Stability of multi-step Method

Linear-multistep method:

$$\rho(E)u_{j-k+1} - h\,\sigma(E)\,u'_{j-k+1} = 0$$

Characteristic equation:

$$\rho(\xi) - \bar{h}\,\sigma(\xi) = 0 \quad , \quad \bar{h} = \lambda h \; , \quad \text{roots } \xi_{ih} \,, \; i = 1,2 \cdots k$$

Reduced characteristic equation:

$$\rho(\xi) = 0 \qquad \text{roots } \xi_i \,, \; i = 1, 2, \cdots k.$$

Growth parameter:

$$k_i \simeq \frac{\sigma(\xi_i)}{\xi_i\,\rho'(\xi)} \; ; \quad i = 1, 2, \cdots k.$$

$$\xi_{ih} = \xi_i\left(1 + \bar{h}k_i + O(|\bar{h}|^2)\right) \quad i = 1, 2, \cdots k.$$

Error equation

$$\varepsilon_j = c_1\xi_{1h}^j + c_2\xi_{2h}^j + \cdots + c_k\xi_{kh}^j + \frac{T}{\bar{h}\,\rho'(1)}$$

Solution of test problem using l.m.s method:

$$u_j = C_1\xi_{1h}^j + C_2\xi_{2h}^j + \cdots + C_k\xi_{kh}^j$$

Exact solution of the test problem $y' = \lambda y$:

$$y = c\,e^{\lambda t}.$$

Some more observations:

Not $$\xi_{ih}^{j} = \xi_i^{j}\left[1+\bar{h}K_i + O(|\bar{h}|^2)\right]^{j}$$
$$\simeq \xi_i^{j}\, e^{\bar{h}K_i j} \qquad i = 1,\dots,k.$$

For a consistent method, $\xi_1 = 1$, $K_1 = 1$, then

$$\xi_{1h}^{j} \simeq e^{j\bar{h}}$$

Here the root ξ_{1h} approximate the solution of the diff. equation $y' = \lambda y$. This root is called principal root and the remaining $(k-1)$ roots are called the extraneous roots. Therefore, for a convergent method it is essential that the principal root is dominant. This leads to a definition of relative stability of a multistep method.

Definitions: The multistep method is said to be

- Stable if $|\xi_i| < 1$, $i \neq 1$.
- Unstable if $|\xi_i| > 1$ for some i or there is a multiple root of $\rho(\xi) = 0$ of magnitude unity.
- weakly stable or conditionally stable: if ξ_i's are simple and if more than one of these roots have modulus unity.
- absolutely stable: if $\exists\ h_0 > 0$ such that $|\xi_{ih}| < 1$, $i = 1,2,\dots k$ $\forall\, h \leq h_0$.
- A-stable: if the interval of absolutely stability is $(-\infty, 0)$
- relative stability if $|\xi_{ih}| < |\xi_{1h}|$, $i = 2,3,\dots k$.

The region of * stability is defined to be the set of points in the λh-plane for which the method is ~~relatively~~ * stable.

Ex: Find the interval of absolute stability for the third order Adams Moulton method

$$u_{j+1} = u_j + \frac{h}{12}\left(5\, u'_{j+1} + 8\, u'_j - u'_{j-1}\right)$$

Sol: Applying the method on the test equation

$y' = \lambda y$, $\lambda < 0$, we obtain

$$u_{j+1} = u_j + \frac{h}{12}\left[5\lambda u_{j+1} + 8\lambda u_j - \lambda u_{j+1}\right]$$

$$\Rightarrow \left[1 - \frac{5\lambda h}{12}\right] u_{j+1} - \left[1 + \frac{8\lambda h}{12}\right] u_j + \frac{\lambda h}{12} u_{j-1} = 0$$

The characteristic equation:

$$\left(1 - \frac{5\lambda h}{12}\right)\xi^2 - \left(1 + \frac{8}{12}\lambda h\right)\xi + \frac{\lambda h}{12} = 0$$

Substituting $\xi = \frac{1+z}{1-z}$ and simplifying

$$\left(1 - \frac{5\lambda h}{12}\right)\left(\frac{1+z}{1-z}\right)^2 - \left(1 + \frac{8}{12}\lambda h\right)\left(\frac{1+z}{1-z}\right) + \frac{\lambda h}{12} = 0$$

$\vdots$

$$\nu_0 z^2 + 2z\nu_1 + \nu_2 = 0$$

where $\nu_0 = 1 - \frac{5\lambda h}{12} + 1 + \frac{8}{12}\lambda h + \frac{\lambda h}{12} = 2 + \frac{\lambda h}{3}$

