Linear Algebra

(Lecture 6)

Problem Set 1.

Basis and dimension.

<u>Theorem</u>: Let V be a vector space that is generated by a set G with exactly n vectors and let L be a subset of V containing m linearly independent vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n-m$ vectors such that $L \cup H$ generate V.

<u>Proof</u>: Proof is by induction on m.

Let $m = 0$; $L = \phi$, take $H = G$.

Let us suppose that the theorem is true for some integer $m \geqslant 0$.

We want to prove that the theorem is true for $m+1$.

Let $V = \{v_1, v_2, \dots, v_{m+1}\}$ be a linearly independent subset of V.

$\Rightarrow \{v_1, v_2, \dots, v_m\}$ is also linearly independent.

We can apply the theorem on this set.

The induction hypothesis says $m \leq n$, & there exists a subset

$\{u_1, u_2, \cdots, u_{n-m}\}$ of G such that

$\{v_1, v_2, \cdots, v_m\} \cup \{u_1, u_2, \cdots, u_{n-m}\}$ generate the vector space V.

This means that there are scalars

$a_1, \cdots, a_m, b_1, \cdots, b_{n-m}$ such that

$$v_{m+1} = a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_{n-m} u_{n-m} \quad (1)$$

Notice that $n - m > 0$.

If not, then v_{m+1} will be a linear combination of $v_1, \cdots, v_m$ which contradicts the assumption that L is a linearly independent set.

$\Rightarrow n \geq m+1$.

Further, at least one of the b_i's has to non-zero.

If not, we arrive at the same contradiction.

WLOG let $b_1 \neq 0$.

Then from (1)

$$u_1 = -\frac{a_1}{b_1} v \quad \cdot \quad -\frac{a_m}{b_1} v_m + \frac{1}{b_1} v_{m+1}$$

$$-\frac{b_2}{b_1} u_2 + \cdots - \frac{b_{n-m}}{b_1} u_{n-m}$$

let

$H = \{u_2, \dots, u_{n-m}\}$.

Then $u_1 \in \operatorname{span}\{L \cup H)$

Trivially, $\{v_1, \dots, v_m, u_2, \dots, u_{n-m}\}$

$\subseteq \operatorname{Span}\{L \cup H)$

$\Rightarrow \{v_1, \dots, v_m, u_1, \dots, u_{n-m}\} \subseteq \operatorname{span}(L \cup H)$

$\Rightarrow \operatorname{span}\{v_1, \dots, v_m, u_1, \dots, u_{n-m}\} \subseteq \operatorname{span}\{L \cup H\}$

$\Rightarrow V \subseteq \operatorname{span}\{L \cup H\}$ This step from induction.

This proves the theorem. ∎

<u>Example</u>: $V = \mathbb{R}^3$

$$G = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$L = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Can you construct H as in the theorem so that $L \cup H = \mathbb{R}^3$.

$$H_0 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$H_0 \cup L = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\operatorname{span}\{H_0 \cup L\} \neq \mathbb{R}^3$$

$$H_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$H_1 \cup L = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

This H_1 works as $\operatorname{span}(H_1 \cup L) = \mathbb{R}^3$

$$H_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

This H_2 also works as

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^3$$

Corollary: Let V be a vector space with dimension n. Then every linearly independent subset of V can be extended to a basis of V.

Exercise:

Let $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$.

Complete it to a basis of $\mathbb{R}^2$.

one option $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\}$ for $a, b \in \mathbb{R}$ and $b \neq 0$.