LECTURE 9 (Friday 3-4 p.m. 5.2.16)

Fixed Point Iteration Continued

x_0 is a fixed point of a function g

$g(x_0) = x_0$ / $g(x) = x$.

(*) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \longrightarrow$ N.R. method.

Fixed point iteration method

(**) $x_{n+1} = g(x_n)$; $n = 0, 1, 2, 3, \ldots$

Given $f(x) = 0 \longrightarrow (1)$

Write (1) as $x = g(x)$ so that

$|g'(x)| < 1 \quad \forall x \in [a, b]$ (prescribed).

or, $|g'(x_0)| < 1$; $x_0 \longrightarrow$ given initial approximation.

Comparing (*) & (**) we see that –

NR method is a particular case of Fixed point iteration scheme.

Q. Find a numercial approximation to the intersection between the line $y = x$ & the curve $y = \dfrac{1.984}{\ln(x)}$, starting from $x_0 = 2.4$ at $x = 0$, by performing a convergent FP iteration scheme. Show x_n for $n = 1, 2, 3$.

Sol. To solve $x = \dfrac{1.984}{\ln(x)}$ by FP iteration.

If $g(x) = \dfrac{1.984}{\ln(x)}$ find $|g'(x)|_{x=2.4} = 1.078 > 1$

So we can't take $g(x)$ as $\dfrac{1.984}{\ln(x)}$.

$\ln(x) = \dfrac{1.984}{x} \Rightarrow x = e^{\frac{1.984}{4}} = g_1(x)$

Now, $|g_1'(2.4)| = 0.787 < 1$.

P.T.O

So, $x_{n+1} = g_1(x_n)$

$$\Rightarrow x_{n+1} = e^{\frac{1.984}{x_n}} \quad ; n = 0, 1, 2.$$

$x_1 = 2.286, \quad x_2 = 2.382, \quad x_3 = 2.3 \ldots$

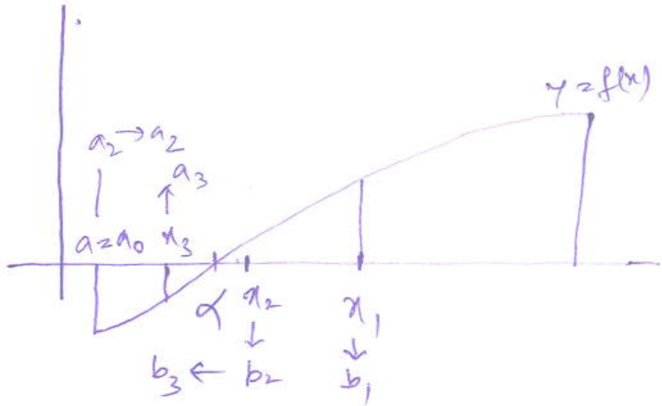
Bisection method

$y = f(x)$

$f(\alpha) = 0$ that means α is an exact root.

Given $[a, b]$

While $a = a_0$, $b = b_0$



$f(a) < 0$, $f(b) > 0$, so that $f(a)f(b) < 0$.

$$x_1 = \frac{a_0 + b_0}{2}$$

Now check — wheather $f(a_0)f(x_1) < 0$?

Ans: Yes $\Rightarrow$ $a_0 \to a_1$, $x_1 \to b_1$;

then $x_2 = \dfrac{a_1 + b_1}{2}$

Ans: No $\Rightarrow$ $f(b_0)f(x_1) \leq 0$

If $f(b_0)f(x_1) = 0$, then x_1 is the exact root otherwise

$f(b_0)f(x_1) < 0 \Rightarrow x_2 = \dfrac{x_1 + b_0}{2}$

Next, Is $f(a_1)f(x_2) < 0$? If Yes then $a_1 \to a_2$, $x_2 \to b_2$

and $x_3 = \dfrac{a_2 + b_2}{2}$

And so on...

Q. Find the root of $f(x) = 10^x + x - 4 = 0$ correct to 3 decimal-places lying in the interval $[.5, .55]$

Ans—

n	a_n	b_n	x_{n+1}	$f(x_{n+1})$
0	.5	.55	0.525	−ve
1	.525	.55	0.5375	−ve
2	.5375	.55	0.5438	+ve
3	.5375	.5438	0.54065	+ve
4	.5375	.54065	0.53906	−ve
5	0.53904	.54065	0.53984	+ve
6	.53904	.53984	0.5394	+ve
7	.53904	.5394	0.539	

Since x_7 & x_8 are matching till 3rd decimal place, therefore the root is 0.539. Correct to 3 decimal places.

- Bisection method is linearly convergent.
  i.e. its order of convergence is 1.

$$|\alpha - x_{n+1}| \leq C|\alpha - x_n|^p$$

$$|\alpha - x_{n+1}| \leq C|\alpha - x_n|$$

$$|\alpha - x_1| \leq C|\alpha - x_0|$$

$$|\alpha - x_2| \leq C|\alpha - x_1| \leq C^2|\alpha - x_0|$$

$$|\alpha - x_n| \leq C^n|\alpha - x_0|$$

$\therefore$ As $n \to \infty$, $x_n \to \alpha$ if $0 < C < 1$.

Note – Finite differences will be attached with the next lecture.