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FOURIER SERIES FOR FUNCTIONS WITH COMPLEX VALUES. APPLICATIONS

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Abstract
Fourier series are fundamental in mathematics for representing functions through simpler sine and cosine terms. This paper focuses on utilizing complex Fourier series to approximate complex-valued functions whose graphs are curves (parametrically defined). Therefore, a MATLAB program was developed. A Python script using the OpenCV2 library for precise extraction of curve points' coordinates from a picture was elaborated. As an application, another MATLAB program was created; it is capable of generating words by leveraging a custom contour library for letters. This demonstrates the practical application of Fourier series in digital image processing, pattern recognition, and typography.
Key words: *Fourier series, complex valued functions, numerical approximation, MATLAB, Python.*

1. INTRODUCTION

Fourier series are a fundamental tool in mathematics for analysing and representing periodic functions. By decomposing a complex waveform into a sum of simpler sine and cosine functions, Fourier series allow for a deeper understanding and manipulation of signals. This paper delves into the theoretical framework and practical implications of applying Fourier series to functions with complex values. These applications are crucial in fields such as signal processing, electrical engineering, and applied mathematics, where they provide powerful methods for analysing and reconstructing complex signals and waveforms.

2. CONTENT

The first focus is on the general form of Fourier series. In that scope, let us consider a continuous function, $f:[0,1]\to\mathbb{R}$, with $f(0)=f(1)$, or a continuous function $f:\mathbb{R}\to\mathbb{R}$ that has a period of T=1. The expansion of such a function into a Fourier series is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cos(2\pi nt) + b_n\, sin(2\pi nt))\,,\ \forall\, t\in[0,1] \tag{1}$$

The a_n and b_n coefficients are defined:

$$a_n = 2\int_0^1 f(t)\, cos(2\pi nt)\, dt\,,\ \mathrm{n}\geq 0 \tag{2}$$

$$b_n = 2\int_0^1 f(t)\, \sin(2\pi nt)\, dt\,,\ \mathrm{n}\geq 1 \tag{3}$$

Now, to get to the complex form of the Fourier series, we begin from Euler's formulas in order to rewrite the cosine and the sine: $cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. By replacing them into (1):

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\frac{e^{2\pi int}+e^{-2\pi int}}{2} + b_n\frac{e^{2\pi int}-e^{-2\pi int}}{2i}\right)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n-ib_n}{2}e^{2\pi int}\right) + \sum_{n=1}^{\infty}\left(\frac{a_n+ib_n}{2}e^{-2\pi int}\right)$$

The following notations will be used so that we may reduce the Fourier series to its complex form: $c_0=\frac{a_0}{2}$, $c_n=\frac{a_n-ib_n}{2}$, $c_{-n}=\frac{a_n+ib_n}{2}$, $n\geq 1$:

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{2\pi int} \tag{4}$$

Because the function that was considered has real values, each c_{-n} coefficient will equal to the conjugate of c_n, respectively: $c_{-n}=\overline{c_n}$ (5)

Let us now consider a function $f : [0,1] \to \mathbb{C}$, with its expansion into a complex Fourier series:

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{2\pi i n t} = \ldots + c_{-2}e^{-2\cdot 2\pi i t} + c_{-1}e^{-1\cdot 2\pi i t} + c_0 e^{0\cdot 2\pi i t} + c_1 e^{1\cdot 2\pi i t} + c_2 e^{2\cdot 2\pi i t} + \ldots \quad (6)$$

Determining the c_n coefficients remains of importance now. We start by integrating the function on the [0,1] interval: $\int_0^1 f(t)dt = \int_0^1 (\ldots + c_{-2}e^{-2\cdot 2\pi i t} + c_{-1}e^{-1\cdot 2\pi i t} + c_0 e^{0\cdot 2\pi i t} + c_1 e^{1\cdot 2\pi i t} + c_2 e^{2\cdot 2\pi i t} + \cdots)dt$

This represents the integral of an infinite sum and we switch the integral with the sum, obtaining:

$$\int_0^1 f(t)dt = \ldots + \int_0^1 c_{-1}e^{-1\cdot 2\pi i t}dt + \int_0^1 c_0 e^{0\cdot 2\pi i t}dt + \int_0^1 c_1 e^{1\cdot 2\pi i t}dt + \int_0^1 c_2 e^{2\cdot 2\pi i t}dt + \ldots$$

Due to the exponentials, each of these separate integrals equals to zero, but one: the integral with the c_0 term, that actually equals to c_0.

To continue the process of determining the coefficients, we now integrate the function multiplied by $e^{-n\cdot 2\pi i t}$ and we use the same procedure to switch the integral with the infinite sum:

$$\int_0^1 f(t)e^{-n\cdot 2\pi i t}dt = \ldots + \int_0^1 c_{-1}e^{-(n+1)\cdot 2\pi i t}dt + \int_0^1 c_0 e^{-n\cdot 2\pi i t}dt + \int_0^1 c_1 e^{(1-n)\cdot 2\pi i t}dt + \cdots + \int_0^1 c_n e^{0\cdot 2\pi i t}dt + \ldots$$

For the same reason as before, all of the integrals that resulted equal to zero, but the one with the c_n term, which brings us to the conclusion that: $c_n = \int_0^1 f(t)e^{-2\pi i n t}dt$ (7)

Unlike the previous case, the c_{-n} coefficients won't necessarily equal to the conjugate of c_n, since the function has now complex values. Additionally, this function can be written using the following form:

$$f(t) = u + iv \quad (8)$$

The graphical representation is a curve in the complex plane and it has a parametric form:

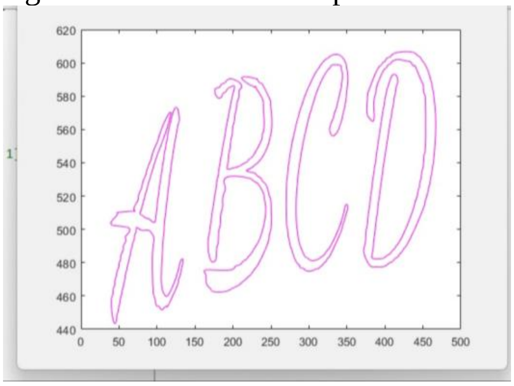
$$\begin{cases} x = u(t) \\ y = v(t) \end{cases}, t \in [0,1] \quad (9)$$

Two aspects need to be taken into consideration when referring to the numerical approximation are:

1. the number N of terms considered from the Fourier series: $f(t) \approx \sum_{-N}^{N} c_n e^{2\pi i n t}$
2. the numerical approximation of the c_n coefficients: $c_n = \int_0^1 f(t)e^{-2\pi i n t}dt$

Knowing the values of the function in some points that belong to its domain, we can approximate the integral using quadrature formulas: $\int_0^1 g(t)dt \approx \sum_{k=0}^{M} \omega_k\, g(t_k)$ (10)

In our program, we have considered the trapezoidal rule for uniform division, for which MATLAB uses the *trapz* procedure. As an application, we started with a Python script to identify the points in (x,y) coordinates corresponding to a given image; we then used the acquired data to approximate the curve in a MATLAB script using Fourier series for complex valued functions.



Approximating different letters' contour by using the Fourier series for complex valued functions

3. CONCLUSIONS

Fourier series are fundamental in mathematics, used to represent complex periodic functions through sine and cosine terms. We explored their theoretical underpinnings, detailing their formulation for both real and complex-valued functions. Numerical aspects such as selecting the number of terms for approximation and employing integration methods ensure precise coefficient computation.

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