Revised Simplex Method

Original simplex method calculates and stores all numbers in the tableau – many are not needed.

→ Revised Simplex Method (more efficient for computing)

Used in all commercially available packages. (e.g. IBM MPSX, CDC APEX III)

$$
\begin{array}{ll}
\textbf{Max} & Z = \underline{c}\,\underline{x} \\
\textbf{subject to} & \underline{A}\,\underline{x} \le \underline{b} \\
 & \underline{x} \ge \underline{0}
\end{array}
$$

Initially constraints become (standard form):

$$
\begin{bmatrix} \underline{A} & \underline{I} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{x}_s \end{bmatrix} = \begin{bmatrix} \underline{b} \\ \ \end{bmatrix}
$$

$\underline{x}_s$ = **slack variables**

Basis matrix: columns relating to basic variables.

$$\underline{B} = \begin{pmatrix} B_{11} & \ldots & \ldots & \ldots & B_{1m} \\ \vdots & \cdot & \cdot & \cdot & \vdots \\ \vdots & \cdot & \cdot & \cdot & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ B_{m1} & \ldots & \ldots & \ldots & B_{mm} \end{pmatrix}$$

(Initially $\underline{B} = \underline{I}$)

Basic variable values: $\underline{x}_B = \begin{pmatrix} x_{B1} \\ \vdots \\ \vdots \\ \vdots \\ x_{Bm} \end{pmatrix}$

At any iteration non-basic variables = 0

$$\underline{B}\,\underline{x}_B = \underline{b}$$

Therefore $\underline{x}_B = \underline{B}^{-1}\underline{b}$ $\quad \underline{B}^{-1} \rightarrow$ **inverse matrix.**

At any iteration, given the original $\underline{b}$ vector and the inverse matrix, $\underline{x}_B$ (current R.H.S.) can be calculated.

$Z = \underline{c}_B\underline{x}_B$ where $\underline{c}_B =$ objective coefficients of basic variables

Step in the Revised Simplex Method

1. **Determine entering variable, X_j, with associated vector $\underline{P}_j$.**

- compute $\underline{Y} = \underline{c}_B \underline{B}^{-1}$
- compute $z_j - c_j = \underline{Y}\,\underline{P}_j - c_j$ for all non-basic variables.
Choose largest negative value (maximisation).
If none, stop.

2. **Determine leaving variable, X_r, with associated vector $\underline{P}_r$.**

- compute $\underline{x}_B = \underline{B}^{-1}\underline{b}$ (current R.H.S.)
- compute current constraint coefficients of entering variable:

$$\underline{\alpha}^j = \underline{B}^{-1}\underline{P}_j$$

X_r is associated with

$$\theta = \min_k \left\{ (\underline{x}_B)_k / \alpha^j_k \,,\ \alpha^j_k > 0 \right\}$$

(minimum ratio rule)

3. **Determine next basis i.e. calculate $\underline{B}^{-1}$**

Go to step 1.

Example:

$$\text{Max } Z = 3X_1 + 5X_2$$

$$\begin{aligned} \text{s.t.} \quad X_1 \qquad & \le 4 \\ 2X_2 & \le 12 \\ 3X_1 + 2X_2 & \le 18 \\ X_1,\ X_2 & \ge 0 \end{aligned}$$

Standard form of constraints:-

$$\begin{aligned} X_1 \qquad + S_1 \qquad\qquad & = 4 \\ 2X_2 \quad + S_2 \qquad & = 12 \\ 3X_1 + 2X_2 \qquad\qquad + S_3 & = 18 \\ X_1,\ X_2,\ S_1, S_2, S_3 & \ge 0 \end{aligned}$$

$$\underline{x}_B = \underline{B}^{-1}\underline{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix}$$

$$\underline{c}_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$Z = 0 \quad 0 \quad 0 \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} = 0$$

First iteration

Step 1

Determine entering variable, X_j, with associated vector $\underline{P_j}$.

