Example: Using shooting method, find the solution of the boundary value problem

$$y'' = 6y^2$$

$$y(0) = 1, \quad y(0.5) = 4/9$$

Assume the initial approximation

$$y'(0) = -1.8$$

$$y'(0) = -1.9$$

and find the solution of the initial value problems using the fourth order Runge-Kutta method with $h = 0.1$. Improve the value of $y'(0)$ using the secant method once. Compare with the exact solution $y(x) = \frac{1}{(1+x)^2}$.

Sol: First we need to solve

$$y'' = 6y^2$$

$$y(0) = 1 \qquad y'(0) = s \quad \text{where} \quad s = -1.8 \ \& \ -1.9$$

set $u = y'$ then.

$$y' = u$$

$$\& \quad u' = 6y^2 \quad \text{with} \quad \left.\begin{array}{l} y(0) = 1 \\ u(0) = s \end{array}\right\} \text{system of first order equations.}$$

$$f_1 = u \ \& \ f_2 = 6y^2$$

Runge Kutta 4th order method:

$$\begin{bmatrix} y_{j+1} \\ u_{j+1} \end{bmatrix} = \begin{bmatrix} y_j \\ u_j \end{bmatrix} + \frac{h}{6}\left(\bar{K}_1 + 2\bar{K}_2 + 2\bar{K}_3 + \bar{K}_4\right)$$

$$\bar{K}_1 = \begin{bmatrix} f_1(x_j, y_j, u_j) \\ f_2(x_j, y_j, u_j) \end{bmatrix} \qquad \bar{K}_2 = \begin{bmatrix} f_1(x_j+\frac{h}{2}, y_j + \frac{1}{2}h\bar{K}_1^{(1)}, u_j + \frac{1}{2}h\bar{K}_1^{(2)}) \\ f_2(x_j+\frac{h}{2}, y_j + \frac{1}{2}h\bar{K}_1^{(1)}, u_j + \frac{1}{2}h\bar{K}_1^{(2)}) \end{bmatrix}$$

$$\bar{K}_3 = \begin{bmatrix} f_1(x_j+\frac{h}{2}, y_j + \frac{h}{2}\bar{K}_2^{(1)}, u_j + \frac{h}{2}\bar{K}_2^{(2)}) \\ f_2(x_j+\frac{h}{2}, y_j + \frac{h}{2}\bar{K}_2^{(1)}, u_j + \frac{h}{2}\bar{K}_2^{(2)}) \end{bmatrix} \qquad \bar{K}_4 = \begin{bmatrix} f_1(x_j+h, y_j + h\bar{K}_3^1, u_j + h\bar{K}_3^2) \\ f_1(x_j+h, y_j + h\bar{K}_3^1, u_j + h\bar{K}_3^2) \end{bmatrix}$$

x	$y(0)=1$, $y'(0)=-1.8$	$y(0)=1$, $y'(0)=-1.9$	$y(0)=1$, $y'(0)=-1.9990$	Y_{exact}
0.1	0.8468	0.8367	0.8266	0.8264
0.2	0.7372	0.7158	0.6947	0.6944
0.3	0.6606	0.6261	0.5922	0.5917
0.4	0.6103	0.5601	0.5108	0.5102
0.5	0.5825	0.5131	0.4453	0.4444

Secont method :

$$s^{(3)} = s^{(2)} - \frac{g(s^{(2)})}{g(s^{(2)}) - g(s^{(1)})} \times (s^{(2)} - s^{(1)}) \quad \text{—— (1)}$$

$$g(s^{(i)}) = y(s^{(i)}; 0.5) - 4/9$$

$$s^{(1)} = -1.8$$

$$s^{(2)} = -1.9$$

(1) $\Rightarrow$ $s^{(3)} = -1.9990$.

SOLVING $g(s)=0$ USING NEWTON-Raphson METHOD

$$s^{(k+1)} = s^{(k)} - \frac{g(s^{(k)})}{g'(s^{(k)})}$$

How to get $g'(s^{(k)})$?

We proceed as follows:

Suppose we want to solve:

$$y'' = f(x, y, y') \quad 0 < x < b \qquad \text{——— (1)}$$

$$a_0 y(a) - a_1 y'(a) = r_1 \qquad \text{——— (1a)}$$

$$b_0 y(b) + b_1 y'(b) = r_2 \qquad \text{——— (1b)}$$

Denote $y_s = y(x,s)$ $\quad y_s' = y'(x,s)$ $\quad y_s'' = y''(x,s)$

Then we consider

$$y_s'' = f(x, y_s, y_s') \qquad \text{——— (2)}$$

$$y_s'(a) = s \;, \quad y_s(a) = \frac{a_1 s + r_1}{a_0} \qquad (2')$$

Note $g(s) = b_0 y_s(b) + b_1 y_s'(b) - r_2$

$$g'(s) = b_0 \underbrace{\boxed{\frac{\partial y_s(b)}{\partial s}}}_{?} + b_1 \underbrace{\boxed{\frac{\partial y_s'(b)}{\partial s}}}_{?}$$

Denote $v = \dfrac{\partial y_s(x)}{\partial s}$

Then, $\boxed{g'(s) = b_0 v(b) + b_1 v'(b)}$

We need to set-up IVP for v.

Diff. (2) w.r.t. s:

$$\frac{\partial}{\partial s} y_s'' = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y_s}\frac{\partial y_s}{\partial s} + \frac{\partial f}{\partial y_s'}\frac{\partial y_s'}{\partial s}$$

as x is indep. of s.

$$= \frac{\partial f}{\partial y_s}\frac{\partial y_s}{\partial s} + \frac{\partial f}{\partial y_s'}\frac{\partial y_s'}{\partial s} \qquad \text{——— (3)}$$

Diff. (2') w.r.t. s:

$$\frac{\partial}{\partial s}\left[y_s'(a)\right] = 1 ; \qquad \frac{\partial}{\partial s}\left[y_s(a)\right] = \frac{a_1}{a_0} \qquad \text{—— } (3')$$

Noting $v = \frac{\partial y_s}{\partial s}$:

$$v' = \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial y_s}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial y_s}{\partial x}\right) = \frac{\partial}{\partial s} y_s'$$

$$v'' = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial s} y_s'\right) = \frac{\partial}{\partial s} y_s''$$

Then (3) $\Rightarrow$

$$v'' = \frac{\partial f}{\partial y_s}(x, y_s, y_s')\, v + \frac{\partial f}{\partial y_s'}(x, y_s, y_s')\, v' \qquad \text{—— } (4)$$

(3') $\Rightarrow$

$$v'(a) = 1 ; \qquad v(a) = \frac{a_1}{a_0} \qquad \text{—— } (4')$$

The differential equation (4) is called the first variational equation. It can be solved step by step along with (2 & 2'). When the computation of one cycle is completed $v(b)$ & $v'(b)$ is available. Then $g'(s)$ is available form

$$g'(s) = b_0 v(b) + b_1 v'(b)$$

REMARK: If the boundary conditions of the first kind are given, then we have

$$a_0 = 1 \quad a_1 = 0$$
$$b_0 = 1 \quad b_1 = 0$$

In this case:

$$g(s) = y(b, s) - \gamma_2$$

$$\& \quad g'(s) = v(b)$$