be rearranged as

$$1 + \frac{K}{s(s^2 + 4s + 5)} = 0 \; ; K = 2\alpha$$

$-\sigma_A = -1.3333$; $\phi_A = 60°, -60°, 180°$; intersection with the $j\omega$-axis at $\pm j\sqrt{5}$; departure angle ϕ_p from complex pole in the upper half of s-plane $= -63.43°$; multiple roots at $s = -1, -1.666$.

The point on $\zeta = 0.5$ line that satisfies the angle criterion is $s_d = -0.63 + j1.09$. The value of K at this point, obtained by magnitude criterion, is 4.32. Therefore $\alpha = 2.16$.
The third pole for $K = 4.32$ is at $s = -2.75$.

7.19 The characteristic equation of the system is

$$s^3 + 9s^2 + 18s + 10\alpha = 0$$

which may be rearranged as

$$1 + \frac{K}{s(s+3)(s+6)} = 0 \; ; K = 10\alpha$$

$-\sigma_A = -3$; $\phi_A = 60º, -60º, 180º$; intersection with the $j\omega$-axis at $\pm j3\sqrt{2}$; multiple roots at $s = -1.268$.
The point on $\zeta = 0.5$ line that satisfies the angle criterion is $s_d = -1 + j1.732$. The value of K at this point is 28. Therefore $\alpha = 2.8$. The third pole for $K = 28$ is at $s = -7$.

7.20 (a) $1 + \dfrac{K}{s(s-2)} = 0$

System is unstable for all $K > 0$.

(b) In practice, the poles and zeros can never exactly cancel since they are determined by two independent pieces of hardware whose numerical values are neither precisely known nor precisely fixed. We should therefore never attempt to cancel poles in the right-half plane, since any inexact cancellation will result in an unstable closed-loop system.

(c)
$$1 + \frac{K(s+1)}{s(s-2)(s+8)} = 0$$

From the Routh array, we find that the loci cross the $j\omega$-axis when $K = 19.2$. For $K > 19.2$, the closed-loop poles are always in the left-half plane and the system is stable.

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{s(s-2)(s+8)}{s(s-2)(s+8)+Ks+K}$$

For $R(s) = 1/s$ and $K > 19.2$ (required for stability),

$$e_{ss} = \lim_{s\to 0} sE(s) = 0$$

For $R(s) = 1/s^2$ and $K > 19.2$,

$$e_{ss} = \lim_{s\to 0} sE(s) = \frac{-16}{K}$$

7.21 $-\sigma_A = -2$; $\phi_A = 60º, 180º, 300º$; intersection with the $j\omega$-axis at $\pm j\sqrt{5}$; multiple roots at $s = -0.473$.
The value of K at the point of intersection of the root locus and $\zeta = 0.3$ line is 7.0.

$$K_v = \lim_{s\to 0} sG(s) = K/5 = 1.4 \; ; \; t_s = 4/\zeta\omega_n = 11.6 \text{ sec.}$$

With the compensator, the characteristic equation becomes

$$1 + \frac{K(10s+1)}{s(100s+1)(s+1)(s+5)} = 0$$

or

$$1 + \frac{K'(s+0.1)}{s(s+0.01)(s+1)(s+5)} = 0 \; ; \; K' = 0.1K$$

The value of K' at the point of intersection of $\zeta = 0.3$ line and the root locus is 6.0. Therefore $K = 60$.

$$K_v = \lim_{s\to 0} sD(s)\, G(s) = K/5 = 12, \; ; \; t_s = 4/\zeta\omega_n = 12.7 \text{ sec.}$$

7.22 $1 + \dfrac{K(s+2.5)}{s(s+1)(s+\alpha)} = 0$

Desired closed-loop pole; $s_d = -1.6 + j4$. Angle criterion at s_d is satisfied when $\alpha = 5.3$. The value of K at s_d is 23.2. The third pole is at $s = -3.1$.

7.23 (a) $1 + \dfrac{K}{s(s+2)} = 0$

$$\zeta = 0.707 \; ; \; t_s = \frac{4}{\zeta\omega_n} = 2. \text{ Therefore, } \zeta\omega_n = 2$$

Proportional control does not meet this requirement.

(b) $1 + \dfrac{K_c + sK_D}{s(s+2)} = 1 + \dfrac{K_D(s + K_c/K_D)}{s(s+2)}$

At the points of intersection of $\zeta = 0.707$ line and $\zeta\omega_n = 2$ line, the angle criterion is satisfied when $K_c/K_D = 4$.
The root locus plot of

$$1 + \frac{K_D(s+4)}{s(s+2)} = 0$$

has circular complex-root branches (refer Problem 7.7). At the point of intersection of $\zeta = 0.707$ line and $\zeta\omega_n = 2$ line, we find by magnitude criterion that $K_D = 2$. Therefore $K_c = 8$.

7.24 $G(s) = \dfrac{K(s+0.1)}{s(s^2+0.8s+4)} \; ; \; K = 4000$

$$D(s) = \frac{s^2 + 0.8s + 4}{(s + 0.384)(s + 10.42)}$$

$D(s)$ improves transient response considerably, and there is no effect on steady-state performance.

7.25 $\zeta = 0.45$ (specified). Let us select the real part of the desired roots as $\zeta\omega_n = 4$ for rapid settling.

The zero of the compensator is placed at $s = -z = -4$, directly below the desired root location. For the angle criterion to be satisfied at the desired root location, the pole is placed at $s = -p = -10.6$. The gain of the compensated system, obtained by magnitude criterion, is 96.5. The compensated system is then

$$D(s)G(s) = \frac{96.5(s + 4)}{s(s + 2)(s + 10.6)}$$

$$K_v = \lim_{s \to 0} sD(s)\, G(s) = 18.2.$$

The K_v of the compensated system is less than the desired value of 20. Therefore, we must repeat the design procedure for a second choice of the desired root.

7.26 Desired root location may be taken as

$$s_d = -1.588 + j3.152$$

It meets the requirements on M_p and settling time. With $D(s) = \frac{s+1}{s+10}$, angle criterion is met. Magnitude criterion gives $K = 147$. The compensated system has $K_v = 2.94$.

7.27 $\zeta = 0.707$; $4/\zeta\omega_n = 1.4 \rightarrow \zeta\omega_n = 2.85$

$$\omega_n = 4\ ;\ \omega_n\sqrt{1-\zeta^2} = 2.85$$

$$s_d = -2.85 + j2.85$$

With $D(s) = \frac{(s+1.5)^2}{(s+3.5)^2}$, angle criterion is satisfied at s_d. The value of K, found by magnitude criterion, is 37.

7.28 From the root locus plot of the uncompensated system, we find that for gain of 1.06, the dominant closed-loop poles are at $-0.33 \pm j\,0.58$. The value of ζ is 0.5 and that of ω_n is 0.66. The velocity error constant is 0.53.

The desired dominant closed-loop poles are $-0.33 \pm j0.58$ with a K_v of 5. The lag compensator

$$D'(s) = \frac{s+0.1}{s+0.001}$$

gives K_v boost of the factor of 10. The angle contribution of this compensator at – 0.33 + j0.58 is about seven degrees. Since the angle contribution is not small, there will be a small change in the new root locus near the desired dominant closed-loop poles. If the damping of the new dominant poles is kept at $\zeta = 0.5$, then the dominant poles from the new root locus are found at $-0.28 \pm j\,0.51$. The undamped natural frequency reduces to 0.56. This implies that the transient response of the compensated system is slower than the original.

The gain at $-0.28 + j0.51$ from the new root locus plot is 0.98. Therefore $K = 0.98/1.06 = 0.925$ and

$$D(s) = \frac{0.925(s+0.1)}{s+0.01}$$

7.29 The dominant closed-loop poles of uncompensated system are located at $s = -3.6 \pm j4.8$ with $\zeta = 0.6$. The value of K is found as 820.

Therefore $K_v = \lim_{s\to 0} sG(s) = 820/200 = 4.1$.

Lag compensator $D(s) = \dfrac{s+0.25}{s+0.025}$ gives a K_v boost of the factor of 10. The angle contribution of this compensator at $-3.6 + j4.8$ is -1.8°, which is acceptable in the present problem.

7.30 (a) $1 + \dfrac{K}{(2s+1)(0.5s+1)} = 1 + \dfrac{K}{(s+0.5)(s+2)} = 0$

From the root locus we find that no value of K will yield $\zeta\omega_n = 0.75$.

(b) $D(s) = K_c\left(1+\dfrac{1}{T_I s}\right) = K_c + K_I/s = K_c\left(\dfrac{s+K_I/K_c}{s}\right)$

$$D(s)G(s) = \frac{K_c(s+K_I/K_c)}{s(s+0.5)(s+2)}$$

At the point corresponding to $\zeta = 0.6$ and $\zeta\omega_n = 0.75$, the angle criterion is satisfied when $K_I/K_c = 0.75$.

The gain at the desired point is $K_c = 2.06$. This yields $K_I = 1.545$.

7.31 $K_v = 20$ demands $K = 2000$. However, using Routh criterion, we find that when $K = 2000$, the roots of the characteristic equation are at $\pm j10$. Clearly the roots of the system when K_v requirement is satisfied are a long way from satisfying the ζ requirement. It will be difficult to bring the dominant roots from the $j\omega$-axis to the $\zeta = 0.707$ line by using a lead compensator. Therefore, we will attempt to satisfy the K_v and ζ requirements by a lag compensator.

From the uncompensated root locus we find that corresponding to $\zeta = 0.707$, the dominant roots are at $-2.9 \pm j2.9$ and the value of K is 236. Therefore necessary ratio of zero to pole of the compensator is

$$\frac{|z|}{|p|} = \frac{2000}{236} = 8.5$$

We will choose $\zeta = 0.1$ and $p = 0.1/9$. For this choice, we find that the angle contribution of the compensator at $-2.9 + j.2.9$ is negligible.

7.32 $D(s)G(s) = \dfrac{K(s+1/\tau_1)(s+1/\tau_2)}{s(s+1)(s+5)(s+1/\alpha\tau_1)(s+\alpha/\tau_2)}$

$$\frac{1}{\tau_1} = 0.05\ ;\ \frac{1}{\tau_2} = 1\ ;\ \frac{1}{\alpha\tau_1} = 0.005\ ;\ \frac{\alpha}{\tau_2} = 10$$

$$D(s)G(s) = \frac{K(s+0.05)}{s(s+0.005)(s+5)(s+10)}$$

For $\zeta = 0.45$, we obtain

$$s_d = -1.571 + j3.119,\ K = 146.3,\ K_v = 29.3,\ \omega_n = 3.49$$

7.33 $D(s) = K_c + K_D s = K_c\left(1 + \dfrac{K_D}{K_c}s\right)$

(a) $D(s)G(s) = \dfrac{K_c\left(1 + \dfrac{K_D}{K_c}s\right)}{s^2+1}$

$$s^2 + 1 + K_c + K_D s = (s + 1 + j\sqrt{3})\,(s + 1 - j\sqrt{3})$$

This equation gives $K_c = 3$ and $K_D = 2$

$$K_p = \lim_{s\to 0} D(s)G(s) = 3.$$

(b) With PID controller

$$D(s) = \frac{K(s+0.2)(s+1.3332)}{s}$$

the angle criterion at $s_d = -1 + j\sqrt{3}$ is satisfied. The gain K is 2.143.

7.34 (a) Uncompensated system:

$$M_p = 20\% \rightarrow \zeta = 0.45$$

At the point of intersection of $\zeta = 0.45$ line and the root locus, $\zeta\omega_n$ = 1.1 and $K = 56$

(b) $t_s = 4/\zeta\omega_n = 3.6$ sec

$$K_v = \lim_{s\to 0} sG(s) = 2.07$$

(c) Compensated system:

$$M_p = 15\% \rightarrow \zeta = 0.55$$

$$t_s = \frac{3.6}{2.5} = 1.45 \;;\; \zeta\omega_n = 4/1.45 = 2.75$$

$$\omega_n = \frac{2.75}{0.55} = 5 \;;\; s_d = -2.75 + j4.18$$

$$K_v \geq 20$$

Lead-lag compensated system:

$$D_1(s)\, D_2(s)\, G(s) = \frac{K(s+3)(s+0.9)}{s(s+0.15)(s+3)(s+9)(s+13.6)} \;;\; K = 506$$

7.35 $G(s) = \dfrac{K}{s(s+2+KK_t)}$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{s(s+2+KK_t)}{s(s+2+KK_t)+K}$$

For $R(s) = 1/s^2$,

$$e_{ss} = \frac{2+KK_t}{K} \leq 0.35$$

$$\zeta \geq 0.707 \;;\; t_s \leq 3 \text{ sec.}$$

The characteristic equation of the system is

$$s^2 + 2s + KK_t s + K = 0$$

which may be rearranged as

$$1 + \frac{KK_t s}{s^2 + 2s + K} = 0$$

The locus of the roots as K varies (set $KK_t = 0$) is determined from the following equation:

$$1 + \frac{K}{s(s+2)} = 0$$

For $K = 20$, the roots are $-1 \pm j4.36$.

Then the effect of varying KK_t is determined from the locus equation

$$1 + \frac{KK_t s}{(s+1+j4.36)(s+1-j4.36)} = 0$$

Note that complex root branches follow a circular path.

At the intersection of the root locus and $\zeta = 0.707$ line, we obtain $KK_t = 4.3$. The real part of the point of intersection is $\sigma = 3.15$, and therefore the settling time is 1.27 sec.

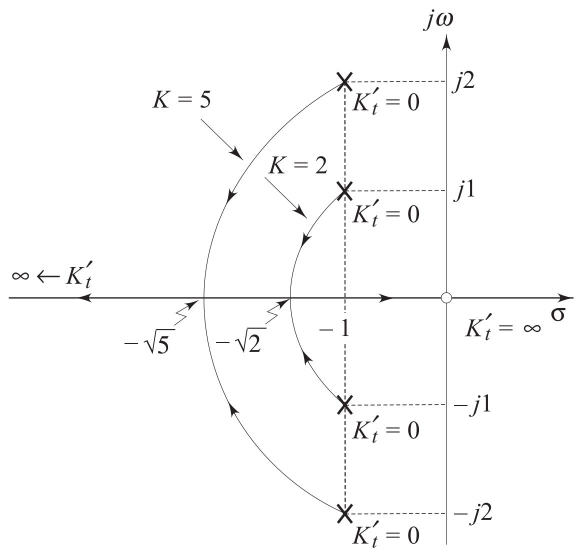
7.36 The characteristic equation of the system is

$$1 + \frac{KK_t s}{s^2 + 2s + K} = 0$$

or

$$1 + \frac{KK_t s}{(s+1+j\sqrt{(K-1})(s+1-j\sqrt{(K-1)})} = 0$$

The root contour plotted for various values of K with $K_t' = = KK_t$ varying from 0 to ∞ are shown in the figure below.



7.37 $G(s) = \dfrac{AK}{s(s+1)(s+5)+KK_t s}$

The characteristic equation is

$$s(s + 1)\,(s + 5) + KK_t\,(s + A/K_t) = 0$$

which may be rearranged as

$$1 + \frac{KK_t(s + A/K_t)}{s(s+1)(s+5)} = 0$$

The angle criterion at $s = -1 + j2$ is satisfied when $A/K_t = 2.5$. Let us take $A = 2$ and $K_t = 0.8$.

$$K_v = \lim_{s\to 0} sG(s) = \frac{AK}{5+KK_t}$$

With $\quad K = 10,\ K_v = 1.54$

The closed-loop transfer function of the system is

$$\frac{Y(s)}{R(s)} = \frac{20}{(s+1+j2)(s+1-j2)(s+4)}$$

7.38 The characteristic equation of the system is

$$s(s + 1)\,(s + 5) + KK_t\,s + AK = 0$$

or

$$s(s+1)(s+5)+\beta s+\alpha=0\ ;\ \beta=KK_t\ ;\ \alpha=AK$$

The root locus equation as a function of α with $\beta = 0$ is

$$1+\frac{\alpha}{s(s+1)(s+5)}=0$$

Sketch a root locus plot for α varying from 0 to ∞. The points on these loci become the open-loop poles for the root locus equation

$$1+\frac{\beta s}{s(s+1)(s+5)+\alpha}=0$$

For some selected values of α, the loci for β are sketched (refer Fig. 7.44).

