# Math 381 Graph Theory: Final Review

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# Contents

# 1 Definitions

## Basic Properties

**Definition 1.1** (Degree)**.** The number of edges incident on a vertex $v$ denoted $d(v)$. The minimum degree of a graph G is denoted $\delta(G)$ and the maximum degree is denoted $\Delta(G)$. $\diamond$

**Definition 1.2** (Cubic)**.** A graph is cubic if it is 3-regular. $\diamond$

**Definition 1.3** (Complement)**.** The complement of a graph $G$, denoted $G'$, is a graph with the vertex set of G such that if vertices $\{v, w\} \in G'$ are adjacent if and only if they are not adjacent in G. $\diamond$

**Definition 1.4** (Self-complementary)**.** A graph is self-complementary if it is isomorphic to its complement. $\diamond$

**Definition 1.5** (Distance)**.** Length of a shortest path between two vertices. $\diamond$

**Definition 1.6** (Diameter)**.** Maximum distance of a graph. $\diamond$

**Definition 1.7** (Bridge)**.** An edge whose deletion increases the number of components. $\diamond$

**Definition 1.8** (Wiener Index)**.** The Wiener Index of a connected graph is the sum of the distance between every pair of vertices. $\diamond$

## Basic Subgraphs

**Definition 1.9** (Walk)**.** A non-empty alternating sequence of vertices and edges in a graph. $\diamond$

**Definition 1.10** (Trail)**.** A walk with no repeated edges. $\diamond$

**Definition 1.11** (Path)**.** A walk with no repeated vertices. $\diamond$

**Definition 1.12** (Component)**.** A subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in its supergraph. $\diamond$

## Types of Graphs

**Definition 1.13** (Bipartite Graph)**.** A graph whose vertices can be divided into two independent sets $U$ and $V$ such that every edge connects a vertex in $U$ to a vertex in $V$. A bipartite graph is a graph without any odd length cycles. A bipartite graph is a graph with a chromatic number of 2. ◇

**Definition 1.14** (Forest)**.** An acyclic graph. ◇

**Definition 1.15** (Tree)**.** A connected acyclic graph. ◇

**Definition 1.16** (Spanning Tree)**.** A spanning tree of a graph G is a subgraph of G with the same vertex set that is a tree. The number of spanning trees in a graph G is denoted $\tau(G)$. ◇

**Definition 1.17** (Eulerian Graph)**.** A graph containing a cycle where every edge is visited once. A graph with vertices of only even degree. ◇

**Definition 1.18** (Hamiltonian Graph)**.** A graph containing a cycle where every vertex is visited once. ◇

## Advanced Properties

**Definition 1.19** (Chromatic Number)**.** The minimum number of colors needed to color the vertices of a graph G denoted $\chi(G)$. ◇

**Definition 1.20** (Chromatic Number)**.** The minimum number of colors needed to color the edges of a graph G denoted $\chi'(G)$. ◇

**Definition 1.21** (Planar Graph)**.** A graph that can be drawn on the plane without crossings. ◇

**Definition 1.22** (Clique)**.** A set of pairwise adjacent vertices. (A complete subgraph) ◇

**Definition 1.23** (Clique Number)**.** The size of the largest clique in graph $G$, denoted $\omega(G)$. ◇

**Definition 1.24** (Matching / Independent Edge Set)**.** A set of pairwise non-adjacent edges. ◇

**Definition 1.25** (Matching Number)**.** The size of the largest matching. ◇

**Definition 1.26** (Independent Vertex Set)**.** A set of pairwise non-adjacent vertices. ◇

**Definition 1.27** (Independence Number)**.** The size of the largest independent vertex set denoted by $\alpha$. ◇

**Definition 1.28** (Girth)**.** Length of the shortest cycle in a graph. In an acyclic graph, the girth is $\infty$. $\diamond$

**Definition 1.29** (Neighborhood)**.** In a graph $G$, the neighborhood of an element $W$ of $G$, denoted $N_G(W)$ is the set of all vertices in $G$ that are adjacent to some element of $W$. $\diamond$

## Advanced Subgraphs

**Definition 1.30** (Line Graph)**.** The line graph of a graph $G$, denoted $L(G)$ is the graph with the vertex set $E(G)$ and vertices adjacent if their corresponding edges in $G$ are adjacent. $\diamond$

**Definition 1.31** (Minor)**.** The minor of a graph is a graph formed by deleting edges, deleting vertices, and contracting edges. $\diamond$

# 2 Theorems and Formulas

## Basic

**Theorem 2.1** (Degree Sum Formula). *For a graph $G$ with $n$ vertices and $m$ edges,*

$$\sum_{i=1}^{n} d_i = 2m$$

.

