Exploring Time-based vs. Move-based Hedging Strategies for a European Put Option: Implementations of Delta Hedging and Delta-Gamma Hedging

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November 13, 2022

Assignment 2 for Dr. Sebastian Jaimungal
In Partial Fulfillment of Coursework for MMF1928F/ STA2503F

Abstract

In this report, we will investigate hedging in a discrete time setting within the Black-Scholes framework. We consider a newly traded short put position, and we compare time-based and move-based hedge re-balancing approaches for both delta hedging and delta-gamma hedging strategies. For delta hedging, we trade in the underlying asset and in the bank account to maintain delta neutral holdings for both time-based and move-based (relative to deviation in delta) approaches. For delta-gamma hedging, we trade in the underlying asset in the bank account and in a call option (of twice the maturity) to maintain delta- and gamma-neutral holdings for both time-based and move-based (relative to deviation in delta) approaches. We quantitatively compare the re-balancing approaches, for each delta and delta-gamma hedging, by simulating the profit and loss distribution of each strategy (and assuming the underlying asset follows the Black-Scholes model). We further go on to observe and discuss the effect of adjusting the bandwidth size for the move-based approach, for each of our hedging strategies. For each of the profit and loss distributions in this report, we also monitored the condition value-at-risk (CVaR) and determined the price to charge in each of the scenarios so that the CVaR is at the desired maximum (0.02) at a relevant confidence level (90%).

Contents

1 Introduction

Hedging in the context of Finance and trading refers to taking investment positions in certain instruments, with the goal of mitigating the risk exposure to price movements of assets in a portfolio. As the name suggests, dynamic hedging involves monitoring the hedge position as the market condition evolves. Here we introduce the concept of rebalancing, which refers to tilting the weights of assets in a hedge portfolio to ensure a better match with the risk exposure over time.

Throughout this report, we attempt to study two of the most commonly seen hedging strategies: Delta hedging and Delta-Gamma hedging. We will also demonstrate their effect in capturing the risk associated with our investment portfolio.

In the world of financial derivatives, Delta measures the directional risk associated with price movements in the underlying. Simply put, it tracks the change in our portfolio value per $1 change in the asset price. When a portfolio is Delta neutral, it is insensitive to the asset price movements. This, however, may not be sufficient as we are still exposed to the changes in Delta itself, described by Gamma. As a result, Delta-Gamma hedge is implemented to guard against both factors.

We will investigate discrete time hedging practices within the Black-Scholes model. As we build our hedging strategies, we consider two approaches with respect to rebalancing our holdings, time-based and move-based. We will also consider the scenarios of rebalancing for both delta hedging and delta-gamma hedging.

2 Model Methodology

2.1 Model Set Up

We consider an asset $S = (S_t)_{t \geq 0}$ that follows the Black-Scholes model. We further consider a scenario we have just sold an at-the-money put on asset S, and we plan to hedge this position. In building a strategy, transaction costs of trades will be considered and we are able to trade in:

- the bank account
- the asset, S
- a defined call option, on asset S

Working within the Black-Scholes model, we will assume that:

- volatility and risk-free rate remain constant
- the price process of the asset is lognormally distributed.
- the continuously compounded return over a given time is normally distributed
- able to purchase split amounts

To understand how to build a hedging strategy, first we define the tradable assets in our Market. Within the Black-Scholes framework, we define the risk-free bank account, denoted M, and we understand the risky asset, S, follows a geometric brownian motion, such that:

$$dB_t = rB_t dt$$
$$dS_t = S_t(\mu dt + \sigma dW_t)$$

2.1.1 Simulating Stock Prices and Bank Account Discounting

To build and test our hedging strategies, we will run 10,000 simulations of the asset, given the price process $dS_t = S_t(\mu dt + \sigma dW_t)$. We will test the hedging strategies (which are detailed later in this report) by reflecting on the profit and loss distribution of each strategy.

