Lecture - 6 — Thursday — 28.1.16 — 3-5 p.m.

(No class was there on 22.1.16 due to Kshitij)

Eigenvector, Eigenvalues.

$A \to n \times n$ matrix.

$$A\underset{\sim}{v} = \lambda \underset{\sim}{v}$$

$\underset{\sim}{v} \to$ eigen vector ($\neq \underset{\sim}{0}$)

$\lambda \to$ eigenvalue.

$|\lambda I - A| = 0 \to$ characteristic equation.

$(\lambda I - A)\underset{\sim}{v} = \underset{\sim}{0}$.

$\lambda = 5 \quad E_{\lambda=5}$ = eigenspace corresponding to $\lambda = 5$

$= \{\underset{\sim}{v} + \underset{\sim}{0}\}$ ($\underset{\sim}{v} \to$ eigen vectors corresponding to $\lambda = 5$.)

$g_{\lambda=5} \to$ geometric multiplicity
= dimension of eigen space $E_{\lambda=5}$

$a_{\lambda=5} \to$ algebraic multiplicity
= multiplicity of λ as a root.

$a_\lambda \geq g_\lambda$, always.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \lambda = 0, 0 \quad a_{\lambda=0} = 2, \; g_{\lambda=0} = 1$$

Cayley-Hamilton theorem.

Every square matrix, say $A_{n\times n}$ is a zero of its characteristic polynomial. If $a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n = 0$ be the characteristic equation for A, then A will satisfy

$$a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI_n = \underset{\sim}{0}.$$

Ex. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ characteristic equation of A : $\lambda^2 = 0$

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \underset{\sim}{0}.$$

Ex.

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow *$$

Characteristic earuation

$$(\lambda-5)^2(\lambda-1) = 0$$

$$\Rightarrow (\lambda^2 - 10\lambda + 25)(\lambda - 1) = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 35\lambda - 25 = 0$$

(i) Verify Cayley-Hamilton theorem for the matrix "A" given in (*).

(ii) Compute "A^3" using Cayley-Hamilton theorem for the matrix "A".

Similar Matrices.

$A \rightarrow n \times n$ matrix.

An $n \times n$ matrix B is said to be similar to the matrix A, if

$$B = P^{-1}AP \longrightarrow (1)$$

for some non-singular matrix P. Premultiply (1) by P & postmultiply (1) by P^{-1}. Then get

$$PBP^{-1} = PP^{-1}APP^{-1} = IAI = A.$$

$$A = PBP^{-1} = (P^{-1})^{-1}B(P^{-1})$$

Let $Q = P^{-1}$, $A = Q^{-1}BQ$, for some nonsingular Q.

$\Rightarrow$ A is similar to B.

Thus if B is similar to A, A is similar to B.

We can say A & B are similar matrices if $B = P^{-1}AP$ holds.

Theorem. If A & B are similar matrices, then they have same determinant, rank, trace, characteristic polynomial and eigenvalues.

Definition. A square matrix A is said to be diagonisable if it is similar to a diagonal matrix D.

ie $A = P^{-1}DP$

or, $D = PAP^{-1}$.

Theorems helpful for diagonalisation.

Theorem 1. The eigenvectors corresponding to distinct eigenvalues are linearly independent.

$A. \longrightarrow \lambda_1 = 1, \quad \lambda_2 = -5, \quad \lambda_3 = 3$

$\lambda_1 = 1, \ (a, a, 2a) \to a\,(1, 1, 2)$

$\lambda_2 = 2, \ (a, 0, -a) \to a\,(1, 0, -1)$

$\lambda_3 = 3, \ (2a, 3a, 0) \to a\,(2, 3, 0)$

$(1,1,2), (1,0,-1), (2,3,0)$ are linearly independent.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$\lambda = 1, 5, 5$

