Lesson 1

Introduction

Ordinary Differential Equation

An ordinary differential equation (ODE) is an equation stating a relationship between a function of a single independent variable and the derivatives of this functions with respect to the independent variable. For example:

$$\psi(t, y, y' \ldots, y^m)$$

The order of an ODE is the order of the highest order derivative in the differential equation.

If no product of the dependent variable $y(t)$ with itself or any of its derivatives occurs, then the equation is called linear, otherwise it is non-linear.
Examples are

$$\begin{array}{ll} y'' + y = 0 & \text{linear} \\ y' + y^2 = 0 & \text{nonlinear} \\ y' + t^2 y = 0 & \text{linear} \\ y'' + \sin(y) = 0 & \text{nonlinear} \end{array}$$

The general first order ODE is of the form

$$\frac{dy}{dt} = f(t, y)$$

A general solution of an ODE of order m contains m arbitrary constants that can be determined by prescribing m conditions. There are two different classes of of ODE, depending on the type of auxiliary conditions specified.

Initial and Boundary Value Problem (IVP & BVP)

If all the auxiliary conditions are specified at the same value of the independent variable and the solution is to be marched forward from that initial point, the differential equation together with the specified condition is called an IVP.

If the auxiliary conditions are specified at two different values of the independent variable, the end point or at the boundaries of the domain of interest, the differential equation is called boundary value problem. For example:

$$y'' + P(t,y)y' + Q(t,y)y = F(t) \quad y(t_0) = c_1 \text{ and } y'(t_0) = c_2 \qquad (IVP)$$

$$y'' + P(t,y)y' + Q(t,y)y = F(t) \quad y(t_1) = d_1 \text{ and } y(t_2) = d_1 \qquad (BVP)$$

Reduction of higher order equations to the system of first order differential equations

Suppose that an n-th order equation can be solved for the n-th derivative, i.e., it can be written in the form:

$$y^{(n)} = f(t, y', y'', \ldots, y^{n-1})$$

This equation can now be written in a system of first order differential equations by a standard change of variables:

$$\begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \\ &\vdots \\ y_n &= y^{n-1}. \end{aligned}$$

Then, the resulting first-order system is the following:

$$
\begin{aligned}
y_1' &= y' = y_2 \\
y_2' &= y'' = y_3 \\
y_3' &= y''' = y_4 \\
&\vdots \\
y_n' &= y^n = f(t, y_2, y_3, \ldots, y_n).
\end{aligned}
$$

In vector form this can simply be written as

$$\mathbf{y}' = \mathbf{f}(\mathbf{t}, \mathbf{y})$$

where $\mathbf{y} = [y_1, y_2, \ldots, y_n]^T$ and $\mathbf{f} = [y_2, y_3, \ldots y_n, f]^T$.

Let us assume that the initial values for the nth order problem are given as

$$y(t_0) = y_0, \;\; y'(t_0) = y_1, \ldots, y^{n-1}(t_0) = y_{n-1}$$

Clearly, it follows

$$\mathbf{y}(\mathbf{t_0}) = [y_0, y_1, \ldots, y_{n-1}]^T.$$

Example 1.1 *Convert the second order IVP into a system of first order IVP*

$$2y'' - 5y' + y = 0$$

$$y(0) = 6; \quad y'(0) = -1;$$

Sol: Let

$$y_1 = y \quad y_2 = y'.$$

It follows then

$$
\begin{aligned}
y_1' &= y_2 \\
y_2' &= -\frac{1}{2}y_1 + \frac{5}{2}y_2
\end{aligned}
$$

and

$$y_1(0) = 6; \qquad y_2(0) = -1$$

Remark 1.2 *The methods of solution of first order initial value problem may be used to solve the system of first order initial value problems and the nth order initial value problem.*

Suggested Readings

A. Quarteroni, R. Sacco, F. Saleri (2007). Numerical Mathematics. Second Edition. Springer Berlin.

M.K. Jain, S.R.K. Iyengar, R.K. Jain (2009). Numerical Mathematics. Fifth Edition. New age international publishers, New Delhi.