Assignment - 7

1) (i) $f = x^2 + y - z - 4$, $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$
$= 2x\hat{i} + \hat{j} - \hat{k}$

$\hat{n} = \frac{\nabla f}{|\nabla f|}\Big]_{(2,0,0)} = \frac{4\hat{i} + \hat{j} - \hat{k}}{3\sqrt{2}}$.

(ii) $f = x^2 + 2y^2 + 3z^2$, $\nabla f = 2x\hat{i} + 4y\hat{j} + 6z\hat{k}$

$\hat{n} = \frac{\nabla f}{|\nabla f|}\Big|_{(\sqrt{10},0,0)} = \hat{i}$.

2) (i) $f(x,y) = e^x \cos y$, $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = e^x \cos y\,\hat{i} - e^x \sin y\,\hat{j}$

$\nabla f|_{(0,\pi/4)} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

The directional derivative in the given direction,
$= \nabla f \cdot \hat{n}$
$= \left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}\right) \cdot \frac{(\hat{i} + 3\hat{j})}{\sqrt{10}} = -\frac{2}{\sqrt{20}} = -\frac{1}{\sqrt{5}}$.

(ii) $f(x,y,z) = e^x + yz$
$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = e^x\hat{i} + z\hat{j} + y\hat{k}$

$\nabla f|_{(1,1,1)} = (e\hat{i} + \hat{j} + \hat{k})$

Directional derivative in the given direction is
$= \nabla f \cdot \hat{n} = (e\hat{i} + \hat{j} + \hat{k}) \cdot \frac{(\hat{i} - \hat{j} + \hat{k})}{\sqrt{3}} = \frac{e}{\sqrt{3}}$.

(iii) $f(x,y,z) = \frac{1}{x^2 + y^2 + z^2}$

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

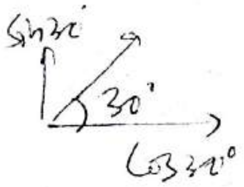
$= \frac{-2x\hat{i} - 2y\hat{j} - 2z\hat{k}}{(x^2 + y^2 + z^2)^2}$

$\nabla f|_{(2,3,1)} = \frac{-4\hat{i} - 6\hat{j} - 2\hat{k}}{14^2}$

The directional derivative in the given direction,
$= \nabla f \cdot \hat{n} = \left(\frac{-4\hat{i} - 6\hat{j} - 2\hat{k}}{196}\right) \cdot \frac{(\hat{i} + \hat{j} - 2\hat{k})}{\sqrt{6}} = -\frac{1}{16\sqrt{6}}$.

$$\frac{\ldots}{(x^2+y^2)^2}\,\hat i + \frac{x\,y}{(x^2+y^2)^2}\,\hat j$$

$$\nabla f|_{(0,1)} = 0\hat i - \hat j$$



the directional derivative

$$\nabla f\cdot\hat n = (0\hat i - \hat j)\cdot\left(\frac{\sqrt3}{2}\hat i + \frac{1}{2}\hat j\right) = -\frac{1}{2}.$$

2) (i) $\vec r = x\hat i + y\hat j + z\hat k$,

$$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x}\hat i + \left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right)\hat j - \frac{1}{r^2}\frac{\partial r}{\partial z}\hat k$$

$$= -\frac{1}{r^2}\cdot\frac{x}{r}\hat i - \frac{1}{r^2}\frac{y}{r}\hat j - \frac{1}{r^2}\frac{\partial r}{\partial z}\hat k$$

$$= -\frac{1}{r^3}(x\hat i + y\hat j + z\hat k) = -\frac{\vec r}{r^3}.$$

(ii) $$\nabla(\log|r|) = \nabla\left(\log(x^2+y^2+z^2)^{1/2}\right)$$

$$= \nabla\left(\frac{1}{2}\log(x^2+y^2+z^2)\right)$$

$$= \frac{1}{2}\sum\frac{\partial}{\partial x}\left(\log(x^2+y^2+z^2)\right)\hat i$$

$$= \frac{1}{\cancel{2}}\sum\frac{1}{(x^2+y^2+z^2)}\cdot \cancel{2}x\,\hat i = \sum\frac{x}{(x^2+y^2+z^2)}\hat i$$

