Date
14/08/2017

Lecture 9

Defn :- The Dirac δ-function $\delta(t)$ is defined as having the following properties:

$$\delta(t) = 0, \ \forall t, \ t \neq 0 \quad \rightarrow (1)$$

$$\& \int_{-\infty}^{\infty} h(t)\, \delta(t)\, dt = h(0) \quad \rightarrow (2)$$

for any continuous function $h(t)$ in $(-\infty, \infty)$.

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(1) $\mathcal{L}^{-1}\left(\frac{s^2}{s^2+1}\right)$

We have, $\dfrac{s^2}{s^2+1} = 1 - \dfrac{1}{(s^2+1)}$

Using the linearity property of Inverse L.T gives

$$\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\} = \mathcal{L}^{-1}\{1\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \delta(t) - \sin t. \quad //$$

This function is sinusoidal with a unit impulse at $t=0$.

Q2) $\mathcal{L}^{-1}\left\{\frac{s^3}{s^2+1}\right\}$

Soln:- We see that

$$\frac{s^3}{s^2+1} = s - \frac{s}{(s^2+1)}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s^3}{s^2+1}\right\} = \mathcal{L}^{-1}\{s\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= \delta'(t) - \cos t,$$

where $\delta'(t)$ is the first derivative of the Dirac-δ-function.

The Dirac-δ-function can also be thought as the limiting case of a <u>top hat</u> function of unit area.

i.e., $$\delta(t) = \mathop{Lt}_{T\to\infty} T_p(t).$$

where $$T_p(t) = \begin{cases} 0, & t \le -1/T \\ T/2, & -1/T < t < 1/T \\ 0, & t \ge 1/T. \end{cases}$$

$$\int_{-\infty}^{\infty} h(t) \mathop{Lt}_{T\to\infty} T_p(t)\, dt$$

$$= \mathop{Lt}_{T\to\infty} \int_{-\infty}^{\infty} h(t)\, T_p(t)\, dt$$ (how?)

$$= h(0)$$

$\boxed{h(t) = h(0) + \epsilon(t),\ |\epsilon(t)| \to 0 \text{ as } T\to\infty.}$

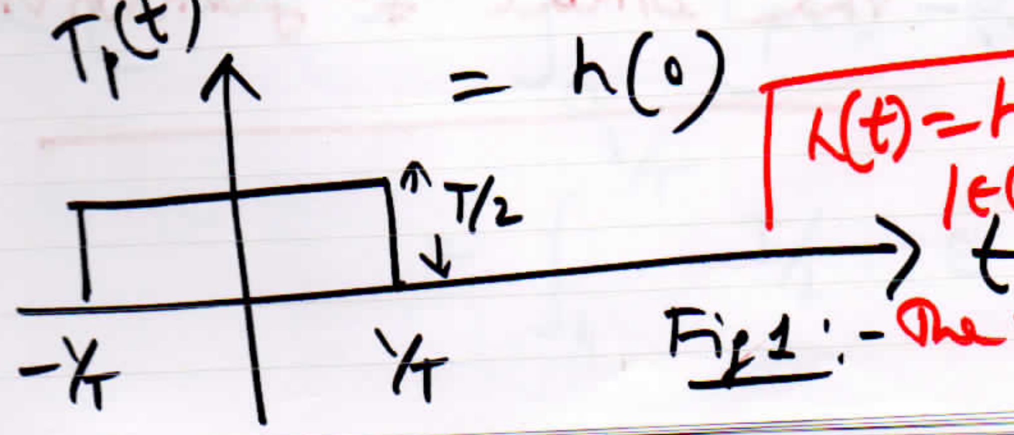


Fig 1:- The top hat f^n.

Equivalent condⁿ to eqⁿ(2)

$$\int_{0_-}^{\infty} h(t)\,\delta(t)\,dt = h(0),$$

$$\& \quad \int_{-\infty}^{0+} h(t)\,\delta(t)\,dt = h(0).$$

we know,

$$\boxed{\int_{-\infty}^{\infty} \delta(t)\,dt = 1}$$

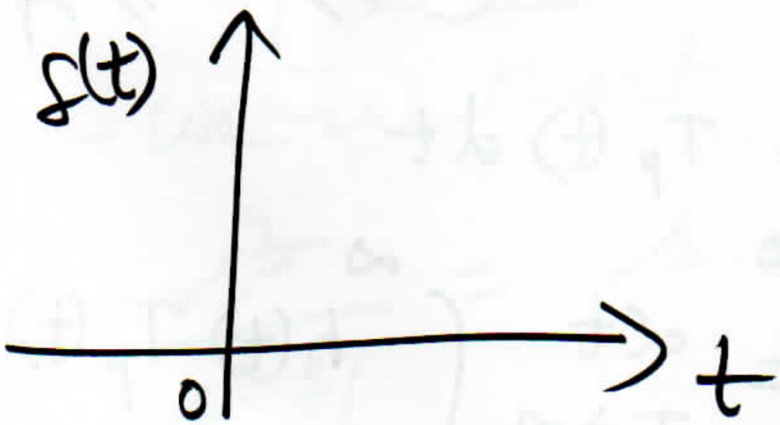


Fig 2 :- The Dirac δ- function

e

What is the L.T of $\delta(t)$?

