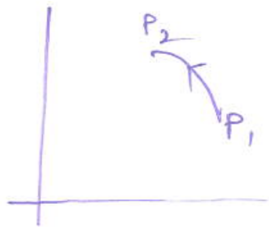
Line Integrals

LECTURE - 20

We know about $\int_a^b f(x)\,dx$, $\int_c^d g(y)\,dy$.

Suppose to compute $\int_C V_1(x,y)\,dx + V_2(x,y)\,dy$

C is an arc of a circle $x^2+y^2=2$ from $P_1(\sqrt{2},\sqrt{2})$ to $\cancel{P_2(\sqrt{3},1)}$ $P_2(1,\sqrt{3})$



$x = 2\cos\theta$, $y = 2\sin\theta$.

$dx = -2\sin\theta\,d\theta$, $dy = 2\cos\theta\,d\theta$.

$$I = \int_{\theta=\pi/4}^{\pi/3} \Big\{ V_1(2\cos\theta, 2\sin\theta)(-2\sin\theta)\,d\theta + V_2(2\cos\theta, 2\sin\theta)(2\cos\theta)\,d\theta \Big\}$$

In general, to compute -

$$\int_C V_1(x,y,z)\,dx + V_2(x,y,z)\,dy + V_3(x,y,z)\,dz$$

Over some path C of some curve,

express the curve parametrically.

i.e. $\quad C: r = r(t)\ ;\ t_1 \le t \le t_2$

In previous examples, $C: \vec{r} = \vec{r}(\theta)$; $\pi/4 \le \theta \le \frac{\pi}{3}$

$= 2\cos\theta\,\hat{i} + 2\sin\theta\,\hat{j}$.

Then,

$$I = \int_{\theta_1}^{\theta_2} \left\{ V_1(t)\frac{dx}{dt} + V_2(t)\frac{dy}{dt} + V_3(t)\frac{dz}{dt} \right\} dt.$$

Parametric representation of a straight line -

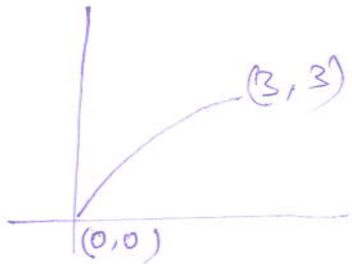
$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0} = t \quad , -\infty < t < +\infty$$

Ex

Compute $\int_C -y\,dx + x\,dy$ along $y^2 = 3x$ from the pt. $(3,3)$, to the pt. $(0,0)$

$\therefore C: \vec{r} = \frac{t^2}{3}\hat{i} + t\hat{j}$

$0 \le t \le 3.$ taking $y = t$, $x = \frac{t^2}{3}$



$$I = \int_{t=3}^{0} \left(-\frac{2t^2}{3}\,dt + \frac{t^2}{3}\,dt\right)$$

$$= \frac{1}{3}\int_0^3 t^2\,dt = \frac{3^3}{9} = 3.$$

Ex

Find the work that is done by a force

$\vec{F} = (x+y)\hat{i} + xy\hat{j} - z^2\hat{k}$

acting on a particle that moves along the line segment from $(0,0,0)$ to $(1,3,1)$ & then along " " " $(1,3,1)$ to $(2,-1,4)$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1+C_2} (x+y)\,dx + xy\,dy - z^2\,dz$$

Where the path C_1 —

$$\frac{x-0}{1-0} = \frac{y-0}{3-0} = \frac{z-0}{1-0} = t$$

$x = t,\ y = 3t,\ z = t$

$0 \le t \le 1$

And the path C_2

$$\frac{x-1}{2-1} = \frac{y-3}{-1-3} = \frac{z-1}{4-1}$$

$x = t+1,\ y = -4t+3,$

$z = 1+3t.$

$0 \le t \le 1.$

$$I_1 = \int_{C_1} (x+y)\,dx + xy\,dy - z^2\,dz.$$

$$= \int_{t=0}^{1} 4t\,dt + 3t^2 \cdot 3\,dt - t^2\,dt = \int_{t=0}^{1} (4t + 8t^2)\,dt$$

$$= \left(2 + \frac{8}{3}\right) = \frac{14}{3}.$$

$$I_2 = \int_0^1 (-3t+4)\,dt + (t+1)(-4t+3)(-4)\,dt - 3(1+3t)^2\,dt$$

$$= \int_0^1 (4-3t)\,dt + 4\int_0^1 (4t^2 + 4t - 3t - 3)\,dt - 3\left[\frac{(1+3t)^3}{9}\right]_0^1$$

$$= \left[\frac{(4-3t)^2}{6}\right]_1^0 - \frac{1}{9}(4^3 - 1) + 4\left(\frac{4t^3}{3} + \frac{t^2}{2} - 3t\right)_0^1$$

$$= \frac{4^2-1}{6} - \frac{63}{9} + 4\left(\frac{4}{3} + \frac{1}{2} - 3\right)$$

$$= \frac{5}{2} - 21 + \frac{16}{3} + 2 - 12 = \frac{47}{6} - 31 = -\cancel{\frac{47}{6}} - \frac{139}{6}$$

