MATH 327: Problem Set #3

Due on February 13, 2017 at 1:10pm

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Problem 1

Five men and five women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X denote the highest ranking achieved by a woman (for instance, $X = 2$ if the top-ranked person was male and the next-ranked person was female). Find $P\{X = i\}, i = 1, 2, 3, ..., 8, 9, 10$.

Solution

$$
\begin{aligned}
P\{1\} &= \frac{5}{10} && = .5 \\
P\{2\} &= \frac{5}{10}\frac{5}{9} && = .28 \\
P\{3\} &= \frac{5}{10}\frac{4}{9}\frac{5}{8} && = .14 \\
P\{4\} &= \frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{5}{7} && = .06 \\
P\{5\} &= \frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{5}{6} && = .02 \\
P\{6\} &= \frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{1}{6}\frac{5}{5} && = .00 \\
P\{7\} &= \frac{5}{10}\frac{4}{9}\frac{3}{8}\frac{2}{7}\frac{1}{6}\frac{0}{5} && = 0 \\
P\{8\} & && = 0 \\
P\{9\} & && = 0 \\
P\{10\} & && = 0
\end{aligned}
$$

Problem 2

Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X ?

Solution:
The difference must be between $-n$ and n because you can have 0 heads and n tails, or n heads and 0 tails. For each additional head we toss, we can toss one less tail and vice versa so the difference can only be incremented or decremented by 2. Therefore, we can only have

$$X \in \{-n, -n+2, -n+4, \ldots, n-4, n-2, n\}$$

or

$$X = -n + 2(k), \forall k \in \mathbb{N} \leq n$$

Problem 3

In Problem 2, if the coin is assumed fair, for n = 3, what are the probabilities associated with the values that X can take on?
Solution:

$$X = -n + 2(k), \forall k \in \mathbb{N} \leq n$$

$$X = -3 + 2(k), \forall k \in \mathbb{N} \leq 3$$

$$k = 0, X = -3 + 2(0) = -3$$

$$k = 1, X = -3 + 2(1) = -1$$

$$k = 2, X = -3 + 2(2) = 1$$

$$k = 3, X = -3 + 2(3) = 3$$

There are $2^3 = 8$ ways to toss 3 coins, therefore the probabilites associated with each value of X are

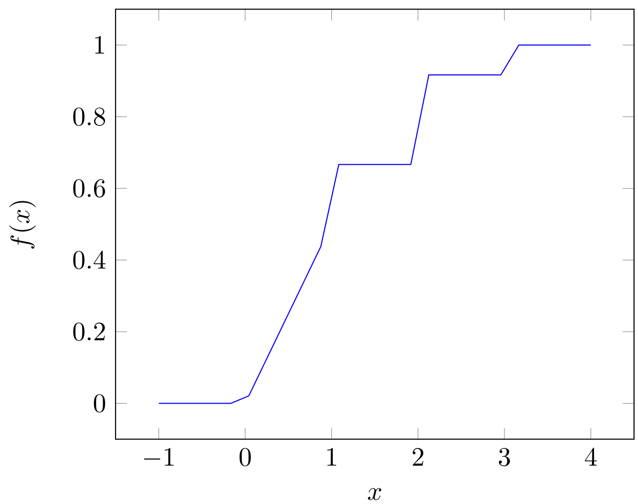
$$\frac{3}{8}, \frac{1}{8}, \frac{1}{8}, \frac{3}{8}$$

Problem 4

The distribution function of the random variable X is given

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \leq x < 1 \\ \frac{2}{3} & 1 \leq x < 2 \\ \frac{11}{12} & 2 \leq x < 3 \\ 1 & 3 \leq x \end{cases}$$

(a) Plot this distribution function.



(b) What is $P\{X > \frac{1}{2}\}$?

$$P\{X > \frac{1}{2}\} = 1 - F(\frac{1}{2}) = 1 - \frac{\frac{1}{2}}{2} = \frac{3}{4}$$

(c) What is $P\{2 < X \leq 4\}$?

$$P\{2 < X \leq 4\} = F(4) - F(2)) = 1 - \frac{11}{12} = \frac{1}{12}$$

(d) What is $P\{X < 3\}$?

$$P\{X < 3\} = \lim_{t \to 0} F(3 - t) = \frac{11}{12}$$

(e) What is $P\{X = 1\}$?

$$P\{X = 1\} = F(1) - \lim_{t \to 0} F(1 - t) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

Problem 5

Suppose the random variable X has probability density function

$$f(x) = \begin{cases} cx^3, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of c.

$$\int_0^1 cx^3 dx = 1$$
$$\frac{c}{4}x^4\Big|_0^1 = 1$$
$$\frac{c}{4}(1-0) = 1$$
$$c = 4$$

(b) Find $P\{.4 < X < .8\}$.

$$4\int_{.4}^{.8} x^3 dx = \frac{4}{4}x^4\Big|_{.4}^{.8} = .8^4 - .4^4 = .38$$

Problem 9

A set of five transistors are to be tested, one at a time in a random order, to see which of them are defective. Suppose that three of the five transistors are defective, and let N_1 denote the number of tests made until the first defective is spotted, and let N_2 denote the number of additional tests until the second defective is spotted. Find the joint probability mass function of N_1 and N_2.