$E_{\lambda=1} = \{(c, c, 0) : c \in \mathbb{R}\}$

$E_{\lambda=5} = \{(b, -b, c) : b, c \in \mathbb{R}\}$

$(x_1, x_2, x_3) \in E_{\lambda=1}$

$\hookrightarrow = c\,\underbrace{(1, 1, 0)}_{v_1}$

$(x_1, x_2, x_3) \in E_{\lambda=5}$

$= b\underbrace{(1,-1,0)}_{v_2} + c\underbrace{(0,0,1)}_{v_3}.$

v_1, v_2, v_3 are linearly independent

<u>Theorem</u> If an $n \times n$ matrix A has 'n' distinct eigen values, then it is similar to a diagonal matrix, whose elements are the eigen values of A.

if eigen values of A are $1, 2, 3$, then A is similar to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

<u>Theorem.</u> If $g_\lambda = a_\lambda$ for each eigenvalue of A, then A is similar to a diagonal matrix or A is diagonalizable.

<u>How to diagonalize a matrix A?</u>

<u>steps.</u> 1. Find eigen values of $A_{n\times n}$, say $\lambda_1, \lambda_2, \ldots, \lambda_m$; $m \le n$.

2. Find geometric multiplicity of each eigen value.

3. If $g_\lambda = a_\lambda$ for each λ, then A is diagonalizable.

Otherwise, stop.

4. How to proceed $\rightarrow$ shown by an example.

$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $\quad \lambda = 1, 5, 5$

$a_{\lambda=1} = 1, \quad g_{\lambda=1} = 1$

$a_{\lambda=5} = 2, \quad g_{\lambda=5} = 2.$

$P \rightarrow$ is a matrix formed by the linearly independent eigen vectors.

$$P = \begin{array}{ccc} \underset{\sim}{v}_{\lambda=1} & \underset{\sim}{v}_{\lambda=5} & \underset{\sim}{v}_{\lambda=5} \\ \end{array}$$
$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$$

$$P = \begin{array}{ccc} \underset{\sim}{v}_{\lambda=5} & \underset{\sim}{v}_{\lambda=1} & \underset{\sim}{v}_{\lambda=5} \\ \end{array}$$
$$P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$$

$$P = \begin{array}{ccc} \underset{\sim}{v}_{\lambda=5} & \underset{\sim}{v}_{\lambda=1} & \underset{\sim}{v}_{\lambda=5} \\ \end{array}$$
$$P = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

$$(P \mid I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 : R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \to 2R_1 - R_2} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \to \frac{R_2}{2}]{R_1 \to \frac{R_1}{2}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise. Check $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

Eigenvalues for special matrices.

Let $A_{n\times n}$ be a matrix. Then A is symmetric if $A^T = A \Rightarrow a_{ij} = a_{ji}$

$$\begin{pmatrix} 1 & -5 & 2 \\ -5 & 2 & 3 \\ 2 & 3 & 6 \end{pmatrix} \qquad \begin{pmatrix} -1 & -5+i & 1 \\ -5+i & i & 2 \\ 1 & 2 & 9-i \end{pmatrix}$$

symmetric matrix

$A_{n\times n}$ is skew symmetric if $A^T = -A$, $a_{ji} = -a_{ij}$

put $j = i$, $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

$$\begin{pmatrix} 0 & -5 & 2 \\ 5 & 0 & 6 \\ -2 & -6 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2-i & 6i \\ -2+i & 0 & 3 \\ -6i & -3 & 0 \end{pmatrix}$$

skew symmetric matrix.

	symmetric	skew-symmetric	orthogonal
Real	$A^T = A$	$A^T = -A$	$A^T A = I = A A^T$
	Hermitian.	skew-Hermitian.	unitary.
Complex	$A^* = A$ $\Rightarrow \bar{A}^T = A.$ $\Rightarrow \bar{A} = A^T$	$A^* = -A$ $\Rightarrow \bar{A}^T = -A$ $\Rightarrow \bar{A} = -A^T$	$A^* A = I = AA^*$ $A^* = A^{-1}$

Theorem. Eigen values of a Hermitian (symmetric) matrix are real, eigenvalues of a skew-Hermitian (skew-symmetric) matrix are purely imaginary or zero, eigenvalue λ of a unitary (orthogonal) matrix is such that $|\lambda| = 1$.

Proof. A is Hermitian $\Rightarrow A^* = A \Rightarrow A^T = \bar{A}$
(symmetric) $(A^T = A)$.