- compute $\underline{Y} = \underline{c_B B^{-1}}$
- compute $z_j - c_j = \underline{Y}\,\underline{P_j} - c_j$ for all non-basic variables.

$$\underline{Y} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$z_1 - c_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - 3 = -3$$

and similarly for $z_2 - c_2 = -5$

Therefore X_2 is entering variable.

Step 2

Determine leaving variable, X_r, with associated vector $\underline{P}_r$.

- **compute $\underline{x}_B = \underline{B}^{-1}\underline{b}$ (current R.H.S.)**
- **compute current constraint coefficients of entering variable:**

$$\underline{\alpha}^j = \underline{B}^{-1}\underline{P}_j$$

X_r is associated with

$$\theta = \operatorname*{Min}_{k} \left\{ (\underline{x}_B)_k / \alpha^j_k ,\ \alpha^j_k > 0 \right\}$$

$$\underline{x}_B = \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} \qquad \underline{\alpha}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

$$\theta = \text{Min} \{ -,\ 12/2,\ 18/2 \}$$
$$= 12/2$$

therefore S_2 leaves the basis.

Step 3

Determine new $\underline{B}^{-1}$

$$\underline{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} \qquad \underline{B}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Solution after one iteration:

$$\underline{x}_B = \underline{B}^{-1}\underline{b}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix}$$

Go to step 1

Step 1 (second iteration)

Compute $\underline{Y} = \underline{c}_B\underline{B}^{-1}$

$$\underline{Y} = \begin{bmatrix} 0 & 5 & 0 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 5/2 & 0 \end{bmatrix}$$

- compute $z_j - c_j = \underline{Y}\,\underline{P_j} - c_j$ for all non-basic variables (X_1 and S_2):-

$$X_1:\ z_1 - c_1 = \begin{bmatrix} 0 & 5/2 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - 3 = -3$$

$$S_2:\ z_4 - c_4 = \begin{bmatrix} 0 & 5/2 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 0 = 5/2$$

Therefore X_1 enters the basis.

<u>Step 2</u>

Determine leaving variable.

$$\underline{x}_B = \begin{pmatrix} 4 \\ 6 \\ 6 \end{pmatrix} \qquad \underline{\alpha}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & ½ & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\theta = \text{Min}\ \{ 4/1 ,\ - ,\ 6/3 \}$$
$$= 6/3$$

therefore S_3 leaves the basis.

Step 3

Determine new $\underline{B}^{-1}$

$$\underline{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 3 \end{pmatrix} \qquad \underline{B}^{-1} = \begin{pmatrix} 1 & 1/3 & -1/3 \\ 0 & 1/2 & 0 \\ 0 & -1/3 & 1/3 \end{pmatrix}$$

Solution after two iterations:

$$\underline{x}_B = \underline{B}^{-1}\underline{b}$$

$$= \begin{pmatrix} 1 & 1/3 & -1/3 \\ 0 & 1/2 & 0 \\ 0 & -1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 4 \\ 12 \\ 18 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$$

Go to step 1

Step 1

- compute $\underline{Y} = \underline{c_B}\underline{B}^{-1}$

$$\underline{Y} = \begin{bmatrix} 0 & 5 & 3 \end{bmatrix} \underline{B}^{-1} = \begin{bmatrix} 0 & 3/2 & 1 \end{bmatrix}$$

- compute $z_j - c_j = \underline{Y}\,\underline{P_j} - c_j$ for all non-basic variables (S_2 and S_3):-

$$S_2\text{:}\ z_4 - c_4 = \begin{bmatrix} 0 & 3/2 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 0 = 3/2$$

$$S_3\text{:}\ z_5 - c_5 = \begin{bmatrix} 0 & 3/2 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - 0 = 1$$

No negatives. Therefore stop.

Optimal solution:

$S_1^* = 2$
$X_2^* = 6$
$X_1^* = 2$
$$Z^* = \underline{c_B}\,\underline{x_B} = \begin{bmatrix} 0 & 5 & 3 \end{bmatrix} \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = 36$$