MATH 327: Problem Set #3

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Problem 21

Five men and five women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X denote the highest ranking achieved by a woman (for instance, $X = 2$ if the top-ranked person was male and the next-ranked person was female). Compute the expected value of the random variable X.
Solution

$$
\begin{aligned}
P\{1\} &= \frac{5}{10} && = .5 \\
P\{2\} &= \frac{5}{10}\frac{5}{9} && = .28 \\
P\{3\} &= \frac{5}{10}\frac{4}{9}\frac{5}{8} && = .14 \\
P\{4\} &= \frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{5}{7} && = .06 \\
P\{5\} &= \frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{5}{6} && = .02 \\
P\{6\} &= \frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{1}{6}\frac{5}{5} && = .00 \\
P\{7\} &= \frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{1}{6}\frac{0}{5} && = 0 \\
P\{8\} & && = 0 \\
P\{9\} & && = 0 \\
P\{10\} & && = 0
\end{aligned}
$$

Problem 22

Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times.
If the coin is assumed fair, for n = 3, what are the probabilities associated with the values that X can take on?
Compute the exected value of the random variable X.
Solution:

$$X = -n + 2(k), \forall k \in \mathbb{N} \leq n$$

$$X = -3 + 2(k), \forall k \in \mathbb{N} \leq 3$$

$$k = 0, X = -3 + 2(0) = -3$$

$$k = 1, X = -3 + 2(1) = -1$$

$$k = 2, X = -3 + 2(2) = 1$$

$$k = 3, X = -3 + 2(3) = 3$$

There are $2^3 = 8$ ways to toss 3 coins, therefore the probabilites associated with each value of X are

$$\frac{3}{8}, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}$$

Problem 23

Each night different meteorologists give us the probability that it will rain the next day. To judge how well these people predict, we will score each of them as follows: If a meteorologist says that it will rain with probability p, then he or she will receive a score of

$$\text{score} = \begin{cases} 1-(1-p)^2 & \text{if it does rain} \\ 1-p^2 & \text{if it does not rain} \end{cases}$$

We will then keep track of scores over a certain time span and conclude that the meteorologist with the highest average score is the best predictor of weather. Suppose now that a given meteorologist is aware of this and so wants to maximize his or her expected score. If this individual truly believes that it will rain tomorrow with probability p, what value of p should he or she assert so as to maximize the expected score?

Problem 25

A total of 4 buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus

(a) Which of $E[X]$ or $E[Y]$ do you think is larger? Why?

(b) Compute $E[X]$ and $E[Y]$.

Problem 27

The density function of X is given by

$$\text{f(x)} = \begin{cases} a+bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = \frac{3}{5}$, find a and b.

Problem 30

Suppose that X has density function

$$\text{f(x)} = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute $E[X^n]$

(a) by computing the density of X^n and then using the definition of expectation

(b) by using Proposition 4.5.1

  (a) If X is a discrete random variable with probability mass function p(x), then for any real-valued function g,

$$E[g(X)] = \sum_x g(x)p(x)$$

(b) If X is a continuous random variable wit probability density function f(x), then for any real-valued function g,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Problem 32

If $E[X] = 2$ and $E[X^2] = 8$, calculate

(a) $E[2 + 4X)^2$

(b) $E[X^2 + (X + 1)^2$

Problem 34

If X is a continuous random variable having distribution function F, then its median is defined as that value for m for which

$$F(m) = \frac{1}{2}$$

Find the median of the random variables with density function

(a) $f(x) = e^{-x}, x \geq 0$

(b) $f(x) = 1, 0 \leq x \leq 1$

Problem 36

We say that m_p is the $100p$ percentile of the distribution function F if

$$F(m_p) = p$$

Find m_p for the distribution having density function

$$f(x) = 2e^{-2x}, x \geq 0$$

Problem 38

Compute the expectation and variance of the number of successes in n independent trials, each of which results in a success with probability p. Is independence necessary?

Problem 44

Let X_i denote the percentage of votes cast in a given election that are for candidate i, and suppose that X_1 and X_2 have a joint density function

$$f_{X_1,X_2}(x,y) = \begin{cases} 3(x+y), & \text{if } x \geq 0, y \geq 0, 0 \leq x+y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the marginal densities of X_1 and X_2.

(b) Find $E[X_i]$ and $\text{Var}(X_i)$ for $i = 1, 2$.

Problem 45

A product is classified according to the number of defects it contains and the factory that produces it. Let X_1 and X_2 be the random variables that represent the number of defects per unit (taking on possible values of 0, 1, 2, or 3) and the factory number (taking on possible values 1 or 2), respectively. The entries in the table represent the joint possibility mass function of a randomly chosen product.

