Contents

1 Definitions

Basic Properties

Definition 1.1 (Degree)**.** The number of edges incident on a vertex v denoted $d(v)$. The minimum degree of a graph G is denoted $\delta(G)$ and the maximum degree is denoted $\Delta(G)$. $\diamond$

Definition 1.2 (Cubic)**.** A graph is cubic if it is 3-regular. $\diamond$

Definition 1.3 (Complement)**.** The complement of a graph G, denoted G', is a graph with the vertex set of G such that if vertices $\{v, w\} \in G'$ are adjacent if and only if they are not adjacent in G. $\diamond$

Definition 1.4 (Self-complementary)**.** A graph is self-complementary if it is isomorphic to its complement. $\diamond$

Definition 1.5 (Distance)**.** Length of the shortest path between two vertices. $\diamond$

Definition 1.6 (Diameter)**.** Maximum distance of a graph. $\diamond$

Definition 1.7 (Bridge)**.** An edge whose deletion increases the number of components. $\diamond$

Definition 1.8 (Wiener Index)**.** The Wiener Index of a connected graph is the sum of the distance between every pair of vertices. $\diamond$

Basic Subgraphs

Definition 1.9 (Walk)**.** A non-empty alternating sequence of vertices and edges in a graph. $\diamond$

Definition 1.10 (Trail)**.** A walk with no repeated edges. $\diamond$

Definition 1.11 (Path)**.** A walk with no repeated vertices. $\diamond$

Definition 1.12 (Component)**.** A subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in its supergraph. $\diamond$

Types of Graphs

Definition 1.13 (Bipartite Graph)**.** A graph whose vertices can be divided into two independent sets U and V such that every edge connects a vertex in U to a vertex in V. A bipartite graph is a graph without any odd length cycles. A bipartite graph is a graph with a chromatic number of 2. ◇

Definition 1.14 (Forest)**.** An acyclic graph. ◇

Definition 1.15 (Tree)**.** A connected acyclic graph. ◇

Definition 1.16 (Spanning Tree)**.** A spanning tree of a graph G is a subgraph of G with the same vertex set that is a tree. The number of spanning trees in a graph G is denoted $\tau(G)$. ◇

Definition 1.17 (Eulerian Graph)**.** A graph containing a cycle where every edge is visited once. A graph with vertices of only even degree. ◇

Definition 1.18 (Hamiltonian Graph)**.** A graph containing a cycle where every vertex is visited once. ◇

Advanced Properties

Definition 1.19 (Chromatic Number)**.** The minimum number of colors needed to color the vertices of a graph G denoted $\chi(G)$. ◇

Definition 1.20 (Chromatic Number)**.** The minimum number of colors needed to color the edges of a graph G denoted $\chi'(G)$. ◇

Definition 1.21 (Planar Graph)**.** A graph that can be drawn on the plane without crossings. ◇

Definition 1.22 (Clique)**.** A set of pairwise adjacent vertices. (A complete subgraph) ◇

Definition 1.23 (Clique Number)**.** The size of the largest clique in graph G, denoted $\omega(G)$. ◇

Definition 1.24 (Matching / Independent Edge Set)**.** A set of pairwise non-adjacent edges. ◇

Definition 1.25 (Matching Number)**.** The size of the largest matching. ◇

Definition 1.26 (Independent Vertex Set)**.** A set of pairwise non-adjacent vertices. ◇

Definition 1.27 (Independence Number)**.** The size of the largest independent vertex set denoted by α. ◇

Definition 1.28 (Girth)**.** Length of the shortest cycle in a graph. In an acyclic graph, the girth is ∞. ◇

Definition 1.29 (Neighborhood)**.** In a graph G, the neighborhood of an element W of G, denoted $N_G(W)$ is the set of all vertices in G that are adjacent to some element of W. ◇

Advanced Subgraphs

Definition 1.30 (Line Graph)**.** The line graph of a graph G, denoted $L(G)$ is the graph with the vertex set $E(G)$ and vertices adjacent if their corresponding edges in G are adjacent. ◇

Definition 1.31 (Minor)**.** The minor of a graph is a graph formed by deleting edges, deleting vertices, and contracting edges. ◇

2 Theorems and Formulas

Basic

Theorem 2.1 (Degree Sum Formula). *For a graph G with n vertices and m edges,*

$$\sum_{i=1}^{n} d_i = 2m$$

.