Does it exist?

We take $h(t) = e^{-st}$

$h(0) = e^0 = 1.$

$$\int_{-\infty}^{\infty} \delta(t) \underbrace{e^{-st}} dt$$

$$= \int_{0-}^{\infty} \delta(t) e^{-st} dt = 1 = h(0).$$

We are given

$$T_p(t) = \begin{cases} 0, & t \le -1/T \\ T/2, & -1/T < t < 1/T \\ 0, & t \ge 1/T. \end{cases}$$

$$\mathcal{L}[T_p(t)] = \int_0^{\infty} \underbrace{T_p(t)} . e^{-st} dt$$

$$= \int_0^{1/T} T/2 . e^{-st} dt.$$

$$= \left[-\frac{T}{2s} e^{-st} \right]_0^{1/T}$$

$$= \left[\frac{T}{2s} - \frac{T}{2s} e^{-s/T} \right]$$

As $T \to \infty$

$$\therefore \; e^{-s/T} \approx 1 - s/T + O\left(\frac{1}{T^2}\right)$$

Hence,
$$\frac{T}{2s} - \frac{T}{2s} e^{-s/T}$$

$$\approx \cancel{\frac{T}{2s}} - \frac{T}{2s}\left(\cancel{1} - s/T + O\left(1/T^2\right)\right)$$

$$\approx \frac{1}{2} + O\left(\frac{1}{T}\right).$$

$$\to 1/2 \text{ as } T \to \infty$$

But, $\mathcal{L}\{\delta(t)\} = 1$.

This means we have
to reduce the width
of the top hat f^n so that
it lies bet^n 0 & $1/T$
EX (not $-1/T$ & $1/T$).
& increasing the height
from $T/2$ to T in order
to preserve unit area.
(how?).

The f^n $\delta(t - t_0)$ represents
an impulse that is centred
at time $t = t_0$.
This can be considered
as the limit of the f^n
$h(t)$, displaced top hat f^n

$$k(t) = \begin{cases} 0, & t \le t_0 - \frac{1}{2T} \\ T/2, & t_0 - \frac{1}{2T} < t < t_0 + \frac{1}{2T} \\ 0, & t \geq t_0 + \frac{1}{2T} \end{cases}$$

as $T \to \infty$.

ie, $\delta(t - t_0) = \underset{T \to \infty}{\mathrm{Lt}}\, h(t)$

$$\int_{-\infty}^{\infty} h(t)\, \delta(t - t_0)\, dt = h(t_0).$$

& that provided $t_0 > 0$,

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$$

letting $t_0 \to 0$, leads to

$$\mathcal{L}\{\delta(t)\} = 1$$

The property of $\delta(t)$ to pick out a particular function value, known ~~as~~ as the <u>filtering property</u> / sifting property, since

$$\int_{-\infty}^{\infty} h(t)\cdot\delta(t-t_0)\,dt = h(t_0).$$

with $h(t) = e^{-st} f(t)$

& $t_0 = a \geq 0$,

$$\boxed{\mathcal{L}\{\underbrace{\delta(t-t_0).}_{f(t)}\} = e^{-as} f(a)}$$

Ex

i.e. $\boxed{\mathcal{L}\{\delta(t-t_0)\, f(t)\} = e^{-as} f(a)}$

The value of the integral

$$\int_{-\infty}^{t} \delta(u-u_0)\,du$$

$$= \begin{cases} 0, & u_0 > t \\ 1, & u_0 < t \end{cases}$$

$$= \begin{cases} 0, & t < u_0 \\ 1, & t > u_0 \end{cases}$$

$$= H(t-u_0)$$

where H is the Heaviside's Unit step function.

i.e., $\boxed{\delta(u-u_0) = H'(u-u_0)}$

i.e., stated as — The Impulse f^n is the derivative of the Heaviside Unit step f^n.

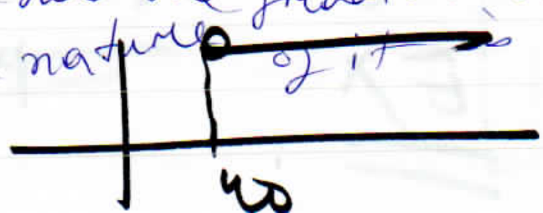
Except at $u = u_0$, the statement is equivalent to saying that the derivative of unity is zero, which is obviously true.

