Real Analysis, Assignment-1

August 28, 2018

1. Let E_1 and E_2 be two bounded nonempty subsets of $\mathbb{R}$, and let $E_1 + E_2 :=\{a+b : a \in E_1, b \in E_2\}$. Prove that $\sup(E_1+E_2) = \sup E_1 + \sup E_2$ and $\inf(E_1+E_2) = \inf E_1 + \inf E_2$.
2. Let S be a nonempty bounded set in $\mathbb{R}$
   (a) Let $a > 0$, and let $aS := \{as : s \in S\}$. Prove that $\inf(aS) =$ a inf S, $\sup(aS) =$ a sup S.
   (b) Let $b < 0$, and let $bS := \{bs : s \in S\}$. Prove that $\inf(bS) =$ b sup S, $\sup(bS) =$ b inf S.
3. If $y > 0$, show that $\exists\ n \in \mathbb{N}$ such that $\frac{1}{2^n} < y$.
4. Show that $\sup\{\frac{1}{n} : n \in \mathbb{N}\} = 1$.
5. Show that a nonempty finite set $S \subseteq \mathbb{R}$ contains its supremum.
6. Let $S \subseteq \mathbb{R}$ and suppose that $s^* :=\sup S$ belongs to S. If $u \notin S$, show that $\sup(S \cup \{u\}) = \sup\{s^*, u\}$.
7. Given a function $f : D \to \mathbb{R}$. We say f is bounded above if the set $f(D) = \{f(x) : x \in D\}$ is bounded above in $\mathbb{R}$. Similarly we can define bounded below functions.
   (a) If $f(x) \leq g(x), \forall x \in D$, show that $\sup\limits_{x\in D} \mathrm{f}(x) \leq \inf\limits_{x\in D} \mathrm{g}(x)$. Does it give any relation between $\sup f(D)$ and $\inf g(D)$?. Justify your results.
   (b) If $f(x) \leq g(y), \forall x, y \in D$, show that $\sup\limits_{x\in D} \mathrm{f}(x) \leq \inf\limits_{y\in D} \mathrm{g}(y)$.
8. Show that set of algebraic numbers is countable.
9. Using rational density theorem , show that if x and y are real numbers with $x < y$, then $\exists$ an irrational number z such that $x < z < y$.
10. Show that if $x \in N_\epsilon(a)$(neighborhood of a), a $\in \mathbb{R}$ for every $\epsilon > 0$, then $x = a$.
11. A metric space is called separable if it contains a countable dense subsets. Show that $\mathbb{R}^n$ is countable.
12. Prove that every separable metric space has a countable base.
13. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

14. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact.

15. Prove that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint open intervals. (use Ex.11)

16. Show that if $\mathbb{R}^k = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed subset of $\mathbb{R}^k$, then at least one of F_n has a nonempty interior.

17. If A is compact, then show that sup A and inf A belongs to A. Give an example of a noncompact set A such that both sup A and inf A belongs to A.

18. If $B_1,\ B_2,\ B_3,\ \cdots,\ B_n$ is a finite open cover of a compact set $A \neq \phi$, can the union $B_1 \cup B_2 \cup \cdots \cup B_n = A$?.

19. Let A be open set. Show that if a finite number of points are removed from A, the remaining set is still open. Is the same true if a countable number of points are removed?

20. Let $K \subset \mathbb{R}$ consists of 0 and the number $\frac{1}{n}$, for $n = 1, 2, 3, \cdots$. Prove that K is compact directly from the definition(without using the Heine-Borel theorem).