$\nu_1 = 2 - \lambda h \qquad \nu_2 = -\lambda h$

Using Routh-Hurwitz Criterion:

$\nu_0 > 0 \Rightarrow \lambda h > -6$

$\nu_1 > 0 \Rightarrow \lambda h < 2$

$\nu_2 > 0 \Rightarrow \lambda h < 0$

Interval of absolute stability $\lambda h \in (-6, 0)$

Example: Discuss the relative and absolute stability of the second order Adams-Bashforth method

$$u_{j+1} = u_j + \frac{h}{2}(3u'_j - u'_{j-1})$$

Sol: Applying the given multistep method to the test equation $y' = \lambda y$:

$$u_{j+1} = u_j + \frac{h}{2}\left(3\lambda u_j - \lambda u_{j-1}\right)$$

$$\Rightarrow \quad u_{j+1} - \left(1 + \frac{3}{2}\lambda h\right)u_j + \frac{\lambda h}{2}u_{j-1} = 0 \quad \text{——} (1)$$

Characteristic equation

$$\xi^2 - \left(1 + \frac{3}{2}\bar{h}\right)\xi + \frac{\bar{h}}{2} = 0$$

Its roots:

$$\xi = \frac{\left(1 + \frac{3}{2}\bar{h}\right) \pm \sqrt{\left(1+\frac{3}{2}\bar{h}\right)^2 - 2\bar{h}}}{2}$$

$$= \frac{1}{4}\left[(2+3\bar{h}) \pm \sqrt{4 + 9\bar{h}^2 + 12\bar{h} - 8\bar{h}}\right]$$

$$= \frac{1}{4}\left[(2+3\bar{h}) \pm \sqrt{4 + 4\bar{h} + 9\bar{h}^2}\right]$$

$$= \frac{1}{4}\left[2 + 3\bar{h} \pm 2\left(1 + \bar{h} + \frac{9}{4}\bar{h}^2\right)^{1/2}\right]$$

$$= \frac{1}{4}\left[2 + 3\bar{h} \pm 2\left(1 + \frac{\bar{h}}{2} + \frac{9}{8}\bar{h}^2 + \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\bar{h}^2 + \cdots\right)\right]$$

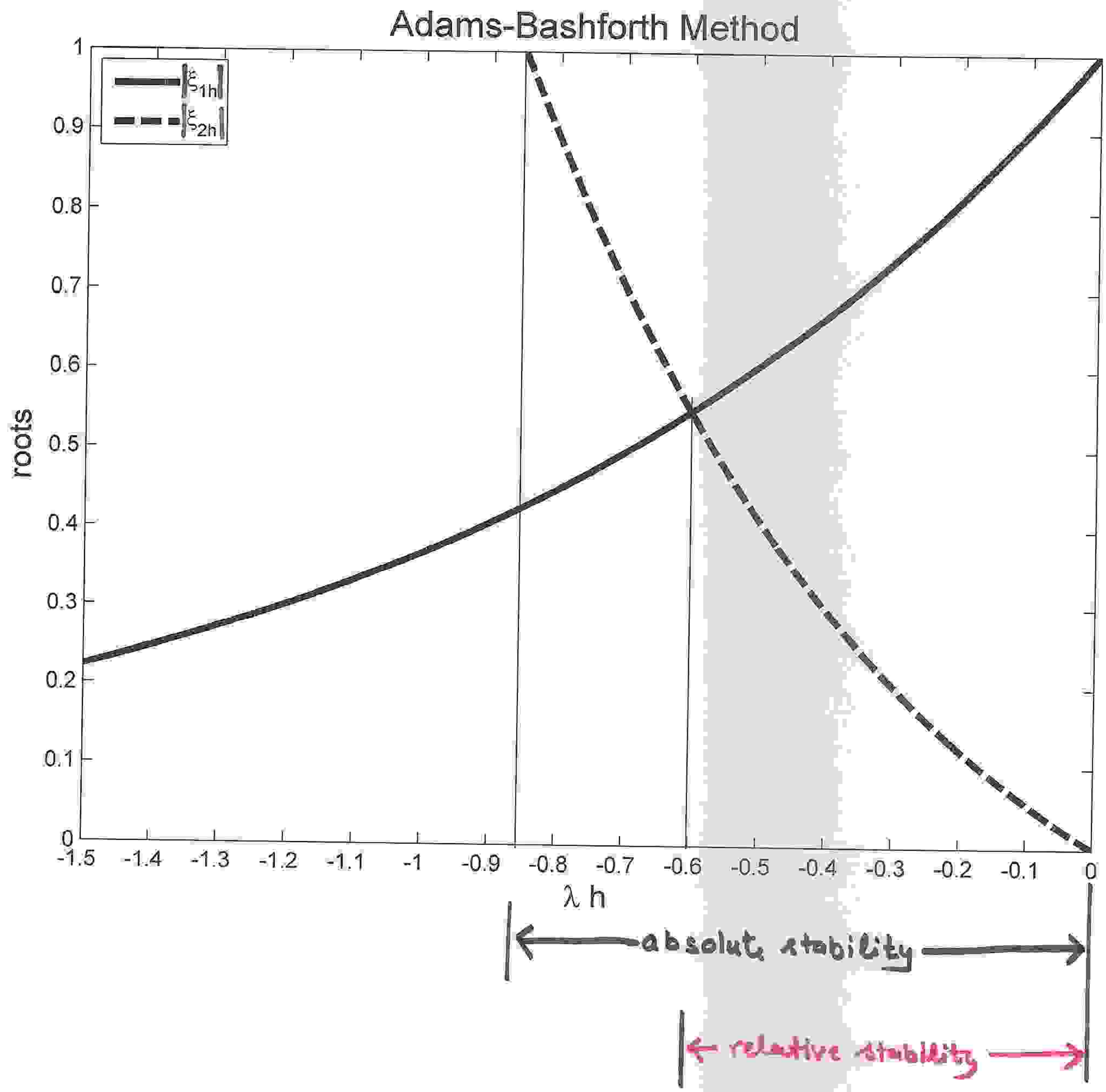
$$= \frac{1}{4}\left[2 + 3\bar{h} \pm 2\left(1 + \frac{\bar{h}}{2} + \bar{h}^2 \cdots\right)\right]$$

