Improper Integrals

Integral Calculus $\longrightarrow$
1. Improper Integral
2. Leibnitz rule (Differentiation under the sign of integration)
3. Beta - Gama functions
4. Multiple integrals.

$\int_a^b f(x)\,dx \longrightarrow$ proper integral if a & b are finite & f(x) is defined & bounded in the interval $[a, b]$.

Thus, $\int_a^b f(x)$ is improper, if —

Type-1

Either a or b or both are infinite

as for example — $\int_2^\infty \frac{dx}{x^2}$; $\int_0^\infty e^{-x}\,dx$; $\int_{-\infty}^\infty \frac{dx}{a^2+x^2}$

Type-2

f(x) fails to be bounded at one or more points in $[a, b]$.

as for example — $\int_0^1 \frac{dx}{x(x+1)}$; $\int_4^5 \frac{dx}{x(x-5)}$;

$\int_4^5 \frac{dx}{(x-4)(x-5)}$; $\int_2^3 \frac{dx}{x(x-2.5)}$.

And there may be integrals which is a combination of type 1 and type 2.

as — $\int_{-\infty}^\infty \frac{x^2\,dx}{(x-1)}$.

(1) $\int_a^{\infty} f(x)\,dx$ is said to be convergent, if -

$$\lim_{B\to\infty}\int_a^B f(x)\,dx \text{ exists.}$$

If $\underset{B\to\infty}{Lt}\int_a^B f(x)\,dx = \pm\infty$, then $\int_a^{\infty} f(x)\,dx$ diverges.

(2) $\int_{-\infty}^{b} f(x)\,dx$ converges, if $\underset{A\to-\infty}{Lt}\int_A^b f(x)\,dx$ exists, ~~and if~~ $\int_{-\infty}^{b} f(x)$ diverges, if

$$\underset{A\to-\infty}{Lt}\int_A^b f(x)\,dx = \pm\infty.$$

(3) $\int_{-\infty}^{\infty} f(x)\,dx$ converges if

$\int_{-\infty}^{c} f(x)\,dx$ & $\int_c^{\infty} f\,dx$ both the integrals converge.

i.e. $\underset{A\to-\infty}{Lt}\int_A^c f\,dx$ & $\underset{B\to\infty}{Lt}\int_c^B f\,dx$ both exist.

If one of the above limits diverges, then $\int_{-\infty}^{\infty} f(x)\,dx$ will diverge.

Examples

$$\int_a^\infty \frac{dx}{x^p} = \lim_{B\to\infty} \int_a^B \frac{dx}{x^p}$$

$$\int_a^B \frac{dx}{x^p} = \left.\frac{x^{-p+1}}{1-p}\right|_a^B = \begin{cases} \dfrac{B^{1-p} - a^{1-p}}{1-p}, & p \neq 1 \\ \log B - \log a, & p = 1. \end{cases}$$

Now, $\underline{p > 1}$ $\quad m = p - 1 > 0.$

$$\int_a^B \frac{dx}{x^p} = \frac{B^{-m} - a^{-m}}{-m} = \frac{1}{m}\left[\frac{1}{a^m} - \frac{1}{B^m}\right]$$

Which implies to the value $\frac{1}{ma^m}$ as $B \to \infty$.

$$\therefore \int_a^B \frac{dx}{x^p} = \frac{1}{(p-1)a^{p-1}}$$

$\underline{p < 1}$ $\quad m = 1 - p > 0.$

$$\int_a^B \frac{dx}{x^p} = \frac{B^m - a^m}{m} \to \infty \text{ as } B \to \infty.$$

$\underline{p = 1}$ $\quad \displaystyle\int_a^B \frac{dx}{x^p} = \ln B - \ln a \to \infty$ as $B \to \infty$.

$\displaystyle\int_{(a>0)}^\infty \frac{dx}{x^p}$, this integral → converges for $p > 1$
→ diverges for $p \leq 1$.

