Instructions:

- This assignment is meant to help you grok certain concepts we will use in the course. Please don't copy solutions from any sources.
- Avoid verbosity.
- Questions marked with * are relatively difficult. Don't be discouraged if you cannot solve them right away!
- The assignment needs to be written in latex using the attached tex file. The solution for each question should be written in the solution block in space already provided in the tex file. **Handwritten assignments will not be accepted.**

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1. Suppose, a transformation matrix A, transforms the standard basis vectors of R^3 as follows :

$$\begin{bmatrix}1\\0\\0\end{bmatrix} => \begin{bmatrix}6\\4\\2\end{bmatrix} ; \begin{bmatrix}0\\1\\0\end{bmatrix} => \begin{bmatrix}1\\2\\8\end{bmatrix} ; \begin{bmatrix}0\\0\\1\end{bmatrix} => \begin{bmatrix}1\\5\\7\end{bmatrix}$$

   (a) If the volume of a hypothetical parallelepiped in the un-transformed space is $100 units^3$ what will be volume of this parallelepiped in the transformed space?

Solution:

the volume of parallelepiped

$$volume(U) := \int_U f(u)\, du$$

Let dT(u) signifies the Jacobian matrix of T at the point u.

As T is a linear transformation it follows that

$dT(u) = T$ so change variable for multiple variable implies

$$\int_{T(U)} f(v)\, dv = \int_U f(T(u))\, |det(\, dT(u)\,)|\, du$$

So the volume of the transformed parallelepiped is

$$volume(\, T(U)\,) = \int_{T(U)} dv = \int_U |det(\, dT(u)\,)|\, du = \int_U |det(T)|\, du$$

as $|det(T)|$ is constant so

$$\int_U |det(T)|\, du = |det(T)| \int_U du = |det(T)|\, volume(U)$$

so $volume(\, T(U)\,) = volume(U)\, |det(T)|$

$$|det(T)| = |det \left(\begin{bmatrix} 6 & 1 & 1 \\ 4 & 2 & 5 \\ 2 & 8 & 7 \end{bmatrix} \right) |$$

$$= |6 \cdot \det \begin{pmatrix} 2 & 5 \\ 8 & 7 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 4 & 5 \\ 2 & 7 \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 4 & 2 \\ 2 & 8 \end{pmatrix} |$$

$$= |6\,(14 - 40) - 1 \cdot\ (28 - 10) + 1 \cdot\ (32 - 4)|$$

$$= |6\,(-26) - 1 \cdot\ 18 + 1 \cdot\ 28|$$

$$= |-146| = 146$$

so $volume(\, T(U)\,) = volume(U)\, |det(T)|$

$$\boxed{= 100 * 146 \quad unit^3 = 14600 \quad unit^3}$$

(b) What will be the volume if the transformation of the basis vectors is as follows :

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} => \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} ; \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} => \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} ; \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} => \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Solution:

$$\text{so } volume(\,T(U)\,) = volume(U)\,|det(T)|$$

$$|det(T)| = |det\left(\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\right)|$$

$$= |1 \cdot \det\begin{pmatrix}5 & 6\\8 & 9\end{pmatrix} - 2 \cdot \det\begin{pmatrix}4 & 6\\7 & 9\end{pmatrix} + 3 \cdot \det\begin{pmatrix}4 & 5\\7 & 8\end{pmatrix}|$$

$$= |1 \cdot (45 - 48) - 2\,(36 - 42) + 3\,(32 - 35)\,|$$

$$= |1 \cdot (-3) - 2\,(-6) + 3\,(-3)\,|$$

$$= |0| = 0$$

$$\text{so } volume(\,T(U)\,) = volume(U)\,|det(T)|$$

$$\boxed{= 100 * 0 \quad unit^3 = 0 \quad unit^3}$$

(c) Comment on the uniqueness of the second transformation.

Solution: As the volume of second transformation becomes zero as Transformed side(H,W,B) of the parallelopiped all become parallel to each other .

2. If R^3 is represented by following basis vectors : $\begin{bmatrix}4\\2\\1\end{bmatrix}$, $\begin{bmatrix}-3\\-1\\5\end{bmatrix}$, $\begin{bmatrix}0\\2\\7\end{bmatrix}$

(a) Find the representation of the vector $\begin{pmatrix}2 & -1 & 6\end{pmatrix}^T$ (as represented in standard basis) in the above basis.