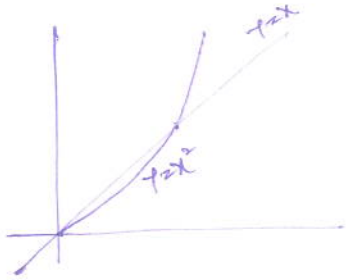
$$\therefore I = I_1 + I_2 = \frac{14}{3} - \frac{139}{6} = +\cancel{\frac{27}{4}} = \cancel{\frac{9}{2}}$$

$$= \frac{28-139}{6} = \boxed{-\frac{37}{2} \quad \text{Ans}}$$

Line integrals independent of path.

Ex let $\vec{F} = y^2 \hat{i} + x^2 \hat{j}$

Evaluate $\int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{r}$ along (1) $y = x$
(2) $y = x^2$



1) Parametric form $x = t, y = t$

$$\int_{t=0}^{1} 2t^2 \, dt = \frac{2}{3}$$

2) $$\int_0^1 t^4 \, dt + 2t^3 \, dt = \frac{1}{5} + \frac{2}{3} = \frac{13}{15}$$

taking $\vec{F} = y\hat{i} + x\hat{j}$

$$\int_{(0,0)}^{(1,1)} y\,dx + x\,dy = \int_{(0,0)}^{(1,1)} d(xy) = [xy]_{(0,0)}^{(1,1)} = 1.$$

Now consider $\vec{F} = y^2\hat{i} + 2xy\,\hat{j}$.
Show that the calculations for (1) and (2)
answers will be same.

Reason - $$\int_{(0,0)}^{(1,1)} \vec{F} \cdot dr = \int_{(0,0)}^{(1,1)} y^2\,dx + 2xy\,dy$$

$$= \int_{(0,0)}^{(1,1)} d(xy^2) = 1.$$

In 3D, $$\int_{P_1}^{P_2} F_1(x,y,z)\,dx + f_2\,dy + f_3\,dz$$

is independent of path, if $\vec{F} = \vec{\nabla} \phi$

In that case. $$I = \int_{P_1}^{P_2} \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = \phi(P_2) - \phi(P_1)$$

Ex - 1

Compute $\int_C \vec{F} \cdot d\vec{r}$

$$\vec{F} = \sin x \,\hat{i} + \cos y \,\hat{j} + z x \,\hat{k}.$$

Take (1) $C: \vec{r}(t) = t^3 \hat{i} - t^2 \hat{j} + t \hat{k}, \quad 0 \le t \le 1$

(2) C: St. line joining $(0,0,0)$ & $(1,1,1)$

Ans 1. $\frac{6}{5} - \cos(1) - \sin(1)$.

Ans - 2

$$\int \sin x \, dx + \int \cos y \, dy + \int z x \, dz.$$

$$= \int_0^1 \sin t \, dt + \int_0^1 \cos t \, dt + \int_0^1 t^2 \, dt$$

[taking $x = t$, $y = t$, $z = t$ the parametric form of the straight line].

$$= -\cos t \Big|_0^1 + \sin t \Big|_0^1 + \frac{1}{3} = -\cos 1 + 1 + \sin 1 + \frac{1}{3}$$

$$= \frac{4}{3} + \cos 1 + \sin 1.$$

Note that

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & \cos y & z x \end{vmatrix} = \hat{i}(0-0) + \hat{j}(0-z) + \hat{k}(0) = -z\hat{j} \neq 0.$$

$\therefore \vec{F} \neq \vec{\nabla}\phi$. [$\because \vec{F}$ is not conservative]

Q. Given $\vec{F} = e^y \hat{i} + x e^y \hat{j} + (z+1) e^z \hat{k}$

① Show that $\vec{F}$ is conservative.

② find ϕ : $\vec{F} = \vec{\nabla}\phi$.

③ Hence compute $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \left[e^y \hat{i} + x e^y \hat{j} + (z+1) e^z \hat{k}\right] \cdot d\vec{r}$

$$= \int_{(0,0,0)}^{(1,1,1)} e^y\, dx + x e^y\, dy + (z+1) e^z\, dz.$$

Ans.

