Nyström Methods (Explicit)

$\rho(\xi) = \xi^{k-2}(\xi^2-1)$ and $\sigma(\xi)$ is of order $(k-1)$.

For k=3: $\rho(\xi) = \xi(\xi^2-1)$

$$\rho(\xi) = [(\xi-1)+1][(\xi-1+1)^2-1]$$

$$= [(\xi-1)+1][(\xi-1)^2+1+2(\xi-1)-1]$$

$$= (\xi-1)^3+3(\xi-1)^2+2(\xi-1)$$

Now:
$$\frac{\rho(\xi)}{\ln(\xi)} = \frac{2(\xi-1)+3(\xi-1)^2+(\xi-1)^3}{\ln(1+(\xi-1))}$$

$$= \frac{2(\xi-1)+3(\xi-1)^2+(\xi-1)^3}{(\xi-1)-\frac{1}{2}(\xi-1)^2+\frac{1}{3}(\xi-1)^3+\cdots}$$

$$= \left[2+3(\xi-1)+(\xi-1)^2\right]\left[1-\frac{1}{2}(\xi-1)+\frac{1}{3}(\xi-1)^2-\cdots\right]^{-1}$$

$$= \left[2+3(\xi-1)+(\xi-1)^2\right]\left[1+\frac{1}{2}(\xi-1)-\frac{1}{3}(\xi-1)^2 + \frac{1}{4}(\xi-1)^2+\cdots\right]$$

$$= 2+4(\xi-1)+(\xi-1)^2\left[1+\frac{3}{2}-\frac{1}{6}\right]+O(\xi-1)^3$$

$$= 2+4(\xi-1)+(\xi-1)^2\left[\frac{7}{3}\right]$$

$$= \frac{7}{3}\xi^2-\frac{2}{3}\xi+\frac{1}{3}.$$

The numerical method becomes

$$\rho(E)\, u_{j-2} - h\,\sigma(E)\, u'_{j-2} = 0$$

$$\Rightarrow (E^3 - E)\, u_{j-2} - h\left[\frac{7}{3}E^2 - \frac{2}{3}E + \frac{1}{3}\right] u'_{j-2} = 0$$

$$\Rightarrow u_{j+1} = u_{j-1} - \frac{h}{3}\left[7u'_j - 2u'_{j-1} + u'_{j-2}\right]$$

The order of the method is 3.

Adams-Moulton Methods (IMPLICIT)

$\rho(\xi) = \xi^{k-1}(\xi - 1)$ and $\sigma(\xi)$ is of degree k.

For $k=2$: $\rho(\xi) = \xi(\xi-1) = (\xi-1) + (\xi-1)^2$

Note that

$$\frac{\rho(\xi)}{\ln \xi} = \left[1 + (\xi-1)\right]\left[1 + \frac{1}{2}(\xi-1) - \frac{1}{3}(\xi-1)^2 + \frac{1}{4}(\xi-1)^2 - \cdots\right]$$

$$= \left[1 + (\xi-1)\right]\left[1 + \frac{1}{2}(\xi-1) - \frac{1}{12}(\xi-1)^2 + \cdots\right]$$

$$= 1 + \frac{3}{2}(\xi-1) + (\xi-1)^2\left(\frac{1}{2} - \frac{1}{12}\right) + O(\xi-1)^3$$

$$= 1 + \frac{3}{2}(\xi-1) + \frac{5}{12}(\xi-1)^2 + O(\xi-1)^3$$

Therefore we have

$$\sigma(\xi) = 1 + \frac{3}{2}(\xi-1) + \frac{5}{12}(\xi-1)^2$$

$$= \frac{5}{12}\xi^2 + \frac{2}{3}\xi - \frac{1}{12}$$

The desired Adams-Moulton method is

$$\rho(E)u_{j-1} - h\sigma(E)u'_{j-1} = 0$$

$$\Rightarrow (E^2-E)u_{j-1} - \frac{h}{12}[5E^2+8E-1]u'_{j-1} = 0$$

$$\Rightarrow \quad u_{j+1} = u_j + \frac{h}{12}[5u'_{j+1} + 8u'_j - u'_{j-1}]$$

The order of the method is 3.

Milne-Simpson Methods: (IMPLICIT)

$$\rho(\xi) = \xi^{k-2}(\xi^2-1) \quad , \sigma(\xi) \text{ is of order } k.$$

For $k=2$: $\rho(\xi) = \xi^2-1 = 2(\xi-1)+(\xi-1)^2$

Now: $$\frac{\rho(\xi)}{\ln(\xi)} = [2+(\xi-1)][1+\tfrac{1}{2}(\xi-1)-\tfrac{1}{12}(\xi-1)^2+\dots]$$

$$= 2+2(\xi-1)+\tfrac{1}{3}(\xi-1)^2+0\times(\xi-1)^3 + O(\xi-1)^4$$

$$\Rightarrow \sigma(\xi) = \frac{1}{3}(\xi^2+4\xi+1)$$

The method is given by $\rho(E)u_{j-1} - h\sigma(E)u'_{j-1} = 0$

$$\Rightarrow \quad u_{j+1} = u_{j-1} + \frac{h}{3}[u'_{j+1} + 4u'_j + u'_{j-1}]$$

The order of the method is 4.