$$= \frac{\vec r}{r^2}.$$

iii) $$\nabla(r^n) = \nabla\left((x^2+y^2+z^2)^{n/2}\right)$$

$$= \sum\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}\hat i = \sum \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot 2x\,\hat i$$

$$= \sum n(x^2+y^2+z^2)^{\frac{n-2}{2}}x\,\hat i = n r^{n-2}\vec r.$$

3) (i) $$\nabla\times F = \begin{vmatrix}\hat i & \hat j & \hat k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_x & f_y & f_z\end{vmatrix}$$ Let $F = f_x\hat i + f_y\hat j + f_z\hat k$

$$= \hat i\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) - \hat j\left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}\right) + \hat k\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)$$

Now $\nabla \cdot (\nabla \times F)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) - \hat{j}\left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}\right) + \hat{k}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 f_z}{\partial x\partial y}} - \cancel{\frac{\partial^2 f_y}{\partial x\partial z}} - \cancel{\frac{\partial^2 f_z}{\partial y\partial x}} + \cancel{\frac{\partial^2 f_x}{\partial y\partial z}} + \cancel{\frac{\partial^2 f_y}{\partial z\partial x}} - \cancel{\frac{\partial^2 f_x}{\partial z\partial y}}$$

$= 0$ [Assuming Commutativity of Partial derivative]

ii) $\operatorname{div}(F\times G) = \sum \hat{i}\cdot\frac{\partial}{\partial x}(F\times G)$

$$= \sum \hat{i}\cdot\left(\frac{\partial F}{\partial x}\times G + F\times\frac{\partial G}{\partial x}\right)$$

$$= \sum\left(\hat{i}\times\frac{\partial F}{\partial x}\right)\cdot G + \sum \hat{i}\cdot\left(F\times\frac{\partial G}{\partial x}\right)$$

$$= \sum\left(\hat{i}\times\frac{\partial F}{\partial x}\right)\cdot G - \left(\left(\sum \hat{i}\times\frac{\partial G}{\partial x}\right)\cdot F\right)$$

$$= \operatorname{curl} F\cdot G - \operatorname{curl} G\cdot F.$$

⑤ (i) $F = 2xz^2\hat{i} + \hat{j} + xy^3z\hat{k}$

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & 1 & xy^3z \end{vmatrix} = \hat{i}(3xy^2z) - \hat{j}(y^3z - 4xz) + \hat{k}(0-0)$$

$$= 3xy^2z\,\hat{i} - (y^3z - 4xz)\hat{j} + 0\hat{k}.$$

(ii) $\vec{F}\times\nabla f$

Now $\nabla f = \hat{i}\frac{\partial}{\partial x}(x^2y) + \hat{j}\frac{\partial}{\partial y}(x^2y) + \hat{k}\frac{\partial}{\partial z}(x^2y)$

$$= 2xy\hat{i} + x^2\hat{j} + 0\hat{k}.$$

$$F\times\nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2xz^2 & 1 & xy^3z \\ 2xy & x^2 & 0 \end{vmatrix} = \hat{i}(-x^3y^3z) - \hat{j}(0 - 2x^2y^4z) + \hat{k}(2x^3z^2 - 2xy).$$

(iii) $F \cdot \nabla g = 4x^2 y + x$

(3). $$\int_C (y\,dx + x\,dy) = \iint_S \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y) \right) dx\,dy$$

$$= \underline{0} \quad \underline{(\text{Green's Th}^{m})}$$

<u>Alternative</u>:

$$\int_{t=0}^{1} \left\{ \sin^9\left(\frac{\pi t}{2}\right) \cdot 9 t^8 dt + t^9 \cdot d\left(\sin^9\left(\frac{\pi t}{2}\right)\right) \right\}$$

$$= \int_{t=0}^{1} \left\{ \sin^9 \frac{\pi t}{2} \cdot 9 t^8 dt + t^9 \cdot 9 \sin^8 \frac{\pi t}{2} \cdot \cos\frac{\pi t}{2} \cdot \pi/2 \, dt \right\}$$

