Indian Institute of Technology Kharagpur
Department of Mathematics
Course: Linear Algebra
Autumn Semester 2018
Problem Set 2

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Notation:
V is a vector space over an arbitrary filed $\mathbb{F}$.
$\mathbb{R}$ denotes the field of real numbers.
$M_{m\times n}(\mathbb{F})$ denotes the vector space of all matrices of size $m \times n$ with entries from the field $\mathbb{F}$.
$\mathcal{C}(\mathbb{R},\mathbb{R})$ denotes the real vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
$P_n(\mathbb{R})$ denotes the real vector space of all polynomials upto degree n.
$P(\mathbb{R})$ denotes the real vector space of all polynomials

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1. Check whether the following functions are linear or not. In case, the functions are linear, find $nullity(T)$ and $rank(T)$, and check if the functions are one-to-one and onto.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1, a_1 - a_2, a_2)$.

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (a_1, a_1 - a_2 + a_3, a_2)$.

   (c) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1, 1, a_2)$.

   (d) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$.

   (e) $T : M_{2\times 3}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

   $$T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = \begin{bmatrix} a_{11} + a_{21} & a_{12} + a_{22} \\ a_{13} + a_{23} & 0 \end{bmatrix}$$

2. Let $T : P(\mathbb{R}) \to P(\mathbb{R})$ defined as

   $$T(f(x)) = \int_0^x f(t)dt$$

   Show that T is linear and one-to-one but not onto.

3. Let V be the vector space of all real sequences. Define $T_\ell, T_r : V \to V$ by

   $$T_r(a_1, a_2, \ldots) = (a_2, a_3, \ldots), \quad \text{and} \quad T_\ell(a_1, a_2, \ldots) = (0, a_1, a_1, \ldots).$$

   Prove that both the functions T_ℓ and T_r are linear. Prove further that T_ℓ is one-to-one but not onto while T_r is onto but not one-to-one.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation. Show that there exist scalars $a, b, c \in \mathbb{R}$ such that $T(x_1, x_2, x_3) = ax_1 + bx_2 + cx_3$ for all $(x_1, x_2, x_3) \in \mathbb{R}^3$. Can you generalize this result to a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ for $n, m \in \mathbb{N}$.

5. For all the linear transformations in Problem (1), write down the matrix representation of the linear transformation by fixing an ordered basis for each of the vector spaces. (Consider the cases where the vector spaces are finite dimensional only.)

6. Let $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ be ordered basis for $M_{2\times 2}(R)$ and $\{1, x, x^2\}$ be ordered basis for $P_2(\mathbb{R})$. Then represent the following linear transformations using matrices.

   (a) $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by

   $$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + (b + c)x + dx^2.$$

   Hence compute $T\left(\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\right)$.

   (b) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(A) = A^T$.

   (c) $T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = trace(A + A^T)$.

7. Let $T_j : P(\mathbb{R}) \to P(\mathbb{R})$ be a linear transformation defined as $T_j(f(x)) = \dfrac{d^j}{dx^j} f(x)$. Then show that for any $n \in \mathbb{N}$, the set $\{T_1, T_2, \ldots, T_n\}$ is linearly independent in $\mathcal{L}(V)$.

8. Let V, W, Z be vector spaces over a filed $\mathbb{F}$ and let $T : V \to W$ and $U : W \to Z$ be linear.

   (a) Prove that UT is one-to-one, then so is T. Must U also be one-to-one?

   (b) Prove that UT is onto, then so is U. Must T also be onto?

   (c) Prove that if U and T are one-to-one and onto, then UT is also one-to-one and onto.

9. Let $\sim$ mean "is isomorphic to". Prove that $\sim$ is an equivalence relation on the class of vector spaces over a given field $\mathbb{F}$.

10. Let $B \in \mathbb{R}^{n\times n}$ be an invertible matrix. Define $\Phi_B : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ as $\Phi_B(A) = B^{-1}AB$ for $A \in M_{n\times n}(\mathbb{R})$. Prove that Φ_B is an isomorphism.