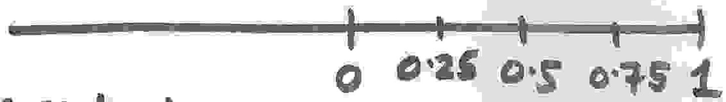
EXAMPLE: Solve the boundary value problem

$$y'' + (1+x^2)y + 1 = 0 \qquad y(\pm 1) = 0$$

with step length $h = 0.25$. Use a second order method.

Solution: Replacing x by $-x$ the BVP remains unchanged. Thus the solution of the problem is symmetrical about y-axis. Therefore we solve the above problem in the domain $[0, 1]$.



The second order method gives the difference equation:

$$\frac{1}{h^2}\left[y_{n+1} - 2y_n + y_{n-1}\right] + (1+x_n^2)\,y_n + 1 = 0$$

$$-y_{n-1} + \left[2 - (1+x_n^2)h^2\right]y_n - y_{n+1} = h^2$$

$$n = 0, 1, 2, 3$$

$\underline{n=0}$:

$$-y_{-1} + \left[2 - \frac{1}{16}\right]y_0 - y_1 = (0.25)^2$$

Since $y_{-1} = y_1$

$$\frac{31}{16}y_0 - 2y_1 = \frac{1}{16} \quad \text{——} \quad (1)$$

$\underline{n=1}$: $$-y_0 + \left[2 - \left(1+\frac{1}{16}\right)\frac{1}{16}\right]y_1 - y_2 = \frac{1}{16}$$

$$\Rightarrow -y_0 + \left[2 - \frac{17}{256}\right]y_1 - y_2 = \frac{1}{16}$$

$$\Rightarrow -y_0 + \left[\frac{495}{256}\right]y_1 - y_2 = \frac{1}{16} \quad \text{——} \quad (2)$$

$\underline{n=2}$:

$$-y_1 + \left[2 - \left(1 + \frac{4}{16}\right)\frac{1}{16}\right] y_2 - y_3 = \frac{1}{16}$$

$$\Rightarrow \quad -y_1 + \left[2 - \frac{20}{256}\right] y_2 - y_3 = \frac{1}{16}$$

$$\Rightarrow \quad -y_1 + \frac{492}{256} y_2 - y_3 = \frac{1}{16}$$

$\underline{n=3}$:

$$-y_2 + \left[2 - \left(1 + \frac{9}{16}\right)\frac{1}{16}\right] y_3 - y_4 = \frac{1}{16}$$

$$\Rightarrow \quad -y_2 + \left[2 - \frac{25}{256}\right] y_3 - y_4 = \frac{1}{16}$$

$$\Rightarrow \quad -y_2 + \frac{487}{256} y_3 - y_4 = \frac{1}{16}$$

$$\Rightarrow \quad -y_2 + \frac{487}{256} y_3 = \frac{1}{16}$$

<u>In matrix form:</u>

$$\begin{bmatrix} \frac{31}{16} & -2 & 0 & 0 \\ -1 & \frac{495}{256} & -1 & 0 \\ 0 & -1 & \frac{492}{256} & -1 \\ 0 & 0 & -1 & \frac{487}{256} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution using Gauss elimination:

$y_0 = 0.9415$

$y_1 = 0.8808$

$y_2 = 0.6992$

$y_3 = 0.4004$

Example: Use a second order method for the solution of the boundary value problem

$$y'' = xy + 1 \qquad x \in [0,1]$$

$$y'(0) + y(0) = 1 \qquad y(1) = 1.$$

with the step length $h = 0.25$.

