4.4 Matrix Tree Theorem

We will now state a result in matrix theory, called the Cauchy Binet Theorem, and use it to prove the famous theorem called the Matrix-Tree Theorem (Kirchhoff 1847). To do so, note that for an $m \times n$ matrix A and $S \subset \{1, 2, \ldots, m\}$, $T \subset \{1, 2, \ldots, n\}$, the notation $A[S|T]$ denotes the sub-matrix of A that is determined by the rows that correspond to the elements of S and the columns that correspond to the elements of T. For example, if

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 1 & 3 & 1 & 0 \\ 1 & 2 & 4 & 4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix}$$

then $B = A[\{1,3\} \mid \{2,3,5\}]$. Also, the notation $A(S|T)$ corresponds to the matrix $A[S^c|T^c]$, where for any subset W of Z, W^c denotes the complement of W in Z. Thus,

$$A(\{1,3\} \mid \{2,3,5\}) = A[\{2\}|\{1,4\}] = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

We also write $A[i|j]$ and $(A(i|j))$ in place of $A[\{i\}|\{j\}]$ and $(A(\{i\}|\{j\}))$.

We now use the above notation to prove a result related to the edge incidence matrix of a connected graph.

Lemma 4.4.1. *Let X be a connected graph on n vertices and m edges. Also, let Q be its edge incidence matrix. If $T \subset \{1, 2, \ldots, m\}$ with $|T| = n-1$ then $\det(Q[1, 2, \ldots, n-1|T\,]) = \pm 1$ if and only if the subgraph Y of X, consisting of edges that correspond only to the elements of T, forms a spanning tree of X.*

Proof. Note that X is a connected graph and hence $m \geq n-1$. Thus, there always exists subsets T of $\{1, 2, \ldots, m\}$ with $|T| = n-1$.

Now, suppose that Y is not a tree. Then the condition that Y has n vertices and $n-1$ edges implies that Y is disconnected. Consider a component, say $\tilde{Y}$, of Y that does not contain the vertex n. Note that one can relabel the vertices and edges of X so that $Q[1, 2, \ldots, n-1|T\,]) = \begin{bmatrix} Q_1 & \mathbf{0} \\ \mathbf{0} & Q_2 \end{bmatrix}$, where Q_1 is the edge incidence matrix of $\tilde{Y}$. That is, the vertices and edges of $\tilde{Y}$ appear in the first block of the above partitioned matrix. Since Q_1 has exactly two non-zero entries 1 and -1, if we add all the rows of $Q[1, 2, \ldots, n-1|T\,]$ corresponding to the rows of Q_1, then they will add to the zero vector and hence $\det(Q[1, 2, \ldots, n-1|T\,]) = 0$.

Now, let us assume that Y is a tree. Then we need to show that $\det\left(Q[\{1, 2, \ldots, n-1\}\,|T]\right) = \pm 1$. To do so, we relabel the vertices of Y, other than the n-th vertex, say v_n, as follows:
Let $T = \{t_1, t_2, \ldots, t_{n-1}\}$. As Y is a tree, it has at least two vertices of degree 1. Let u_1 be a vertex that is distinct from v_n with $\deg(u_1) = 1$. Without loss of generality, let t_1 be the edge incident with the vertex u_1. Now, use the graph $Y \setminus u_1$ to obtain a vertex u_2 distinct from v_n. Also, let t_2 be the edge incident with u_2. We continue this process till we have obtained

the vertices $u_1, u_2, \ldots, u_{n-1}$, all distinct from v_n, with corresponding edges $\{t_1, t_2, \ldots, t_{n-1}\}$. Observe that this relabeling of the vertices and edges of Y determines a matrix, say E, which is a permutation of the rows and columns of the matrix $Q[\{1, 2, \ldots, n-1\} \mid T]$. Also, observe that the matrix E has been constructed in such a way that all its diagonal entries are either 1 or -1. Thus, $\det(E) = \pm 1$ and hence $\det\left(Q[\{1, 2, \ldots, n-1\} \mid T]\right) = \pm 1$. Thus, we have obtained the required result. ■

To prove our main result of this section, called the matrix tree theorem, we need the following result that gives the determinant of product of two rectangular matrices.

