2.2 Pigeonhole Principle Continued ...

Example 2.2.1. *1. Let $\{a_1, a_2, \ldots, a_{mn+1}\}$ be a sequence of distinct $mn+1$ real numbers. Then prove that this sequence has a subsequence of either $(m+1)$ numbers that is strictly increasing or $(n+1)$ numbers that is strictly decreasing.*

Observation: *The statement is NOT TRUE if there are exactly mn numbers. For example, consider the sequence $4, 3, 2, 1, 8, 7, 6, 5, 12, 11, 10, 9$ of $12 = 3 \times 4$ distinct numbers. This sequence neither has an increasing subsequence of 4 numbers nor a decreasing subsequence of 5 numbers. Also, observe that if we take any number different from $1, 2, \ldots, 12$ and place it at any position in the above sequence then it can be verified that there is either an increasing subsequence of 4 numbers or a decreasing subsequence of 5 numbers. For example, if we place 7.5,*

(a) "before the number 4" or "between the numbers 8 and 7" or "after the number 9", then there is a decreasing subsequence of length 5.

(b) "between the numbers 4 and 8" or between the numbers 7 and 9, then there is an increasing subsequence of length 4.

Proof. Let T be the given sequence. That is, $T = \{a_k\}_{k=1}^{mn+1}$ and define

$$\ell_i = \max_s \{s : \text{ an increasing subsequence of length } s \text{ exists starting with } a_i\}.$$

Then there are $mn+1$ positive integers $\ell_1, \ell_2, \ldots, \ell_{mn+1}$. If there exists a $j, 1 \leq j \leq mn+1$, such that $\ell_j \geq m+1$, then by definition of ℓ_j, there exists an increasing sequence of length $m+1$ starting with a_j and thus the result follows. So, on the contrary assume that $\ell_i \leq m$, for $1 \leq i \leq mn+1$.

That is, we have $mn+1$ numbers $(\ell_1, \ldots, \ell_{mn+1})$ and all of them have to be put in the boxes numbered $1, 2, \ldots, m$. So, by the generalized pigeonhole principle, there are at least $\left\lceil \dfrac{mn+1}{m} \right\rceil = n+1$ numbers (ℓ_i's) that lies in the same box. Therefore, let us assume that there exist numbers $1 \leq i_1 < i_2 < \cdots < i_{n+1} \leq mn+1$, such that

$$\ell_{i_1} = \ell_{i_2} = \cdots = \ell_{i_{n+1}}. \tag{2.1}$$

That is, the length of the largest increasing subsequences of T starting with the numbers $a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}}$ are all equal. We now claim that $a_{i_1} > a_{i_2} > \cdots > a_{i_{n+1}}$.

We will show that $a_{i_1} > a_{i_2}$. A similar argument will give the other inequalities and hence the proof of the claim. On the contrary, let if possible $a_{i_1} < a_{i_2}$ (recall that a_i's are distinct) and consider a largest increasing subsequence $a_{i_2} = \alpha_1 < \alpha_2 < \cdots < \alpha_{i_2}$ of T, starting with a_{i_2}, that has length ℓ_{i_2}. This subsequence with the assumption that $a_{i_1} < a_{i_2}$ gives an increasing subsequence

$$a_{i_1} < a_{i_2} = \alpha_1 < \alpha_2 < \cdots < \alpha_{\ell_{i_2}}$$

of T, starting with a_{i_1}, of length $\ell_{i_2}+1$. So, by definition of ℓ_i's, $\ell_{i_1} \geq \ell_{i_2}+1$. This gives a contradiction to the equality, $\ell_{i_1} = \ell_{i_2}$, in Equation (2.1). Hence the proof of the example is complete. ∎

2. *Prove that there exist two powers of* 3 *whose difference is divisible by* 2011.

   Proof. Consider the set $S = \{1 = 3^0, 3, 3^2, 3^3, \ldots, 3^{2011}\}$. Then $|S| = 2012$. Also, we know that when we divide positive integers by 2011 then the possible remainders are $0, 1, 2, \ldots, 2010$ (corresponding to exactly 2011 boxes). So, if we divide the numbers in S with 2011, then by pigeonhole principle there will exist at least two numbers $0 \leq i < j \leq 2011$, such that the remainders of 3^j and 3^i, when divided by 2011, are equal. That is, 2011 divides $3^j - 3^i$. Hence, this completes the proof.

   Observe that this argument also implies that "there exists a positive integer ℓ such that 2011 *divides* $3^\ell - 1$*" or "there exists a positive power of* 3 *that leaves a remainder* 1 *when divided by* 2011*" as* $\gcd(3, 2011) = 1$. ∎

3. *Prove that there exists a power of three that ends with* 0001.

   Proof. Consider the set $S = \{1 = 3^0, 3, 3^2, 3^3, \ldots\}$. Now, let us divide each element of S by 10^4. As $|S| > 10^4$, there exist $i > j$ such that the remainders of 3^i and 3^j, when are divided by 10^4, are equal. But $\gcd(10^4, 3) = 1$ and thus, 10^4 divides $3^\ell - 1$. That is, $3^\ell - 1 = s \cdot 10^4$ for some positive integer s. That is, $3^\ell = s \cdot 10^4 + 1$ and hence the result follows. ∎