7.39

$$e^{-s}=-\frac{(s-2)}{s+2}$$

$$1-\frac{K(s-2)}{(s+1)(s+2)}=0=1-F(s)$$

$$F(s)=\frac{K(\sigma+j\omega-2)}{(\sigma+j\omega+1)(\sigma+j\omega+2)}$$

$$=\frac{K(\sigma-2+j\omega)}{(\sigma+1)(\sigma+2)-\omega^2+j\omega(2\sigma+3)}$$

$$\angle F(s)=\tan^{-1}\frac{\omega}{\sigma-2}-\tan^{-1}\frac{\omega(2\sigma+3)}{(\sigma+1)(\sigma+2)-\omega^2}$$

$$=\tan^{-1}\left\{\frac{\dfrac{\omega}{\sigma-2}-\dfrac{\omega(2\sigma+3)}{(\sigma+1)(\sigma+2)-\omega^2}}{1+\dfrac{\omega}{\sigma-2}\left[\dfrac{\omega(2\sigma+3)}{(\sigma+1)(\sigma+2)-\omega^2}\right]}\right\}$$

$\angle F(s) = 0°$ if

$$\frac{\omega}{\sigma-2}-\frac{\omega(2\sigma+3)}{(\sigma+1)(\sigma+2)-\omega^2}=0$$

Manipulation of this equation gives

$$(\sigma-2)^2+\omega^2=12$$

Centre = (2, 0) and radius = $\sqrt{12}$ = 3.464

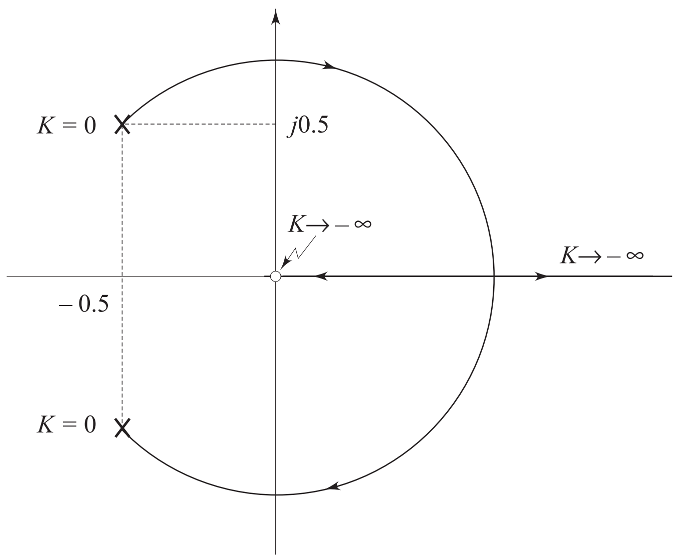
It can easily be verified that at every point on this circle, $\angle F(s) = 0°$. Revisiting Example 7.14 will be helpful.

7.40 The proof runs parallel to that in Example 7.2, with a change in angle criterion as $K \leq 0$.

Centre at zero at $s = 0$

Radius = $\sqrt{\alpha^2 + \beta^2} = \sqrt{(0.5)^2 + (0.5)^2} = 0.707$

The root locus plot is shown in the figure below.



7.41 $G(s) = \dfrac{20.7(s+3)}{s(s+2)(s+8)}$

(i) $\alpha = \alpha_0 + \Delta\alpha = 8 + \Delta\alpha$

The characteristic equation is

$s(s + 2)\ (s + 8) + \Delta\alpha\, s(s + 2) + 20.7\ (s + 3) = 0$

When $\Delta\alpha = 0$, the roots are (may be determined by root-locus method or the Newton-Raphson method)

$$\lambda_{1,2} = -2.36 \pm j2.48,\ \lambda_3 = -5.27$$

The root locus for $\Delta\alpha$ is determined using the root locus equation

$$1 + \frac{\Delta\alpha\ s(s+2)}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = 0$$

The angle of departure at $s = \lambda_1$, is – 80º.

Near $s = \lambda_1$, the locus may be approximated by a line drawn from – 2.36 + j2.48 at an angle of – 80º. For a change of $\Delta\lambda_1 = 0.2 \angle -80^\circ$ along the departure line, $\Delta\alpha$ is determined by magnitude criterion:

$$\Delta\alpha = 0.48$$

Therefore the sensitivity at λ_1 is

$$S_{\alpha+}^{\lambda_1} = \frac{\Delta\lambda_1}{\Delta\alpha/\alpha_0} = \frac{0.2\angle -80^\circ}{0.48/8} = 3.34 \angle -80^\circ$$

(ii) Sensitivity of the root $s = \lambda_1$ to a change in zero at $s = -3$ is determined as follows.

$$\beta = \beta_0 + \Delta\beta = 3 + \Delta\beta$$

The characteristic equation is

$$s(s+2)(s+8) + 20.7(s + 3 + \Delta\beta) = 0$$

or

$$1 + \frac{20.7\Delta\beta}{(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = 0$$

The angle of departure of λ_1 is 50º.

For a change of $\Delta\lambda_1 = 0.2 \angle +50^\circ$, we obtain $\Delta\beta = 0.21$.

Therefore

$$S_{\beta}^{\lambda_1} = \frac{\Delta\lambda_1}{\Delta\beta/\beta_0} = \frac{0.2\angle +50^\circ}{0.21/3} = 2.84 \angle +50^\circ$$

The sensitivity of the system to the pole can be considered to be less than the sensitivity to the zero because of the direction of departure from the pole at $s = \lambda_1$.

7.42 (a) The characteristic equation is

$$1 + \frac{K(s+5)}{s(s+2)(s+3)} = 0$$

It can easily be verified from the root locus plot that for $K = 8$ the closed-loop poles are

$$\lambda_{1,2} = -0.5 \pm j\,3.12,\ \lambda_3 = -4$$

(b) $s(s + 2 + \delta)(s + 3) + 8(s + 5) = 0$

or

$$1 + \frac{\delta s(s+3)}{s(s+2)(s+3)+8(s+5)} = 0$$

or
$$1 + \frac{\delta s(s+3)}{s(s-\lambda_1)(s-\lambda_2)(s-\lambda_3)} = 0$$

Case *I*: $\delta > 0$

From the root locus plot we find that the direction of departure from the pole at $s = \lambda_1$, is away from the $j\omega$-axis.

Case *II*: $\delta < 0$

From the root locus plot we find that the direction of departure from the pole at $s = \lambda_1$ is towards the $j\omega$-axis. Therefore the variation $\delta < 0$ is dangerous.

CHAPTER 8 THE NYQUIST STABILITY CRITERION AND STABILITY MARGINS

8.1 (a) Revisiting Example 8.1 will be helpful.

(i) $1\angle\, 0^\circ$ (ii) $0\angle -180^\circ$ (iv) $\dfrac{\sqrt{\tau_1\tau_2}}{\tau_1+\tau_2}\angle -90^\circ$

(b) Revisiting Example 8.2 will be helpful.

(i) $-\tau - j\infty$ (ii) $0\angle -180^\circ$

(c) Revisiting Example 8.3 will be helpful.

(i) $\infty\angle -180^\circ$ (ii) $0\angle -270^\circ$

(d) (i) $-(\tau_1+\tau_2) - j\infty$ (ii) $0\angle -270^\circ$

(iii) $\dfrac{\tau_1\tau_2}{\tau_1+\tau_2}\angle -180^\circ$

(e) (i) $\infty\angle -180^\circ$ (ii) $0\angle\, 0^\circ$ (iv) $\dfrac{\sqrt{\tau_1\tau_2}\,(\tau_1\tau_2)}{\tau_1+\tau_2}\angle -90^\circ$

8.2 (a) Number of poles of $G(s)$ in right half s-plane, $P = 1$
Number of clockwise encirclements of the critical point, $N = 1$
Z = number of zeros of $1 + G(s)$ in right half s-plane $= N + P = 2$
The closed-loop system is unstable.

(b) $P = 2$

Number of counterclockwise encirclements of the critical point = 2

Therefore N = number of clockwise encirclements = – 2
$Z = N + P = -2 + 2 = 0$
The closed-loop system is stable.

(c) $P = 0$
N = (Number of clockwise encirclements of the critical point – number of counterclockwise encirclements of the critical point) = 0
$Z = N + P = 0$; the closed-loop system is stable.

8.3 The polar plot of $G(j\omega)$ for $\omega = 0^+$ to $\omega = +\infty$ is given in Fig. P8.3b. Plot of $G(j\omega)$ for $\omega = -\infty$ to $\omega = 0^-$ is the reflection of the given polar plot with respect to the real axis. Since

$$G(j\omega)\big|_{\omega=0} = \infty\angle -180^\circ,$$

$G(s)$ has double pole at the origin. The map of Nyquist contour semicircle $s = \rho e^{j\phi}$, $\rho \to 0$, ϕ varying from -90° at $\omega = 0^-$ through 0° to $+90^\circ$ at $\omega = 0^+$, into the Nyquist plot is given by $\infty\angle -2\phi$ (an infinite semicircle traversed clockwise).

With this information, Nyquist plot for the given system can easily be drawn.

(i) $P = 0$, $N = 0$; $Z = N + P = 0$
The closed-loop system is stable

(ii) $P = 1$, $N = 0$; $Z = N + P = 1$
The closed-loop system is unstable

(iii) $P = 0$, $N = 0$; $Z = N + P = 0$
The closed-loop system is stable.

8.4 (a) The key points to the polar plot are:

$$G(j\omega)H(j\omega)\big|_{\omega=\infty} = 18\angle\ 0^{\circ}$$

$$G(j\omega)H(j\omega)\big|_{\omega=\infty} = 0\angle -270^{\circ}$$

The intersections of the polar plot with the axes of $G(s)H(s)$-plane can easily be ascertained by identifying the real and imaginary parts of $G(j\omega)H(j\omega)$. When we set Im $[G(j\omega)H(j\omega)]$ to zero, we get $\omega = 4.123$ and

$$G(j\omega)H(j\omega)\big|_{\omega=4.123} = -1.428$$

Similarly, setting Re $[G(j\omega)H(j\omega)]$ to zero, we get intersection with the imaginary axis.

Based on this information, a rough sketch of the Nyquist plot can easily be made. From the Nyquist plot, we find that $N = 2$. Since $P = 0$, we have $Z = N + P = 2$, i.e., two closed-loop poles in right half s-plane.

(b) The key points of the polar plot are (refer Problem 8.1d):

$$G(j\omega)H(j\omega)\big|_{\omega=0} = -3 - j\,\infty$$

$$G(j\omega)H(j\omega)\big|_{\omega=\infty} = 0\angle -270^{\circ}$$

Intersection with the real axis at $\omega = 1/\sqrt{2}$;

$$G(j\omega)H(j\omega)\big|_{\omega=\frac{1}{\sqrt{2}}} = -\frac{4}{3}$$

The map of Nyquist contour semicircle $s = \rho e^{j\phi}$, $\rho \to 0$, ϕ varying from -90° at $\omega = 0^{-}$ through 0° to $+90^{\circ}$ at $\omega = 0^{+}$, into the Nyquist plot is given by $\infty\angle -\phi$ (an infinite semicircle traversed clockwise).

Based on this information, a rough sketch of Nyquist plot can easily be made.

$$P = 0,\ N = 2.\ \text{Therefore } Z = N + P = 2$$

(c) $G(j\omega)H(j\omega)|_{\omega=0} = 2\angle -180°$

$G(j\omega)H(j\omega)|_{\omega=\infty} = 0\angle -90°$

No more intersections with real/imaginary axis. From the Nyquist plot, we find that $N = -1$. Since $P = 1$, we have $Z = N + P = 0$; the closed-loop system is stable.

(d) $G(j\omega)H(j\omega)|_{\omega=0} = 0\angle\ 90°$; $G(j\omega)H(j\omega)|_{\omega=1} = 2.6\angle 161°$;

$G(j\omega)H(j\omega)|_{\omega=2} = 2.6\ \angle\ 198°$; $G(j\omega)H(j\omega)|_{\omega=10} = 0.8\angle -107°$

$G(j\omega)H(j\omega)|_{\omega=\infty} = 0\angle -90°$

The Nyquist plot is given in Fig. P8.2b.

$P = 2$, $N = -2$; $Z = N + P = 0$

(e) $G(j\omega)H(j\omega)|_{\omega=0} = \infty\angle -180°$

$G(j\omega)H(j\omega)|_{\omega=\infty} = 0\ \angle -90°$

No intersections with real/imaginary axis.
The map of Nyquist contour semicircle $s = \rho e^{j\phi}$, $\rho \to 0$, ϕ varying from $-90°$ at $\omega = 0^-$ through $0°$ to $+90°$ at $\omega = 0^+$, into the Nyquist plot is given by $\infty\angle -2\phi$ (an infinite semicircle traversed clockwise).
With this information, Nyquist plot for the given system can easily be drawn.

$P = 0$, $N = 0$; $Z = N + P = 0$

(f) $G(j\omega)H(j\omega)|_{\omega=0} = \dfrac{1}{100}\angle 0°$

$G(j\omega)H(j\omega)|_{\omega=10^-} = \infty\angle 0°$

Consider the Nyquist contour shown in figure below. For the semi-circle $s = j10 + \rho e^{j\phi}$, $\rho \to 0$, ϕ varying from $-90°$ at $\omega = 0^-$ through $0°$ to $+90°$ at $\omega = 10^+$,

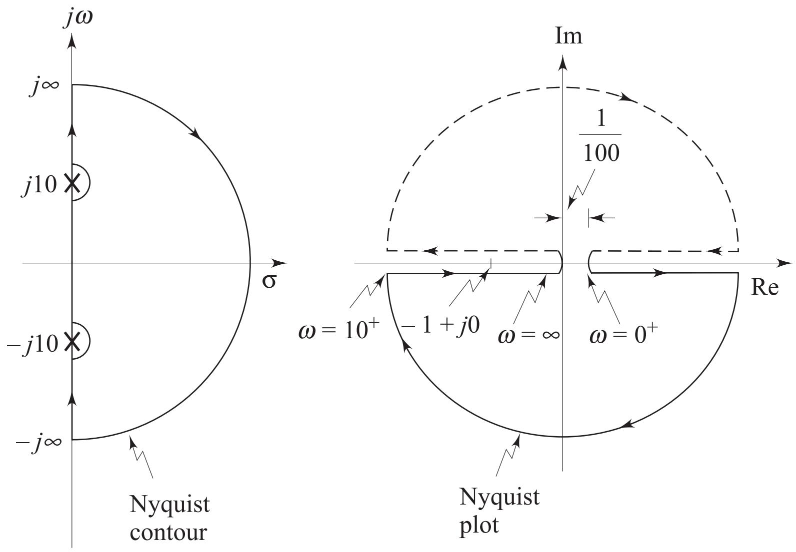
$$G(s)H(s) = \lim_{\rho\to 0} \frac{1}{(j10+\rho e^{j\phi})^2 + 100} = \lim_{\rho\to 0}\frac{1}{\rho e^{j\phi}(j20+\rho e^{j\phi})}$$

$$= \lim_{\rho\to 0}\frac{1}{\rho}e^{-j\phi}$$

It is an infinite semicircle from $\omega = 10^-$ to $\omega = 10^+$ traversed clockwise.

$G(j\omega)H(j\omega)|_{\omega=\infty} = 0\angle -180°$

The Nyquist plot is shown in the figure below.
The Nyquist plot passes through $-1 + j0$ point. The Nyquist criterion is not applicable.



(g) $G(j\omega)H(j\omega)\big|_{\omega=0} = -8 + j\infty$

$G(j\omega)H(j\omega)\big|_{\omega=\infty} = 0^\circ \angle -90^\circ$

$G(j\omega)H(j\omega)\big|_{\omega=\sqrt{3}} = -2 + j0$

No intersection with the imaginary axis.

The map of Nyquist contour semicircle $s = \rho e^{j\phi}$, $\rho \to 0$, ϕ varying from -90° at $\omega = 0^-$ through 0° to $+90^\circ$ $\omega = 0^+$, into the Nyquist plot is given by $\infty\angle -\phi$ (an infinite semicircle traversed clockwise).

With this information, a rough sketch of Nyquist plot can easily be made (refer Fig. P8.2a).

$P = 1$, $N = -1$. Therefore $Z = N + P = 0$.

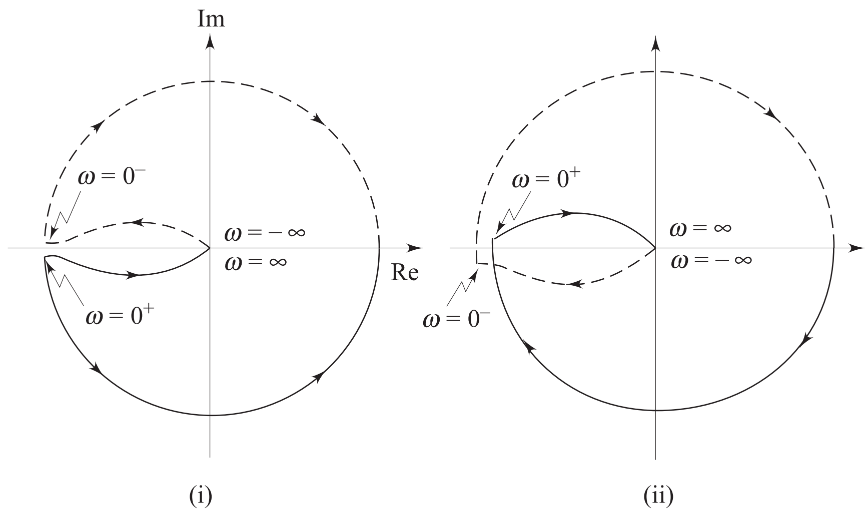
8.5 $G(j\omega)H(j\omega) = \dfrac{-K(1+\omega^2\tau_1\tau_2)}{\omega^2(1+\omega^2\tau_1^2)} + \dfrac{j\omega K(\tau_1 - \tau_2)}{\omega^2(1+\omega^2\tau_1^2)}$

(i) For $\tau_1 < \tau_2$, the polar plot of $G(j\omega)H(j\omega)$ is entirely in the third quadrant as shown in figure below.

$P = 0, N = 0$. Therefore $Z = N + P = 0$; the closed-loop system is stable.

(ii) For $\tau_1 > \tau_2$, the polar plot of $G(j\omega)H(j\omega)$ is entirely in the second quadrant, as shown in figure below.

$P = 0$, $N = 2$. Therefore $Z = N + P = 2$; the closed-loop system is unstable.



8.6 (a) $G(j\omega)\big|_{\omega=0} = 4\angle 0^\circ$; $G(j\omega)\big|_{\omega=\infty} = 0 \angle -270^\circ$

Intersection with the real axis at – 0.8.
The critical point $-1 + j0$ will be encircled by the Nyquist plot if $K > 1/0.8$. Since $P = 0$, we want net encirclements of the critical point to be zero for stability. Therefore K must be less than 5/4.

