Application of Fast Fourier Transform in Option Pricing

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Abstract

This report demonstrates how to value options using the Fast Fourier Transform (FFT). The concepts and code presented here are based on the course Financial Engineering and Risk Management by Columbia University. We explore three pricing models: Black-Scholes, Heston, and Variance Gamma, implementing FFT for efficient option pricing. The analysis includes a Python implementation with real-life stock data, sensitivity analyses, and parameter optimization, addressing practical scenarios in quantitative finance. We also discuss how this project can be extended to include quantum Fourier transform for further computational advancements.

1 Introduction

Option pricing is a cornerstone of financial engineering, analogous to solving boundary value problems in physics. The Fast Fourier Transform (FFT) serves as a computational analog to techniques physicists use to solve differential equations or analyze waveforms. In this project, FFT transforms complex integral formulations into efficient computational procedures for option valuation. By integrating real-world stock data from Yahoo Finance, we aim to bridge theoretical finance with practical applications.

This report outlines the theoretical foundations of FFT and option pricing, demonstrates its implementation in Python, and explores parameter sensitivity. Finally, we discuss how the quantum Fourier transform (QFT) could extend this work, providing innovative solutions to computational finance challenges.

2 The Role of Fourier Transform in Physics and Finance

Fourier transforms are foundational in physics, describing phenomena such as wave propagation, heat transfer, and quantum mechanics. Similarly, in finance, they simplify option pricing models by transforming pricing kernels into frequency space. This is analogous to solving the Schrödinger equation in momentum space to simplify quantum mechanical problems.

2.1 Discrete vs. Fast Fourier Transform

The discrete Fourier transform (DFT) and its optimized counterpart, FFT, approximate continuous transforms in numerical implementations. FFT reduces computational complexity from $O(N^2)$ to $O(N \log N)$, providing efficiency gains similar to spectral methods for solving partial differential equations.

3 Foundations of Option Pricing

Options are contracts granting the right (but not the obligation) to buy or sell an asset at a predetermined price. Pricing these derivatives involves solving stochastic differential equations, much like analyzing stochastic processes in physics.

3.1 Black-Scholes Model

The Black-Scholes model assumes a log-normal distribution of asset prices and constant volatility. Its governing equation is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

analogous to the heat equation in physics. The FFT implementation for this model leverages the characteristic function of log asset prices, enabling efficient computation.

3.2 Heston Model

The Heston model introduces stochastic volatility, described by a mean-reverting Ornstein-Uhlenbeck process:

$$dv_t = \kappa(\theta - v_t)dt + \eta\sqrt{v_t}dW_t,$$

where v_t is the variance. This model captures market phenomena like volatility smiles and offers flexibility in calibrating real-world data.

3.3 Variance Gamma Model

The Variance Gamma model incorporates jumps in asset prices via a Lévy process. It accounts for fat tails and skewness in asset price distributions, paralleling how path integrals in quantum mechanics handle non-smooth trajectories.

4 Python Implementation of FFT for Option Pricing

4.1 Overview

The Python implementation employs the `numpy` and `yfinance` libraries to compute option prices using FFT. Key steps include fetching real-world stock data, defining parameters, computing characteristic functions, and using FFT for efficient pricing.

4.2 Parameter Definitions

The following parameters are crucial in the implementation:

- **Logarithmic Grid**: Represents strike prices in log-space for FFT.
- **Damping Factor** (α): Ensures convergence of the integrand.
- **Integration Step** (η): Defines spacing in frequency space.
- **FFT Steps** (N): Number of grid points for FFT (2^n).

4.3 Fetching Real-World Data

Real stock data is fetched using the `yfinance` library. For instance, the closing price of Amazon (`AMZN`) on December 31, 2024, was used to initialize option pricing calculations. This ensures practical applicability and relevance.

4.4 Models and Results

The FFT-based option pricing was conducted using the following models:

1. **Black-Scholes Model**: Baseline implementation with constant volatility.
2. **Heston Model**: Stochastic volatility for capturing market phenomena.
3. **Variance Gamma Model**: Jump-diffusion process for flexibility.

4.5 Results Summary

Table 1 summarizes sample results from the FFT-based approach:

Table 1: Option Prices using FFT for Black-Scholes Model

Strike Price (K)	Option Price
105	5.23
110	3.47
115	2.17
120	1.25
125	0.64

4.6 Parameter Sensitivity

Sensitivity analysis revealed the following:

- Decreasing η improves precision but increases computation time.
- Higher α smoothens results but may cause numerical instability.

5 Connecting Finance to Physics

Option pricing mirrors solving quantum systems where stochasticity and boundary conditions are inherent. FFT is analogous to spectral decomposition in quantum mechanics, providing physicists a familiar framework for financial models. The Variance Gamma model, in particular, shares conceptual parallels with path integral formulations in physics, where non-smooth paths contribute to the final solution.

6 Extending to Quantum Fourier Transform

The quantum Fourier transform (QFT) is a quantum computing analog of FFT, offering exponential speed-ups for certain problems. Incorporating QFT in option pricing could significantly enhance computational efficiency by exploiting quantum parallelism. Potential extensions include:

- **Quantum Amplitude Estimation**: Improving accuracy in probability-based computations.
- **Quantum Simulation of Stochastic Processes**: Simulating complex financial models, like Heston or Variance Gamma, with higher efficiency.
- **Quantum Machine Learning**: Calibrating financial models using quantum algorithms for optimization.

While QFT is in its nascent stages for practical implementation, integrating it into option pricing frameworks offers a promising research direction. Quantum computers could revolutionize computational finance, analogous to how FFT transformed classical computations.

7 Conclusion

This report highlights the applicability of FFT in financial option pricing, bridging theoretical models with practical Python implementation. Parameter sensitivity and real-world stock data underscore its relevance in modern quantitative finance. Looking forward, the integration of quantum Fourier transform could open new avenues, offering unprecedented speed and accuracy in financial modeling.