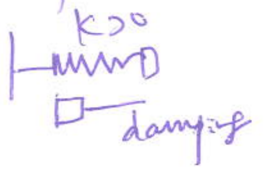
Why 2nd order ODEs ? — (Stability of 2nd order vs higher order)

$\textcircled{a_0}$

application areas where ODE with constant coeffs.:

① Simple pendulum $L\ddot{\theta} + g\theta = 0$

② damped harmonic oscillator



particle position $x = x(t)$

restoring force $f_r(x) = -kx$

friction force $f_f(v) = -\gamma v = -\gamma\dot{x}(t)$

external force $f_e(x)$.

Newton's II law $m\ddot{x}(t) = f_r + f_f + f_e$

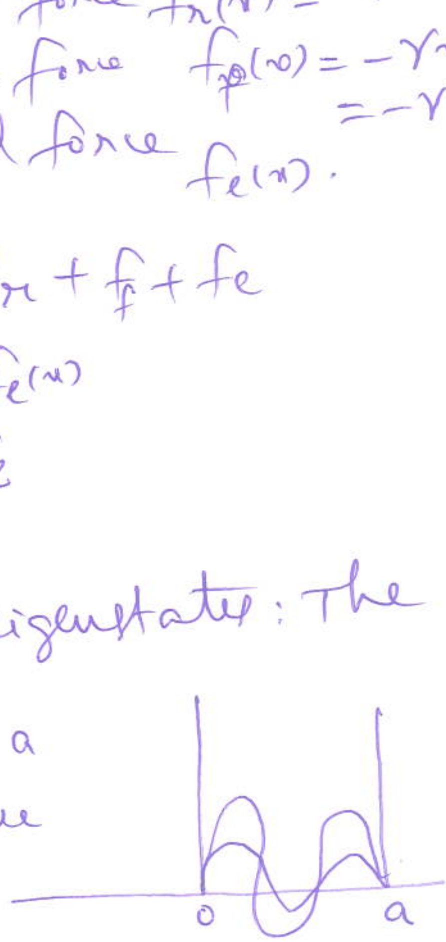
$$m\ddot{x} = -kx - \gamma\dot{x}(t) + f_e(x)$$

$$\Rightarrow \ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}f_e$$

ODE with variable coeffs

① Quantization of energy eigenstates: The infinite potential well

$$U(x) = \begin{cases} 0, & 0 < x < a \\ \infty, & \text{otherwise} \end{cases}$$



$$-\frac{\hbar^2}{2m}\frac{d^2x}{dx^2} + U(x)x = Ex$$

wave function

if $U(x) = f(x)$, we have $x'' + g(x)x = 0$.

② Laplace equation $\bar{v} = \nabla\phi$, $\nabla\cdot\bar{v} = \nabla^2\phi = 0$

(r, θ, ϕ) Legendre

③ Helmholtz $(\nabla^2 - \lambda^2)f = 0$

Bessel fn.

<u>aim:</u> to develop series solutions of the form $\sum_{n=0}^{\infty} a_n x^n$ to given 2nd order ODE with constant coefficients

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad (1)$$

linear 2nd order ODE with variable coefficients

we look for solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad a_n \text{- coefficients}$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

An infinite series of the form $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is called a "power series" about $x = x_0$

$$\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2!\,x^2 + 3!\,x^3 + \cdots$$

$$\sum_{0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

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$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

converges when $|x|<1$

diverges $|x|>1$

$R=1$ radius of convergence

<u>Ex:</u> practice tests of convergence

<u>Q:</u> Can we <u>approximate</u> the solution of a given linear 2nd order ODE with variable coeffts? by a "power series"

Let us try with $y''+y=0$ — (X)

general soln: $y(x) = a_1 \cos x + a_2 \sin x$

Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

adjust the index of the 1st sum

$n \to n+2$

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} + a_n \right] x^n = 0$$

to be valid for any given x.