Solution: let X is the vector in standard basis matrix

$$TX = \begin{pmatrix} 2 & -1 & 6 \end{pmatrix}^T$$

$$\implies T^{-1}TX = T^{-1}\begin{pmatrix} 2 & -1 & 6 \end{pmatrix}^T$$

$$\implies X = T^{-1}\begin{pmatrix} 2 & -1 & 6 \end{pmatrix}^T$$

so find the inverse of Transformation Matrix

$$T = \begin{bmatrix} 4 & -3 & 0 \\ 2 & -1 & 2 \\ 1 & 5 & 7 \end{bmatrix} \quad is$$

$$\implies \begin{bmatrix} 4 & -3 & 0 & | & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 2 & | & -\frac{1}{2} & 1 & 0 \\ 1 & 5 & 7 & | & 0 & 0 & 1 \end{bmatrix} \quad R_2 \leftarrow R_2 - \frac{1}{2} \cdot R_1$$

$$\implies \begin{bmatrix} 4 & -3 & 0 & | & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 2 & | & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{23}{4} & 7 & | & -\frac{1}{4} & 0 & 1 \end{bmatrix} \quad R_3 \leftarrow R_3 - \frac{1}{4} \cdot R_1$$

$$\implies \begin{bmatrix} 4 & -3 & 0 & | & 1 & 0 & 0 \\ 0 & \frac{23}{4} & 7 & | & -\frac{1}{4} & 0 & 1 \\ 0 & \frac{1}{2} & 2 & | & -\frac{1}{2} & 1 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\implies \begin{bmatrix} 4 & -3 & 0 & | & 1 & 0 & 0 \\ 0 & \frac{23}{4} & 7 & | & -\frac{1}{4} & 0 & 1 \\ 0 & 0 & \frac{32}{23} & | & -\frac{11}{23} & 1 & -\frac{2}{23} \end{bmatrix} \quad R_3 \leftarrow R_3 - \frac{2}{23} \cdot R_2$$

$$\implies \begin{bmatrix} 4 & -3 & 0 & | & 1 & 0 & 0 \\ 0 & \frac{23}{4} & 7 & | & -\frac{1}{4} & 0 & 1 \\ 0 & 0 & \frac{32}{23} & | & -\frac{11}{23} & 1 & -\frac{2}{23} \end{bmatrix}$$

reduce it to reduced row echelon form we get

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{17}{32} & -\frac{21}{32} & \frac{3}{16} \\ 0 & 1 & 0 & | & \frac{3}{8} & -\frac{7}{8} & \frac{1}{4} \\ 0 & 0 & 1 & | & -\frac{11}{32} & \frac{23}{32} & -\frac{1}{16} \end{bmatrix}$$

$$\text{so } T^{-1} = \begin{bmatrix} \frac{17}{32} & -\frac{21}{32} & \frac{3}{16} \\ \frac{3}{8} & -\frac{7}{8} & \frac{1}{4} \\ -\frac{11}{32} & \frac{23}{32} & -\frac{1}{16} \end{bmatrix}$$

Solution:

$$\text{so the representation of the vector } \begin{pmatrix} 2 & -1 & 6 \end{pmatrix}^T$$
$$\text{(as represented in standard basis) in the above basis is}$$
$$\begin{bmatrix} \frac{17}{32} & -\frac{21}{32} & \frac{3}{16} \\ \frac{3}{8} & -\frac{7}{8} & \frac{1}{4} \\ -\frac{11}{32} & \frac{23}{32} & -\frac{1}{16} \end{bmatrix} \begin{bmatrix} 2 & -1 & 6 \end{bmatrix}^T$$
$$= \begin{bmatrix} \frac{17}{32}\cdot 2 + \left(-\frac{21}{32}\right)(-1) + \frac{3}{16}\cdot 6 \\ \frac{3}{8}\cdot 2 + \left(-\frac{7}{8}\right)(-1) + \frac{1}{4}\cdot 6 \\ \left(-\frac{11}{32}\right)\cdot 2 + \frac{23}{32}(-1) + \left(-\frac{1}{16}\right)\cdot 6 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{91}{32} \\ \frac{25}{8} \\ -\frac{57}{32} \end{bmatrix}$$
$$so \quad \begin{pmatrix} 2 & -1 & 6 \end{pmatrix}^T = \frac{91}{32}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} + \frac{25}{8}\begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix} + \frac{-57}{32}\begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}$$

(b) We know that, orthonormal basis simplifies this to a great extent. What would be the representation of vector $\begin{pmatrix} 2 & -1 & 6 \end{pmatrix}^T$ in the orthogonal basis represented by : $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

Solution:

$$T = \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

augmanted matrix is

$$\implies \begin{bmatrix} 2 & -2 & 1 & | & 1 & 0 & 0 \\ 2 & 1 & -2 & | & 0 & 1 & 0 \\ 1 & 2 & 2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & -2 & 1 & | & 1 & 0 & 0 \\ 0 & 3 & -3 & | & -1 & 1 & 0 \\ 1 & 2 & 2 & | & 0 & 0 & 1 \end{bmatrix} \quad R_2 \leftarrow R_2 - 1 \cdot R_1$$

$$\implies \begin{bmatrix} 2 & -2 & 1 & | & 1 & 0 & 0 \\ 0 & 3 & -3 & | & -1 & 1 & 0 \\ 0 & 3 & \frac{3}{2} & | & -\frac{1}{2} & 0 & 1 \end{bmatrix} \quad R_3 \leftarrow R_3 - \frac{1}{2} \cdot R_1$$

$$\implies \begin{bmatrix} 2 & -2 & 1 & | & 1 & 0 & 0 \\ 0 & 3 & -3 & | & -1 & 1 & 0 \\ 0 & 0 & \frac{9}{2} & | & \frac{1}{2} & -1 & 1 \end{bmatrix} \quad R_3 \leftarrow R_3 - 1 \cdot R_2$$