① $\text{Curl}\, \vec{F} = \vec{0}$

② $\vec{F} = \vec{\nabla}\phi$;

$$\frac{\partial\phi}{\partial x} = e^y, \quad \frac{\partial\phi}{\partial y} = x e^y, \quad \frac{\partial\phi}{\partial z} = (z+1) e^z$$

$$\Rightarrow \phi = x e^y + z e^z + c.$$

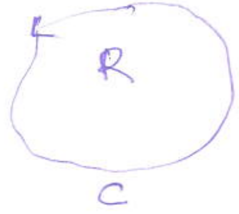
③ $$\int_{(0,0,0)}^{(1,1,1)} F_1\, dx + F_2\, dy + F_3\, dz$$

$$= \int \left(\frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz\right)$$

$$= \int d\phi = \left[\phi\right]_{(0,0,0)}^{(1,1,1)} = 2e$$

Green's theorem in plane

Suppose R is a simply connected region bounded by a curve C (taken in anticlockwise sense).

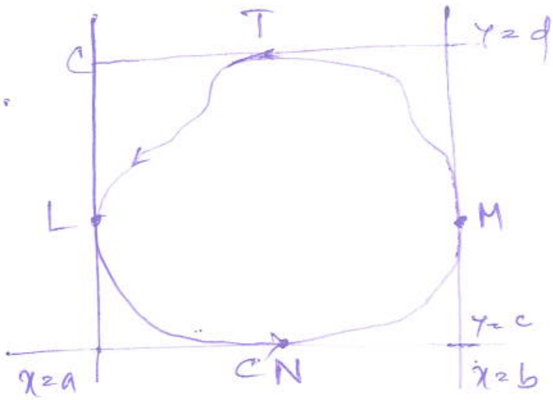


then $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$P(x,y)$ and $Q(x,y)$ are two functions defined in the region R such that

1) P, Q are continuous in R.
2) possess continuous 1^{st} order partial derivatives in R.

Proof



$R: \phi_1(y) \le x \le \phi_2(y)\ ;\ c \le y \le d.$

TLN NMT

$R: \psi_1(x) \le y \le \psi_2(x)\ ;\ a \le x \le b.$

LNM MTL

$$\oint_C P\,dx = \int_{LNM} P(x, \psi_1(x))\,dx + \int_{MTL} P(x, \psi_2(x))\,dx$$

$$= \int_{x=a}^{b} P(x, \psi_1(x))\,dx + \int_{x=b}^{a} P(x, \psi_2(x))\,dx \quad \text{--- (1)}$$

$$\oint_C Q(x,y)\,dy = \int_{TLN} Q(\phi_1(y), y)\,dy + \int_{NMT} Q(\phi_2(y), y)\,dy.$$

P.T.O

$$= \int_{y=d}^{c} Q\left(\phi_1(y), y\right) dy + \int_{c}^{d} Q\left(\phi_2(y), y\right) dy \longrightarrow (2)$$

$$(1) + (2) \Rightarrow \oint_C P\,dx + Q\,dy.$$

$$= \int_{x=a}^{b} \left[P(x, \psi_1(x)) - P(x, \psi_2(x))\right] dx$$

$$+ \int_{y=c}^{d} \left[Q(\phi_2(y), y) - Q(\phi_1(y), y)\right] dy \longrightarrow (3)$$

$$= L.H.S$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \int_{y=c}^{d} \left(\int_{x=\phi_1(y)}^{\phi_2(y)} \frac{\partial Q}{\partial x} dx\right) dy - \int_{x=a}^{b} \left[\int_{y=\psi_1(x)}^{\psi_2(x)} \frac{\partial P}{\partial y} dy\right] dx.$$

$$= \int_{y=c}^{d} \left[Q(\phi_2(y), y) - Q(\phi_1(y), y)\right] dy$$

$$- \int_{x=a}^{b} \left[P(x, \psi_2(x)) - P(x, \psi_1(x))\right] dx$$

$$= \int_{x=a}^{b} \left\{P(x, \psi_1(x)) - P(x, \psi_2(x))\right\} dx,$$

$$+ \int_{y=c}^{d} \left\{Q(\phi_2(y), y) - Q(\phi_1(y), y)\right\} dy$$

$= R.H.S$ Hence $L.H.S = R.H.S$

① Verify Green's theorem -

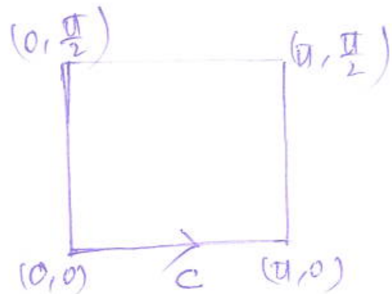
You've to compute both the line integral & the double integral & to show that the values are same.

