Date
07/08/2017

$\boxed{\text{Lecture 7}}$

various forms of $u(t-a)$ or $H(t-a)$ or $u_a(t)$

1) $u(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a. \end{cases}$

2) $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a. \end{cases}$

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}, \quad (s>0)$$

3) $u(t-a) = \begin{cases} 1, & t > a \\ 0, & t \leq a. \end{cases}$

**Ex 1** / Determine the Laplace Transform of the sine function switched on at time $t = 3$.

**Sol$^n$**:- $s(t) = \begin{cases} \sin t, & t \geq 3 \\ 0, & t < 3. \end{cases}$

$\Rightarrow s(t) = u(t-3)\sin t$.

Now, $\sin(t) = \sin(t-3+3)$

$= \sin(t-3)\cos 3 + \cos(t-3)\sin 3$

Now, $\mathcal{L}\{u(t-3)\sin(t)\} = \mathcal{L}\{\sin(t-3)\cos 3 \cdot u(t-3) + \cos(t-3)\sin 3\, u(t-3)\}$

$$\therefore \mathcal{L}\{S(t)\} = e^{-3s} \cdot \frac{1}{(s^2+1)} \cos 3 + e^{-3s} \cdot \sin 3 \cdot \frac{s}{(s^2+1)}$$

$$= \boxed{\{\cos 3 + s \sin 3\} \cdot \frac{e^{-3s}}{s^2+1}}$$

---

$+$ $-$, $\times$, $\div$, $\int$ $\frac{d}{dx}$

## Inverse Laplace Transform

Def$^n$:- If $f(t)$ has the Laplace transform $F(s)$, that is $\mathcal{L}\{f(t)\} = F(s)$ then the Inverse Laplace transform is defined by

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \quad (t \geq 0)$$

& is unique apart from the null functions.

eg, $\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin\omega t,\ t \geq 0.$

Q) Can there be some other functions $f(t) \neq \sin\omega t$.

with $\mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = f(t)$?.

Is the inverse unique.

Soln :- Yes, let

$$f(t) = \begin{cases} \sin\omega t, & t > 0 \\ 1, & t = 0. \end{cases}$$

but $\mathcal{L}\{f(t)\} = \frac{\omega}{s^2+\omega^2}$.

Since altering a function at a single point (or even at a finite no. of points), does not alter the value of the Laplace (Riemann) integral.

---

$\therefore \mathcal{L}^{-1}\{F(s)\}$ can be more than one $f^n$. Infact, there are infinitely, many such $f^ns$.. , when considering $f^ns$ with discontinuities.

—

Th - 1/ The Inverse Laplace transform is linear.

i.e., $\mathcal{L}^{-1}\{a F_1(s) + b F_2(s)\}$

$= a \mathcal{L}^{-1}\{F_1(s)\} + b \mathcal{L}^{-1}\{F_2(s)\}$

where $a$ & $b$ are constants.

Since Laplace Transform is linear, we have for suitably well-behaved f^ns $f_1(t)$ & $f_2(t)$,

$\mathcal{L}\{a f_1(t) + b f_2(t)\}$
$= a \mathcal{L}\{f_1(t)\} + b \mathcal{L}\{f_2(t)\}$

$= a\,F_1(s) + b\,F_2(s).$

Taking the Inverse Laplace Transform of this expression gives

$$a f_1(t) + b f_2(t)$$

$$= \mathcal{L}^{-1}\{a\,F_1(s) + b\,F_2(s)\}$$

$$\Rightarrow a\,\mathcal{L}^{-1}\{F_1(s)\} + b\,\mathcal{L}^{-1}\{F_2(s)\}$$

$$= \mathcal{L}^{-1}\,a\{F_1(s)\} + b\{F_2(s)\}$$

---

## Null Functions

If $N(t)$ is a function of $t$ such that for all $t > 0$,

$$\boxed{\int_0^t \mathcal{N}(u)\,du = 0.}$$

we call $\mathcal{N}(t)$ is a

null function

e.g., The function

$$f(t) = \begin{cases} 1, & t = \frac{1}{2} \\ -1, & t = 1 \\ 0, & \text{otherwise.} \end{cases}$$

is a null function (how?)