reducing to reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \\ 0 & 1 & 0 & | & -\frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ 0 & 0 & 1 & | & \frac{1}{9} & -\frac{2}{9} & \frac{2}{9} \end{bmatrix}$$

$$\text{so } T^{-1} = \begin{bmatrix} \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \\ -\frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{1}{9} & -\frac{2}{9} & \frac{2}{9} \end{bmatrix}$$

so the representation of the vector $\begin{pmatrix} 2 & -1 & 6 \end{pmatrix}^T$

(as represented in standard basis) in the above basis is

$$\begin{bmatrix} \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \\ -\frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{1}{9} & -\frac{2}{9} & \frac{2}{9} \end{bmatrix} \begin{bmatrix} 2 & -1 & 6 \end{bmatrix}^T$$

Solution:

$$= \begin{bmatrix} \frac{2}{9} \cdot 2 + \frac{2}{9}(-1) + \frac{1}{9} \cdot 6 \\ \left(-\frac{2}{9}\right) \cdot 2 + \frac{1}{9}(-1) + \frac{2}{9} \cdot 6 \\ \frac{1}{9} \cdot 2 + \left(-\frac{2}{9}\right)(-1) + \frac{2}{9} \cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{9} \\ \frac{7}{9} \\ \frac{16}{9} \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 6 \end{bmatrix}^T = \frac{8}{9}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \frac{7}{9}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + \frac{16}{9}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

(c) Comment on the advantages of having orthonormal basis.

Solution: in this case X can be easily obtained by

$$T = \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

is orthogonal to make it orthonormal we have to divide every column by it's magnitude

$$T_N = \begin{bmatrix} \frac{2}{\sqrt{(2^2+2^2+1^2)}} & \frac{-2}{\sqrt{((-2)^2+1^2+2^2)}} & \frac{1}{\sqrt{(1^2+(-2)^2+2^2)}} \\ \frac{2}{\sqrt{(2^2+2^2+1^2)}} & \frac{1}{\sqrt{((-2)^2+1^2+2^2)}} & \frac{-2}{\sqrt{(1^2+(-2)^2+2^2)}} \\ \frac{1}{\sqrt{(2^2+2^2+1^2)}} & \frac{2}{\sqrt{((-2)^2+1^2+2^2)}} & \frac{2}{\sqrt{(1^2+(-2)^2+2^2)}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{9} & \frac{-2}{9} & \frac{1}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{-2}{9} \\ \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \end{bmatrix}$$

$$T_N^T = \begin{bmatrix} \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \\ -\frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{1}{9} & -\frac{2}{9} & \frac{2}{9} \end{bmatrix}$$

$$so \quad T^{-1} = T_N^T$$

3. Consider a square matrix A such that the sum of the entries of every column of A is the same number c. Prove that c is an eigenvalue of transpose of A.

Solution: let

$$A = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix}$$

Characteristic equation $A - \lambda I = 0$

$$\implies \begin{bmatrix} x_{11}-\lambda & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22}-\lambda & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn}-\lambda \end{bmatrix} = 0$$

$$\implies \begin{bmatrix} x_{11}+x_{21}+\cdots+x_{n1}-\lambda & x_{12}+x_{22}+\cdots+x_{n2}-\lambda & \cdots & x_{1n}+x_{2n}+\cdots+x_{nn}-\lambda \\ x_{21} & x_{22}-\lambda & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn}-\lambda \end{bmatrix} = 0$$

$$using R_1 \leftarrow R_1 + R_2 + \cdots R_n$$

$$x_{11}+x_{21}+\cdots+x_{n1} = c$$

$$x_{12}+x_{22}+\cdots+x_{n2} = c$$

$x_{1n}+x_{2n}+\cdots+x_{nn} = c$ as given we could write

$$\begin{bmatrix} c-\lambda & c-\lambda & \cdots & c-\lambda \\ x_{21} & x_{22}-\lambda & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn}-\lambda \end{bmatrix} = 0$$

$$\implies (c-\lambda) \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_{21} & x_{22}-\lambda & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn}-\lambda \end{bmatrix} = 0$$

so $(c-\lambda) = 0$ is a solution

$so \quad \lambda = c$ so c is a eigen value of A

$$|A^T - \lambda I| = 0 => |(A-\lambda I)^T| = 0$$

determinant do not change for transpose Matrix

so eigen value do not change for Transpose of matrix

$\boxed{\textbf{so c is a Eigen value of } A^T \textbf{(proved)}}$

4. Let C be a 2×2 matrix. If the trace of matrix C is 0, then what can you say about matrix C^n where n is a positive integer?