Ex 1

Evaluate by Green's th.

$$\oint_C e^{-x}(\sin y\, dx + \cos y\, dy)$$

$C \rightarrow$ is the boundary of the rectangle with vertices $(0,0)$, $(\pi, 0)$, $(\pi, \frac{\pi}{2})$, $(0, \frac{\pi}{2})$



Here $P = e^{-x} \sin y$

$Q = e^{-x} \cos y$

$$\frac{\partial Q}{\partial x} = -e^{-x}\cos y, \quad \frac{\partial P}{\partial y} = e^{-x}\cos y$$

$$\oint P\,dx + Q\,dy = 2\int_{y=0}^{\pi/2}\int_{x=0}^{\pi} -e^{-x}\cos y\, dx\, dy$$

$$= 2\left(e^{-\pi} - 1\right)$$

Exercise

Verify Green's theorem in the plane for

$$\oint_C \left[(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy\right]$$

$C \rightarrow$ boundary of the region enclosed by $x=0$, $y=0$, $x+y=1$

Exercise

Verify Green's theorem when $P = xy + y^2$, $Q = x^2$

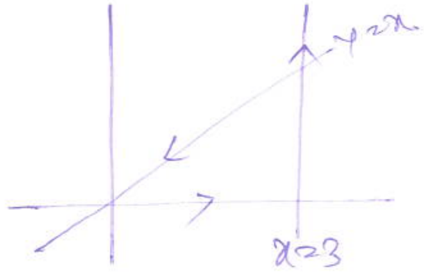
C = closed curve of the region bounded by

$y = x$, $y = x^2$.

Ex Evaluate -

$$\oint_C \left[(y^2 - x^2)\,dx + (x^2+y^2)\,dy\right]$$

$C \to$ triangle bounded by $y = 0$, $x = 3$, $y = x$.



$\Rightarrow$ ~~Over $x=3$, $y=0$~~

Over $y = 0$

$$\oint_C (y^2-x^2)\,dx + (x^2+y^2)\,dy$$

$$= \int_{x=0}^{3} -x^2\,dx \quad \left[\text{Since } y=0 \text{ and } dy=0\right]$$

over $x = 3$

$$\int_C (y^2-x^2)\,dx + (x^2+y^2)\,dy = \int_{y=0}^{3} (9+y^2)\,dy$$

Now take the parametric form of $y = x$ which will contribute $\displaystyle\int_{t=3}^{0} 2t^2\,dt$

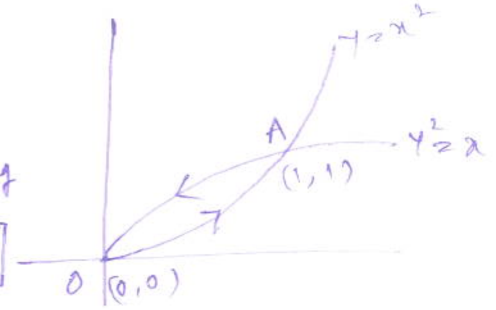
$$\therefore \oint_C \left[(y^2-x^2)\,dx + (x^2+y^2)\,dy\right]$$

$$= \int_{t=3}^{0} 2t^2\,dt + \int_{x=0}^{3} -x^2\,dx + \int_{y=0}^{3} (9+y^2)\,dy = 9$$

Now $\displaystyle\int_{x=0}^{3}\int_{y=0}^{x} (2x-2y)\,dy\,dx = \int_0^3 \left[2xy - y^2\right]_0^x dx = 9$

Ex. Verify $\oint_C (2xy - x^2)\,dx + (x^2+y^2)\,dy$

$C \rightarrow$ bounded by the curves $y = x^2$, $y^2 = x$.



Ans

$$AO = \int_{t=1}^{0} (2t^2 \cdot t - t^4)\,2t\,dt + (t^4+t^2)\,dt \quad \left[\begin{array}{l}\text{taking}\\ y=t\\ x=t^2\end{array}\right]$$

$$= -\int_0^1 5t^4\,dt + 2\int_0^1 t^5\,dt - \int_0^1 t^2\,dt$$

$$= -1 + \frac{2}{6} - \frac{1}{3} = -1$$

$$OA = \int_0^1 (2t \cdot t^2 - t^2)\,dt + (t^2+t^4)\,2t\,dt \quad \left[\begin{array}{l}\text{taking}\\ x=t\\ y=t^2\end{array}\right]$$

$$= 1 - \frac{1}{3} + \frac{2}{6} = 1$$

$$\therefore \oint_C = 0$$

Again $\iint 2x - 2x\,dx\,dy = 0$ [verified]

Exercise ~~Evaluate by Green's theorem~~

~~$\oint_C e^{-x}(\sin y\,dx + \cos$~~

Exercise Compute $\oint_C (P\,dx + Q\,dy)$ using suitable theorem of vector calculus.

$P = xe^{-y^2}$, $Q = -x^2 y e^{-y^2} + \frac{1}{x^2+y^2}$, C: Bounded by $|x| \le a$, $|y| \le a$,