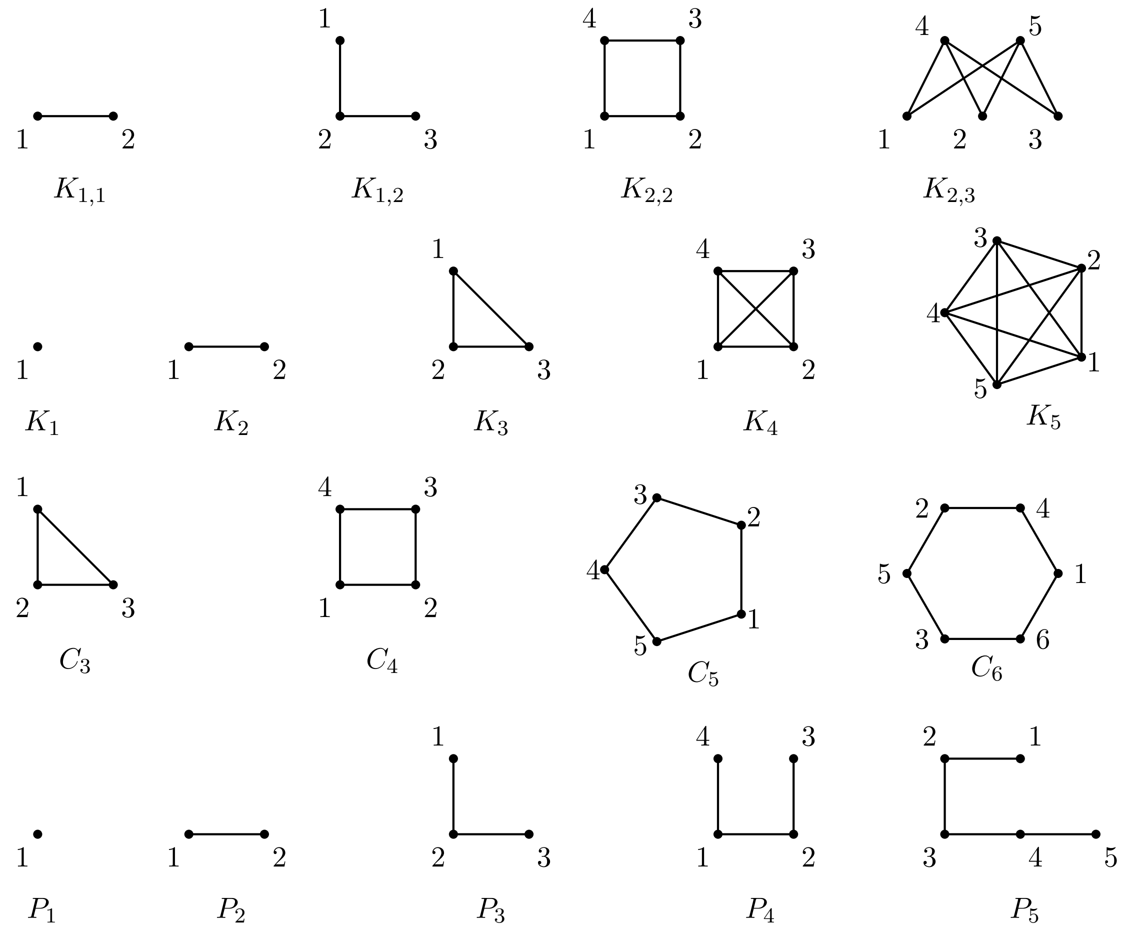

Figure 4.5: Some well known family of graphs

4.2 Graph Operations

A graph $Y = (V', E')$ is said to be a *subgraph* of a graph $X = (V, E)$ if $V' \subset V$ and $E' \subset E$. One also states this by saying that the graph X is a *supergraph* of the graph Y. A subgraph $Y = (V', E')$ of $X = (V, E)$ is said to be a *spanning subgraph* if $V' = V$ and is called an *induced subgraph* if, for each $u, v \in V' \subset V$, the edge $\{u, v\} \in E'$ whenever $\{u, v\} \in E$. In this case, the set V' is said to induce the subgraph Y and this is denoted by writing $Y = X[V']$. With the definitions as above, one observes the following:

1. Every graph is its own subgraph.
2. If Z is a subgraph of Y and Y is a subgraph of X then Z is also a subgraph of X.
3. A single vertex of a graph is also its subgraph.
4. A single edge of a graph together with its incident vertices is also its subgraph.

