3.9 Polya Inventory Theorem

In this section, the ideas of the previous subsection are generalized. This generalization allows us to count the distinct number of necklaces even if there are not sufficient number of beads of each color. To do this, each element of C is assigned a *weight*, that in turn gives weight to each color pattern. This weight may be a number, a variable or in general, an element of a commutative ring with identity. The setup for our study remains the same. To start with, we have the following definitions.

Definition 3.9.1 (Weight of a color pattern)**.** *Let A be a commutative ring with identity (the elements of A are called weights). Let $w : C \longrightarrow A$ be a map that assigns weights to each color. Then, the weight of a color pattern $\phi : X \longrightarrow C$, with respect to the weight function w is given by $w(\phi) = \prod\limits_{x \in X} w\big(\phi(x)\big)$.*

Fix $g \in G$. Then we have seen that g fixes a color pattern $\phi \in \Omega$ if and only if ϕ colors the elements in a given cycle of g with the same color. Similarly, for each fixed $g \in G$ and $\phi \in \Omega$, one has

$$w(g \circledast \phi) = \prod_{x \in X} w\big(g \circledast \phi(x)\big) = \prod_{x \in X} w\big(\phi(g^{-1} \star x)\big) = \prod_{y \in X} w\big(\phi(y)\big) = w(\phi), \tag{3.1}$$

as $\{g \star x : x \in X\} = X$ (see Remark 3.6.2). That is, for a fixed $\phi \in \Omega$, the weight of each element of $\mathcal{O}(\phi) = \{g \circledast \phi : g \in G\}$ is the same and it equals $w(\phi)$. That is, $w(\phi) = w(\psi)$, whenever $\psi = g \circledast \phi$, for some $g \in G$.

Example 3.9.2. *Let X consist of the set of faces of a cube, G be the group of symmetries of the cube and let C consist of two colors 'Red' and 'Blue'. Thus, if the weights R and B are assigned to the two elements of C then the weight*

1. *B^6 corresponds to "all faces being colored Blue";*
2. *R^2B^4 corresponds to "any two faces being colored 'Red' and the remaining four faces being colored 'Blue';*
3. *R^3B^3 corresponds to "any three faces being colored 'Red' and the remaining three faces being colored 'Blue' and so on.*

Example 3.9.2 indicates that different color patterns need not have different weights. We also need the following definition to state and prove results in this area.

Definition 3.9.3 (Pattern Inventory)**.** *Let G be a group acting on the set Ω (the set of color patterns) and let $w : C \longrightarrow A$ be a weight function. The pattern inventory, denoted I, under the action of G on Ω, with respect to w, is the sum of the weights of the orbits. That is, $I = \sum\limits_{\Delta} w(\Delta)$, where the sum runs over all the distinct orbits Δ obtained by the action of G on Ω.*

With the above definitions, we are ready to prove the Polya's Enumeration Theorem. To do so, we first need to prove the weighted Burnside's Lemma. This Lemma is the weighted version of the Burnside's Lemma 3.7.3.

Lemma 3.9.4. *With the definitions and notations as above,*

$$I = \sum_{\Delta} w(\Delta) = \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{\phi \in \Omega \\ g \circledast \phi = \phi}} w(\phi),$$

where the sum runs over all the distinct orbits Δ obtained by the action of G on Ω.

Proof. As G acts on Ω, for each $\alpha \in \Omega$, the application of Lemma 3.6.7 gives $|G_\alpha| \cdot |\mathcal{O}(\alpha)| = |G|$. Since Δ is an orbit under the action of G, for each $\phi \in \Delta$, $|G_\phi| \cdot |\Delta| = |G|$. Also, by definition, $w(\Delta) = w(\phi)$, for all $\phi \in \Delta$. Thus,

$$w(\Delta) = w(\phi) = \frac{1}{|\Delta|} \sum_{\phi \in \Delta} w(\phi) = \sum_{\phi \in \Delta} \frac{1}{|\Delta|} w(\phi) = \sum_{\phi \in \Delta} \frac{|G_\phi|}{|G|} w(\phi) = \frac{1}{|G|} \sum_{\phi \in \Delta} |G_\phi| \cdot w(\phi).$$

