Lecture-3

12/1/17

Vector space $(V)$

Subspace $(W)$ $\quad W = \{\underset{\sim}{w}_1, \underset{\sim}{w}_2, \ldots\}$

If $W \subset V$, & identity element of $V$ is $\underset{\sim}{0}_V$.

To check, 1) $\underset{\sim}{0}_V \in W$

2) $c_1\underset{\sim}{w}_1 + c_2\underset{\sim}{w}_2 \in W$.

Linear dependence and independence of vectors

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Let

$S = \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_n\}$, be a set of vectors

$S$ is linearly independent (l.i). or the vectors $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_n$ are l.i, if

$$c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 + \cdots + c_n\underset{\sim}{v}_n = \underset{\sim}{0} \longrightarrow (1)$$

holds when all $c_i = 0$; $i = 1, 2, \ldots, n$

$\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_n$ are said to be linearly dependent, when (1) holds for at least one non-zero $c_i$.

$$\left[\begin{aligned} & c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 \\ & 2(1,3) + (-2,-6) \\ &= (2,6) + (-2,-6) \\ &= (0,0) \end{aligned}\right]$$

**Ex.** Let $\underset{\sim}{v}_1 = (1, 2, 5)$, $\underset{\sim}{v}_2 = (0, 1, 3)$

Form, $c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 = \underset{\sim}{0}$

$\therefore c_1(1,2,5) + c_2(0,1,3) = (0,0,0) \longrightarrow (1)$

$(c_1, 2c_1, 5c_1) + (0, c_2, 3c_2) = (0,0,0)$

or, $(c_1, 2c_1 + c_2, 5c_1 + 3c_2) = (0,0,0)$

$c_1 = 0$, $2c_1 + c_2 = 0$, $5c_1 + 3c_2 = 0$

$2.0 + c_2 = 0 \Rightarrow c_1 = 0 = c_2$

$\boxed{\begin{aligned} &\therefore (1) \text{ holds when } c_1 = 0 \ \& \ c_2 = 0 \\ &\therefore \underset{\sim}{v}_1, \underset{\sim}{v}_2 \text{ are l.i} \end{aligned}}$

$$\begin{pmatrix} \underset{\sim}{v}_1 \\ \underset{\sim}{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \end{pmatrix}.$$

Rule: Form a matrix whose rows are given vectors. Bring the matrix in echelon form. In echelon form, if the no. of ~~vectors~~ non-zero rows is equal to no. of given vectors, then the vectors are linearly independent, otherwise they are linearly dependent (l.d.).

Ex. Check linear dependence / independence of the vectors (a) $\underset{\sim}{v}_1 = (1, 1)$, $\underset{\sim}{v}_2 = (5, 5)$

Note, $\underset{\sim}{v}_2 = 5\underset{\sim}{v}_1$

(b) $\underset{\sim}{v}_1 = (2, 3, -4)$, $\underset{\sim}{v}_2 = (-1, -2, 6)$, $\underset{\sim}{v}_3 = (3, 4, -2)$.

Note. $\underset{\sim}{v}_3 = 2\underset{\sim}{v}_1 + \underset{\sim}{v}_2$

Sol. (a) $c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 = \underset{\sim}{0}$.

$c_1(1, 1) + c_2(5, 5) = (0, 0) \longrightarrow (1)$

or, $(c_1 + 5c_2, c_1 + 5c_2) = (0, 0)$.

$$\left.\begin{aligned} c_1 + 5c_2 &= 0 \\ c_1 + 5c_2 &= 0 \end{aligned}\right\} \Rightarrow c_1 + 5c_2 = 0.$$

$c_2 = 1$, $c_1 = -5$

$c_2 = -1$, $c_1 = 5$

$\therefore$ (1) holds for infinitely many non-zero values of $c_1, c_2$.

$\therefore$ (1, 1) & (5, 5) are l.d.

(b). $c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 + c_3\underset{\sim}{v}_3 = \underset{\sim}{0}$

$c_1(2, 3, -4) + c_2(-1, -2, 6) + c_3(3, 4, -2) = (0, 0, 0)$

| | | |
|---|---|---|
| $2c_1 - c_2 + 3c_3 = 0$ | $2c_1 - c_2 + 3c_3 = 0$ | $2c_1 - c_2 + 3c_3 = 0$ |
| $3c_1 - 2c_2 + 4c_3 = 0$ | $c_2 + c_3 = 0$. | $c_2 + c_3 = 0$ |
| $-4c_1 + 6c_2 - 2c_3 = 0$ | $4c_2 + 4c_3 = 0$ | $0 = 0$ |

