Solution of Tutorial Problems set-II

Note: All these problems can be solved using the results of Chapter-2.

[**0.0.1**] ***Exercise*** Find a necessary and sufficient condition for $\langle x, y\rangle = \sum_{i=1}^{n} \alpha_i x_i y_i$ to be an inner product on $\mathbb{R}^n$.

Sol. We assume that $\langle x, y\rangle = \sum_{i=1}^{n} \alpha_i x_i y_i$ is an inner product on $\mathbb{R}^n$. Take e_i. Then $\langle e_i, e_i\rangle = \alpha_i > 0$ for $i = 1, \ldots, n$.

Converse: Assume that $\alpha_i > 0$ for $i = 1, \ldots, n$. To show $\langle x, y\rangle = \sum_{i=1}^{n} \alpha_i x_i y_i$ is an inner product on $\mathbb{R}^n$.

1(a). $\langle x, x\rangle = \sum_{i=1}^{n} \alpha_i x_i^2 > 0$ as $\alpha_i > 0$ for $i = 1, \ldots, n$.

1(b). $\langle x, x\rangle = \sum_{i=1}^{n} \alpha_i x_i^2 = 0 \implies x_i = 0$ for $i = 1, \ldots, n$.

2. It is trivial.

3. It is trivial.

4. It is trivial.

[**0.0.2**] ***Exercise*** Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2×2 matrix with real entries. Let $f_A : \mathbb{R}^2 \to \mathbb{R}$ be a map defined by $f_A(x, y) = y^t Ax$, where $x, y \in \mathbb{R}^2$. Show that f_A is an inner product on $\mathbb{R}^2$ if and only if $A = A^t$, $a_{11} > 0$, $a_{22} > 0$ and $det(A) > 0$.

Sol. We first assume that f_A is an inner product. Using definition of inner product, we have $f_A(x, x) > 0$ for all non-zero $x \in \mathbb{R}^2$. Then $f_A(e_1, e_1) = e_1^t Ae_1 = (1, 0) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} > 0 \implies$ $a_{11} > 0$. Using e_2, we can show that $a_{22} > 0$.

Using definition of inner product we have $f_A(x, y) = \overline{f_A(y, x)} = f_A(y, x)$ as this is a real inner product space. Therefore we have

$y^t Ax = x^t Ay$

$\implies (y^t Ax)^t = x^t Ay$

$\implies x^t A^t y = x^t Ay$

$\implies x^t(A^t - A)y = 0$. This is true for all $x, y \in \mathbb{R}^2$.

Take $x = (1,0)^t$ and $y = (0,1)^t$. Then we have $(1,0)\begin{pmatrix} 0 & a_{21} - a_{12} \\ a_{12} - a_{21} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$. This implies that $a_{21} - a_{12} = 0 \implies a_{12} = a_{21}$. Hence $A = A^t$.

To prove $\text{DET}\,(A) > 0$, we take $x = (a_{22}, -a_{12})$.

Since $f_A(x,x) > 0$, we have $(a_{22}, -a_{12})\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} a_{22} \\ -a_{12} \end{pmatrix} > 0$

Then $a_{22}\left(a_{11}a_{22} - a_{12}^2\right) > 0$.

$\left(a_{11}a_{22} - a_{12}^2\right) > 0$ as $a_{22} > 0$.

Hence $det(A) > 0$.

We now prove the converse.

1(a). $f_A(x,x) = (x_1, x_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$= a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \text{ (as } a_{12} = a_{21})$$

$$= a_{11}(x_1 + \tfrac{a_{12}}{a_{11}x_2})^2 + \tfrac{a_{11}a_{22} - a_{12}^2}{a_{11}}x_2^2 \text{ (as } a_{11} > 0)$$

$$> 0$$

1(b). $f_A(x,x) = 0$. Using $a_{11}(x_1 + \frac{a_{12}}{a_{11}x_2})^2 + \frac{a_{11}a_{22} - a_{12}^2}{a_{11}}x_2^2$, we have $x_1 = 0$ and $x_2 = 0$.

2. It is trivial.

3. It is trivia.

4. It is trivia.

[**0.0.3**] ***Exercise*** Let $\mathbb{V}$ be a finite-dimensional vector space and let $B = \{u_1, \ldots, u_n\}$ be a basis for $\mathbb{V}$. Let $\langle x, y\rangle$ be an inner product on $\mathbb{V}$. If $c_1, \ldots, c_n$ are any n scalars, show that there is exactly one vector x in $\mathbb{V}$ such that $\langle x, u_i\rangle = c_i$ for $i = 1, \ldots, n$.

Sol. This solution will be sent later.

