The Functions

```
Clear["Global`*"]
```

$$\text{nMomPDF}[\theta\_, \theta0\_, \sigma\_, \tau\_, k\_] := \frac{\sqrt{\pi}}{2^k\,\text{Gamma}\left[k+\frac{1}{2}\right]}\left(\frac{(\theta-\theta0)^2}{\sigma^2\,\tau}\right)^k * \frac{1}{\text{Sqrt}\left[2*\text{Pi}*\sigma^2*\tau\right]} * \text{Exp}\left[-\frac{1}{2}*\frac{(\theta-\theta0)^2}{\sigma^2*\tau}\right] \;(*\text{ Note that } \frac{1}{(-1+2\ k)!!} == \frac{\sqrt{\pi}}{2^k\text{Gamma}\left[k+\frac{1}{2}\right]} *)$$

Check equivalence to Johnson Rossell 2012 JASA

```
(* My version if Ak=Ik (identity)*)
```

$$\text{MVNormMomPDF}[\theta1\_, \theta2\_, A\_, \sigma\_, \tau\_, r\_, p\_] := ((2*r-1)!!)^{-p}\,(2*\pi)^{-p/2}\left(\tau*\sigma^2\right)^{-r*p-p/2} * \text{Det}[A]^{1/2} * \text{Exp}\left[-\frac{1}{2*\tau*\sigma^2} * \text{Transpose}[\{\theta1, \theta2\}].A.\{\theta1, \theta2\}\right] * \theta1^{2*r} * \theta2^{2*r};$$

```
(* Johnson Rossell version *)
```

$$\text{MVNormMomPDF}[\theta1\_, \theta2\_, A\_, \sigma\_, \tau\_, r\_, p\_] := dp*(2\,\pi)^{-p/2} * \left(\tau\,\sigma^2\right)^{-r\,p-p/2} * \text{Det}\left[A_p\right]^{1/2} * \text{Exp}\left[\frac{-1}{2\,\tau\,\sigma^2}\,\beta'\,A_p\,\beta\right] * \text{Product}\left[\beta_i{}^{2\,r}, \{i, 1, p\}\right]$$

```
(* Same density. Penalty term from paper if Ak = Ik is:  *)
```

$$d[p\_] = ((2\,r-1)!!)^{-p}$$

Compute Marginal Likelihood accodring to Johnson Rossell 2012 JASA

$(*C_{(k)}=X_{(k)}{}^{T}\ X_{(k)}+(1/\tau)\ A_{(k)}*)$ (* Xk real matrix of n times k with n>k *)

```
Ck = Transpose[Xk].Xk + (1 / τ) * Ak; (* for now Ak is Identity matrix of dimension k,
τ is some positive real number *)
```

$(*\tilde{\beta}_{(k)}=C_{(k)}{}^{-1}\ X_{(k)}{}^{T}\ y_{(n)}*)$

```
μ = Inverse[Ck].Transpose[Xk].y; (* data vector of size n *)
```

$(*R_{(k)}=y_{(n)}{}^{T}\ (I_{(n)}-X_{(k)}\ C_{(k)}{}^{-1}\ X_{(k)}{}^{T})\ y_{(n)}*)$

```
Rk = Transpose[y].(IdentityMatrix[n] - Xk.Inverse[Ck].Transpose[Xk]).y;
```

```
(* start algoithm *)
(* step 1 *)
```

$$E\left[\prod_{i=1}^{k}\beta_{k_i}{}^{2\,r}\right] := \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{i=1}^{k}\beta_{k_i}{}^{2\,r}\right)\phi\left[\beta_k;\ \mu,\ \Sigma^{-1}\right] d\beta_{k_1}\ldots d\beta_{k_k}$$

```
(* Allgemein gilt: p(a)p(b... z|a) *)
(* p(a,b,c,...z)=p(a)p(b|a)p(c|b,a)... p(z|...) *)
```