Theorem 4.4.2 (Cauchy Binet Theorem). *Let A and B be two $m \times n$ and $n \times m$ matrices, respectively, for some positive integers m and n with $m \le n$. Then*

$$\det(AB) = \sum_T \det\left(A[\{1, 2, \ldots, m\} \mid T]\right) \det\left(B[T \mid \{1, 2, \ldots, m\}]\right),$$

where the summation runs over all subsets T of $\{1, 2, \ldots, n\}$ with $|T| = m$.

Now, let $A = \begin{bmatrix} 0 & -1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 \\ 1 & 4 \\ -4 & 1 \end{bmatrix}$. Then $\det(AB) = \det\left(\begin{bmatrix} -17 & 8 \\ 0 & 5 \end{bmatrix}\right) = -85$ and by Theorem 4.4.2, $\det(AB)$ equals

$$\det\left(\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -4 & 1 \end{bmatrix}\right) + \det\left(\begin{bmatrix} -1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}\right) = 12 - 80 - 17.$$

Before proceeding further, consider the graph X given in Figure 4.10. Let L be its Laplacian matrix. Then verify that all the cofactors of L are equal and it equals 14. Note that Figure 4.10 also contains 14 labeled spanning trees of X (the spanning trees in Figure 4.10 have not been labeled as such).

Theorem 4.4.3 (Matrix Tree Theorem). *Let X be a connected labeled graph and let L be its Laplacian matrix. Then all the cofactors of L are equal and their common value equals the number of spanning trees of X.*

Proof. Recall that $L(i|j)$ denotes the sub-matrix of L that is obtained from L by removing the i-th row and the j-th column. Also, recall that if we add the 2-nd, 3-rd and so on till n-th row of $L = (\ell_{ij})$ to the first row of L then the first row of L equals the zero vector.

Thus, if we add all the rows, except the first row, of $L(1|1)$ to the first row of $L(1|1)$, then the first row of $L(1|1)$ equals $-(\ell_{12}, \ell_{13}, \ldots, \ell_{1n})$. Thus, $\det(L(1|1)) = -\det(L(2|1))$. That is, the cofactors of ℓ_{11} and ℓ_{21} are equal. A similar argument applied to $L(i|i)$ will imply that the cofactors of ℓ_{ii} and ℓ_{ki} are equal, for all i and k. Now, using the symmetry of L, it follows that all the cofactors of L are equal.

Let us now prove that this number equals the number of spanning trees of X. To do so, let us assume that $|E| = m$ and consider $L(n|n) = (QQ^t)(n|n)$. This corresponds to removing the n-th row of Q and n-th column of Q^t. Thus, using the Cauchy Binet Theorem 4.4.2,

$$\begin{aligned}
\det(L(n|n)) &= \det\big((QQ^t)(n|n)\big) \\
&= \sum_T \det\left(Q[\{1,2,\ldots,n-1\}\,|T]\right)\ \det\left(Q^t[T|\{1,2,\ldots,n-1\}]\right) \\
&= \sum_T \left(\det\left(Q[\{1,2,\ldots,n-1\}\,|T]\right)\right)^2,
\end{aligned}$$

where the summation runs over all subsets T of $\{1,2,\ldots,m\}$ with $|T| = n-1$. Thus, we need to show that $\det\left(Q[\{1,2,\ldots,n-1\}\,|T]\right) = \pm 1$, whenever the sub-graph Y of X, consisting of edges that correspond only to the elements of T, forms a tree (a spanning tree).

But this holds true as has been shown in Lemma 4.4.1. Thus, the required result follows. ■

We end this section with a result of Cayley which gives the number of labeled trees.

Corollary 4.4.4 (Cayley 1897)**.** *The number of labeled trees on n vertices equals n^{n-2}.*

Proof. Note that the collection of all labeled trees on n vertices basically corresponds to the spanning trees of the complete graph K_n. So, let us apply the matrix tree theorem (Theorem 4.4.3) to the graph K_n. In this case, the Laplacian matrix of K_n equals

$$L(K_n) = L = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix}$$

and it can be easily verified that $\det\big(L(1|1)\big) = n^{n-2}$. ■

For more results on topics that relate matrices with graphs, the readers are advised to see the excellent books by Bapat [1] and Cvetokiv *et al* [3].