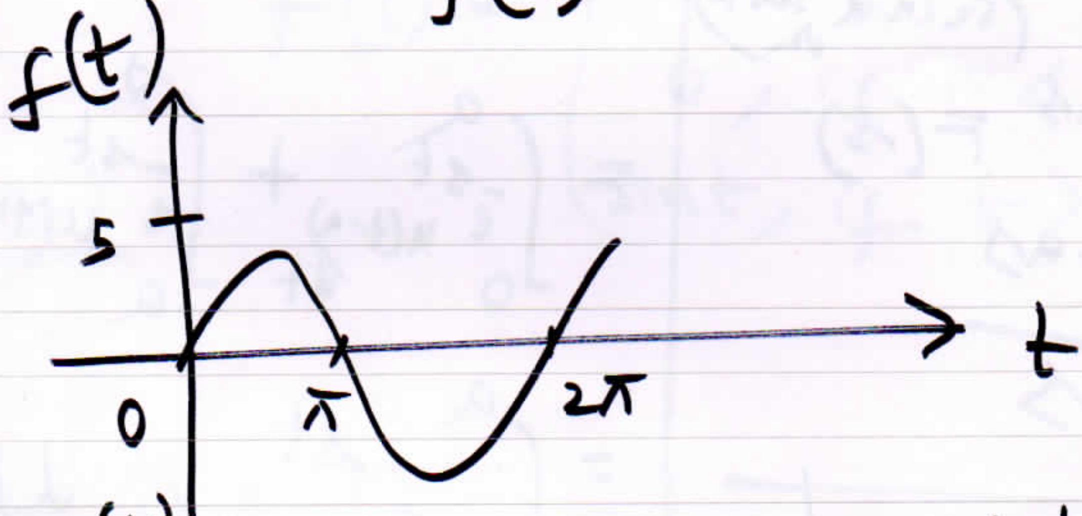
Date 01/08/2017

Lecture 6

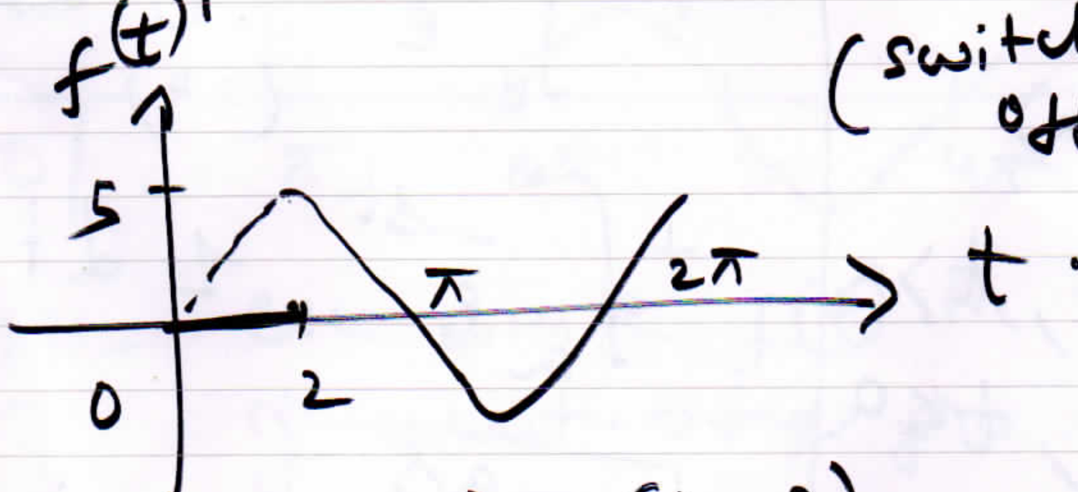
Unit step Function :-

$u(t-a)$ or $H(t-a)$

$$f(t) = 5 \sin t$$

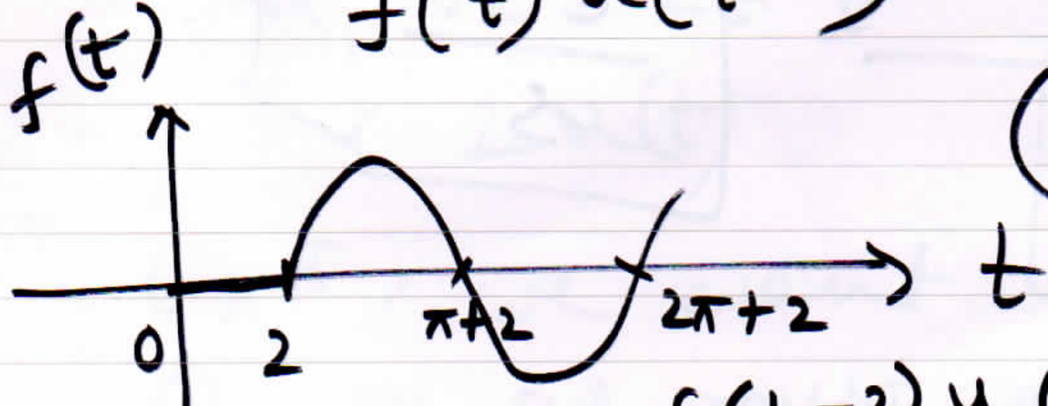


(switching off ε on)



$f(t)\, u(t-2)$

(shift)



$f(t-2)\, u(t-2)$

Ex/

$$\mathcal{L}\{u(t-a)\}$$ (by def.

$$= \int_0^\infty e^{-st} \cdot u(t-a)\,dt$$

$$\mathcal{L}\{1 \cdot u(t-a)\}$$ (second shift th)

$$= e^{-as} F(s)$$

$$= \frac{e^{-as}}{s}$$

$$= \int_0^a e^{-st} u(t-a)\,dt + \int_a^\infty e^{-st} u(t-a)\,dt$$

