Tutorial Problems set-II

Note: All these problems can be solved using the updated slides.

[**0.0.1**] ***Exercise*** Find a necessary and sufficient condition for $\langle x, y\rangle = \sum_{i=1}^{n} \alpha_i x_i y_i$ to be an inner product on $\mathbb{R}^n$.

[**0.0.2**] ***Exercise*** Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2×2 matrix with real entries. Let $f_A : \mathbb{R}^2 \to \mathbb{R}$ be a map defined by $f_A(x, y) = y^t Ax$, where $x, y \in \mathbb{R}^2$. Show that f_A is an inner product on $\mathbb{R}^2$ if and only if $A = A^t$, $a_{11} > 0$, $a_{22} > 0$ and $det(A) > 0$.

[**0.0.3**] ***Exercise*** Let $\mathbb{V}$ be a finite-dimensional vector space and let $B = \{u_1, \ldots, u_n\}$ be a basis for $\mathbb{V}$. Let $\langle x, y\rangle$ be an inner product on $\mathbb{V}$. If $c_1, \ldots, c_n$ are any n scalars, show that there is exactly one vector x in $\mathbb{V}$ such that $\langle x, u_1\rangle = c_i$ for $i = 1, \ldots, n$.

[**0.0.4**] ***Exercise*** Let $(\mathbb{V}, \langle,\rangle)$ be an inner product space. Show that $\langle x, y\rangle = 0$ for all $y \in \mathbb{V}$, then $x = 0$.

[**0.0.5**] ***Exercise*** Show that $\langle x, y\rangle = \sum_{i=1}^{n} \overline{x_i} y_i$ is not an inner product on $\mathbb{C}^n$.

[**0.0.6**] ***Exercise*** Let $(\mathbb{V}, \langle,\rangle)$ be a finite inner product space. Prove that for $v \in \mathbb{V} - \{0\}$, the set $W = \{w \in \mathbb{V} : \langle w, v\rangle = 0\}$ is a subspace of $\mathbb{V}$ of dimension $\text{DIM}\,\mathbb{V} - 1$.

[**0.0.7**] ***Exercise*** Decide which of the following functions define an inner product $\mathbb{C}^2$. For $x = (x_1, y_1)$, $y = (y_1, y_2)$.

1. $\langle x, y\rangle = x_1\overline{y_2}$
2. $\langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2}$
3. $\langle x, y\rangle = x_1 y_1 + x_2 y_2$
4. $\langle x, y\rangle = 2x_1\overline{y_1} + i(x_2\overline{y_1} - x_1\overline{y_2}) + 2x_2\overline{y_2}$