FUNCTIONS OF SEVERAL REAL VARIABLES

DEF. If to each point (x, y) of a certain part of the x-y plane, there corresponds a real value z according to some given rule $f(x,y)$, then $f(x,y)$ is called a real valued function of two variables x & y. It is written as

$$z = f(x,y) \quad ; \quad (x,y) \in \mathbb{R}^2, \; z \in \mathbb{R}$$

$x, y \rightarrow$ independent variables

$z \rightarrow$ dependent variable

A real valued function of n-variables is defined as

$$z = f(x_1, x_2, \ldots, x_n) \; ; \; (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \; z \in \mathbb{R} \quad -(1)$$

The function given in (1) is called an explicit function, whereas a function defined by $\Phi(z, x_1, x_2, \ldots, x_n) = 0$ is called an implicit function.

FUNCTION OF TWO VARIABLES:

$$z = f(x,y)$$

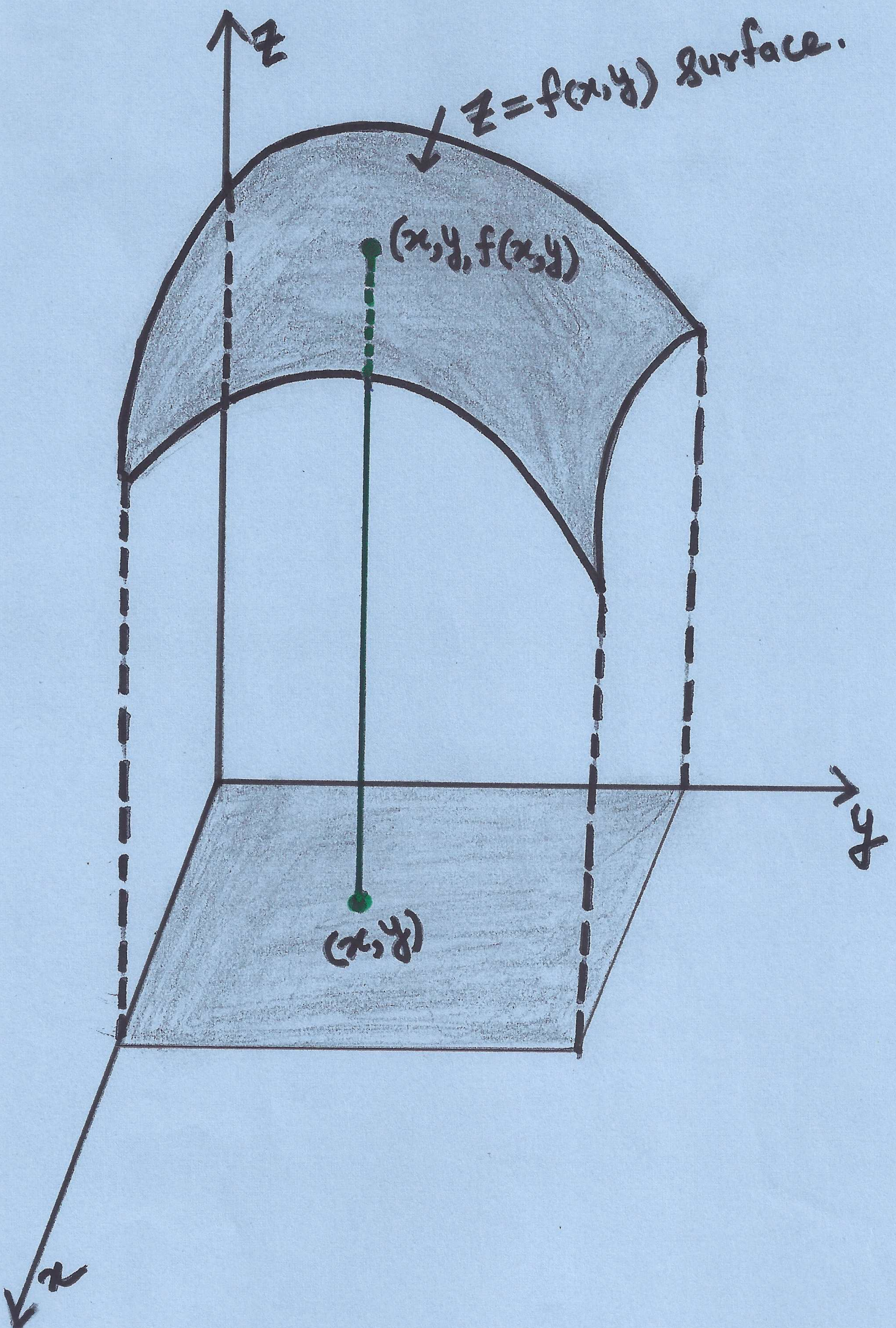
The set of point (x,y) in the x-y plane for which $f(x,y)$ is defined is called domain of definition of the function and is denoted by D. The collection of the corresponding values of z is called the range of the function f.