Theorem 2.2 (Hand Shaking Lemma). *Every finite undirected graph has an even number of vertices with odd degree.*

Theorem 2.3 (Konig's Characterization Theorem). *A graph is bipartite if and only if it has no odd cycles.*

Theorem 2.4 (Cayley's Formula). *There exist n^{n-2} labelled trees on n vertices.*
$\tau = n^{n-2} \forall$ *complete graphs on n vertices where $n \geq 2$*

Hamiltonicity

Theorem 2.5 (Ore's Theorem). *A graph on $n \geq 3$ vertices is Hamiltonian if, for every pair of non-adjacent vertices, the sum of their degrees is n or greater.*

Theorem 2.6 (Dirac's Theorem). *A simple graph on $n \geq 3$ vertices is Hamiltonian if every vertex has degree $\frac{n}{2}$ or greater.*

Coloring

Theorem 2.7 (Brooks' Theorem). *A graph with maximum degree Δ can be colored with Δ colors, except for two cases, complete graphs and odd cycles which require $\Delta + 1$ colors.*

Theorem 2.8 (Vizing's Theorem). *For a simple graph G,*

$$\Delta \leq \chi'(G) \leq \Delta + 1$$

Theorem 2.9 (Konig's Theorem). *For a bipartite graph G,*

$$\chi'(G) = \Delta(G)$$

Theorem 2.10 (Mycielski's Theorem). *There exist triangle-free graphs with arbitrarily high χ.*

Theorem 2.11 (Whitney's Theorem). *Two connected graphs are isomorphic if their line graphs are isomorphic.*

Theorem 2.12 (Stanley's Theorem). *The number of acyclic orientations of a graph is the value of its chromatic polynomial $P(k)$ where $k = -1$.*

Theorem 2.13 (Four-Color Theorem). *Any planar graph has a chromatic number less than or equal to 4.*

Planarity

Theorem 2.14 (Kuratowski's Theorem)**.** *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ are minors of G.*

Theorem 2.15 (Euler's Formula)**.** *For a planar graph G with n vertices, m edges, and f faces,*

$$n - m + f = 2$$

Theorem 2.16. *If G is a simple, planar graph, then $m \leq 3n - 6$.*

Theorem 2.17. *If G is a bipartite, simple, planar graph, then $m \leq 2n - 4$.*

Matching

Theorem 2.18 (Hall's Theorem)**.** *Let $G = (A \cup B, E)$ be a bipartite graph. G has a matching covering A if and only if*

$$|N_G(X)| \geq |X| \forall X \subseteq A$$

Theorem 2.19 (Tutte's Theorem)**.** *A graph G has a perfect matching if and only if $\forall X \subseteq V(G)$, and the number of odd components of $(G - X) \leq |X|$.*

Miscellaneous

Theorem 2.20 (Menger's Theorem)**.** *A graph G with at least $k + 1$ vertices is k-connected if and only if any two vertices of G are joined by at least k paths, no two of which have any other vertices in common. (min edge cut = max edge-independent paths)*

Aka: Let G be a graph and let u, v be two non-adjacent vertices. The maximum number of pairwise internally disjoint uv-paths is equal to the minimum number of vertices to be deleted so that there does not exist any uv-path remaining.

Theorem 2.21 (Robbins' Theorem)**.** *A connected graph is orientable if and only if it contains no bridges.*

3 Procedures

Prufer Codes

Yo where dat smallest leaf at? Oh there it is, bet. Whats it connected to? Oh that one, lemme write that down. Aite now lemme delete that smallest leaf. (repeat until 2 vertices left)

Kruskal's Algorithm

Append each subsequent lowest weighted edge to the spanning tree as long as it doesn't create a cycle. When you hit all vertices, you have constructed the minimum spanning tree.

Prim's Algorithm

Start at a vertex. Construct a minimum spanning tree by appending the minimum weighted edge adjacent to your current minimum spanning tree until all vertices are reached.

Havel-Hakimi Algorithm

Sort in increasing order, remove maximum degree x, decrement x previous degrees by 1, repeat.

Kirchhoff's Matrix-Tree Theorem

For a loopless, connected graph G on n vertices, $V(G) = \{v_1, v_2, ...v_n\}$, $M = (m_{ij})$ an $n \times n$ matrix such that $m_{ii} = d(v_i)$, and $m_{ij} = -($number of edges connecting v_i and $v_j)$, then $\tau(G) =$ the cofactor of any element of M.

Chromatic Polynomial

$$P_G(k) = P_{G-e}(k) - P_{\frac{G}{e}(k)}$$

For trees:

$$P_G(k) = k(k-1)^{n-1}$$