Solution:

$$C = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

from characteristic equation we find

$$\det \begin{bmatrix} \alpha - \lambda & \beta \\ \gamma & \delta - \lambda \end{bmatrix} = 0$$

$$\lambda^2 - (\alpha + \delta)\lambda + \alpha\delta - \beta\gamma = 0$$

putting C in place of λ we get(Caley -Hamiltonian Theorem)

$$C^2 - Trace(C)C - \det(C)I_{2\times 2} = 0$$

$$C^2 = -det(C)I_{2\times 2} \quad as \quad Trace(C) = 0$$

if n is even

$$\boxed{C^n = (-1)^{\frac{n}{2}} \det(C)^{\frac{n}{2}} I_{2\times 2}}$$

if n is odd

$$\boxed{C^n = (-1)^{\frac{n-1}{2}} \det(C)^{\frac{n-1}{2}} C_{2\times 2}}$$

5. *If c and d are two real numbers then the exponential of $c + d$ is the product of the exponential of c with the exponential of d i.e

$$e^{c+d} = e^c e^d$$

(a) If we replace them with square matrices C and D, does the equality still holds ? Prove it, if yes, else provide a counterexample.

Solution: NO.
from Taylor series

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \quad where \quad t \in \mathbb{R} \quad and \quad A \in \mathbb{C}^{n \times n}$$

$$e^{A} = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad if \quad t = 1$$

$$C = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad then \quad C^k = 0 \quad for \quad n \geq 2 \quad so$$

$$e^{tC} = I + tC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad then \quad D^k = 0 \quad for \quad n \geq 2 \quad so$$

$$e^{tD} = I + tD = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

Let A=C+D so $A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$so \quad A^2 = -I, A^3 = -A, A^4 = I, A^5 = A, A^6 = -I$ etc,so that

$$A^{2k} = (-1)^k I \quad and \quad A^{2k+1} = (-1)^k A$$

$$e^{tA} = \left(\sum_{k=0}^{\infty} \frac{t^{2k} A^{2k}}{2k!} \right) + \left(\sum_{k=0}^{\infty} \frac{t^{2k+1} A^{2k+1}}{(2k+1)!} \right)$$

$$= \left(\sum_{k=0}^{\infty} \frac{t^{2k} (-1)^k I}{2k!} \right) + \left(\sum_{k=0}^{\infty} \frac{t^{2k+1} (-1)^k A}{(2k+1)!} \right)$$

$$= cos(t) I + sin(t) A = \begin{bmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{bmatrix}$$

$$while$$

$$e^{tC} e^{tD} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} 1 - t^2 & -t \\ t & 1 \end{bmatrix}$$

$$so \quad e^{t(c+D)} \neq e^{tC} e^{tD}$$

$$similarly \quad \boxed{e^{(c+D)} \neq e^{C} e^{D}}$$

(b) Are there any special conditions on C and D under which it will always hold ?

Solution: from Taylor series

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \quad where \quad t \in \mathbb{R} \quad and \quad A \in \mathbb{C}^{n\times n}$$

$$e^{A} = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad if \quad t = 1$$

$$e^{C+D} = \sum_{k=0}^{\infty} \frac{(C+D)^k}{k!} \text{ putting } A = C + D \tag{1}$$

$$e^{C}e^{D} = \sum_{k=0}^{\infty} \frac{(C)^k}{k!} * \sum_{k=0}^{\infty} \frac{(D)^k}{k!} \tag{2}$$

let in case of $(P+Q)^2 \quad (P^2 + PQ + QP + Q^2) \neq (P^2 + 2PQ + Q^2) \quad as \quad PQ \neq QP$

where P,Q are matrix and matrix multiplications are not commutative

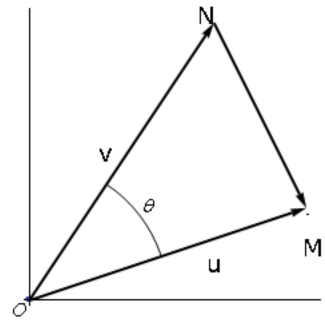
as we expand eq(1) we will get $(C+D)^k$ terms which coud Generate CD as well as DC

But in case of eq(2) we only get CD Terms no possibility of DC terms so

$e^{(C+D)} \neq e^c e^D$ **But if CD=DC** $e^{(C+D)} = e^c e^D$

6. Prove that if u, v are nonzero vectors in R^2, then $< u, v >=\| u \| \| v \| cos\theta$ where θ is the angle between u and v (thinking of u and v as arrows with initial point at the origin).

Solution:



From the above triangle MON ,the length of each side is only magnitude of the vector along that side.
the Law of Cosine tell us that

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\, cos\theta \tag{1}$$

using property of dot product we can write

$$\begin{aligned} \|\vec{u}-\vec{v}\|^2 &= (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) \\ &= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} \\ &= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \end{aligned} \tag{2}$$

equating both eq(2) and eq(1) we get

$$\begin{aligned} \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\, cos\theta \\ -2\vec{u}\cdot\vec{v} &= -2\|\vec{u}\|\|\vec{v}\|\, cos\theta \\ \vec{u}\cdot\vec{v} &= \|\vec{u}\|\|\vec{v}\|\, cos\theta \\ \boxed{< u, v > = \| u \| \| v \|\ cos\theta}\ &\textbf{(proved)} \end{aligned}$$

7. **Linear Dependence Lemma**
   If the list of vectors $(u_1, \ldots, u_n)$ is linearly dependent in some vector space V, and $u_1 \neq \mathbf{0}$, then prove that there exists an index $i \in \{2, \ldots, n\}$ such that $u_i \in span(u_1, \ldots, u_{i-1})$, and if u_i is removed, the span of the list remains unchanged.

