Linear Algebra

Lecture 9

(Lecture 8 : class test)

Linear Transformations.

Let V and W be vector spaces over a field $\mathbb{F}$. Then a function $T : V \to W$ is called as linear transformation if for $x, y \in V$ and $\alpha \in \mathbb{F}$,

$$T(\alpha x + y) = \alpha T(x) + T(y)$$

Ex: If $V = W = \mathbb{F}^n$

$$T(x) = A_T x \qquad \text{for some } A_T \in \mathbb{F}^{n \times n}.$$

Ex: Let $V = C(\mathbb{R})$ = the real real vector space of all continuous functions on $\mathbb{R}$.

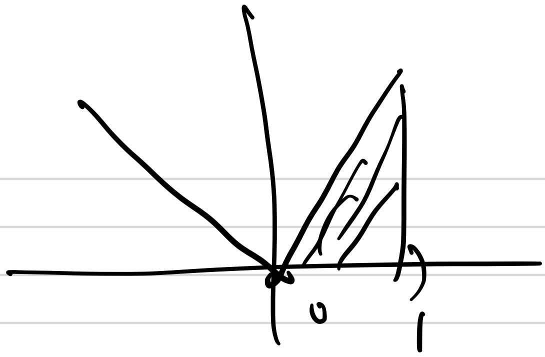
$$T : V \to \mathbb{R}$$

$$T(f) = \int_a^b f(x)\,dx \qquad \text{for some } a, b \in \mathbb{R}$$

Set $a=0$, $b=1$

$f(x) = |x|$

$T(f) = 1/2$



Is T a linear transformation??

$$T(\alpha f_1 + f_2) = \int_a^b (\alpha f_1(x) + f_2(x))\, dx$$

??

$$= \alpha \int_a^b f_1(x) + \int_a^b f_2(x)$$

$$= \alpha T(f_1) + T(f_2)$$

Trivial Examples of linear transformation

1) $V = W$ $\quad T(x) = x \quad \forall x \in V.$

2) $T: V \to W$ $\quad T(x) = 0 \quad \forall x \in V.$

Definitions:

Let V and W be vector spaces over $\mathbb{F}$. Let $T: V \to W$ be a linear transformation.

We define null-space of T, denoted as $N(T)$, as

$$N(T) = \{x \in V \mid T(x) = \underset{\in W}{0}\}$$

We define Range-space of T, denoted as $R(T)$, as

$$R(T) = \{T(x) \mid x \in V\}$$

Note: $N(T) \subseteq V$ and $R(T) \subseteq W$.

Theorem: Let V and W be vector spaces over $\mathbb{F}$, and $T: V \to W$ be a linear transformation. Then $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

<u>Proof:</u>

To prove $N(T)$ is a subspace

Let $x_1, x_2 \in N(T)$ and $\alpha \in \mathbb{F}$.

To show. $\alpha x_1 + x_2 \in N(T)$

$\Rightarrow \quad T(\alpha x_1 + x_2) = 0$ definition of $N(T)$

$\Rightarrow \quad \alpha \underbrace{T(x_1)}_{0} + \underbrace{T(x_2)}_{0} = 0$ linearity of T

Assumption that $x_1, x_2 \in N(T)$.

This proves $N(T)$ is a subspace of V.

Further $0 \in N(T)$.

To prove $R(T)$ is a subspace of W.

Let $y_1, y_2 \in R(T)$ and $\alpha \in \mathbb{F}$.

To show: $\alpha y_1 + y_2 \in R(T)$

Since y_1 & $y_2 \in R(T)$, there exist $x_1, x_2 \in V$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$

Observe: $T(\alpha x_1 + x_2) = \alpha T(x_1) + T(x_2)$

$$= \alpha y_1 + y_2$$

$\Rightarrow \alpha y_1 + y_2 \in R(T)$

$R(T)$ is a subspace of W.

and clearly $R(T)$ also contains 0.

Note that $0 \in N(T)$, this $0 \in V$,

$0 \in R(T)$, this $0 \in W$.