The additional information which is given is a

Investigate

quantification of the nature of the unit jump in $H(u-u_0)$

We know that the gradient there is infinite. But the nature of it is embodied in the second integral cond^n in the defining the delta f^n.



It is possible to define a whole family of derivatives $\delta'(t)$, $\delta''(t)$, etc.

where all these derivatives are zero everywhere except at $\boxed{\underline{t=0}}$

The key to keeping rigorous is to use this property.

$$\int_{-\infty}^{\infty} h(t)\,\delta(t)\,dt = h(0).$$

$$\int_{-\infty}^{\infty} h(t)\,\delta'(t)\,dt = -h'(0)$$

(Integrating by parts)

Ex

$$\int_{-\infty}^{\infty} h(t)\, \delta^{(n)}(t)\, dt = (-1)^n h^n(0) \longrightarrow (*)$$

where $h(t)$ is appropriately differentiable.

Also, $\mathcal{L}\{\delta^{(n)}(t)\}$

$$= \int_{-\infty}^{\infty} e^{-st}\, \delta^{(n)}(t)\, dt.$$

$$= \int_{0-}^{\infty} e^{-st}\, \delta^{(n)}(t)\, dt$$

[by $(*)$]

$$= (-1)^n . (-1)^n . s^n$$

$$\left[\begin{array}{l} h(t) = e^{-st} \\ h^n(t) = (-1)^n s^n e^{-st} \end{array} \right.$$

$$= s^n.$$

$$\left[h^n(0) = (-1)^n s^n \right.$$

$\Rightarrow$

$$\mathcal{L}\{\delta^n(t)\} = s^n$$

$$\Rightarrow \mathcal{L}^{-1}\{s^n\} = \delta^n(t)$$

Hence, for all these generalized f^ns, the cond^{n}s for the validity of Initial value theorem is violated (how?) &

Final value theorem is satisfied / valid but useless (how?)

Periodic functions

Def^n:- If $f(t)$ is a function that obeys the rule

$$f(t) = f(t+\tau),$$

for some real τ for all values of t, then $f(t)$ is called a periodic function with period τ.

Th-15/ Let $f(t)$ have period $T > 0$, so that $f(t) = f(t+T)$. Then

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

i.e., if a periodic function f has period T, $T>0$, then $f(t+T)=f(t)$. The Laplace transform of a periodic f^n can be obtained by integration over one period.

proof:— $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt$$

$$+ \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots + \int_{(n-1)T}^{nT} e^{-st} f(t)\,dt$$

$$+ \cdots$$

provided the series on the R.H.S is convergent. This is assured (how?) since the f^n $f(t)$ satisfies the cond^n for the existence by construction :

We consider the integral

$$\int_{(n-1)T}^{nT} e^{-st} f(t)\,dt$$

:

we substitute $u = t - (n-1)T$

$$\Rightarrow t = u + (n-1)T$$

$$\therefore dt = du$$

$$= \int_{u=0}^{T} e^{-s(u+\overline{n-1}T)} \, \underbrace{f(u+(n-1)T)}\, du$$

when $t = nT$, $u = T$

$t = (n-1)T$, $u = 0$

$$= e^{-s(n-1)T} \int_0^T e^{-su} f(u)\, du \quad n = 1, 2, \ldots$$

[since f has period T] (how?)

which gives

$$\int_0^\infty e^{-st} f(t)\, dt$$

$$= \left(1 + e^{-sT} + e^{-2sT} + \dots\right) \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}} \qquad |e^{-sT}| < 1.$$

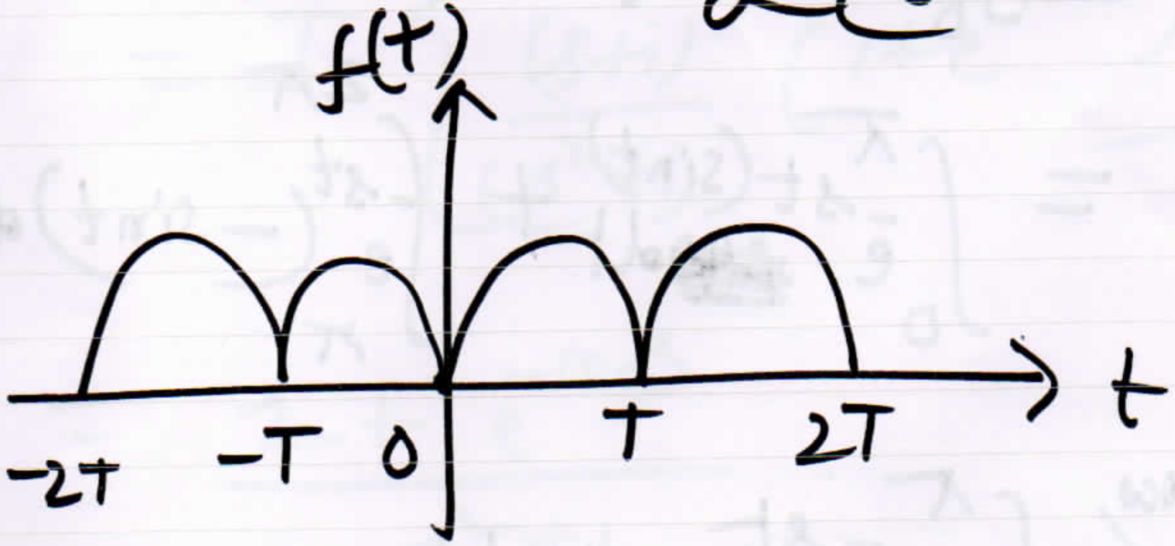
[as $1 + r + r^2 + \dots = \frac{1}{1-r}$, $|r| < 1$] G. Progression