Example 2

$$\int_0^\infty \sin x \, dx \longrightarrow \text{diverges.}$$

Since,

$$\lim_{B\to\infty} \int_0^B \sin x \, dx = \lim_{B\to\infty} \cos x \Big|_B^0 = \left[1 - \cos B\right]_{B\to\infty}$$

$\Rightarrow$ does not exists.

Example - 3

$$\int_{-\infty}^\infty e^{-|x|} \, dx.$$

$$= \int_{-\infty}^0 e^{x} \, dx + \int_0^\infty e^{-x} \, dx.$$

$$= \lim_{A\to -\infty} \int_A^0 e^{x} \, dx + \lim_{B\to\infty} \int_0^B e^{-x} \, dx$$

$$= \lim_{A\to -\infty} e^{x}\Big|_A^0 + \lim_{B\to\infty} e^{-x}\Big|_B^0$$

$$= 1 - \underbrace{\lim_{A\to -\infty} e^{A}}_{= 0} + 1 - \underbrace{\lim_{B\to\infty} e^{-B}}_{= 0}$$

$$= 2$$

But $\int_{-\infty}^\infty e^{-x} \, dx$ diverges $\Rightarrow$

$$= e^{-x}\Big|_0^{-\infty} = \lim_{A\to\infty} e^{-A} - 1 = e^{-(-\infty)} - 1$$

$$= \infty$$

Tests for convergence / divergence

1. Comparison test (inequality)

(i) $f(x), g(x), h(x)$ are continuous in $a \le x < \infty$.

(ii) $0 < f(x) \le g(x)$ in $a \le x < \infty$.

(iii) $\int_a^\infty g(x)\,dx$ converges.

then $\int_a^\infty f(x)\,dx$ converges.

(iv) $0 < h(x) \le f(x)$ in $a \le x < \infty$

& $\int_a^\infty h(x)\,dx$ diverges.

Then, $\int_a^\infty f(x)\,dx$ will also diverge.

2. Limit comparison test.

(i) f, g are continuous in $a \le x < \infty$.

(ii) $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = l$

Now, three cases may arise —

(a) l is finite $\neq 0$ then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converges or diverges together.

(b) If $l = 0$ then

$\int_a^\infty g(x)\,dx$ convergent $\Rightarrow \int_a^\infty f(x)\,dx$ converge.

(c) If l is infinite then, $\int_a^\infty g(x)\,dx \rightarrow$ diverge

$\Rightarrow \int_a^\infty f(x)\,dx$ diverge.

(3) μ - test

Let $\displaystyle\lim_{x \to \infty} x^{\mu} f(x) = l$

case - I. l finite $\neq 0$.

Then if $\mu > 1$, $\int_a^{\infty} f\, dx$ will converge

if $\mu \leq 1$, $\int_a^{\infty} f\, dx$ will diverge.

case - II $l = 0$.

Then if $\mu > 1$, $\int_a^{\infty} f\, dx$ will converge.

case - III $l = \infty$

If $\mu \leq 1$, $\int_a^{\infty} f\, dx$ will diverge.

Example

$$\int_2^{\infty} \frac{x^2\, dx}{\sqrt{x^7 + 1}}, \quad f(x) = \frac{x^2}{\sqrt{x^7+1}}$$

$$= \frac{x^2}{x^{7/2}\sqrt{1 + \frac{1}{x^7}}}$$

$$= \frac{1}{x^{3/2}\sqrt{1 + \frac{1}{x^7}}}$$

$$\frac{1}{x^7} > 0 \Rightarrow 1 + \frac{1}{x^7} > 1 \Rightarrow \frac{1}{\sqrt{1+\frac{1}{x^7}}} < 1$$

$$\Rightarrow \underbrace{\frac{1}{x^{3/2}\sqrt{1+\frac{1}{x^7}}}}_{f(x)} < \underbrace{\frac{1}{x^{3/2}}}_{g(x)}$$

Now $\int_2^{\infty} \frac{dx}{x^{3/2}}$ converges $\left[\because \frac{3}{2} > 1\right]$.