$$u(t-a) = \begin{cases} 1, & t>a \\ 0, & t<a \end{cases}$$

$$= \int_0^a e^{-st} \cdot 0 \cdot dt \quad (=0)$$

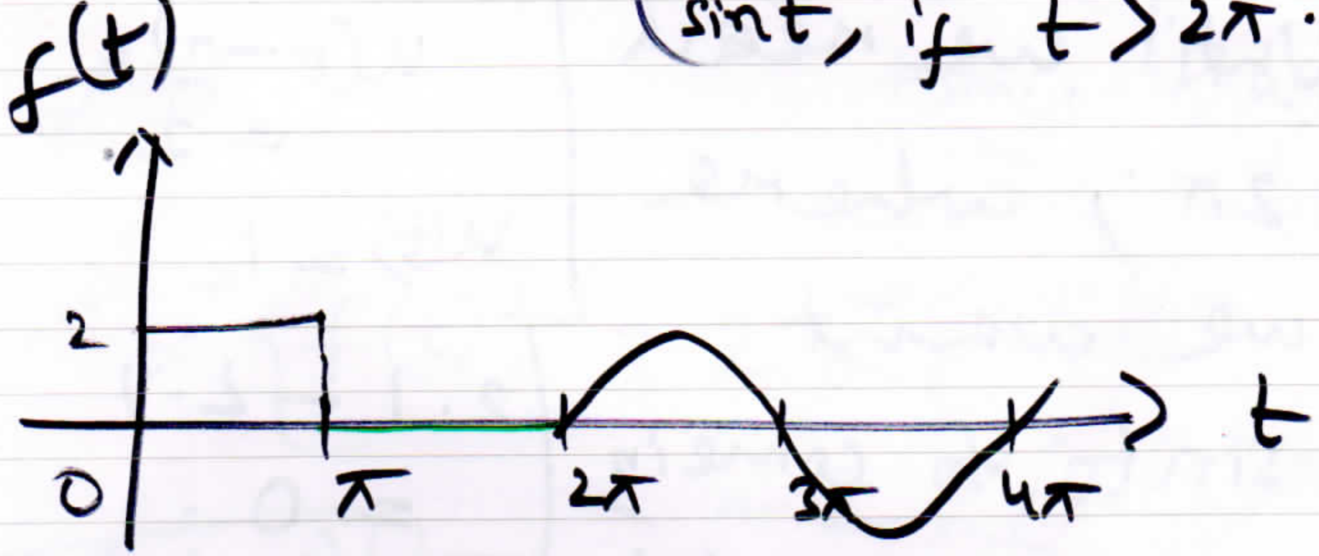
$$+ \int_a^\infty e^{-st} \cdot 1\,dt$$

$$= \boxed{\frac{e^{-as}}{s}}$$ $(s>0)$

Application of Unit step function (Uses of)

Q) Find the transform of the f^n

$$f(t) = \begin{cases} 2, & \text{if } 0 < t < \pi \\ 0, & \text{if } \pi < t < 2\pi \\ \sin t, & \text{if } t > 2\pi. \end{cases}$$



Sol^n:- We write $f(t)$ in terms of unit step f^{n}.