(b) $G(s) = \dfrac{4(1+s)}{s^2(1+0.1s)}$ = ¢ $\dfrac{K'(\tau_2 s+1)}{s^2(\tau_1 s+1)}$; $\tau_1 < \tau_2$

The Nyquist plot for this transfer function has already been given in the Solution of Problem 8.5. From this plot we see that the closed-loop system is table for all K.

(c) $G(s) = \dfrac{4(1+0.1s)}{s^2(1+s)} = \dfrac{K'(\tau_2 s+1)}{s^2(\tau_1 s+1)}$; $\tau_1 > \tau_2$

The Nyquist plot for this transfer function has already been given in the Solution of Problem 8.5. From this plot we see that the closed-loop system is unstable for all K.

(d) $G(j\omega) = \dfrac{e^{-0.8j\omega}}{j\omega+1}$

$= \dfrac{1}{1+\omega^2}$ [(cos $0.8\omega - \omega$ sin 0.8ω) – j(sin $0.8\omega + \omega$ cos 0.8ω)]

The imaginary part is equal to zero if

$\sin 0.8\omega + \omega \cos 0.8\omega = 0$

This gives

$\omega = -\tan 0.8\omega$

Solving this equation for smallest positive value of ω, we get ω = 2.4482.

$G(j\omega)\big|_{\omega=0} = 1\angle\, 0^\circ$; $G(j\omega)\big|_{\omega=2.4482} = -\,0.378 + j0$

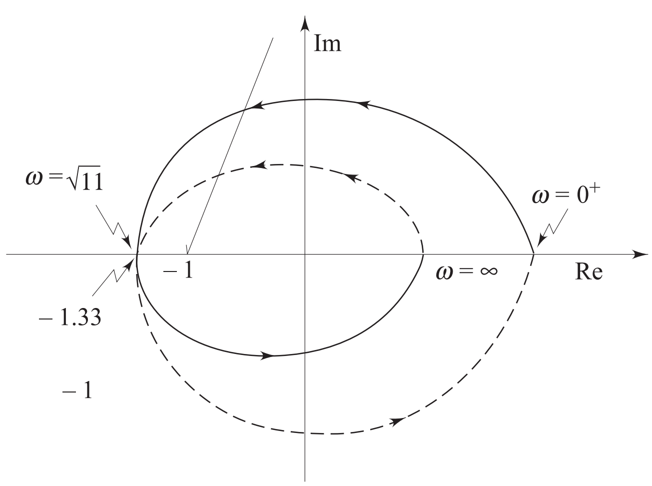
$G(j\omega)\big|_{\omega=\infty} = 0$

The polar plot will spiral into the $\omega \to \infty$ point at the origin (refer Fig. 8.39).

The critical value of K is obtained by letting $G(j\,2.4482)$ equal – 1. This gives K = 2.65. The closed-loop system is stable for K < 2.65.

8.7 The figure given below shows the Nyquist plot of $G(s)H(s)$ for K = 1. As gain K is varied, we can visualize the Nyquist plot in this figure expanding (increased gain) or shrinking (decreased gain) like a balloon.

Since P = 2, we require N = – 2 for stability, i.e., the critical point must be encircled ccw two times. This is true if – 1.33 K < – 1 or K > 0.75.

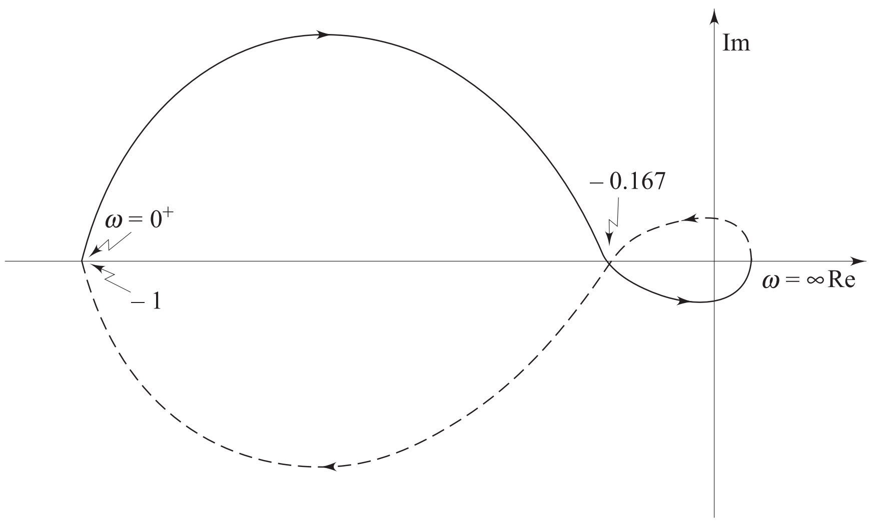


8.8 $\left.\frac{1}{K}G(j\omega)H(j\omega)\right|_{\omega=0} = -1 + j0$; $\left.\frac{1}{K}G(j\omega)H(j\omega)\right|_{\omega=\infty} = 0.05 + j0$

$$\left.\frac{1}{K}G(j\omega)H(j\omega)\right|_{\omega=0.62} = -0.167 + j0$$

The Nyquist plot is shown in the figure below. If the critical point lies inside the larger loop, $N = 1$. Since $P = 1$, we have $Z = N + P = 2$ and the closed-loop system is unstable.

If the critical point lies inside the smaller loop, $N = -1$, $Z = N + P = 0$ and the closed-loop system is stable. Therefore, for stability
$-0.167K < -1$ or $K > 6$.



8.9 Eliminating the minor-loop we obtain forward-path transfer function of an equivalent single-loop system.

$$G(s) = \frac{K(s+0.5)}{s^3 + s^2 + 1}$$

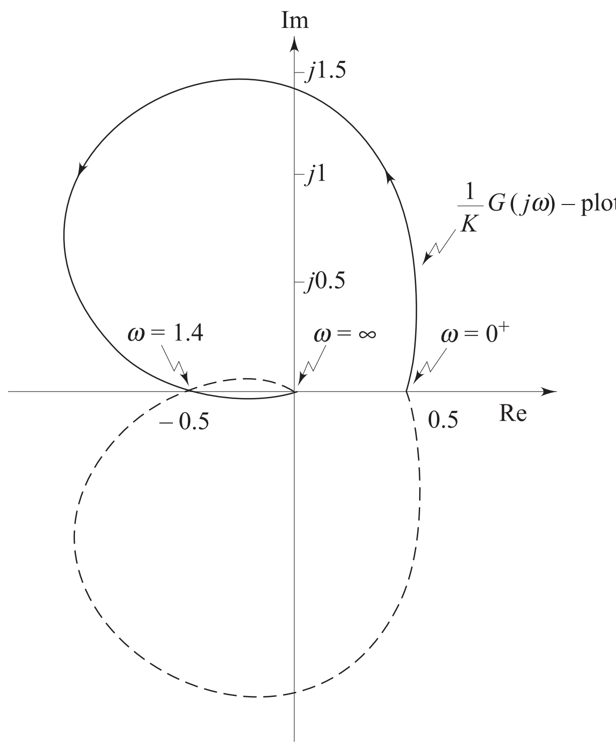
By Routh criterion, we find that the polynomial $(s^3 + s^2 + 1)$ has two roots in the right half s-plane. Therefore $P = 2$.

$$\left.\frac{1}{K}G(j\omega)\right|_{\omega=0} = 0.5\angle 0^\circ \text{ ; } \left.\frac{1}{K}G(j\omega)\right|_{\omega=\infty} = 0\angle -180^\circ$$

$$\left.\frac{1}{K}G(j\omega)\right|_{\omega=1.4} = -0.5 + j0$$

The polar plot intersects the positive imaginary axis; the point of intersection can be found by setting real part of $\frac{1}{K}G(j\omega)$, equal to zero.

With this information, Nyquist plot can easily be constructed. From the Nyquist plot we find that the critical point is encircled twice in ccw if $-0.5K < -1$ or $K > 2$. For this range of K, the closed loop system is stable. The figure given below illustrates this result.



8.10 Fig. P8.10a:

$$G(s) = \frac{Ke^{-2s}}{s(s+1)(4s+1)}$$

$$\left.\frac{1}{K}G(j\omega)\right|_{\omega=0} = \infty\angle -90^\circ; \quad \left.\frac{1}{K}G(j\omega)\right|_{\omega=0.26} = -2.55 + j0$$

$$\left.\frac{1}{K}G(j\omega)\right|_{\omega=\infty} = 0$$

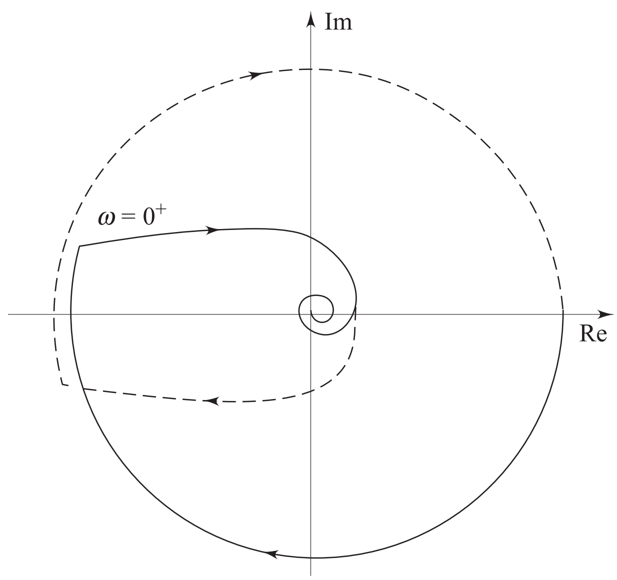
The polar plot will spiral into the $\omega \to \infty$ point at the origin (refer Fig. 8.39).
It is found that the maximum value of K for stability is 1/2.55 = 0.385

Fig. P8.10b:

$$D(s)G(s) = \frac{Ke^{-2s}}{s^2(4s+1)}$$

$$\frac{1}{K}D(j\omega)G(j\omega)\Big|_{\omega=0} = \infty\angle -180^\circ\ ;\ \frac{1}{K}D(j\omega)G(j\omega)\Big|_{\omega=\infty} = 0$$

The polar plot will spiral into the $\omega \to \infty$ point at the origin. A rough sketch of the Nyquist plot is shown in figure below. It is observed that the system is unstable for all values of K.



8.11 Revisiting Example 8.15 will be helpful.

$1 + G(s) = 0$

may be manipulated as

$$\frac{e^{-s\tau_D}}{s(s+1)} = -1 \text{ or } G_1(s) = -e^{s\tau_D}\ ;\ G_1(s) = \frac{1}{s(s+1)}$$

The polar plot of $G_1(j\omega)$ intersects the polar plot of $-e^{j\omega\tau_D}$ at a point corresponding to $\omega = 0.75$ at the $G(j\omega)$-locus. At this point, the phase of the polar plot of $-e^{j\omega\tau_D}$ is found to be 52°. Therefore,

$$0.75\,\tau_D = 52\,(\pi/180)$$

This gives

$$\tau_D = 1.2 \text{ sec}$$

8.12 $G(j\omega)|_{\omega=0} = -2.2. - j\infty;\ G(j\omega)|_{\omega=\infty} = 0\angle -270^{\circ}$

$G(j\omega)|_{\omega=15.9} = -0.182 + j0;\quad G(j\omega)|_{\omega=6.2} = 1\ \angle -148.3^{\circ}$

$GM = 1/0.182 = 5.5$; $\Phi M = 31.7^{\circ}$

$\omega_g = 6.2$ rad/sec ; $\omega_{\phi} = 15.9$ rad/sec.

8.13 (a) This result can easily proved using Routh criterion.

(b) General shape of the Nyquist plot of given transfer function is as shown in Fig. P8.2c. When $K = 7$, the polar plot intersects the negative real axis at the point $-1.4 + j0$. The resulting Nyquist plot encircles the critical point once in clockwise direction and once in counter-clockwise direction. Therefore $N = 0$. Since $P = 0$, the system is stable for $K = 7$. Reducing the gain by a factor of 1/1.4 will bring the system to the verge of instability. Therefore $GM = 0.7$.

The phase margin is found as $+10^{\circ}$.

8.14 (a) The system is type-0; so the low-frequency asymptote has a slope of 0 dB/decade, and is plotted at dB = 20 log 25 = 27.96.

Asymptote slope changes to –20 dB/decade at the first corner frequency $\omega_{c1} = 1$; then to – 40 dB/decade at $\omega_{c2} = 10$, and to – 60 dB/decade at $\omega_{c3} = 20$.

Asymptotic crossing of the 0 dB-axis is at $\omega_g = 16$.

(b) The system is type-1; so the low-frequency asymptote has a slope of –20 dB/decade and is drawn so that its extension would intersect the 0 dB-axis at $\omega = K = 50$. The asymptote slope changes to – 40 dB/decade at the first corner frequency at $\omega_{c1} = 1$; then to – 20 dB/decade at $\omega_{c2} = 5$, and back to – 40 dB/decade at $\omega_{c3} = 50$. The asymptotic gain crossover is at $\omega_g = 10.2$.

(c) The system is type-2; the low-frequency asymptote is drawn with a slope of – 40 dB/decade, and is located such that its extension would intersect the 0 dB-axis at $\omega = \sqrt{K} = \sqrt{500} = 22.36$. The asymptote slope changes to – 60 dB/decade at the first corner frequency $\omega_{c1} = 1$; then to – 40 dB/decade at $\omega_{c2} = 5$, to – 20 dB/decade at $\omega_{c3} = 10$ and to – 40 dB/decade at $\omega_{c4} = 50$. The asymptotic gain crossover is at $\omega_g = 10$.

(d) The system is type-1, and the low-frequency asymptote has a slope of – 20 dB/decade and is drawn so that its extension would cross the

0 dB-axis at $\omega = K = 50$. The asymptote slope changes to –40 dB/decade at $\omega_{c1} = 10$; then to –20 dB/decade at $\omega_{c2} = 20$; to –40 dB/decade at $\omega_{c3} = 50$; and to – 80 dB/decade at $\omega_{c4} = 200$ (note that the corner frequency for the complex poles is $\omega_{c4} = \omega_n = 200$).
The asymptotic gain crossover is at $\omega_g = 26$.

8.15 (a) The low-frequency asymptote has a slope of – 20 dB/decade and passes through the point ($\omega = 1$, dB = 20). The asymptote slope changes to – 40 dB/decade at $\omega_c = 10$.
Compensate the asymptotic plot by – 3 dB at $\omega = 10$, by – 1dB at $\omega = 5$ and by – 1 dB at $\omega = 20$. The compensated magnitude plot crosses the 0 dB line at $\omega_g = 7.86$.

The phase is computed from

$\angle G(j\omega) = -90^\circ - \tan^{-1} 0.1\omega$

The phase shift at $\omega_g = 7.86$ is –128.2º, and the resulting phase margin is – 128.2º – (– 180º) = 51.8º. Because the phase curve never reaches – 180º line, the gain margin is infinity.

(b) Low-frequency asymptote has a slope of – 20 dB/decade and passes through the point ($\omega = 1$, dB = 20)
The asymptote slope changes to –60 dB/decade at the corner frequency $\omega_c = 10$. Compensate the asymptotic plot by – 6 dB at $\omega = 10$, by – 2 dB at $\omega = 5$ and – 2 dB at $\omega = 20$. The compensated magnitude plot crosses the 0 dB line at $\omega_g = 6.8$. The phase is computed from

$\angle G(j\omega) = -90^\circ - 2\tan^{-1} 0.1\omega$

The phase shift at $\omega_g = 6.8$ is – 158.6º. Therefore, the phase margin is 180º – 158.6º = 21.4º. At a frequency of $\omega_\phi = 10$, the phase shift is – 180º; the gain margin is 6 dB.

(c) The low-frequency asymptote has a slope of –20 dB/decade and passes through the point ($\omega = 0.1$, dB = 20log 200 = 46). The slope changes to – 40 dB/decade at the first corner frequency $\omega_{c1} = 2$, and back to – 20 dB/decade at the second corner frequency $\omega_{c2} = 5$. Compensate the asymptotic plot by – 3 dB at $\omega = 2$, by – 1 dB at $\omega = 1$, by – 1 dB at $\omega = 4$, by + 3 dB at $\omega = 5$, by + 1 dB at $\omega = 2.5$ and by + 1 dB at $\omega = 10$. The compensated magnitude plot crosses the 0 dB line at $\omega_g = 9$. The phase shift at this frequency is – 106.6º. Therefore the phase margin is 73.4º.

Phase never reaches – 180º line; the gain margin is infinity.

(d) Reconsider the bode plot for system of Problem 8.15c.
The time-delay factor $e^{-0.1s}$ will not change the magnitude plot; it will change only the phase characteristics. For the system with dead-time,

ω_g = 9 rad/sec; ΦM = 22º; ω_ϕ = 13.65 rad/sec; GM = 4.2 dB

(e) $$G(s) = \frac{40}{(s+2)(s+4)(s+5)} = \frac{1}{\left(\frac{1}{2}s+1\right)\left(\frac{1}{4}s+1\right)\left(\frac{1}{5}s+1\right)}$$

The low-frequency asymptote coincides with 0 dB line; its slope changes to – 20 dB/decade at ω_{c1} = 2, to – 40 dB/decade at ω_{c2} = 4, and to – 60 dB/decade at ω_{c3} = 5. Compensated magnitude plot can easily be obtained by applying corrections.
The phase plot crosses the – 180º line at ω_ϕ = 7 rad/sec, and the gain margin is 20 dB. Since the gain never reaches 0 dB (ω_g = 0), the phase margin is infinity.