A is skew-Hermitian $\Rightarrow A^* = -A \Rightarrow A^T = -\bar{A}$
(skew-symmetric) $(A^T = -A)$.

(orthogonal) $\Rightarrow A^T A = A^T A = I$

A is unitary $\Rightarrow A^* A = I \Rightarrow (\bar{A})^T A = I$
$\Rightarrow A^T \bar{A} = I$, also $\bar{A} A^T = I$.

Let A be an $n \times n$ matrix.
λ be an eigen value & $\underset{\sim}{v}$ be the corresponding eigen vector of A. $(\underset{\sim}{v} \neq \underset{\sim}{0})$.

$\therefore \quad A\underset{\sim}{v} = \lambda \underset{\sim}{v} \longrightarrow (1)$

let $\underset{\sim}{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \because \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow$ atleast one of x_i's $\neq 0$.

$$P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise. Check $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

Eigenvalues for special matrices.

Let $A_{n\times n}$ be a matrix. Then A is symmetric if $A^T = A \Rightarrow a_{ij} = a_{ji}$

$$\begin{pmatrix} 1 & -5 & 2 \\ -5 & 2 & 3 \\ 2 & 3 & 6 \end{pmatrix} \qquad \begin{pmatrix} -1 & -5+i & 1 \\ -5+i & i & 2 \\ 1 & 2 & 9-i \end{pmatrix}$$

symmetric. matrix

$A_{n\times n}$ is skew symmetric if $A^T = -A$, $a_{ji} = -a_{ij}$

put $j = i$, $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

$$\begin{pmatrix} 0 & -5 & 2 \\ 5 & 0 & 6 \\ -2 & -6 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2-i & 6i \\ -2+i & 0 & 3 \\ -6i & -3 & 0 \end{pmatrix}$$

skew symmetric matrix.

	symmetric	skew-symmetric	orthogonal
Real	$A^T = A$	$A^T = -A$	$A^T A = I = A A^T$
	Hermitian.	skew-Hermitian.	unitary.
Complex	$A^* = A$ $\Rightarrow \bar{A}^T = A.$ $\Rightarrow \bar{A} = A^T$	$A^* = -A$ $\Rightarrow \bar{A}^T = -A$ $\Rightarrow \bar{A} = -A^T$	$A^* A = I = AA^*$ $A^* = A^{-1}$

Theorem. Eigen values of a Hermitian (symmetric) matrix are real, eigenvalues of a skew-Hermitian (skew-symmetric) matrix are purely imaginary or zero, eigenvalue λ of a unitary (orthogonal) matrix is such that $|\lambda| = 1$.

Proof. A is Hermitian $\Rightarrow A^* = A \Rightarrow A^T = \bar{A}$
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A is unitary $\Rightarrow A^* A = I \Rightarrow (\bar{A})^T A = I$
$\Rightarrow A^T \bar{A} = I$, also $\bar{A} A^T = I$.

Let A be an $n \times n$ matrix.
λ be an eigen value & $\underset{\sim}{v}$ be the corresponding eigen vector of A. $(\underset{\sim}{v} \neq \underset{\sim}{0})$.

$$\therefore \quad A\underset{\sim}{v} = \lambda \underset{\sim}{v} \longrightarrow (1)$$

Let $\underset{\sim}{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \because \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow$ atleast one of x_i's $\neq 0$.

Premultiply both sides of (1) by $\bar{\underset{\sim}{v}}^T$

$$\bar{\underset{\sim}{v}}^T A \underset{\sim}{v} = \bar{\underset{\sim}{v}}^T \lambda \underset{\sim}{v} = \lambda \bar{\underset{\sim}{v}}^T \underset{\sim}{v}$$

Notice, $\bar{\underset{\sim}{v}}^T \underset{\sim}{v} = [\bar{x}_1 \ \bar{x}_2 \ldots \bar{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [\sum |x_i|^2] = \sum_{i=1}^{n} |x_i|^2 \neq 0$
$= m$ (say).

So, $\lambda m = \underset{1\times n}{\bar{\underset{\sim}{v}}^T} \underset{n\times n}{A} \underset{\sim}{v}_{n\times 1}$

R.H.S. will give us a 1×1 matrix i.e. some number z.

