Stochastic Process

Probability and Statistics

Name	Definition
Joint probability distribution	$F(x,y) = P(X \le x, Y \le y)$ $= \sum_{x_i \le x} \sum_{y_i \le y} p(x_i, y_i)$ $F_X(x) = \lim_{y\to\infty} F(x,y)$ $F_Y(y) = \lim_{x\to\infty} F(x,y)$ if x nd y are independent $F(x,y) = F_X(x).F_Y(y)$
Expectation	$E(X) = \sum_{X=x_i} x_i f_X(x_i)$
Conditional Probability	$P(A\|B) = \frac{P(A \cap B)}{P(B)}$ $P(A \cap B \cap C) = P(C\|A \cap B).P(B\|A).P(A)$
Law of total Probability	$P(B) = \sum_{i}^{\infty} P(B\|A_i).P(A_i)$

Definitions

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Stochastic Process	A SP is a family of random variables $\{X(t), t \in T\}$ defined on a probability space indexed by parameter t where $t \in T$
States	The values of s v/s $X(t)$ are called states and set of all possible values of states is called $\boldsymbol{State\ Space(E)}$
Types of SP	Discrete State, continuous parameter; Discrete State, discrete parameter; Continuous State, discrete parameter; Continuous State, continuous parameter;
Markov Inequality	Let $X \ge 0$, for any positive constant λ. Then $E(X) \ge \lambda \times P(X \ge \lambda)$
Maximal Inequality for non-negative Martingale	Martingale has constant mean, then we can use markov inequality $E(X_i) = E(X_0) \forall i$ $P(X_n \ge \lambda) \le \frac{E(X_0)}{\lambda}$
n-step Transition Probability	$P_{ij}^n = P(X_{m+n} = j\|X_m = i)$ $= P(X_n = j\|X_o = i)$
Chapman Kolomgrov Equation	$p_{ij}^{m+n} = \sum_{k \in E} p_{ik}^n \times p_{kj}^m$

Classification of States	$\{X_n\} = Markov\ Chain, E = State\ Space$ i) $i \to j$ State j is accessible from state i $p_{ij}^n > 0$ ii) $i \leftrightarrow j$ (state I and j communicate) if $i \to j$ and $j \to i$ iii) Markov chain is irreducible if all states communicate with each other. Otherwise reducible
Period of State i	$d(i) = \gcd I^+ = \{1,2, \dots\}$, $'n'$ such that $p_{ii}^n > 0$ $if\ p_{ii}^n = 0\ \forall n \geq 1 \to d(i) = 0$
Recurrence Time Probability	$f_{ii}^n = f_i^n = p(X_n = i, X_k \neq i, \forall k = 1,2, \dots, n-1 \mid X_0 = i)$ Probability of first visit of state I in 'n' steps $f_i = \sum_{k=1} f_i^k$
Recurrent and Transient States	Recurrent: If $f_i = 1$ i.e return to state 'i' is certain Transient: if $f_i < 1$ return to state i is uncertain.
Mean Recurrence time	$m_i = \sum_n n.f_i^n$ If $m_i = \infty$ then the state i is called recurrent null If $m_i < \infty$ then state I is non-null recurrent or positive recurrent.
Mean time at Transient State	Let $P_T = p_{ij}$ for $\forall i, j \in J$ s_{ij} =expected time periods that a MC is in state j starting from state i $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ and $I_{n,j} = 1\ if\ X_n = j$ $s_{ij} = \delta_{ij} + \sum_{k=1}^{t} p_{ik} S_{kj}$ $S = (I - P_T)^{-1}$ $v_i = \sum_{j=0}^{r-1} s_{ij}$
Conversion of TPM to Q,R,0,1	Arrange Transient states in 0… r-1 and absorbing states as r…N then. U =probability of absorption. Rows represent transient states and Columns represents absorbing states in order of S & R $P = \begin{bmatrix} Q_{r\times r} & R_{r\times(N-r)} \\ 0_{(N-r)\times r} & 1_{(N-r)\times(N-r)} \end{bmatrix}$ $S = [s_{ij}] = (I - Q)^{-1}$ $U = SR$
Probability of ever transition	$e_{ij} = f_{ij}$ =probability that MC will ever make transition into state I from state j $e_{ij} = f_{ij} = \frac{s_{ij} - \delta_{ij}}{s_{jj}}$