$c_3 \to$ free variable (for which you can assign arbitrary values)

$c_3 = c, \quad c_2 = -c_3 = -c.$

$2c_1 - c_2 + 3c_3 = 0.$

$2c_1 = c_2 - 3c_3 \Rightarrow c_1 = \frac{c_2}{2} - \frac{3c_3}{2}.$

$\therefore \; c_1 = -\frac{c}{2} - \frac{3c}{2} = -2c.$

$c_1 = 2c, \quad c_2 = -c, \quad c_3 = c.$

$2c_1 + 3c_3 + c_2 = 0.$

$c_3 + c_2 = 0.$ $\qquad c_2 \to$ free variable.

no. of free variables = no. of unknowns − no of equations in the reduced form.

$2c_1 - c_2 + 3c_3 = 0.$ $\qquad$ coeff. matrix

$3c_1 - 2c_2 + 4c_3 = 0.$

$-4c_1 + 6c_2 - 2c_3 = 0.$

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 3 & -2 & 4 \\ -4 & 6 & -2 \end{pmatrix}.$$

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 3 & -2 & 4 \\ -4 & 6 & -2 \end{vmatrix} = 0$$

A homogeneous system of $n$ equations in $n$ unknowns has <u>infinitely many</u> (only zero) solutions if and only if <u>det A = 0</u> (det A ≠ 0).

**Theorem 1.** In a set of vectors, say $S = \{v_1, v_2, \ldots, v_n\}$ if any of the vectors is a linear combination of the other vectors then the set of vectors is linearly dependent.

**Theorem 2.** Any set containing a single non-zero vector is l.i. / a single non-zero vector is itself l.i.

$$S = \{\underset{\sim}{v}_1\} \; ; \; \underset{\sim}{v}_1 \neq 0 \, .$$

$$c\underset{\sim}{v}_1 = \underset{\sim}{0} \, . \rightarrow (1)$$

$\because \underset{\sim}{v}_1 \neq \underset{\sim}{0} \; \therefore$ (1) is possible only when $c_1 = 0 \; \therefore \underset{\sim}{v}_1$ is l.i.

[Note: $c\underset{\sim}{v} = \underset{\sim}{0}$ if either $c = 0$ or $\underset{\sim}{v} = 0$]

**Theorem 3.** Any set of vectors containing the zero vector is a linearly dependent set.

$$S = \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3, \ldots, \underset{\sim}{0}, \ldots \underset{\sim}{v}_n\}$$

Then, $c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2 + \cdots + c_r\underset{\sim}{0} + \cdots + c_n\underset{\sim}{v}_n = \underset{\sim}{0}$

holds even if all $c_i$'s $= 0$ but $c_r \neq 0$

$\therefore S$ is l.d.

Ex. Check whether the functions $\underset{\sim}{f}_1 = \sin^2 x$, $\underset{\sim}{f}_2 = \cos^2 x$, $\underset{\sim}{f}_3 = 3$ are l.i or l.d.

Note: $3\sin^2 x + 3\cos^2 x - 3 = 0$

i.e. $3.\underset{\sim}{f}_1 + 3\underset{\sim}{f}_2 - 1.\underset{\sim}{f}_3 = 0$.

$\underset{\sim}{f}_3 = 3\underset{\sim}{f}_1 + 3\underset{\sim}{f}_2$. $\quad \underset{\sim}{f}_1 = \frac{\underset{\sim}{f}_3}{3} - \underset{\sim}{f}_2$

$f_2 = -\underset{\sim}{f}_1 + \frac{1}{3}\underset{\sim}{f}_3$.

## Wronskian of functions

$f_1, f_2, \ldots, f_n$ are continuous & $(n-1)$ times differentiable and are defined in some interval $I$.

Wronskian of $f_1, f_2, \ldots, f_n$ is defined by.

$$W = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

If $W = 0$ identically for all values of $x \in I$
$f_1, f_2, \ldots, f_n$ are l.d.

Otherwise (if $W \neq 0$) $f_1, f_2, \ldots, f_n$ are l.i.

$f_1 = \sin^2 x$, $f_2 = \cos^2 x$, $f_3 = 3$.

$$W = \begin{vmatrix} \sin^2 x & \cos^2 x & 3 \\ 2\sin x\cos x & -2\cos x\sin x & 0 \\ 2\cos 2x & -2\cos 2x & 0 \end{vmatrix} = 0,$$ $f_1, f_2, f_3$ are l.d.

Find whethere the functions are l.i / l.d.

$1, e^x, e^{2x}$ in the interval $(-\infty, \infty)$.

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = 2e^{3x} \neq 0$$

$\therefore 1, e^x, e^{2x}$ are l. i.

linear span.