We also have the following graph operations.

Definition 4.2.1. *1. Let $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ be two graphs. Then the* *(a) union of X_1 and X_2, denoted $X_1 \cup X_2$, is a graph whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$.*

(b) intersection of X_1 and X_2, denoted $X_1 \cap X_2$, is a graph whose vertex set is $V_1 \cap V_2$ and the edge set is $E_1 \cap E_2$.

2. *Let $X = (V, E)$ be a graph and fix a subset $U \subset V$ and $F \subset E$. Then*

   (a) $X \setminus U$ denotes a subgraph of X that is obtained by removing all the vertices $v \in U$ from X and all edges that are incident with some vertex $v \in U$.

   (b) $X \setminus F$ denotes a subgraph of X that is obtained by the removal of each edge $e \in F$.

   Thus, note that $X \setminus F$ is a spanning subgraph of X, whereas $X \setminus U$ is an induced subgraph of X on the vertex set $V \setminus U$. If $U = \{v\}$ and $F = \{e\}$, then one writes $X \setminus v$ in place of $X \setminus \{v\}$ and $X \setminus e$ in place of $X \setminus \{e\}$.

3. *Let $X = (V, E)$ be a graph and let v be a vertex such that $v \notin V$. Also, suppose that there exist $x, y \in V$ such that $e = \{x, y\} \notin E$. Then*

   (a) $X + v$ is a graph that is obtained from X by including the vertex v and joining it to all other vertices of X. That is, $X + v = (V', E')$, where $V' = V \cup \{v\}$ and $E' = E \cup \{\{v, u\} : u \in V\}$.

   (b) $X + e$ is a graph that is obtained from X by joining the edge e. That is, $X + e = (V', E')$, where $V' = V$ and $E' = E \cup \{e\}$.

   (c) Let $X = (V, E)$ and $Y = (V', E')$ be two graphs with $V \cap V' = \emptyset$. Then the

      i. join of X and Y, denoted $X + Y = (V_1, E_1)$, is a graph having $V_1 = V \cup V'$ and $E_1 = E \cup E' \cup \{ \{u, v\} : u \in V, v \in V'\}$.

      ii. cartesian product of X and Y, denoted $X \times Y = (V_1, E_1)$, is a graph having $V_1 = V \times V'$ and whose edge set consists of all elements $\{(u_1, u_2), (v_1, v_2)\}$, where either $u_1 = v_1$ and $\{u_2, v_2\} \in E'$ or $u_2 = v_2$ and $\{u_1, v_1\} \in E$.

See Figure 4.6 for examples related to the above graph operations.

The graph operations defined above lead to the concepts of what are called connected components, cut-vertices, cut-edge/bridges and blocks in a graph. We define them now.

Definition 4.2.2. *Let $X = (V, E)$ be a graph. Then*

1. *a connected component (or in short component) of X is a connected induced subgraph $Y = (V', E')$ of X such that if Z is any connected subgraph of X that has Y as its subgraph then $Y = Z$.*

2. *a vertex $v \in V$ is called a cut-vertex if the number of components in $X \setminus v$ increases.*