$$\Rightarrow \quad (n+2)(n+1)\, a_{n+2} + a_n = 0$$

$$\Rightarrow \quad a_{n+2} = -\frac{a_n}{(n+1)(n+2)}, \quad n = 0, 1, 2 \ldots$$

Recurrence relation

$a_2 = \dfrac{-a_0}{1\cdot 2}$; $a_4 = \dfrac{-a_2}{3\cdot 4} = \dfrac{a_0}{1\cdot 2\cdot 3\cdot 4}$

$n=0$ $\qquad$ $n=2$

$a_6 = -\dfrac{a_4}{5\cdot 6} = \dfrac{-a_0}{6!}$

$a_3 = \dfrac{-a_1}{2\cdot 3}$; $a_5 = \dfrac{-a_3}{4\cdot 5} = \dfrac{a_1}{2\cdot 3\cdot 4\cdot 5}$

$n=1$ $\qquad$ $n=3$

similarly.

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - - - \right)$$

$$+ a_1\left(x - \frac{x^3}{3!} - - - \right)$$

say, $y(0) = 1$, $y'(0) = 0$

$$y = a_0 \cos x + a_1 \sin x$$

$$\Rightarrow a_0 =$$

$$a_1 =$$

<u>Example 2</u> $y'' - 2xy' + y = 0$

obtain $y(x)$ as a power series about $x = 0$.

let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ & determine a_n.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$\downarrow$ $\downarrow$

$n \to n+2$ $\cancel{n \to n+1}$

a_5

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+1)(n+2)a_{n+2} - (2n-1)a_n\right]x^n = 0$$

$$\Rightarrow \quad a_{n+2} = \frac{(2n-1)a_n}{(n+1)(n+2)}, \quad n=0,1,2\ldots$$

$n=0,\quad a_2 = -\dfrac{a_0}{1\cdot 2}$

$n=1,\quad a_3 = \dfrac{1\cdot a_1}{2\cdot 3}$

$n=2,\quad a_4 = \dfrac{3\cdot a_2}{3\cdot 4} = -\dfrac{3\,a_0}{1\cdot 2\cdot 3\cdot 4} = -3\dfrac{a_0}{4!}$

$n=3,\quad a_5 = \dfrac{5\,a_3}{4\cdot 5} = 1\cdot\dfrac{5\,a_1}{2\cdot 3\cdot 4\cdot 5}$

$$a_{2n} = \frac{-3\cdot 7\cdot 11\cdot\,\cdots\,(4n-5)\,a_0}{(2n)!}$$

$$a_{2n+1} = \frac{1\cdot 5\cdot 9\cdot\,\cdots\,(4n-3)\,a_1}{(2n+1)!}$$

Find y as a power series about $x=0$ of $y'' + xy' + y = 0$

(26)

$$y' = \sum_{0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{0}^{\infty} n(n-1) a_n x^{n-2}.$$

$$y'' + xy' + y = 0$$

$$\Rightarrow \underbrace{\sum_{0}^{\infty} n(n-1) a_n x^{n-2}}_{n \to n+2.} + x\sum_{0}^{\infty} n a_n x^{n-1} + \sum a_n x^n = 0$$

$$\Rightarrow \sum_{0}^{\infty} (n+1)(n+2) a_{n+2} x^n + \sum_{0}^{\infty} n a_n x^n + \sum a_n x^n = 0$$

$$\therefore \sum_{0}^{\infty} \left((n+1)(n+2) a_{n+2} + (n+1) a_n\right) x^n = 0$$

$$\Rightarrow \cancel{(n+1)}(n+2) a_{n+2} + \cancel{(n+1)} a_n = 0$$

$$\Rightarrow (n+2) a_{n+2} = -a_n.$$

$$\therefore a_{n+2} = \frac{-a_n}{n+2}$$

$$a_{2n} = \frac{(-1)^n}{2^n n!} \cancel{x^{2n}}; \quad a_{2n+1} = \frac{(-2)^n n!}{(2n+1)!} \cancel{x^{2n+1}}$$