Since we know that the price process of the asset S is a geometric brownian motion, we know that it is also lognormally distributed.

Therefore, let $f(t, S_t) = log(S_t)$ and

$$\delta_t f = 0 \qquad \delta_s f = \frac{1}{S_t} \qquad \delta_{ss} f = -\frac{1}{S_t^2}$$

By Itô's lemma we have that

$$df_t = \delta_t f(t,S)|_{S=S_t} + \delta_s f(t,S)|_{S=S_t} dS_t + \delta_{ss} f(t,S)|_{S=S_t} d[S,S]_t$$
$$\text{where } dS_t = \mu S_t dt + \sigma S_t dW_t$$
$$\text{and } d[S,S]_t = \sigma^2 S_t^2 dt$$

Therefore we have that

$$\begin{aligned} df_t &= \frac{1}{S_t}\mu(S_t dt + \sigma S_t dW_t) - \frac{1}{2S_t^2}\sigma^2 S_t^2 dt \\ &= (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t \end{aligned}$$

Now we integrate both sides over the domain $[t, t + \Delta t]$

$$\begin{aligned} \int_t^{t+\Delta t} df_t &= (\mu - \frac{\sigma^2}{2}) \int_t^{t+\Delta t} dt + \sigma \int_t^{t+\Delta t} dW_t \\ f_{t+\Delta t} - ft &= (\mu - \frac{\sigma^2}{2})\Delta t + \sigma(W_{t+\Delta t} - W_t) \end{aligned}$$

Here we note that $W_{t+\Delta t} - W_t = W_{\Delta t}$. Since $W_{\Delta t} \overset{\mathbb{P}}{\sim} N(0, \Delta t)$, we can see that that $W_{\Delta t} \overset{d}{=} \sqrt{\Delta t}$ Z, where Z $\overset{\mathbb{P}}{\sim} N(0, 1)$. Therefore we have that

$$log(\frac{S_{t+\Delta t}}{S_t}) = (\mu - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}\text{Z}, \text{ where Z} \sim N(0, 1)$$

And finally, we get that the stock price at the next timestep in the discrete process, t+Δt, can be found with

$$S_{t+\Delta t} = S_t e^{(\mu - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}Z}, \text{ where Z} \sim N(0, 1)$$

In the Black-Scholes model, we consider the bank account, M, to be a risk-free asset that earns a risk-free rate and therefore

$$M_{t+\Delta t} = M_t e^{r\Delta t}$$

2.1.2 Greeks

Now that we have spoken about the price processes of the tradable assets in our market, let's consider how to price the contingent claims we are concerned with. Where the payoff of the call option is given by $(S_t - K)^+$ and the payoff of a put option is given by $(K - S_t)^+$, through the Black-Scholes pricing model, we see that a contingent claim can be priced as a function of time, t, and the price of the underlying, S, at that time t:

$$C(t,S) = S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$$
$$P(t,S) = Ke^{-r(T-t)}\Phi(-d_-) - S\Phi(-d_+) \quad \text{where } d_\pm = \frac{ln(\frac{S}{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Every call and put option has inherent risks associated with factors that affect the price of the derivative. The most significant of these have been identified and classified as the Greeks ($\Delta\ \Gamma\ V\ \Theta\ \rho$) and happen to be associated with the partial derivatives of the solution to the Black-Scholes PDE respective to their related variable. For our purposes, we will be concerned with delta, Δ, and gamma, Γ, where:

$$\Delta^{call}(t,S) = \delta_S C(t,S) = \Phi(d_+) \ \in [0,1]$$
$$\Delta^{put}(t,S) = \delta_S P(t,S) = -\Phi(-d_+) = \Phi(d_+) - 1 \ \in [-1,0]$$

$$\begin{aligned}\Gamma^{call}(t,S) = \delta_S^2 C(t,S) = \Gamma^{put}(t,S) &= \delta_S^2 P(t,S) \\ &= \frac{\phi(d_+)}{S\sigma\sqrt{T-t}} > 0\end{aligned}$$

Being the first derivative of the option value concerning the underlying asset price, Δ provides information about how much an option's price is expected to change as a result of small changes in the price of the underlying. Being the second derivative, Γ provides information about the rate of change of an option's Δ with changes in the price of the underlying. Hedging with Γ helps to reduce exposure to large movements in the price of the underlying.