For $0 < t < \pi$,

$2u(t)$

$$\left[u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \right]$$

$t > \pi$, we want 0, so we must subtract the step f^n $2u(t-\pi)$

Then we have

$2u(t) - 2u(t-\pi)$

$= 0$, when $\underline{\underline{t > \pi}}$

Until we reach 2π; where we want sint to come in & so we add $u(t-2\pi)$.

$u(t-\pi) = 1.$

$u(t) = 1.$

$2.1 - 2.1$

$= 0.$

Together.

$f(t) = 2u(t) - 2u(t-\pi) \underline{\underline{+ \sin t \cdot u(t-2\pi)}}$

$t > 2\pi$

$2 - 2 + \sin t$

$= \sin t.$

the last term

$\sin t \cdot u(t-2\pi)$

$= u(t-2\pi) \cdot \sin(t-2\pi)$

[since $\sin t$ is periodic]

so that $f(t) = 2u(t) - 2u(t-\pi)$

$\cancel{\mathcal{L} f}$ $+ \sin(t-2\pi) \cdot u(t-2\pi)$

$$\therefore \mathcal{L}\{f(t)\} = 2\mathcal{L}\{u(t)\} - 2\mathcal{L}\{u(t-\pi)\} + \mathcal{L}\{\sin(t-2\pi) \cdot u(t-2\pi)\}$$

$$\boxed{\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}}$$

$$= 2/s - 2 \cdot \frac{e^{-\pi s}}{s}$$

(how?)

$$+ \frac{e^{-2\pi s}}{(s^2+1)}$$

Ex/ (Inverse Transform)

Find the inverse Laplace transform $f(t)$ of

$$F(s) = \frac{2}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{4e^{-2s}}{s} + \frac{s e^{-\pi s}}{s^2+1}$$

Soln:- Without the exponential functions, the four terms of $F(s)$ could have the inverses $2t$, $-2t$, -4, $\cos t$.

Hence,

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{4e^{-2s}}{s} + \frac{s e^{-\pi s}}{s^2+1}\right\}$$

[by second shifting the]

$$= 2t - 2(t-2)u(t-2) - 4u(t-2) + \cos(t-\pi)u(t-\pi)$$

$$\mathcal{L}^{-1}\left\{e^{-2s} \cdot 2/s^2\right\}$$

$$= \underbrace{f(t-2)} \cdot u(t-2)$$

$$= (t-2)\, u(t-2).$$

$$\mathcal{L}^{-1}\left\{e^{-\pi s} \cdot \frac{s}{s^2+1}\right\}$$

$$= f(t-\pi) \cdot u(t-\pi)$$

$$= \cos(t-\pi) \cdot u(t-\pi)$$

eti.

$$f(t) = 2t - 2t\,u(t-2) + \cancel{4u(t-2)} - \cancel{4u(t-2)} - \cos t\; u(t-\pi).$$

$$= 2t - 2t\,u(t-2) - \cos t \cdot u(t-\pi)$$

$$\Rightarrow f(t) = \begin{cases} 2t, & \text{if } 0 < t < 2 \\ 2t - 2t = 0, & \text{if } 2 < t < \pi \end{cases}$$

$$f(t) = \begin{cases} 2t, & \text{if } 0 < t < 2 \\ 2t - 2t = 0, & \text{if } 2 < t < \pi \\ 2t - 2t - \cos t = -\cos t, & \text{if } t > \pi \end{cases}$$

