PREDICTOR - CORRECTOR METHODS:

If a predictor method (explicit method) is used to predict a value of $u_{n+1}^{(0)}$ and this value is taken as the starting approximation of the iteration for obtaining u_{n+1} using the corrector method (Implicit method), such combination of methods are called Predictor-corrector methods.

Suppose we want to use the implicit method

$$u_{j+1} = h b_0 f_{j+1} + \underbrace{\sum_{i=1}^{k} \left(a_i u_{j-i+1} + h b_i f_{j-i+1} \right)}_{=: C}$$

OR

$$u_{j+1} = h b_0 f_{j+1} + C$$

We first use the explicit (predictor) method for predicting $u_{j+1}^{(0)}$ and then use the implicit (corrector) method iteratively until the convergence is obtained.

P: Predict some value $u_{j+1}^{(0)}$

E: Evaluate $f(x_{j+1}, u_{j+1}^{(0)})$

C: Correct $u_{j+1}^{(1)} = h b_0 f(x_{j+1}, u_{j+1}^{(0)}) + C$

E: Evaluate $f(x_{j+1}, u_{j+1}^{(1)})$

C: Correct $u_{j+1}^{(2)} = h b_0 f(x_{j+1}, u_{j+1}^{(1)}) + C$

The sequence of operations PECECE....CE is denoted by $P(EC)^m E$ and is called a predictor corrector method.

Examples:

Modified Euler Method

$$P: \quad u_{j+1} = u_j + h f_j \qquad \text{(Euler method)}$$

$$C: \quad u_{j+1} = u_j + \frac{h}{2}(f_j + f_{j+1}) \qquad \text{(Euler-Cauchy)}$$

The Adams-Bashforth-Moulton method

$$P: \quad u_{j+1} = u_j + \frac{h}{2}(3u'_j - u'_{j-1})$$

$$C: \quad u_{j+1} = u_j + \frac{h}{12}(5u'_{j+1} + 8u'_j - u'_{j-1})$$

Example: Solve the IVP

$$\frac{dy}{dx} = x + y, \qquad y = 1 \text{ when } x = 0$$

with $h = 0.1$ on the interval $[0, 0.2]$ using the P-C method:

$$P: \quad u_{j+1} = u_j + h f_j$$

$$C: \quad u_{j+1} = u_j + \frac{h}{2}(f_j + f_{j+1})$$

as $P(EC)^2E$.

Sol: $P: \quad u_1^{(0)} = u_0 + h f_0$

$$= 1 + 0.1 \times (0 + 1) = 1.1$$

$E. \quad f(x_1, u_1^{(0)}) = (0.1 + 1.1) = 1.2$

$C: \quad u_1^{(1)} = u_0 + \frac{h}{2}(f_0 + f_1)$

$$u_1^{(1)} = 1 + \frac{0.1}{2}(1 + 1.2) = \underline{1.11.}$$

E: $f(x_1, u_1^{(1)}) = 0.1 + 1.11 = 1.21$

C: $u_1^{(2)} = u_0 + \frac{h}{2}\left(f(x_1, \underline{u_1^{(1)}}) + f(x_0, u_0)\right)$

$$= 1 + \frac{0.1}{2}(1.21 + 1) = 1.1105$$

$$u_2^{(0)} = u_1 + h f_1 = 1.1105 + 0.1(0.1 + 1.1105)$$

$$= 1.2316$$

$$u_2^{(1)} = u_1 + \frac{h}{2}\left(f(x_2, u_2^{(0)}) + f(x_1, u_1)\right)$$

$$= 1.1105 + \frac{0.1}{2}\left((0.2 + 1.2316) + (0.1 + 1.1105)\right)$$

$$= 1.2426$$

$$u_2^{(2)} = 1.1105 + \frac{0.1}{2}\left((0.2 + 1.2426) + 1.2105\right)$$

$$= 1.2432.$$

Hence,

$$y(0.1) \approx 1.1105$$

$$y(0.2) \approx 1.2432.$$

Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

Find $y(0.1)$, $y(0.2)$, $y(0.3)$ using a sixth order Taylor's series method.

