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1 Coloring

Vertex Coloring

Theorem 1.1 (Brook's Theorem). *In a connected graph in which every vertex has at most Δ neighbors, the vertices can be colored with only Δ colors, except for two cases, complete graphs and cycle graphs of odd length, which require $\Delta + 1$ colors.*

1.0.1 Chromatic Polynomial

$$P_G(k) = P_{G_1}(k) + P_{G_2}(k)$$

The first coefficient is always 1.
The degree of the first term is the $(|V|)$.
The second coefficient is always $-(|E|)$.
The final (constant) coefficient is always 0.

Definition 1.2. The chromatic polynomial of a complete graph K_n on n vertices is

$$P_{K_n} = k(k-1)(k-2)...(k-(n-1))$$

◇

Definition 1.3. The chromatic polynomial of a tree T_n on n vertices is

$$P_{T_n} = k(k-1)^{n-1}$$

◇

Edge Coloring

Definition 1.4. The chromatic index of a graph, χ', is ... ◇

number of edges in $L(G) = \sum_{i=1}^{n} \binom{d_i}{2}$

Theorem 1.5. $\chi'(G) = \chi(L(G))$

Theorem 1.6 (Vizing's Theorem). *The chromatic index of simple undirected graph is either Δ or $\Delta + 1$.*

Theorem 1.7 (König's Line Coloring Theorem).

Continued line graph derivations of connected graphs

1. Paths
2. Cycles
3. $K_{1,3}$
4. All others grow

Theorem 1.8 (Whitney's Theorem)**.** *Two connected graphs on at least 4 vertices are isomorphic if and only if their line graphs are isomorphic.*

Proposition 1.9. *If H is a line graph, then*

$$\omega(H) \leq \chi(H) \leq \omega(H) + 1$$

Proposition 1.10. *If G is a graph, then*

$$\omega(L(G)) \leq \chi'(H) \leq \omega(L(G)) + 1$$

Planar Duality

Definition 1.11 (Dual Graph)**.** The dual graph of a plane graph G is a graph that has a vertex for each face of G. The dual graph has an edge whenever two faces of G are separated from each other by an edge. ◇

Theorem 1.12. *A graph has the same number of edges as its dual.*

Theorem 1.13. *A graph with n vertices and f faces has a dual with f vertices and n faces.*

Proposition 1.14. *Wheels are self-dual.*

Proposition 1.15. *Duality is an involution.*

2 Matching

Definition 2.1 (Matching)**.** A matching in a graph is a set of pairwise non-adjacent edges. ◇

Definition 2.2 (Matching Number)**.** The matching number of a graph G, denoted $\nu(G)$, is the size of a largest matching. ◇

Proposition 2.3. $\nu(G) \leq \frac{n}{2}$

Definition 2.4 (Perfect Matching)**.** A perfect matching is a matching of size $\frac{n}{2}$. ◇

$$\nu(K_n) = \left\{ \begin{array}{cc} \frac{n}{2}, & \text{n is even} \\ \frac{n-1}{2}, & \text{n is odd} \end{array} \right\}$$

$$(m \leq n)\nu(K_{m,n}) = m$$

Questions

- What can you say about nu(G) if G is Hamiltonian? n/2 or (n-1)/2
- What can you say about nu(G) if G is cubic and Hamiltonian? n/2
- What can you say about Chi(G) if G has no triangles? Nothing (Mycielski's theorem)
- Does every regular graph have a perfect matching?
- Will the properties: regular, even num of vertices, and connected force the existence of perfect matching?

Definition 2.5. An independent set in a graph is a set of pairwise non-adjacent vertices. ◇

Definition 2.6. An independence number of a graph, G, is the size of the largest independent set. This is denoted by $\alpha(G)$. ◇