X_1 \ X_2	1	2
0	$\frac{1}{8}$	$\frac{1}{16}$
1	$\frac{1}{16}$	$\frac{1}{16}$
2	$\frac{3}{16}$	$\frac{1}{8}$
3	$\frac{1}{8}$	$\frac{1}{4}$

(a) Find the marginal probability distributions of X_1 and X_2.

(b) Find $E[X_1]$, $E[X_2]$, $\mathrm{Var}(X_1)$, $\mathrm{Var}(X_2)$, and $\mathrm{Cov}(X_1, X_2)$.

Problem 46

Find $\mathrm{Corr}(X_1, X_2)$ for the random variables of Problem 44.

Problem 52

If X_1 and X_2 have the same probability distribution function, show that

$$\mathrm{Cov}(X_1 - X_2, X_1 + X_2) = 0$$

Note that independence is not being assumed.

Problem 53

Suppose that X has density function

$$f(x) = e^{-x}, x > 0$$

Compute the moment generating function of X and use your result to determine its mean and variance. Check your answer for the mean by a direct calculation.

Problem 57

Let X and Y have respective distribution functions F_X and F_Y , and suppose that for some constants a and $b > 0$,

$$F_X(x) = F_Y(\frac{x-a}{b})$$

(a) Determine $E[X]$ in terms of $E[Y]$.

(b) Determine Var(X) in terms of Var(Y).

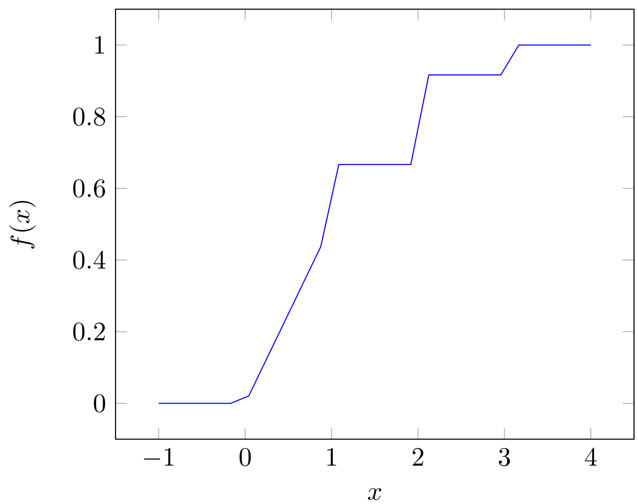
Hint: X has the same distribution as what other random variable?

Problem 58

The distribution function of the random variable X is given

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \leq x < 1 \\ \frac{2}{3} & 1 \leq x < 2 \\ \frac{11}{12} & 2 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

(a) Plot this distribution function.



(b) What is $P\{X > \frac{1}{2}\}$?

$$P\{X > \frac{1}{2}\} = 1 - F(\frac{1}{2}) = 1 - \frac{\frac{1}{2}}{2} = \frac{3}{4}$$

(c) What is $P\{2 < X \leq 4\}$?

$$P\{2 < X \leq 4\} = F(4) - F(2)) = 1 - \frac{11}{12} = \frac{1}{12}$$

(d) What is $P\{X < 3\}$?

$$P\{X < 3\} = \lim_{t \to 0} F(3 - t) = \frac{11}{12}$$

(e) What is $P\{X = 1\}$?

$$P\{X = 1\} = F(1) - \lim_{t \to 0} F(1 - t) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

Problem 59

Suppose the random variable X has probability density function

$$f(x) = \begin{cases} cx^3, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c.

$$\begin{aligned} \int_0^1 cx^3 dx &= 1 \\ \frac{c}{4}x^4\Big|_0^1 &= 1 \\ \frac{c}{4}(1-0) &= 1 \\ c &= 4 \end{aligned}$$

(b) Find $P\{.4 < X < .8\}$.

$$4\int_{.4}^{.8} x^3 dx = \frac{4}{4}x^4\Big|_{.4}^{.8} = .8^4 - .4^4 = .38$$

Problem 9

A set of five transistors are to be tested, one at a time in a random order, to see which of them are defective. Suppose that three of the five transistors are defective, and let N_1 denote the number of tests made until the first defective is spotted, and let N_2 denote the number of additional tests until the second defective is spotted. Find the joint probability mass function of N_1 and N_2.

Solution: Let a 0 represent a working transistor, and let a 1 represent a defective transistor. Each ordering of 5 transistors with 3 defective and 2 working can be represented as a 5 bit binary number. All possible orderings are as follows.

