TAYLOR'S FORMULA (Approximations of differentiable functions by polynomials)

Assume that the function f has all derivatives up to the $(n+1)$th order in some interval containing the point $x=a$.

We wish to find a polynomial $P_n(x)$ of degree n, such that

$$P_n(a) = f(a), \quad P_n'(a) = f'(a), \quad P_n''(a) = f''(a), \ldots, P_n^{(n)}(a) = f^{(n)}(a) \quad -(i)$$

Note that its is expected that the polynomial is 'in some sense' close to the function f al least in the neighbourhood of $x=a$.

Polynomial Construction: consider a polynomial in powers of $(x-a)$ with undetermined coefficients:

$$P_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots + C_n(x-a)^n \quad -(ii)$$

We now define the coefficients $C_0, C_1, \cdots C_n$ so that the conditions (i) are satisfied. First, we calculate the derivatives of $P_n(x)$ as

$$P_n'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots + nC_n(x-a)^{n-1}$$

$$P_n''(x) = 2C_2 + 3\cdot 2\cdot C_3(x-a) + \cdots + n\cdot(n-1)(x-a)^{n-2}$$

$$\vdots$$

$$P_n^{(n)}(x) = n\cdot(n-1)\cdots 2\cdot 1.\ (x-a)^0$$

Using conditions (i), we get

$$C_0 = f(a), \quad C_1 = f'(a), \quad C_2 = \frac{f''(a)}{2\cdot 1} \quad \cdots \quad C_n = \frac{f^{(n)}(a)}{n(n-1)\cdots}$$

subst. in (ii), we obtain:

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\underline{|2}} f''(a) + \cdots + \frac{(x-a)^n}{\underline{|n}} f^{(n)}(a)$$

Denoting $R_n(x)$ the difference between the values of the given function $f(x)$ and the constructed polynomial $P_n(x)$:

$$R_n(x) = f(x) - P_n(x)$$

How to evaluate the remainder $R_n(x)$?

Let us write the remainder in the form

$$R_n(x) = \frac{(x-a)^{n+1}}{\lfloor n+1} Q(x)$$

Alternate proof
Advanced calculus
P.M. Fitzpatrick.

Now we define an auxiliary function of t as

$$F(t) = f(x) - f(t) - \frac{(x-t)}{\lfloor 1} f'(t) - \frac{(x-t)^2}{\lfloor 2} f''(t) - \cdots$$

$$- \frac{(x-t)^n}{\lfloor n} f^{(n)}(t) - \frac{(x-t)^{n+1}}{\lfloor n+1} Q \quad , \quad t \in [a, x]$$

Note that $F(a) = 0$ & $F(x) = 0$ and all other conditions of Rolle's theorem are satisfied for $F(t)$. Then we have

$F'(c) = 0$ for some $c \in (a, x)$

$$\Rightarrow \left[-\frac{(x-t)^n}{\lfloor n} f^{(n+1)}(t) + \frac{(x-t)^n}{\lfloor n} Q \right]_{t=c} = 0$$

$$\Rightarrow Q = f^{(n+1)}(c)$$

Therefore $R_n(x) = \dfrac{(x-a)^{n+1}}{\lfloor n+1} f^{(n+1)}(c)$, $c \in (a, x)$ LAGRANGE FORM OF REMAINDER

Finally $\boxed{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \cdots + \frac{(x-a)^n}{\lfloor n} f^{(n)}(a) + R_n(x)}$

This is called Taylor's formula of the function $f(x)$.

REMARKS:

1. Since c lies between x & a, the remainder may be represented in the following form:

$$R_n(x) = \frac{(x-a)^{n+1}}{\lfloor n+1} f^{(n+1)}\big(a+\theta(x-a)\big), \quad \theta \in (0,1)$$

2. If, we set $a=0$ in the Taylor's formula of the function $f(x)$, then it is called Maclaurin's formula.

3. In the Taylor's formula, if the remainder $R_n(x) \to 0$ as $n \to \infty$, then

$$f(x) = f(a) + (x-a)\,f'(a) + \cdots + \frac{(x-a)^n}{\lfloor n} f^{(n)}(a) + \cdots$$

is called Taylor's series. For $a=0$, it is called Maclaurin's series

SLIDE (3')

EXAMPLE: Obtain the Taylor's formula for the function $f(x) = \sin x$ about the point $x=0$.
Show that the remainder term goes to zero as $n \to \infty$ and write down the Taylor's series exponsion of $f(x)$.
Approximate $\sin 30^\circ$ with the Taylor's polynomial of degree 3 and estimate the error using remainder form.
Verify the error estimate with the exact error.

