4.7 Euler's Theorem for Planar Graphs

We will now use a result of Euler, proved for a convex polyhedron, to prove that the graphs K_5 and $K_{3,3}$ are non-planar. The theorem states that for any convex polyhedron, the sum of the number of vertices and the number of faces equals the number of edges plus two. This result also holds for a planar graph.

Let $X = (V, E)$ be a tree on n vertices. Then $|E| = n - 1$. Also, observe that X is a planar graph and in any planar embedding of X, there is only one face. Hence, we do see that the Euler condition, $|V| - |E| +$ No. of faces $= n - (n-1) + 1 = 2$, is satisfied for any tree. It can also be verified that the result holds for a cycle, C_n as in this case the number of vertices equals the number of edges and the number of faces equals 2 (inside face and the outside face of the cycle).

To understand the theorem, consider a graph X and its planar embedding. The planar embedding divides the plane into regions/faces. Figure 4.16 gives two examples of planar graphs. We also label the faces with f_i's to understand different faces.

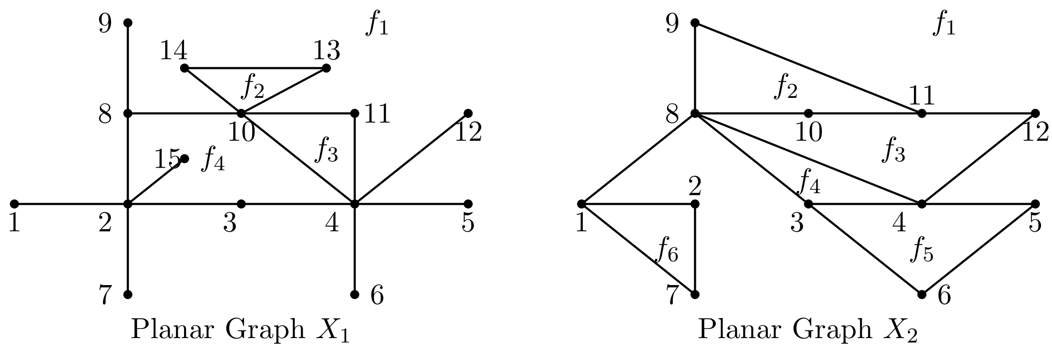

Figure 4.16: Planar graphs with labeled faces to understand the Euler's theorem

The faces of the planar graph X_1 and their corresponding edges are listed below.

Face	Corresponding Edges
f_1	$\{9,8\},\{8,9\},\{8,2\},\{2,1\},\{1,2\},\{2,7\},\{7,2\},\{2,3\},\{3,4\},\{4,6\},\{6,4\},\{4,5\},$ $\{5,4\},\{4,12\},\{12,4\},\{4,11\},\{11,10\},\{10,13\},\{13,14\},\{14,10\},\{10,8\},\{8,9\}$
f_2	$\{10,13\},\{13,14\},\{14,10\}$
f_3	$\{4,11\},\{11,10\},\{10,4\}$
f_4	$\{2,3\},\{3,4\},\{4,10\},\{10,8\},\{8,2\},\{2,15\},\{15,2\}$

From the table, we observe that each edge of X_1 appears in two faces. This can be easily observed for the faces that don't have pendant vertices (see the faces f_2 and f_3). In faces f_1 and f_4, there are a few edges which are incident with a pendant vertex. Observe that the edges that are incident with a pendant vertex, *e.g.*, the edges $\{2,15\},\{8,9\}$ and $\{1,2\}$ etc., appear twice when traversing a particular face. This observation leads to the proof of Euler's theorem for planar graphs which is the next result.

Theorem 4.7.1. *Let $X = (V, E)$ be a connected planar graph. Then the number of faces, denoted N_f, in any planar embedding of X satisfies*

$$|V| - |E| + N_f = 2.$$

Proof. We prove the result by induction on N_f, the number of faces of the graph. Let $N_f = 1$. Then it can be easily observed that X cannot have a subgraph that is isomorphic to a cycle. For if, X has a subgraph isomorphic to a cycle then in any planar embedding of X, $N_f \geq 2$. Therefore, X is a connected graph without cycle and hence X is a tree and this case has already been verified in the second paragraph of this section.