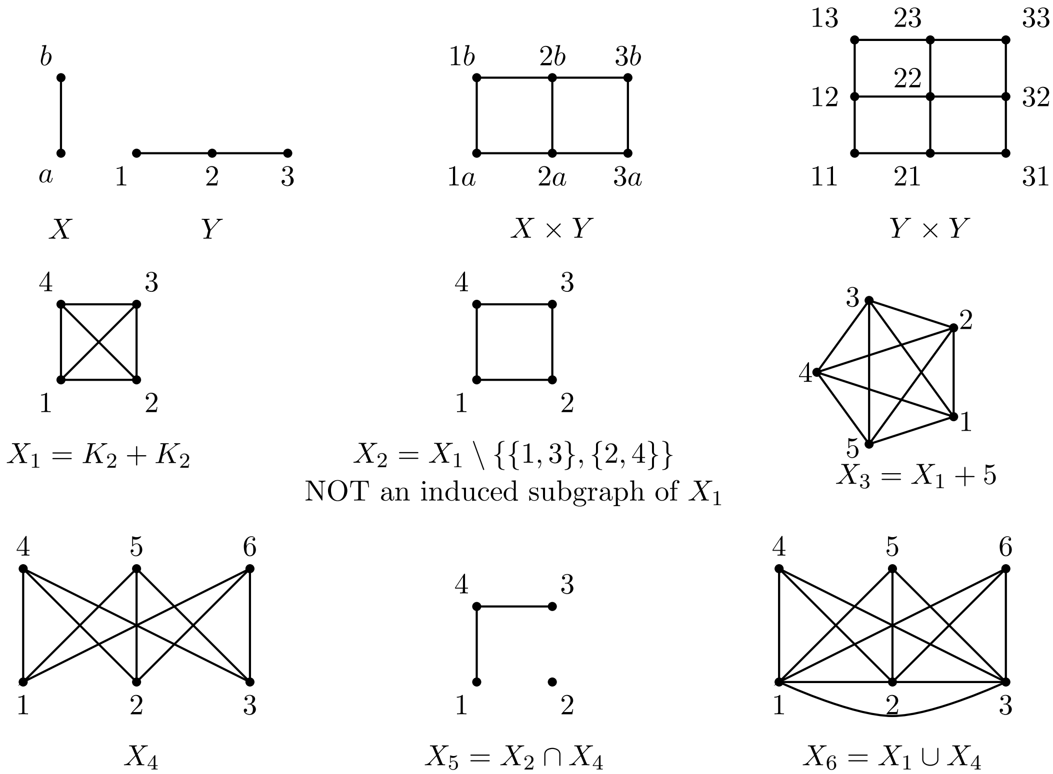

Figure 4.6: Examples of graph operations

3. *an edge $e \in E$ is called a cut-edge/bridge if the number of components in $X \setminus e$ increases.*

4. *a block of X is a maximal induced connected subgraph of X having no cut-vertex.*

5. *a clique of X is a maximal induced complete subgraph of X.*

We also need the following important definitions related with graph operations.

Definition 4.2.3. *Let $X = (V, E)$ be a graph with $|V| = n$. Then*

1. *the complement graph of X is the graph $Y = (V', E')$, denoted $\overline{X}$, such that $V' = V$ and $E' = E(K_n) \setminus E$, where $E(K_n)$ is the edge set of the complete graph K_n.*

2. *the line graph of X is the graph $Y = (V', E')$, denoted $\mathcal{L}(X)$, such that $V' = E$ and any two elements of V' are joined by an edge if they have a common vertex of X incident to them.*

See Figure 4.8 for examples of line and complement graphs. Observe that if $X = (V, E)$ is the complement of the graph $Y = (V', E')$ then $V = V'$. If $|V| = n$, then the set E can be obtained by removing those edges of K_n that are also edges of Y. In other words, the two graphs are complementary to each other. A graph X is said to be *self-complementary* if $X = \overline{X}$. For example, the path P_4, on 4 vertices, and the cycle C_5, on 5 vertices, are self-complementary graphs.

