The Functions

```
Clear["Global`*"]
```

$$\text{nMomPDF}[\theta\_,\ \theta0\_,\ \sigma\_,\ \tau\_,\ k\_] := \frac{\sqrt{\pi}}{2^{k}\ \text{Gamma}\left[k+\frac{1}{2}\right]}\left(\frac{(\theta-\theta0)^{2}}{\sigma^{2}\ \tau}\right)^{k} *$$

$$\frac{1}{\text{Sqrt}\left[2*\text{Pi}*\sigma^{2}*\tau\right]} * \text{Exp}\left[-\frac{1}{2}*\frac{(\theta-\theta0)^{2}}{\sigma^{2}*\tau}\right]\ \text{(* Note that } \frac{1}{(-1+2\ k)\,!!} == \frac{\sqrt{\pi}}{2^{k}\text{Gamma}\left[k+\frac{1}{2}\right]}\ \text{*)}$$

Check equivalence to Johnson Rossell 2012 JASA

```
(* My version if Ak=Ik (identity)*)
```

$$\text{MVNormMomPDF}[\theta1\_,\ \theta2\_,\ A\_,\ \sigma\_,\ \tau\_,\ r\_,\ p\_] :=$$

$$((2*r-1)\,!!)^{-p}\ (2*\pi)^{-p/2}\ \left(\tau*\sigma^{2}\right)^{-r*p-p/2}*\text{Det}[A]^{1/2}*$$

$$\text{Exp}\left[-\frac{1}{2*\tau*\sigma^{2}}*\text{Transpose}[\{\theta1,\ \theta2\}].A.\{\theta1,\ \theta2\}\right]*\theta1^{2*r}*\theta2^{2*r};$$

```
(* Johnson Rossell version *)
```

$$\text{MVNormMomPDF}[\theta1\_,\ \theta2\_,\ A\_,\ \sigma\_,\ \tau\_,\ r\_,\ p\_] :=$$

$$\text{dp}*(2\ \pi)^{-p/2}*\left(\tau\ \sigma^{2}\right)^{-r\ p-p/2}*\text{Det}[A_{p}]^{1/2}*\text{Exp}\left[\frac{-1}{2\ \tau\ \sigma^{2}}\ \beta'\ A_{p}\ \beta\right]*\text{Product}\left[\beta_{i}^{2\ r},\ \{i,\ 1,\ p\}\right]$$

```
(* Same density. Penalty term from paper if Ak = Ik is:  *)
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$$d[p\_] = ((2\ r-1)\,!!)^{-p}$$

Compute Marginal Likelihood accordring to Johnson Rossell 2012 JASA

$(*C_{(k)}=X_{(k)}^{T}\ X_{(k)}+(1/\tau)\ A_{(k)}*)$ (* Xk real matrix of n times k with n>k *)

```
Ck = Transpose[Xk].Xk + (1 / τ) * Ak;
(* for now Ak is Identity matrix of dimension k,
τ is some positive real number *)
```

$(*\tilde{\beta}_{(k)}=C_{(k)}^{-1}\ X_{(k)}^{T}\ y_{(n)}*)$

```
βtk = Inverse[Ck].Transpose[Xk].y; (* data vector of size n *)
```

$(*R_{(k)}=y_{(n)}^{T}\ (I_{(n)}-X_{(k)}\ C_{(k)}^{-1}\ X_{(k)}^{T})\ y_{(n)}*)$

```
Rk = Transpose[y].(IdentityMatrix[n] - Xk.Inverse[Ck].Transpose[Xk]).y;
```

$$E[\beta\_] := \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{i=1}^{k}\beta_{k_i}{}^{2\ r}\right)\phi\left[\beta_{k};\ \beta tk,\ \sigma^{2}\ \text{Solve}[Ck]\right]\ d\beta_{k_1}\cdots d\beta_{k_k}$$

(* β_{k_i} is the k_i-th element of the k-dimensional β-vector and ϕ is a
multivariate normal PDF with mean βtk and variance σ^2Solve[Ck] *)

(* ϕ is a multivariate normal pdf with mean βtk
and variance $\sigma^2 C_k^{-1}$ wher σ is some positve real number *)