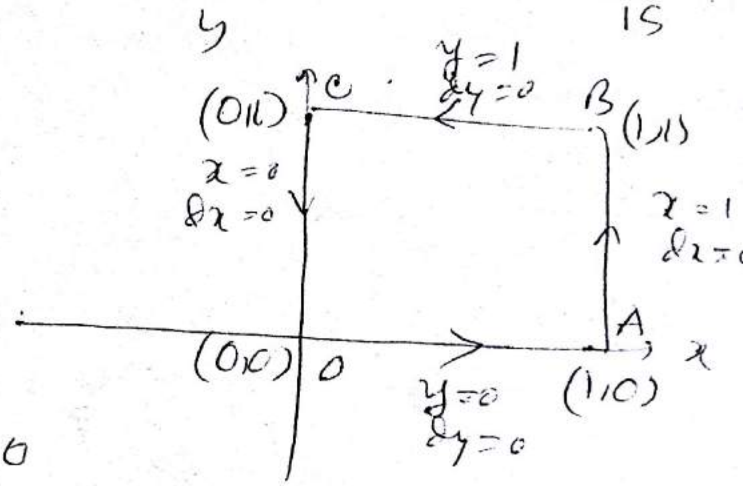
(4). Parametric form $x = t,\ y = t^2,\ z = 1$.

$$\int_{t=0}^{1} (t^2 dt + t^3 d(t^2) + 0) = \int_{t=0}^{1} t^2 dt + 2t^4 dt$$

$$= \frac{1}{3} + \frac{2}{5} = \frac{5+6}{15}$$

$$= \frac{11}{15}$$

(5).



$$\int_C F \cdot dr = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CA}$$

$$= \int_{x=0}^{1} x^2 dx + \int_{y=0}^{1} y\,dy + \int_{x=1}^{0} x^2 dx + \int_{y=1}^{0} 0$$

$$= \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{1/3}$$

$$= \frac{1}{2}.$$

9. If $\nabla\times F = 0$, then by stokes th we ...

$$\oint_C F\cdot dr = \iint_S (\nabla\times F)\cdot n\,ds = 0$$

thus the condn is sufficient

To prove that the condn is necessary we assume that $\oint_C F\cdot dr = 0$ for every closed path.

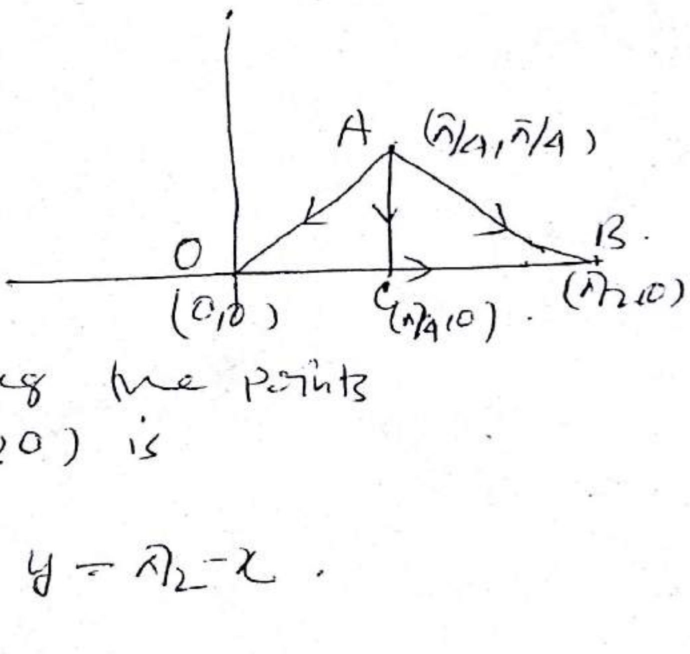
If $\nabla\times F \neq 0$ identically, then $\nabla\times F = 0$ at some point and then from continuity, $\nabla\times F \neq 0$ in some region about the point. Now choose a small plane surface S in this region, the normal to the plane having direction $\nabla\times F$.

Then using stokes thm we have,

$$\oint_C F\cdot dr = \iint_S (\nabla\times F)\cdot ds > 0$$ which is a contradiction

Hence $\nabla\times F = 0$ identically.