2.1.3 Option Hedging

Option hedging involves mitigating the risk of an existing trade by entering into a position (or positions) that would offset the risk of loss. In other words, if the value of one position decreases, then the value of the other would increase and have a net effect of reducing exposure to loss. For our purposes, we focus on delta and delta-gamma hedging.

A delta hedging strategy mitigates the exposure to the risk of price movement in the underlying asset. To hedge price movement risk, one would enter an offsetting position equal to Δ of the holdings in the underlying asset. If the right amount of the underlying is traded, then holdings can be made delta neutral, where Δ is set to zero. At that moment, (reasonable) changes in the price of the underlying will not have a net effect on the value of your holdings, all else equal.

A delta-gamma hedging strategy mitigates both the exposure to the risk of price movement of the underlying asset and the risk of price movement of the derivative. As we delta-gamma hedge the portfolio, we trade in a second derivative position to hedge the gamma of our holdings and trade in the underlying to hedge the delta of our new overall holdings (which now includes two derivatives positions). When we delta-gamma hedge, the holdings become delta neutral and gamma neutral for that moment.

This raises the question of how often to rebalance the holdings. We will examine both a time-based approach and a move-based approach and compare the two.

2.2 Time-Based Strategy

The time-based approach involves rebalancing holdings in a consistent and predetermined manner, such as daily, weekly, quarterly, etc. More specifically, at the end of some time increments over the investment horizon, we try to create a hedge portfolio that, combined with the option, will create a portfolio that removes instantaneous risk by matching the hedge positions with the option being hedged to some extent. This report demonstrates delta and delta-gamma hedging, rebalancing daily with an investment horizon of $\frac{1}{4}$ years.

2.2.1 Time-based Delta Hedging

Recall that the delta of an option is the first derivative of the option value concerning the underlying asset price. In our example, it is the rate of change in the put price concerning the underlying stock price. Note that because a put option is in the money when the stock price falls below the strike price, $\Delta^{put} \in [0, -1]$. Naturally, the delta of a stock is 1 because $\frac{dS}{S} = 1 = \Delta_S$. On a daily basis, the delta hedging strategy aims to create a portfolio that is delta neutral, in other words, $\Delta_p = 0$.
At $t = 0$, We will create a hedge for a short put position by holding a short position in the underlying stock, $\alpha_0 = \Delta_0^{put}$. We will have a cash flow from this transaction denoted by M. Clearly this portfolio is delta-neutral, $\Delta_0^{portfolio} = \alpha_0 - \Delta_0^{put} = 0$. And in the transaction,

$$M_0 = f^{put}(0, S_0) - \alpha_0 * S_0 - \phi_{equity} * |\alpha_0|$$

and as time to maturity shortens ($0 < t < T$),

we rebalance holdings each day where

$$M_t = M_{t-1}e^{r\Delta t} - (\alpha_t - \alpha_{t-1})S_t - |\alpha_t - \alpha_{t-1}|\phi_{equity}$$

and when time reaches maturity (t = T),

we liquidate holdings and calculate profit/loss as

$$M_T = M_{T-1}e^{r\Delta t} + (\alpha_{T-1})S_T - |\alpha_{T-1}|\phi_{equity} - (K - S_T)^+$$

2.2.2 Time-based Delta-Gamma Hedging

Recall that the gamma of an option is the first derivative of the delta or the second derivative of the option value concerning the underlying asset price. Also the gamma $\in \mathbb{R}^+$ for long derivative holdings, and gamma $\in \mathbb{R}^-$ for short derivative holdings. In a delta-gamma hedging strategy, we will trade in a second derivative and hedge to make the holdings gamma neutral. In our example, this would be entering a position in a call option where $\Gamma_0^{put} = \beta_0 \Gamma_0^{call}$ and β is our holdings in the call option. We would then trade in the underlying asset to make the overall holdings delta neutral.

