

ANALYTIC FUNCTION: If the derivative $f'(z)$ exists at all points z of a domain D (open), then $f(z)$ is said to be analytic in D .

The terms regular, and holomorphic are also used for analytic.

A function $f(z)$ is said to be analytic at a point z_0 if there exists a neighbourhood $|z - z_0| < \delta$ at all points of which $f(z)$ exists.

CAUCHY - RIEMANN EQUATIONS (C-R Equations)

A necessary condition that $f(z) = u(x, y) + i v(x, y)$ be analytic in a domain D is that u & v satisfy C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (1)}$$

in D .

Moreover, if the partial derivatives in (1) are continuous in D then the C-R equations are sufficient conditions for analyticity of $f(z)$ in D .

PROOF: Necessary conditions (C-R Equations)

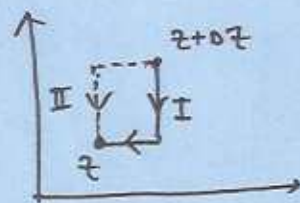
Assume $f'(z)$ exists at z .

We need to prove $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Begin with

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{[u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y)] - [u(x, y) + i v(x, y)]}{\Delta x + i \Delta y}.$$

Since $f'(z)$ exist, the right hand side limit should be the same along all path $\Delta z \rightarrow 0$



Along path I: $\Delta y \rightarrow 0$ and then $\Delta x \rightarrow 0$

$$\Rightarrow f'(z) = \lim_{\Delta x \rightarrow 0} \left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right]$$

$$\Rightarrow f'(z) = u_x + i v_x \quad \text{--- (1)}$$

Along path II:

$$f'(z) = \lim_{\Delta y \rightarrow 0} \left[\frac{u(x, y+\Delta y) - u(x, y)}{i \Delta y} + \frac{v(x, y+\Delta y) - v(x, y)}{\Delta y} \right]$$

$$\Rightarrow f'(z) = v_y - i u_y \quad \text{--- (2)}$$

Note that existence of $f'(z)$ implies existence of u_x, u_y, v_x, v_y .

Comparing (1) & (2):

$$u_x = v_y \quad \& \quad v_x = -u_y.$$

NOTE 1: If we know the existence of the derivative, we can use the following formula

$$f'(z) = u_x + i v_x = v_y - i u_y.$$

Not 2: The C-R equations are necessary condition for f to be differentiable at a point. If they are not satisfied, then $f'(z)$ does not exist at this point.

If the C-R equations hold at a point z , then f may or may not be differentiable at z .

Example: let $f(z) = \bar{z} = x - iy$ (Nowhere differentiable)

$$\Rightarrow u(x,y) = x \quad \& \quad v(x,y) = -y$$

$$u_x = 1 \quad v_y = -1$$

\Rightarrow C-R equations do not hold at any point and therefore f is not differentiable at any point. (C-R eq. is a necessary condition)

Example: let $f(z) = z \operatorname{Re}(z)$. (Differentiable at origin but not analytic)

$$= (x+iy)x$$

$$= x^2 + ixy$$

$$u(x,y) = x^2 \quad v(x,y) = xy$$

$$u_x = 2x \quad v_y = x$$

$$u_y = 0 \quad v_x = y$$

C-R equations do not hold at any point except $z=0$. (C-R eq. is a necessary condition)

\Rightarrow f is not differentiable at z if $z \neq 0$, but may have a derivative at 0. Differentiability at $z=0$ needs to be checked.

$$f'(0) = \lim_{\Delta z \rightarrow 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta z \operatorname{Re}(\Delta z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \operatorname{Re}(\Delta z) = 0.$$

\Rightarrow The function is differentiable at $z=0$.

Example: $f(z) = |z|^2$
 $= x^2 + y^2$

(Diff at origin but not analytic)

$$\Rightarrow u(x,y) = x^2 + y^2 \quad v(x,y) = 0$$

$$\Rightarrow u_x = 2x \quad u_y = 2y$$

$$v_x = 0 \quad v_y = 0$$

C-R equations are satisfied only at $z=0$, nowhere else. (C-R equations are a necessary condition)

$\Rightarrow f$ is not differentiable at z if $z \neq 0$, but may have a derivative at $z=0$. In fact this function is differentiable at $z=0$, since

$$f'(0) = \lim_{\Delta z \rightarrow 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{|\Delta z|^2}{\Delta z} = \lim_{\Delta z \rightarrow 0} \overline{\Delta z} = 0.$$

Example: Show that $f(z) = \sqrt{|xy|}$ is not differentiable at the origin, although C-R equations are satisfied at the point. (C-R equations are not suff. for differentiability)

Sol: $u(x,y) = \sqrt{|xy|} \quad v(x,y) = 0$

At the origin:

$$\frac{\partial u}{\partial x} = \lim_{x \rightarrow 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \rightarrow 0} \frac{0 - 0}{x} = 0.$$

$$\frac{\partial u}{\partial y} = \lim_{y \rightarrow 0} \frac{u(0,y) - u(0,0)}{y} = 0$$

$$\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$$

\Rightarrow C-R equations are satisfied.

Again: consider

$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} = \lim_{z \rightarrow 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

Take $z \rightarrow 0$ along the path $y = mx$, then

$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} = \lim_{x \rightarrow 0} \frac{\sqrt{|mx^2|}}{(1 + im)x} = \lim_{x \rightarrow 0} \frac{\sqrt{|m|}}{1 + im}$$

which depends on $m \Rightarrow f'(0)$ does not exist.

Hence $f(z)$ is not differentiable at the origin although C-R equations are satisfied.