Solution: Let a 0 represent a working transistor, and let a 1 represent a defective transistor. Each ordering of 5 transistors with 3 defective and 2 working can be represented as a 5 bit binary number. All possible orderings are as follows.

00111
01011
01101
10011
10101
10110

Let f(x, y) be the probability mass function of N_1 and N_2 such that x is a value of N_1 and y is a value of N_2.

$$f(x,y) = \begin{cases} \frac{2}{5}\frac{1}{4} = \frac{1}{10} & x = 3, y = 1 \\ \frac{2}{5}\frac{3}{4}\frac{1}{3} = \frac{1}{10} & x = 2, y = 2 \\ \frac{2}{5}\frac{3}{4}\frac{2}{3} = \frac{2}{10} & x = 2, y = 1 \\ \frac{3}{5}\frac{2}{4}\frac{1}{3} = \frac{1}{10} & x = 1, y = 3 \\ \frac{3}{5}\frac{2}{4}\frac{2}{3} = \frac{2}{10} & x = 1, y = 2 \\ \frac{3}{5}\frac{2}{4} = \frac{3}{10} & x = 1, y = 1 \\ 0 & otherwise \end{cases}$$

Problem 10

The joint probability density function of X and Y is given by

$$f(x,y) = \frac{6}{7}(x^2 + \frac{xy}{2}), 0 < x < 1, 0 < y < 2$$

(a) Verify that this is indeed a joint density function. The integral of a PDF must be 1.

$$\frac{6}{7}\int_0^2 \int_0^1 (x^2 + \frac{xy}{2})\, dx\, dy$$
$$\frac{6}{7}\int_0^2 (\frac{x^3}{3} + \frac{y}{2}\frac{x^2}{2})\Big|_0^1\, dy$$
$$\frac{6}{7}\int_0^2 (\frac{1^3}{3} + \frac{y}{2}\frac{1^2}{2})\, dy$$
$$\frac{6}{7}\int_0^2 (\frac{1}{3} + \frac{y}{4})\, dy$$
$$\frac{6}{7}(\frac{y}{3} + \frac{y^2}{8})\Big|_0^2$$
$$\frac{6}{7}(\frac{2}{3} + \frac{2^2}{8}) = \frac{6}{7}(\frac{7}{6}) = 1$$

(b) Compute the density function of X.

$$\frac{6}{7}\int_0^2 (x^2 + \frac{xy}{2})\, dy$$
$$\frac{6}{7}(yx^2 + \frac{xy^2}{4})\Big|_0^2$$
$$\frac{12x^2}{7} + \frac{6x}{7}$$
$$\frac{6x(2x+1)}{7}$$

(c) Find $P\{X > Y\}$.

$$\frac{6}{7}\int_0^1 \int_0^x (x^2 + \frac{xy}{2})\, dy\, dx$$
$$\frac{6}{7}\int_0^1 (yx^2 + \frac{xy^2}{4})\Big|_0^x\, dx$$
$$\frac{6}{7}\int_0^1 (x^3 + \frac{x^3}{4})\, dx$$
$$\frac{6}{7}(\frac{x^4}{4} + \frac{x^4}{16})\Big|_0^1$$
$$\frac{6}{7}(\frac{1^4}{4} + \frac{1^4}{16})$$
$$\frac{6}{7}(\frac{5}{16})$$
$$\frac{15}{56}$$

Problem 12

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} xe^{-(x+y)}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the denisty of X.

$$\begin{aligned} \int_0^\infty f(x,y)\,dy \\ \int_0^\infty (xe^{-(x+y)})\,dy \\ -xe^{-(x+y)}\Big|_0^\infty \\ -xe^{-(x+\infty)} + xe^{-(x+0)} \\ xe^{-x} \end{aligned}$$

(b) Compute the density of Y.

$$\begin{aligned} \int_0^\infty f(x,y)\,dx \\ \int_0^\infty (xe^{-(x+y)})\,dx \\ (x+1)(-e^{-(x+y)})\Big|_0^\infty \\ (\infty+1)(-e^{-(\infty+y)}) - (0+1)(-e^{-(0+y)}) \\ e^{-y} \end{aligned}$$

(c) Are X and Y independent?
Yes. (Density of X) $\times$ (density of Y) = joint density of X and Y.

Problem 13

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} 2, & 0 < x < y, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the density of X.

$$\int_0^1 f(x,y)\,dy$$
$$\int_0^1 2\,dy$$
$$2y\Big|_0^1$$
$$2(1) - 2(0)$$
$$2$$

(b) Compute the density of Y.

$$\int_0^y f(x,y)\,dx$$
$$\int_0^y 2\,dx$$
$$2x\Big|_0^y$$
$$2(y) - 2(0)$$
$$2y$$

(c) Are X and Y independent?
No. (Density of X) $\times$ (density of Y) $\neq$ joint density of X and Y.