(10) We have to show that



$$\int_{AB} = \int_{AC} + \int_{CB}.$$

The eqn of the line joining the points $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ to $(\pi/2, 0)$ is

$$\frac{y-\pi/4}{\pi/4 - 0} = \frac{x-\pi/4}{\pi/4 - \pi/2} \quad \text{or,} \quad y = \pi/2 - x.$$

$$\int_{AB} = \int_{x=\pi/4}^{\pi/2} \left(1-\sin x \sin(\pi/2 - x)\right)dx + \left(1+\cos x\cos(\pi/2 - x)\right)\,d(\pi/2 - x).$$

$$= \int_{x=\pi/4}^{\pi/2} -\sin 2x = -\frac{1}{2}.$$

$$\int_{AC} = \int_{y=\pi/4} \left(1+\frac{1}{\sqrt{2}}\cos y\right) dy$$

Along AC.
$x = \pi/4$
$dx = 0$.

$$= -\pi/4 - \frac{1}{2}.$$

Along $$\int_{CB} = \int_{\pi/4}^{\pi/2} dx$$

Along CB
$y = 0$
$dy = 0$

$$= \pi/2 - \pi/4 = \pi/4.$$

Hence $$\int_{AC} + \int_{CB} = -\cancel{\pi/4} - \frac{1}{2} + \cancel{\pi/4}$$
$$= -\frac{1}{2}.$$

Hence the integral is path independent.

D $\nabla \times \vec{F} =$
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy-3x^2z^2 & -2x^2 & -2x^3z \end{vmatrix}$$

$$= 0\hat{i} - \hat{j}\left(-6x^2z + 6x^2z\right) + \hat{k}\left(-4x + 4x\right)$$
$$= 0\hat{i} + 0\hat{j} + 0\hat{k}.$$

Hence the force field is conservative.

Hence $\oint_C F \cdot dr$ is independent of the curve C or any curve

D (i) $\nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+2yz & y^2 \end{vmatrix} = \hat{i}(2y-2y) - \hat{j}(0-0) + \hat{k}(2x-2x)$$
$$= 0\hat{i} + 0\hat{j} + 0\hat{k}.$$

Hence field is conservative force field.

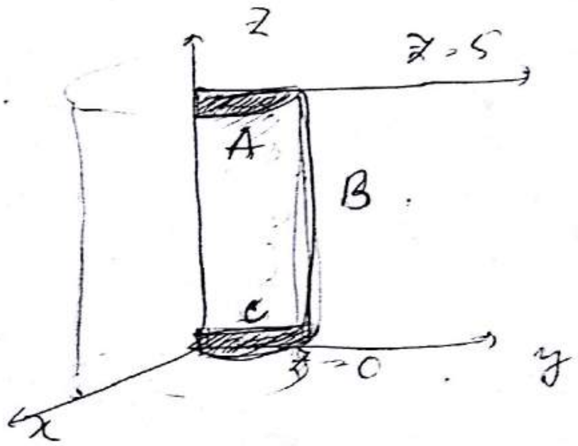
iii. $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$

$= \hat{i}(0-0) - \hat{j}(3z^2-3z^2) + \hat{k}(2x-2x)$

$= 0\hat{i} + 0\hat{j} + 0\hat{k}$.

Hence force field is conservative.

3.



Projecting S on xz plane

$$\iint \vec{F}\cdot\hat{n}\, ds$$

$$\iint_R \vec{F}\cdot\hat{n} \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

A normal to $x^2+y^2=16$ is

$$\hat{n} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4(x^2+y^2)}} = \frac{x\hat{i}+y\hat{j}}{4}.$$

$$\therefore \frac{dx\,dz}{\hat{n}\cdot\hat{j}} = \frac{dx\,dz}{y/4}.$$

Hence the required integral

$$= \int_{z=0}^{5}\int_{x=0}^{4}\left(z\hat{i} - x\hat{j} + 3y^2z\hat{k}\right)\left(\frac{x\hat{i}+y\hat{j}}{4}\right)\frac{dx\,dz}{y/4}.$$

$$= \int_{z=0}^{5}\int_{x=0}^{4}\left(zx - x\sqrt{16-x^2}\right)\frac{dx\,dz}{\sqrt{16-x^2}}$$