Def. $S = \{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_n}\}$ be a set of vectors. By linear span of S we mean a set L(S) which is the set of all linear combinations of vectors in S.

$$\therefore L(S) = \{c_1\underset{\sim}{v_1} + c_2\underset{\sim}{v_2} + \dots + c_n\underset{\sim}{v_n} : c_i \in R, \underset{\sim}{v_i} \in S\}$$

$S = \{(2,3)\}$

**Note.** $\boxed{L(S) = \{(2c, 3c)\}}$

If $(x,y) \in L(S)$, $x = 2c$, $y = 3c$

$\frac{x}{2} = \frac{y}{3}$

$2(2,3) = (4,6)$, $-1(2,3) = (-2,-3)$

$0.(2,3) = (0,0)$..

$3x - 2y = 0$ → a line passing through $(2,3)$ & $(0,0)$

$L(S) = \{(4,6), (-2,-3), (0,0) \dots\}$

$\therefore L(S)$ represents a line passing through $(2,3), (0,0)$

$S = \{(1,1), (-2,0)\}$

$c_1(1,1) + c_2(-2,0) = (c_1 - 2c_2, c_1)$

$L(S) = \{(c_1 - 2c_2, c_1)\}$
$= \{(1,1), (3,1), (-8,0), \dots\}$

$S = \{ v_1, v_2 \} \subset \mathbb{R}^2$ or $\mathbb{R}^3$.

$L(S) \to$ plane ~~passes~~ containing $v_1, v_2$ and $(0,0)$.

$S = \{ v_1 \} \subset \mathbb{R}^2 / \mathbb{R}^3$

$L(S) \to$ line passing through origin.

<u>Thm</u>:- {Deletion theorem} If a vector space $V$ be spanned by a linearly dependent set $S = \{ v_1, v_2, \ldots, v_n \}$, then $V$ can also be spanned by a smaller subset of $\{ v_1, v_2, \ldots, v_n \}$. i.e we can delete some vectors from the l.d set $S$ to get the same vector space $V$

<u>Ex.</u> $S = \{ v_1 = (1,2,0), v_2 = (3,-1,1), v_3 = (4,1,1) \}$

<u>Note.</u> $v_3 = v_1 + v_2$.

$$L(S) = \{ c_1 v_1 + c_2 v_2 + c_3 v_3 \}$$

$$\underline{L(v_1, v_2, v_3)} \quad \{ c_1 v_1 + c_2 v_2 + c_3 (v_1 + v_2) \}$$

$$= \{ (c_1 + c_3) v_1 + (c_2 + c_3) v_2 \} \qquad S_1 = (v_1, v_2)$$

$$= \{ d_1 v_1 + d_2 v_2 \} = L(S_1)$$

$$L(S) = \{ c_1 (v_3 - v_2) + c_2 v_2 + c_3 v_3 \}$$

$$= \{ (c_2 - c_1) v_2 + (c_1 + c_3) v_3 \} = L(S_2); \quad S_2 = (v_3, v_3)$$

~~Basis~~

<u>Thm</u>. $L(S)$ is a vector space which is the smallest subspace of $V$ (of which $S$ is a subset) containing $S$.

i.e if $W$ is any other subspace of $V$, then

$$W \supset L(S) \supset S.$$

## Basis

A set $S = \{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots, \underset{\sim}{v_n}\}$ is said to be a basis for a vector space $V$ if.

1) $S$ is l. i
2) $S$ spans $V$ i. e $L(S) = V$.
 i.e every element of $V$ is a linear combination of elements of $S$.

$\mathbb{R}^3$ ; $(2,3,4) \in \mathbb{R}^3$.

$$(2,3,4) = 2(1,0,0) + 3(0,1,0) + 4(0,0,1).$$
$$= 2\underset{\sim}{e_1} + 3\underset{\sim}{e_2} + 4\underset{\sim}{e_3}$$

$$\begin{pmatrix} \underset{\sim}{e_1} \\ \underset{\sim}{e_2} \\ \underset{\sim}{e_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$(x,y,z) \in \mathbb{R}^3$

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

$\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_3}$ are l. i

$\Rightarrow$ every element $(x,y,z) \in \mathbb{R}^3$ can be expressed as a linear combination of the vectors $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_3}$

$\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ form <u>a</u> basis for $\mathbb{R}^3$.

There may be many bases for a vector space.

Ex Show that —

$(1, 2, 1)$, $(0, 2, 3)$, $(0, 0, 4)$ also form a basis for $\mathbb{R}^3$.

