MA 51002: Measure Theory and Integration
Mid Semester Examination (Spring 2016)
Time: 2 Hours, Full Marks: 30, Number of students = 85.

Answer all the eight problems. Numbers at the right hand side after each question denote marks. No clarification will be entertained during the examination.

(1) For two non-empty sets of real numbers A and B, define $A + B$ to be the set of all sums $a + b$ where $a \in A$ and $b \in B$. Show that if A is open, then $A + B$ is open. Show that if A and B are compact, then $A + B$ is compact. Prove of disprove whether A and B are closed implies $A + B$ is closed or not. [2+2+1]

(2) Let $f = p + g$, where p is a polynomial of odd degree and g is a bounded continuous function on the real line. Show that there is at least one solution to $f(x) = 0$. [3]

(3) Show that : $\chi_{A \cup B}(x) = \max(\chi_A(x), \chi_B(x)) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$ where χ_A denotes the characteristic function of the set A. [3]

(4) Prove that if E_1 is measurable and E_2 differs from E_1 by a set of measure zero, then E_2 is also measurable. [3]

(5) Prove that for every measurable set $E \subset (0, 1)$ there exists a Borel set $B \in \mathcal{B}$ such that $m(E \Delta B) = 0$. [4]

(6) Define a metric space $(\mathcal{X}, d)$ of equivalence classes of measurable sets $E \subset (0, 1)$ with $d(E, F) = m(E \Delta F)$. Show that the Lebesgue measure m is continuous with respect to this metric, i.e. if $d(E_n, E) \to 0$, then $m(E_n) \to m(E)$. [4]

(7) Prove that there exists a non-measurable set. [4]

(8) Prove that every Borel set is measurable. How about the converse? Explain. [2+2]