In general,

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases}$$

is the same as

$$f(t) = g(t) - g(t)\,u(t-a) + h(t)\,u(t-a)$$

s'lly, $f(t) = \begin{cases} 0, & 0 \leq t < a \\ g(t), & a \leq t \leq b \\ 0, & t \geq b \end{cases}$

is

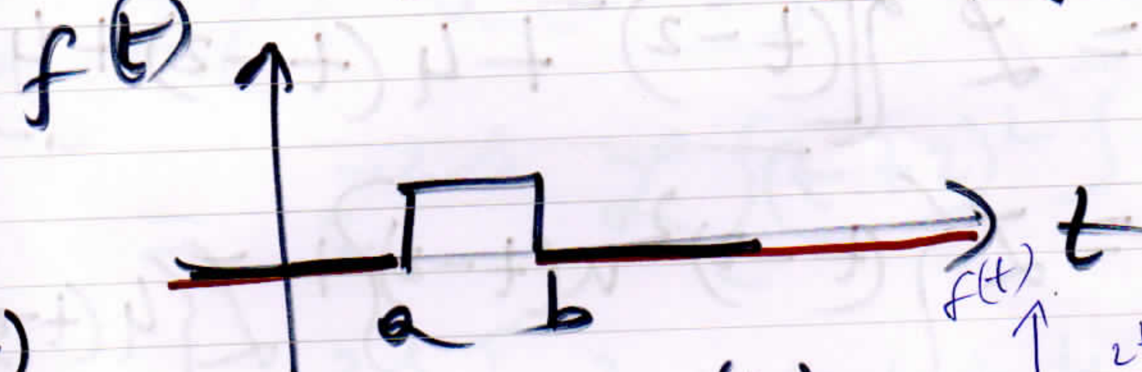
$$f(t) = g(t)\left[u(t-a) - u(t-b)\right]$$

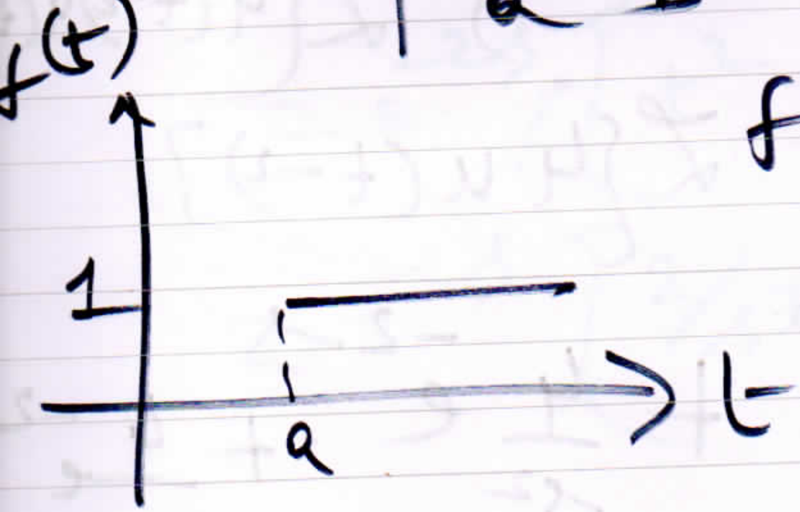
defined on $(-\infty, \infty)$, $u(t-a)$