Ex. $\quad z = \sqrt{1 - x^2 - y^2}$

Since z is real, we have $1 - x^2 - y^2 \geq 0 \Rightarrow x^2 + y^2 \leq 1$.

$\Rightarrow$ Domain : $D = \{(x,y) : x^2 + y^2 \leq 1\}$ 

Range: set of all real positive numbers between 0 & 1.



GEOMETRIC REPRESENTATION OF A FUNCTION OF TWO VARIABLES

DEF.

- DISTANCE BETWEEN THE TWO POINTS:

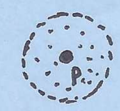
$$|PQ| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

- NEIGHBOURHOOD OF A POINT:

Consider a point $P(x_0, y_0)$

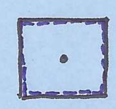
δ-neighbourhood of P $\left(N_\delta(P) \text{ or } N(P,\delta)\right)$

$$:= \left\{(x,y) : \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\right\}$$



OR

$$N_\delta(P) = \left\{(x,y) : \underset{(<)}{x_0 - \delta \leq} \underset{(<)}{x \leq} x_0 + \delta,\ \underset{(<)}{y_0 - \delta \leq} \underset{(<)}{y \leq} y_0 + \delta\right\}$$



- OPEN DOMAIN:

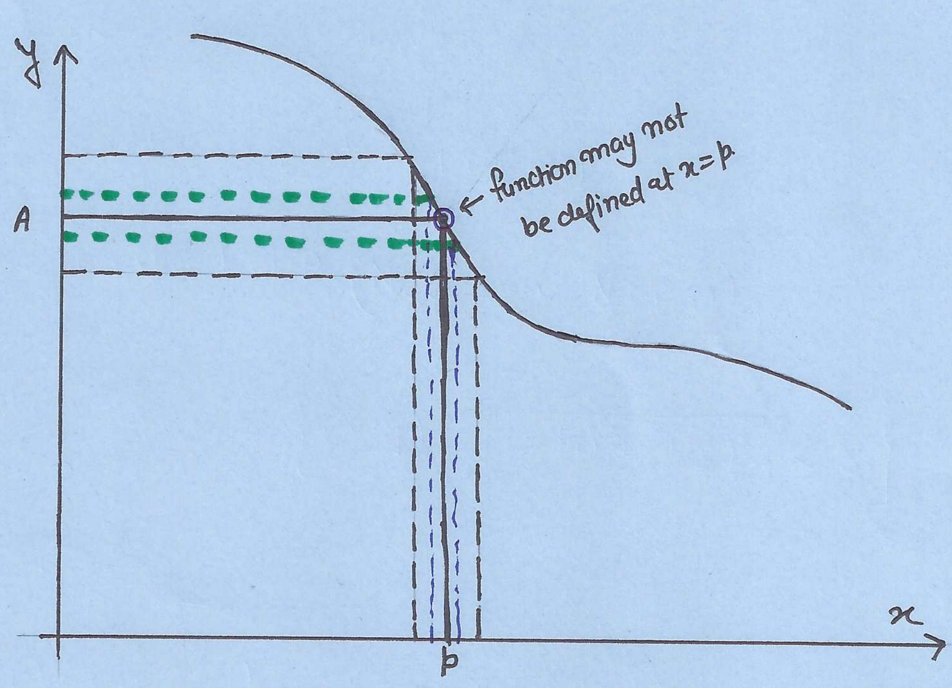
A domain D is open, if there exists a number $\delta > 0$ corresponding to every point p in D such that all point in δ-neighboured of p are in D.

- BOUNDED DOMAIN: D is bounded if $\exists$ a number (finite & positive) M such that D can be enclosed within a circle with radius M and centre at origin.
- CLOSED REGION: A closed region is a bounded domain with its boundary.
- BOUNDED FUNCTIONS: A function $f(x,y)$ defined in some domain D in R^2 is bounded, if there exists a real number (finite) M, such that $|f(x,y)| \leq M$ for all $(x,y) \in D$..

