SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATIONS

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots, y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots, y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots, y_n)$$

I. METHOD OF ELIMINATION

EXAMPLE: SOLVE

$$\frac{dx}{dt} = 7x - y$$

$$\frac{dy}{dt} = 2x + 5y$$

Denoting $\frac{d}{dt} \equiv D$:

$$(D-7)x + y = 0 \quad \text{—— (i)}$$

$$-2x + (D-5)y = 0 \quad \text{—— (ii)}$$

multiplying (i) by 2 and operating (ii) by $(D-7)$ and then adding the two equations:

$$2y + (D-7)(D-5)y = 0$$

$$\Rightarrow (D^2 - 12D + 37)y = 0$$

auxiliary equation $m^2 - 12m + 37 = 0$

Its roots are $6 \pm i$

$$\text{C.F.} = \boxed{y = e^{6t}(c_1 \cos t + c_2 \sin t)}$$

$$\Rightarrow \quad Dy = 6e^{6t}(c_1 \cos t + c_2 \sin t) + e^{6t}(-c_1 \sin t + c_2 \cos t)$$

$$= e^{6t}\left[(6c_1 + c_2)\cos t + (6c_2 - c_1)\sin t\right]$$

$$(ii) \Rightarrow \quad x = \frac{1}{2}\left[Dy - 5y\right]$$

$$\Rightarrow \quad \boxed{x = \frac{1}{2} e^{6t}\left[(c_1 + c_2)\cos t + (c_2 - c_1)\sin t\right]}$$

II. METHOD OF DIFFERENTIATION

EXAMPLE: Determine the general solutions for x and y for

$$\frac{dx}{dt} - y = t \qquad \text{—(i)}$$

$$\frac{dy}{dt} + x = 1 \qquad \text{—(ii)}$$

Differentiating (i) w.r.t. t and replacing $\frac{dy}{dt}$ from (ii), we get

$$\frac{d^2x}{dt} - (1 - x) = 1$$

$$\Rightarrow \quad \frac{d^2x}{dt} + x = 2$$

$$C.F. = c_1 \cos t + c_2 \sin t$$

$$P.I. = \frac{1}{D^2+1} \cdot 2$$

$$= 2.$$

$$\Rightarrow \boxed{x = c_1 \cos t + c_2 \sin t + 2}$$

(i) $\Rightarrow$ $y = \frac{dx}{dt} - t$

$$\boxed{y = -c_1 \sin t + c_2 \cos t - t}$$

EXAMPLE: Solve $\frac{dy_1}{dx} = y_1 + y_2 + x$ — (i)

$$\frac{dy_2}{dx} = -4y_1 - 3y_2 + 2x \quad \text{— (ii)}$$

Differentiating (i): $\frac{d^2y_1}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx} + 1$

$$\Rightarrow \frac{d^2y_1}{dx} = \frac{dy_1}{dx} + (-4y_1 - 3y_2 + 2x) + 1$$

$$= \frac{dy_1}{dx} - 4y_1 - 3\underbrace{\left(\frac{dy_1}{dx} - y_1 - x\right)}_{\text{from (i)}} + 2x + 1$$

$$= -2\frac{dy_1}{dx} - y_1 + 5x + 1$$

$$\Rightarrow \frac{d^2y_1}{dx} + 2\frac{dy_1}{dx} + y_1 = 5x + 1.$$

auxiliary equation $m^2 + 2m + 1 = 0$

$$\Rightarrow m = -1, -1.$$

$$C.F. = (c_1 + c_2 x)\, e^{-x}$$

$$P.I. = \frac{1}{D^2+2D+1} \cdot 5x+1$$

$$= (D+1)^{-2}(5x+1)$$

$$= (1-2D+\cdots)(5x+1)$$

$$= (5x+1) - 2(5)$$

$$= 5x-9$$

$$\boxed{y_1 = (c_1 + c_2 x)e^{-x} + 5x - 9}$$

form(i)

$$y_2 = \frac{dy_1}{dx} - y_1 - x$$

$$= -c_1 e^{-x} + c_2\left(-x e^{-x} + e^{-x}\right) + 5$$

$$\quad - c_1 e^{-x} - c_2 x e^{-x} - 5x + 9 - x$$

$$= -2c_1 e^{-x} - 2c_2 x e^{-x} + c_2 e^{-x} - 6x + 14$$

$\Rightarrow$

$$\boxed{y_2 = e^{-x}(-2c_1 - 2c_2 x + c_2) - 6x + 14}$$

III Method of undetermined coefficients:

