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Lecture 4

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Th-7 (Laplace Transform of the derivative of any f^n of order n)
H.W

Let $f(t)$ & its derivatives $f'(t), f''(t), \ldots, f^{(n-1)}(t)$ be continuous f^ns for all $t \geq 0$, satisfying $|f(t)| \leq Me^{kt}$ for some K & M, & let the derivative $f^n(t)$ be piece-wise continuous on every finite interval in the range $t \geq 0$. Then the

Then the Laplace transform of $f^n(t)$ exists when $s>k$ & is given by

$$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

Ex1/ Let $f(t) = t^2$. Derive $\mathcal{L}(f)$ from $\mathcal{L}(1)$.

Sol^n:- Here, $f(t) = t^2$, $f(0) = 0$

$f'(t) = 2t$, $f'(0) = 0$

$f''(t) = 2$,

We obtain from the formula

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s\underset{(=0)}{f(0)} - \underset{(=0)}{f'(0)}$$

$$\Rightarrow \mathcal{L}(2) = s^2 \mathcal{L}(f)$$

Also, $\mathcal{L}(2) = 2\mathcal{L}(1)$. $= 2/s$

$$\Rightarrow 2/s = s^2 \mathcal{L}(f)$$

$$\Rightarrow \mathcal{L}(f) = \mathcal{L}(t^2) = 2/s^3$$

C.W

Ex 2/ Derive the L.T of $\cos \omega t$

Sol^n:- Let $f(t) = \cos \omega t$.

Use $\mathcal{L}(f'') = s^2\mathcal{L}(f) - s f(0) - f'(0)$

$$\Rightarrow \mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$$

$\mathcal{L}(\sin \omega t) = ? = \omega/(s^2+\omega^2)$

H.W

EX3/ Let $f(t) = \sin^2 t$.

Find $\mathcal{L}(f) = \frac{2}{s(s^2+4)}$

EX4/ Let $f(t) = t \sin \omega t$.

Find $\mathcal{L}(f) = \frac{2\omega s}{(s^2+\omega^2)^2}$

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§/ Laplace transform of the integral of a function

Th-8/ [Integration of $f(t)$]

Let $F(s)$ be the Laplace transform of $f(t)$. If $f(t)$ is piece-wise continuous & satisfies $|f(t)| \leq Me^{kt}$, then

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s} F(s)$$

$(s>0, s>k)$

$\rightarrow (1)$

or, if we take the inverse Transform on both sides of (1), we get

$$\int_0^t f(\tau)\, d\tau = \mathcal{L}^{-1}\left(\frac{F(s)}{s}\right)$$

proof :- Suppose that $f(t)$ is piece-wise continuous & satisfies $|f(t)| \le Me^{kt}$ $\rightarrow (2)$

for some k & M.

Clearly, if (2) holds for some negative k, it also holds for positive k, & we may assume that k is positive.

yyy

Then the integral

$$g(t) = \int_0^t f(\tau)\, d\tau$$

is continuous, & by using ② we obtain (EX)

$$|g(t)| = \left|\int_0^t f(\tau)\, d\tau\right|$$

$$\le \int_0^t |f(\tau)|\, d\tau$$

$$\le M \int_0^t e^{k\tau}\, d\tau \quad \text{(how?)}$$

$$= M \frac{(e^{kt}-1)}{k}$$

$$\le \frac{M}{k} e^{kt}.$$

EX

$e^{kt}-1 < e^{kt}$

how?

$\because e^{kt} > 0$

$\therefore e^{kt-1} < e^{kt}$

$\because k > 0,\ t > 0$

$kt > 0$

$\Rightarrow e^{kt} > 1$

$\Rightarrow e^{kt}-1 > 0$

This shows that $g(t)$ also satisfies an inequality of the form ②

Also, $g'(t) = f(t)$

[This is known as the Fundamental theorem of the Calculus],

except for points at which $f(t)$ is discontinuous.

Hence, $g'(t)$ is piece-wise continuous on each finite ~~int~~ interval & by $\boxed{\text{Th}-2}$

$$\mathcal{L}(f(t)) = \mathcal{L}\{g'(t)\}$$

$$= s\,\mathcal{L}\{g(t)\} - g(0) \quad (s>k)$$

Here, clearly $g(0) = 0$ (how?)

so that

$$\mathcal{L}(f) = s\,\mathcal{L}(g)$$

$$\Rightarrow \mathcal{L}(g) = \frac{\mathcal{L}(f)}{s}$$

$$\Rightarrow \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

$$\Rightarrow \int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\}$$

Ex/ Let $\mathcal{L}(f) = \dfrac{1}{s(s^2+\omega^2)}$.

Find $f(t)$.

Sol^n :- We know from the table,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \frac{\sin \omega t}{\omega}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s^2+\omega^2)}\right\} = \frac{1}{\omega}\int_0^t \sin \omega \tau . d\tau$$

$$= \frac{1}{\omega^2}\left(1 - \cos \omega t\right).$$

Again, $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{(s^2+\omega^2)}\right\}$ (how ??)

$$= \frac{1}{\omega^2}\int_0^t (1 - \cos \omega \tau)\, d\tau$$

$$= \frac{1}{\omega^2}\left(t - \frac{\sin \omega t}{\omega}\right) = \frac{1}{\omega^2}\left[\tau - \frac{\sin \omega \tau}{\omega}\right]_0^t$$

Th-9/ (Integration of Transforms)

If $\mathcal{L}\{f(t)\} = F(s)$,

then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du,$$

assuming that

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} \to 0 \text{ as } s \to \infty$$

or, $\underset{t\to 0^+}{Lt} \frac{f(t)}{t}$ exists.

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e.g., The f^n

$$Si(t) = \int_0^t \frac{\sin u}{u}\,du$$

defines the sine integral function $\xrightarrow{\text{occurs}}$ in the study of optics