Improper Integral

Type II. $\int_a^b f(x)\,dx$; a, b finite but f becomes unbounded at $x = c$, $a \le c \le b$.

as for example $\int_1^2 \frac{dx}{x - 1.5}$, $\int_4^5 \frac{dx}{x(x-4)}$

$\int_{a^+}^b f(x)dx$ converges, if the limit

$$\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f\,dx \text{ exists.}$$

$\int_a^{b^-} f(x)\,dx$ converges, if the limit

$$\lim_{\epsilon \to 0^-} \int_a^{b-\epsilon} f\,dx \text{ exists.}$$

When f be unbounded at $x = c \in (a, b)$

then $\int_a^b f(x)\,dx$ converges, if both the limits

$$\lim_{\epsilon \to 0^+} \int_a^{c-\epsilon} f(x)\,dx \quad \& \quad \lim_{\delta \to 0^+} \int_{c+\delta}^b f\,dx \text{ exists.}$$

Consider $I = \int_a^b \frac{dx}{(x-a)^p}$

Note: If $p \le 0$, then I is proper.

$$I = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b \frac{dx}{(x-a)^p} = \lim_{\epsilon \to 0} \left[\frac{(x-a)^{-p+1}}{1-p} \Bigg|_{a+\epsilon}^b \right] \quad p \neq 1$$

$$= \lim_{\epsilon \to 0} \left[\frac{(b-a)^{1-p} - \epsilon^{1-p}}{1-p} \right] \begin{cases} = \dfrac{(b-a)^{1-p}}{p}, & 1-p > 0 \\ = \text{does not exist}, & 1-p < 0 \end{cases}$$

P.T.O

Now if $p = 1$

$$I = \lim_{\epsilon \to 0} \int_{a+\epsilon}^{b} \frac{dx}{x-a} = \lim_{\epsilon \to 0} \Big[\ln|x-a| \Big]_{a+\epsilon}^{b}$$

$$= \lim_{\epsilon \to 0} \Big[\ln|b-a| - \ln|\epsilon| \Big] = \infty$$

Conclusion

$$I = \int_a^b \frac{dx}{(x-a)^p}$$

- $p \leq 0$: I proper
- $p \geq 1$: I diverges
- $0 < p < 1$: I converges

Test for convergence

1. (inequality) comparison test.

let (i) $f(x), g(x), h(x)$ are continuous in $a < x \leq b$

(ii) $f(x), g(x), h(x)$ keep same sign in $a < x \leq b$

(iii) $0 \leq f(x) \leq g(x)$

$0 \leq h(x) \leq f(x)$

(iv) $\int_{a^+}^{b} g(x)\, dx$. converges $\Rightarrow \int_{a^+}^{b} f(x)$ converge

(v) $\int_{a^+}^{b} h(x)$ diverges $\Rightarrow \int_{a^+}^{b} f(x)$ diverge.

If $\int_a^\infty \left| \frac{\sin x}{x^3} \right| dx$ converges then $\int_a^\infty \frac{\sin x}{x^3} dx$ converge.

Limit comparison test

1) $f(x), g(x)$ keep same sign in $a < x \le b$.
2) $f(x), g(x)$ continuous in $(a, b]$
3) $\lim_{x \to a} \frac{f(x)}{g(x)} = l$.

l finite $\neq 0$

$\int_{a^+}^{b} f(x)\,dx$, $\int_{a^+}^{b} g(x)\,dx$ converge/diverge together.

$l = 0$

If $\int_{a^+}^{b} g\,dx$ converges, $\int_{a^+}^{b} f\,dx$ will converge.

$l = \infty$

If $\int_{a^+}^{b} g\,dx$ diverges, $\int_{a^+}^{b} f\,dx$ will diverge.

μ - test

1) $f(x)$ keeps the same sign in $a < x \le b$
2) $f(x)$ is continuous in $(a, b]$
3) $\lim_{x \to a} (x-a)^{\mu} f(x) = l$.

Case 1

$l =$ finite $\neq 0$

$0 < \mu < 1$ $\int_{a^+}^{b} f\,dx$ converges

$\mu \ge 1$, $\int_{a^+}^{b} f\,dx$ diverges

Case 2 $l = 0$.

$0 < \mu < 1$, $\int_{a^+}^{b} f\,dx$ converges

Case 3, $l = \infty$

$\mu \ge 1$, $\int_{a^+}^{b} f\,dx$ diverges.