(f) From the Bode plots of $G(j\omega)$ we find that gain crossover frequency ω_g = 3.16 rad/sec, and the phase margin is –33º. The phase plot is asymptotic to the –180º line in the low-frequency range; it never reaches – 180º
Therefore the GM = – ∞

8.16 $$G(s) = \frac{K/5}{s(s+1)\left(\frac{1}{5}s+1\right)}$$

From the Bode plot of the system sketched for K = 10, we find that ΦM = 21º, GM = 8 dB.
Increasing the gain from K = 10 to K = 100 shifts the 0 dB axis down by 20 dB. The phase and gain margins for the system with K = 100 are

ΦM = – 30º, GM = – 12 dB

Thus the system is stable for K = 10 but unstable for K = 100.

8.17 (a) $$G(s) = \frac{K/40}{\left(\frac{1}{2}s+1\right)\left(\frac{1}{4}s+1\right)\left(\frac{1}{5}s+1\right)}$$

Since $G(s)$ has all poles in left half s-plane, the open-loop system is stable. Hence the closed-loop system will be stable if the frequency response has a gain less than unity when the phase is 180º.
When K = 40, the gain margin is 20 dB (refer Problem 8.15e). Therefore, an increase in gain of +20 dB (i.e., by a factor of 10) is possible before the system becomes unstable. Hence K < 400 for stability.

(b) From the Bode plot of $G(j\omega)$ with $K = 1$, we find that the phase crossover frequency $\omega_\phi = 15.88$ rad/sec and gain margin = 34.82 dB. This means that the gain can be increased by 34.82 dB (i.e., by a factor of 55) before the system becomes unstable. Hence $K < 55$ for stability.

(c) From the Bode plot of $G(j\omega)$ with $K = 1$, we find that $\omega_\phi = 0.66$ rad/sec and $GM = 4.5$ dB. Thus the critical value of K for stability is 1.67, i.e., $K < 1.67$ for the system stability.

8.18 From the Bode plot of $G(j\omega)$, it can easily be determined that (a) when $\tau_D = 0$, the $GM = 12$ dB, and $\Phi M = 33°$; and (b) when $\tau_D = 0.04$ sec, the $GM = 2.5$ dB and $\Phi M = 18°$. The relative stability of the system reduces due to the presence of dead-time.

(c) The value of τ_D for the system to be on verge of instability is obtained by setting the phase margin equal to zero, i.e.,

$$G_1(j\omega)\Big|_{\omega=\omega_g} - \frac{\omega_g \tau_D \times 180°}{\pi} = -180°$$

where $G_1(j\omega)$ is the system without dead-time, and ω_g is the gain crossover frequency. From the Bode plot of $G_1(j\omega)$, we get $\omega_g = 7.4$ rad/sec and $\angle G_1(j\omega_g) = -147°$.

Therefore,

$$-147° - \frac{7.4\tau_D \times 180°}{\pi} = -180°$$

$$\text{or } \tau_D = 0.078 \text{ sec}$$

8.19 Since the systems have minimum-phase characteristics, zeros (if any) are in left half s-plane.

(a) The low-frequency asymptote has a slope of – 20 dB/decade and its extension intersects the 0 dB-axis at $\omega = 4$. The system is therefore type-1; $4/s$ is a factor of the transfer function.

The asymptote slope changes from – 20 dB/decade to 0 dB/decade at $\omega = 2$ (this corresponds to the factor $(1 + s/2)$), then to – 20 dB/decade at $\omega = 10$ (this corresponds to the factor $(1 + s/10)$). The transfer function

$$G(s) = \frac{4\left(1+\frac{1}{2}s\right)}{s\left(1+\frac{1}{10}s\right)}$$

(b) The low-frequency asymptote has a slope of + 20 dB/decade and it

intersects 0 dB-axis at $\omega = 0.2$. The system is therefore type-0; Ks is a factor of the transfer function with 20 log $0.2K = 0$ or $K = 5$.
The low-frequency asymptote has a magnitude of 20 dB at a frequency ω_{c1}, where

$$20 \log (5\omega_{c1}) = 20. \text{ This gives } \omega_{c1} = 2.$$

The transfer function

$$G(s) = \frac{5s}{\left(1+\frac{1}{2}s\right)\left(1+\frac{1}{10}s\right)\left(1+\frac{1}{30}s\right)}$$

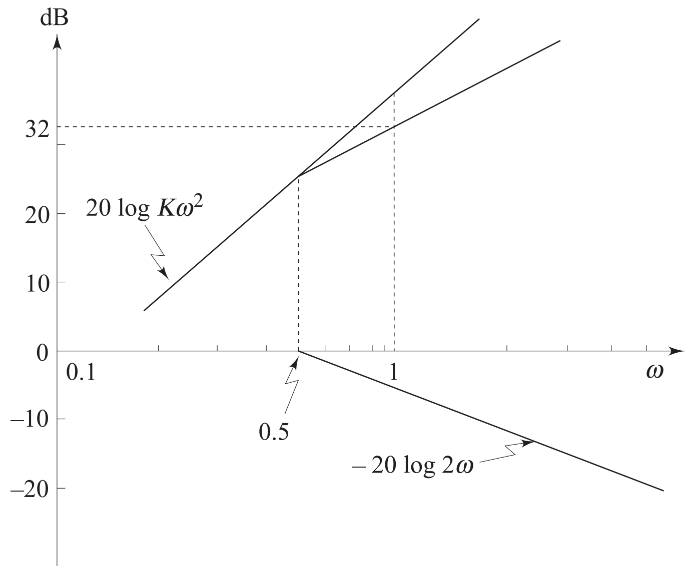
(c) The low-frequency asymptote has a slope of – 6 dB/octave with a magnitude of – 9 dB at $\omega = 1$. The system is therefore type-1; K/s is a factor of the transfer function with 20 log $(K/\omega) = -9$ at $\omega = 1$. This gives $K = 0.355$.
The transfer function

$$G(s) = \frac{0.355(1+s)\left(1+\frac{1}{20}s\right)}{s\left(1+\frac{1}{40}s\right)}$$

(d) The low-frequency asymptote has a slope of – 20dB/decade with a magnitude of 40 dB at $\omega = 2.5$. The system is therefore type-1; K/s is a factor of the transfer function with 20 log $(K/2.5) = 40$. This gives $K = 250$.

$$G(s) = \frac{250}{s\left(1+\frac{1}{2.5}s\right)\left(1+\frac{1}{40}s\right)}$$

(e) The low-frequency asymptote has a slope of +12 dB/octave; the system is therefore type-0 and Ks^2 is a factor of the transfer function. The first corner frequency $\omega_{c1} = 0.5$. The slope of the asymptote changes to +6 dB/decade at this corner frequency. Therefore $1/(2s + 1)$ is a factor of the transfer function.
The asymptotic plot of $Ks^2/(2s + 1)$ is shown in the figure below.



From this figure, we find that

$$(20 \log K\omega^2 - 20 \log 2\omega)\Big|_{\omega = 1} = 32$$

This gives $K = 79.6$.

The transfer function

$$G(s) = \frac{79.6s^2}{(1+2s)(1+s)(1+0.2s)}$$

8.20 The low-frequency asymptote has a slope of 20 dB/decade. Ks is a factor of the transfer function with $20\log K = 30$. This gives $K = 31.623$. The transfer function

$$G(s) = \frac{31.623s}{(1+s)\left(1+\dfrac{1}{5}s\right)\left(1+\dfrac{1}{20}s\right)}$$

(i) $20 \log (31.623\omega_{g1}) = 0$; this gives $\omega_{g1} = 0.0316$

(ii) $20 \log (31.623\omega_{g2}) - 20 \log \omega_{g2} - 20 \log \left(\dfrac{1}{5}\omega_{g2}\right)$

$- 20 \log \left(\dfrac{1}{20}\omega_{g2}\right) = 0$; this gives $\omega_{g2} = 56.234$

8.21 The low-frequency asymptote has a slope of – 6 dB/octave and has a magnitude of 11 dB at $\omega = 3$. K/s is therefore a factor of the transfer function with $20 \log (K/3) = 11$. This gives $K = 10.64$.
The transfer function

$$G(s) = \frac{10.64}{s\left(\frac{1}{3}s+1\right)\left(\frac{1}{8}s+1\right)}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}\frac{1}{3}\,\omega - \tan^{-1}\frac{1}{8}\,\omega$$

$$\angle G(j\omega) = -180^\circ \text{ at } \omega = 4.9$$

The gain obtained from asymptotic magnitude plot at $\omega = 4.9$ is

$$20 \log \frac{10.64}{4.9} - 20\log\frac{4.9}{3} = 2.5 \text{ dB}$$

Therefore, $GM = -2.5$ dB

8.22 Revisit Review Example 8.3.

$$G(s) = \frac{5\left(1+\frac{1}{10}s\right)}{s\left(1+\frac{1}{2}s\right)\left[1+\frac{0.6}{50}s+\frac{1}{2500}s^2\right]}$$

8.23 From the asymptotic magnitude plot we find that the system is type-0 with a double pole at $s = -3$. The transfer function obtained from the magnitude plot is

$$G(s) = \frac{0.1}{\left(\frac{1}{3}s+1\right)^2}$$

$\angle G(j\omega)$ vs ω may be compared with the given phase characteristics to check the accuracy of identification of corner frequency at $\omega_c = 3$.

CHAPTER 9 FEEDBACK SYSTEM PERFORMANCE BASED ON THE FREQUENCY RESPONSE

9.1 For derivation of the result, refer Section 9.3; Eqn. (9.9).

9.2 For derivation of the result, refer Section 9.3; Eqn. (9.18).

9.3 For derivation of the result, refer Section 9.3; Eqns (9.15) – (9.16).

9.4 The relations

$$\zeta^4 - \zeta^2 + \frac{1}{4M_r^2} = 0 \;; \quad \omega_r = \omega_n\sqrt{1-2\zeta^2}$$

yield $\zeta = 0.6$ and $\omega_n = 21.8$

Note that $M_r - \zeta$ relation gives two values of ζ for $M_r = 1.8$; $\zeta = 0.6$ and $\zeta = 0.8$. We select $\zeta = 0.6$ as damping ratio larger than 0.707 yields no peak above zero frequency.

The characteristic equation of the given system is

$$s^2 + as + K = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

This gives $K = 475$ and $a = 26.2$

$$t_s = \frac{4}{\zeta\omega_n} = 0.305 \text{ sec}$$

$$\omega_b = \omega_n\left[1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right]^{1/2}$$

$$= 25.1 \text{ rad/sec}$$

9.5 From the response curve, we find

$$M_p = 0.135, \; t_p = 0.185 \text{ sec}$$

$$\zeta^2 = \frac{(ln\, M_p)^2}{(ln\, M_p)^2+\pi^2} \text{ gives } \zeta = 0.535 \text{ for } M_p = 0.135$$

$$\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.185 \text{ gives } \omega_n = 20$$

The corresponding frequency response performance indices are:

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.11 \;; \; \omega_r = \omega_n\sqrt{1-2\zeta^2} = 13.25$$

$$\omega_b = \omega_n\left[1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right]^{1/2} = 24.6$$

9.6 $\zeta \approx 0.01\phi_m$

The characteristic equation of the system is

$\tau s^2 + (1 + KK_t)s + K = 0$

This gives

$$\omega_n = \sqrt{K/\tau}\ ;\ \zeta = \frac{1+KK_t}{2\sqrt{K\tau}} = 0.01\phi_m$$

$$K = \frac{1}{K_t}\left[2\times10^{-4}\phi_m^2\tau - K_t \pm 2\times10^{-2}\phi_m\sqrt{\tau}\sqrt{10^{-4}\phi_m^2\tau - K_t}\right]$$

9.7 The polar plot of $G(j\omega)$ crosses the real axis at $\omega = 1/\sqrt{\tau_1\tau_2}$. The magnitude $|G(j\omega)|$ at this frequency is given by

$$|G(j\omega)| = \frac{K\tau_1\tau_2}{\tau_1+\tau_2}$$

Therefore

$$G_m \times \frac{K\tau_1\tau_2}{\tau_1+\tau_2} = 1$$

This gives

$$K = \frac{1}{G_m}\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)$$

9.8 (a) It can be determined from the Bode plot that $\omega_g = 1$ rad/sec and $\Phi M = 59.2^\circ$

(b) $$\Phi M = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}}$$

$$\Phi M = 59.2^\circ \rightarrow \zeta = 0.6$$

$$\omega_g = \omega_n\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}$$

This gives $\omega_n = 1.4$

Therefore,

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 9.48\%$$

$$t_s = \frac{4}{\zeta\omega_n} = 4.76 \text{ sec.}$$

9.9 The low-frequency asymptote has a slope of – 20 dB/decade and intersects the 0 dB-axis at $\omega = 8$; $8/s$ is a factor of the transfer function. The forward-path transfer function can easily be identified as

$$G(s) = \frac{8\left(1+\frac{1}{2}s\right)\left(1+\frac{1}{4}s\right)}{s\left(1+\frac{1}{8}s\right)\left(1+\frac{1}{24}s\right)\left(1+\frac{1}{36}s\right)}$$

The phase curve can now be constructed.

From the phase curve and the asymptotic magnitude curve, we obtain

$\Phi M = 50°$; $GM = 24$ dB

Phase margin of 50º gives $\zeta = 0.48$ which corresponds to $M_p \approx 18\%$.

9.10 (a) From the Bode plot, we get $\Phi M = 12°$

(b) For a phase margin of 50º, we require that $G(j\omega)\,H(j\omega) = 1 \angle -130°$ for some value of ω. From the phase curve of $G(j\omega)H(j\omega)$, we find that $\angle G(j\omega)H(j\omega) \simeq -130°$ at $\omega = 0.5$. The magnitude of $G(j\omega)H(j\omega)$ at this frequency is approximately 3.5. The gain must be reduced by a factor of 3.5 to achieve a phase margin of 50º.

(c) ΦM of 50º gives $\zeta = 0.48$ which corresponds to $M_p \simeq 18\%$.

9.11 $G(j\omega) = \dfrac{K(j\omega+2)}{(j\omega)^2}$

$$\angle G(j\omega) = \tan^{-1}\frac{\omega}{2} - 180° = -130°$$

This equation gives $\omega = 2.3835$

The magnitude of $G(j\omega)$ must be unity at $\omega = 2.3835$.

$$\left|\frac{K(j\omega+2)}{(j\omega)^2}\right|_{\omega=2.3835} = 1$$

This equation gives $K = 1.826$.

Since the phase curve never reaches – 180º line, the gain margin = ∞.

9.12 (a) From the open-loop frequency response table, we find that $\omega_\phi = 10$ rad/sec and $GM = 20\log\dfrac{1}{0.64} = 3.88$ dB; $\omega_g = 8$ rad/sec and $\Phi M = 10°$.

(b) For the desired gain margin of 20 dB, we must decrease the gain by (20 – 3.88) = 16.12 dB. It means that magnitude curve must be lowered by 16.12 dB, i.e., the gain must be changed by a factor of β, given by 20 log β = – 16.12. This gives β = 0.156.

(c) To obtain a phase margin of 60º, we must first determine the frequency at which the phase angle of $G(j\omega)$ is – 120º, and then adjust the gain so that $|G(j\omega)|$ at this frequency is 0 dB. From the Bode plot, we find that lowering the magnitude plot by 16.65 dB gives the desired phase margin. It equivalently means that the gain should be changed by a factor of α where 20 log α = – 16.65. This gives α = 0.147.

9.13 A 9.48% overshoot implies ζ = 0.6. For this damping, required phase margin is 59.2º.

$$G(s) = \frac{100K}{s(s+36)(s+100)}$$

$$= \frac{K/36}{s\left(\frac{1}{36}s+1\right)\left(\frac{1}{100}s+1\right)}$$

Make a Bode plot for say K = 3.6.

From the plot we find that at ω = 14.8 rad/sec, $\angle G(j\omega)$ = – 120.8º. At ω = 14.8, the gain is – 44.18 dB. The magnitude curve has to be raised to 0 dB at ω = 14.8 to yield the required phase margin. The gain should be changed by a factor of α where 20 log α = 44.18. This gives α = 161.808. Therefore,

$$K = 3.6 \times 161.808 = 582.51$$

$$\omega_g = 14.8 = \omega_n \sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}$$

This gives

$$\omega_n = 20.68$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.19 \text{ sec}$$

9.14 (a) From the Bode plot, it can easily be determined that GM = 5 dB and ΦM = 30º. The gain crossover frequency ω_g = 0.83 rad/sec.

(b) The magnitude plot should be lowered by 5 dB to obtain a gain margin of 10 dB. This is achieved by reducing the gain by 5 dB.

(c) The gain at ω = 1.27 is –4.8 dB. Therefore, the gain should be increased by 4.8 dB to obtain a gain crossover frequency of 1.27 rad/sec.

(d) Gain at the frequency which gives $\angle G(j\omega)$ = – 135º, should be 0 dB.

From the Bode plot we find that this requires reducing the gain by 3.5 dB.

9.15 The requirements on ΦM and ω_g are satisfied if we increase the gain to the limit of zero GM with a phase margin $\geq$ 45º. From the Bode plot we find that K = 37.67 meets this requirement.

9.16 (a) From the Bode plot we find that 0 dB crossing occurs at a frequency of 0.47 rad/sec with a phase angle of – 145º. Therefore, the phase margin is 35º. Assuming a second-order approximation, $\Phi M = 35^{\circ} \rightarrow \zeta = 0.33$, M_p = 33%.

(b) The zero dB crossing occurs with a phase angle of – 118º. Therefore the phase margin is 62º. $\Phi M = 62^{\circ} \rightarrow \zeta = 0.64$, M_P = 7.3%.

