Def: A vector is a tuple of numbers.
Def: The dimension is the size of this tuple.
Def: A matrix is a 2-dimensional grid of numbers.
Def: $\mathbb{R}^{m\times n}$ denotes all $m \times n$ matrices with field $\mathbb{R}$.
Def: $\mathbb{C}^{m\times n}$ denotes all $m \times n$ matrices with field $\mathbb{C}$.
Def: vector-matrix product: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$ (Inner dimensions must agree. Outer dimensions remain.)
$\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$: Express matrix-vector multiplication as a linear combination of the matrix.
Def: Inner product $< \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} >= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$
Def: Matrix Multiplication: $A \in \mathbb{R}^{m\times k}, B \in \mathbb{R}^{k\times n}$, $C = AB \in \mathbb{R}^{m\times n}$ and $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$.
Def: span of vectors $v_1, v_2, \ldots, v_n$ is the set of vectors that can be obtained as linear combination of the vectors $v_1, v_2, \ldots, v_n$:
$span\{v_1, v_2, \ldots, v_n\} = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ for $a_1, a_2, \ldots, a_n$ are scalars.
Def: A set of verctor $v_1, v_2, \ldots, v_n$ is said to be lienarly independent if none of these vectors can be expressed as a linear combination of others.

Solving a system of linear equations
Def: Augmented matrix - append b to A, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$.
Def: A pivot in a row is its leftmost non-zero element.
Note: if the column spafe of A, $\mathcal{C}(A)$ is LD, there is no soln or infinite number solns.
Matrix Multiplication: $AB = C$ iff $C_{ik} =< A_i^T, B_k >$.
Application: find the walk of length n from node i to j in a graph. The answer will be A_{ij}^n, where A is the adjacency matrix of the graph.
Thm: If $A^{-1}A = I$ and $A\tilde{A}^{-1} = I$, then $A^{-1} = \tilde{A}^{-1}$
Def: A is symmetric iff $A^T = A$.
Def: A matrix whose transpose is also its inverse is called orthogonal. $A^TA = I \implies < col\ i, col\ j >= \delta_{ij}$.
Def: length of a vector. $\|v\| = \sqrt{< v, v >}$. (Also known as L2 norm).
Def: Angle between two vectors. $\frac{<v,w>}{\|v\|\|w\|} = \cos\theta$.

Def: Permutation is the operation where in each element in the pre-image appears exactly once in the image.
Properties of a permutation matrix:

- square matrix
- there is a one '1' per every row/col

- all rows are distinct, all columns are distinct.

Suppose A and B are both permutation matrices of the same dimension. AB is another permutation.
Permutations don't change the length of the vector and the between vectors.
Def: Transpose $A^T_{ij} = A_{ji}$. A does NOT need to be square.
Thm: If A is a permutation matrix, then its transpose is its inverse. Moreover, A is also an orthogonal matrix.

Rotation: matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, one can show that apply a rotation matrix to a vector does NOT alter its length. Rotation matrices are orthogonal.

Vector Space
Def: A vector Space over $\mathbb{R}$ is a set V with rules

1. closed under vector addition.
2. closed under scalar multiplication.

Def: If V is a vector space, S is a subspace of V if

1. $\forall v, w \in S : v + w \in S$
2. $\forall a \in \mathbb{R}, \forall v \in S : av \in S$
3. $\vec{0} \in S$

Def: Column Space of matrix A is the set of all linear combination of columns of A.
Consider $A_{n\times n}$ a square matrix, A is invertible iff $\mathcal{C}(A) = \mathbb{R}^n$.
Def: Null space of a matrix A: $\mathcal{N}(A) = \{x | Ax = 0\}$.
Def: Column rank is the number of linearly independent column.
Def: A collection of vectors $U = \{v_1, v_2, \ldots, v_n\}$ is said to be linearly independent if $\sum_{i=1}^n a_i v_i = \vec{0}$ only has $a_1 = a_2 = \cdots = a_n = 0$ as the solution. Moreover, $span\{U\} = \{\sum_{i=1}^n a_i v_i\}$.
Def: A set of vectors U is said to be a basis for a vector space V if

1. $span(U) = V$.
2. U is LI.

Def: Two vectors (in the same vector space) are orthogonal if the angle between them is 90°.
Def: Two subspaces $V, W \subseteq \mathbb{R}^n$ are orthogonal if $\forall v \in V \forall w \in W : v^T w = 0$.
Def: Given $V \subseteq \mathbb{R}^n$, its orthogonal complement is $V^\perp = \{w \in \mathbb{R}^n | w^T v = 0 \ \forall v \in V\}$.
Def: orthogonal decomposition If V and W form a pair of subspaces that are orthogonal complements of each other in $\mathbb{R}^n$, then any vector $z \in \mathbb{R}^n$ can be written as $z = v + w$, where $v \in V, w \in W$.

Def: projection of v onto w: $proj_w(v) = \frac{<v,W>}{<w,w>} w = aw$

Def: Gram-Shmidts procedure $\begin{cases} w_1 = v_1 \\ w_n = v_n - \sum_{i=1}^{n-1} proj_{w_i} v_n \end{cases}$ and finally normalize all vector w.

Linear transformation and determinant

Def: Suppose V and W are vector spaces. $\mathcal{L} : V \to W$ is a linear transformation if $\mathcal{L}(\alpha v + \beta w) = \alpha\mathcal{L}(v) + \beta\mathcal{L}(w)\ \forall \alpha, \beta \in \mathbb{R}, \forall v, w \in V$.

Determinant of a square matrix - to understand how the volume of a set S changes under the transformation Φ. $\det(AB) = \det(A) \cdot \det(B)$.

Thm: Consider matrix $A \in \mathbb{R}^{m \times n}$. $[\mathcal{N}(A)]^{\perp} = \mathcal{C}(A^T)$.

Thm: Consider $V \subseteq \mathbb{R}^n$. $\dim(V) + \dim(V^{\perp}) = n$.

Thm: If $A \in \mathbb{R}^{m \times n}$ then $Rank(A) + nullity(A) = n$, where $nullity(A) = \dim(\mathcal{N}(A))$.

Thm: Suppose matrix A has both a left inverse and a right inverse, then they are the same, and are unique.

Thm: One right inverse (if it exists) is $A^{-R} = A^T(AA^T)^{-1}$.

Thm: One left inverse (if it exists) is $A^{-l} = (A^T A)^{-1} A^T$.

Thm: Consider $A \in \mathbb{R}^{m \times n}$. A has a right inverse iff it has full row rank.

Thm: Consider $A \in \mathbb{R}^{m \times n}$. A has a left inverse iff it has full column rank.

Def: projection on a vector space. Let V be a subspace of $\mathbb{R}^n$ and $w \in \mathbb{R}^n$. denot $V_w = proj_V w$

Def: Given a subspace $V \subseteq \mathbb{R}^n$, we can project w on V as follows

1. $V_w \in V$.
2. $(w - V_w) \perp V$.

Def: TODO: place holder for definition for projection matrix.

Def:SVD: $A \in \mathbb{R}_r^{m \times n}$ where $r \leq \min\{m, n\}$. We can decompose A into $A = U\Sigma V^T$, where $U \in \mathbb{R}^{m \times n}, V \in \mathbb{R}^{n \times n}$ and $\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$, where $S = diag(\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_r)$. Moreover, U and V are orthogonal ($U^T U = V^T V = I$)