At $t = 0$, we will create a delta-gamma hedge for a short put position by holding a long call position, $\beta = \Gamma_0^{put} / \Gamma_0^{call}$, and a short position in the underlying stock, $\alpha_0 = \Delta_0^{put} + \beta_0 \Delta_0^{call}$. Clearly this portfolio is delta-gamma-neutral, since $\Gamma^{portfolio} = -\Gamma_0^{put} + \beta_0 \Gamma_0^{call} = 0$ and $\Delta_0^{portfolio} = \alpha_0 - \Delta_0^{put} - \beta_0 \Delta_0^{call} = 0$. The cash flow from the transactions is reflected in the bank account, M. We can see that at the initial time, the amount of money in the bank account, M_0, is the difference between the initial price of the put option and the cost of entering the hedging positions, including transaction costs:

$$M_0 = f^{put}(0, S_0) - \alpha_0 S_0 - \phi_{equity}|\alpha_0| - \beta_0 * f^{call}(0, S_0) - \phi_{option}|\beta_0|$$

and as time to maturity shortens ($0 < t < T$),
we rebalance holdings each day where

$$\begin{aligned} M_t = M_{t-1}e^{r\Delta t} - (\alpha_t - \alpha_{t-1})S_t - |\alpha_t - \alpha_{t-1}|\phi_{equity} \\ -(\beta_t - \beta_{t-1})f^{call}(t, S_t) - |\beta_t - \beta_{t-1}|\phi_{option} \end{aligned}$$

and when time reaches maturity (t = T),
we liquidate holdings and calculate profit/loss as

$$\begin{aligned} M_T = M_{T-1}e^{r\Delta t} + (\alpha_{T-1})S_T - |\alpha_{T-1}|\phi_{equity} + (\beta_{T-1})f^{call}(T, S_T) \\ -|\beta_{T-1}|\phi_{option} - (K - S_T)^+ \end{aligned}$$

2.3 Move-Based Strategy

The move-based approach differs from the time-based approach by the strategy of re-balancing the holdings to re-hedge for risk. What if re-balancing daily is too much and an unnecessary expenditure of transaction costs? Instead, in this move-based approach, we will monitor the delta on a daily basis and rebalance only when it has moved far enough. We will use a band of 0.1 around the current delta position such that when $\Delta^{put}_{new} = \Delta^{put}_0 \pm 5$, we rebalance and establish $\Delta^{put}_{new} \pm 5$ as the new bands. Note that in move-based strategies, Δt is the time it takes for the Δ^{put} to deviate from the last noted value by the bandwidth size (divided by 2). Later in the report, we will also speak to the effect of increasing or decreasing the size of the band. We will also consider this strategy in the context of both delta hedging and delta-gamma hedging.

2.3.1 Move-based Delta Hedging

Now we don't move by timestep of 1 each day, but rather by Δt, where rebalancing occurs as the Δ^{put} deviates to a boundary band at time, $t_{last\ rebalance} + \Delta t$.

$$M_0 = f^{put}(0, S_0) - \alpha_0 * S_0 - \phi_{equity} * |\alpha_0|$$

$$\begin{aligned} &\text{and as time to maturity shortens } (0 < t < T), \text{ we rebalance holdings when} \\ &\Delta^{put} \text{ deviates to a boundary band at time, t = t}_{last\ rebalance}\text{+}\Delta t \\ &M_t = M_{t-\Delta t}e^{r\Delta t} - (\alpha_t - \alpha_{t-\Delta t})S_t - |\alpha_t - \alpha_{t-\Delta t}|\phi_{equity} \end{aligned}$$

$$\begin{aligned} &\text{and when time reaches maturity (t = T),} \\ &\text{we liquidate holdings and calculate net profit/loss as} \\ &M_T = M_{T-\Delta t}e^{r\Delta t} + (\alpha_{T-\Delta t})S_T - |\alpha_{T-\Delta t}|\phi_{equity} - (K - S_T)^+ \end{aligned}$$

