SET 3

Indian Institute of Technology
Department of Mathematics
Autumn End Semester Examination-2012
Subject Name: Mathematics I
Subject No: MA10001
No. of students:1400 Time: 3 hrs F.M. 50

Instructions: Answer ALL questions. Numerals in right margin indicate marks. Answer each question in a NEW page and all the parts of the same question TOGETHER. Write question SET NUMBER on the top of your answer script.

1. (a) Evaluate the integral $\oint \frac{z-1}{z^2+1}\,dz$ around each curve in counter clockwise direction,

$$\text{(i) } |z-i|=1 \quad \text{(ii) } |z-1|=1 \quad \text{(iii) } |z+i|=1$$

(b) Find the value of the integral $\oint_{|z|=1} \frac{e^{3z}}{(4z-\pi i)^3}\,dz$. Also name the singularity of the integrand at $z=\frac{\pi i}{4}$.

(c) Find Laurent series expansion of $f(z)=\frac{1}{(z+3)(z+1)}$ in the region $1<|z|<3$. [3+2+3]

2. Find the general solution of the following differential equations.

(a) $y\sin x dx + (y^3 - 2y^2\cos x + \cos x)dy = 0$

(b) $(x^2+y^2+2x)dx + 2ydy = 0$

(c) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x + xe^x$

[3+2+3]

3. (a) Solve $y'' - 3y' + 2y = \frac{e^x}{1+e^x}$ using method of variation of parameters.

(b) Solve the following system of differential equations,

$$(5D+4)x - (2D+1)y = e^{-t}, \; (D+8)x - 3y = 5e^{-t}$$

(c) Find the general solution of $x^3y''' + 3x^2y'' + xy' + 8y = 65\cos(\log x)$. [3+2+3]

4. (a) Given that $\sin u = \phi(x,y)$, where ϕ is a homogeneous function of degree 2. Using Euler's Theorem express $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$ as a function of u.

(b) Find the extreme values of $x-2y+2z$ on the surface $x^2+y^2+z^2=1$ using Lagrange multiplier method. (Given that there is no point of inflection.)

(c) Test the continuity of the complex valued function

$$f(z) = \frac{Re(z)}{Im(z) - 1} + i\frac{(Re(z + i\bar{z}))(Im(z - i\bar{z}))}{|z|^2}$$

at $z = i$ and $z = 0$. [3+3+3]

5. (a) Find the radius of curvature of the curve $y = 4\sin x - \sin 2x$ at $x = \pi/2$.
   (b) Find $\frac{dy}{dx}$ if $x^y + y^x = a^b$.
   (c) If $z = z(x, y)$ and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, then write $\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}$ in terms of the partial derivatives of z with respect to x and y.
   (d) Discuss the continuity of the function $f : R^2 \to R$ at $(0, 0)$, where

$$f(x, y) = \frac{x^2 - x\sqrt{y}}{x^2 + y}, \quad \text{for} \quad (x, y) \neq (0, 0) \quad \text{and} \quad 0 \quad \text{at} \quad (x, y) = (0, 0).$$

[2+2+2+2]

6. (a) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ such that $u - v = (x - y)(x^2 + 4xy + y^2)$, then find $f(z)$ in terms of z.

   (b) Find the value of the integral $\int_0^{1+i}(x - y + ix^2)\, dz$
   (i) Along the straight line from $z = 0$ to $z = 1 + i$
   (ii) Along the real axis from 0 to 1 and then from 1 to 1+i.

   (c) Applying Cauchy's Integral Theorem for multiply connected domain find the value of $\oint_C \frac{2z-3}{z^2-3z-18}\, dz$, where $C : |z| = 8$ in counter clockwise direction. [3+3+3]

****************THE END****************