Consider

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}\right\} \text{--- (1)}$$

We seek a particular solution.

$$x_1 = \alpha_1 e^{kt}, \quad x_2 = \alpha_2 e^{kt}, \quad \ldots, x_n = \alpha_n e^{kt}$$

It is required to determine the constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ and k in such a way that the functions $\alpha_1 e^{kt}, \alpha_2 e^{kt}, \ldots \ldots \alpha_n e^{kt}$ satisfy the above system of differential equations.

(1) $\Rightarrow$

$$\begin{aligned}
k\alpha_1 e^{kt} &= (a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n)e^{kt} \\
k\alpha_2 e^{kt} &= (a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n)e^{kt} \\
&\vdots \\
k\alpha_n e^{kt} &= (a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nn}\alpha_n)e^{kt}
\end{aligned}$$

$$\Rightarrow \left.\begin{aligned} (a_{11}-K)\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n &= 0 \\ a_{21}\alpha_1 + (a_{21}-K)\alpha_2 + \cdots + a_{2n}\alpha_n &= 0 \\ &\vdots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + (a_{nn}-K)\alpha_n &= 0 \end{aligned}\right\} \quad (2)$$

For a nontrivial solution of the above system

$$\begin{vmatrix} a_{11}-K & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21}-K & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & (a_{nn}-K) \end{vmatrix} = 0$$

This equation is called auxiliary equation of the system (1)

Suppose the roots of the auxiliary equation are real and distinct say, $K_1, K_2, \ldots K_n$.

For each root K_i:

evaluate

$$\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_n^{(i)}$$

For the root K_i, obtain the following solution of the system:

$$x_1^{(i)} = \alpha_1^{(i)} e^{K_i t}, \quad x_2^{(i)} = \alpha_2^{(i)} e^{K_i t}, \ldots, x_n^{(i)} = \alpha_n^{(i)} e^{K_i t}$$

$i = 1, 2, \ldots n.$

<u>General solution:</u>

$$x_1 = \sum_{i=1}^{n} c_i \alpha_1^{(i)} e^{K_i t}$$

$$x_2 = \sum_{i=1}^{n} c_i \alpha_2^{(i)} e^{K_i t}$$

$$\vdots$$

$$x_n = \sum_{i=1}^{n} c_i \alpha_n^{(i)} e^{K_i t}$$

EXAMPLE: $$\frac{dx_1}{dt} = 2x_1 + 2x_2$$
$$\frac{dx_2}{dt} = x_1 + 3x_2$$

auxiliary equation:

$$\begin{vmatrix} 2-K & 2 \\ 1 & 3-K \end{vmatrix} = 0$$

$$\Rightarrow \quad 6 - 5K + K^2 - 2 = 0$$
$$\Rightarrow \quad K^2 - 5K + 4 = 0$$
$$\Rightarrow \quad (K-4)(K-1) = 0 \Rightarrow K_1 = 1, \; K_2 = 4.$$

For: $K_1 = 1$:

Solving the system:

$$\alpha_1 + 2\alpha_2 = 0$$

Choose $\alpha_1 = 1 \Rightarrow \alpha_2 = -1/2$

Solution: $x_1^{(1)} = e^t \qquad x_2^{(1)} = -\frac{1}{2}e^t$

For $K_2 = 4$: system:

$$-2\alpha_1 + 2\alpha_2 = 0$$
$$\Rightarrow \alpha_1 = \alpha_2 = 1 \text{ (choose)}$$

Solution $x_1^{(2)} = e^{4t} \qquad x_2^{(2)} = e^{4t}$

General solution:

$$x_1 = c_1 e^t + c_2 e^{4t}$$
$$x_2 = -\frac{1}{2}c_1 e^t + c_2 e^{4t}$$