2.3.2 Move-based Delta-Gamma Hedging

Now we don't move by timestep of 1 each day, but rather by Δt, where rebalancing occurs as the Δ^{put} deviates to a boundary band at time, t+Δt.

$$M_0 = f^{put}(0, S_0) - \alpha_0 S_0 - \phi_{equity}|\alpha_0| - \beta_0 * f^{call}(0, S_0) - \phi_{option}|\beta_0|$$

$$\begin{aligned} &\text{and as time to maturity shortens } (0 < t < T), \text{ we rebalance holdings when} \\ &\Delta^{put} \text{ deviates to a boundary band at time, t = t}_{last\ rebalance}\text{+}\Delta t \\ &M_t = M_{t-\Delta t}e^{r\Delta t} - (\alpha_t - \alpha_{t-\Delta t})S_t - |\alpha_t - \alpha_{t-\Delta t}|\phi_{equity} \\ &-(\beta_t - \beta_{t-\Delta t})f^{call}(t, S_t) - |\beta_t - \beta_{t-\Delta t}|\phi_{option} \end{aligned}$$

$$\begin{aligned} &\text{and when time reaches maturity (t = T),} \\ &\text{we liquidate holdings and calculate net profit/loss as} \\ &M_T = M_{T-\Delta t}e^{r\Delta t} + (\alpha_{T-\Delta t})S_T - |\alpha_{T-\Delta t}|\phi_{equity} + (\beta_{T-\Delta t})f^{call}(T, S_T) \\ &-|\beta_{T-\Delta t}|\phi_{option} - (K - S_T)^+ \end{aligned}$$

3 Results

Using the methodology and basis detailed in the above sections, we now perform the analysis. We consider an asset $S = (S_t)_{t \geq 0}$ that follows the Black-Scholes model with $S_0 = \$100,\ \sigma = 20\%,\ \mu = 10\%$ and constant risk-free rate, r = 2%

We further consider we have just sold an at-the-money put (maturity of $\frac{1}{4}$ year and strike price of $100), on the asset S, and we plan to hedge this position. In building a strategy, we are able to trade in:

- the bank account
- the asset, S
- a call option, on asset S, with a strike of 100 and maturity of $\frac{1}{2}$ year

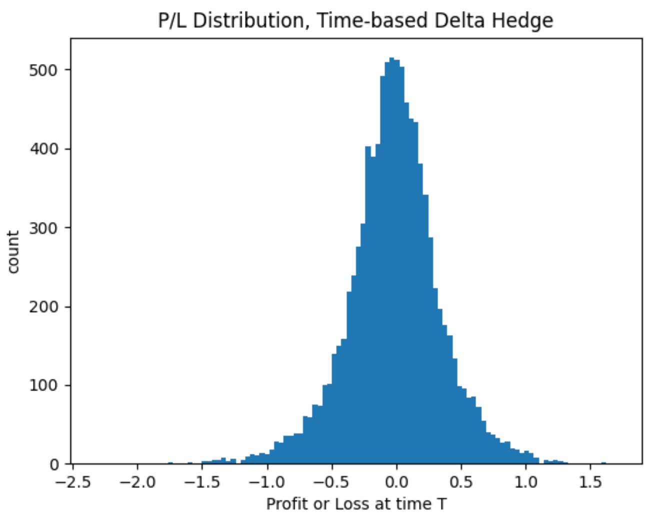
  Note that we will also take transaction costs into account, which are 0.005$ and 0.010$ for each unit of the asset traded and for each unit of options traded, respectively.