Let $F_g = \{\phi \in \Omega : g \circledast \phi = \phi\}$. Then $\sum\limits_{\phi \in \Omega} \sum\limits_{g \in G_\phi} w(\phi) = \sum\limits_{g \in G} \sum\limits_{\phi \in F_g} w(\phi)$ and hence

$$\begin{aligned} I &= \sum_{\Delta} w(\Delta) = \sum_{\Delta} \frac{1}{|G|} \sum_{\phi \in \Delta} |G_\phi| \cdot w(\phi) = \frac{1}{|G|} \sum_{\Delta} \sum_{\phi \in \Delta} |G_\phi| \cdot w(\phi) = \frac{1}{|G|} \sum_{\phi \in \Omega} |G_\phi| \cdot w(\phi) \\ &= \frac{1}{|G|} \sum_{\phi \in \Omega} \sum_{g \in G_\phi} w(\phi) = \frac{1}{|G|} \sum_{g \in G} \sum_{\phi \in F_g} w(\phi). \qquad \blacksquare \end{aligned}$$

We are now in a position to prove the Polya's Enumeration Theorem. Before doing so, recall that F_g consists precisely of those color schemes which color each cycle of g with just one color (see the argument used in the second paragraph in the proof of Theorem 3.8.5).

Theorem 3.9.5 (Polya's Enumeration Theorem)**.** *With the definitions and notations as above,*

$$I = \sum_{\Delta} w(\Delta) = P_G(x_1, x_2, \ldots, x_n),$$

where the sum runs over all the distinct orbits Δ obtained by the action of G on Ω and $x_i = \sum\limits_{c \in C} w(c)^i$, is the i^{th} power sum of the weights of the colors. In particular, if weight of each color is 1, $I = P_G(|C|, |C|, \ldots, |C|)$.

Proof. Using the weighted Burnside Lemma 3.9.4, we need to prove that

$$\sum_{g \in G} \sum_{\phi \in F_g} w(\phi) = \sum_{g \in G} x_1^{\ell_1(g)} x_2^{\ell_2(g)} \cdots x_n^{\ell_n(g)},$$

where $\ell_i(g)$ is the number of cycles of length i in the cycle representation of g.

Now, fix a $g \in G$. Suppose g has exactly t disjoint cycles, say $g_1, g_2, \ldots, g_t$. As F_g consists precisely of those color schemes which color each cycle of g with just one color, we just need to determine the weight of such a color pattern. To do so, for $1 \le i \le t$, define X_i to be that subset of X whose elements form the cycle g_i. Then, it is easy to see that $X_1, X_2, \ldots, X_t$ defines a partition of X. Also, the condition that x and $g \star x$ belong to the same cycle of g, implies that $w(\phi(s_i)) = w(\phi(g \star s_i))$, for each $s_i \in X_i, 1 \le i \le t$. Thus, for each $\phi \in F_g$,

$$w(\phi) = \prod_{x \in X} w(\phi(x)) = \prod_{i=1}^{t} \prod_{x \in X_i} w(\phi(x)) = \prod_{i=1}^{t} w(\phi(s_i))^{|X_i|}.$$

Note that if we pick a term from each factor in $\prod\limits_{i=1}^{t} \left(\sum\limits_{c \in C} w(c)^{|X_i|} \right)$ and take the product of these terms, we obtain all the terms of $\prod\limits_{i=1}^{t} \left(\sum\limits_{c \in C} w(c)^{|X_i|} \right)$. All these terms also appear in $\sum\limits_{\phi \in F_g} \prod\limits_{i=1}^{t} w(\phi(s_i))^{|X_i|}$ because as ϕ is allowed to vary over all elements of F_g, the images $\phi(s_i)$, for $1 \le i \le t$, take all values in C. The argument can also be reversed and hence it follows that

$$\sum_{\phi \in F_g} w(\phi) = \sum_{\phi \in F_g} \prod_{i=1}^{t} w(\phi(s_i))^{|X_i|} = \prod_{i=1}^{t} \left(\sum_{c \in C} w(c)^{|X_i|} \right).$$