Example: Prove that the function $f(z) = u + iv$ where

$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is continuous and C-R equations are satisfied at the origin, yet $f'(z)$ does not exist there.

Sol: $f(z) = \frac{x^3 - y^3 + i(x^3 + y^3)}{x^2 + y^2}, z \neq 0.$

$$\Rightarrow u(x, y) = \frac{x^3 - y^3}{x^2 + y^2} \quad v(x, y) = \frac{x^3 + y^3}{x^2 + y^2}$$

Continuity of $f(z)$ at $z = 0$:

$$\lim_{z \rightarrow 0} u(x, y) = \lim_{r \rightarrow 0} r(\cos^3 \theta - \sin^3 \theta) = 0$$

$$\lim_{z \rightarrow 0} v(x,y) = \lim_{r \rightarrow 0} r(\cos^3 \theta + \sin^3 \theta) = 0.$$

Hence $f(z)$ is continuous at $z=0$.

Also, at the origin:

$$\frac{\partial u}{\partial x} = \lim_{x \rightarrow 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \rightarrow 0} \frac{x-0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y \rightarrow 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \rightarrow 0} -\frac{y}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \rightarrow 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \rightarrow 0} \frac{x}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y \rightarrow 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \rightarrow 0} \frac{y}{y} = 1.$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

\Rightarrow C-R equations are satisfied at the origin.

Now:
$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} = \lim_{z \rightarrow 0} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}$$

Along $y = mx$:

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{(x^3 - m^3 x^3) + i(x^3 + m^3 x^3)}{(x^2 + m^2 x^2)(x + imx)} \\ &= \frac{(1 - m^3) + i(1 + m^3)}{(1 + m^2)(1 + im)} \quad \text{depends on } m. \end{aligned}$$

$\Rightarrow f'(0)$ does not exist.

HARMONIC FUNCTIONS:

A function $u(x,y)$ which satisfies the Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in a domain D is said to be harmonic in D .

Th.: If $f(z) = u(x,y) + i v(x,y)$ is analytic in a domain D , then u & v satisfy Laplace's equation $u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$.

Proof:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial y} \left(-\frac{\partial v}{\partial x} \right) \quad \text{C-R equations.} \\ &= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0 \end{aligned}$$

Given $f(z)$ analytic, Existence of u_{xx}, u_{yy}, \dots & continuity of higher order derivatives are obvious.

Th.: If u be harmonic on a domain D , then for some v , $u + i v$ defines an analytic function for $z = x + i y$ in D .

Not.: u & v are called harmonic conjugate of each other.

CONSTRUCTION OF ANALYTICAL FUNCTION

Example: Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic and find v such that $f(z) = u + iv$ is analytic.

Sol: $\frac{\partial u}{\partial x} = -e^{-x}(x \sin y - y \cos y) + e^{-x}(\sin y)$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(x \sin y - y \cos y) - e^{-x}(\sin y) - e^{-x} \sin y \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y + y \sin y - \cos y)$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}[-x \sin y + \sin y + y \cos y + \sin y] \quad \text{--- (2)}$$

$$\text{(1) \& (2)} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u \text{ is harmonic.}$$

From C-R equations:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x} \sin y - x e^{-x} \sin y + y e^{-x} \cos y \quad \text{--- (3)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(x e^{-x} \cos y + y e^{-x} \sin y - e^{-x} \cos y) \quad \text{--- (4)}$$

Integrating (3): $v = -e^{-x} \cos y + x e^{-x} \cos y + e^{-x} [y \sin y - \int \sin y dy] + F(x)$

$$= -e^{-x} \cos y + x e^{-x} \cos y + e^{-x} y \sin y + e^{-x} \cos y + F(x)$$
$$= x e^{-x} \cos y + y e^{-x} \sin y + F(x)$$

Now calculate

$$\frac{\partial v}{\partial x} = -x e^{-x} \cos y + e^{-x} \cos y - y e^{-x} \sin y + F'(x) \quad \text{--- (5)}$$

$$\text{(4) \& (5)} \Rightarrow F'(x) = 0 \Rightarrow F(x) = C: \text{ (constant)}$$

$$\Rightarrow v = x e^{-x} \cos y + y e^{-x} \sin y + C$$

Example: Find an analytic function $f(z) = u(x,y) + i v(x,y)$, given that

$$v = \frac{x}{x^2+y^2} + \cosh x \cos y$$

$$\frac{\partial v}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \sinh x \cos y$$

$$\frac{\partial v}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} - \cosh x \sin y$$

C-R equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} - \cosh x \sin y$$

$$u = \int \frac{-2xy}{(x^2+y^2)^2} \cdot dx - \int \cosh x \sin y \, dx + F(y)$$

$$= -y \cdot \left(-\frac{1}{(x^2+y^2)} \right) - \sin y \cdot \sinh x + F(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} - \sinh x \cos y + F'(y)$$

Again C-R equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ gives.

$$\cancel{\frac{1}{x^2+y^2}} - \cancel{\frac{2y^2}{(x^2+y^2)^2}} - \cancel{\sinh x \cos y} + F'(y) = -\cancel{\frac{1}{x^2+y^2}} + \cancel{\frac{2x^2}{(x^2+y^2)^2}} + \cancel{\sinh x \cos y}$$

$$\Rightarrow F'(y) = 0 \Rightarrow F(y) = C.$$

$$\text{So } u = \frac{y}{x^2+y^2} - \sin y \sinh x + C$$

$$f(z) = u + iv = \frac{y}{x^2+y^2} - \sin y \sinh x + i \left(\frac{x}{x^2+y^2} + \cosh x \cos y \right) \quad \left(\begin{array}{l} C=0 \\ \text{for simplicity} \end{array} \right)$$