This is true because $T(0) = 0$

Example:

V is a vector space over $\mathbb{R}$.

$$T(x) = cx \qquad \forall\, x \in V$$

$N(T) = \{0\}$ if $c \neq 0$	$N(T) = V$ if $c = 0$
$R(T) = V$ if $c \neq 0$	$R(T) = \{0\}$ if $c = 0$

Example: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x,y) = \begin{pmatrix} x+y \\ 2x+2y \end{pmatrix}$

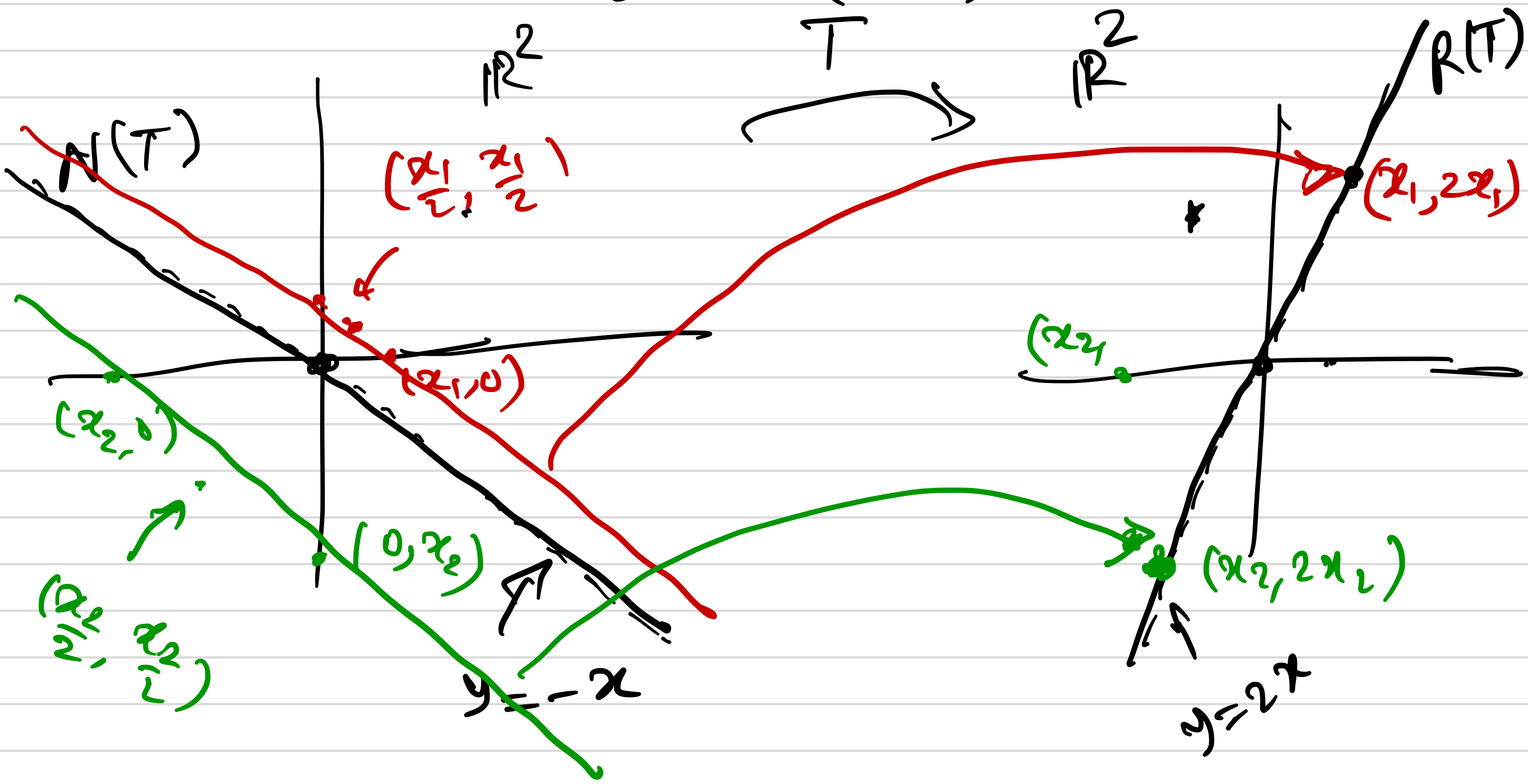
$$N(T) = \{(x,y) \in \mathbb{R}^2 \mid x = -y\}$$