[**0.0.4**] ***Exercise*** Let $(\mathbb{V}, \langle , \rangle)$ be an inner product space. Show that $\langle x, y\rangle = 0$ for all $y \in \mathbb{V}$, then $x = 0$.

Sol. Given that that $\langle x, y\rangle = 0$ for all y in $\mathbb{V}$. To show that $x = 0$. Since $\langle x, y\rangle = 0$ for all $y \in \mathbb{V}$, then $\langle x, x\rangle = 0$ as x is an element in $\mathbb{V}$. Using definition of inner product, $\langle x, x\rangle = 0 \implies x = 0$.

[**0.0.5**] ***Exercise*** Show that $\langle x, y\rangle = \sum_{i=1}^{n} \overline{x_i} y_i$ is not an inner product on $\mathbb{C}^n$.

Sol. This is not an inner product on $\mathbb{C}^n$. It does not satisfy homogeneity property, that is $\langle \alpha x, y\rangle \neq \alpha\langle x, y\rangle$. For example, take $x = (1, 0, 0, \ldots, 0)$, $y = (1, 0, 0, \ldots, n)$ and $\alpha = i$. Then $\langle \alpha x, y\rangle = -i$ and $\alpha\langle x, y\rangle = i$ They are not equal.

[**0.0.6**] ***Exercise*** Let $(\mathbb{V}, \langle , \rangle)$ be a finite dimensional inner product space. Prove that for $v \in \mathbb{V} - \{0\}$, the set $\mathbb{W} = \{w \in \mathbb{V} : \langle w, v\rangle = 0\}$ is a subspace of $\mathbb{V}$ of dimension $\textsc{dim}\, \mathbb{V} - 1$.

Sol. The definition of $\mathbb{W}$ says that $\mathbb{W} = \{v\}^{\perp}$. Hence $\mathbb{W}$ is a subspace of $\mathbb{V}$. To find the dimension of $\mathbb{W}$, we use the following fact. Let S be a subset of $\mathbb{V}$, then $S^{\perp} = (LS(S))^{\perp}$. Using this fact $\{v\}^{\perp} = LS(\{v\})^{\perp} = \mathbb{W}$. Then $\mathbb{V} = \mathbb{W} + LS(\{v\})$ (internal direct sum). We know that $\textsc{dim}\,(LS(\{v\})) = 1$. Hence $\textsc{dim}\, \mathbb{W} = \textsc{dim}\, \mathbb{V} - 1$.

[**0.0.7**] ***Exercise*** Decide which of the following functions define an inner product $\mathbb{C}^2$. For $x = (x_1, y_1)$, $y = (y_1, y_2)$.

1. $\langle x, y\rangle = x_1 \overline{y_2}$
2. $\langle x, y\rangle = x_1 \overline{y_1} + x_2 \overline{y_2}$
3. $\langle x, y\rangle = x_1 y_1 + x_2 y_2$
4. $\langle x, y\rangle = 2x_1 \overline{y_1} + i(x_2 \overline{y_1} - x_1 \overline{y_2}) + 2x_2 \overline{y_2}$

Sol.

1. Not an inner product. Take $x = (1, 0)$. $\langle x, x\rangle = 0$ but x is not equal to zero.
2. Yes, inner product.
3. Not an inner product. Conjugate symmetry does not satisfy.
4. Not an inner product. Conjugate symmetry does not satisfy. Take $x = (1, i)$ and $y = (i, 1)$.

[**0.0.8**] ***Exercise*** Let $\mathbb{V} = \mathbb{P}_3(x)$ be a subspace of real polynomials of degree at most 3. Equip $\mathbb{V}$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx$$

.

1. Find the orthogonal complement of the subspace of scalar polynomials.

2. Apply the Gram Schmidt process to the basis $\{1, x, x^2, x^3\}$.

Sol. 1. To find the orthogonal complement of of the subspace of scalar polynomials (scalar polynomial means zero degree polynomial).

Let $\mathbb{W}$ be the orthogonal complement of the subspace of scalar polynomials.

Let $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be an arbitrary element in $\mathbb{W}$. Then $\langle 1, P(x)\rangle = 0 \implies \int_0^1 P(x)dx = 0$

$\int_0^1 (a_0 + a_1x + a_2x^2 + a_3x^3)dx = 0$

$\implies a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0$

$\implies 12a_0 + 6a_1 + 4a_2 + 3a_3 = 0$

$\implies a_0 = \frac{-6a_1 - 4a_2 - 3a_3}{12}$.

$P(x) = \frac{-6a_1 - 4a_2 - 3a_3}{12} + a_1x + a_2x^2 + a_3x^3$

$= a_1(x - 1/2) + a_2(x^2 - 1/3) + a_3(x^3 - 1/4)$

This says that $P(x)$ is a linear combination of $x - 1/2$, $x^2 - 1/3$ and $x^3 - 1/4$.