$$\int_{z=0} \int_{x=0} \frac{zx}{\sqrt{16-x^2}}\,dx\,dz - \int_{z=0} \int_{x=0} x\,dx\,dz$$

$$= S_1 - S_2$$

$$S_1 = \int_{z=0}^{5} \int_{x=0}^{4} \frac{zx}{\sqrt{16-x^2}}\,dx\,dz$$

$16 - x^2 = t^2$

$-2x\,dx = 2t\,dt$

$$= 5 \int_{4}^{0} -\frac{t\,dt}{t} = 5 \times 4$$

$$= \underline{20}.$$

$$S_2 = \int_{z=0}^{5} dz \int_{x=0}^{4} x\,dx = 5 \times \frac{4^2}{2} = 5 \times 8$$

$$= 40.$$

$$\therefore S_1 - S_2 = 20 - 40 = -20.$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix}$$

$$= \hat{i}(-x+2x) - \hat{j}(-y-0) + \hat{k}(1-2z-1)$$

$$= x\hat{i} + y\hat{j} - 2z\hat{k}$$

A unit normal to the surface is

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4(x^2+y^2+z^2)}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}.$$

Taking Regⁿ above the xy plane we get

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{z/a}.$$

$$\iint_S (\nabla\times F)\cdot n\,ds = \iint_R (x^2+y^2-2z^2)\cdot\frac{a}{z}\,dx\,dy$$

But $z = \sqrt{a^2-x^2-y^2}$ in the xy plan

$$= \iint_R (x^2+y^2) - 2(a^2-x^2-y^2)\cdot\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$$= \iint_R 3(x^2+y^2) - 2a^2\cdot\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy.$$

where R is the area over the circle

$$x^2+y^2=a^2.$$

$$= a \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \frac{(3r^2 - 2a^2)}{\sqrt{a^2-r^2}} \, r\,dr\,d\theta .$$

Now let $a^2 - r^2 = z^2$

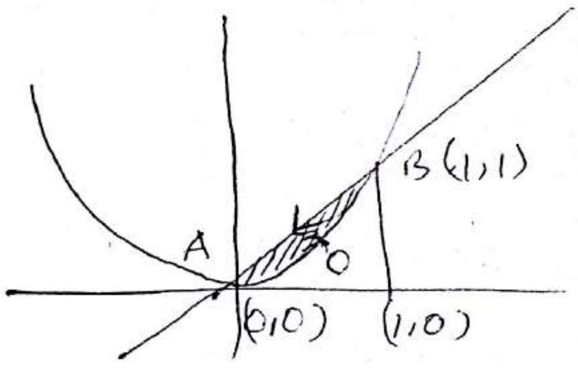
$-\cancel{2}r\,dr = \cancel{2}z\,dz$

$$= 2\pi \int_{z=0}^{a} - \frac{(3a^2 - 3z^2 - 2a^2)}{\cancel{z}} \cancel{z}\,dz$$

$$= -2\pi \int_{z=0}^{a} (a^2 - 3z^2)\,dz$$

$$= -2\pi \left(a^2 z - z^3\right)\Big|_0^a = 0 .$$

15.



Counterclockwise

Along AOB, $y = x^2$, $dy = 2x\,dx$.

$$\int_{x=0}^{1} (x^3 + x^4)\,dx + x^2 \cdot 2x\,dx$$

$$= \int_0^1 (3x^3\,dx + x^4\,dx)$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along BA, $y = x$, $dy = dx$,