So, $z = \bar{\underset{\sim}{v}}^T A \underset{\sim}{v}$

Also, we've $m \neq 0$, is real.

$\therefore \lambda = \frac{z}{m}$ will be real/purely imaginary.

<u>Case I.</u> A is Hermitian.

$$A^T = \bar{A}$$

$$\bar{z} = \overline{\bar{\underset{\sim}{v}}^T A \underset{\sim}{v}} = \underset{\sim}{v}^T \bar{A} \bar{\underset{\sim}{v}} = \underset{\sim}{v}^T A^T \bar{\underset{\sim}{v}} = (\bar{\underset{\sim}{v}}^T A \underset{\sim}{v})^T$$

$= z^T = z$ ($\because$ z is a number it is invariant under transposition.)

i.e. $a - ib = a + ib$

$\Rightarrow b = 0 \quad \therefore z = a$, real

So λ is real in this case.

<u>Case II</u> Try it!

Case III A is unitary.

$$\Rightarrow A^T\bar{A} = \bar{A}A^T = I$$

$$A\underset{\sim}{v} = \lambda \underset{\sim}{v} \longrightarrow (1)$$

Take transpose on both sides of (1)

$$\underset{\sim}{v}^T A^T = \lambda \underset{\sim}{v}^T \longrightarrow (2)$$

Take conjugate on both sides of (2)

$$\overline{\underset{\sim}{v}^T A^T} = \overline{\lambda \underset{\sim}{v}^T}$$

$$\Rightarrow \bar{\underset{\sim}{v}}^T \bar{A}^T = \bar{\lambda} \bar{\underset{\sim}{v}}^T \longrightarrow (3).$$

Multiply both sides of (3) by both sides of (1).

$$\bar{\underset{\sim}{v}}^T \bar{A}^T A \underset{\sim}{v} = \lambda\bar{\lambda}\, \bar{\underset{\sim}{v}}^T \underset{\sim}{v}$$

or $\bar{\underset{\sim}{v}}^T A^* A \underset{\sim}{v} = \lambda\bar{\lambda}\, \bar{\underset{\sim}{v}}^T \underset{\sim}{v}$

or $\bar{\underset{\sim}{v}}^T_{1\times n} I_{n\times n} \underset{\sim}{v}_{n\times 1} = \lambda\bar{\lambda}\, m$

or $\bar{\underset{\sim}{v}}^T \underset{\sim}{v} = |\lambda|^2 m$

or $m = |\lambda|^2 m$

or $|\lambda|^2 = 1 \quad (\because m \neq 0).$

$$\Rightarrow |\lambda| = +1.$$

Orthogonal matrix.

$$A^T A = I \quad \text{or} \quad A^T A = I$$

i.e. if $A^T = A^{-1}$.

Take determinant. $|A^T|\,|A| = 1$

$$\Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1.$$

$\Rightarrow$ An orthogonal matrix is always nonsingular.

Theorems.

1. If A & B are two symmetric matrices, then $A \pm B$ is also symmetric.

2. If A & B are both symmetric matrices of the same order, then AB is symmetric if and only if $AB = BA$.

3. If A be an $n \times n$ matrix, then the matrices AA^t & A^tA are symmetric.

4. Any square matrix A (real / complex) can be expressed as a sum of symmetric matrix $\frac{A+A^T}{2}$ & a skew symmetric matrix $\frac{A-A^T}{2}$.

$$\therefore \quad A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).$$

5. If A is an orthogonal matrix then $\det A = \pm 1$ i.e. A is always non-singular.

6. If A is an orthogonal matrix, then A^{-1} is also orthogonal.

7. If A & B are orthogonal matrices of the same order then AB & BA are orthogonal.

Complex Matrices.

A matrix A whose elements are taken from the field $\mathbb{C}$ (whose elements are complex) is a complex matrix.

$$A = P + iQ \; ; \quad P, Q \text{ are real matrices.}$$

Here we define.

$$A^* = \bar{A}^T \text{ (Conjugate transpose)}.$$