$$F(x) = \exp(x)$$

$$P_0(x) = 1$$

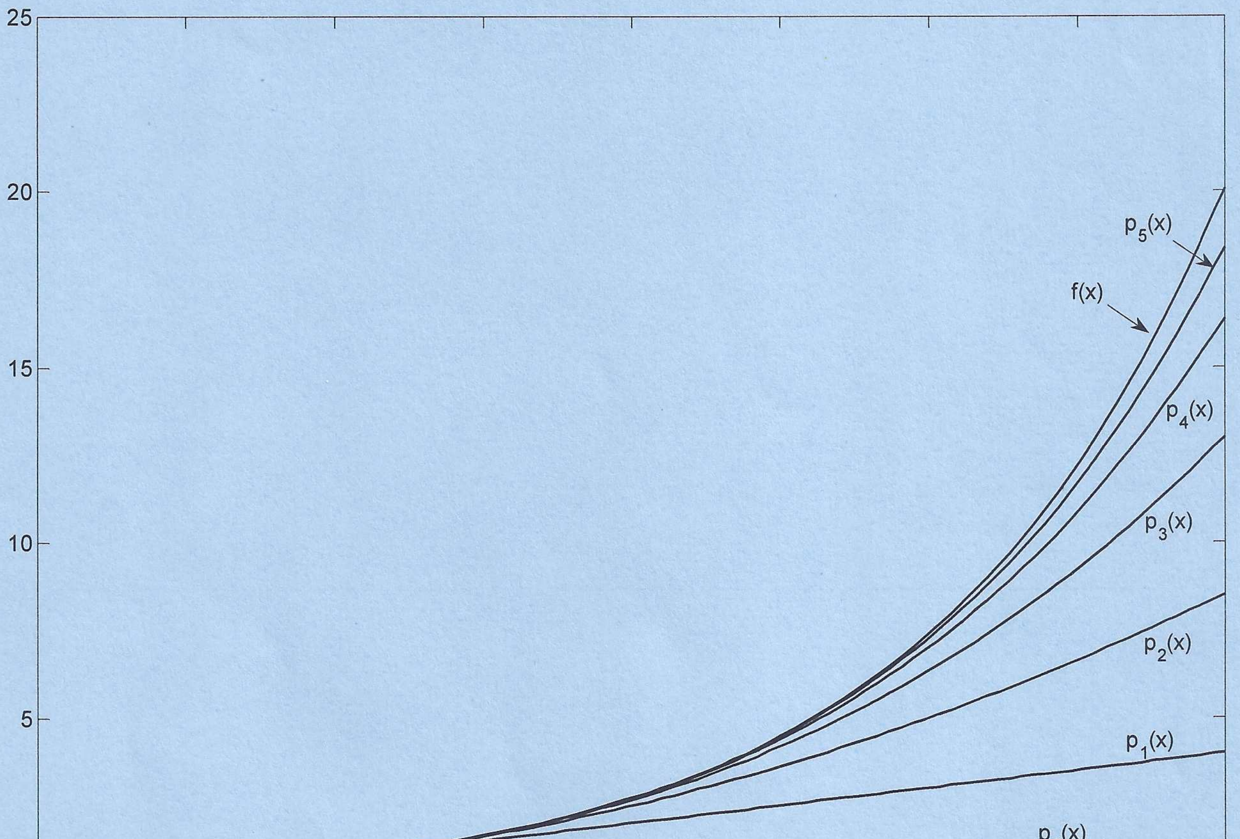
$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + x^2/2$$

$$P_3(x) = 1 + x + x^2/2 + x^3/6$$

$$P_4(x) = 1 + x + x^2/2 + x^3/6 + x^4/24$$

$$P_5(x) = 1 + x + x^2/2 + x^3/6 + x^4/24 + x^5/120$$



SOLUTION:

TAYLOR'S FORMULA

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(IV)}(x) = \sin x$	$f^{(IV)}(0) = 0$
$f^{(V)}(x) = \cos x$	$f^{(V)}(0) = 1$
$\vdots$	
$f^{(2n)}(x) = (-1)^n \sin x$	$f^{(2n)}(0) = 0$
$f^{(2n+1)}(x) = (-1)^n \cos x$	$f^{(2n+1)}(0) = (-1)^n$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{2n}}{(2n)!} f^{(2n)}(0) + \frac{x^{2n+1}}{(2n+1)!} f^{(2n+1)}(0)$$

$$+ \frac{x^{2n+2}}{(2n+2)!} f^{(2n+2)}(c), \quad c \in (0,x)$$

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}(-1)^n + \frac{x^{2n+2}}{(2n+2)!} f^{(2n+2)}(c).$$