$x^2 y'' - y = 0$ obtain power series

sln. about $x = 0$

$$x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$n(n-1) a_n - a_n = 0$$

$$a_n (n(n-1) - 1) = 0$$

$x y'' - y' = 0$ / one can't develop a power series of the form $y = \sum_0^{\infty} a_n x^n$

other hand,

$$y = \sqrt{x}\left(1 + \frac{1}{2} x - \frac{1}{8} x^2 + \cdots\right)$$

can't be represented by $y = \sum_0^{\infty} a_n x^n$

$$y = x^{1/2} \sum_{n=0}^{\infty} a_n x^n$$

$$\boxed{x^2 y'' + x y' + y = 0}$$

b2

$$y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0 \quad \text{— A}$$

$$y'' + p(x)\, y' + q(x)\, y = 0$$

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0$$

$$y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0$$

$$y'' + 2x y' + y = 0 \quad \text{— B}$$

$$\frac{Q}{P} = 2x\,; \quad \frac{R}{P} = 1 \quad \text{— B}$$

$$p(x) = \frac{Q}{P} = \frac{1}{x}$$

$$q(x) = \frac{R}{P} = \frac{1}{x^2} \quad \text{— A}$$

Classification of 2nd order ODE with variable coeffts.

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0 \quad \text{— (1)}$$

$$\Leftrightarrow \quad y'' + p(x)\, y' + q(x)\, y = 0$$

where $p(x) = \frac{Q}{P}$; $q(x) = \frac{R}{P}$

if $p(x)$ and $q(x)$ are finite at $x = x_0$, then $x = x_0$ is called an "ordinary point" of (1).

eg. $y'' + x^3 y' + x y = 0 \quad (x = 0)$

$$y'' + (x-2)^4 y' + (x-2)^5 y = 0$$

$x = 0$

$x = 2$

$$y'' + \frac{(x-2)^3}{x} y' + \frac{(x-2)}{x^2} y = 0$$

$x = 0$ is not an ordinary point

$$y'' + \frac{1}{x^3} y' + \frac{1}{x^4} y = 0$$

$x = 0$ not an ordinary point

can't develop series $\sum_0^\infty a_n x^n$ but $\sum_0^\infty a_n (x-1)^n$

by

$$y'' + p(x)\, y' + q(x)\, y = 0$$

if
$$y \sim A_n (x - x_0)^n \sim x^n$$
$$y' \sim (x - x_0)^{n-1}$$
$$y'' \sim (x - x_0)^{n-2}$$

$x - x_0$ to x

$$x^2 y'' \sim \cancel{x - x_0}\ x^n$$
$$x y' \sim x^n$$

hence linear combination of $x^2 y''$, xy', y must be zero

$$y'' + \frac{(\cdot)}{x} y' + \frac{(\cdot)}{x^2} y = 0$$

$$y'' + p(x)\, y' + q(x)\, y = 0$$

if $x\,p(x)$ and $x^2 q(x)$ are (analytic) finite at $x = 0$, then $x = 0$ is called a regular singular point

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0$$

$$\rightsquigarrow \quad y'' + \frac{Q}{P} y' + \frac{R}{P} y = 0$$

if $\frac{Q}{P}$, $\frac{R}{P}$ are finite at $x = x_0$

then $x = x_0$ is an ordinary point

~~if~~ otherwise, $x = x_0$ is a singular point

eg. $y'' + x y' + x^2 y = 0$ $\quad x = 0$ ordinary point

$y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0$, $\quad x = 0$ singular point

further, if $x \frac{Q}{P}$ and $x^2 \frac{R}{P}$ are finite at $x = x_0$, then $x = x_0$ is called a regular singular point

