Def: A <u>vector</u> is a tuple of numbers.
Def: The <u>dimension</u> is the size of this tuple.
Def: A <u>matrix</u> is a 2-dimensional grid of numbers.
Def: $\mathbb{R}^{m\times n}$ denotes all $m \times n$ matrices with field $\mathbb{R}$.
Def: $\mathbb{C}^{m\times n}$ denotes all $m \times n$ matrices with field $\mathbb{C}$.
Def: vector-matrix product: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$ (Inner dimensions must agree. Outer dimensions remain.)
$\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$: Express matrix-vector multiplication as a linear combination of the matrix.
Def: <u>Inner product</u> $< \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} >= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$
Def: <u>Matrix Multiplication:</u> $A \in \mathbb{R}^{m\times k}, B \in \mathbb{R}^{k\times n}$, $C = AB \in \mathbb{R}^{m\times n}$ and $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$.
Def: <u>span</u> of vectors $v_1, v_2, \ldots, v_n$ is the set of vectors that can be obtained as linear combination of the vectors $v_1, v_2, \ldots, v_n$:
$span\{v_1, v_2, \ldots, v_n\} = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ for $a_1, a_2, \ldots, a_n$ are scalars.
Def: A set of verctor $v_1, v_2, \ldots, v_n$ is said to be <u>lienarly independent</u> if none of these vectors can be expressed as a linear combination of others.

<u>Solving a system of linear equations</u>
Def: Augmented matrix - append b to A, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$.
Def: A <u>pivot</u> in a row is its leftmost non-zero element.
Note: if the column spafe of A, $\mathcal{C}(A)$ is LD, there is no soln or infinite number solns.
<u>Matrix Multiplication</u>: $AB = C$ iff $C_{ik} =< A_i^T, B_k >$.
Application: find the walk of length n from node i to j in a graph. The answer will be A_{ij}^n, where A is the adjacency matrix of the graph.
Thm: If $A^{-1}A = I$ and $A\tilde{A}^{-1} = I$, then $A^{-1} = \tilde{A}^{-1}$
Def: A is <u>symmetric</u> iff $A^T = A$.
Def: A matrix whose transpose is also its inverse is called <u>orthogonal</u>. $A^TA = I \implies < col\ i, col\ j >= \delta_{ij}$.
Def: <u>length</u> of a vector. $\|v\| = \sqrt{< v, v >}$. (Also known as L2 norm).
Def: <u>Angle</u> between two vectors. $\frac{<v,w>}{\|v\|\|w\|} = \cos\theta$.

Def: <u>Permutation</u> is the operation where in each element in the pre-image appears exactly once in the image.
Properties of a permutation matrix:

- square matrix
- there is a one '1' per every row/col

- all rows are distinct, all columns are distinct.

Suppose A and B are both permutation matrices of the same dimension. AB is another permutation.
Permutations don't change the length of the vector and the between vectors.
Def: Transpose $A^T_{ij} = A_{ji}$. A does NOT need to be square.
Thm: If A is a permutation matrix, then its transpose is its inverse. Moreover, A is also an orthogonal matrix.

Rotation: matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, one can show that apply a rotation matrix to a vector does NOT alter its length. Rotation matrices are orthogonal.

Vector Space
Def: A vector Space over $\mathbb{R}$ is a set V with rules

1. closed under vector addition.
2. closed under scalar multiplication.

Def: If V is a vector space, S is a subspace of V if

1. $\forall v, w \in S : v + w \in S$
2. $\forall a \in \mathbb{R}, \forall v \in S : av \in S$
3. $\vec{0} \in S$

Def: Column Space of matrix A is the set of all linear combination of columns of A.
Consider $A_{n\times n}$ a square matrix, A is invertible iff $\mathcal{C}(A) = \mathbb{R}^n$.
Def: Null space of a matrix A: $\mathcal{N}(A) = \{x | Ax = 0\}$.
Def: Column rank is the number of linearly independent column.