```
MargMVNormPDF[] :=
```

$$((2\ r-1)\,!!)^{-k}*(2\ \pi)^{-n/2}*\tau^{-k/2-r\ k}*\left(\sigma^{2}\right)^{-n/2-r\ k}*\left(\frac{\text{Det}[Ak]}{\text{Det}[Ck]}\right)^{\frac{1}{2}}*\text{Exp}\left[-\frac{Rk}{2\ \sigma^{2}}\right]*E[\beta]$$

Find Constant for Ak = g($X'X$)$^{-1}$, i.e. g-Prior

(* Univariate Case *)

$$\text{Integrate}\left[\frac{\theta^{2k}}{\tau^{k}} * \frac{1}{\text{Sqrt}[2*\text{Pi}*\tau]} * \text{Exp}\left[-\frac{1}{2} * \frac{\theta^{2}}{\tau}\right],\ \{\theta, -\infty, \infty\},\ \text{Assumptions} \to \{\tau > 0,\ k \in \text{PositiveIntegers}\}\right]$$

Out[ ]=

$$\frac{2^{-1+k}\left(1+(-1)^{2k}\right)\text{Gamma}\left[\frac{1}{2}+k\right]}{\sqrt{\pi}}$$

In[ ]:= (* multivariate Case *)

$$\text{Integrate}\left[(2*\pi)^{-p/2}\,\tau^{-r*p-p/2} * \text{detA}^{1/2} * \text{Exp}\left[-\frac{1}{2*\tau} * \text{Transpose}[\{\theta1, \theta2\}].\begin{pmatrix} a & b \\ b & c \end{pmatrix}.\{\theta1, \theta2\}\right] * \theta1^{2*r} * \theta2^{2*r},\ \{\theta1, -\infty, \infty\},\ \text{Assumptions} \to \{\tau > 0,\ k \in \text{PositiveIntegers},\ \theta2 \in \text{Reals},\ r \in \text{PositiveIntegers},\ p \in \text{PositiveIntegers},\ \text{detA} \in \text{Reals},\ a > 0,\ c > 0,\ b \in \text{Reals},\ \text{detA} > 0\}\right]$$

Out[ ]=

$$2^{-\frac{1}{2}-\frac{p}{2}+r}\, a^{-1-r}\sqrt{\text{detA}}\; e^{-\frac{c\,\theta2^{2}}{2\tau}}\,\pi^{-p/2}\,\theta2^{2r}\,\tau^{r-p\left(\frac{1}{2}+r\right)} \left(\left(1+(-1)^{2r}\right)\sqrt{a\,\tau}\;\text{Gamma}\left[\frac{1}{2}+r\right] \times \text{Hypergeometric1F1}\left[\frac{1}{2}+r, \frac{1}{2}, \frac{b^{2}\,\theta2^{2}}{2\,a\,\tau}\right] + \sqrt{2}\left(-1+(-1)^{2r}\right) b\,\theta2\;\text{Gamma}[1+r] \times \text{Hypergeometric1F1}\left[1+r, \frac{3}{2}, \frac{b^{2}\,\theta2^{2}}{2\,a\,\tau}\right]\right)$$

$$\text{Integrate}\left[2^{-\frac{1}{2}-\frac{p}{2}+r}\, a^{-1-r}\sqrt{\text{detA}}\; e^{-\frac{c\,\theta2^{2}}{2\tau}}\,\pi^{-p/2}\,\theta2^{2r}\,\tau^{r-p\left(\frac{1}{2}+r\right)} \left(\left(1+(-1)^{2r}\right)\sqrt{a\,\tau}\;\text{Gamma}\left[\frac{1}{2}+r\right] \times \text{Hypergeometric1F1}\left[\frac{1}{2}+r, \frac{1}{2}, \frac{b^{2}\,\theta2^{2}}{2\,a\,\tau}\right] + \sqrt{2}\left(-1+(-1)^{2r}\right) b\,\theta2\;\text{Gamma}[1+r] \times \text{Hypergeometric1F1}\left[1+r, \frac{3}{2}, \frac{b^{2}\,\theta2^{2}}{2\,a\,\tau}\right]\right),\ \{\theta2, -\infty, \infty\},\ \text{Assumptions} \to \{\tau > 0,\ k \in \text{PositiveIntegers},\ \theta2 \in \text{Reals},\ r \in \text{PositiveIntegers},\ p \in \text{PositiveIntegers},\ \text{detA} \in \text{Reals},\ a > 0,\ c > 0,\ b \in \text{Reals},\ \text{detA} > 0\}\right]$$

(* b^2<a*c is assured if A is positive definite *)

Out[ ]=

$$2^{-\frac{p}{2}+2r}\left(1+(-1)^{2r}\right) a^{-1-r}\,\pi^{-p/2}\,\tau^{r-p\left(\frac{1}{2}+r\right)}\left(\frac{\tau}{c}\right)^{\frac{1}{2}+r}\sqrt{a\,\text{detA}\,\tau}\;\text{Gamma}\left[\frac{1}{2}+r\right]^{2}\,\text{Hypergeometric2F1}\left[\frac{1}{2}+r, \frac{1}{2}+r, \frac{1}{2}, \frac{b^{2}}{a\,c}\right] \quad \text{if } b^{2} < a\,c$$

In[ ]:= $2^{2 r+1-\frac{p}{2}} a^{-\frac{1}{2}-r} d^{-\frac{1}{2}-r} \pi^{-p/2} \tau^{2 r+1-p \left(\frac{1}{2}+r\right)} \sqrt{detA}$
$Gamma\left[\frac{1}{2}+r\right]^2 Hypergeometric2F1\left[\frac{1}{2}+r, \frac{1}{2}+r, \frac{1}{2}, \frac{b^2}{a c}\right]$

Out[ ]= $2^{1-\frac{p}{2}+2 r} a^{-\frac{1}{2}-r} d^{-\frac{1}{2}-r} \sqrt{detA}\, \pi^{-p/2} \tau^{1+2 r-p \left(\frac{1}{2}+r\right)}$
$Gamma\left[\frac{1}{2}+r\right]^2 Hypergeometric2F1\left[\frac{1}{2}+r, \frac{1}{2}+r, \frac{1}{2}, \frac{b^2}{a c}\right]$

Try General