First, we will consider the delta hedging strategy for our short put position. We simulated 10,000 scenarios where we followed each of a time-based re-balancing and a move-based rebalancing strategy. Each simulation results in a terminal profit/loss and we plotted the results of the profit and loss distribution in the histograms below.

Apart from the distribution, we are also interested in assessing the expected shortfall of our hedged portfolio. The risk metric we applied is the conditional value at risk (CVaR).

$$CVaR = \frac{1}{1-C} \int_{-1}^{VaR} xp(x)dx$$

where c denotes the cut-off point in the distribution. In our analysis, the VaR break-point is set at 90%. i.e. the first quantile in the left tail. Based on the CVaR, we are able to calculate the adjusted price to charge the client.



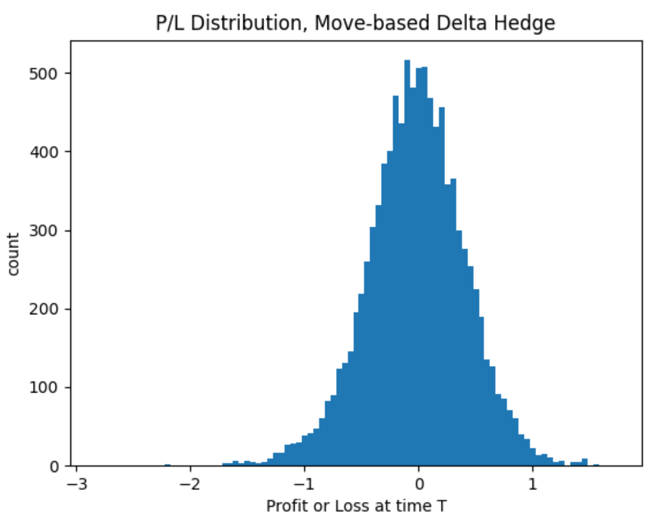
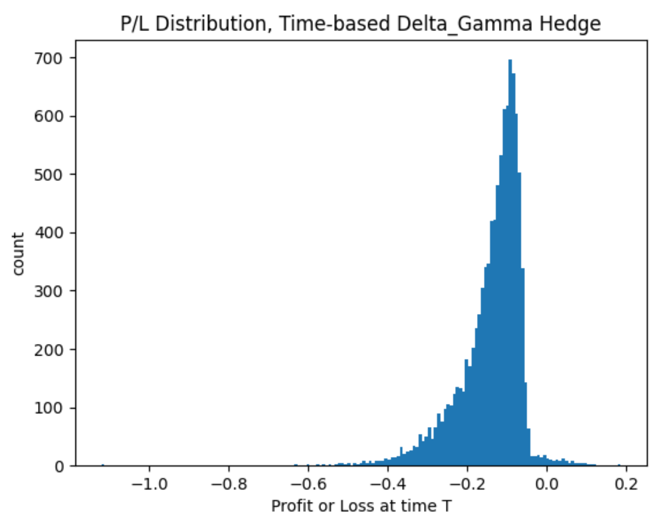


Table 1: Distribution Dynamics for Delta hedge

Type	Mean	Standard Deviation	VaR	CVaR	Adjusted Price
Time based	\$(0.0184)	\$0.3683	\$(0.4548)	\$(0.7037)	\$4.4137
Move based	\$(0.0182)	\$0.4383	\$(0.5539)	\$(0.8093)	\$4.5188

Based on our calculation, both distributions yield a negative mean with a similar magnitude. CVaR deteriorates for move-based delta hedging, leading to a higher adjusted price to charge the client.

Next, we will consider the delta-gamma hedging strategy for our short put position. For each of the time-based and move-based approaches, we simulated 10,000 scenarios and similarly plotted the results of the profit and loss distribution in the histograms below.