$$\phi\left[\beta_k;\ \mu,\ \Sigma^{-1}\right] = \phi_1\left[\beta_1;\ \mu_1,\ \left(\Sigma^{-1}\right)_{1,1}\right]\phi_{\{2:k\}}\left[(\beta_{2:k};)\ \overline{\mu}_{2:k},\ \overline{\Sigma}^{-1}\right]$$

$$E\left[\prod_{i=1}^{k}\beta_{k_i}{}^{2\,r}\right] := \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{i=1}^{k}\beta_{k_i}{}^{2\,r}\right)\phi_1\left[\beta_1;\ \mu_1,\ \left(\Sigma^{-1}\right)_{1,1}\right]\phi_{\{2:k\}}\left[(\beta_{2:k};)\ \overline{\mu}_{2:k},\ \overline{\Sigma}^{-1}\right] d\beta_{k_1}\ldots d\beta_{k_k}$$

(* rearange terms *)

$$E\left[\prod_{i=1}^{k}\beta_{k_i}{}^{2r}\right] := \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{i=2}^{k}\beta_{k_i}{}^{2r}\right)\phi_{\{2:k\}}\left[(\beta_{2:k};)\,\overline{\mu}_{2:k},\,\overline{\Sigma}^{-1}\right]\int_{-\infty}^{\infty}\beta_{k_1}{}^{2r}\,\phi_1\left[\beta_1;\ \mu_1,\,\left(\Sigma^{-1}\right)_{1,1}\right]d\beta_{k_1}\ldots d\beta_{k_k}$$

(* solve integral by MGF $\left[\mu_1, \left(\Sigma^{-1}\right)_{1,1}\right]$ *)

$$E\left[\prod_{i=1}^{k}\beta_{k_i}{}^{2r}\right] := \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{i=2}^{k}\beta_{k_i}{}^{2r}\right)\phi_{\{2:k\}}\left[(\beta_{2:k};)\,\overline{\mu}_{2:k},\,\overline{\Sigma}^{-1}\right]E\left[\beta_{k_1}{}^{2r}\right]d\beta_{k_2}\ldots d\beta_{k_k}$$

(* Compute conditional mean $\overline{\mu}_{2:k}$ and conditional VCV-matrix $\overline{\Sigma}^{-1}$ using a = $E\left[\beta_{k_1}{}^{2r}\right]$*)

$$E\left[\prod_{i=1}^{k}\beta_{k_i}{}^{2r}\right] := E\left[\beta_{k_1}{}^{2r}\right]\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\prod_{i=2}^{k}\beta_{k_i}{}^{2r}\right)\phi_{\{2:k\}}\left[(\beta_{2:k};)\,\overline{\mu}_{2:k},\,\overline{\Sigma}^{-1}\right]d\beta_{k_2}\ldots d\beta_{k_k}$$

(* back to step one *)

(* *)

(* at the end we finish with getting a product of conditional expectations *)

$$E\left[\prod_{i=1}^{k}\beta_{k_i}{}^{2r}\right] = E\left[\beta_{k_1}{}^{2r}\right]*E\left[\beta_{k_2}{}^{2r}\,\middle|\,\beta_{k_1}{}^{2r} = E\left[\beta_{k_1}{}^{2r}\right]\right]\ldots*E\left[\beta_{k_k}{}^{2r}\,\middle|\,\prod_{i=1}^{k-1}\beta_{k_i}{}^{2r} = E\left[\prod_{i=1}^{k-1}\beta_{k_i}{}^{2r}\right]\right]$$

(* β_{k_i} is the k_i-th element of the k-dimensional β-vector and φ is a multivariate normal PDF with mean βtk and variance σ^2Solve[Ck] *)

(* φ is a multivariate normal pdf with mean βtk and variance $\sigma^2 C_k{}^{-1}$ wher σ is some positve real number *)

MargMVNormPDF[] :=

$$((2r-1)!!)^{-k}*(2\pi)^{-n/2}*\tau^{-k/2-rk}*\left(\sigma^2\right)^{-n/2-rk}*\left(\frac{\mathrm{Det}[Ak]}{\mathrm{Det}[Ck]}\right)^{\frac{1}{2}}*\mathrm{Exp}\left[-\frac{Rk}{2\,\sigma^2}\right]*E[\beta]$$