$\therefore \int_2^{\infty} \frac{x^2\, dx}{\sqrt{x^7+1}}$ converges.

In other way —

taking $f(x) = \dfrac{1}{x^{3/2}\sqrt{1+\frac{1}{x^7}}}$, $g(x) = \dfrac{1}{x^{3/2}}$

$$\frac{f(x)}{g(x)} = \frac{1}{\sqrt{1+\frac{1}{x^7}}} \to 1 \text{ as } x \to \infty.$$
$$\neq 0$$

$\therefore \int_2^\infty g(x)\, dx$ & $\int_2^\infty f(x)\, dx$ converges / diverges together.

$\because \int_2^\infty \dfrac{dx}{x^{3/2}}$ converges, $\therefore \int_2^\infty \dfrac{x^2\, dx}{\sqrt{x^7+1}}$ converges.

(2) $\int_2^\infty \dfrac{x^3\, dx}{\sqrt{x^7+1}}$

$$f(x) = \frac{1}{x^{-3} x^{7/2}\sqrt{1+\frac{1}{x^7}}} = \frac{1}{x^{1/2}\sqrt{1+\frac{1}{x^7}}}$$

Now, choose $g(x) = \dfrac{1}{x^{1/2}}$

$$\frac{f(x)}{g(x)} = \frac{1}{\sqrt{1+\frac{1}{x^7}}} \to 1 \neq 0 \text{ as } x \to \infty$$

$\therefore \int_2^\infty f(x)\, dx$, $\int_2^\infty g(x)\, dx$ converge / diverge together.

and since $\int_2^\infty \dfrac{dx}{x^{1/2}}$ diverges [as $p = \frac{1}{2} < 1$]

$\therefore \int_2^\infty \dfrac{x^3\, dx}{\sqrt{x^7+1}}$ diverges.

Ex $I = \int_1^\infty \frac{\sin x}{x^{3/2}}\,dx$. it is absolutely convergent.

i.e $\int_1^\infty \frac{|\sin x|}{x^{3/2}}\,dx$ & I both converge.

Note.- $0 \le |\sin x| \le 1$, $x \in [1, \infty)$

$$\therefore \frac{|\sin x|}{x^{3/2}} \le \frac{1}{x^{3/2}}$$

$$= f \qquad\qquad = g$$

Now, $\int_1^\infty \frac{dx}{x^{3/2}}$ converges

$\therefore \int_1^\infty \frac{|\sin x|}{x^{3/2}}\,dx$ converges.

$\Rightarrow \int_1^\infty \frac{\sin x}{x^{3/2}}\,dx$ converges

$\therefore I$ is absolutely convergent.

Ex $I = \int_0^\infty \frac{\sin x}{x}\,dx$ converges

but $\int_0^\infty \frac{|\sin x|}{x}\,dx$ diverges.

$\therefore I$ is conditionally convergent.

Ex

$$\int_1^\infty \frac{dx}{\sqrt{1+x^3}}$$

$$f(x) = \frac{1}{x^{3/2}\sqrt{1+\frac{1}{x^3}}}$$

μ- test:

$$\underset{x\to\infty}{Lt}\, x^\mu f(x) = \underset{x\to\infty}{Lt}\, x^{3/2} \frac{1}{x^{3/2}\sqrt{1+\frac{1}{x^3}}} = 1$$

$\mu = \frac{3}{2} > 1$ & Limit is finite.

so, $\int_1^\infty \frac{dx}{\sqrt{1+x^3}}$ will converge.

Ex

$$\int_1^\infty \frac{7e^{-x}-1}{\sqrt[3]{1+2x^2}}\, dx$$

$$f(x) = \frac{7e^{-x}-1}{x^{2/3}\left(\frac{1}{x^2}+2\right)^{1/3}}$$

$$\underset{x\to\infty}{Lt}\, x^\mu f(x) = \underset{x\to\infty}{Lt}\, x^{2/3} f(x) = -\frac{1}{2^{1/3}}$$

$\because$ the limit is finite & $\mu < 1$,

then I will diverge.