$$R(T) = \{(x,y) \in \mathbb{R}^2 \mid y = 2x\}$$

Let $\begin{pmatrix} x \\ 2x \end{pmatrix} \in R(T)$ then

$$T\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix}$$

$$T\begin{pmatrix} x/2 \\ x/2 \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix}$$



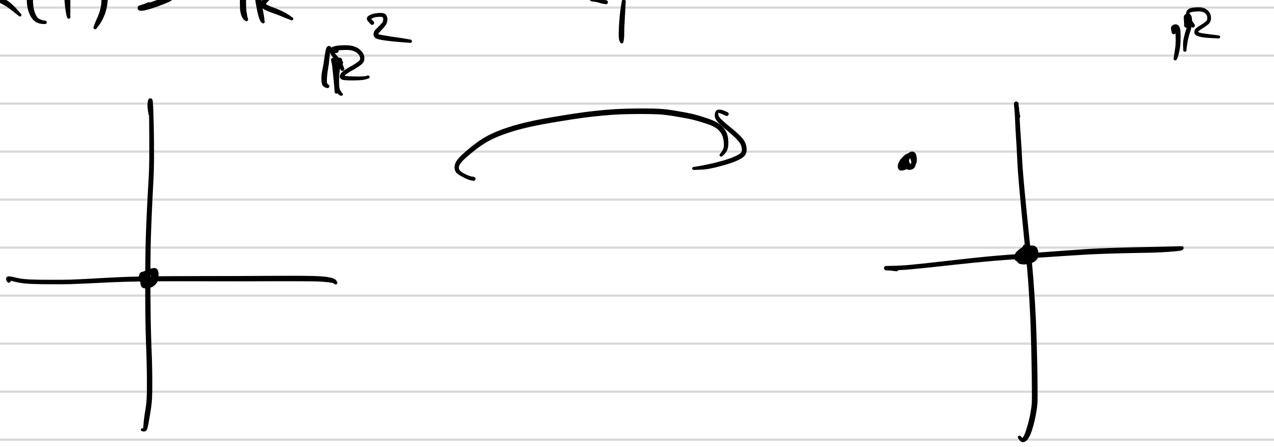
Example:

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T(x,y) = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$$

$$N(T) = \{0\} = \{(0,0)\}$$

$$R(T) = \mathbb{R}^2$$



Theorem: Let V and W be vector spaces over $\mathbb{F}$ and $T: V \to W$ be a linear transformation. Let $\{v_1, v_2, \dots, v_n\}$ be a basis for V, then

$$R(T) = \operatorname{Span}\{T(v_1), T(v_2), \dots, T(v_n)\}$$

Proof:

We want to prove

$R(T) = \text{span}\{T(v_1), \ldots, T(v_n)\}$

Notice that $T(v_i) \in R(T)$ for $i = 1, 2, \ldots, n$

Since $R(T)$ is a subspace of W,

$\text{span}\{T(v_1), \ldots, T(v_n)\} \subseteq R(T)$

Now to show $R(T) \subseteq \text{span}\{T(v_1), \ldots, T(v_n)\}$

Take $w \in R(T)$. There exists $v \in V$

such $w = T(v)$.

Since $\{v_1, \ldots, v_n\}$ is a basis for V,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

$$\Rightarrow T(v) = T(\alpha_1 v_1 + \cdots + \alpha_n v_n)$$

$$\Rightarrow w = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n)$$

$$\Rightarrow w \in \text{span}\{T(v_1), T(v_2), \ldots, T(v_n)\}$$