eg. $x^2 y'' - 2x y' - y = 0$

$$y'' - \frac{2}{x} y' - \frac{1}{x^2} y = 0$$

$\frac{Q}{P} = \frac{-2}{x}$; $\frac{R}{P} = \frac{-1}{x^2}$

$x \frac{Q}{P} = -2$; $x^2 \frac{R}{P} = -1$

$x = 0$ is a regular singular point

$$x y'' - x y' + y = 0$$

$$\sim \quad y'' - y' + \frac{1}{x} y = 0$$

$$\frac{Q}{P} = -1 \implies x \frac{Q}{P} \sim -x$$

$$\frac{R}{P} = \frac{1}{x} \implies x^2 \frac{R}{P} \sim x$$

$x=0$ is a regular singular point

$$(x-1)^2 y'' + \frac{(x-1)}{x^5} y' + y = 0$$

$$y'' + \frac{1}{x^5(x-1)} y' + \frac{1}{(x-1)^2} y = 0$$

$$\frac{Q}{P} = \frac{1}{x^5(x-1)} \implies (x-1)\frac{Q}{P} = \frac{1}{x^5}$$

$$\frac{R}{P} = \frac{1}{(x-1)^2} \implies (x-1)^2 \frac{R}{P} = 1$$

$\implies x=1$ is a regular singular point

Frobenius method to obtain series solution

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look for solution of the ~~*~~ form

$$y(x) = x^m \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{m+n}$$

$$= x^m (a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$= x^{m+1} (a_1 + a_2 x + \cdots) \text{ if } a_0 = 0$$

$$= x^l (b_0 + b_1 x + \cdots)$$

hence $y = \sum_0^{\infty} a_n x^{m+n}$, $a_0 \neq 0$

Example $4xy'' + 2y' - y = 0$ ✓

$$y'' + \frac{1}{2x} y' - \frac{1}{4x} y = 0$$

$x = 0$ is a singular point

$x \frac{1}{2x} = 1/2$

$x^2 \frac{1}{4x} = \frac{x}{4}$

bounded at $x=0$, hence $x = 0$ is a regular singular point.

$$4xy'' + 2y' - y = 0$$

look for $y(x) = \sum_{n=0}^{\infty} a_n x^{m+n}$

$$y' = \sum_{n=0}^{\infty} (m+n) a_n x^{m+n-1}$$

$$y'' = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n x^{m+n-2}$$

$$4xy'' = \sum_{n=0}^{\infty} 4(m+n)(m+n-1) a_n x^{m+n-1}$$

$$\sum_{n=0}^{\infty} 4(m+n)(m+n-1) a_n x^{m+n-1} + \sum_{n=0}^{\infty} 2(m+n) a_n x^{m+n-1} - \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$\sum_{n=-1}^{\infty} 4(m+n)(m+n+1) a_{n+1} x^{m+n} + \sum_{n=-1}^{\infty} 2(m+n+1) a_{n+1} x^{m+n} - \sum_{n=0}^{\infty} a_n x^{m+n} = 0$$

$$4(m-1) m a_0 x^{m-1} + 2 m a_0 x^{m-1} + \sum_{n=0}^{\infty} \left\{ [4(m+n)(m+n+1) + 2(m+n+1)] a_{n+1} - a_n \right\} x^{m+n} = 0$$

$$2(2m-1)m\,a_0\,x^{m-1}$$
$$+\sum_{n=0}^{\infty} x^{m+n}\left[2(m+n+1)(2m+2n+1)a_{n+1}-a_n\right]$$
$$=0$$

should be an identity for all x,

and for some m.

equate equal powers to zero

$$\boxed{2m(2m-1)a_0=0} \text{ — } A$$

$$2(m+n+1)(2m+2n+1)a_{n+1}-a_n=0$$
$$n=0,1,2\ldots$$
— B

since $a_0\neq 0$, A $\Rightarrow$ $m(2m-1)=0$

$\Rightarrow$ $m=0$ or $1/2$

and $$a_{n+1}=\frac{a_n}{2(m+n+1)(2m+2n+1)}$$
$$n=0,1,2\ldots$$

coefficient of a_0, say $f(m)=0$

is called indicial equation

b10

for $\underline{m=0}$

$$a_{n+1} = \frac{a_n}{2(m+n+1)(2m+2n+1)}, \quad n=0,1,2\ldots$$

$$\Rightarrow \quad a_{n+1} = \frac{a_n}{2(n+1)(2n+1)}, \quad n=0,1,2\ldots$$

$$a_1 = \frac{a_0}{2}$$

$$a_2 = \frac{a_1}{2\cdot 2\cdot 3} = \frac{a_0}{2\cdot 2\cdot 2\cdot 3} = \frac{a_0}{(2\cdot 2)!}$$