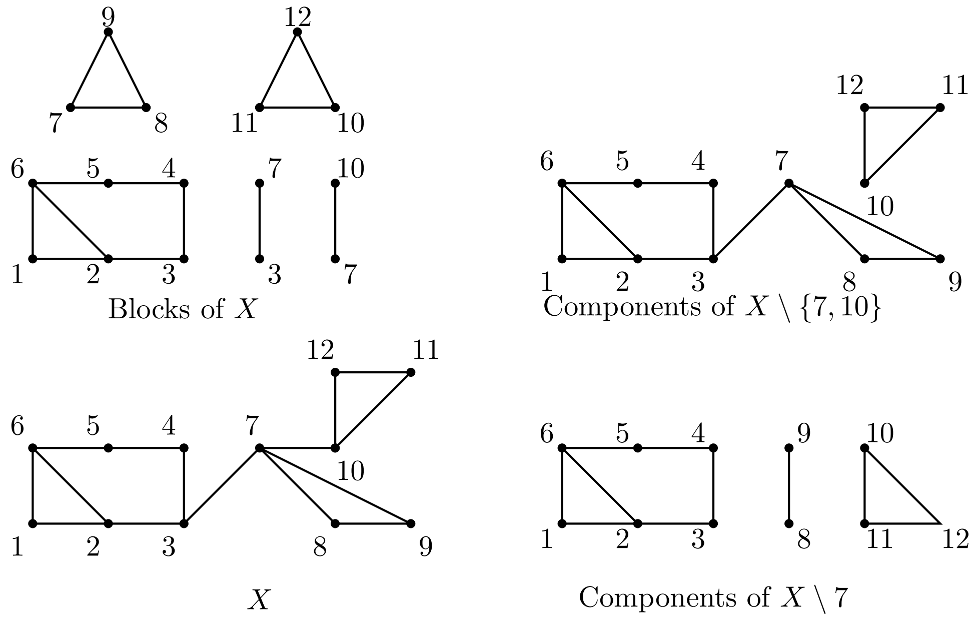

Figure 4.7: Examples of cut-vertex, bridge, block and connected components

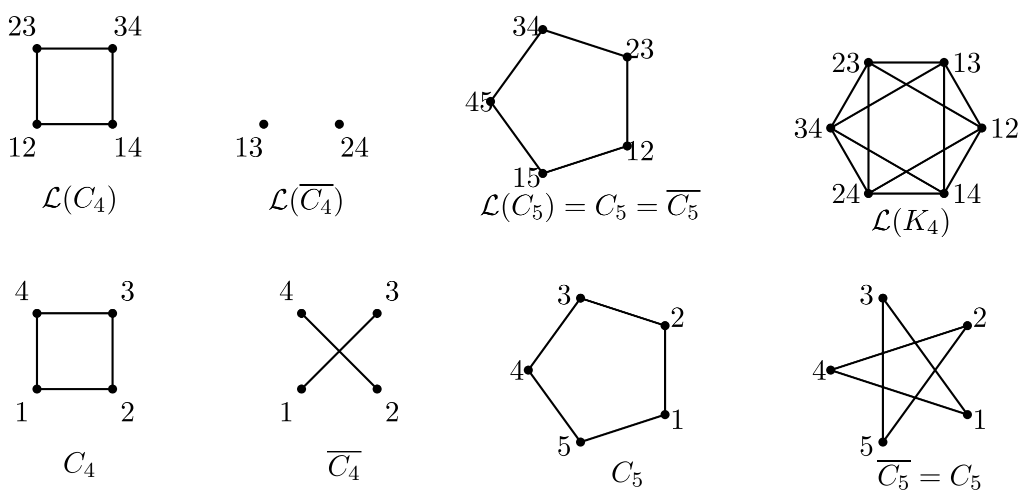

Figure 4.8: Line graphs and complement graphs

4.2.1 Characterization of trees

Recall that a graph X is called a tree if it is connected and has no cycle. A collection of trees is called a forest. That is, a graph is a forest if it has no cycle. Also, recall that every tree is a bipartite graph. We now prove that the following statements that characterize trees are equivalent.