In general, any $f^n$ which is zero at all but a countable set of points is a null function.

# Uniqueness of Inverse L.T

Since, we know that the Laplace transform of a null function $N(t)$ is zero (why?) it is clear that

if $\mathcal{L}\{f(t)\} = F(s)$,

then also $\mathcal{L}\{f(t) + N(t)\} = F(s)$.

From this, it follows that we can have two different f^ns with the same L.T.

eg; The two different fns

$$f_1(t) = e^{-3t},$$

$$f_2(t) = \begin{cases} 0, & t = 1 \\ e^{-3t}, & \text{otherwise} \end{cases}$$

$$= f_1(t) + N(t) \quad (\text{Ex})$$

have the same L.T

ie, $\dfrac{1}{(s+3)}$.

---

Note:- If we allow null functions, we see that the inverse L.T is not unique

It is unique, however if we disregard the null functions.
[which do not in general arise in cases of physical interest].

* * * * *

Th-2 (Lerch's theorem)

If we restrict ourselves to functions $f(t)$, which are sectionally / piecewise continuous in every finite interval $0 \leq t \leq N$ & of exponential order for $t > N$

then the inverse L.T

of $F(s)$ ie;

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

is unique.

(ie, Distinct continuous $f^{ns}$ on $[0,\infty)$ have distinct L.T).

---

Inverse L.T

Differ. $\int$ —

$\mathcal{L}\{f(t)\}$ , $\mathcal{L}^{-1}\{f(t)\}$

$s \xrightarrow{\text{No guarantee.}} t$

Ex

One necessary condⁿ

function of $s$ i.e., $F(s)$ $\to 0$ as $s \to \infty$.

(Partial Fractions)

Ex1/ Use Partial fractions to determine

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\}.$$

Solⁿ:- We note that

$$\frac{a}{s^2-a^2} = \frac{1}{2}\left[\frac{1}{(s-a)} - \frac{1}{(s+a)}\right]$$

We have

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$- \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left(e^{at} - e^{-at}\right) \text{ (how?)}$$

$$= \sinh(at).$$

---

Ex2/ Determine the value of $\mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\}$.

Sol:- we have,

$$\frac{s^2}{(s+3)^3} = \frac{A}{(s+3)} + \frac{B}{(s+3)^2} + \frac{C}{(s+3)^3}$$

(Ex)

$$= \frac{1}{s+3} - \frac{6}{(s+3)^2} + \frac{9}{(s+3)^3}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s^2}{(s+3)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$- \mathcal{L}^{-1}\left\{\frac{6}{(s+3)^2}\right\}$$

$$+ \mathcal{L}^{-1}\left\{\frac{9}{(s+3)^3}\right\}$$

$$= e^{-3t} - 6t \cdot e^{-3t} + \frac{9}{2} \cdot t^2 \cdot e^{-3t}.$$

[Using First shifting theorem]

---

Q) Determine the following inverse L.T

H.W ✓ (a) $\mathcal{L}^{-1}\left\{\frac{(s+3)}{s(s-1)(s+2)}\right\}$

Ans: $-3/2 + 4/3 e^{t} + \frac{1}{6} e^{-2t}$

H.W ✓ (b) $\mathcal{L}^{-1}\left\{\frac{(s-1)}{s^2+2s-8}\right\}$ Ans: $\frac{1}{6} e^{2t} + 5/6 e^{-4t}$

H.W ✓ (c) $\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s+5}\right\}$

Ans: $3e^{t}\cos(2t) + 5e^{t}\sin(2t)$.

C.W ✓ (d) $\mathcal{L}^{-1}\left\{\frac{e^{-7s}}{(s+3)^3}\right\}$.