9.17 The 0 dB crossing occurs at ω = 0.8, with a phase angle of – 140º when τ_D = 0, and – 183º when τ_D = 1. Therefore phase margin of the system without dead-time is 40º; and with dead-time added, the phase margin becomes – 3º. We find that with the dead-time added, the system becomes unstable. Therefore, the system gain must be reduced in order to provide a reasonable phase margin.

We find from the Bode plot that in order to provide a phase margin of 30º, the gain would have to be decreased by 5 dB, i.e., by a factor of β where 20 log β = 5. This gives β = 1.78.

We find that the dead-time necessitates the reduction in loop gain is order to obtain a stable response. The cost of stability is the resulting increase in the steady-state error of the system as the loop gain is reduced.

9.18 Make a polar plot of

$$G(j\omega) = \frac{50/18}{j\omega(1+j\omega/3)(1+j\omega/6)}$$

$$G(j\omega)\big|_{\omega=0} = -\frac{25}{18} - j\infty;\ G(j\omega)\big|_{\omega=\infty} = 0 \angle -270^{\circ}$$

The polar plot intersects the negative real axis at ω = 4.24.

The polar plot is tangential to the M = 1.8 circle (refer Eqn. (9.22)). Therefore, the resonance peak M_r = 1.8.

The bandwidth of a system is defined as the frequency at which the magnitude of the closed-loop frequency response is 0.707 of its magnitude at ω = 0. For the system under consideration, closed-loop gain is unity at ω = 0; therefore, bandwidth is given by the frequency at which the M = 0.707 circle intersects the polar plot. This frequency is found to be 3.61 rad/sec. Therefore ω_b = 3.61

$$\zeta^4 - \zeta^2 + \frac{1}{4M_r^2} = 0 \; ; \; M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100$$

$$\omega_b = \omega_n \sqrt{\left[(1-2\zeta^2) + \sqrt{4\zeta^4 - 2\zeta^2 + 2}\,\right]}$$

$$t_s = 4/\zeta\omega_n$$

The $M_r - \zeta$ relation gives $\zeta = 0.29$, 0.96 for $M_r = 1.8$. We select $\zeta = 0.29$ because damping ratio larger than 0.707 yields no peak above zero frequency. $\zeta = 0.29$ corresponds to $M_p = 38.6\%$.

For $\zeta = 0.29$ and $\omega_b = 3.61$, we get $\omega_n = 2.4721$. The setting time t_s = 5.58 sec.

9.19 The polar plot is tangential to $M = 1.4$ circle (refer Eqn. (9.22)) at a frequency $\omega = 4$ rad/sec. Therefore $M_r = 1.4$ and $\omega_r = 4$.

$$\zeta^4 - \zeta^2 + \frac{1}{4M_r^2} = 0$$

$$\omega_r = \omega_n \sqrt{1-2\zeta^2}$$

$$M_r = 1.4 \rightarrow \zeta = 0.39.$$

The corresponding $M_p = 26.43\%$.

$\omega_r = 4$ and $\zeta = 0.39 \rightarrow \omega_n = 4.8$. The settling time $t_s = 2.14$ sec.

The $M_r - \zeta$ relationship has been derived for standard second-order systems with zero-frequency closed-loop gain equal to unity. The answer is based on the assumption that zero-frequency closed-loop gain for the system under consideration is unity.

9.20 From the Bode plot, we find that

$$\omega_g = 8.3 \text{ rad/sec} \; ; \; \omega_\phi = 14.14 \text{ rad/sec};$$

$$GM = 8.2 \text{ dB} \; ; \; \Phi M = 27.7^\circ$$

For each value of ω, the magnitude and phase of $G(j\omega)$ are transferred from the Bode plot to the Nichols chart. The resulting dB vs phase curve is tangential to $M = 6.7$ dB contour of the Nichols chart at $\omega = 9$ rad/sec. Therefore $M_r = 6.7$ dB and $\omega_r = 9$. The dB vs phase curve intersects the – 3 dB contour of the Nichols chart at $\omega = 13.6$. Therefore bandwidth ω_b = 13.6 rad/sec.

The following points are worth noting in the use of Nichols chart.

(i) The frequency parameters (ω_r or ω_b) can be easily determined by transferring the dB and/or phase data from the dB vs phase curve to the Bode plot.

(ii) The open-loop dB vs phase plot may not be tangential to one of the constant-M contours of the Nichols chart. One must do a little interpolation. In the present problem, M_r = 6.7 dB has been obtained by interpolation.

9.21 Plot the given frequency response on a dB scale against phase angle. From the dB vs phase curve, we obtain,

GM = 7 dB ; ΦM = 17°

The dB vs phase curve when transferred on Nichols chart gives M_r = 10 dB; ω_r = 2.75 rad/sec; ω_b = 4.2 rad/sec

Note that the –3dB bandwidth definition is applicable to systems having unity closed-loop gain at $\omega = 0$. We have made this assumption for the system under consideration.

9.22 For parts (a) and (b) of this problem, refer Problem 9.10.

(c) Since the data for –3dB contour of the Nichols chart has been provided, we don't require the Nichols chart for bandwidth determination. On a linear scale graph sheet, dB vs phase cure of the open-loop frequency response is plotted with data coming from Bode plot. On the same graph sheet, the – 3dB contour using the given data is plotted.

The intersection of the two curves occurs at ω = 0.911 rad/sec. Therefore bandwidth ω_b = 0.911.

(d)
$$\Phi M = \tan^{-1} \frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1}-2\zeta^2}}$$

$$\omega_b = \omega_n \sqrt{\left[1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}\right]}$$

$$\Phi M = 50^\circ \rightarrow \zeta = 0.48 \; ; \; M^p \approx 18\%$$

$$\omega_b = 0.911 \text{ and } \zeta = 0.47 \rightarrow \omega_n = 0.7025 \; ; \; t_s = 11.93 \text{ sec.}$$

9.23 The frequency-response data from the Bode plot of $G(j\omega)$ when transferred to the Nichols chart, gives the following result.

M_r = 1.4 ; ω_r = 6.9 rad/sec

From the relations

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \; ; \; \omega_r = \omega_n \sqrt{1-2\zeta^2}$$

we obtain

ζ = 39 ; ω_n = 8.27 rad/sec

Note that the given type-0 system has zero-frequency closed-loop gain = 54/(25 + 54) = 0.6835. The approximation of the transient response of the system by $\zeta = 0.39$ and $\omega_n = 8.27$ has large error since the correlations $(M_r, \omega_r) \rightarrow (\zeta, \omega_n)$ used above, have been derived from systems with unity zero-frequency closed-loop gain.

9.24 (a) Transfer the frequency-response data from the Bode plot of $G(j\omega)$ to the Nichols chart.

Shifting the dB vs phase curve down by 5.4 dB makes this curve tangent to the $M = 20 \log 1.4$ contour. Therefore gain must be changed by a factor of β, where $20 \log \beta = -5.4$. This gives $\beta = 0.54$.

(b) The gain-compensated system intersects the -3 dB contour of the Nichols chart at $\omega = 9.2$. Therefore bandwidth $\omega_b = 9.2$ rad/sec.

Note that the -3dB bandwidth definition given by Eqn. (9.2b) is applicable to systems having unity closed-loop gain at $\omega = 0$. The given system does not satisfy this requirement. The answer, therefore, is an approximation of the bandwidth.

9.25 (a) From the Bode plot we find the phase crossover frequency $\omega_\phi = 2.1$ and $GM = 14$ dB. For the gain margin to be 20 dB, the magnitude plot is to be brought down by 6 dB. The value of K corresponding to this condition is given by the relation

$20 \log K = -6$

which gives $K = 0.5$

(b) From the Bode plot it is observed that $\Phi M = 60°$ will be obtained if gain crossover frequency ω_g is 0.4 rad/sec. This is achieved if magnitude plot is brought down by 7 dB. This condition corresponds to $K = 0.446$.

(c) From the dB-phase plot on the Nichols chart, we find that for this plot to be tangent to $M = 1$ dB contour, the dB-phase curve must be brought down by 4 dB. This condition corresponds to $K = 0.63$. The corresponding value of ω_r (read off from Bode plot) is 0.5 rad/sec.

The value of bandwidth (ω_b) is the frequency at which the dB-phase plot intersects the -3dB contour. It is found that $\omega_b = 1$ rad/sec.

(d) The dB vs phase plot on the Nichols chart when raised by 2.4 dB, intersects the -3 dB contour at $\omega = 1.5$ rad/sec. This condition corresponds to $K = 1.35$.

9.26 Plot the given frequency response on a dB scale against phase angle. From this dB vs phase curve, we obtain

$$GM = 16.5 \text{ dB} \;;\; \Phi M = 59°$$

Transfer the dB vs phase curve on the Nichols chart. We find that this curve when raised by about 5 dB, touches the M = 20 log1.4 contour of the Nichols chart. Therefore the gain should be increased by a factor of β, where 20 log β = 5. This gives β = 1.75.

The gain-compensated system has GM = 11.6 dB and ΦM = 42.5º.

9.27 The dB vs phase plot becomes tangential to 20 log1.4 dB contour on Nichols chart if the plot is lowered by about 11 dB. This means that the gain K must be reduced by a factor of α where 20 log (1/α) = –11. This gives α = 3.5.

Phase margin of the gain-compensated system is 42.5º

$$M_r = 1.4 \rightarrow \zeta = 0.387 \ ; \ \Phi M = 42.5^\circ \rightarrow \zeta = 0.394$$

The following comments may be carefully noted.

(i) Zero-frequency closed-loop gain has been assumed to be unity.

(ii) One is usually safe if the lower of the two values of ζ is utilized for analysis and design purposes.

9.28 $$S_G^M(j\omega) = \frac{1}{1+G(j\omega)} = \frac{G^{-1}(j\omega)}{1+G^{-1}(j\omega)}$$

Magnitude of $S_G^M(j\omega)$ can be obtained by plotting $G^{-1}(j\omega)$ on the Nichols chart.

The dB vs phase plot of $G^{-1}(j\omega)$ is tangent to M = 20 log 2.18 dB contour of Nichols chart. Therefore

$$\left|S_G^M(j\omega)\right|_{\max} = 2.18$$

The peak occurs at ω = 7 rad/sec.

CHAPTER 10 COMPENSATOR DESIGN USING BODE PLOTS

10.1 (a) From the Bode plot of $G(j\omega)H(j\omega)$, we find that GM = 6 dB and ΦM = 17º

(b) Now the gain and phase of the compensator with K_c = 1 are added to the Bode plot. From the new Bode plot we find that the gain must be raised by 1.5 dB to have a gain margin of 6 dB.
20 log K_c = 1.5 gives K_c = 1.2

10.2 (a) From the Bode plot of $G(j\omega)H(j\omega)$ we find that ω_g = 4.08 rad/sec; ΦM = 3.9º; GM = 1.6 dB

(b) Now the gain and phase of the compensator are added to the Bode plot. From the new Bode plot, we find that
ω_g = 5 rad/sec; ΦM = 37.6º; GM = 18 dB
The increase in phase and gain margins implies that lead compensation increases margin of stability. The increase in ω_g implies that lead compensation increases speed of response.

10.3 (a) $K_v = \lim\limits_{s \to 0} sD(s)G(s) = K = 12$

(b) From the Bode plot and Nichols chart analysis of the system with K=12, we find that ΦM = 15º; ω_b = 5.5 rad/sec; M_r = 12 dB; ω_r = 3.5 rad/sec

(c) The phase margin of the uncompensated system is 15º. The phase lead required at the gain crossover frequency of the compensated magnitude curve = 40º – 15º + 5º = 30º

$$\alpha = \frac{1-\sin 30^\circ}{1+\sin 30^\circ} = 0.334$$

The frequency at which the uncompensated system has a magnitude of $-20 \log (1/\sqrt{\alpha}) = -4.8$ dB is 4.6 rad/sec. Selecting this frequency as gain crossover frequency of the compensated magnitude curve, we set

$$\frac{1}{\sqrt{\alpha}\,\tau} = 4.6$$

The transfer function of the lead compensator becomes

$$D(s) = \frac{12(0.376s+1)}{0.128s+1}$$

(d) ΦM = 42º; ω_b = 9 rad/sec; M_r = 3 dB;

ω_r = 4.6 rad/sec

10.4 Lead compensator $D(s) = \dfrac{10(0.5s+1)}{0.1s+1}$ meets the phase margin and steady-state accuracy requirements. These requirements are also met by the lag compensator $D(s) = \dfrac{10(10s+1)}{100s+1}$

In this particularly simple example, specifications could be met by either compensation. In more realistic situations, there are additional performance specifications such as bandwidth and there are constraints on loop gain. Had there been additional specifications and constraints, it would have influenced the choice of compensator (lead or lag).

105. The settling time and peak overshoot requirements on performance may be translated to the following equivalent specifications:

$$\zeta = 0.45 \;\; ; \;\; \omega_n = 2.22$$

ζ is related to the phase margin by the relation

$$\Phi M \approx \frac{\zeta}{0.01} = 45^\circ$$

ω_n is related to the bandwidth ω_b by the relation

$$\omega_b = \omega_n \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right]^{1/2}$$

For a closed-loop system with $\zeta = 0.45$, we estimate from this relation

$$\omega_b = 1.33\ \omega_n$$

Therefore, we require a closed-loop bandwidth $\omega_b \simeq 3$. The gain K of the compensator

$$D(s) = \frac{K(\tau s+1)}{\alpha\tau s+1}$$

may be set a value given by $K = \omega_n^2$. This gives $K \simeq 5$. To provide a suitable margin for settling time, we select $K = 10$.

The phase margin of the uncompensated system is 0º because the double integration results in a constant 180º phase lag. Therefore, we must add a 45º phase lead at the gain crossover frequency of the compensated magnitude curve. Evaluating the value of α, we have

$$\alpha = \frac{1-\sin 45^\circ}{1+\sin 45^\circ} = 0.172$$

To provide a margin of safety, we select $\alpha = 1/6$.

The frequency at which the uncompensated system has a magnitude of – 20 log $(1/\sqrt{\alpha})$ = – 7.78 dB is 4.95 rad/sec. Selecting this frequency as the gain crossover frequency of the compensated magnitude curve, we set

$$\frac{1}{\sqrt{\alpha}\tau} = 4.95$$

The closed-loop system with the compensator

$$D(s) = \frac{10(0.083s+1)}{0.5s+1}$$

satisfies the performance specifications.

10.6 $K_v = \dfrac{A}{e_{ss}} = \dfrac{A}{0.05A} = 20$

$K = 20$ realizes this value of K_v.

From the Bode plot of

$$G(j\omega) = \frac{20}{j\omega(j0.5\omega+1)}$$

we find that the phase margin of the uncompensated system is 18º. The phase lead required at the gain crossover frequency of the compensated magnitude curve = 45º – 18º + 3º = 30º.

$$\alpha = \frac{1-\sin 30^\circ}{1+\sin 30^\circ} = 0.334$$

The frequency at which the uncompensated system has a magnitude of – 20 log $(1/\sqrt{\alpha})$ = – 4.8 dB is 8.4 rad/sec. Selecting this frequency as the gain crossover frequency of the compensated magnitude curve, we set

$$\frac{1}{\sqrt{\alpha}\tau} = 8.4$$

Therefore the compensator

$$D(s) = \frac{0.2s+1}{0.0668s+1}$$

The phase margin of the compensated system is found to be 43.7º. If we desire to have exactly a 45º phase margin, we should repeat the steps with a decreased value of α.

From the Nichols chart analysis of the compensated and uncompensated systems, we find that the lead compensator has increased the bandwidth from 9.5 rad/sec to 12 rad/sec.

10.7 From the Bode plot of

$$G(j\omega) = \frac{20}{j\omega(j0.5\omega+1)}$$

we find that the phase margin of the uncompensated system is 18º. To realize a phase margin of 45º, the gain crossover frequency should be moved to ω_g' where the phase angle of the uncompensated system is: – 180º + 45º + 5º = – 130º.

From the Bode plot of uncompensated system, we find that ω_g' = 1.5. The attenuation necessary to cause ω_g' to be the new gain crossover frequency is 20 dB. The β parameter of the lag compensator can now be calculated.

$$20 \log \beta = 20$$

This gives β = 10. Placing the upper corner frequency of the compensator a decade below ω_g', we have $1/\tau$ = 0.15.

Therefore, the lag compensator

$$D(s) = \frac{6.66s+1}{66.6s+1}$$

As a final check, we numerically evaluate the phase margin and the bandwidth of the compensated system. It is found that that ΦM = 45º and ω_b = 2.5 rad/sec.

10.8 From the steady-state requirement, we set K = 100. From the Bode plot of

$$G(j\omega) = \frac{100}{j\omega(j0.1\omega)(j0.2\omega+1)}$$

we find that the phase margin is – 40º, which means that the system is unstable. The rapid decrease of phase of $G(j\omega)$ at the gain crossover frequency ω_g = 17 rad/sec, implies that single-stage lead compensation may be ineffective for this system. (The reader should, in fact, try a single-stage lead compensator). For the present system, in which the desired K_v is 100, a phase lead of more than 85º is required. For phase leads greater than 60º, it is advisable to use two or more cascaded stages of lead compensation (refer Fig. 10.15). The design approach may be that of achieving a portion of the desired phase margin improvement by each compensator stage.

We first add a single-stage lead compensator that will provide a phase lead of about 42.5º, i.e., the compensator will improve the phase margin to about 2.5º.

Since the phase curve of the Bode plot of the uncompensated system at the gain crossover frequency has a large negative slope, the value of $\alpha = 0.19$ given by

$$\alpha = \frac{1-\sin 42.5^{\circ}}{1+\sin 42.5^{\circ}}$$

will not yield a phase margin of 2.5º. In fact, when we set $\alpha = 0.08$ and $\tau = 0.1$, the single-stage compensator improves the phase margin to 3º.