So, let us assume that the result is true whenever a planar embedding results in $N_f \leq n$. Now, let X be a connected planar graph whose embedding results in $N_f(X) = n + 1$. Now, choose an edge that is not a cut-edge, say e. Then $X \setminus e$ is still a connected graph. Also, the edge e is incident with two separate faces and hence it's removal will combine the two faces and thus $X \setminus e$ has only n faces. Thus, by induction hypothesis,

$$|V(X \setminus e)| - |E(X \setminus e)| + N_f(X \setminus e) = 2.$$

This in turn implies that $|V(X)| - |E(X)| + N_f(X) = 2$ as $|V(X \setminus e)| = |V(X)|$, $|E(X \setminus e)| = |E(X)| - 1$ and $N_f(X \setminus e) = N_f(X) + 1$.

Thus, by the principle of mathematical induction, the result holds. ■

Theorem 4.7.1 implies that the number of faces of a planar graph is independent of the embedding as it is just a function of the number of vertices and the number of edges of a graph, which are graph invariants. This observation leads to the next result and hence the proof is omitted.

Corollary 4.7.2. *Let X be a connected planar graph. Then $N_f(X)$, the number of faces of X, is independent of the planar embedding.*

Theorem 4.7.1 can be used to obtain quite a few inequalities involving the number of vertices and the number of edges of a planar graph. We state them next and use them to prove that the graphs K_5 and $K_{3,3}$ are non-planar.

Corollary 4.7.3. *Let X be a connected planar graph having n vertices and m edges. If $n \geq 3$, then $m \leq 3n - 6$.*

Proof. Without loss of generality, let us assume that X is a simple connected planar graph with $n \geq 3$. Then a face of X can be enclosed with at least 3 edges. As each edge is incident with exactly two faces, one has $2m = 2|E| \geq 3N_f$. Thus, using Euler's theorem, we have

$$2 = |V| - |E| + N_f = n - m + N_f \leq n - m + \frac{2}{3}m$$

and hence the required result follows. ■

Corollary 4.7.4. *The complete graph K_5 is non-planar.*

Proof. Suppose there exists a planar embedding of K_5. Then the Corollary 4.7.3 implies that

$$10 = |E(K_5)| \leq 3|V(K_5)| - 6 = 3 \times 5 - 6 = 9,$$

a contradiction. Thus, the required result follows. ■

An idea similar to the idea used in the proof of Corollary 4.7.3 gives another prove of the non-planarity of $K_{3,3}$.

Corollary 4.7.5. *The complete bipartite graph $K_{3,3}$ is non-planar.*

Proof. Suppose there exists a planar embedding of $K_{3,3}$. Note that the condition that $K_{3,3}$ is bipartite (each cycle of $K_{3,3}$ consists of at least 4 edges), in each planar embedding of $K_{3,3}$, each face of of the embedding will be enclosed with at least 4 edges.

Then, as in the proof of Corollary 4.7.2, one has $2m = 2|E| \geq 4N_f$. Thus, using Euler's theorem, we have

$$2 = |V| - |E| + N_f \leq n - m + \frac{m}{2} \leq 6 - 9 + \frac{9}{2} = \frac{3}{2},$$

a contradiction and hence the required result follows. ■

Corollary 4.7.6. *Let X be a simple connected planar graph. Then X has a vertex of degree less than or equal to* 5*.*

Proof. Let $\delta = \min\{\deg(v) : v \in V(X)\}$. Then we need to show that $\delta \leq 5$.

If $|V(X)| \leq 2$, then there is nothing to prove. So, let us assume that $|V(X)| \geq 3$. Thus, by definition $2|E| = \sum\limits_{v \in V(X)} \deg(v) \geq \delta|V(X)|$. Now, using Corollary 4.7.3, one has $\delta|V(X)| \leq 2\big(3|V(X)| - 6\big)$ or equivalently, $(6-\delta)|V(X)| \geq 12$ and hence $\delta \leq 5$. Thus, one has the required result. ■

This result is used in the proof of the five color theorem. This result is out of the scope of this book. The interested readers are advised to see the book "Graph Theory" by Harary [6].