1. Group Action and Sylow's Theorem

(1) Prove or disprove: An abelian group is simple if it has prime order.
(2) Prove that a group of prime power order is not simple.
(3) Determine the class equation for each of the following groups: (a) Quaternion group Q_8, (b) the Klein 4-group, (c) the dihedral group D_4, (d) the group of upper triangular matrices in $GL_2(\mathbb{F}_3)$, $(e) S_5$. .
(4) Determine all groups having atmost three conjugacy classes.
(5) Let N be a normal subgroup of a group G. Suppose $|N| = 5$ and $|G|$ is odd. Prove that $N \subseteq Z(G)$.
(6) Let G be a finite group with a proper subgroup H whose index is the smallest prime number p dividing $|G|$. Show that H is a normal subgroup of G.
(7) Let G be a group of order $2m$ where m is odd. Show that G has a normal subgroup of index 2.
(8) Find the number of Sylow p-subgroups of S_p where p is a prime number. Hence deduce Wilson's Theorem in number theory: p divides $(p-1)! + 1$.
(9) Let G be a group of order $p^\alpha m$ such that $p \nmid m$. Then G has an unique Sylow p-subgroup P if and only if P is normal in G.
(10) How many elements of order 5 are there in a group of order 20 ?
(11) Show that a group of order 30 is not simple.
(12) Classify group of order 8.
(13) Show that the subgroup of strictly upper triangular matrices in $GL_n(\mathbb{F}_p)$ is a Sylow p-subgroup.
(14) Show that the number of Sylow p-subgroups of $GL_2(\mathbb{F}_p)$ is $p + 1$.
(15) Prove that group of order pq are not simple where p, q are prime numbers.
(16) Classify group of order 55.