Ex

$$I = \int_1^\infty \frac{dx}{\sqrt{1+x^3}}$$

$$f(x) = \frac{1}{\sqrt{1+x^3}} = \frac{1}{x^{3/2}\sqrt{1+\frac{1}{x^3}}}.$$

choose $g(x) = \frac{1}{x^{3/2}}$.

$$\therefore \frac{f(x)}{g(x)} = \frac{1}{\sqrt{1+\frac{1}{x^3}}} \longrightarrow 1 \text{ as } x \longrightarrow \infty.$$

$\therefore \int_1^\infty f\, dx$ and $\int_1^\infty g\, dx$ coverges or diverges together.

$\because \int_1^\infty \frac{dx}{x^{3/2}}$ converges, $\therefore$ I converges.

Ex $\int_1^\infty \frac{7e^{-x}-1}{\sqrt[3]{1+2x^2}}\, dx$

$f = \frac{7e^{-x}-1}{x^{2/3}\left(\frac{1}{x^2}+2\right)^{1/3}}$, Now choose $g(x) = \frac{1}{x^{2/3}}$

$$\therefore \frac{f(x)}{g(x)} = \frac{7e^{-x}-1}{\left(\frac{1}{x^2}+2\right)^{1/3}}$$

$$\therefore \lim_{x\to\infty} \frac{f(x)}{g(x)} \longrightarrow -\frac{1}{2^{1/3}}$$

$\int_1^\infty \frac{dx}{x^{2/3}}$ diverges

$\therefore$ I will diverge

Ex

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

$$= \underbrace{\int_0^1 \frac{\sin x}{x}\,dx}_{\text{has finite value}} + \int_1^\infty \frac{\sin x}{x}\,dx$$

Since, in the case of improper integral we are concern only with infinite discontinuities of the integrals, and here we can see, in the first integral, it has finite value, as for example

$$f = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases} \quad \text{or} \quad f(x) = x, \; 0 < x < 3$$
$$= x^2, \; 3 \leq x < 4$$

both the functions have finite jump, so at this point no need to consider these integrals from improper integrals point of view.

So, we have to check only the second integral, wheather it converge or not.

And by applying Cauchy's test we can see —

$$\int_a^\infty \frac{\sin x}{x}\,dx \quad \text{converges.}$$

So $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

$\boxed{\text{Note}}$ —

$$f(x) = x, \; 0 < x < 3$$
$$= x^2, \; 3 \leq x < 4$$

$\int_0^4 f(x)\,dx$ exists.

$$= \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx = \int_0^3 x\,dx + \int_3^4 x^2\,dx.$$

Ex Prove that

$$\int_0^\infty \frac{|\sin x|}{x}\,dx.$$ does not converge.

$$k\pi \le x \le (k+1)\pi \quad , \quad k = 0, 1, 2, 3, \ldots$$

$$x \le (k+1)\pi$$

$$\frac{1}{x} \geq \frac{1}{(k+1)\pi}.$$

$$\int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx \geq \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{(k+1)\pi}\,dx = \frac{2}{(k+1)\pi}$$

$$\therefore \int_0^\infty \frac{|\sin x|}{x}\,dx$$

$$= \left[\int_0^\pi + \int_\pi^{2\pi} + \int_{2\pi}^{3\pi} + \cdots\right] \frac{|\sin x|}{x}\,dx.$$

$$\geq \frac{2}{\pi} + \frac{2}{2\pi} + \frac{2}{3\pi} + \cdots$$

$$= \frac{2}{\pi}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{R}\right]$$

$$\int_0^\infty \frac{|\sin x|}{x}\,dx \geq$$ any large finite value

$$\rightarrow \infty$$

$\rightarrow$ diverges.

But in the above problem we have seen $\int_0^\infty \frac{\sin x}{x}\,dx$ in convergent.

So, $\int_0^\infty \frac{\sin x}{x}\,dx$ in <u>conditionally</u> <u>convergent</u>.