$$\xi_{1h} = \frac{1}{4}\left[2 + 3\bar{h} + 2 + \bar{h} + 2\bar{h}^2 + \cdots\right] = \frac{1}{4}\left[4 + 4\bar{h} + 2\bar{h}^2\right]$$

$$= 1 + \bar{h} + \bar{h}^2/2 + \cdots \approxeq e^{\bar{h}}$$

$$\xi_{2h} = \frac{1}{4}\left[2 + 3\bar{h} - 2 - \bar{h} - 2\bar{h}^2 \cdots\right] = \frac{\bar{h}}{2} - \frac{\bar{h}^2}{2}$$

$$= \frac{1}{2}\bar{h}e^{-\bar{h}}$$


Adams-Bashforth Method
$|\xi_{1h}|$
$|\xi_{2h}|$
roots
λ h
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-1.5
-1.4
-1.3
-1.2
-1.1
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
absolute stability
relative stability

Stability of Milne-Simpson method: (absolute stability)

$$u_{j+1} = u_{j-1} + \frac{h}{3}\left[u'_{j+1} + 4u'_j + u'_{j-1}\right]$$

applying the method to the test problem $y' = \lambda y$:

$$\left[1 - \frac{\lambda h}{3}\right] u_{j+1} - 4\frac{\lambda h}{3} u_j - \left(1 + \frac{\lambda h}{3}\right) u_{j-1} = 0$$

or $$\left[1 - \frac{\bar{h}}{3}\right] u_{j+1} - \frac{4}{3}\bar{h}\, u_j - \left(1 + \frac{\bar{h}}{3}\right) u_{j-1} = 0$$

Characteristic equation

$$\left(1 - \frac{\bar{h}}{3}\right)\xi^2 - \frac{4}{3}\bar{h}\,\xi - \left(1 + \frac{\bar{h}}{3}\right) = 0$$

subst. $\xi = \frac{1+z}{1-z}$:

$$\left(1 - \frac{\bar{h}}{3}\right)(1 + z^2 + 2z) - \frac{4}{3}\bar{h}\,(1 - z^2) - \left(1 + \frac{\bar{h}}{3}\right)(1 - 2z + z^2) = 0$$

$$\Rightarrow \quad \frac{2}{3}\bar{h} z^2 + 4z - 2\bar{h} = 0$$

or $$\frac{\bar{h}}{3} z^2 + 2z - \bar{h} = 0$$

Ac. to the Routh Hurwitz criterion:

$$\bar{h} > 0 \qquad 2 > 0 \qquad -\bar{h} > 0$$

never satisfied.

The method is nowhere absolute stable. $\xi^2 - 1 = 0$

However, the reduced characteristic equation gives $\xi = \pm 1$ and hence the method is weakly stable.