$\because \int_1^3 \frac{dx}{\sqrt{x-1}} = \int_1^3 \frac{dx}{(x-1)^{1/2}}$ converges,

then $\int_1^3 \frac{dx}{\sqrt{x^3-1}}$ will also converge.

EX

$$I = \int_1^2 \frac{\sqrt{x}}{\log x}\, dx.$$

Let $g(x) = \frac{1}{x-1}$, $\frac{f}{g} = \frac{\sqrt{x}\,(x-1)}{\log x}$

$$\therefore \underset{x\to 1}{Lt} \frac{f}{g} = \underset{x\to 1}{Lt} \frac{\sqrt{x}\,(x-1)}{\log x} = \underset{x\to 1}{Lt} \frac{x^{3/2} - x^{1/2}}{\log x} \quad \left(\frac{0}{0}\right)$$

$$= \underset{x\to 1}{Lt} \frac{\frac{3}{2} x^{1/2} - \frac{1}{2} x^{-1/2}}{1/x} = 1.$$

$\int_1^2 \frac{dx}{x-1}$ diverges, $p = 1$.

$\therefore \int_1^2 \frac{\sqrt{x}\,dx}{\log x}$ diverges.

EX

$$\int_0^1 \frac{\log x}{\sqrt{x}}\, dx.$$

$\underset{x\to 0}{Lt}\, x^{\mu} \frac{\log x}{\sqrt{x}}$ choose $\mu = \frac{3}{4}$.

$$\underset{x\to 0}{Lt}\, x^{3/4 - 2/4} \log x = 0$$

$\because \mu < 1 \quad \therefore \int_0^1 \frac{\log x}{\sqrt{x}}\, dx$ converge.

Ex Apply comparison test to check the convergence of

$$I = \int_1^3 \frac{dx}{\sqrt{x^3-1}}.$$

$\cancel{1-x^3}$ $x^3-1 = (x-1)(x^2+x+1)$

$x > 1 \quad \therefore \quad 1+x+x^2 > 1+1+1 = 3.$

$$\therefore \frac{1}{\sqrt{x^3-1}} < \frac{1}{\sqrt{(x-1)\cdot 3}} = \frac{1}{\sqrt{3}\sqrt{x-1}}$$

$$0 < \frac{1}{\sqrt{x^3-1}} \leq \frac{1}{\sqrt{3}\sqrt{x-1}}$$

Now, $\int_1^3 \frac{dx}{\sqrt{3}\sqrt{x-1}} \to$ converges $\left[\int_{a^+}^b \frac{dx}{(x-a)^\mu}, \text{ Here } \mu = 1/2 < 1\right]$

So, by comparison test I converges.

$$f(x) = \frac{1}{\sqrt{x^3-1}} = \frac{1}{\sqrt{x-1}\sqrt{1+x+x^2}}$$

$$g(x) = \frac{1}{\sqrt{x-1}}, \quad \lim_{x\to 1} \frac{f}{g} = \lim_{x\to 1} \frac{1}{\sqrt{1+x+x^2}}$$

$$= \frac{1}{\sqrt{3}} \neq 0$$

$$\therefore \int_1^3 \frac{dx}{\sqrt{x^3-1}} \text{ . converge.}$$

Absolute convergence

If $f(x)$ changes sign in $(a, b]$, you cannot apply comparison tests.

Thm If $\int_{a^+}^{b} |f(x)|\, dx$ converges, $\int_{a}^{b} f(x)\, dx$ converges.

converse is not true.

i.e. $\int_{a^+}^{b} f(x)\, dx$ converges $\not\Rightarrow$ $\int_{a^+}^{b} |f(x)|\, dx$ converges.

$\int_0^1 \frac{\sin \frac{1}{x}}{x^{3/2}}\, dx$ converges,

but $\int_0^1 \frac{|\sin \frac{1}{x}|}{x^{3/2}}\, dx$ diverges.

$\int_0^1 \frac{\sin 1/x}{x^p}$ converges absolutely, when $0<p<1$.

$\int_0^1 \frac{|\sin 1/x|}{x^p}\, dx$, $0 \le \underbrace{\frac{|\sin 1/x|}{x^p}}_{f(x)} \le \underbrace{\frac{1}{x^p}}_{g(x)}$