Solution: Discretization at $x = x_n$ gives.

$$-\left(\frac{y_{n-1} - 2y_n + y_{n+1}}{h^2}\right) + x_n y_n + 1 = 0$$

or

$$-y_{n-1} + (2 + x_n h^2) y_n - y_{n+1} = -h^2 \qquad \text{— ①}$$

$$n = 0, 1, 2, 3$$

BC:

$$\frac{y_1 - y_{-1}}{2h} + y_0 = 1 \Rightarrow y_{-1} = y_1 + 2h y_0 - 2h$$

$$\& \quad y_4 = 1.$$

At $n = 0$; ① $\Rightarrow$

$$-y_{-1} + (2) y_0 - y_1 = -\frac{1}{16}$$

$$\Rightarrow -\left(y_1 + 2\cdot\frac{1}{4} y_0 - 2\cdot\frac{1}{4}\right) + 2y_0 - y_1 = -\frac{1}{16}$$

$$\frac{3}{2} y_0 - 2y_1 = -\frac{1}{16} - \frac{1}{2} = -\frac{9}{16} \qquad \text{— ①}$$

$n = 1$:

$$-y_0 + \left(2 + \frac{1}{4}\cdot\frac{1}{16}\right) y_1 - y_2 = -\frac{1}{16}$$

$$-y_0 + \frac{129}{64} y_1 - y_2 = -\frac{1}{16} \qquad \text{— ②}$$

$\underline{n=2:}$

$$-y_1+\left(2+\frac{2}{4}\cdot\frac{1}{16}\right)y_2-y_3=-\frac{1}{16}$$

$$\Rightarrow -y_1+\frac{65}{32}y_2-y_3=-\frac{1}{16} \quad \text{— ③}$$

$\underline{n=3:}$

$$-y_2+\left(2+\frac{3}{4}\cdot\frac{1}{16}\right)y_3-y_4=-\frac{1}{16}$$

$$\Rightarrow -y_2+\frac{131}{16}y_3-\underbrace{y_4}_{=1}=-\frac{1}{16}$$

$$\Rightarrow -y_2+\frac{131}{16}y_3=\frac{15}{16} \quad \text{— ④}$$

In matrix form:

$$\begin{bmatrix} \frac{3}{2} & -2 & 0 & 0 \\ -1 & \frac{129}{64} & -1 & 0 \\ 0 & -1 & \frac{65}{32} & -1 \\ 0 & 0 & -1 & \frac{131}{64} \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}=-\frac{1}{16}\begin{bmatrix} 9 \\ 1 \\ 1 \\ -15 \end{bmatrix}$$

Using Gauss-elimination or Thomas algorithm we get:

$y_0=-7.4616$ $\qquad$ $y_1=-5.3150$

$y_2=-3.1889$ $\qquad$ $y_3=-1.0999$

Nonlinear second order differential equations:

$$u'' = f(x,u) \qquad a < x < b \qquad \text{———(1)}$$

subject to the BCs:

$$u(a) = \gamma_1 \qquad u(b) = \gamma_2$$

A second order finite difference leads to:

$$u_{j-1} - 2u_j + u_{j+1} = h^2 f(x_j, u_j) \; ; \quad j = 1, 2, \cdots, N \quad \text{—(ii)}$$

with $u_0 = \gamma_1 \qquad u_{N+1} = \gamma_2$

The system of equations (ii) can be solved using Newton's method or by any other iteration method.

A simple iterative scheme:

$$u_{j-1}^{[s+1]} - 2u_j^{[s+1]} + u_{j+1}^{[s+1]} = h^2 f(x_j, u_j^{[s]})$$

$$j = 1, 2, \cdots, N.$$

This is a system of linear equations which can be solved by any known method.

Newton-Raphson-Method:

The system of equations (ii) can be written in the form

$$F(u_1, u_2, \cdots, u_N) =: F(u) = 0$$

where $F = [F_1, F_2, \cdots, F_N]^T$.

and $u = [u_1, u_2, \cdots, u_N]^T$.

Compute the Jacobian

$$J(u_1, u_2, \cdots u_N) = \frac{\partial F}{\partial u} = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \cdots & \frac{\partial F_1}{\partial u_N} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \cdots & \frac{\partial F_2}{\partial u_N} \\ \vdots & & & \\ \frac{\partial F_N}{\partial u_1} & \frac{\partial F_N}{\partial u_2} & \cdots & \frac{\partial F_N}{\partial u_N} \end{bmatrix}$$