Therefore $D_1(s) = \dfrac{0.1s+1}{0.008s+1}$

Notice that the single-stage lead compensator not only improves the phase margin, but also reduces the slope of the phase curve of the gain crossover frequency. By adding another stage of the compensator with $D_2(s) = D_1(s)$, we obtain

$$D_2(s)D_1(s)G(s) = \frac{100(0.1s+1)}{s(0.2s+1)(0.008s+1)^2}$$

Note that pole and zero at $s = -10$ cancel each other.

The final compensated system has a phase margin of 45º.

10.9 $G(s) = \dfrac{2500K}{s(s+25)}$

$K = 1$ satisfies the steady-state performance requirement. The phase margin of the uncompensated system, read at the gain crossover frequency $\omega_g = 47$ rad/sec, is 28º.

The phase lead required at the gain crossover frequency of the compensated magnitude curve = 45º – 28º + 8º = 25º

$$\alpha = \frac{1-\sin 25^{\circ}}{1+\sin 25^{\circ}} = 0.405$$

The frequency at which the uncompensated system has a magnitude of $-20 \log (1/\sqrt{\alpha}) = -3.93$ dB is 60 rad/sec. Selecting this frequency as gain crossover frequency of the compensated magnitude curve, we set

$$\frac{1}{\sqrt{\alpha}\,\tau} = 60$$

The transfer function of the lead compensator is

$$D(s) = \frac{0.026s+1}{0.01s+1}$$

The phase margin of the compensated system is found to be 47.5º. By Nichols chart analysis, it is found that the lead compensator reduces the resonance peak from 2 to 1.25 and increases the bandwidth from 74 to 98 rad/sec.

10.10 The phase margin of the uncompensated system

$$G(s) = \frac{2500K}{s(s+25)};\ K = 1$$

read at the gain crossover frequency ω_g = 47 rad/sec is 28º. To realize a phase margin of 45º, the gain crossover frequency should be moved to ω'_g where the phase angle of the uncompensated system is:

– 180º + 45º = – 135º.

From the Bode plot of the uncompensated system we find that ω'_g = 25. Since the lag compensator contributes a small negative phase when the upper corner frequency of the compensator is placed at 1/10 of the value of ω'_g, it is a safe measure to choose ω'_g at somewhat less than 25 rad/sec, say, 20 rad/sec.

The attenuation necessary to cause ω'_g = 20 to be the new gain crossover frequency is 14 dB.

The β parameter of the lag compensator can now be calculated.

20 log β = 14. This gives β = 5

Placing the upper corner frequency of the compensator a decade below ω'_g, we have $1/\tau = 2$. Therefore the lag compensator

$$D(s) = \frac{0.5s+1}{2.5s+1}$$

From the Bode plot–Nichols chart analysis we find that the compensated system has

$$\Phi M = 51^\circ;\ M_r = 1.2 \text{ and } \omega_b = 27.5 \text{ rad/sec.}$$

10.11 $K_v = \lim_{s\to 0} sG(s) = 5$

From the Bode plot we find that the uncompensated system has a phase margin of – 20º; the system is therefore unstable.

We attempt a lag compensator. This choice is based on the observation made from Bode plot of uncompensated system that there is a rapid decrease in phase of $G(j\omega)$ near ω_g. Lead compensator will not be effective for this system.

To realize a phase margin of 40º, the gain crossover frequency should be moved to ω_g' where the phase angle of the uncompensated system is:

$$-180^\circ + 40^\circ + 12^\circ = -128^\circ$$

From the Bode plot of the uncompensated system, it is found that ω_g' = 0.5 rad/sec. The attenuation necessary to cause ω_g' to be the new gain crossover frequency is 20 dB. The β parameter of the lag compensator can now be calculated.

20 log β = 20. This gives β = 10

Since we have taken a large safety margin of 12°, we can place the upper corner frequency of the compensator at 0.1 rad/sec, i.e., $1/\tau = 0.1$. Thus the transfer function of the lag compensator becomes

$$D(s) = \frac{10s+1}{100s+1}$$

From the Bode plot of compensated system we find that the phase margin is about 40° and the gain margin is about 11 dB.

10.12 It easily follows that K = 30 satisfies the specification on K_v. From the Bode plot of

$$G(j\omega) = \frac{30}{j\omega(j0.1\omega+1)(j0.2\omega+1)}$$

we find that gain crossover frequency is 11 rad/sec and phase margin is – 24º. Nichols chart analysis gives ω_b = 14 rad/sec.

If lead compensation is employed, the system bandwidth will increase still further, resulting in an undesirable system which will be sensitive to noise. If lag compensation is attempted, the bandwidth will decrease sufficiently so as to fall short of the specified value of 12 rad/sec, resulting in a sluggish system. These facts can be verified by designing lead and lag compensators. We thus find that there is need to go in for lag-lead compensation.

Since the full lag compensator will reduce the system bandwidth excessively, the lag section of the lag-lead compensator must be designed so as to provide partial compensation only. The lag section, therefore, should move the gain crossover frequency to a value higher than the gain crossover frequency of the fully lag-compensated system. We make a choice of new gain crossover frequency as ω_g' = 3.5 rad/sec. The attenuation necessary to cause ω_g' to be the new gain crossover frequency is 18.5 dB. This gives the β parameter of the lag section as

20 log β = 18.5; β = 8.32, say, 10

Placing the upper corner frequency of the lag section at $1/\tau_1 = 1$, we get the transfer function of the lag section as

$$D_1(s) = \frac{\tau_1 s + 1}{\beta\tau_1 s + 1} = \frac{s+1}{10s+1}$$

It is found that the lag-section compensated system has a phase margin of 24º.

We now proceed to design the lead section. The implementation of the lag-lead compensator is simpler if α and β parameters of the lead and lag compensators, respectively, are related as $\alpha = 1/\beta$. Let us first make this choice. If our attempt fails, we will relax this constraint on α.

The frequency at which the lag-section compensated system has a magnitude of $-20 \log (1/\sqrt{\alpha}) = -10$ dB is 7.5 rad/sec. Selecting this frequency as gain crossover frequency of the lag-lead compensated magnitude curve, we set

$$\frac{1}{\sqrt{\alpha}\,\tau_2} = 7.5$$

The transfer function of the lead-section becomes

$$D_2(s) = \frac{\tau_2 s + 1}{\alpha\tau_2 s + 1} = \frac{0.422s+1}{0.0422s+1}$$

The analysis of the lag-lead compensated system gives $\Phi M = 48^\circ$ and $\omega_b = 13$ rad/sec.

10.13 $\zeta^2 = \dfrac{(ln\ M_p)^2}{(ln\ M_p)^2 + \pi^2}$

$$M_p = 0.2 \rightarrow \zeta = 0.456$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$t_p = 0.1 \rightarrow \omega_n = 35.3$$

$$\omega_b = \omega_n\,[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\,]^{1/2}$$

$$= 46.576$$

$$\Phi M = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{4\zeta^4+1} - 2\zeta^2}}$$

$$\simeq 48^\circ$$

In order to meet the specification of $K_v = 40$, K must be set at 1440. From the Bode plot of

$$G(j\omega) = \frac{144000}{j\omega(j\omega+36)(j\omega+100)}$$

we find that the gain crossover frequency is 29.7 rad/sec and phase margin is 34º.

The phase lead required at the gain crossover frequency of the compensated magnitude curve = 48º – 34º + 10º = 24º

$$\alpha = \frac{1-\sin 24^\circ}{1+\sin 24^\circ} = 0.42$$

The frequency at which the uncompensated system has a magnitude of $-20 \log (1/\sqrt{\alpha}) = -3.77$ dB is 39 rad/sec. Selecting this frequency as gain crossover frequency of the compensated magnitude curve, we set

$$\frac{1}{\sqrt{\alpha}\tau} = 39$$

The transfer function of the lead compensator becomes

$$D(s) = \frac{1440(0.04s+1)}{0.0168s+1}$$

It can easily be verified by Nichols chart analysis that the bandwidth of the compensated system exceeds the requirement. We assume the peak time specification is met. This conclusion about the peak time is based on a second-order approximation that should be checked via simulation.

10.14 $K = 2000$ meets the requirement on K_v. We estimate a phase margin of 65º to meet the requirement on ζ.

(a) From the Bode plot we find that the uncompensated system with K = 2000 has a phase margin of zero degrees. From the Bode plot we observe that there is a rapid decrease of phase at the gain crossover frequency. Since the requirement on phase lead is quite large, it is not advisable to compensate this system by a single-stage lead compensator (refer Fig. 10.15).

(b) Allowing 10º for the lag compensator, we locate the frequency at which the phase angle of the uncompensated system is: $-180^\circ + 65^\circ + 10^\circ = -105^\circ$. This frequency is equal to 1.5 rad/sec. The gain crossover frequency ω_g' should be moved to this value. The necessary attenuation is 23 dB. The β parameter of the lag compensator can now be calculated.

$$20 \log \beta = 23. \text{ This gives } \beta = 14.2$$

Placing the upper corner frequency of the compensator one decade below ω_g', we have $1/\tau = 0.15$. Therefore the lag compensator

$$D(s) = \frac{6.66s + 1}{94.66s + 1}$$

The phase margin of the compensated system is found to be 67º.

(c) The bandwidth of the compensated system is found to be ω_b = 2.08 rad/sec.

10.15 Approximate relation between ϕ_m and ζ is $\zeta \approx 0.01\phi_m$

$\zeta = 0.4 \rightarrow \phi_m = 40º$

Let us try lag compensation.

The realize a phase margin of 40º, the gain crossover frequency should be moved to ω_g' where the phase angle of the uncompensated system is: – 180º + 40º + 10º = – 130º. From the Bode plot of the uncompensated system we find that, ω_g' = 6. The attenuation necessary to cause ω_g' to be the new gain crossover frequency is 9 dB.

20 log β = 9. This gives $\beta \simeq 3$

Placing the upper corner frequency of the lag compensator two octaves below ω_g', we have $1/\tau = 1.5$.

The lag compensator

$$D(s) = \frac{0.67s + 1}{2s + 1}$$

Nichols chart analysis of the compensated system gives ω_b = 11 rad/sec. Using second-order approximation

$$\omega_b = \omega_n \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right]^{1/2}$$

we get $\omega_n = 10$

Therefore $t_s = 4/\zeta\omega_n = 1$ sec

Note that the zero-frequency closed-loop gain of the system is nonunity. Therefore, the use of – 3 dB bandwidth definition, and the second-order system correlation between ω_b and ζ & ω_n may result in considerable error. Final design must be checked by simulation.

10.16 (a) From the Bode plot of uncompensated system we find that $\Phi M = 0.63º$

(b) Phase margin of the system with the second-order compensator is 9.47º. There is no effect of the compensator on steady-state performance of the system.

(c) The lead compensator

$$D(s) = \frac{1 + 0.0378s}{1 + 0.0012s}$$

meets the requirements on relative stability.

10.17 $K_v = 4.8$

CHAPTER 11 HARDWARE AND SOFTWARE IMPLEMENTATION OF COMMON COMPENSATORS

11.1 (a) $\dfrac{E(s)}{Y(s)} = -\dfrac{R_D}{R+1/C_D s} = -\dfrac{T_D s}{\alpha T_D s+1}$; $\alpha = R/R_D$; $T_D = R_D C_D$

(b) $\dfrac{E(s)}{Y(s)} = -\dfrac{R}{R_1 + R_2/(1+sR_2C)} = -\dfrac{K_c(T_D s+1)}{\alpha T_D s+1}$

$$K_c = \frac{R}{R_1+R_2} \ ; \ \alpha = \frac{R_1}{R_1+R_2} \ ; \ T_D = R_2 C$$

11.2 $Z_1(s) = R_2 + R_1/(1 + R_1Cs)$; $Z_2(s) = R_1 + R_2$

$$\frac{E_2(s)}{E_1(s)} = -\frac{Z_2(s)}{Z_1(s)}[-1]$$

$$= \frac{\tau s+1}{\alpha\tau s+1};\ \tau = R_1 C,\ \alpha = \frac{R_2}{R_1+R_2} < 1$$

Refer Section 11.2 (Fig. 11.4) for the Bode plot and filtering properties of the lead compensator.

11.3 $Z_1(s) = R_1 + R_2$; $Z_2(s) = R_2 + R_1/(1 + R_1Cs)$

$$\frac{E_2(s)}{E_1(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{\tau s+1}{\beta\tau s+1};\ \tau = \frac{R_1R_2}{R_1+R_2}C,\ \ \beta = \frac{R_1+R_2}{R_2} > 1$$

Add an inverting amplifier of gain unity. Refer Section 11.2 (Fig. 11.7) for the Bode plot and filtering properties of the lag compensator.

11.4 $Z_1(s) = (R_1C_1s + 1)R_3/[(R_1 + R_3)C_1s + 1]$

$Z_2(s) = (R_2C_2s + 1)R_4/[(R_2 + R_4)C_2s + 1]$

$$\frac{E_2(s)}{E_1(s)} = -\frac{Z_2(s)}{Z_1(s)}\left[-\frac{R_6}{R_5}\right] = \frac{K_c\left(s+\dfrac{1}{\tau_1}\right)\left(s+\dfrac{1}{\tau_2}\right)}{\left(s+\dfrac{1}{\alpha\tau_1}\right)\left(s+\dfrac{1}{\beta\tau_2}\right)}$$

$$\tau_1 = (R_1 + R_3)C_1 \ ; \ \tau_2 = R_2C_2 \ ; \ \alpha = \frac{R_1}{R_1+R_3} \ ; \ \beta = \frac{R_2+R_4}{R_2} \ ;$$

$$K_c = \frac{R_2 R_4 R_6}{R_1 R_3 R_5}\left(\frac{R_1 + R_3}{R_2 + R_4}\right)$$

11.5 $Z_1(s) = R_1/(1 + R_1Cs)$; $Z_2(s) = R_2 + 1/C_2s$

$$\frac{E_2(s)}{E_1(s)} = -\frac{Z_2(s)}{Z_1(s)}\left[-\frac{R_4}{R_3}\right] = K_c\left[1 + \frac{1}{T_1 s} + T_D s\right]$$

$$K_c = \frac{R_4(R_1C_1 + R_2C_2)}{R_3R_1C_2} = 39.42;\; T_I = R_1C_1 + R_2C_2 = 3.077$$

$$T_D = \frac{R_1R_2C_1C_2}{R_1C_1 + R_2C_2} = 0.7692$$

These equations give

$R_1 = R_2 = 153.85$ kΩ; $R_4 = 197.1$ kΩ

11.6 $Z_1(s) = R_1/(1 + R_1C_1s)$; $Z_2(s) = R_2/(1 + R_2C_2s)$

$$\frac{E_2(s)}{E_1(s)} = -\frac{Z_2(s)}{Z_1(s)}\left[-\frac{R_4}{R_3}\right]$$

$$= \frac{R_4R_2}{R_3R_1}\left(\frac{R_1C_1s+1}{R_2C_2s+1}\right) = 2.51\left(\frac{0.345s+1}{0.185s+1}\right)$$

This equation gives

$R_1 = 34.5$ kΩ; $R_2 = 18.5$ kΩ; $R_4 = 45.8$ kΩ

11.7 By Trapezoidal rule for integration:

$$\int_0^{kT} e(t)dt = T\left[\frac{e(0)+e(T)}{2} + \frac{e(T)+e(2T)}{2} + \ldots + \frac{e(k-1)T + e(kT)}{2}\right]$$

$$= T\left[\sum_{i=1}^{k}\frac{e(i-1)T + e(iT)}{T}\right]$$

By backward-difference approximation for derivatives:

$$\left.\frac{de(t)}{dt}\right|_{t=kT} = \frac{e(kT) - e((k-1)T)}{T}$$

A difference equation model of the PID controller is, therefore, given by

$$u(k) = K_c \left\{ e(k) + \frac{T}{T_I}\sum_{i=1}^{k}\frac{e(i-1)+e(i)}{2} + \frac{T_D}{T}[e(k) - e(k-1)] \right\}$$

11.8 Taking sampling frequency twenty times the closed-loop natural frequency (which is a measure of bandwidth), we have

$$\omega_s = 20\omega_n = \frac{2\pi}{T} \text{ ; this gives } T = 0.5 \text{ sec}$$

Settling time $t_s = 4/\zeta\omega_n = 13$

$\frac{1}{10}$th of t_s is 1.3 sec; $T = 0.5$ sec is therefore a safe choice.

$$\frac{U(s)}{E(s)} = \frac{2.2(s+0.1)}{s+0.01}$$

$$\dot{u}(t) + 0.01\, u(t) = 2.2\, \dot{e}(t) + 0.22\, e(t)$$

$$\frac{1}{T}[u(k) - u(k-1)] + 0.01\, u(k) = \frac{2.2}{T}\, [e(k) - e(k-1)] + 0.22\, e(k)$$

This gives

$$u(k) = \frac{2}{2.01}\, u(k-1) + \frac{4.62}{2.01}\, e(k) - \frac{4.4}{2.01}\, e(k-1)$$

Configuration of the digital control scheme is shown in Fig. 11.19.

