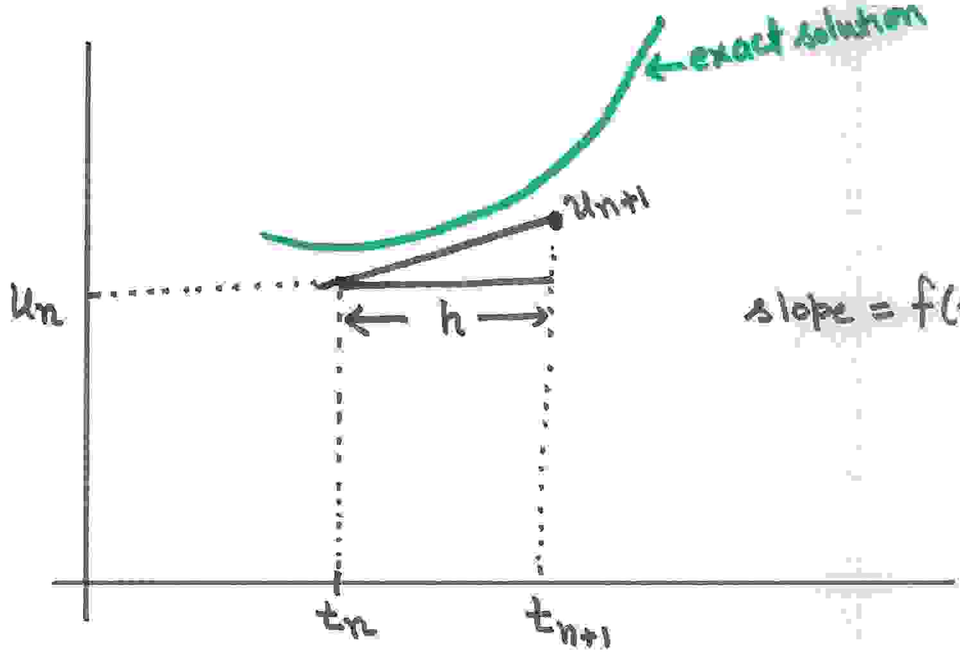
Euler Method:

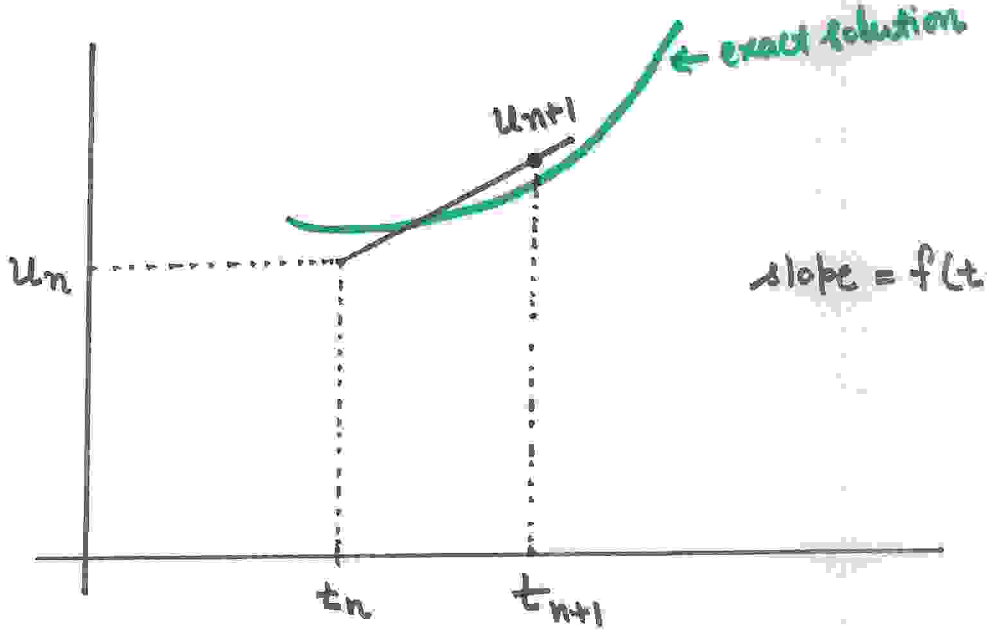
$$u_{n+1} = u_n + h\, f(t_n, u_n)$$



$$\text{slope} = f(t_n, u_n) = \frac{u_{n+1} - u_n}{h}$$

Backward Euler Method:

$$u_{n+1} = u_n + h\, f(t_{n+1}, u_{n+1})$$



$$\text{slope} = f(t_{n+1}, u_{n+1}) = \frac{u_{n+1} - u_n}{h}$$

Since evaluation of u_{n+1} requirs u_{n+1}, backward Euler method is an implicit method.

For the solution of nonlinear equation $u_{n+1} = u_n + h f(t_{n+1}, u_{n+1})$, we can use Newton-Raphson method.

Example: Solve the IVP

$$u' = -2tu^2 \qquad u(0) = 1$$

with $h = 0.2$ on the interval $[0, 0.4]$ using the backward Euler method.