If $f(t) = 1$, then

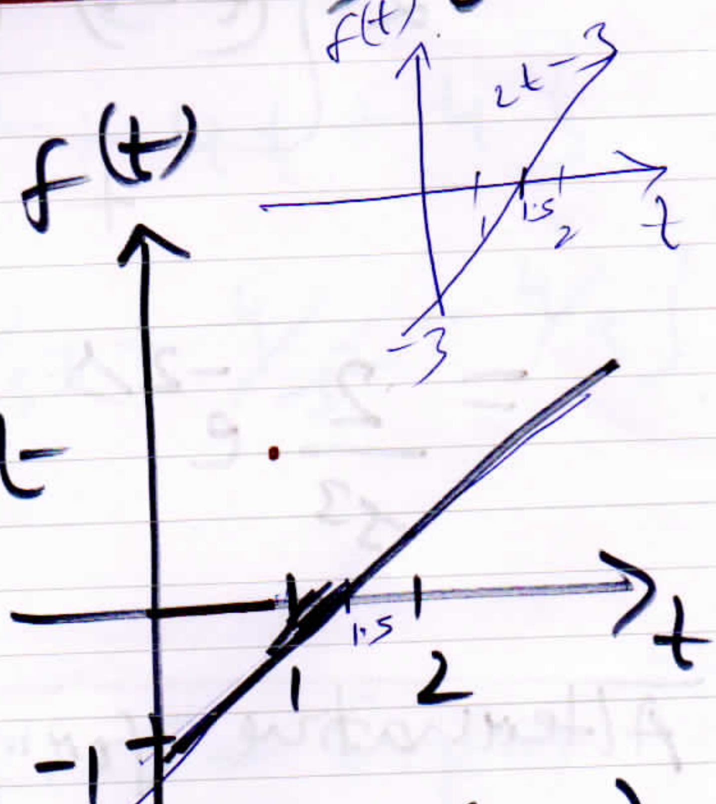
$$f(t) = u(t-a) - u(t-b)$$

→ boxcar f^n

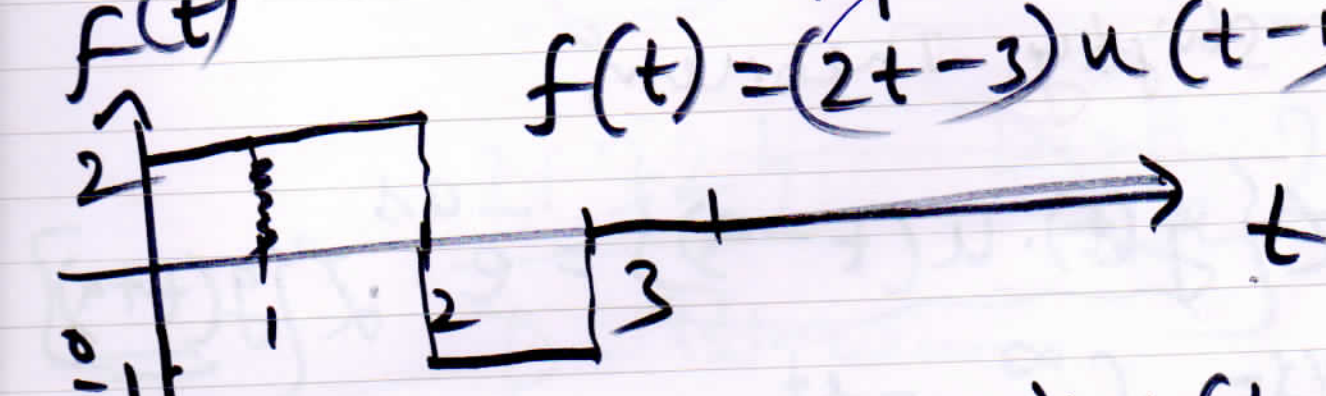




$u(t-a)$



$$f(t) = (2t-3)\,u(t-1)$$



$$f(t) = 2 - 3u(t-2) + u(t-3)$$

$$\mathcal{L}\left\{g(t)\,u(t-a)\right\}$$

$$= \mathcal{L}\left\{t^2 \cdot u(t-2)\right\}$$

$$= \mathcal{L}\left\{\left[(t-2)^2 + 4(t-2) + 4\right] u(t-2)\right\}$$

$$= \mathcal{L}\left\{(t-2)^2 u(t-2)\right\} + \mathcal{L}\left\{4(t-2)u(t-2)\right\}$$
$$+ \mathcal{L}\left\{4\,u(t-2)\right\}$$

$$= \frac{2}{s^3} \cdot e^{-2s} + \frac{4}{s^2} e^{-2s} + \frac{4}{s} e^{-2s}$$

Alternative form of Second shifting theorem

$$\mathcal{L}\left\{g(t) \cdot u(t-a)\right\} = e^{-as} \mathcal{L}\left\{g(t+a)\right\}$$

$$L.H.S = \int_a^\infty e^{-st} \cdot g(t)\, dt \cdot \left[u = t-a\right.$$

7. $\mathcal{L}\{t^2 u(t-2)\}$

$$= e^{-2s}\,\mathcal{L}\{(t+2)^2\}$$

$$= e^{-2s}\,\mathcal{L}\{t^2+4t+4\}$$

$$= e^{-2s}\left[2/s^3 + 4/s^2 + 4/s\right].$$

Sat
~~Fri~~day: 5 – 6 pm
Thurs

N 324 Dept: of Mathematics.

D. Brown.