Starting with a suitable estimate $u^{[0]}$, we define

$$u^{[s+1]} = u^{[s]} + \Delta u^{[s]} \qquad s = 0, 1, 2 \cdots$$

where $\Delta u^{[s]}$ is the solution of

$$J(u^{[s]}) \Delta u^{[s]} = -F(u^{[s]}) \qquad s = 0, 1, 2 \cdots$$

Example: Solve the boundary value problem

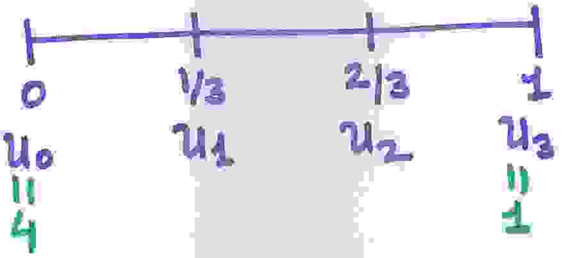
$$u'' = \frac{3}{2} u^2$$

$$u(0) = 4 \qquad u(1) = 1$$

with $h = \frac{1}{3}$. Use a second order finite difference method for its solution.

<u>Sol</u>:

The second order finite difference approximation



$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = \frac{3}{2} u_j^2$$

$$\Rightarrow u_{j-1} - 2u_j + u_{j+1} = \frac{3}{2} \cdot \frac{1}{3} \times \frac{1}{3} \cdot u_j^2$$

$$= \frac{1}{6} u_j^2 \qquad j = 1, 2.$$

<u>For $j = 1$</u>:

$$u_0 - 2u_1 + u_2 = \frac{1}{6} u_1^2$$

Using B.C.

$$u_1^2 + 12u_1 - 6u_2 - 24 = 0$$

<u>For $j = 2$</u>:

$$u_1 - 2u_2 + u_3 = \frac{u_2^2}{6}$$

using B.C.

$$u_2^2 - 6u_1 + 12u_2 - 6 = 0.$$

So we solve:

$$F_1 = u_1^2 + 12u_1 - 6u_2 - 24$$

$$F_2 = u_2^2 - 6u_1 + 12u_2 - 6$$

$$J = \begin{bmatrix} 2u_1 + 12 & -6 \\ -6 & 2u_2 + 12 \end{bmatrix}$$

Therefore

$$J^{[s]} \Delta u^{[s]} = -F(u^{[s]})$$

$$\Rightarrow \begin{bmatrix} 2u_1^{[s]} + 12 & -6 \\ -6 & 2u_2^{[s]} + 12 \end{bmatrix} \begin{bmatrix} \Delta u_1^{[s]} \\ \Delta u_2^{[s]} \end{bmatrix} = - \begin{bmatrix} (u_1^{[s]})^2 + 12u_1^{[s]} - 6u_2^{[s]} - 24 \\ (u_2^{[s]})^2 - 6u_1^{[s]} + 12u_2^{[s]} - 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta u_1^{[s]} \\ \Delta u_2^{[s]} \end{bmatrix} = -\frac{1}{D} \begin{bmatrix} 2u_2^{[s]} + 12 & 6 \\ 6 & 2u_1^{[s]} + 12 \end{bmatrix} \begin{bmatrix} (u_1^{[s]})^2 + 12u_1^{[s]} - 6u_2^{[s]} - 24 \\ (u_2^{[s]})^2 - 6u_1^{[s]} + 12u_2^{[s]} - 6 \end{bmatrix}$$

$$D = \left[2u_1^{[s]} + 12\right]\left[2u_2^{[s]} + 12\right] - 36$$

then $\begin{bmatrix} u_1^{[s+1]} \\ u_2^{[s+1]} \end{bmatrix} = \begin{bmatrix} u_1^{[s]} \\ u_2^{[s]} \end{bmatrix} + \begin{bmatrix} \Delta u_1^{[s]} \\ \Delta u_2^{[s]} \end{bmatrix} \quad s = 0, 1, 2, \ldots$

Taking $u_1^{(0)} = u_2^{(0)} = 1$

$u_1^{[1]} = 2.4500$ $\qquad u_2^{[1]} = 1.5500$

$u_1^{[2]} = 2.2969$ $\qquad u_2^{[2]} = 1.4691$

$u_1^{[3]} = 2.2950$ $\qquad u_2^{[3]} = 1.4679$