Find Constant for Ak = g$(X'X)^{-1}$, i.e. g-Prior

(* Univariate Case *)

$$\mathrm{Integrate}\left[\frac{\theta^{2k}}{\tau^k}*\frac{1}{\mathrm{Sqrt}[2*\mathrm{Pi}*\tau]}*\mathrm{Exp}\left[-\frac{1}{2}*\frac{\theta^2}{\tau}\right],\ \{\theta,-\infty,\infty\},\ \mathrm{Assumptions}\to\{\tau>0,\ k\in\mathrm{PositiveIntegers}\}\right]$$

Out[∘]=
$$\frac{2^{-1+k}\left(1+(-1)^{2k}\right)\mathrm{Gamma}\left[\frac{1}{2}+k\right]}{\sqrt{\pi}}$$

In[ ]:=
```
(* multivariate Case *)
Integrate[(2 * π)^(-p/2) τ^(-r*p-p/2) * detA^(1/2) *
  Exp[-1/(2 * τ) * Transpose[{θ1, θ2}].{{a, b}, {b, c}}.{θ1, θ2}] * θ1^(2*r) * θ2^(2*r), {θ1, -∞, ∞},
 Assumptions → {τ > 0, k ∈ PositiveIntegers, θ2 ∈ Reals, r ∈ PositiveIntegers,
   p ∈ PositiveIntegers, detA ∈ Reals, a > 0, c > 0, b ∈ Reals, detA > 0}]
```

Out[ ]=
$$2^{-\frac{1}{2}-\frac{p}{2}+r}\, a^{-1-r} \sqrt{\text{detA}}\, e^{-\frac{c\,\theta 2^2}{2\tau}} \pi^{-p/2}\, \theta 2^{2r}\, \tau^{r-p\left(\frac{1}{2}+r\right)} \left(\left(1+(-1)^{2r}\right)\sqrt{a\,\tau}\, \text{Gamma}\left[\frac{1}{2}+r\right] \text{Hypergeometric1F1}\left[\frac{1}{2}+r, \frac{1}{2}, \frac{b^2\,\theta 2^2}{2\,a\,\tau}\right] + \sqrt{2}\left(-1+(-1)^{2r}\right) b\,\theta 2\, \text{Gamma}[1+r]\, \text{Hypergeometric1F1}\left[1+r, \frac{3}{2}, \frac{b^2\,\theta 2^2}{2\,a\,\tau}\right]\right)$$

```
Integrate[2^(-1/2-p/2+r) a^(-1-r) Sqrt[detA] E^(-c θ2^2/(2 τ)) π^(-p/2) θ2^(2 r) τ^(r-p (1/2+r))
  ((1 + (-1)^(2 r)) Sqrt[a τ] Gamma[1/2 + r] Hypergeometric1F1[1/2 + r, 1/2, b^2 θ2^2/(2 a τ)] +
   Sqrt[2] (-1 + (-1)^(2 r)) b θ2 Gamma[1 + r] Hypergeometric1F1[1 + r, 3/2, b^2 θ2^2/(2 a τ)]), {θ2, -∞, ∞},
 Assumptions → {τ > 0, k ∈ PositiveIntegers, θ2 ∈ Reals, r ∈ PositiveIntegers,
   p ∈ PositiveIntegers, detA ∈ Reals, a > 0, c > 0, b ∈ Reals, detA > 0}]
(* b^2<a*c is assured if A is positive definite *)
```

Out[ ]=
$$2^{-\frac{p}{2}+2r}\left(1+(-1)^{2r}\right) a^{-1-r}\, \pi^{-p/2}\, \tau^{r-p\left(\frac{1}{2}+r\right)} \left(\frac{\tau}{c}\right)^{\frac{1}{2}+r} \sqrt{a\,\text{detA}\,\tau}\; \text{Gamma}\left[\frac{1}{2}+r\right]^2 \text{Hypergeometric2F1}\left[\frac{1}{2}+r, \frac{1}{2}+r, \frac{1}{2}, \frac{b^2}{a\,c}\right] \text{ if } b^2 < a\,c$$