$$a_3 = \frac{a_2}{2\cdot 3\cdot 5} = \cancel{\frac{a_0}{2\cdot 2\cdot 2\cdot 3\cdot 2\cdot 4\cdot 7}}$$

$$= \frac{a_0}{2\cdot 2\cdot 2\cdot 3\cdot 2\cdot 3\cdot 5} = \frac{a_0}{(2\cdot 3)!}$$

$$a_{n+1} = \frac{a_0}{(2\cdot(n+1))!}$$

$m = 1/2$

$$a_{n+1} = \frac{a_n}{2(2n+3)(n+1)}, \quad n = 0, 1, \ldots$$

$$a_1 = \frac{a_0}{2\cdot 3}$$

$$a_2 = \frac{a_1}{2\cdot 5\cdot 2} = \cancel{\frac{a_0}{(2\cdot 2+1)!}}\ \frac{a_0}{(2\cdot 2+1)!}$$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^n}{(2n)!} + a_0 x^{1/2} \sum_{n=0}^{\infty} \frac{x^n}{(2n+1)!}$$

$$= A x^0 \sum_{n=0}^{\infty} \frac{x^n}{(2n)!} + B x^{1/2} \sum_{n=0}^{\infty} \frac{x^n}{(2n+1)!}$$

indicial equation may invite~~s~~ equal roots, in which case, one may not get a 2nd linearly independent solution

Use the method of Frobenius to find general sdn. of

$$xy'' + y' - xy = 0, \quad a_0 \neq 0 \qquad \circledast$$

Let $y = \sum_{n=0}^{\infty} a_n x^{m+n}$,

$$y' = \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2}$$

$\circledast \Rightarrow$

$$\sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-1} + \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1} - \sum_{n=0}^{\infty} a_n x^{m+n+1} = 0$$

adjust index: to get x^{m+n}

$n \to n+1$ in Σ_1 & Σ_2

$n \to n-1$ in Σ_3

$$\sum_{n=-1}^{\infty} a_{n+1}(m+n+1)(m+n)\,x^{m+n} + \sum_{n=-1}^{\infty} a_{n+1}(m+n+1)\,x^{m+n} - \sum_{n=1}^{\infty} a_{n-1}\,x^{m+n} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty} a_{n+1}(m+n+1)^2 x^{m+n} - \sum_{n=1}^{\infty} a_{n-1}\,x^{m+n} = 0$$

$$\Rightarrow a_0\, m^2 x^{m-1} + a_1 (m+1)^2 x^m + \sum_{n=1}^{\infty}\left[(m+n+1)^2 a_{n+1} - a_{n-1}\right] x^{m+n} = 0$$

$$\Rightarrow a_0\, m^2 = 0\ ; \quad a_1 (m+1)^2 = 0$$

$$a_{n+1} = \frac{a_{n-1}}{(m+n+1)^2}, \quad n = 1, 2, \ldots$$

$\because a_0 \neq 0$, $m = 0$ is a repeated root

also, $a_1 = 0$

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$$a_{n+1} = \frac{a_{n-1}}{(n+1)^2}$$

$a_2 = \dfrac{a_0}{2^2}$ $\qquad$ $a_3 = \dfrac{a_1}{3^2} = 0$

$a_4 = \dfrac{a_2}{4^2} = \dfrac{a_0}{2^2 \cdot 4^2}$ $\qquad$ similarly $a_5, a_7, a_9 \cdots a_{2n+1} = 0$