```
In[ ]:= Integrate[(2*π)^(-p/2)*τ^(-r*p-p/2)*
   Det[A]^(1/2)*Exp[-(1/(2*τ))*{θ1, θ2}.A.{θ1, θ2}]*
   θ1^(2*r)*θ2^(2*r), {θ1, -∞, ∞}, {θ2, -∞, ∞},
  Assumptions → {Element[A, Matrices[{2, 2}, Reals, Symmetric[{1, 2}]]],
    τ > 0, r ∈ PositiveIntegers, p ∈ PositiveIntegers}]
```

Out[ ]= $(2\pi)^{-p/2} \tau^{-p\left(\frac{1}{2}+r\right)} \sqrt{Det[A]}$
$Integrate\left[e^{-\frac{\{\theta1,\theta2\}.A.\{\theta1,\theta2\}}{2\tau}} \theta1^{2r} \theta2^{2r}, \{\theta1, -\infty, \infty\}, \{\theta2, -\infty, \infty\}, Assumptions \rightarrow\right.$
$\left.\{A \in Matrices[\{2, 2\}, \mathbb{R}, Symmetric[\{1, 2\}]], \tau > 0, r \in \mathbb{Z} \,\&\&\, r > 0, p \in \mathbb{Z} \,\&\&\, p > 0\}\right]$

In[ ]:= $Simplify\left[Hypergeometric2F1\left[\frac{1}{2}+r, \frac{1}{2}+r, \frac{1}{2}, \frac{b^2}{a d}\right],\right.$
$\left. Assumptions \rightarrow \{r \in PositiveIntegers, a > 0, d > 0, b \in Reals\}\right]$

Out[ ]= $Hypergeometric2F1\left[\frac{1}{2}+r, \frac{1}{2}+r, \frac{1}{2}, \frac{b^2}{a d}\right]$

Via Moment:

```
In[ ]:= μ = {μ1, μ2};(*For a 2-dimensional case*)
A = {{σ1^2, ρ*σ1*σ2}, {ρ*σ1*σ2, σ2^2}};(*Covariance matrix*)

Moment[MultinormalDistribution[μ, A], {2 k}]
```

Out[ ]= $Moment\left[MultinormalDistribution\left[\{\mu1, \mu2\}, \left\{\left\{\sigma1^2, \rho\,\sigma1\,\sigma2\right\}, \left\{\rho\,\sigma1\,\sigma2, \sigma2^2\right\}\right\}\right], \{2 k\}\right]$