In[ ]:=
$$2^{2r+1-\frac{p}{2}}\, a^{-\frac{1}{2}-r}\, d^{-\frac{1}{2}-r}\, \pi^{-p/2}\, \tau^{2r+1-p\left(\frac{1}{2}+r\right)} \sqrt{\text{detA}}\; \text{Gamma}\left[\frac{1}{2}+r\right]^2 \text{Hypergeometric2F1}\left[\frac{1}{2}+r, \frac{1}{2}+r, \frac{1}{2}, \frac{b^2}{a\,c}\right]$$

Out[ ]=
$$2^{1-\frac{p}{2}+2r}\, a^{-\frac{1}{2}-r}\, d^{-\frac{1}{2}-r} \sqrt{\text{detA}}\, \pi^{-p/2}\, \tau^{1+2r-p\left(\frac{1}{2}+r\right)}\; \text{Gamma}\left[\frac{1}{2}+r\right]^2 \text{Hypergeometric2F1}\left[\frac{1}{2}+r, \frac{1}{2}+r, \frac{1}{2}, \frac{b^2}{a\,c}\right]$$

Try General

In[ ]:=
```
Integrate[(2 * π) ^ (-p / 2) * τ^ (-r * p - p / 2) * Det[A] ^ (1 / 2) *
  Exp[- (1 / (2 * τ)) * {θ1, θ2}.A.{θ1, θ2}] * θ1^ (2 * r) * θ2^ (2 * r), {θ1, -∞, ∞},
 {θ2, -∞, ∞}, Assumptions → {Element[A, Matrices[{2, 2}, Reals, Symmetric[{1, 2}]]],
   τ > 0, r ∈ PositiveIntegers, p ∈ PositiveIntegers}]
```

Out[ ]=

$$(2\pi)^{-p/2}\,\tau^{-p\left(\frac{1}{2}+r\right)}\sqrt{\mathrm{Det}[A]}$$

$$\text{Integrate}\left[e^{-\frac{\{\theta1,\theta2\}.A.\{\theta1,\theta2\}}{2\tau}}\,\theta1^{2r}\,\theta2^{2r},\ \{\theta1,-\infty,\infty\},\ \{\theta2,-\infty,\infty\},\ \text{Assumptions}\to \{A\in\text{Matrices}[\{2,2\},\mathbb{R},\text{Symmetric}[\{1,2\}]],\ \tau>0,\ r\in\mathbb{Z}\,\&\&\,r>0,\ p\in\mathbb{Z}\,\&\&\,p>0\}\right]$$

In[ ]:=

$$\text{Simplify}\left[\text{Hypergeometric2F1}\left[\frac{1}{2}+r,\ \frac{1}{2}+r,\ \frac{1}{2},\ \frac{b^2}{a\,d}\right],\ \text{Assumptions}\to\{r\in\text{PositiveIntegers},\ a>0,\ d>0,\ b\in\text{Reals}\}\right]$$

Out[ ]=

$$\text{Hypergeometric2F1}\left[\frac{1}{2}+r,\ \frac{1}{2}+r,\ \frac{1}{2},\ \frac{b^2}{a\,d}\right]$$

Via Moment:

In[ ]:=
```
μ = {μ1, μ2};(*For a 2-dimensional case*)
A = {{σ1^2, ρ * σ1 * σ2}, {ρ * σ1 * σ2, σ2^2}};(*Covariance matrix*)

Moment[MultinormalDistribution[μ, A], {2 k}]
```

Out[ ]=

$$\text{Moment}\left[\text{MultinormalDistribution}\left[\{\mu1,\mu2\},\ \left\{\left\{\sigma1^2,\ \rho\,\sigma1\,\sigma2\right\},\ \left\{\rho\,\sigma1\,\sigma2,\ \sigma2^2\right\}\right\}\right],\ \{2\,k\}\right]$$