**Theorem 2.2** (Hand Shaking Lemma). *Every finite undirected graph has an even number of vertices with odd degree.*

**Theorem 2.3** (Konig's Characterization Theorem). *A graph is bipartite if and only if it has no odd cycles.*

**Theorem 2.4** (Cayley's Formula). *There exist $n^{n-2}$ labelled trees on $n$ vertices.*
$\tau = n^{n-2} \forall$ *complete graphs on $n$ vertices where $n \geq 2$*

## Hamiltonicity

**Theorem 2.5** (Ore's Theorem). *A graph on $n \geq 3$ vertices is Hamiltonian if, for every pair of non-adjacent vertices, the sum of their degrees is $n$ or greater.*

**Theorem 2.6** (Dirac's Theorem). *A simple graph on $n \geq 3$ vertices is Hamiltonian if every vertex has degree $\frac{n}{2}$ or greater.*

## Coloring

**Theorem 2.7** (Brooks' Theorem). *A graph with maximum degree $\Delta$ can be colored with $\Delta$ colors, except for two cases, complete graphs and odd cycles which require $\Delta + 1$ colors.*

**Theorem 2.8** (Vizing's Theorem). *For a simple graph $G$,*

$$\Delta \leq \chi'(G) \leq \Delta + 1$$

**Theorem 2.9** (Konig's Theorem). *For a bipartite graph $G$,*

$$\chi'(G) = \Delta(G)$$

**Theorem 2.10** (Mycielski's Theorem). *There exist triangle-free graphs with arbitrarily high $\chi$.*

**Theorem 2.11** (Whitney's Theorem). *Two connected graphs are isomorphic if their line graphs are isomorphic.*

**Theorem 2.12** (Stanley's Theorem). *The number of acyclic orientations of a graph is the value of its chromatic polynomial $P(k)$ where $k = -1$.*

**Theorem 2.13** (Four-Color Theorem). *Any planar graph has a chromatic number less than or equal to 4.*

## Planarity

**Theorem 2.14** (Kuratowski's Theorem). *A graph $G$ is planar if and only if neither $K_5$ nor $K_{3,3}$ are minors of $G$.*

**Theorem 2.15** (Euler's Formula). *For a planar graph $G$ with $n$ vertices, $m$ edges, and $f$ faces,*

$$n - m + f = 2$$

**Theorem 2.16.** *If $G$ is a simple, planar graph, then $m \leq 3n - 6$.*

**Theorem 2.17.** *If $G$ is a bipartite, simple, planar graph, then $m \leq 2n - 4$.*

## Matching

**Theorem 2.18** (Hall's Theorem). *Let $G = (A \cup B, E)$ be a bipartite graph. $G$ has a matching covering $A$ if and only if*

$$|N_G(X)| \geq |X| \forall X \subseteq A$$

**Theorem 2.19** (Tutte's Theorem). *A graph $G$ has a perfect matching if and only if $\forall X \subseteq V(G)$, and the number of odd components of $(G - X) \leq |X|$.*

## Miscellaneous

**Theorem 2.20** (Menger's Theorem). *A graph $G$ with at least $k + 1$ vertices is $k$-connected if and only if any two vertices of $G$ are joined by at least $k$ paths, no two of which have any other vertices in common. (min edge cut = max edge-independent paths)*

*Aka: Let $G$ be a graph and let $u, v$ be two non-adjacent vertices. The maximum number of pairwise internally disjoint $uv$-paths is equal to the minimum number of vertices to be deleted so that there does not exist any $uv$-path remaining.*

**Theorem 2.21** (Robbins' Theorem). *A connected graph is orientable if and only if it contains no bridges.*

# 3 Procedures

**Prufer Codes**

Yo where dat smallest leaf at? Oh there it is, bet. Whats it connected to? Oh that one, lemme write that down. Aite now lemme delete that smallest leaf. (repeat until 2 vertices left)

**Kruskal's Algorithm**

Append each subsequent lowest weighted edge to the spanning tree as long as it doesn't create a cycle. When you hit all vertices, you have constructed the minimum spanning tree.

**Prim's Algorithm**

Start at a vertex. Construct a minimum spanning tree by appending the minimum weighted edge adjacent to your current minimum spanning tree until all vertices are reached.

**Havel-Hakimi Algorithm**

Sort in increasing order, remove maximum degree x, decrement x previous degrees by 1, repeat.

**Kirchhoff's Matrix-Tree Theorem**

For a loopless, connected graph $G$ on $n$ vertices, $V(G) = \{v_1, v_2, ...v_n\}$, $M = (m_{ij})$ an $n \times n$ matrix such that $m_{ii} = d(v_i)$, and $m_{ij} = -$(number of edges connecting $v_i$ and $v_j$), then $\tau(G) =$ the cofactor of any element of M.

**Chromatic Polynomial**

$$P_G(k) = P_{G-e}(k) - P_{\frac{G}{e}(k)}$$

For trees:

$$P_G(k) = k(k-1)^{n-1}$$