11.9 (a) $\dot{u}(t) + au(t) = K\,\dot{e}(t) + Kb\, e(t)$

By backward-difference approximation:

$$\frac{u(k)-u(k-1)}{T} + au(k) = \frac{K}{T}\, [e(k) - e(k-1)] + Kbe(k)$$

By backward-rectangular rule for integration:

$$u(t) = u(0) - a\int_0^t u(\tau)d\tau + Ke(t) - Ke(0) + Kb\int_0^t e(\tau)d\tau$$

$$u(k) = u(k-1) - aT\, u(k) + K\, e(k) - K\, e(k-1) + Kb\, T\, e(k)$$

By both the approaches of discretization, we get the following computer algorithm.

$$u(k) = \frac{1}{1+aT}u(k-1) + \frac{K(1+bT)}{1+aT}e(k) - \frac{K}{1+aT}e(k-1)$$

(b) $u(k) = u(k-1) - aT\left[\dfrac{u(k)+u(k-1)}{2}\right] + Ke(k) - Ke(k-1)$

$$+ KbT\left[\frac{e(k)+e(k-1)}{2}\right]$$

$$= \frac{1-\frac{aT}{2}}{1+\frac{aT}{2}}\,u(k-1) + \frac{K\left(1+\frac{bT}{2}\right)}{1+\frac{aT}{2}}\,e(k) - \frac{K\left(1-\frac{bT}{2}\right)}{1+\frac{aT}{2}}\,e(k-1)$$

11.10 For a choice of $T = 0.015$ sec, sampling frequency $\omega_s = 2\pi/T = 418.88$ rad/sec which is about 11 times the closed-loop bandwidth. This is a safe choice.

$$\frac{U(s)}{E(s)} = \frac{\frac{s^2}{(36)^2} + \frac{1.12s}{36} + 1}{\left(\frac{s}{740}+1\right)^2}$$

$$1.826 \times 10^{-6}\,\ddot{u}(t) + 2.7 \times 10^{-3}\,\dot{u}(t) + u(t)$$

$$= 7.7 \times 10^{-4}\,\ddot{e}(t) + 0.03\,\dot{e}(t) + e(t)$$

Using the results of Eqns (11.36) – (11.37), we get

$$1.826 \times 10^{-6}\left[\frac{u(k)-2u(k-1)+u(k-2)}{T^2}\right]$$

$$+ 2.7 \times 10^{-3}\left[\frac{u(k)-u(k-1)}{T}\right] + u(k)$$

$$= 7.7 \times 10^{-4}\left[\frac{e(k)-2e(k-1)+e(k-2)}{T^2}\right]$$

$$+ 0.03\left[\frac{e(k)-e(k-1)}{T}\right] + e(k)$$

This gives

$$1.1881\,u(k) - 0.1962\,u(k-1) + 0.0081\,u(k-2)$$

$$= 6.42\,e(k) - 8.84\,e(k-1) + 3.42\,e(k-2)$$

By direct digital design, we will be able to achieve the desired closed-loop performance using longer value of sampling interval.

11.11 $\dfrac{B(s)}{Y(s)} = \dfrac{K_t s}{\tau s + 1}$

$$\tau \dot{b}(t) + b(t) = K_t\, \dot{y}(t)$$

$$b(t) = b(0) - \frac{1}{\tau}\int_0^t b(\theta)d\theta + \frac{K_t}{\tau}[y(t) - y(0)]$$

$$b(k) = b(k-1) - \frac{T}{\tau}\left[\frac{b(k)+b(k-1)}{2}\right] + \frac{K_t}{\tau}[y(k) - y(k-1)]$$

$$= \frac{1 - T/2\tau}{1 + T/2\tau}\, b(k-1) + \frac{K_t/\tau}{1+T/2\tau} y(k) - \frac{K_t/\tau}{1+T/2\tau} y(k-1)$$

11.12 The set-point control, proportional action and derivative action are all included in the op amp circuit of Fig. 11.43. Only the integral action is to be added. Connect the integral-action unit of Fig. 11.41 in cascade with the proportional action unit of Fig. 11.43; the output e_4 of the proportional-action unit of Fig. 11.43 becomes input to the integral-action unit (output e_3).

The resulting op amp circuit is governed by the following relation (refer Examples 11.6 and 11.7);

$$E_3(s) = K_c\,(1 + T_D s)\left(1 + \frac{1}{T_I s}\right)[Y(s) - R(s)]$$

$$K_c = \frac{R_2}{R_2'}\ ;\ T_D = R_D C_D\ ;\ T_I = R_I\, C_I$$

11.13 $E_2(s) = Y(s) - R(s)$; $E_3(s) = -\,T_D\, sY(s)$; $T_D = R_D C_D$

$$E_4(s) = K_c\,[E_3(s) - E_2(s)]\ ;\ K_c = \frac{R_2}{R_2'}$$

$$= K_c\,[R(s) - (1 + T_D s)\, Y(s)]$$

11.14 $K_{cu} = 1.2$, $T_u = 4.5$ min

From the tuning rules given in Table 11.1:

$$K_c = 0.45K_{cu} = 0.54 \text{ or } \frac{1}{0.54} \times 100 = 185\% \text{ PB}$$

$$T_I = T_u/1.2 = 3.75 \text{ min or } \frac{1}{3.75} = 0.266 \text{ repeats/min}$$

(b) K_c: 110% PB ; T_I: 0.355 repeats/min ; T_D: 0.45 min

11.15 From the process reaction curve (refer Section 11.9) we obtain

$$\tau_D = 6 \text{ sec}, \ \tau = 12 \text{ sec}, K = 1$$

From tuning rules given in Table 11.2;

$$K_c = 1.8, T_I = 19.98 \text{ sec}$$

11.16 The model obtained from process reaction curve (refer Section 11.9):

$$K = 1 \ ; \ \tau = 1.875 \ ; \ \tau_D = 1.375$$

From tuning rules given in Table 11.2:

(a) $K_c = 1.36$

(b) $K_c = 1.23$; $T_I = 4.58$ min

11.17 $\tau_{ND} = 0.2$ for process *A*, 0.5 for process *B*, and 0.5 for Process *C*. Process *A* is more controllable than processes *B* and *C*, which are equally controllable.

11.18 (a) From the tuning rules given in Table 11.2:

$$K_c' = 3 \ ; \ T_I' = 4 \text{ min} \ ; \ T_D' = 1 \text{ min}$$

(b) $\tau_{CD} = \tau_D + \frac{1}{2} T = 2 \text{ min} + \frac{1}{2} \times 8 \text{ sec} = 2.067 \text{ min}.$

Replacing τ_D by τ_{CD} in the tuning formulas, we obtain

$$K_c' = 2.9 \ ; \ T_I' = 4.13 \text{ min} \ ; \ T_D' = 1.03 \text{ min}$$

11.19 Replace τ_D in tuning formulas of Table 11.2 by $\tau_{CD} = \tau_D + \frac{1}{2} T$. Increase in sampling time demands reduced K_c and increased T_I and T_D.

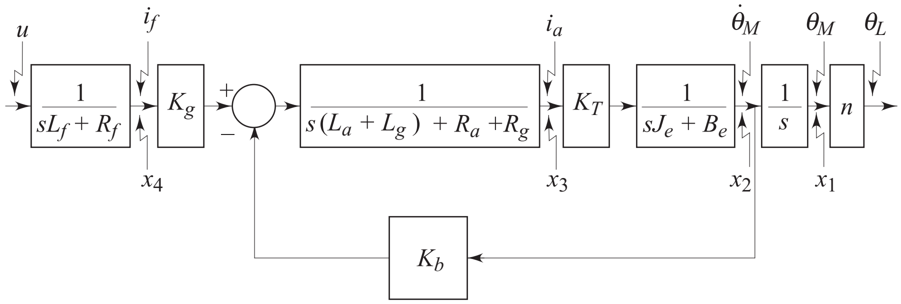
11.20 From tuning rules given in Table 11.1:

$$K_c = 2.25 \ ; \ T_I = 28.33 \text{ sec}$$

Correction to account for sampling is not required since the test has directly been conducted on the digital loop.

CHAPTER 12 CONTROL SYSTEM ANALYSIS USING STATE VARIABLE METHODS

12.1



$$J_e = n^2 J = 0.4 \; ; \; B_e = n^2 B = 0.01$$

$$\dot{x}_1 = x_2 \; ; \; 0.4\dot{x}_2 + 0.01x_2 = 1.2x_3$$

$$0.1\dot{x}_3 + 19x_3 = 100\,x_4 - 1.2x_2 \; ; \; 5\dot{x}_4 + 21x_4 = 4$$

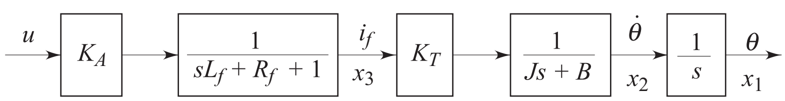
or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + bu$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.025 & 3 & 0 \\ 0 & -12 & -190 & 1000 \\ 0 & 0 & 0 & -4.2 \end{bmatrix}; \; \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.2 \end{bmatrix}$$

$$y = \theta_L = nx_1 = 0.5x_1$$

12.2



$$\dot{x}_1 = x_2 \; ; \; 0.5\dot{x}_2 + 0.5x_2 = 10x_3$$

$$20\dot{x}_3 + 100x_3 = 50u \quad ; \quad y = x_1$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 20 \\ 0 & 0 & -5 \end{bmatrix}; \; \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 2.5 \end{bmatrix}; \; \mathbf{c} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

12.3 $x_1 = \theta_M$; $x_2 = \dot{\theta}_M$, $x_3 = i_a$; $y = \theta_L$

$$\dot{x}_1 = x_2$$

$$2\dot{x}_2 + x_2 = 38\, x_3$$

$$2\dot{x}_3 + 21x_3 = e_a - 0.5\, x_2$$

$$e_a = k_1(\theta_R - \theta_L) - k_2\,\dot{\theta}_M = k_1\theta_R - \frac{k_1}{20}x_1 - k_2x_2$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.5 & 19 \\ -\frac{k_1}{40} & -\frac{(k_2+0.5)}{2} & -\frac{21}{2} \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \frac{k_1}{2} \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} \frac{1}{20} & 0 & 0 \end{bmatrix}$$

12.4 $x_1 = \omega$; $x_2 = i_a$

$$J\dot{\omega} + B\omega = K_T\, i_a$$

$$R_a i_a + L_a \frac{di_a}{dt} = e_a - K_b\omega$$

$$e_a = K_c e_c = K_c\,[k_1\,(e_r - K_t\,\omega) - k_2\, i_a\}$$

$$\mathbf{A} = \begin{bmatrix} -\frac{B}{J} & \frac{K_T}{J} \\ \frac{-(k_1K_tK_c + K_b)}{L_a} & \frac{-(R_a + k_2K_c)}{L_a} \end{bmatrix};$$

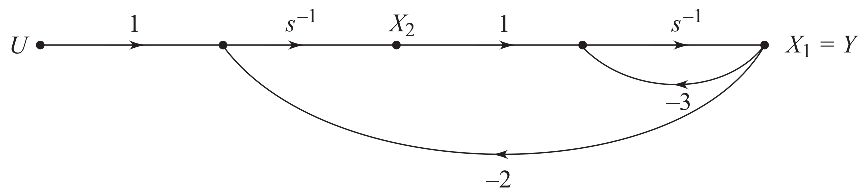
$$\mathbf{b} = \begin{bmatrix} 0 \\ \frac{k_1K_c}{L_a} \end{bmatrix};\ \boldsymbol{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

12.5

$$\overline{\mathbf{A}} = \mathbf{P}^{-1}\,\mathbf{AP} = \begin{bmatrix} -11 & 6 \\ -15 & 8 \end{bmatrix}$$

$$\overline{\mathbf{b}} = \mathbf{P}^{-1}\,\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix};\ \bar{\mathbf{c}} = \mathbf{cP} = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

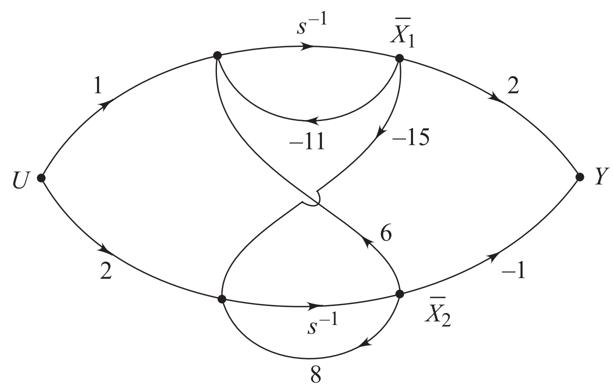
$$\frac{Y(s)}{U(s)} = \frac{P_1\Delta_1}{\Delta} = \frac{s^{-2}}{1-(-3s^{-1}-2s^{-2})} = \frac{1}{s^2+3s+2}$$



$$\frac{Y(s)}{U(s)} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4}{\Delta}$$

$$= \frac{2s^{-1}(1-8s^{-1}) + 15s^{-2}(1) + (-2s^{-1})[1+11s^{-1}] + 2s^{-1}(6s^{-1})2[1]}{1-[-11s^{-1}+8s^{-1}+(-15s^{-1})(6s^{-1})]+[(-11s^{-1})(8s^{-1})]}$$

$$= \frac{1}{s^2+3s+2}$$



12.6 $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\overline{\mathbf{A}} = \mathbf{P}^{-1}\,\mathbf{AP} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$; $\overline{\mathbf{b}} = \mathbf{P}^{-1}\,\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|\lambda\mathbf{I} - \mathbf{A}| = |\lambda\mathbf{I} - \overline{\mathbf{A}}| = \lambda^2$

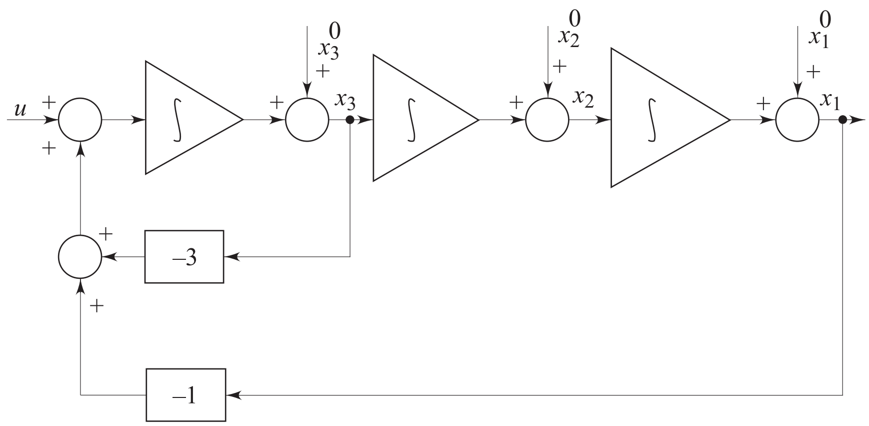
12.7 $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{x}^0 + (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}\,U(s)$

$= \mathbf{G}(s)\mathbf{x}^0 + \mathbf{H}(s)\,U(s)$

$$\mathbf{G}(s) = \frac{1}{\Delta}\begin{bmatrix} s(s+3) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -1 & s^2 \end{bmatrix}; \mathbf{H}(s) = \frac{1}{\Delta}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}$$

$\Delta = s^3 + 3s^2 + 1$

12.8



12.9 Taking outputs of integrators as state variables, we get (x_1 being the output of rightmost integrator),

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2x_2 + x_3 \\
\dot{x}_3 &= -x_3 - x_2 - y + u \\
y &= 2x_1 - 2x_2 + x_3
\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & -2 \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \mathbf{c} = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix}$$

12.10 (a)

$$\begin{aligned}
G(s) &= \mathbf{c}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b} \\
&= \frac{s+3}{(s+1)(s+2)}
\end{aligned}$$

(b)

$$G(s) = \frac{1}{(s+1)(s+2)}$$

12.11

$$\begin{aligned}
\dot{x}_1 &= -3x_1 + 2x_2 + [-2x_1 - 1.5x_2 - 3.5x_3] \\
\dot{x}_2 &= 4x_1 - 5x_2 \\
\dot{x}_3 &= x_2 - r
\end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} -5 & 0.5 & -3.5 \\ 4 & -5 & 0 \\ 0 & 1 & 0 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$$G(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b} = \frac{14}{(s+1)(s+2)(s+7)}$$

12.12 (a) x_1 = output of lag $1/(s + 2)$

x_2 = output of lag $1/(s + 1)$

$$\dot{x}_1 + 2x_1 = x_2;\ \dot{x}_2 + x_2 = -x_1 + u$$

$$y = x_2 + (-x_1 + u)$$

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} -1 & 1 \end{bmatrix};\ d = 1$$

(b) x_1 = output of lag $1/(s + 2)$

x_2 = output of lag $1/s$

x_3 = output of lag $1/(s + 1)$

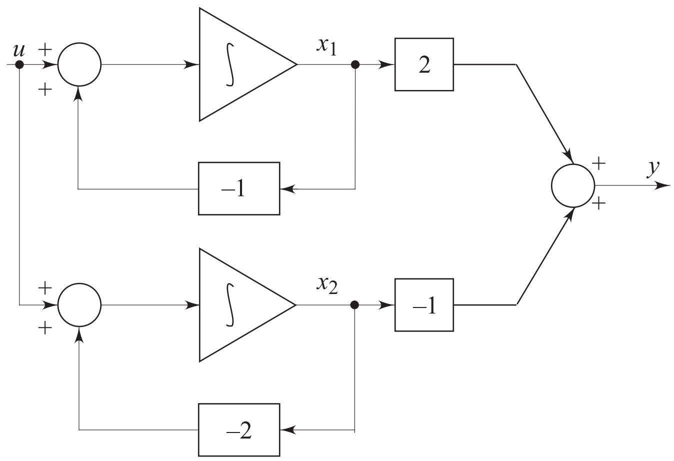
$$\dot{x}_1 + 2x_1 = y;\ \dot{x}_2 = -x_1 + u$$

$$\dot{x}_3 + x_3 = -x_1 + u;\ y = x_2 + x_3$$

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

12.13 (i) $$\frac{Y(s)}{U(s)} = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-1}{s+2}$$

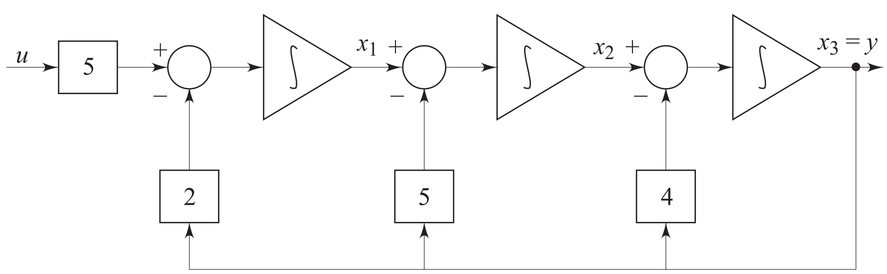
$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 2 & -1 \end{bmatrix}$$



(ii) $\dfrac{Y(s)}{U(s)} = \dfrac{\beta_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3} = \dfrac{5}{s^3 + 4s^2 + 5s + 2}$

From Eqns (12.46), the second companion form of the state model is given below.

