Integration

Date 27/2/19

(Shakarchi & Stein
Real Analysis)

Recap:

Lebesgue measure ← outer measure + σ additivity

additivity ⟷ continuous

Qn) Given a bijection b/w the interval $[0,1]$ & the Cantor set. i.e. show they have the same cardinality.

Cantor set → Uncountable
→ Closed, Perfect
→ Bijection to $[0,1]$
→ Has measure 0.

Qn) Given an example of 2 sets A & B s.t. $m(A) = m(B) = 0$ but $m(A+B) \geq 0$.

$$\bigcup_{i,j} \left[(a_i + B) \cup (A + b_j) \right] \subseteq A+B$$

Riemann Integration: $\int_a^b f(x)\,dx$ f: bdd.

partition a,b as P.

$$m_i = \inf_{x \in (x_{i-1}, x_i)} f(x)$$

$$M_i = \sup_{x \in (x_{i-1}, x_i)} f(x)$$

$$U(P,f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

$$L(P,f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

Date __/__/__

$$U(f) = \overline{\int_a^b} f(x)\,dx = \inf_P U(P,f)$$

$$L(f) = \underline{\int_a^b} f(x)\,dx = \sup_P L(P,f)$$

$U(f) = L(f) \Rightarrow f$ is Riemann Integrable

Riemann Sum

$$S(P,f) = \sum_{i=1}^{n} f(x^*)(x_i - x_{i-1})$$

$$x_{i-1} < x^* < x_i$$

Riemann Integrable iff the set of discontinuities have lebesgue mea Zero.

eg) $f_n \to f$

f_n is integrable but

f is not integrable when?

eg) Integral of $f \neq$ integral of f_n.

Consider

$$f_n(x) = \begin{cases} 1, & x \in \{q_1, \dots q_n\} \\ 0, & \text{otherwise} \end{cases}$$

Riemann integrable

q_n is an enumeration of $\mathbb{Q}$.

$\downarrow n \to \infty$

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0 & x \in \mathbb{I} \end{cases}$$ Dirichlet fun.

Date __/__/__

$$f_n(x) = \begin{cases} n, & 0 < x < 1/n \\ 0, & 1/n \le x < 1 \end{cases}$$

$$\int_0^1 f_n(x)\,dx = 1, \quad \forall n$$

But $\quad f(x) = 0 \quad \forall x \in [0,1]$

$$\int_0^1 f(x)\,dx = 0$$

$$\int f \ne \int f_n$$

Overcome Issues.

Ex But if $f_n \to f$ uniformly $(f_n \rightrightarrows f)$ where $f_n \in R[a,b]$ i.e. Riemann Integrable

Prove $f \in R$ and $\int f_n \to \int f$

Ex $\sup |f_n(x) - f(x)| \to 0$ as $n \to \infty$ $\Leftrightarrow$ Uniform convergence

$\|f_n - f\|$

$$\left|\int f_n - \int f\right| \le \left|\int_a^b (f_n - f)\right| \le \int_a^b |f_n - f|$$

$$\le (b-a)\,\|f_n - f\| \to 0 \text{ as } n \to \infty$$

i.e. $\int f_n = \int f$

Date __/__/__

Fundamental Theorem of Integral Calculus.

If you have an integrable fn

(1) $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F'(x) = f(x)$

F exists if f is continuous,
i.e. $f \in C[a,b]$

Hint: use MVT. (mean value)

Show $U(f) = L(f)$
$\quad\quad\quad\quad \parallel$
$\quad\quad F(b) - F(a)$

(2) $F(x) = \int_a^x f(x)\,dx$

If f is Riemann Integrable
$\Rightarrow$ F is continuous.
If f is contin F is differ $F'(x) = f(x)$

Proof?

Example $\frac{d}{dx}\left[\int_{x^2}^{x} e^{-t^2}\,dt\right] = ?$

Given (2),

$F(x) = \int_0^x e^{-t^2}\,dt \qquad f'(x) = e^{-x^2}$

$\frac{d}{dx}\left[\int_{x^2}^{0} e^{-t^2}\,dt + \int_0^x e^{-t^2}\,dt\right]$

Date ___ / ___ / ___

$$= -e^{-x^4} 2x + e^{-x^2}$$

<u>Proof</u>

1: Consider a partition P of $[a,b]$

$a = x_0 \quad x_1 \quad \dots \quad x_{n-1} \quad b = x_n$

$[x_{k-1}, x_k]$

$$F(x_k) - F(x_{k-1}) = (x_k - x_{k-1}) F'(c)$$
$$= (x_k - x_{k-1}) f(c)$$
$$c \in [x_k, x_{k-1}]$$

$$\sum (x_k - x_{k-1}) m_k \le \sum (x_k - x_{k-1}) f(c)$$

$$\le \sum M_k (x_k - x_{k-1})$$

$$L(f) \le F(b) - F(a) \le U(f)$$

But f is integrable.

$$\Rightarrow L(f) = U(f) = F(b) - F(a)$$

2. $$|F(x+h) - F(x)| = \left| \int_a^{x+h} f(x) - \int_a^x f(x) \right|$$

$$= \int_x^{x+h} f(x)\, dx \xrightarrow{h \to 0} 0$$

$$\le M \int_x^{x+h} dx = hM \to 0$$

as $h \to 0$

Date __/__/__

$$\lim_{h\to 0} \frac{1}{h}\int_a^{a+h} f(x)\,dx - f(a)$$

$$= \lim_{h\to 0} \frac{1}{h}\int_a^{a+h} \big(f(x) - f(a)\big)\,dx$$

$\leq \varepsilon/_h \cdot h$ taking modulus and use continuity at a.

Choose h for ε.

Show

$$\lim_{h\to 0} \frac{F(c+h) - F(c)}{h} \text{ Exists}$$

28/2 ① Darboux integral $\leftarrow \int_a^b f(x)\,dx$

(ii) Riemann "

Fundamental thm of Integral Calc

(i) $f \in R[a,b]$, if $\exists$ a diff. function F s.t. $F'(x) = f(x)$, then
$\int_a^b f(x)\,dx = F(b) - F(a)$

(ii) $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

f is $R[a,b]$ then $F(x) = \int_a^x f(t)\,dt$

Moreover if f is $C[a,b]$
then $F(x) = \int_a^x f(t)\,dt$

diff & $F'(x) = f(x)$