(d) $\dfrac{e^{-7s}}{(s-3)^3}$

we note that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^3}\right\} = \frac{1}{2}t^2 \cdot e^{3t}$$

[how ?]

so, $\mathcal{L}^{-1}\left\{\dfrac{e^{-7s}}{(s-3)^3}\right\}$

$$= \begin{cases} \frac{1}{2}(t-7)^2 \cdot e^{3(t-7)}, & t>7 \\ 0, & 0 \le t \le 7. \end{cases}$$

(how?)

$$= \frac{1}{2}u(t-7)(t-7)^2 e^{3(t-7)}$$

# Limiting Theorems

d.eq$^n$ $y(t)$ . Numerical

$\mathcal{L}^{-1}\{Y(t)\}$ Inversion techniques

↓

Control engineers

↓

Sometimes insight into the behaviour of the sol$^n$. can be deduced without actually solving the d.eq$^n$.

by studying the asymptotic character of $F(s)$ for small $s$ or large $s$.

Th-13 (Initial value Theorem)

If $f(t)$ & $f'(t)$ are Laplace transformable & $F(s)$ is the Laplace transform of $f(t)$, then the behaviour of $f(t)$ in the neighbourhood of $t=0$ corresponds to the behaviour of $s\,F(s)$ in the neighbourhood of $s=\infty$. Mathematically,

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\,F(s)$$

(The L.H.S is $f(0)$ of course

or, $f(0+)$ if $\lim_{t \to 0} f(t)$ is

not unique.

Proof :- We know that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \quad \rightarrow (1)$$

However, if $f'(t)$ obeys the usual criteria for the existence of the L.T, that is $f'(t)$ is of exponential order & is piece-wise continuous, then

$$\left| \int_0^\infty e^{-st} f'(t)\, dt \right|$$
$$\leq \int_0^\infty \left| e^{-st} f'(t) \right| dt$$

$$\leq \int_0^\infty e^{-st} |f'(t)|\, dt$$

$$\leq \int_0^\infty e^{-st} \cdot e^{Mt}\, dt$$

$$= \int_0^\infty e^{-(s-M)t}\, dt$$

$$= \left[ \frac{e^{-(s-M)t}}{M-s} \right]_0^\infty$$

$$\left| \int_0^\infty e^{-st} f'(t)\, dt \right| \leq -\frac{1}{(M-s)}$$

$$\therefore \lim_{s \to \infty} \int_0^\infty e^{-st} f'(t)\, dt = 0$$

Thus letting $s \to \infty$ in (1), we get

(& assuming that $f(t)$ is continuous at $t=0$)

$$\therefore 0 = \lim_{s \to \infty} sF(s) - f(0)$$

$$\Rightarrow \boxed{\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)}$$

Note :- If $f(t)$ is not continuous at $t=0$, the reqd. theorem still holds but we must use the result :-

If $f(t)$ fails to be

Proof of Final value Theorem :-

Again, we have —

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0) \rightarrow (1)$$

This time writing the integral out explicitly the limit of the integral as $s \to 0$ is :

$$\lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt = \int_0^\infty f'(t)\,dt$$

$$= \lim_{p\to\infty} \int_0^p f'(t)\,dt$$

$$= \lim_{p\to\infty} \left[f(t)\right]_0^p = \lim_{p\to\infty} \left[f(p) - f(0)\right]$$

$$= \lim_{t\to\infty} f(t) - f(0).$$

The limit of the R.H.S as $s \to 0$ is

$$\lim_{s\to 0} sF(s) - f(0).$$

Thus, $\lim_{t\to\infty} f(t) - \cancel{f(0)} = \lim_{s\to 0} sF(s) - \cancel{f(0)}$

$$\Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

If $f(t)$ is not continuous, the result still holds, but we proceed as before.

— x —

continuous at $t=0$,

but $\lim\limits_{t \to 0} f(t) = f(0+)$

exists (but is not equal to $f(0)$, which may or may not exist), then

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0+).$$

H.W

## Th-/ (Final Value Theorem)

If $f(t)$ & $f'(t)$ are Laplace transformable & $F(s)$ is the L.T of $f(t)$, then the behaviour of $f(t)$ in the neighbourhood of $t=\infty$ corresponds

corresponds to the behaviour of $sF(s)$ in the neighbourhood of $s=0$. Mathematically,

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

**Ex 1/** Illustrate initial & final value theorems for the function

$$f(t) = 3e^{-2t}.$$

**Soln:-** We have $f(t) = 3e^{-2t}$

$$\therefore F(s) = \mathcal{L}\{f(t)\} = \left(\frac{3}{s+2}\right).$$