00111
01011
01101
10011
10101
10110

Let f(x, y) be the probability mass function of N_1 and N_2 such that x is a value of N_1 and y is a value of N_2.

$$f(x,y) = \begin{cases} \frac{2}{5}\frac{1}{4} = \frac{1}{10} & x = 3, y = 1 \\ \frac{2}{5}\frac{3}{4}\frac{1}{3} = \frac{1}{10} & x = 2, y = 2 \\ \frac{2}{5}\frac{3}{4}\frac{2}{3} = \frac{2}{10} & x = 2, y = 1 \\ \frac{3}{5}\frac{2}{4}\frac{1}{3} = \frac{1}{10} & x = 1, y = 3 \\ \frac{3}{5}\frac{2}{4}\frac{2}{3} = \frac{2}{10} & x = 1, y = 2 \\ \frac{3}{5}\frac{2}{4} = \frac{3}{10} & x = 1, y = 1 \\ 0 & otherwise \end{cases}$$

Problem 10

The joint probability density function of X and Y is given by

$$f(x,y) = \frac{6}{7}(x^2 + \frac{xy}{2}), 0 < x < 1, 0 < y < 2$$

(a) Verify that this is indeed a joint density function. The integral of a PDF must be 1.

$$\frac{6}{7}\int_0^2 \int_0^1 (x^2 + \frac{xy}{2})\,dx\,dy$$
$$\frac{6}{7}\int_0^2 (\frac{x^3}{3} + \frac{y}{2}\frac{x^2}{2})\Big|_0^1\,dy$$
$$\frac{6}{7}\int_0^2 (\frac{1^3}{3} + \frac{y}{2}\frac{1^2}{2})\,dy$$
$$\frac{6}{7}\int_0^2 (\frac{1}{3} + \frac{y}{4})\,dy$$
$$\frac{6}{7}(\frac{y}{3} + \frac{y^2}{8})\Big|_0^2$$
$$\frac{6}{7}(\frac{2}{3} + \frac{2^2}{8}) = \frac{6}{7}(\frac{7}{6}) = 1$$

(b) Compute the density function of X.

$$\frac{6}{7}\int_0^2 (x^2 + \frac{xy}{2})\,dy$$
$$\frac{6}{7}(yx^2 + \frac{xy^2}{4})\Big|_0^2$$
$$\frac{12x^2}{7} + \frac{6x}{7}$$
$$\frac{6x(2x+1)}{7}$$

(c) Find $P\{X > Y\}$.

$$\frac{6}{7}\int_0^1 \int_0^x (x^2 + \frac{xy}{2})\,dy\,dx$$
$$\frac{6}{7}\int_0^1 (yx^2 + \frac{xy^2}{4})\Big|_0^x\,dx$$
$$\frac{6}{7}\int_0^1 (x^3 + \frac{x^3}{4})\,dx$$
$$\frac{6}{7}(\frac{x^4}{4} + \frac{x^4}{16})\Big|_0^1$$
$$\frac{6}{7}(\frac{1^4}{4} + \frac{1^4}{16})$$
$$\frac{6}{7}(\frac{5}{16})$$
$$\frac{15}{56}$$

Problem 12

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} xe^{-(x+y)}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the denisty of X.

$$\int_0^\infty f(x,y)\,dy$$
$$\int_0^\infty (xe^{-(x+y)})\,dy$$
$$-xe^{-(x+y)}\Big|_0^\infty$$
$$-xe^{-(x+\infty)} + xe^{-(x+0)}$$
$$xe^{-x}$$

(b) Compute the density of Y.

$$\int_0^\infty f(x,y)\,dx$$
$$\int_0^\infty (xe^{-(x+y)})\,dx$$
$$(x+1)(-e^{-(x+y)})\Big|_0^\infty$$
$$(\infty+1)(-e^{-(\infty+y)}) - (0+1)(-e^{-(0+y)})$$
$$e^{-y}$$

(c) Are X and Y independent?
Yes. (Density of X) $\times$ (density of Y) $=$ joint density of X and Y.

Problem 13

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} 2, & 0 < x < y, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the density of X.

$$\begin{aligned} \int_0^1 f(x,y)\,dy \\ \int_0^1 2\,dy \\ 2y\Big|_0^1 \\ 2(1) - 2(0) \\ 2 \end{aligned}$$

(b) Compute the density of Y.

$$\begin{aligned} \int_0^y f(x,y)\,dx \\ \int_0^y 2\,dx \\ 2x\Big|_0^y \\ 2(y) - 2(0) \\ 2y \end{aligned}$$

(c) Are X and Y independent?
No. (Density of X) $\times$ (density of Y) $\neq$ joint density of X and Y.