Hence $\mathbb{W} = \text{LS}(\{x - 1/2, x^2 - 1/3, x^3 - 1/4\})$.

The set of scalar polynomials is equal the $\mathbb{R}$ and we know the dimension of $\mathbb{R}$ is 1.

We also know that $\mathbb{P}_3(x) = \mathbb{R} \oplus \mathbb{W}$. Hence $\text{DIM}\, \mathbb{W} = 3$.

Therefore $\{x - 1/2, x^2 - 1/3, x^3 - 1/4\}$ is basis of $\mathbb{W}$.

2. Consider $u_1 = 1, u_2 = x, u_3 = x^2, u_4 = x^3$.

$v_1 = u_1 = 1$

$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = x - 1/2$.

$v_3 = u_3 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = x^2 - x + 1/6$.

$v_4 = u_4 - \frac{\langle u_4, v_3 \rangle}{\langle v_3, v_3 \rangle} v_3 - \frac{\langle u_4, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle u_4, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}.$

[**0.0.9**] ***Exercise*** Find an inner product on $\mathbb{R}^2$ such that $\langle e_1, e_2 \rangle = 2$.

Sol. Exercise 0.0.2 helps you to solve Exercise 0.0.9. If you are able to find a symmetric matrix A with each diagonal entry is positive and $det(A) > 0$ such that $e_1^t A e_2 = 2$ then you are done and your desire inner product will be $\langle x, y \rangle = y^t A x$. Take $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$. You can easily check that A is symmetric, each diagonal entry of A is positive and $det(A) > 0$. Notice that $e_1^t A e_2 = 2$.

Hence your desire inner product is $\langle x, y \rangle = y^t A x = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1y_1 + 2x_2y_1 + 2x_1y_2 + 3x_2y_2$.

[**0.0.10**] ***Exercise*** Let $\mathbb{V}$ be the space of all $n \times n$ over $\mathbb{R}$ with the inner product $\langle A, B \rangle = trace(AB^t)$. Find the orthogonal complement of the subspaces of diagonal matrices.

[**0.0.11**] ***Exercise*** Let $(\mathbb{V}, \langle , \rangle)$ be an IPS. Let $\alpha, \beta \in \mathbb{V}$. Then show that $\alpha = \beta$ if and only if $\langle \alpha, \gamma \rangle = \langle \beta, \gamma \rangle$ for all $\gamma \in \mathbb{V}$.

Sol. First we assume that $\langle \alpha, \gamma \rangle = \langle \beta, \gamma \rangle$ for all $\gamma \in \mathbb{V}$. Then $\langle \alpha - \beta, \gamma \rangle = 0$ for all $\gamma \in \mathbb{V}$. Using Exercise 0.0.4, we have $\alpha - \beta = 0$. Hence $\alpha = \beta$.

Now we assume that alpha=beta, that is $\alpha - \beta = 0$. Then $\langle \alpha - \beta, \gamma \rangle = 0$ for all $\gamma \in \mathbb{V}$. This implies that $\langle \alpha, \gamma \rangle = \langle \beta, \gamma \rangle$ for all $\gamma \in \mathbb{V}$.

[**0.0.12**] ***Exercise*** Apply Gram Schmidt process to the vectors $u_1 = (1, 0, 1)$, $u_2 = (1, 0, -1)$ and $u_3 = (0, 3, 4)$ to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product.

Sol. $v_1 = u_1 = (1, 0, 1)$.

$v_2 = u_2$

$v_3 = u_3 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle u_3, \rangle v_1}{\langle v_1, \rangle v_1} v_1.$

$= (0, 3, 4) - \frac{-4}{2}(1, 0, -1) - \frac{4}{2}(1, 0, 1).$

$= (0, 3, 4) + 2(1, 0, -1) - 2(1, 0, 1)$

$= (0, 3, 4) + (0, 0, -4)$

$= (0, 3, 0).$

$v_1 = (1,0,1)$, $v_2 = (1,0,-1)$ and $v_3 = (0,3,0)$ are orthogonal.

[**0.0.13**] ***Exercise*** Consider the inner product $\langle x, y\rangle = y^t AX$ on $\mathbb{R}^3$ where $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$. Find an orthonormal basis B of $S := \{(x_1, x_2, x_3) : \ x_1 + x_2 + x_3 = 0\}$ and then extend it to an orthonormal basis of $\mathbb{R}^3$.

