Tutorial Sheet-2 (Hints and Answers)

MATHEMATICS-II (MA10002)

1. (a), (b), (d) The given set is a basis of the corresponding vector space. First show that the set is linearly independent. Then show that it spans the given vector space.

   (c) The given set is not a basis of the corresponding vector space. The set is linearly dependent.

2. $S = \{(1,6,0,0,0),(0,0,-2,1,0),(0,0,-3,0,1)\}$ is a basis of U. U can be written as $U = \{(z_1, 6z_1, -2z_4 - z_5, z_4, z_5) : z_1, z_4, z_5 \in \mathbb{C}\} = span\ S$. Now try to show that S is linearly independent and spans U.

3. (a) $S = \{(-5,1,3)\}$ is a basis of U. Dimension of U is 1. Try to show that U can be written as $U = \{(x,y,z) \in \mathbb{R}^n : x+2y+z=0, -3y+z=0\} = \{(-5y, y, 3y) : y \in \mathbb{R}\}$.

   (b) $S = \{(1,0,-1,3,0),(0,1,-1,0,0),(0,0,0,7,1)\}$ is a basis of U. Dimension of U is 3. Try to show that U can be written as $U = \{(x_1, x_2, -x_1-x_2, 3x_1+7x_5, x_5) : x_1, x_2, x_5 \in \mathbb{R}\}$.

4. (a) $dim(U \cap W) = 2$ and $dim(U+W) = 4$. $U \cap W = \{a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathbb{P}_3 : a_0 + a_1 + a_2 + a_3 = 0, a_1 + 2a_2 + 3a_3 = 0\} = span\{1-2x+x^2, 2-3x+x^2\}$. $U + W = \{(a_0+a_1x+a_2x^2+a_3x^3)+(b_0+b_1x+b_2x^2+b_3x^3) \in \mathbb{P}_3 : a_0+a_1+a_2+a_3 = 0, b_1+2b_2+3b_3 = 0 = span\{-1+x, -1+x^2, -1+x^3, 1, -2x+x^2, -3x+x^3\} = span\{-1+x, -1+x^2, -1+x^3, 1\}$.

   (b) (i) $S = \{x, -\frac{1}{3}x^2, x^3, -\frac{1}{5}x^4\}$ is a basis of U and $dimU = 4$. U can be written as $U = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \in \mathbb{P}_4 : 2a_0 + \frac{2a_2}{3} + \frac{2a_4}{5} = 0\} = spanS$.

   (ii) $S_1 = \{1, x, -\frac{1}{3}x^2, x^3, -\frac{1}{5}x^4\}$ is a basis for $\mathbb{P}_4$.

5. (a), (c), (d) T is a linear transformation. Check the property $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$ for scalars α, β and vectors v_1, v_2.

   (b) T is not a linear mapping. Check that $T(\alpha v_1) \neq \alpha T(v_1)$.

6. Take $\phi(z) = \overline{z},\ z \in \mathbb{C}$. Then $\phi(z+w) = \phi(z) + \phi(w)$, for $z, w \in \mathbb{C}$ but $\phi(i(1+i)) = -1-i \neq 1+i = i\phi(1+i)$.

7. (a) $\mathcal{N}(T) = span\{(1,0,-1)\},\ \mathcal{R}(T) = span\{(1,2,1),(1,1,2)\}$. $dim\mathcal{N}(T) = 1,\ dim\mathcal{R}(T) = 2$ and $dim\mathcal{N}(T) + dim\mathcal{R}(T) = dim\mathbb{R}^3$. $\mathcal{N}(T) = \{(x,y,z) \in \mathbb{R}^3 : x+y+z=0, 2x+y+2z = 0, x+2y+z=0\} = \{(x,y,z) \in \mathbb{R}^3 : x+z=0, y=0\}$ and $\mathcal{R}(T) = \{(x+y+z, 2x+y+$

$2z, x+2y+z) : x, y, z \in \mathbb{R}\} = span\{(1,2,1),(1,1,2),(1,2,1)\}$.

(b)$\mathcal{N}(T)$= Set of all skew symmetric matrices in $M_{2\times 2}$, $\mathcal{R}(T)$= Set of all symmetric matrices in $M_{2\times 2}$. $dim\mathcal{N}(T) = \frac{2(2-1)}{2} = 1$, $dim\mathcal{R}(T) = \frac{2(2+1)}{2} = 3$ and $dim\mathcal{N}(T) + dim\mathcal{R}(T) = dim M_{2\times 2}$.

(c) $\mathcal{N}(T) = span\{(1,-1)\}$, $\mathcal{R}(T) = span\{(1,1)\}$. $dim\mathcal{N}(T) = 1$, $dim\mathcal{R}(T) = 1$ and $dim\mathcal{N}(T) + dim\mathcal{R}(T) = dim\mathbb{R}^2$.

(d) $\mathcal{N}(T) = span\{(1,0,-1),(0,1,-1)\}$, $\mathcal{R}(T) = \mathbb{R}$. $dim\mathcal{N}(T) = 2$, $dim\mathcal{R}(T) = 1$ and $dim\mathcal{N}(T) + dim\mathcal{R}(T) = dim\mathbb{R}^3$.

8. (a)$T(x,y,z) = 8x-3y-2z,\ (x,y,z) \in \mathbb{R}^3$. Try to express $(x,y,z) = \alpha(1,1,1)+\beta(0,1,-2)+\gamma(0,0,1)$ for some α, β, γ, then operate T on both side.

   (b)$T(x,y,z) = (x+2y+z, -x+z, y+z),\ (x,y,z) \in \mathbb{R}^3$. Try to express $(x,y,z) = \alpha e_1 + \beta e_2 + \gamma e_3$ for some α, β, γ, then operate T on both side.

   (c) $T(x,y,z) = (\frac{x+y+z}{4}, \frac{x+y+z}{4}, \frac{x+y+z}{4}),\ (x,y,z) \in \mathbb{R}^3$. Try to express $(x,y,z) = \alpha(2,1,1)+\beta(1,2,1)+\gamma(1,1,2)$ for some α, β, γ, then operate T on both side.

9. (a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Try to find the values on the basis vectors and express those in terms of that basis.

   (b) $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.Try to find the values on the basis vectors and express those in terms of that basis.

   (c) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Try to find the values on the basis vectors and express those in terms of that basis.

10. (i) Check $dim\mathcal{N}(T) = dim\mathbb{R}^2$. Then apply rank nullity theorem.

    (ii) $dim\mathcal{N}(T) = 3$. Then apply rank nullity theorem to show that $dim\mathcal{R}(T) = 5$.

    (iii) Apply rank nullity theorem.

    (iv) Find $dim\mathcal{N}(T)$ and apply rank nullity theorem.