Indian Institute of Technology Kharagpur
End Autumn Semester Examination 2015
Department of Mathematics
Sub No: MA 40001/41007 Sub Name: Functional Analysis

Time: Three hours. Full Marks: 50

Answer all questions, the questions are of equal values

1(a). Show that if the closed unit ball $M = \{x : ||x|| \leq 1\}$ is compact in a norm linear space X, then X is finite dimensional.

1(b). Show that l^p is Hilbert space iff $p = 2$.

2(a). Let X be an inner product space and M is a non-empty convex subset which is complete (in the metric induced by the inner product). Show that for every given $x \in X$, $\exists$ a unique $y \in M$ such that

$$\delta = \inf_{z \in Y} ||x - z|| = ||x - y||$$

2(b). Show that an orthonormal set in an inner product space is linearly independent.

3(a). State and prove the Bessel's inequality.

3(b). Show that in an inner product space X, $x \perp y$ iff we have $||x + \alpha y|| = ||x - \alpha y||$ for all scalars α and $x, y \in X$.

4(a). State and prove the Riesz representation theorem in Hilbert spaces.

4(b). If z is any fixed element of an inner product space X, show that $f(x) =< x, z >$ defines a bounded linear functional f on X, of norm $||z||$.

5(a). Let Y be any closed subspace of a Hilbert space H. Then show that $H = Y \oplus Y^{\perp}$.

5(b). Show that the projection map $P : H \mapsto Y$ is an idempotent bounded linear map. Here (Y is a subspace of a Hilbert space H).

6. Using Zabreiko's lemma state and prove the open mapping theorem.

7. State and prove the closed graph theorem.

8. State and prove the Hahn-Banach theorem in Hilbert spaces.

9. If (x_n) in a Banach space X is such that $(f(x_n))$ is bounded for all $f \in X'$, then show that $(||x_n||)$ is bounded.

10. Let $T : D(T) \mapsto Y$ be a linear operator, where $D(T) \subset X$ and X, Y are norm linear spaces. Then show that if T is continuous at a single point, then it is continuous every where.

THE END