Sol. We first find a basis of S. Let (x_1, x_2, x_3) be an arbitrary element in S. Then $x_1 + x_2 + x_3 = 0$. This implies $(x_1, x_2, x_3) = (-x_2 - x_3, x_2, x_3) = x_2(-1,1,0) + x_3(-1,0,1)$.

Notice that $S = \text{LS}(\{(-1,1,0), (-1,0,1)\})$. It is easy to prove that $\{(-1,1,0), (-1,0,1)\}$ is linearly independent. Hence $\{(-1,1,0), (-1,0,1)\}$. Applying Gram Schmidt process on $\{(-1,1,0), (-1,0,1)\}$. Let $u_1 = (-1,1,0)$ and $u_2 = (-1,0,1)$.

$v_1 = u_1 = (-1,1,0)$.

$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$.

We have to calculate $\langle u_2, v_1\rangle$ and $\langle v_1, v_1\rangle$.

$$\langle u_2, v_1\rangle = v_1^t A u_2 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 2$$

$$\langle v_1, v_1\rangle = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1.$$

Then $v_2 = u_2 - 2v_1 = (-1,0,1) - 2(-1,1,0) = (1,-2,1)$

$\{\frac{1}{\sqrt{\langle v_1, v_1\rangle}} v_1, \frac{1}{\sqrt{\langle v_2, v_2\rangle}} v_2\} = \{(-1,1,0), (-1,2,-1)\} \left(\langle v_1, v_1\rangle = 1, \langle v_2, v_2\rangle = -1\right)$ is an orthonormal basis of S.

You can notice that $(1,1,1)$ is orthogonal to $(-1,1,0)$ and $(-1,2,-1)$. Therefore $\{(-1,1,0), (-1,2,-1), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\}$ is an orthonormal set in $\mathbb{R}^3$. They are linearly independent and dimension of $\mathbb{R}^3$ is 3. Then $\{(-1,1,0), (-1,2,-1), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\}$ is an orthonormal basis on $\mathbb{R}^3$.

[**0.0.14**] ***Exercise*** Let $(\mathbb{V}, \langle , \rangle)$ be an IPS. Let $||u|| = \sqrt{\langle u, u\rangle}$ for all $u \in \mathbb{V}$ be the norm induced by $\langle , \rangle$. Then prove that $||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2$.

Sol. **Note:** Before going to solve this problem, I would like to introduce something. $||x|| = \sqrt{\langle x, x\rangle}$.

We have seen that this a norm on $\mathbb{V}$. This is called a norm induced by the inner product $\langle , \rangle$.

This problem says that any norm which is induced by an inner product that norm must satisfy this condition $||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$.

$$||u+v||^2 = \langle u+v, u+v \rangle = \langle u,u \rangle + \langle u,v \rangle + \langle v,u \rangle + \langle v,v \rangle.$$

$$||u-v||^2 = \langle u-v, u-v \rangle = \langle u,u \rangle - \langle u,v \rangle - \langle v,u \rangle + \langle v,v \rangle.$$

After adding them, we have $||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$. This is clled Parallelogram Identity.

Note: The Parallelogram Identity is not true in general for any arbitrary norm.

[**0.0.15**] ***Exercise*** Let $(\mathbb{V}, \langle , \rangle)$ be a finite dimensional IPS. Let $B = \{u_1, u_2, \ldots, u_n\}$ be a basis of $\mathbb{V}$. Then prove that $\langle u, v \rangle = \bar{y}^t A x$ for all $u, v \in \mathbb{V}$ where $x = (x_1, \ldots, x_n)^t$,$y = (y_1, \ldots, y_n)^t$ are coordinates of u and v with respect to basis B and $a_{ij} = \langle u_i, u_j \rangle$.

Sol. Given that $(\mathbb{V}, \langle , \rangle)$ is a finite dimensional IPS and $B = \{u_1, u_2, \ldots, u_n\}$ is a basis of $\mathbb{V}$. Let $u, v \in \mathbb{V}$. Then $x = x_1u_1 + \cdots + x_nu_n$ and $y = y_1u_1 + \cdots + y_nu_n$. Here $x = (x_1, \ldots, x_n)^t$ and $y = (y_1, \ldots, y_n)^t$ are the coordinates of u and v with respect to basis B.

$$\langle u, v \rangle = \langle x_1u_1 + \cdots + x_nu_n, y_1u_1 + \cdots + y_nu_n \rangle$$

$$= \sum_{i,j=1}^{n} x_i \overline{y_j} \langle u_i, u_j \rangle.$$

$= \overline{y}^t A x$ where $a_{ij} = \langle u_i, u_j \rangle$ and $x = (x_1, \ldots, x_n)^t$ and $y = (y_1, \ldots, y_n)^t$.