REMAINDER → 0:

$$|R_n| = \left| \frac{x^{2n+2}}{(2n+2)!} \cdot (-1)^{n+1} \sin c \right| \leq \left| \frac{x^{2n+2}}{(2n+2)!} \right| = \frac{|x|^{2n+2}}{(2n+2)!}$$

$$\lim_{n \to \infty} \frac{|x|^{2n+2}}{(2n+2)!} = ?$$

$$\lim_{n\to\infty} \frac{|x|^{2n+2}}{\lfloor\!\underline{2n+2}}$$

For a fixed x we can always find a N such that

$$|x| < N$$

Consider $2n+2 > N$ and do the following:

$$\frac{|x|^{2n+2}}{\lfloor\!\underline{2n+2}} = \frac{|x|^{2n+2}}{1\cdot 2 \cdot \ldots\ldots \cdot (2n+2)} = \frac{|x|}{1}\cdot\frac{|x|}{2}\cdots\frac{|x|}{N-1}\cdot\frac{|x|}{N}\,\frac{|x|}{N+1}\cdots\frac{|x|}{(2n+2)}$$

Let $\frac{|x|}{N} = q < 1$

Then

$$\frac{|x|^{2n+2}}{\lfloor\!\underline{2n+2}} < \underbrace{\frac{|x|}{1}\,\frac{|x|}{2}\cdots\frac{|x|}{N-1}}\, q\cdot q\cdot \cdots q$$

$$= \frac{|x|^{N-1}}{\lfloor\!\underline{N-1}}\cdot q^{(2n+2)-(N-1)} = \frac{|x|^{N-1}}{\lfloor\!\underline{N-1}}\cdot \underbrace{q}_{<1}{}^{2n-N+3}$$

As $n \to \infty$

$$\frac{|x|^{2n+2}}{\lfloor\!\underline{2n+2}} \to 0 .\quad \text{Hence} \quad \lim_{n\to\infty} |R_n| = 0 .$$

The Taylor's series expansion is given as

$$\sin x = x - \frac{x^3}{\lfloor\!\underline{3}} + \frac{x^5}{\lfloor\!\underline{5}} - \cdots + (-1)^n \frac{x^{2n+1}}{\lfloor\!\underline{2n+1}} + \cdots$$

APPROXIMATION OF SIN 30°:

$$\sin 30^\circ = \sin\frac{\pi}{6} \simeq \frac{\pi}{6} - \left(\frac{\pi}{6}\right)^3 \frac{1}{\lfloor 3}$$

$$= 0.49967417$$

ERROR ESTIMATE:

$$|R_3(x)| = \left|\frac{x^4}{\lfloor 4} f^{(4)}(c)\right| \Rightarrow \left|R_3(x)\Big|_{x=\pi/6}\right| = \left|\frac{\pi^4}{6^4}\cdot\frac{1}{\lfloor 4}\cdot \sin c\right|$$

$$\le \frac{\pi^4}{6^4}\frac{1}{\lfloor 4} = 0.00313.$$

In this case $f^{IV}(0)=0$, so a better error bound may be obtained

$$|R_4(x)| = \left|\frac{x^5}{\lfloor 5} f^{(5)}(c)\right| \Rightarrow \left|R_4\left(\frac{\pi}{6}\right)\right| = \left|\left(\frac{\pi}{6}\right)^5\cdot\frac{1}{\lfloor 5}\cos c\right|$$

$$\le \frac{\pi^5}{6^5}\frac{1}{\lfloor 5} = 0.000327$$

Exact error:

$$= \sin\left(\frac{\pi}{6}\right) - 0.49967417$$

$$= 0.000325.$$

Ex. Find the number of terms that must be retained in the Taylor's polynomial approximation about the point $x=0$ for the function $\cosh x$ in the interval $[0,1]$ such that $|\text{error}| < 0.001$.

Sol:

$$f(x) = \cosh x \qquad\qquad \cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$f'(x) = \sinh x$$

$$f''(x) = \cosh x$$

$$\vdots$$

$$|R_n(x)| = \left|\frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi)\right| \qquad \xi \in (0,1)$$

$$= \frac{|x|^{n+1}}{(n+1)!}\left|f^{(n+1)}(\xi)\right| \leq \frac{1}{(n+1)!}\left|f^{(n+1)}(\xi)\right|$$

Note that $\left|f^{(n+1)}(\xi)\right| \leq \dfrac{e^{\xi}+e^{-\xi}}{2} < \dfrac{e+e^{-1}}{2}$

Now set $\left(\dfrac{e+\frac{1}{e}}{2}\right)\dfrac{1}{(n+1)!} < 0.001 \Rightarrow n \geq 5$

$\Rightarrow$ Minimum six terms are required.

$$P_5 = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \frac{x^5}{5!} f^{(v)}(0) + \frac{x^6}{6!} f^{(vi)}(0).$$