Solution: let A=$(u_1, \ldots, u_n)$ as the linear dependence defination $\exists a_1, a_2, \ldots, a_n$ not all zero such that $a_1u_1 + a_2u_2 + \ldots a_nu_n = 0$
let $i \in \{2, \ldots n\}$ be the largest index such that $a_i \neq 0$.Then

$$u_i = -\frac{a_1}{a_i}u_1 - \ldots - \frac{a_{i-1}}{a_i}u_{i-1} \tag{1}$$

so u_i is a vector space where $(u_1, \ldots, u_{i-1})$ forms the basis

$$\text{so } u_i \in span(u_1, \ldots, u_{i-1})\textbf{(proved)}$$

let $A^* = A \setminus \{u_i\}$,as $A^* \subset A \implies span(A^*) \subset span(A)$
$Let \quad v \in span(A), \quad Then$
$v = b_1u_1 + b_{i-1}u_{i-1} + b_iu_i + b_{i+1}u_{i+1} + \ldots + b_nu_n$
Substitute (1) in $\mathbf{v} \quad for \quad \mathbf{u_i}$ and the sum is in terms of A^*, such that $v \in span(A^*)$
Thus $span(A) \subset span(A^*)$
combining both $span(A) \subset span(A^*)$ and $span(A^*) \subset span(A)$
$\implies span(A) \equiv span(A^*)$
so after removing u_i the span of the list remain unchanged **(proved)**

8. * **Independent Lists Cannot Be Arbitrarily Long**
Prove that for any finite dimensional vector space, the length of any independent list of vectors is always smaller than or equal to the length of any spanning list of vectors. (Hint: you may want to use the Linear Dependence Lemma)

Solution: Let $u_1, u_2, \cdots u_m$ is linearly independent in in vector space V suppose also that $v_1, v_2, \cdots v_n$ spans in V we need to prove that $m \leq n$ in step by step process we add one of u's and remove one of v's.
let $m > n$ and A be the list $v_1, v_2, \cdots v_n$, which spans V. Thus adjoining any vector in V to this list produces a linearly dependent list (as new vector can be written as Linear combination of other vector)
$u_1, v_1, v_2, \cdots v_n$ so by the Linear Dependence Lemma we can remove one of the vs so that the new list A (of length n) consisting of u1 and the remaining vs spans V.

Doing so we will cross ith step we will add all u_i to the list and remove v_i.

The process stops when either u is run out $(m \leq n)$ or we run out of v's $(m > n)$.if $(m > n)$ then span $(v_1, v_2 \ldots, v_n) = V$ and $m > n$ means $v_n \notin span(v_1, v_2, \cdots v_n) = V$, but this is a contradiction to our assumption since $\forall i \quad v_i \in V$
so m must be $m \leq n$.

So the length of any independent list of vectors is always smaller than or equal to the length of any spanning list of vectors.**(proved)**

9. **Cyclic Differences**
Consider the two lists of vectors $A = (u_1, u_2, u_3, \ldots, u_{n-1}, u_n)$, and $B = (u_1 - u_2, u_2 - u_3, u_3 - u_4, \ldots, u_{n-1} - u_n, u_n)$ (the last element is the same as A) in some vector space V. Prove or disprove the following statements:

   1. If A is linearly independent, then so is B.
   2. If A is spanning list, then so is B.

Solution:
1.
in case of B

$$a_1(u_1 - u_2) + a_2(u_2 - u_3) + \ldots + a_{n-1}(u_{n-1} - u_n) + a_n u_n = 0$$

rearranging the equation gives

$$a_1 u_1 + (a_2 - a_1)u_2 + \ldots + (a_n - a_{n-1})u_n = 0$$

As $(u_1, u_2, u_3, \ldots, u_{n-1}, u_n)$ are linearly independent

$$a_1 = 0$$
$$a_i - a_{i-1} = 0 \quad for \quad 2 \leq i \leq n$$
so we claim that $a_i = 0 \quad for \quad 1 \leq i \leq n$
by Induction base case $a_1 = 0$
let assume it's true for some t such that $\quad 2 \leq i < n$
$Then \quad a_{t+1} = a_t = 0$ which completes the proof.
so if $A = (u_1, u_2, u_3, \ldots, u_{n-1}, u_n)$ is linearly Independent then also
$B = (u_1 - u_2, u_2 - u_3, u_3 - u_4, \ldots, u_{n-1} - u_n, u_n)$ is linearly independent