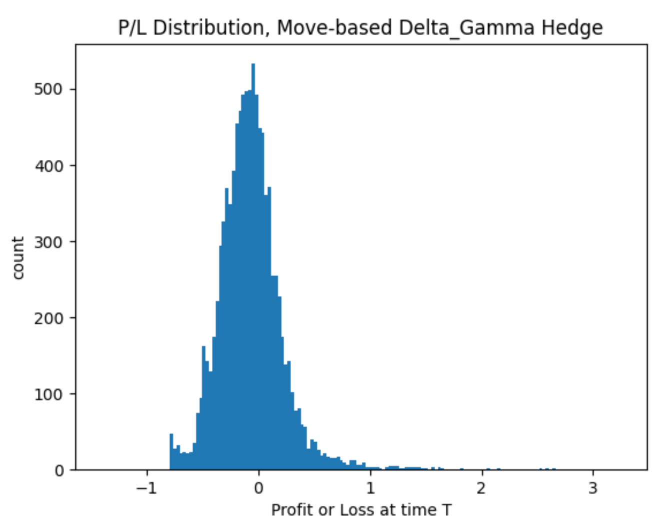


Table 2: Distribution Dynamics for Delta hedge

Type	Mean	Standard Deviation	VaR	CVaR	Adjusted Price
Time based	\$(0.1381)	\$0.0808	\$(0.2399)	\$(0.3131)	\$4.0251
Move based	\$(0.0661)	\$0.2886	\$(0.3806)	\$(0.5093)	\$4.2203

Based on our calculation, both distribution land on a negative mean, where time based delta-gamma hedging shows narrower variance and better CVaR. Similar to that of the Delta hedges, moves based Delta-Gamma result in a higher adjusted price to charge the client.

Here we can clearly see that the move-based delta-gamma hedge is a much more attractive strategy. The time-based approach resulted in an average return of a loss with a small probability of profit. However, the move based followed a desirable distribution. With a mean of around 0, we see that the left and right tails have different characteristics. While the losses do not exceed \$1, we see that the tail distribution of the profit extends in some cases as far as \$2.5.

4 Discussion

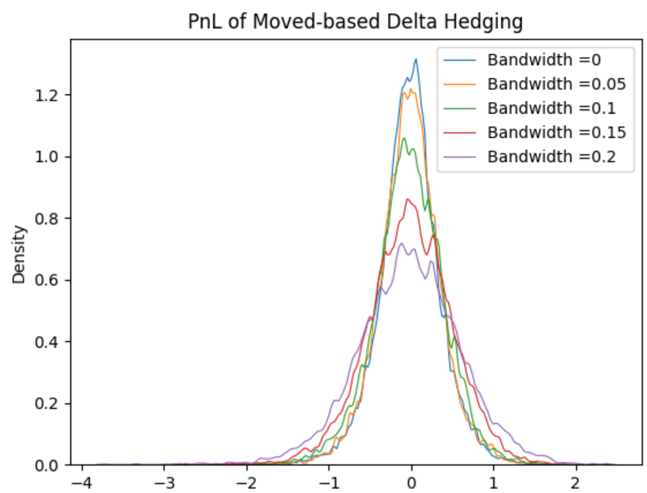
We have determined move-based hedging to be a more effective hedging strategy than the time-based approach. Especially as we consider transaction costs, the key question for our analysis is: When do we need to rebalance?

We know that with both delta and delta-gamma hedging, it may be unnecessary to rebalance at a given time in the investment horizon. This is because, in some cases, the expected effect of risk mitigation by hedging does not outweigh the effect of the transactional cost to rebalance the hedging position. As such, in the plots below, we look closer at the effect of adjusting the bandwidth for both of our move-based strategies.

As we see (and would expect), the profit-loss distribution changes as we adjust the bandwidth of our move-based strategy.