Now, assume that g has $\ell_k(g)$ cycles of length k, $1 \le k \le n$. This means that in the collection $|X_1|, |X_2|, \ldots, |X_t|$, the number 1 appears $\ell_1(g)$ times, the number 2 appears $\ell_2(g)$ times and so on till the number n appears $\ell_n(g)$ times (note that some of the $\ell_i(g)$'s may be zero). Consequently, $\prod\limits_{i=1}^{t} \left(\sum\limits_{c \in C} w(c)^{|X_i|} \right)$ equals $\prod\limits_{k=1}^{n} x_k^{\ell_k(g)}$, as $x_1 = \sum\limits_{c \in C} w(c)$, $x_2 = \sum\limits_{c \in C} w(c)^2$ and so on till $x_n = \sum\limits_{c \in C} w(c)^n$. Hence, $\sum\limits_{\phi \in F_g} w(\phi) = \prod\limits_{k=1}^{n} x_k^{\ell_k(g)}$ and thus, the required result follows. ■

Example 3.9.6. *1. Consider a necklace consisting of 6 beads. If there are 3 color choices, say R, B and G, then determine*

(a) the number of necklaces that have at least one R bead.

(b) the number of necklaces that have three R, two B and one G bead.

Solution: *Recall that D_6 acts on a regular hexagon and its cycle index polynomial equals*

$$P_{D_6}(z_1, z_2, \ldots, z_6) = \frac{1}{12}(z_1^6 + 4z_2^3 + 2z_3^2 + 2z_6 + 3z_1^2 z_2^2).$$

So, for the first part, at least one R needs to be used and the remaining can be any number of B and/or G. So, we define the weight of the color R as x and that of B and G as 1. Therefore, by Polya's Enumeration Theorem 3.9.5,

$$\begin{aligned} I &= \frac{1}{12}\big((x+1+1)^6 + 4(x^2+1+1)^3 + 2(x^3+1+1)^2 \\ &\qquad +2(x^6+1+1) + 3(x+1+1)^2(x^2+1+1)^2\big) \\ &= x^6 + 2x^5 + 9x^4 + 16x^3 + 29x^2 + 20x + 15. \end{aligned}$$

So, the required answer is $1 + 2 + 9 + 16 + 29 + 20 = 77$.

For the second part, define the weights as R, B *and* G *itself. Then*

$$\begin{aligned} I &= \frac{1}{12}\left((R+B+G)^6 + 4(R^2+B^2+G^2)^3 + 2(R^3+B^3+G^3)^2\right. \\ &\qquad \left. +2(R^6+B^6+G^6) + 3(R+B+G)^2(R^2+B^2+G^2)^2\right). \end{aligned}$$

The required answer equals the coefficient of R^3B^2G *in* I, *which equals*

$$\frac{1}{12}\left(\binom{6}{3,2,1} + 3\cdot 2\cdot 2\right) = \frac{1}{12}\left(\frac{6!}{3!2!} + 6\right) = 6.$$

We end this chapter with a few Exercises. But before doing so, we give the following example with which Polya started his classic paper on this subject.

Example 3.9.7. *Suppose we are given* 6 *similar spheres in three different colors, say, three Red, two Blue and one Yellow (spheres of the same color being indistinguishable). In how many ways can we distribute the six spheres on the* 6 *vertices of an octahedron freely movable in space?*

Solution: *Here* $X = \{1,2,3,4,5,6\}$ *and* $C = \{R, B, Y\}$. *Using Example 3.2.1.2b on Page 77 the cycle index polynomial corresponding to the symmetric group of the octahedron that acts on the vertices of the octahedron is given by*

$$\frac{1}{24}\left(z_1^6 + 6z_1^2z_4 + 3z_1^2z_2^2 + 8z_3^2 + 6z_2^3\right).$$

Hence, the number of patterns of the required type is the coefficient of the term R^3B^2Y *in*

$$\begin{aligned} I &= \frac{1}{24}\left((R+B+Y)^6 + 6(R+B+Y)^2(R^4+B^4+Y^4) + 3(R+B+Y)^2(R^2+B^2+Y^2)^2\right. \\ &\qquad \left. +8(R^3+B^3+Y^3)^2 + 6(R^2+B^2+Y^2)^3\right). \end{aligned}$$

Verify that this number equals 3.