Using these values, calculate $y(0.4)$ using the following P-C set

$$P: \quad u_{n+4} = u_n + \frac{4}{3}h[2f_{n+1} - f_{n+2} + 2f_{n+3}]$$

$$C: \quad u_{n+4} = u_{n+2} + \frac{h}{3}[f_{n+2} + 4f_{n+3} + f_{n+4}]$$

Sol: Note that the Taylor's formula is given by

$$y(x) \simeq y(0) + xy'(0) + \frac{x^2}{\lfloor 2}y''(0) + \frac{x^3}{\lfloor 3}y'''(0) + \frac{x^4}{\lfloor 4}y^{(iv)}(0) + \frac{x^5}{\lfloor 5}y^{(v)}(0) + \frac{x^6}{\lfloor 6}y^{(vi)}(0)$$

Given: $y(0) = 1 \quad y'(0) = 0$

From the differential equation:

$$y'' = -xy' - y \Rightarrow y''(0) = -1$$

Diff. the given DE:

$$y''' = -xy'' - 2y' \Rightarrow y'''(0) = 0$$

$$y^{(iv)} = -xy^{(iii)} - 3y'' \Rightarrow y^{(iv)}(0) = 3$$

$$y^{(v)} = -xy^{(iv)} - 4y''' \Rightarrow y^{(v)}(0) = 0$$

$$y^{(vi)} = -xy^{(v)} - 5y^{(iv)} \Rightarrow y^{(vi)}(0) = -15$$

Hence,

$$y(x) \simeq y(0) + \frac{x^2}{\lfloor 2}(-1) + \frac{x^4}{\lfloor 4} \times 3 + \frac{x^6}{\lfloor 6} \cdot (-15)$$

$\Rightarrow \quad y(0.1) \simeq 0.995$

$y(0.2) \simeq 0.9802$

$y(0.3) \simeq 0.956$

However, in order to compute further values using P-C method we need to transfer higher order into a system of first order DEs.

set $y' = z \Rightarrow z' = -(xz + y)$

$$\Rightarrow \begin{bmatrix} y \\ z \end{bmatrix}' = \begin{bmatrix} z \\ -(xz+y) \end{bmatrix} =: \begin{bmatrix} f \\ g \end{bmatrix} \qquad \begin{matrix} y(0) = 1 \\ z(0) = 0 \end{matrix}$$

Once again using Taylor's series:

$$z(x) = y'(x) = -x + \frac{x^3}{2} - \frac{1}{4} x^5$$

$$z(0.1) = -0.0995$$

$$z(0.2) = -0.1960$$

$$z(0.3) = -0.2863$$

$\underline{P}$: $$\begin{bmatrix} y(0.4) \\ z(0.4) \end{bmatrix} = \begin{bmatrix} y(0) \\ z(0) \end{bmatrix} + \frac{4}{3} h \begin{bmatrix} 2f_1 - f_2 + 2f_3 \\ 2g_1 - g_2 + 2g_3 \end{bmatrix}$$

where $h = 0.1$ $\quad f_i = f(x_i, y_i, z_i)$

& $g_i = g(x_i, y_i, z_i)$

$$\Rightarrow \begin{bmatrix} y(0.4) \\ z(0.4) \end{bmatrix} = \begin{bmatrix} 0.9231 \\ -0.3692 \end{bmatrix}.$$

$\underline{C}$: $$\begin{bmatrix} y(0.4) \\ z(0.4) \end{bmatrix} = \begin{bmatrix} y(0.2) \\ z(0.2) \end{bmatrix} + \frac{h}{3} \begin{bmatrix} f_2 + 4f_3 + f_4 \\ g_2 + 4g_3 + g_4 \end{bmatrix}$$

f_4 & g_4 can be evaluated using $y(0.4), z(0.4)$ form ~~p~~ predictor formula.

$$\Rightarrow \begin{bmatrix} y(0.4) \\ z(0.4) \end{bmatrix} = \begin{bmatrix} 0.9232 \\ -0.3692 \end{bmatrix}$$

One can improve these values by repeating corrector formula.