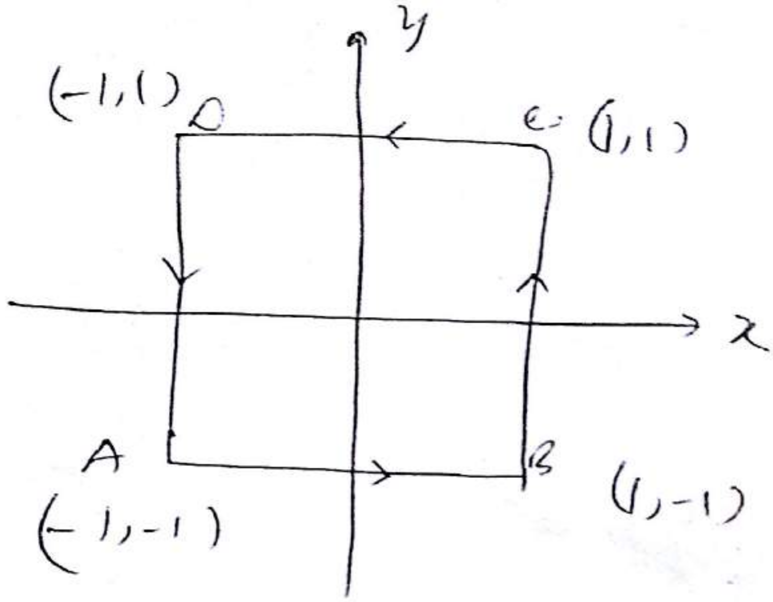
BA:
$$\int_{x=1}^{0} (x^2+x^2)\,dx + x^2\,dx$$
$$= -\int_{x=0}^{1} 3x^2\,dx = -1$$

Hence
$$\int_{AEB} + \int_{BA} = \frac{19}{20} - 1 = -\frac{1}{20}.$$

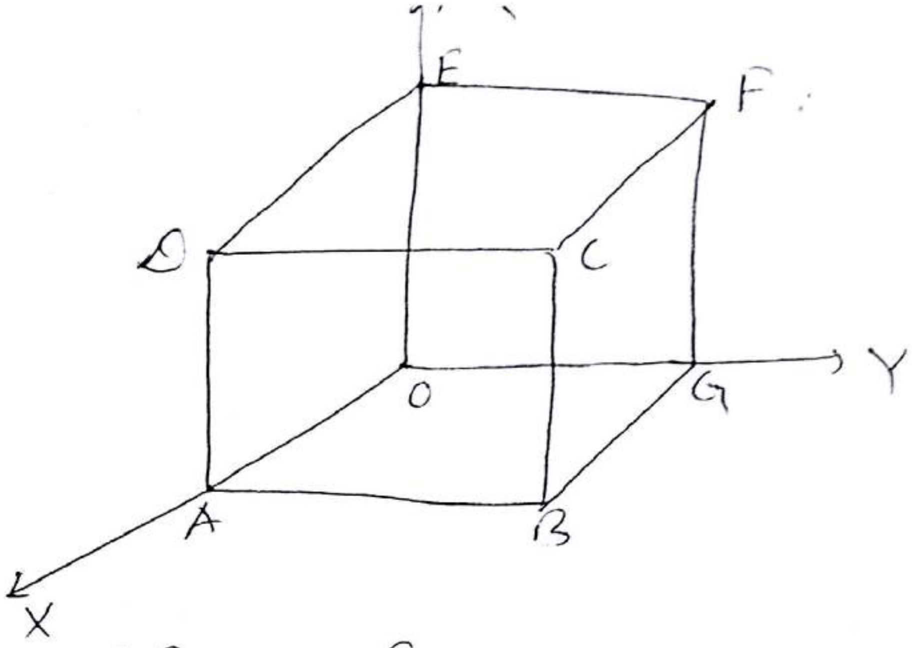
Now apply Green's th^m we have
$$\int_C (M\,dx + N\,dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$
$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (2x - x - 2y)\,dx\,dy$$
$$= \int_{x=0}^{1} (xy - y^2)\Big|_{x^2}^{x}\,dx.$$
$$= \int_{x=0}^{1} (x^4 - x^3)\,dx$$
$$= \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

Hence Green's th^m verified.

⑥



$$\oint_C y\,dx - x\,dy$$
$$= \iint_A \left\{\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right\} dx\,dy$$
$$= -2\iint_A dx\,dy$$
$= -2 \times$ Area of the square
$$= -2 \times 2^2 = -8.$$



$F = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$.

