1. Homomorphism,Isomorphism

(1) Let $(a_1, \dots, a_k), \sigma \in S_n$. Then show that $\sigma(a_1, \dots, a_k)\sigma^{-1} = (\sigma(a_1), \dots, \sigma(a_k))$.

(2) Let $\phi : G \longrightarrow H$ be an isomorphism. Prove that $\phi^{-1} : H \longrightarrow G$ is an isomorphism.

(3) Let $a \in G$ and $i_a : G \longrightarrow G$ be the map $i_a(g) = aga^{-1}$ for all $g \in G$. Show that i_a is an automorphism of G. Let $B(G)$ denote the group of bijections of G. Define $\psi : G \longrightarrow B(G)$ by $\psi(a) = i_a$. Show that ψ is a group homomorphism and $\ker\phi = Z(G)$. Show that the image $I(G)$ of G under ψ is a normal subgroup of $Aut(G)$.

(4) Show that the map $\phi : GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$ defined by $\phi(A) = (A^t)^{-1}$ is an automorphism.

(5) Show that the map $\phi : G \longrightarrow G$ defined by $\phi(x) = x^{-1}$ is an automorphism if and only if G is abelian.

(6) Show that upto isomorphism the only cyclic groups are $\mathbb{Z}/n\mathbb{Z}$ for $n = 0, 1, 2, \dots$

(7) Determine the group of automorphism of $\mathbb{Z}, S_3$ and $\mathbb{Z}/n\mathbb{Z}$.

(8) Give an example of a subgroup of index 3 which is not normal.

(9) Show that the functions $f(x) = 1/x$ and $g(x) = (x-1)/x$ generate, under composition of functions, a group isomorphic to S_3.

(10) Let $\phi : G \longrightarrow G'$ be a surjective group homomorphism. Show that for any normal subgroup N of G, $\phi(N)$ is a normal subgroup of G'.

(11) Prove that the subgroup of upper triangular matrices in $GL_3(\mathbb{F}_2)$ is isomorphic to the dihedral group of order 8.

(12) Let G be an abelian group of odd order. Prove that the map $\phi : G \longrightarrow G$ defined by $\phi(g) = g^2$ for all $g \in G$ is an automorphism.

(13) Show that $GL_2(\mathbb{R})$ is not a normal subgroup of $GL_2(\mathbb{C})$.

(14) Give examples of three groups $G \triangleleft H \triangleleft K$ so that G is not normal in K.

(15) Suppose H and K are subgroups of finite index in the group G with $[G : H] = m$ and $[G : K] = n$. Prove that $l.c.m(m, n) \leq [G : H \cap K] \leq mn$. Deduce that if m and n are relatively prime then $[G : H \cap K] = [G : H][G : K]$.

(16) Let $K < H < G$ be subgroups of a finite group G. Show that $[G : K] = [G : H][H : K]$.

(17) Prove that if H and K are finite subgroups of G whose order are relatively prime then $H \cap K = 1$

(18) Let $H < G$. Prove that the map $x \mapsto x^{-1}$ sends each left coset of H in G onto a right coset of H and gives a bijection between the set of left cosets and the set of right cosets of H in G.

(19) For $n \in \mathbb{N}$, $\phi(n) := \{j \in \mathbb{N} | 1 \leq j \leq n \text{ and } (j, n) = 1\}$ is called the Euler's phi function. Use Lagrange's theorem in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ to prove Fermat's Little Theorem: if p is a prime then $a^p \equiv a(mod\ p)$ for all $a \in \mathbb{Z}$.

(20) Let p be a prime and n be a positive integer. Find the order of $\overline{p}$ in $(\mathbb{Z}/(p^n-1)\mathbb{Z})^\times$ and deduce that $n | \phi(p^n - 1)$ (here ϕ is Euler's function).

(21) Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^{\times}$ to prove Euler's Theorem: $a^{\phi(n)} \equiv 1 (mod\ n)$ for every integer a relatively prime to n, where ϕ is Euler's phi function.