Sol: Backward Euler Method $u_{n+1} = u_n + h f(t_{n+1}, u_{n+1})$

$$\Rightarrow u_{n+1} = u_n + h\left(-2t_{n+1} u_{n+1}^2\right) \quad ; \quad n = 0, 1.$$

OR $$u_{n+1} = u_n - 2h t_{n+1} u_{n+1}^2$$

We can solve the above quadratic equation directly or by NR method as follows: Define

$$F(u_{n+1}) = u_{n+1} - u_n + 2h t_{n+1} u_{n+1}^2$$

$$F'(u_{n+1}) = 1 + 4h t_{n+1} u_{n+1}$$

Thus, NR:

$$u_{n+1}^{(s+1)} = u_{n+1}^{(s)} - \frac{F(u_{n+1}^{(s)})}{F'(u_{n+1}^{(s)})} \quad ; \quad s = 0, 1, 2 \cdots\cdots \quad (*)$$

Take $u_{n+1}^{(0)} = u_n$

For $n = 0$: using $(*)$: $u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, \ldots\ldots$

$$u(0.2) \simeq u_1 = u_1^{(3)} = 0.93070331$$

For $n = 1$: Using $(*)$: $u_2^{(1)}, u_2^{(2)}, u_2^{(3)}, \ldots\ldots$

$$u(0.4) \simeq u_2 = u_2^{(3)} = 0.82247016 .$$

Consistency of Backward Euler Method:

$$\tau_{n+1} = y(t_{n+1}) - y(t_n) - h f(t_{n+1}, y(t_{n+1}))$$

$$= y(t_{n+1}) - y(t_{n+1} - h) - h f(t_{n+1}, y(t_{n+1}))$$

$$= \cancel{y(t_{n+1})} - \left\{ \cancel{y(t_{n+1})} - h \cancel{y'(t_{n+1})} + \frac{h^2}{2!} y''(\xi) \right\} - h \cancel{f(t_{n+1}, y(t_{n+1}))}$$

$$= -\frac{h^2}{2!} y''(\xi), \qquad t_n < \xi < t_{n+1}$$

Thus, $\left| \frac{1}{h} \tau_{n+1} \right| = O(h)$

Similar to Euler method, Backward Euler method is a first order single step method.

Runge-Kutta Methods:

Consider the IVP:

$$\frac{dy}{dt} = f(t,y), \qquad y(t_0) = y_0$$

Integrating the above differential equation from t_j to t_{j+1}:

$$\int_{t_j}^{t_{j+1}} \frac{dy}{dt}\,dt = \int_{t_j}^{t_{j+1}} f(t,y)\,dt$$

Applying mean value theorem in the integral on the R.H.S.

$$y(t_{j+1}) - y(t_j) = h\,f\big(t_j+\theta h,\, y(t_j+\theta h)\big), \quad 0<\theta<1.$$

Different value of θ gives us a new numberical method.

Case-I: $\theta = 0$:

$$u_{j+1} = u_j + h\,\underbrace{f(t_j, u_j)}_{\text{slope at } t_j} \qquad \text{Euler Method}$$

Case-II: $\theta = 1$: $$u_{j+1} = u_j + h\,\underbrace{f(t_{j+1}, u_{j+1})}_{\text{slope at } t_{j+1}} \qquad \text{Backward Euler Method}$$

Case III: $\theta = 1/2$: $$y(t_{j+1}) \simeq y(t_j) + h\,\underbrace{f\left(t_j+\frac{h}{2},\, y\left(t_j+\frac{h}{2}\right)\right)}_{\text{slope at midpoint}}$$

However, $t_{j+1/2}$ is not a nodal point.

How to evaluate (approximate) $f(t_j+h/2,\, y(t_j+h/2))$?

IDEA - 1:

$$y(t_j+\tfrac{h}{2}) \simeq y_j + \tfrac{h}{2} f(t_j, y_j) \rightarrow \text{Euler Method}$$

Then the numerical Method becomes:

$$u_{j+1} = u_j + h f\left(t_j+\tfrac{h}{2},\ u_j + \tfrac{h}{2} f_j\right), \quad f_j = f(t_j, u_j)$$

We can rewrite the above formula as:

Set $k_1 = f_j$

$$k_2 = f\left(t_j+\tfrac{h}{2},\ u_j+\tfrac{h}{2}k_1\right)$$

$$u_{j+1} = u_j + h k_2$$

This Method is called Modified Euler-Cauchy method or Midpoint method.

IDEA-2: $f(t_j+\frac{h}{2}, y(t_j+\frac{h}{2})) = y'(t_j+\frac{h}{2}) \simeq \frac{1}{2}\left[y'(t_j)+y'(t_{j+1})\right]$

$$= \tfrac{1}{2}\left[f(t_j,y_j) + f(t_{j+1}, y_{j+1})\right]$$

Using Euler Method:

$$f(t_j+\tfrac{h}{2}, y(t_j+\tfrac{h}{2})) \simeq \tfrac{1}{2}\left[f(t_j,y_j) + f(t_{j+1}, y_j+hf_j)\right]$$

Then the numerical Method becomes:

$$u_{j+1} = u_j + \tfrac{h}{2}\left[f(t_j,u_j) + f(t_{j+1}, u_j+hf_j)\right]$$

OR: $k_1 = f_j \qquad k_2 = f(t_{j+1}, u_j + h k_1)$

$$u_{j+1} = u_j + \tfrac{h}{2}[k_1+k_2]$$

This method is called Euler-Cauchy method (Heun's method)