— — — :

— — — .

$$a_{2n} = \frac{a_0}{2^2 \cdot 4^2 \cdots (2n)^2}$$

$$= \frac{a_0}{(2^2)^n (n!)^2} = \frac{a_0}{2^{2n}(n!)^2}$$

How to get a 2nd solu?

$$xy'' + y = 0$$

$$a_0\, m(m-1)\, x^{m-1}$$

$$+ \sum_{n=0}^{\infty} \left\{ (m+n)(m+n+1)\, a_{n+1} + a_n \right\} x^{m+n} = 0$$

indicial eqn. $m(m-1) = 0$

$\Rightarrow \quad m = 0,\ 1$

$$a_{n+1} = -\frac{a_n}{(m+n)(m+n+1)}, \quad n = 0, 1, 2 \ldots$$

$\underline{m=1}$

$$a_{n+1} = -\frac{a_n}{\cancel{m+1}\,(n+1)(n+2)}, \quad n = 0, 1 \ldots$$

$n=0 \qquad a_1 = -\dfrac{a_0}{1\cdot 2}$

$n=1 \qquad a_2 = -\dfrac{a_1}{2\cdot 3} = \dfrac{a_0}{1\cdot 2\cdot 2\cdot 3}$

$$a_n = \frac{(-1)^n\, a_0}{(n+1)(n!)^2}$$

$\underline{m=0}$

$$a_{n+1} = -\frac{a_n}{n(n+1)}, \quad n = 0, 1 \ldots$$

$$a_{n+1} = \frac{a_n}{n(n+1)}, \quad n=0,1,\cdots \text{b/c}$$

$$a_1 \cdot 0 = a_0$$

$$x^2 y'' - xy' + (x^2-8)y = 0$$

indicial eqn.: $(m^2-2m-8) = (m+2)(m-4) = 0$

coeff of a_1: $(m^2-9) \neq 0$, $m=-2$, $m=4$

$\Rightarrow a_1 = 0$

recurrence relation

$$a_{n+2} = \frac{-a_n}{(m+n)(m+n+2)-8}$$

$$n=0,1,2 \ldots$$

<u>$m=4$</u> $\quad y_1 = x^4 a_0 \left(1 - \frac{x^2}{4\cdot 10} + \frac{x^4}{4\cdot 10\cdot 2\cdot 8} \cdots \right)$

<u>$m=-2$</u> $\quad a_{n+2} = \frac{-a_2}{n(n-2)-8} = \frac{-a_n}{(n-4)(n+2)}$

b/7

$$xy'' + 4y' - xy = 0$$

$$a_0 m(m+3)x^{m-1} + a_1(m+1)(m+4)x^m$$

$$+ \sum_{n=0}^{\infty} \left[(m+n+2)(m+n+5)a_{n+2} - a_n\right] x^{m+n+1} = 0$$

$\Rightarrow$ indicial eqn $m(m+3) = 0$

$$\Rightarrow m = 0, -3$$

also $a_1 = 0$

Recurrence relation

$$a_{n+2} = \frac{a_n}{(m+n+2)(m+n+5)}, \quad n = 0, 1, \ldots$$

$\underline{m=0}$ (larger)

$\underline{m=-3}$ $\quad a_{n+2} = \dfrac{a_n}{(n-1)(n+2)}, \quad n = 0, 1, \ldots$

$\underline{n=0}$ $\quad a_2 = \dfrac{a_0}{-2}$

$\underline{n=1}$. $\quad a_3 = \dfrac{a_1}{0} \Rightarrow 0 \cdot a_3 = a_1 (=0)$

$$\Rightarrow a_3 \text{ is arbitrary}$$

$$a_4 = \frac{a_2}{}$$

1. obtain power series about $x=0$

$$(1-x^2)\,y'' - 2xy' + 6y = 0$$

2.

$$xy'' + y' + xy = 0$$

obtain Frobenius series about $x=0$

Submit only today, before the end of this class!