Theorem 4.2.4. *Let $X = (V, E)$ be a graph on n vertices and m edges. Then the following statements are equivalent for X.*

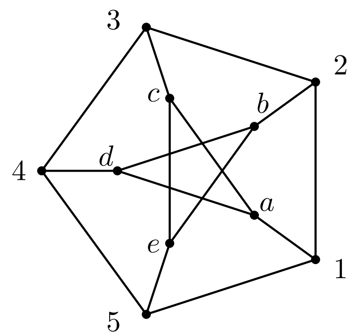

Figure 4.9: Petersen graphs

1. *X is a tree.*

2. *Let u and v be distinct vertices of X. Then there is a unique path from u to v.*

3. *X is connected and $n = m + 1$.*

Proof. 1 implies 2: Since X is connected, for each $u, v \in V$, there is a path from u to v. On the contrary, let us assume that there are two distinct paths P_1 and P_2 that join the vertices u and v. Since P_1 and P_2 are distinct and both start at u and end at v, there exist vertices, say u_0 and v_0, such that the paths P_1 and P_2 take different edges after the vertex u_0 and the two paths meet again at the vertex v_0 (note that u_0 can be u and v_0 can be v). In this case, we see that the graph X has a cycle consisting of the portion of the path P_1 from u_0 to v_0 and the portion of the path P_2 from v_0 to u_0. This contradicts the assumption that X is a tree (it has no cycle).

2 implies 3: Since for each $u, v \in V$, there is a path from u to v, the connectedness of X follows. We need to prove that $n = m + 1$. We prove this by induction on the number of vertices of a graph. The result is clearly true for $n = 1$ or $n = 2$. Let the result be true for all graphs that have n or less than n vertices. Now, consider a graph X on $n + 1$ vertices that satisfies the conditions of Item 2. The uniqueness of the path implies that if we remove an edge, say $e \in E$, then the graph X will become disconnected. That is, $X \setminus e$ will have exactly two components. Let the number of vertices in the two components be n_1 and n_2. Then $n_1, n_2 \leq n$ and $n_1 + n_2 = n + 1$. Hence, by induction hypothesis, the number of edges in $X - e$ equals $(n_1 - 1) + (n_2 - 1) = n_1 + n_2 - 2 = n - 1$ and hence the number of edges in X equals $n - 1 + 1 = n$. Thus, by the principle of mathematical induction, the result holds for all graphs that have a unique path from each pair of vertices.

3 implies 1: It is already given that X is a connected graph. We need to show that X has no cycle. So, on the contrary, let us assume that X has a cycle of length k. Then this cycle has k vertices and k edges. Now, consider the $n - k$ vertices that do not lie of the cycle. Then for each vertex (corresponding to the $n - k$ vertices), there will be a distinct edge incident with it on the smallest path from the vertex to the cycle. Hence, the number of edges will be greater than or equal to $k + (n - k) = n$. A contradiction to the assumption that the number of edges equals $n - 1$. Thus, the required result follows. ∎

As a next result in this direction, we prove that a tree has at least two pendant (end) vertices.

Theorem 4.2.5. *Let X be a non-trivial tree. Then X has at least two vertices of degree* 1.

Proof. Let $X = (V, E)$ with $|V| = n \geq 2$. Then, by Theorem 4.2.4, $|E| = n - 1$. Also, by handshake lemma (Lemma 4.1.3), we know that $2|E| = \sum\limits_{v \in V} \deg(v)$. Thus, $2(n-1) = \sum\limits_{i=1}^{n} \deg(v_i)$. Now, X is connected implies that $\deg(v) \geq 1$, for all $v \in V$ and hence the above equality implies that there are at least two vertices for which $\deg(v) = 1$. This ends the proof of the result. ■

For more results on trees, see the book graph theory by Harary [6].