2.
let $u \in V$. since $u_1, \ldots, u_n$ spans V so we can write $\exists(a_1, \ldots, a_n)$ such that

$$a_1 u_1 + a_2 u_2 + \ldots + a_n u_n = u \tag{1}$$

we want to find coefficient c_i such that

$$c_1(u_1 - u_2) + c_2(u_2 - u_3) + \ldots + c_{n-1}(u_{n-1} - u_n) + c_n u_n = u \tag{2}$$

as $eq(1) = eq(2)$ so R.H.S can written as

$$c_1 u_1 + (c_2 - c_1)v_2 + \ldots + (c_n - c_{n-1})u_n$$

so compairing both the coefficients we can see

Solution:

$$a_1 = c_1$$
$$a_i = c_i - c_{i-1} \quad for \quad 2 \leq i \leq n$$
$$so \quad c_2 = a_2 + c_1$$
$$c_3 = a_3 + c_2 = a_3 + a_2 + a_1$$
so by induction we can say that
$$c_i = \sum_{i=1}^{n} a_i$$
so eq(2) becomes
$$u = a_1 u_1 + (a_2 + a_3)(u_2 - u_3) + \ldots + (\sum_{i=1}^{n} a_i) u_n$$
so as $u \in span(u_1, \ldots u_n)$ so the list

$(u_1 - u_2, u_2 - u_3, u_3 - u_4, \ldots, u_{n-1} - u_n, u_n)$ also spans V

10. Compute and compare the L1 norm and Frobenius norm of the matrices given below.

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 3 & -6 \\ 1 & 8 & 9 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 1 \\ 8 & 3 & -6 \\ 9 & 2 & 1 \end{bmatrix}$$

Solution:

$$\boxed{l_1 \quad norm \quad \|A\|_1 = \max_{1\leqslant j\leqslant n}\sum_{i=1}^{m}|a_{ij}|}$$

$$\boxed{Frobenius \quad norm \quad \|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2}}$$

$$l_1 \quad norm \quad of \quad \begin{bmatrix} 4 & -2 & 1 \\ 2 & 3 & -6 \\ 1 & 8 & 9 \end{bmatrix}$$
$= max((|4| + |2| + |1|), (|-2| + |3| + |8|), (|1| + |-6| + |9|))$
$= max(7, 13, 16)$
$\boxed{= 16}$

$$l_1 \quad norm \quad of \quad \begin{bmatrix} 4 & -2 & 1 \\ 8 & 3 & -6 \\ 9 & 2 & 1 \end{bmatrix}$$
$= max((|4| + |8| + |9|), (|-2| + |3| + |2|), (|1| + |-6| + |1|))$
$= max(21, 7, 8)$
$\boxed{= 21}$

$$Frobenius \quad norm \quad of \quad \begin{bmatrix} 4 & -2 & 1 \\ 2 & 3 & -6 \\ 1 & 8 & 9 \end{bmatrix}$$
$=\sqrt{(|4|^2 + |2|^2 + |1|^2) + (|-2|^2 + |3|^2 + |8|^2) + (|1|^2 + |-6|^2 + |9|^2))}$

$=\sqrt{16 + 4 + 1 + 4 + 9 + 64 + 1 + 36 + 81}$

$=\sqrt{216}$
$\boxed{= 14.6969}$

$$Frobenius \quad norm \quad of \quad \begin{bmatrix} 4 & -2 & 1 \\ 8 & 3 & -6 \\ 9 & 2 & 1 \end{bmatrix}$$
$=\sqrt{(|4|^2 + |8|^2 + |9|^2) + (|-2|^2 + |3|^2 + |2|^2) + (|1|^2 + |-6|^2 + |1|^2))}$

$=\sqrt{16 + 64 + 81 + 4 + 9 + 4 + 1 + 36 + 1}$

$=\sqrt{216}$
$\boxed{= 14.6969}$

11. * **Induced Matrix Norms**

In case you didn't already know, a norm $\|.\|$ is any function with the following properties:

1. $\|x\| \geq 0$ for all vectors x.
2. $\|x\| = 0 \iff x = \mathbf{0}$.
3. $\|\alpha x\| = |\alpha|\|x\|$ for all vectors x, and real numbers α.
4. $\|x + y\| \leq \|x\| + \|y\|$ for all vectors x, y.

Now, suppose we're given some vector norm $\|.\|$ (this could be L2 or L1 norm, for example). We would like to use this norm to measure the size of a matrix A. One way is to use the corresponding induced matrix norm, which is defined as $\|A\| = \sup_x\{\|Ax\| : \|x\| = 1\}$.

E.g.: $\|A\|_2 = \sup_x\{\|Ax\|_2 : \|x\|_2 = 1\}$, where $\|.\|_2$ is the standard L2 norm for vectors, defined by $\|x\|_2 = \sqrt{x^T x}$.

Prove the following properties for an arbitrary induced matrix norm:

(a) $\|A\| \geq 0$.

Solution:

By Defination $\|Ax\| \geq 0$ for any x using $\|x\| \geq 0$, so $\|A\|\|x\| \geq 0 \implies \|A\| \geq 0$

(b) $\|\alpha A\| = |\alpha|\|A\|$ for any real number α.

