

Def: A vector is a tuple of numbers.

Def: The dimension is the size of this tuple.

Def: A matrix is a 2-dimensional grid of numbers.

Def: $\mathbb{R}^{m \times n}$ denotes all $m \times n$ matrices with field \mathbb{R} .

Def: $\mathbb{C}^{m \times n}$ denotes all $m \times n$ matrices with field \mathbb{C} .

Def: vector-matrix product: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$ (Inner dimensions must agree. Outer dimensions remain.)

$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$: Express matrix-vector multiplication as a linear combination of the matrix.

Def: Inner product $< \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} > = [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 +$

$x_2 y_2 + \cdots + x_n y_n$

Def: Matrix Multiplication: $A \in \mathbb{R}^{m \times k}, B \in \mathbb{R}^{k \times n}, C = AB \in \mathbb{R}^{m \times n}$ and $C_{ij} = \sum_{k=1}^n A_{ik} B_{kj}$.

Def: span of vectors v_1, v_2, \dots, v_n is the set of vectors that can be obtained as linear combination of the vectors v_1, v_2, \dots, v_n :

$\text{span}\{v_1, v_2, \dots, v_n\} = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$ for a_1, a_2, \dots, a_n are scalars.

Def: A set of vector v_1, v_2, \dots, v_n is said to be linearly independent if none of these vectors can be expressed as a linear combination of others.

Solving a system of linear equations

Def: Augmented matrix - append \bar{b} to A , where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$.

Def: A pivot in a row is its leftmost non-zero element.

Note: if the column space of A , $\mathcal{C}(A)$ is LD, there is no soln or infinite number solns.

Matrix Multiplication: $AB = C$ iff $C_{ik} = < A_i^T, B_k >$.

Application: find the walk of length n from node i to j in a graph. The answer will be A_{ij}^n , where A is the adjacency matrix of the graph.

Thm: If $A^{-1}A = I$ and $A\tilde{A}^{-1} = I$, then $A^{-1} = \tilde{A}^{-1}$

Def: A is symmetric iff $A^T = A$.

Def: A matrix whose transpose is also its inverse is called orthogonal. $A^T A = I \implies < \text{col } i, \text{col } j > = \delta_{ij}$.

Def: length of a vector. $\|v\| = \sqrt{< v, v >}$. (Also known as L2 norm).

Def: Angle between two vectors. $\frac{< v, w >}{\|v\| \|w\|} = \cos \theta$.

Def: Permutation is the operation where in each element in the pre-image appears exactly once in the image.

Properties of a permutation matrix:

- square matrix
- there is a one '1' per every row/col

- all rows are distinct, all columns are distinct.

Suppose A and B are both permutation matrices of the same dimension. AB is another permutation.

Permutations don't change the length of the vector and the between vectors.

Def: Transpose $A_{ij}^T = A_{ji}$. A does NOT need to be square.

Thm: If A is a permutation matrix, then its transpose is its inverse. Moreover, A is also an orthogonal matrix.

Rotation: matrix $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$, one can show that apply a rotation matrix to a vector does NOT alter its length. Rotation matrices are orthogonal.

Vector Space

Def: A vector Space over \mathbb{R} is a set V with rules

1. closed under vector addition.
2. closed under scalar multiplication.

Def: If V is a vector space, S is a subspace of V if

1. $\forall v, w \in S : v + w \in S$
2. $\forall a \in \mathbb{R}, \forall v \in S : av \in S$
3. $\vec{0} \in S$

Def: Column Space of matrix A is the set of all linear combination of columns of A .

Consider $A_{n \times n}$ a square matrix, A is invertible iff $\mathcal{C}(A) = \mathbb{R}^n$.

Def: Null space of a matrix A : $\mathcal{N}(A) = \{x | Ax = 0\}$.

Def: Column rank is the number of linearly independent column.

Def: A collection of vectors $U = \{v_1, v_2, \dots, v_n\}$ is said to be linearly independent if $\sum_{i=1}^n a_i v_i = \vec{0}$ only has $a_1 = a_2 = \dots = a_n = 0$ as the solution. Moreover, $\text{span}\{U\} = \{\sum_{i=1}^n a_i v_i\}$.

Def: A set of vectors U is said to be a basis for a vector space V if

1. $\text{span}(U) = V$.
2. U is LI.

Def: Two vectors (in the same vector space) are orthogonal if the angle between them is 90° .

Def: Two subspaces $V, W \subseteq \mathbb{R}^n$ are orthogonal if $\forall v \in V \forall w \in W : v^T w = 0$.

Def: Given $V \subseteq \mathbb{R}^n$, its orthogonal complement is $V^\perp = \{w \in \mathbb{R}^n | w^T v = 0 \forall v \in V\}$.

Def: orthogonal decomposition If V and W form a pair of subspaces that are orthogonal complements of each other in \mathbb{R}^n , then any vector $z \in \mathbb{R}^n$ can be written as $z = v + w$, where $v \in V, w \in W$.

Def: projection of v onto w : $proj_w(v) = \frac{\langle v, w \rangle}{\langle w, w \rangle} w = aw$

Def: Gram-Shmidts procedure $\begin{cases} w_1 = v_1 \\ w_n = v_n - \sum_{i=1}^{n-1} proj_{w_i} v_n \end{cases}$ and finally nor-

malize all vector w .

Linear transformation and determinant

Def: Suppose V and W are vector spaces. $\mathcal{L} : V \rightarrow W$ is a linear transformation if $\mathcal{L}(\alpha v + \beta w) = \alpha \mathcal{L}(v) + \beta \mathcal{L}(w) \forall \alpha, \beta \in \mathbb{R}, \forall v, w \in V$.

Determinant of a square matrix - to understand how the volume of a set S changes under the transformation Φ . $\det(AB) = \det(A) \cdot \det(B)$.

Thm: Consider matrix $A \in \mathbb{R}^{m \times n}$. $[\mathcal{N}(A)]^\perp = \mathcal{C}(A^T)$.

Thm: Consider $V \subseteq \mathbb{R}^n$. $\dim(V) + \dim(V^\perp) = n$.

Thm: If $A \in \mathbb{R}^{m \times n}$ then $Rank(A) + nullity(A) = n$, where $nullity(A) = \dim(\mathcal{N}(A))$.

Thm: Suppose matrix A has both a left inverse and a right inverse, then they are the same, and are unique.

Thm: One right inverse (if it exists) is $A^{-R} = A^T(AA^T)^{-1}$.

Thm: One left inverse (if it exists) is $A^{-l} = (A^T A)^{-1} A^T$.

Thm: Consider $A \in \mathbb{R}^{m \times n}$. A has a right inverse iff it has full row rank.

Thm: Consider $A \in \mathbb{R}^{m \times n}$. A has a left inverse iff it has full column rank.