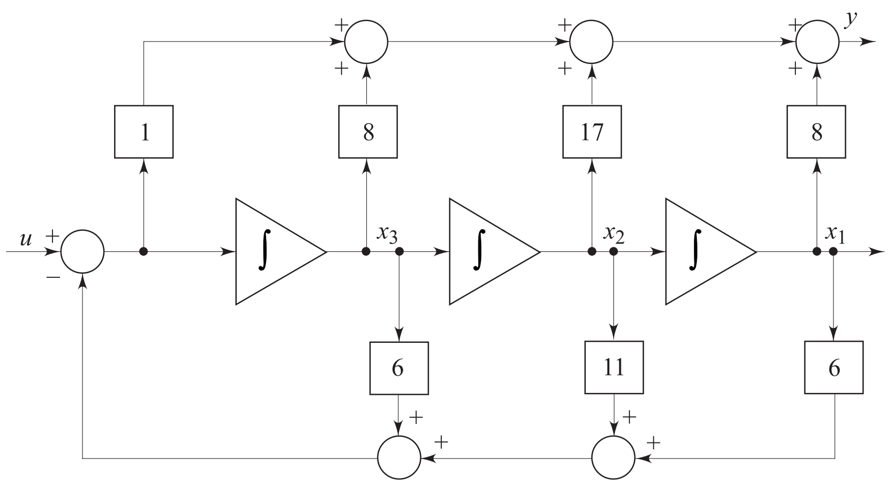
$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -5 \\ 0 & 1 & -4 \end{bmatrix};\ \mathrm{b} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$



(iii) $\dfrac{Y(s)}{U(s)} = \dfrac{\beta_0 s^3 + \beta_1 s^2 + \beta_2 s + \beta_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3} = \dfrac{s^3 + 8s^2 + 17s + 8}{s^3 + 6s^2 + 11s + 6}$

From Eqns (12.44), the state model in second companion form is given below.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 2 & 6 & 2 \end{bmatrix}$$



12.14 (i) $$\frac{Y(s)}{U(s)} = \frac{s+1}{s^3+3s^2+2s} = \frac{\beta_2 s+\beta_3}{s^3+\alpha_1 s^2+\alpha_2 s+\alpha_3}$$

From Eqns (12.46):

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

(ii) $$\frac{Y(s)}{U(s)} = \frac{1}{s^3+6s^2+11s+6} = \frac{\beta_3}{s^3+\alpha_1 s^2+\alpha_2 s+\alpha_3}$$

From Eqns (12.44):

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

(iii) $$\frac{Y(s)}{U(s)} = \frac{s^3+8s^2+17s+8}{s^3+6s^2+11s+6} = 1+\frac{-1}{s+1}+\frac{2}{s+2}+\frac{1}{s+3}$$

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mathbf{c} = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix};\ d = 1$$

12.15 (a) $$\frac{Y(s)}{U(s)} = \frac{1000s + 5000}{s^3 + 52s^2 + 100s} = \frac{\beta_2 s + \beta_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$$

From Eqns (12.44):

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -100 & -52 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 5000 & 1000 & 0 \end{bmatrix}$$

(b) $$\frac{Y(s)}{U(s)} = \frac{1000s + 5000}{s^3 + 52s^2 + 100s} = \frac{50}{s} + \frac{-31.25}{s+2} + \frac{-18.75}{s+50}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -50 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 50 \\ -31.25 \\ -18.75 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

12.16 $$\frac{Y(s)}{U(s)} = \frac{2s^2 + 6s + 5}{(s+1)^2(s+2)} = \frac{1}{(s+1)^2} + \frac{1}{s+1} + \frac{1}{s+2}$$

From Eqns (12.55):

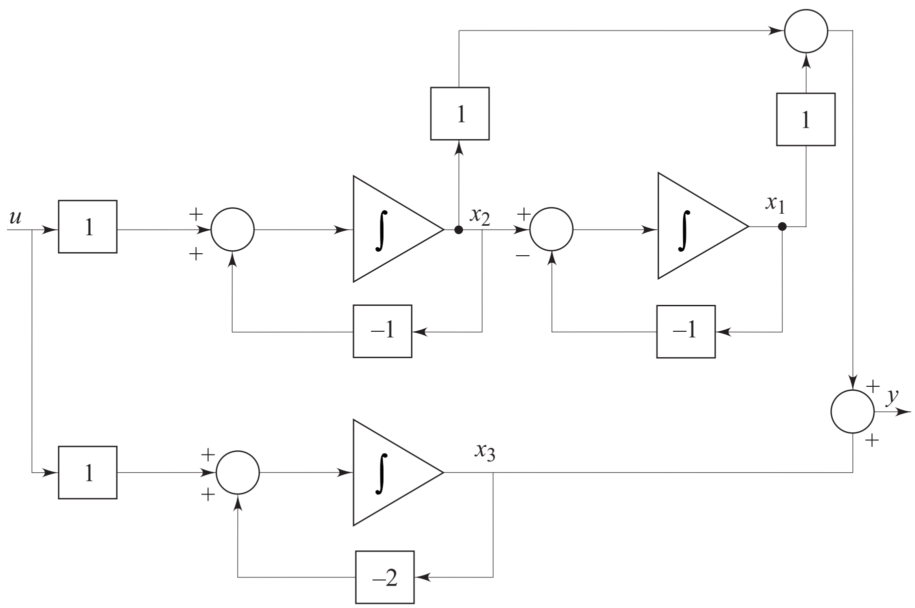
$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{b} u(\tau) d\tau$$

$$e^{-\mathbf{A}t} = \mathscr{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \begin{bmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

$$\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)d\tau = \begin{bmatrix} \int_0^t (t-\tau)e^{-(t-\tau)}d\tau \\ \int_0^t e^{-(t-\tau)}d\tau \\ \int_0^t e^{-2(t-\tau)}d\tau \end{bmatrix} = \begin{bmatrix} 1-e^{-t}-te^{-t} \\ 1-e^{-t} \\ \frac{1}{2}(1-e^{-2t}) \end{bmatrix}$$

$$y = x_1 + x_2 + x_3 = 2.5 - 2e^{-t} - te^{-t} - 0.5\, e^{-2t}$$



12.17 $\mathbf{F} = e^{AT} = \begin{bmatrix} 0.696 & 0.246 \\ 0.123 & 0.572 \end{bmatrix}$

$\mathbf{g} = (e^{-AT} - \mathbf{I})\, \mathbf{A}^{-1}\, \mathbf{b}$

$$= \begin{bmatrix} -0.304 & 0.246 \\ 0.123 & -0.428 \end{bmatrix} \begin{bmatrix} \frac{-3}{4} & \frac{-1}{2} \\ \frac{-1}{4} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -0.021 \\ 0.747 \end{bmatrix}$$

$\mathbf{c} = [2 \quad -4]$; $d = 6$

12.18 $$\frac{Y(s)}{U(s)} = \frac{e^{-0.4s}}{s+1}$$

$$\dot{x}_1(t) = -x_1(t) + u(t-0.4)$$

$$\tau_D = 0.4 \text{ ; therefore, } N = 0,\ \Delta = 0.4,\ m = 0.6$$

$$F = e^{-1} = 0.3679$$

$$g_2 = \int_0^{0.6} e^{-\sigma} d\sigma = 0.4512 \text{ ; } g_1 = e^{-0.6}\int_0^{0.4} e^{-\theta} d\theta = 0.1809$$

$$x_1(k+1) = 0.3679\, x_1(k) + 0.1809\, u(k-1) + 0.4512 u(k)$$

Introduce a new state $x_2(k) = u(k-1)$

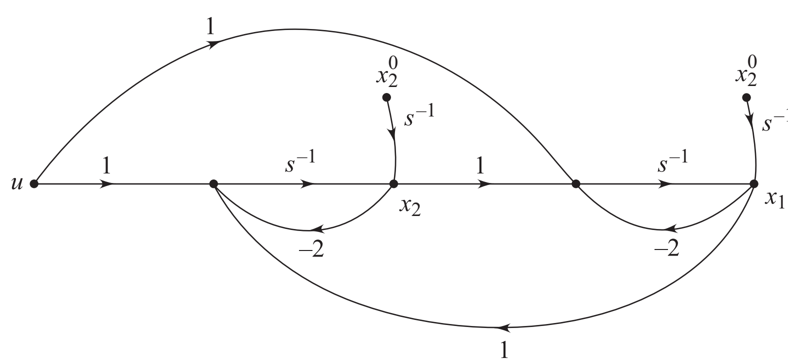
$$\mathbf{x}(k+1) = \mathbf{F}\mathbf{x}(k) = \mathbf{g}u(k)$$

$$y(k) = \mathbf{c}\mathbf{x}(k)$$

$$\mathbf{F} = \begin{bmatrix} 0.3679 & 0.1809 \\ 0 & 0 \end{bmatrix} \text{; } \mathbf{g} = \begin{bmatrix} 0.4512 \\ 1 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

12.19



$$\mathbf{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \text{; } \mathbf{H}(s) = \begin{bmatrix} H_1(s) \\ H_2(s) \end{bmatrix}$$

$$\frac{X_1(s)}{x_1^0} = G_{11}(s) = \frac{s^{-1}(1+2s^{-1})}{1-(-2s^{-1}-2s^{-1}+s^{-2})+(-2s^{-1})(-2s^{-1})}$$

$$= \frac{s^{-1}(1+2s^{-1})}{\Delta} = \frac{1/2}{s+1} + \frac{1/2}{s+3}$$

$$\frac{X_1(s)}{x_2^0} = G_{12}(s) = \frac{s^{-2}(1)}{\Delta} = \frac{1/2}{s+1} + \frac{-1/2}{s+3}$$

$$\frac{X_2(s)}{x_1^0} = G_{21}(s) = \frac{s^{-2}(1)}{\Delta} = \frac{1/2}{s+1} + \frac{-1/2}{s+3}$$

$$\frac{X_2(s)}{x_2^0} = G_{22}(s) = \frac{s^{-1}(1+2s^{-1})}{\Delta} = \frac{1/2}{s+1} + \frac{1/2}{s+3}$$

$$\frac{X_1(s)}{U(s)} = H_1(s) = \frac{s^{-2}(1) + s^{-1}(1+2s^{-1})}{\Delta} = \frac{1}{s+1}$$

$$\frac{X_2(s)}{U(s)} = H_2(s) = \frac{1}{s+1}$$

Zero-input response:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\,\mathbf{x}^0 = \mathscr{L}^{1}\,[\mathbf{G}(s)\mathbf{x}^0]$$

$$= \frac{1}{2}\begin{bmatrix} e^{-t}+e^{-3t} & e^{-t}-e^{-3t} \\ e^{-t}-e^{-3t} & e^{-t}+e^{-3t} \end{bmatrix}\begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$$

Zero-state response:

$$\mathbf{x}(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)\,d\tau = \mathscr{L}^{-1}\,[\mathbf{H}(s)U(s)] = \begin{bmatrix} 1-e^{-t} \\ 1-e^{-t} \end{bmatrix}$$

12.20 $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$; eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -2$

$$Y(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{x}^0 + \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}\;U(s)$$

$$= \frac{s+4}{(s+1)(s+2)} + \frac{1}{s(s+1)(s+2)}$$

$$= \frac{3}{s+1} - \frac{2}{s+2} + \frac{1/2}{s} - \frac{1}{s+1} + \frac{1/2}{s+2}$$

$$y(t) = \frac{1}{2} + 2e^{-t} - \frac{3}{2}e^{-2t}$$

12.21
$$\dot{x}_1 = -3x_1 + 2x_2 + [7r - 3x_1 - 1.5\,x_2]$$
$$\dot{x}_2 = 4x_1 - 5x_2$$
$$\mathbf{A} = \begin{bmatrix} -6 & 0.5 \\ 4 & -5 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 7 \\ 0 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$e^{\mathbf{A}t} = \mathscr{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$
$$= \begin{bmatrix} \frac{1}{3}e^{-4t} + \frac{2}{3}e^{-7t} & \frac{0.5}{3}e^{-4t} - \frac{0.5}{3}e^{-7t} \\ \frac{4}{3}e^{-4t} - \frac{4}{3}e^{-7t} & \frac{2}{3}e^{-4t} + \frac{1}{3}e^{-7t} \end{bmatrix}$$
$$y(t) = x_2(t) = 7\int_0^t \left[\frac{4}{3}e^{-4(t-\tau)} - \frac{4}{3}e^{-7(t-\tau)}\right]d\tau$$
$$= \frac{28}{3}\left[\frac{1}{4}(1 - e^{-4t}) - \frac{1}{7}(1 - e^{-7t})\right]$$

12.22
$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} + te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)$

Given: $x_1(2) = 2$; $t_0 = 1$, $t = 2$

Manipulation of the equation gives

$x_1(2) = 2e^{-1}\,x_1(1) + e^{-1}\,x_2(1) = 2$

If $x_2(1) = 2k$, then $x_1(1) = e^1 - k$

Thus

$\begin{bmatrix} e^1 - k \\ 2k \end{bmatrix}$ is a possible set of states $\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix}$ for any $k \neq 0$

12.23 $\begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix};\ \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

This gives

$$e^{\mathbf{A}t} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

$$= \mathscr{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{2}{s+2} - \frac{2}{s+1} & \frac{2}{s+2} - \frac{1}{s+1} \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

12.24 $\mathbf{V} = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \vdots \\ \mathbf{cA}^{n-1} \end{bmatrix}$ is a triangular matrix with diagonal elements equal to unity; $|\mathbf{V}| = (-1)^n$ for all α_i's. This proves the result.

12.25 $\mathbf{U} = [\mathbf{b} \quad \mathbf{Ab} \ldots \mathbf{A}^{n-1}\mathbf{b}]$ is a triangular matrix with diagonal elements equal to unity; $|\mathbf{U}| = (-1)^n$ for all α_i's. This proves the result.

12.26 (i) $\mathbf{U} = [\mathbf{b} \quad \mathbf{Ab}] = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$; $\rho(\mathbf{U}) = 2$

Completely controllable

$$\mathbf{V} = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}; \ \rho(\mathbf{V}) = 1$$

Not completely observable.

(ii) The system is in Jordan canonical form; it is controllable but not observable.

(iii) The system is in controllable companion form. The given system is therefore controllable. We have to test for observability property only.

$$\rho(\mathbf{V}) = \rho\begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \\ \mathbf{cA}^2 \end{bmatrix} = 3$$

The system is completely observable.

(iv) Given system is in observable companion form. We have to test for controllability property only.

$$\rho(\mathbf{U}) = \rho[\mathbf{b} \;\; \mathbf{Ab} \;\; \mathbf{A}^2\mathbf{b}] = 3$$

The system is completely controllable.

12.27 (i) Observable but not controllable

(ii) Controllable but not observable

(iii) Neither controllable nor observable.

Controllable and observable realization:

$$A = -1 \;;\; b = 1 \;;\; c = 1$$

12.28 (i)
$$G(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b} = \frac{1}{s+2}$$

Given state model is in observable companion form. Since there is a pole-zero cancellation, the state model is uncontrollable.

(ii)
$$G(s) = \frac{s+4}{(s+2)(s+3)}$$

The given state model is in controllable companion form. Since there is a pole-zero cancellation, the model is unobservable.

12.29 (a)
$$|\lambda\mathbf{I} - \mathbf{A}| = (\lambda - 1)\,(\lambda + 2)\,(\lambda + 1)$$

The system is unstable.

(b)
$$G(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}$$
$$= \frac{1}{(s+1)(s+2)}$$

The $G(s)$ is stable

(c) The unstable mode e^t of the free response is hidden from the transfer function representation

12.30 (a)
$$G(s) = \frac{10}{s^2+s} = \frac{\beta_2}{s^2+\alpha_1 s+\alpha_2}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix};\; \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix};\; \mathbf{c} = [10 \quad 0]$$

(b) Let us obtain controllable companion form realization of

$$G(s) = \frac{10(s+2)}{s(s+1)(s+2)} = \frac{10s+20}{s^3+3s^2+2s} = \frac{\beta_2 s+\beta_3}{s^3+\alpha_1 s^2+\alpha_2 s+\alpha_3}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 20 & 10 & 0 \end{bmatrix}$$

(c) Let us obtain observable companion form realization of

$$G(s) = \frac{10(s+2)}{s(s+1)(s+2)}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix};\ \mathbf{b} = \begin{bmatrix} 20 \\ 10 \\ 0 \end{bmatrix};\ \mathbf{c} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$