1.10 Indistinguishable Balls and Distinguishable Boxes

Example 1.10.1. Consider the following three problems which have the same solution.

- 1. Determine the number of distinct strings that can be formed using 3 A's and 6 B's.
- 2. Determine the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 = 6$, where each $x_i \in \mathbb{Z}$ and $0 \le x_i \le 6$.
- 3. Determine the number of ways of placing 6 indistinguishable balls into 4 distinguishable boxes.

Solution: The solution is based on the understanding that all the three problems correspond to forming strings using +'s (or |'s) and 1's (or balls) in place of A'a and B's?

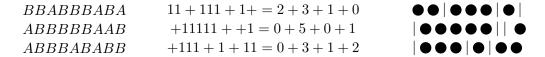


Figure 1.1: Understanding the three problems

Note that the 3 A's are indistinguishable among themselves and the same holds for 6 B's. Thus, we need to find 3 places, from the 9 = 3 + 6 places, for the A's. Hence, the answer is $\binom{9}{3}$. The answer will remain the same as we just need to replace A's with +'s (or |'s) and B's with 1's (or balls) in any string of 3 A's and 6 B's. See Figure 1.1 or note that four numbers can be added using 3 +'s or four adjacent boxes can be created by putting 3 vertical lines or |'s.

We now generalize this example to a general case.

Lemma 1.10.2. Determine the number of

- 1. solutions to the equation $x_1 + x_2 + \cdots + x_n = m$, where each $x_i \in \mathbb{Z}$ and $0 \le x_i \le m$.
- 2. ways to put m indistinguishable balls into n distinguishable boxes.

Proof: Note that the number m or the m balls can be replaced with m 1's or $m \not \star$'s. Once this is done, using the idea in Example 1.10.1.2, we see that it is enough to find the number of distinct strings formed using n-1+'s (or |'s) and m 1's (or $m \not \star$'s). Then the indistinguishability of the +'s (or |'s) and placing them among the 1's (or $\not \star$'s), we get

$$\binom{n-1+m}{m} = \binom{n-1+m}{n-1} = \binom{m+(n-1)}{m}.$$

Remark 1.10.3. Observe that the problems in Lemma 1.10.2 is same as "Determine the number of non-decreasing sequences of length m using the numbers 1, 2, ..., n". Hint: Since we are looking at a non-decreasing sequences, we note that the sequence is determined if we know the number of times a particular number has appeared in the sequence. So, let x_i , for $1 \le i \le n$, denote the number of times the number i has appeared in the sequence.