Solution:

$$\|\alpha A\| = \big\|(\alpha A)x\big\| = \big\|\alpha(Ax)\big\| \leq |\alpha|\|Ax\| \leq |\alpha|\|A\|$$

(c) $\|A + B\| \leq \|A\| + \|B\|$.

Solution:

$$\begin{aligned}
\|A + B\| &= \max_{\|x\|=1}\big\|(A + B)x\big\| \\
&\leq \max_{\|x\|=1}\|Ax\| + \|Bx\| \\
&\leq \max_{\|x\|=1}\|A\|\|x\| + \|B\|\|x\| \\
&\leq \|A\| + \|B\|
\end{aligned}$$

(d) $\|A\| = 0 \iff A = 0$.

Solution: $\|Ax\| = 0 \iff A = 0$ since $\|A\|$ is calculated from the max$\|A\|$ evaluated on the unit sphere

(e) $\|AB\| \leq \|A\|\|B\|$.

Solution: by Defination

$$\begin{aligned}
\|AB\| &= \max_{x\neq 0} \frac{\|ABx\|}{\|x\|} \\
&\leq \max_{x\neq 0} \frac{\|A\|\|Bx\|}{\|x\|} \\
&\leq \max_{x\neq 0} \frac{\|A\|\|B\|\cancel{\|x\|}}{\cancel{\|x\|}} \\
&\boxed{\leq \max_{x\neq 0} \|A\|\|B\| \textbf{ (Proved)}}
\end{aligned}$$

(f) $\|A\|_2 = \sigma_{\max}(A)$, where $\sigma_{\max}$ is the largest singular value.

Solution:

$$\begin{aligned}
\|A\|_2 &= \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \\
&= \sup_{x\neq 0} \frac{\left\|U \sum V^T x\right\|_2}{\|x\|_2} \\
&= \sup_{x\neq 0} \frac{\left\|\sum V^T x\right\|_2}{\|x\|_2} \\
&= \sup_{y\neq 0} \frac{\left\|\sum y\right\|_2}{\|Vy\|_2} \\
&= \sup_{y\neq 0} \frac{\left\|\sum y\right\|_2}{\|y\|_2} \\
&= \sup_{y\neq 0} \frac{(\sum_{i=1}^{n} \sigma_i^2 |y_i|^2)^{\frac{1}{2}}}{(\sum_{i=1}^{n} |y_i|^2)^{\frac{1}{2}}} \leq \sigma_{\max}(A)
\end{aligned}$$

12. Prove that the eigen vectors of a real symmetric(S_{n*n}) matrix are linearly independent and forms a orthogonal basis for R^n.

Solution: let $S^T = S$ have eigen vectors $\vec{u_1}$ and $\vec{u_2}$ for eigen values $\lambda_1 \neq \lambda_2$.
let compute dot product of

$$(S\vec{u_1}) \cdot \vec{u_2} = (\lambda_1 \vec{u_1}) \cdot \vec{u_2} = \lambda_1(\vec{u_1} \cdot \vec{u_2}) \tag{1}$$

on the L.H.S we can write

$$(S\vec{u_1}) \cdot \vec{u_2} = (S\vec{u_1})^T \cdot \vec{u_2} = \vec{u_1}^T S^T \cdot \vec{u_2} = \vec{u_1}^T \cdot (S\vec{u_2}) = \vec{u_1} \cdot (\lambda_2 \vec{u_2}) \tag{2}$$

so from eq(1) and eq(2)

$\lambda_1(\vec{u_1} \cdot \vec{u_2}) = \lambda_2(\vec{u_1} \cdot \vec{u_2})$ since $\lambda_1 \neq \lambda_2$ by hypothesis so $\vec{u_1} \cdot \vec{u_2} = 0$
so for different eigen value symmetric matrix eigen vectors are orthogonal.
for linearly independent , we know that

$$a_1u_1 + a_2u_2 + \cdots + a_ku_k = 0.$$
$$when \quad a_1 = a_2 = \cdots = a_k = 0$$

Let compute the dot product of u_i and the above linear combination for each $1 \leq i \leq k$

$$0 = u_i \cdot 0 = u_i \cdot (a_1u_1 + a_2u_2 + \cdots + a_ku_k)$$
$$= a_1u_i \cdot u_1 + a_2u_i \cdot u_2 + \cdots + a_ku_i \cdot u_k.$$

we know $u_i \cdot u_j = 0 \quad if \quad i \neq j$

so all terms but i-th term are zero so we have

$$0 = a_iu_i \cdot u_i = a_i\|u_i\|^2.$$

as u_i is non zero vector its length $\|u_i\|$ is non zero

$$so \quad a_i = 0$$

this way for $i = 1, 2, \ldots, k$ can colclude that

$$a_1 = a_2 = \cdots = a_k = 0$$

The eigen vectors of a real symmetric(S_{n*n}) matrix are linearly independent and forms a orthogonal basis for R^n

13. If A_{n*n} is a square symmetric matrix. Prove that solution to the equation $\max_x \{x^T Ax \mid \|x\| = 1\}$ is given by the largest eigen value of A, when x is the eigen vector corresponding to largest eigen value.