Different types of Process

Name	Definition
Counting Process	X(t)= # of events at time (0,t] i) X(0)=0 ii) S<t → X(s) ≤ X(t) iii) X(t)-X(s) → #of events in (s,t]
Independent Increments	Events occurring in disjoint time intervals are independent. $X(b_1) - X(a_1), X(b_2) - X(a_2)$ are independent
Stationary Increments	$X(t+h) - X(s+h) \stackrel{\text{def}}{=} X(t) - X(s) = X(t-s)$
Martingales	$X_n: n = 0,1,2 \ldots$ is a martingale if $a)\ E\lvert X_n\rvert < \infty$ $b)\ E(X_{n+1}\lvert X_0, \ldots, X_n) = X_n$ And $E(E(X_{n+1}\lvert X_0, \ldots, X_n)) = E(X_n)$ $E(X_{n+1}) = E(X_n)$ Martingales have constant means
Gambler's Rum game	Rs i to start Aim= $Rs\ N$ Z_i i^th^ bet st $P(Z_i = 1) = p;\ \ p(Z_i = -1) = q = 1 - p$ X_n for time of gambler after n steps/bets $X_n = Z_1 + Z_2 + \cdots + Z_n + i$
1-D random walk (here $\boldsymbol{T_N}$is time to get Rs N, and $\boldsymbol{T_0}$ is the time he gets broke)	$p_{i,i+1} = p$ $p_{i,i-1} = q = 1 - p$ $p_{ij} = 0\ if\ q \neq p + 1\ or\ p - 1$ $p_i^n = \begin{cases} 0\ \ if\ n = 2k+1 \\ a_k = \binom{2k}{k} p^k (1-p)^k\ \ if\ n = 2k \end{cases}$ $P_i = P(T_N < T_0) = E[P(T_n < T_0\lvert Z_1)]$ $P_i = \begin{cases} \dfrac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^N}\ \ if\ \dfrac{q}{p} \neq 1 \\ \dfrac{i}{N}\ \ if\ \dfrac{q}{p} = 1 \end{cases}$
Branching Process (Z_i =# of offspring's of i^th^ individual) $\boldsymbol{X_n = Z_1 + \cdots + Z_{\lvert X_{n-1}\rvert}}$ π_0=population will die out if $X_0 = 1$ p_j=Probability that one parent will give birth to j offspring.	$E(Z_i) = \mu = \sum j p_j$ $V(Z_i) = \sigma^2 = \sum (j-\mu)^2 p_j$ $E(X_n) = \mu^n E(X_0)$ $V(X_n) = \begin{cases} \mu\sigma^2 & \mu = 1 \\ \frac{\sigma^2\mu^{n-1}(1-\mu^n)}{1-\mu} & \mu \neq 1 \end{cases}$ when E(X_0)=1 $\pi_0 = \begin{cases} 1\ \ if\ \mu \leq 1 \\ \sum_{j=0}^{\infty} \pi_o^j p_j \end{cases}$ If $X_0 = k$ then $\pi_0^* = \pi_0^k$
Poisson Process $N(t)$=number of events in interval (0,t] & N(a)-N(b)=N(c)-N(d) if a-b=c-d	$P(N(t+h) - N(t) = j) = \dfrac{e^{-jh}(\lambda h)^j}{j!}$

	$$p_n(t) = P(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$
Exponential Distribution $\boldsymbol{f_X(x) = \lambda e^{-\lambda x}}$**:pdf** $\boldsymbol{F_X(x) = 1 - e^{-\lambda x}}$**:cdf** $\boldsymbol{\bar{F}_X(x) = 1 - F_x(x) = e^{-\lambda x}}$**:Reliability function** $r(t)$**: Failure Reliability function**	$P(X > x) = \bar{F}_X(x)$ $P(X > s + t \mid X > s) = P(X > t) = e^{-\lambda t}$ $r(t) = \lim_{\Delta t \to 0} \frac{P(t \le x \le t + \Delta t \mid x > t)}{\Delta t} = \lambda$
Interval times and Waiting time distribution N(t)=#no of events occurring in (0,t]$\sim pp(\lambda)$ T_1=time of first arrival T_n=interval time between n-1th and nth interval S_n=$\sum_i^n T_i$ is the waiting time for nth results	MGF of T_i: $M_{T_i}(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$ MGF of S_n: $M_{S_n}(t) = \left(1 - \frac{t}{\lambda}\right)^{-n}$ $S_n \sim Gamma(n, \lambda) = \frac{\lambda^n}{(n-1)!} e^{-\lambda t} t^{n-1}$ $E(S_n) = \frac{n}{\lambda}; V(S_n) = \frac{n}{\lambda^2}$

Important Results

Name	Result
	$$P(AB \mid BC) = P(A \mid BC)$$
Expectation on time periods for transient states	Let N is number of time periods such that a process is in state i starting from state i. $$P(N = n) = f_i.f_i.f_i \dots f_i.(1 - f_i) = f_i^{n-1}(1 - f_i)$$ $$E(N) = \frac{1}{1 - f_i}$$
Probability of ultimate absorption **State=** $\{\boldsymbol{0, 1, \dots, N}\}$ $\boldsymbol{0, 1, \dots, r - 1}$ as transient, $\boldsymbol{r, \dots N}$ as reccurent	$$u_{ik} = u_i = P(absorption\ in\ state\ k \mid x_0 = i) = p_{ik} + \sum_{j=0}^{r-1} p_{ij} u_j$$ $$v_i = E(T \mid X_o = i) = 1 + \sum_{j=0}^{r-1} p_{ij} v_j$$