Gauss Divergence thm state that

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div} F\, dv.$$

Now $\operatorname{div} F = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$

$$= 4z - 2y + y = (4z - y).$$

$$\iiint_V \operatorname{div} F\, dv = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z - y)\, dx\, dy\, dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 2z^2 - yz \Big|_{z=0}^{1} dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (2 - y)\, dy\, dx$$

$$= (2 - 1/2)\cdot 1$$

$$= 3/2.$$

P.T.O

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Along ABCD, $n = \hat{i}$, $x = 1$, $dx = 0$

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \int_{y=0}^{1} \int_{z=0}^{1} 4z \, dz \, dy = 2.$$

Along OGFE, $n = -\hat{i}$, $x = 0$, $dx = 0$.

$$\iint_S F \cdot \hat{n} \, ds = 0.$$

Along BGFC, $\hat{n} = \hat{j}$, $y = 1$, $dy = 0$

$$\iint_S F \cdot \hat{n} \, ds = \int_{x=0}^{1} \int_{z=0}^{1} - dx \, dz = -1.$$

Along OADE, $\hat{n} = -\hat{j}$, $y = 0$, $dy = 0$.

$$\iint_S F \cdot \hat{n} \, ds = 0.$$

Along AOGB, $z = 0$, $dz = 0$, $n = -\hat{k}$

$$\iint_S F \cdot \hat{n} \, ds = 0.$$

Along CDEF, $\hat{n} = \hat{k}$, $z = 1$, $dz = 0$.

$$\iint_S F \cdot \hat{n} \, ds = \int_{x=0}^{1} \int_{y=0}^{1} y \, dy \, dx = \frac{1}{2}.$$

Hence

$$\iint_S F \cdot \hat{n} \, ds = \int_{ABCD} + \int_{OGFE} + \int_{BGFC} + \int_{OADE} + \int_{AOGB} + \int_{CDEF}$$

$\ldots\ldots\ldots\ldots/2 = 5/2$

Hence divergence th^m verified.

(18) $F = x^3\hat{i} + x^2y\hat{j} + x^2z\hat{k}$

$ds' = dy\,dz\,\hat{i} + dz\,dx\,\hat{j} + dx\,dy\,\hat{k}$

$$\iint F\cdot ds' = \iint_S F\cdot\hat{n}\,ds$$

$$= \iiint_V \operatorname{div} F\,dv$$

$$= 4\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}}\int_{z=0}^{3}(3x^2+x^2+x^2)\,dx\,d\ldots$$

$$= 12\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} 5x^2\,dy\,dx$$

$$= 12\times 5\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} x^2\,dy\,dx$$

$$= 60\int_{y=0}^{2}\int_{x=0}^{\sqrt{4-y^2}} x^2\,dx\,dy$$

$$= 60\int_{y=0}^{2}\frac{(4-y^2)}{3}\sqrt{4-y^2}\,dy$$

let $y = 2\sin\theta$

$$= 20\int_0^{\pi/2} 4\cos^2\theta\cdot 2\cos\theta\cdot 2\cos\theta\,d\theta$$

$$= 16\times 20 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= \frac{16\times 20}{4} \int_0^{\pi/2} (1+\cos 2\theta)^2 d\theta$$

$$= 80 \int_0^{\pi/2} (1+2\cos 2\theta + \cos^2 2\theta)\, d\theta .$$

$$= 80 \int_0^{\pi/2} \left\{1 + \frac{1}{2}(1+\cos 4\theta)\right\} d\theta$$

$$= 80 \times 3/2 \times \pi/2 .$$

$$= 60\pi$$

(19) $\int_C (-y^3 dx + x^3 dy - z^3 dz)$

$$= \int_C (-y^3\hat{i} + x^3\hat{j} - z^3\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

where $\vec{F} = -y^3\hat{i} + x^3\hat{j} - z^3\hat{k}$.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = 3(x^2+y^2)\hat{k} .$$

A unit normal to the plane $x+y+z=1$ is

$$\hat{n} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}} .$$

Using Stokes th^m we get that,

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, ds = \int_C \vec{F}\cdot d\vec{r}$$

$$\iint_S$$

$$= 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \frac{3(x^2+y^2)}{\cancel{\sqrt{3}}}\,dx\,dy\,\cancel{\sqrt{3}}$$