Solution: using cayley-hamilton theorem the characteristic equation of the Matrix

$$Ax = \lambda x \text{ where } \lambda \text{ is eigen value}$$
$$\text{and x is eigen vector corrosponding to the eigen value } (\lambda_1, \lambda_2, \cdots, \lambda_n)$$
$$max_x\{x^T Ax\} = max_x\{x^T \lambda x\} \text{ so for every eigen value we replace } Ax \text{ with } \lambda x$$
$$= max_x\{\lambda x^T x\} = max_x\{\lambda x^T x\} = max_x\{\lambda \|x\|^2\} = max_x\{\lambda * 1\}$$
$$\text{so it will return the maximum eigen value from}$$
$$\boxed{= max_x\{\lambda_1, \lambda_2, \cdots, \lambda_n\}\textbf{(proved)}}$$

14. Prove that a full rank square matrix A_{n*n} is always similar to some diagonal matrix D_{n*n}.

Solution: As Ais full rank matrx then it's has non zero eigen values $\lambda_1, \lambda_2 \cdots \lambda_n$ and corrosponding eigrn vectors which are linearly independent $x_1, x_2, \cdots, x_n$

let $U = \begin{bmatrix} x_1, x_2, \cdots, x_n \end{bmatrix}$ where $(x_1, x_2, \cdots, x_n)$ are following $\begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{bmatrix}, \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{bmatrix}, \cdots, \begin{bmatrix} x_{n1} \\ x_{n2} \\ \vdots \\ x_{nn} \end{bmatrix}$

and eigen values matrix $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ which is a Diagonal Matrix

$$
\begin{aligned}
AU &= A \begin{bmatrix} x_1, x_2, \cdots, x_n \end{bmatrix} \\
&= \begin{bmatrix} Ax_1, Ax_2, \cdots, Ax_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 x_1, \lambda_2 x_2, \cdots, \lambda_n x_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 x_{11} & \lambda_2 x_{21} & \cdots & \lambda_n x_{n1} \\ \lambda_1 x_{12} & \lambda_2 x_{22} & \cdots & \lambda_n x_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 x_{1n} & \lambda_2 x_{2n} & \cdots & \lambda_n x_{nn} \end{bmatrix} \\
&= \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = UD
\end{aligned}
$$

so $A = UDU^{-1}$ multiplying U^{-1}on rightside of the both in L.H.S and R.H.S
So full rank square matrix A_{n*n} is always similar to some diagonal matrix D_{n*n}**(proved)**.

15. Consider two vectors x and y separated by angle θ. Suppose an orthonormal transformation represented by matrix A_{n*n} is applied to vectors x and y. Find the relation between θ and the angle between the newly transformed vectors Ax and Ay.

Solution:

by defination of dot product

$$x \cdot y = \parallel x \parallel \parallel y \parallel cos(\theta) = x^T y$$

transformed x is Ax

transformed y is Ay

let the new angle between x and y is $\phi \quad so$

$$Ax \cdot Ay = (Ax)^T Ay$$

$$= x^T A^T Ay = x^T y (\text{as A is orthonormal matrix so } A^T A = 1)$$

$$= \parallel x \parallel \parallel y \parallel cos(\phi)$$

$$\boxed{cos(\phi) = cos(\theta)}$$

16. Let $u_1, u_2, ..., u_n$ be a set of n orthonormal vectors. Similarly let $v_1, v_2, ..., v_n$ be another set of n orthonormal vectors.

 (a) Show that $u_1 v_1^T$ is a rank-1 matrix.

Solution:

for matrices u_1 and v_1 it is defined as

$$\text{rank}\, u_1 v_1^T \leq \min\{\text{rank}\, u_1^T, \text{rank}\, v_1^T\}$$

$$\text{rank}\, u_1 v_1^T \leq \text{rank}\, u_1$$

is because the column space of $u_1 v_1^T$ is contained in the column space of u_1.
using rank-nullity theorem, rank $u_1 v_T \leq$ rank v_T so the Null space of v_1^T is also contained in the null space of $u_1 v_1^T$.so $\dim N(v_T) \leq \dim N(u_1 v_1^T)$.so using Nulity theorem we can say that $\boxed{rank(u_1 v_1^T) = rank(u_1) = rank(v_1^T) = 1}$

 (b) Show that $u_1 v_1^T + u_2 v_2^T$ is a rank-2 matrix.

Solution:

$$u_1v_1^T + u_2v_2^T = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}$$

$$= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \text{using the form of SVD of } U \sum V^T$$

so as there are one Identity $I_{2\times 2}$ matrix of rank 2 so maximum rank of $u_1v_1^T + u_2v_2^T$ is 2

(c) Show that $\sum_{i=1}^{n} u_iv_i^T$ is a rank-n matrix.

Solution:

$$\sum_{i=1}^{n} u_iv_i^T = \begin{bmatrix} u_1 \dots u_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \dots u_n \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{I_N} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \text{using the form of SVD of } U \sum V^T$$

so as there are one Identity $I_{n\times n}$ matrix of rank n so maximum rank of $\sum\limits_{i=1}^{n} u_iv_i^T$ is n