$\underset{\sim}{b}_1 \quad \underset{\sim}{b}_2 \quad \underset{\sim}{b}_3$

$$\begin{pmatrix} \underset{\sim}{b}_1 \\ \underset{\sim}{b}_2 \\ \underset{\sim}{b}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow \text{(matrix in echelon form having 3 non-zero rows)}.$$

so, $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3$ are l. i.

To check whether any $(x, y, z)$ can be expressed as a l. c. of $\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3$.

Let, $(x, y, z) = c_1 \underset{\sim}{b}_1 + c_2 \underset{\sim}{b}_2 + c_3 \underset{\sim}{b}_3$.

$$c_1(1, 2, 1) + c_2(0, 2, 3) + c_3(0, 0, 4) = (x, y, z)$$

or, $(c_1, 2c_1 + 2c_2, c_1 + 3c_2 + 4c_3) = (x, y, z)$

$$c_1 = x, \quad 2c_1 + 2c_2 = y, \quad c_1 + 3c_2 + 4c_3 = z$$

$$c_1 = x, \quad c_2 = \frac{y - 2x}{2}$$

$$c_3 = \frac{z - c_1 - 3c_2}{4} = \frac{z - x - \frac{3}{2}(y - 2x)}{4}$$

$$= \frac{4x - 3y + 2z}{8}.$$

$$(x, y, z) = c_1 \underset{\sim}{b}_1 + c_2 \underset{\sim}{b}_2 + c_3 \underset{\sim}{b}_3$$

$$= x(1, 2, 1) + \frac{y - 2x}{2}(0, 2, 3) + \frac{4x - 3y + 2z}{8}(0, 0, 4)$$

Thm. No. of vectors in ~~a basis~~ any basis for a vector space is ~~equal~~ same.

Def. No. of vectors in a basis of a vector space is called <u>dimension</u> of the vector space.

Note. $\because$ $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3$ form a basis of $\mathbb{R}^3$,

$\dim \mathbb{R}^3 = 3$.

$\dim \mathbb{R}^n = n$

$(x_1, x_2, \ldots, x_n) = x_1(1, 0, \ldots, 0)$
$+ x_2(0, 1, 0, \ldots, 0) + x_3(0, 0, 1, 0, \ldots, 0)$
$+ \ldots + x_n(0, 0, \ldots, 0, 1)$.

$= x_1 \underset{\sim}{e}_1 + x_2 \underset{\sim}{e}_2 + \ldots + x_n \underset{\sim}{e}_n$.

$$\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = I_n$$

is in $\rightarrow$ echelon form having $n$ non-zero rows.

Also $n$ = no. of vectors $e_1, \ldots, e_n$.

$\therefore$ $\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_n$ are l.i.

So, $\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_n$ form a basis for $\mathbb{R}^n$.

$\therefore$ $\dim \mathbb{R}^n = n$.

$V^{2\times 3}$ = set of all matrices of order $2\times 3$.

$$= \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \right\}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ d\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= a\,\underset{\sim}{e}_1 + b\,\underset{\sim}{e}_2 + c\,\underset{\sim}{e}_3 + d\,\underset{\sim}{e}_4 + e.\underset{\sim}{e}_5 + f.\underset{\sim}{e}_6$$

$$c_1\underset{\sim}{e}_1 + c_2\underset{\sim}{e}_2 + c_3\underset{\sim}{e}_3 - - - + c_6\underset{\sim}{e}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$c_i = 0$$

$\therefore \underset{\sim}{e}_1, -, \underset{\sim}{e}_6$ form a basis for $V^{2\times 3}$.

dim of $V^{2\times 3} = 6 = 2\times 3$.

dim of $V^{m\times n} = mn$

$V_f$ = set of all functions $f: \mathbb{R} \to \mathbb{R}$.

$\to V_f$ is of infinite dimension.

$P(t) \to$ set of all polynomials is of infinite dimension.

$P_n(t) \to$ set of all polynomials whose degree is $\leq n$.

$$= \{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n\}.$$

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

$$= a_0 \times 1 + a_1 \times t + a_2 \times t^2 + \cdots + a_n t^n$$

$(1, t, t^2, \ldots, t^n)$ is a basis for $P_n(t)$

$\therefore$ dim of $P_n(t) = n + 1$.

$$\boxed{\text{To check whether } S = \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \ldots, \underset{\sim}{v}_n\} \text{ is a basis for } V}$$

dim $V$ is known (say, dim $V = n$).

$\downarrow$

Check either 1) l. independence of $S$

or, 2) whether $L(S) = V$

dim $V$ is not known.

$\downarrow$

To check

1) $S$ is l.i. or not

AND 2) $L(S) = V$ or not.