$x = r\cos\theta$
$y = r\sin\theta$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} 3r^2 \cdot r\,dr\,d\theta$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} 3r^3\,dr\,d\theta$$

$$= 2\pi \times \frac{3}{4} = \frac{3\pi}{2}$$

(20) $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x+3z & x+3y & 2y-3z \end{vmatrix}$

$$= \hat{i}(2-0) - \hat{j}(0-3) + \hat{k}(1)$$

$$= 2\hat{i} + 3\hat{j} + \hat{k}.$$

An outward unit normal to the plane is

$$\hat{n} = \frac{6\hat{i}+3\hat{j}+4\hat{k}}{\sqrt{6^2+3^2+4^2}}$$

$$= \frac{1}{\sqrt{61}}(6\hat{i}+3\hat{j}+4\hat{k})$$

P.T.O

Taking proj along xy plane or xy

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$\hat{n}\cdot\hat{k} = \frac{4}{\sqrt{61}}$$

$$\iint_S \frac{(12+9+4)}{\cancel{\sqrt{61}}} \cdot \frac{\cancel{\sqrt{61}}}{4}\, dx\, dy$$

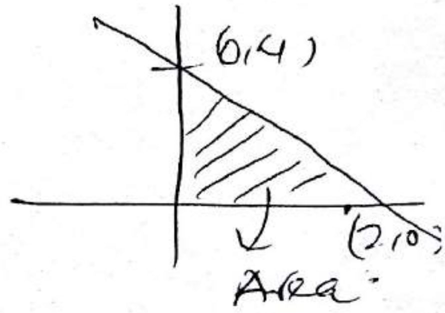
$$= \frac{25}{4} \iint_S dx\, dy$$

$6x+3y=12$

$x/2 + \frac{y}{4} = 1$

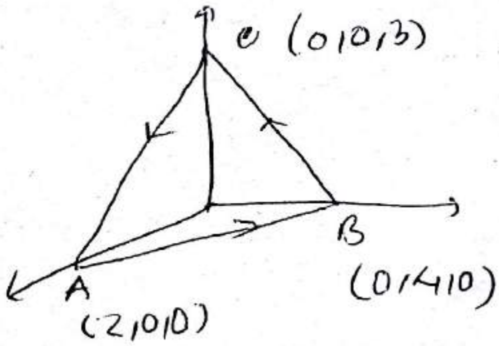
$$= \frac{25}{\cancel{4}} \times \frac{1}{\cancel{2}} \times \cancel{2\times4}$$

$$= 25.$$



Second part

Along AB, $z=0$, $dz=0$,



$$\int \vec{F}\cdot d\vec{r}$$

$$= \int_{x=2}^{0} (3x+\cancel{2z}^{\,0})\,dx + (x+3y)\,dy + \cancel{(2y-3z)\,dz}^{\,0}$$

$$= \int_{x=2}^{0} 3x\,dx + (x+12-6x)(-2\,dx)$$

$$= 3\,x^2/2\Big|_2^0 - 2\int_0^2 (-5x+12)\,dx$$

$$= -6 - 2\left[-6\frac{x^2}{2}\Big|_2^0 + 12x\Big|_2^0\right]$$

$$= -6 - 2\,[10-24]$$

$$= 22.$$

Along BC, $x=0,\ dx=0$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_{y=4}^{0} 3y\,dy + \left(2y - 3\left(\frac{12-3y}{4}\right)\right)\left(-\frac{3}{4}\right)dy$$

$$= \int_{y=4}^{0} \left(-\frac{3}{16}y + \frac{27}{4}\right)dy$$

$$= -\frac{51}{2}.$$

Along CA: $y=0,\ dy=0$ | $z = \frac{6-3x}{2}$.

$$\int_{CA} \vec{F}\cdot d\vec{r} = \int_{x=0}^{2} \left[3x + 3\cdot\left(\frac{6-3x}{2}\right)\right]dx - 3\cdot\frac{6-3x}{2}\cdot\left(-\frac{3}{2}dx\right)$$

$$= \int_{x=0}^{2} \left\{\left(3x - \frac{9}{2}x - \frac{27}{4}x\right) + 9 + \frac{27}{2}\right\}dx$$

$$= \int_{x=0}^{2} \left(-\frac{33}{4}x + \frac{45}{2}\right)dx$$

$$= -\frac{33}{4}\times 2 + \frac{45}{2}\times 2$$

$$= \frac{57}{2}.$$

$$\int_{C} \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CA}$$

$$= 22 - \frac{51}{2} + 57/2$$

$$= 22 + 3 = 25.$$

Hence stokes thm verified.