Mean	Standard Deviation	VaR	CVaR	Adjusted Price	Bandwidth
-0.0204	0.3675	-0.4548	-0.689	4.399	0.00
-0.0154	0.3873	-0.4861	-0.7359	4.4458	0.05
-0.0177	0.4344	-0.5556	-0.8098	4.5192	0.10
-0.0106	0.5119	-0.6420	-0.9227	4.6316	0.15
-0.0251	0.6004	-0.7691	-1.1125	4.8205	0.2

Table 3: Statistics of Delta Hedging with Different Bandwidths

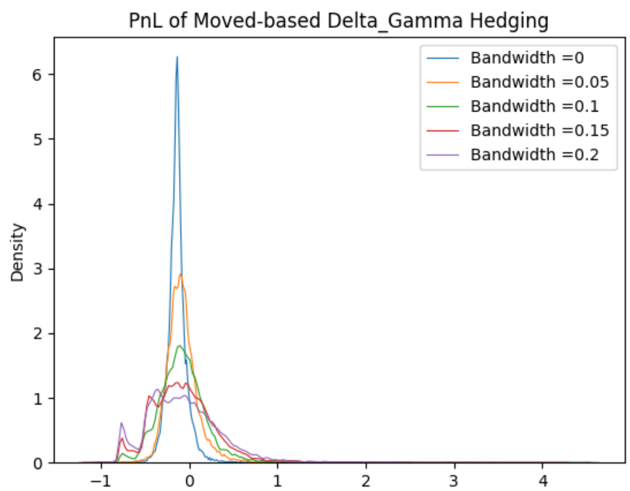


As we delta hedge the put option, we can see that changing the bandwidth has an effect on the dispersion of the distribution. Although the expected value of the portfolio is centred just below zero (slight loss) for each of the scenarios, by increasing the bandwidth we increase exposure to tail events. Also, since the portfolio is being rebalanced less, transaction costs have a smaller impact and result in deeper exposure to the right tail (profit) of the distribution. Note that the bandwidth = 0 mirrors the time-based delta hedging strategy where rebalancing would occur daily.

In 3, we observe that as the bandwidth increases, there is an increase in standard deviation, which leads to higher VaR(Value at Risk) and, thus, a higher CVaR(Conditional Value at Risk). To adjust the CVaR in alignment with the client's threshold of -0.02, we charge a higher price to cover this spread.

Mean	Standard Deviation	VaR	CVaR	Adjusted Price	Bandwidth
-0.1347	0.1158	-0.2567	-0.324	4.0359	0.00
-0.0766	0.1880	-0.2730	-0.3455	4.0573	0.05
-0.0658	0.3122	-0.3875	-0.517	4.2279	0.10
-0.0608	0.3862	-0.4683	-0.612	4.3225	0.15
-0.0596	0.444	-0.5122	-0.689	4.399	0.2

Table 4: Statistics of Delta-Gamma Hedging with Different Bandwidths



As we delta-gamma hedge the put option, we can see that changing the bandwidth has an effect on the dispersion of the distribution. Although the expected value of the portfolio is centred around just below zero (slight loss) for each of the scenarios, by increasing the bandwidth we increase exposure to tail events. Also, since the portfolio is being rebalanced less, transaction costs have a smaller impact and result in deeper exposure to the right tail (profit) of the distribution. Note that the bandwidth = 0 mirrors the time-based delta-gamma hedging strategy where rebalancing would occur daily.

Comparing Tables 3 and 4, there are two key observations. First, notice that as the bandwidth increases, we observe the same trend in all statistics, i.e. larger standard deviation, larger CVaR and higher CVaR adjusted price. On the other hand, there is a significant increase in the average profit and loss, combined with a lower standard deviation. Naturally, this leads to a lower CVaR adjusted price because CVaR corresponding to all bandwidth levels becomes smaller. For a bandwidth of 0.1, we see a \$0.2917 decrease in price in order to achieve the CVaR threshold of -0.02, and this spread widens to \$0.4215 for a bandwidth of 0.02. In other words, we found that delta-gamma hedging earns a higher profit on average through a better balance between the benefits from rebalancing and the costs of it.