Ex 1/ A rectified sine wave is defined by the expression

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ -\sin t, & \pi < t < 2\pi . \end{cases}$$

$$f(t) = f(t+2\pi).$$

Determine $\mathcal{L}\{f(t)\}$.



Sol^n:- The f^n $f(t)$ actually has period π, but for ease of calculation we we take the period as 2π.

With $T = 2\pi$,

$$\mathcal{L}\{f(t)\} = \frac{\int_0^{2\pi} e^{-st} f(t)\,dt}{1 - e^{-s \times 2\pi}}$$

Now, $\int_0^{2\pi} e^{-st} f(t)\,dt$

$$= \int_0^{\pi} e^{-st}(\sin t)\,dt + \int_{\pi}^{2\pi} e^{-st}(-\sin t)\,dt$$

Now, $\int_0^{\pi} e^{-st} \sin t\,dt$

$$= I\left\{\int_0^{\pi} \underbrace{e^{-st+it}}_{= e^{t(i-s)}}\,dt\right\}$$

$$= I\left[\frac{e^{-st+it}}{i-s}\right]_0^{\pi}$$

$$= I\left\{\frac{1}{(i-s)}\left(e^{-s\pi+i\pi}-1\right)\right\}$$

$$= I\left\{\frac{1}{(s-i)}\left(1+e^{-s\pi}\right)\right\}$$

$$= I\left\{\frac{(s+i)}{(s^2+1)}\left(1+e^{-\pi s}\right)\right\}$$

$$= \frac{1+e^{-\pi s}}{s^2+1}.$$

$$\Rightarrow \int_0^{\pi} e^{-st}\sin t\,dt = \frac{1+e^{-\pi s}}{s^2+1}$$

S|||y, $\int_{\pi}^{2\pi} e^{-st}(-\sin t)\, dt$

$$= -\left(\frac{e^{-2\pi s} + e^{-\pi s}}{1+s^2}\right)$$

Hence, we deduce that

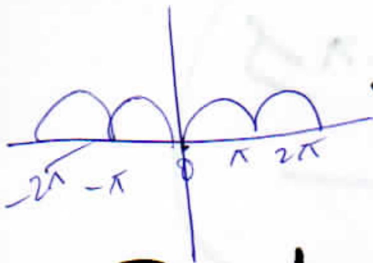
$$\mathcal{L}\{f(t)\} = \frac{1 + 2e^{-\pi s} + e^{-2\pi s}}{(1+s^2).(1-e^{-2\pi s})}$$

$$= \frac{(1+e^{-\pi s})^2}{(1+s^2)(1-e^{-2\pi s})}$$

$$= \frac{1+e^{-\pi s}}{(1+s^2)(1-e^{-\pi s})}$$

Define

C.W $f(t) = \sin t, \quad 0 < t < \pi$



$f(t) = f(t+\pi)$

Find $\mathcal{L}\{f(t)\}$

$$= \frac{\int_0^{\pi} e^{-st} f(t)\,dt}{1 - e^{-\pi s}}$$

$$= \frac{1 + e^{-\pi s}}{(1+s^2)(1 - e^{-\pi s})}$$

Special Functions

D. eq^n with variable co-efficients

$$\mathcal{L}\{t\,f(t)\} = -\,F'(s) \longrightarrow (1)$$

with $f(t) = y' = \frac{dy}{dt}$ &

$$\mathcal{L}\{y'\} = sY(s) - y(0).$$

then $$\mathcal{L}\{t y'\} = -\frac{d}{ds}\{sY - \underbrace{y(0)}_{(=0)}\}$$

$$= -Y - s\frac{dY}{ds}.$$

(2)

$$= X$$