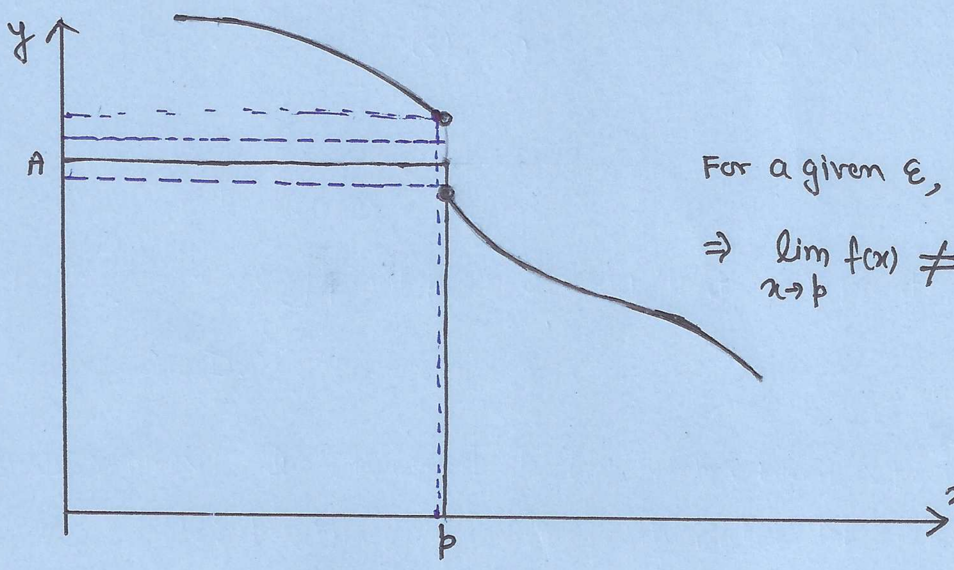
LIMIT OF A FUNCTION

- ONE VARIABLE (RECALL)

  - $\lim\limits_{x \to p} f(x) = A$ means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ whenever $0 < |x - p| < \delta$.



$\lim\limits_{x \to p} f(x) = A$



For a given ε, δ does not exist

$\Rightarrow \lim\limits_{x \to p} f(x) \neq A$

- $\lim\limits_{x \to p} f(x) = A$ means that every neighbourhood $N_\varepsilon(A)$ of A there is some neighbourhood $N_\delta(p)$ such that

$$f(x) \in N_\varepsilon(A) \quad \text{whenever} \quad x \in N_\delta(p) \text{ and } x \neq p.$$

LIMITS (TWO VARIABLES)

Let $z = f(x,y)$ be a function of two variables defined in a domain

D. Let $P(x_0, y_0)$ be a point in D. If for a given real number $\varepsilon > 0$, however small, we can find a real number $\delta > 0$ such that for every point (x,y) in the δ-neighbourhood of $P(x_0,y_0)$

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

then the real number L is called the limit of the function $f(x,y)$ as $(x,y) \to (x_0,y_0)$. Symbolically,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

REMARK:

Note that for the limit to exist, the function $f(x,y)$ may or may not be defined at (x_0,y_0). If $f(x,y)$ is not defined at $P(x_0,y_0)$ then we write

$$|f(x,y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

EXAMPLE: Using ε-δ approach show that

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy}{\sqrt{x^2+y^2}}\right) = 0$$

Sol. For $(x,y) \neq (0,0)$,

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{2xy}{2\sqrt{x^2+y^2}}\right| \leq \frac{x^2+y^2}{2\sqrt{x^2+y^2}} = \underbrace{\frac{1}{2}\sqrt{x^2+y^2}}_{<\frac{1}{2}\delta} < \varepsilon$$

as $(x-y)^2 = x^2+y^2-2xy \geq 0 \Rightarrow x^2+y^2 \geq 2xy$

If we choose $\delta < 2\varepsilon$

then

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta$$

Hence

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 .$$

<u>EXAMPLE</u>: Show that

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{xy}\right) = 0$$

<u>Sol</u>: For $(x,y) \neq (0,0)$

$$\left|(x^2+y^2)\sin\left(\frac{1}{xy}\right) - 0\right| = \left|(x^2+y^2)\sin\left(\frac{1}{xy}\right)\right| \leq \underbrace{(x^2+y^2) \overset{!}{<} \varepsilon}_{\delta^2 < \varepsilon}$$

If we choose $\delta^2 < \varepsilon$ then

$$\left|(x^2+y^2)\sin\left(\frac{1}{xy}\right)\right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta .$$

Hence

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{xy}\right) = 0$$