Composition of two linear transformation.

Let $T_1 : V \to W$, $T_2 : W \to U$ be two linear transformations.

Let $v \in V$, $w \in W$, $u \in U$ and $T_1(v) = w \in W$

$T_2(w) = u \in U$.

$$T_2\left(T_1(v)\right) = T_2(w) = u$$

$$\Rightarrow \left(T_2 \circ T_1\right)v = u \in U.$$

$T_2 \circ T_1$ exists $\not\Rightarrow$ $T_1 \circ T_2$ will also exist.

Example 1. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by
$T_1(x, y) = (y, x)$, $T_2(x, y) = (0, x)$.
Find $T_1 \circ T_2$, $T_2 \circ T_1$, T_1^2, T_2^2.

Solution.
$$\begin{aligned}(T_1 \circ T_2)(x, y) &= T_1 \circ (T_2(x, y)) \\ &= T_1(0, x) \\ &= (x, 0)\end{aligned}$$

$$\begin{aligned}(T_2 \circ T_1)(x, y) &= T_2\left(T_1(x, y)\right) \\ &= T_2(y, x) = 0, y\end{aligned}$$

$\therefore$ $T_1 \circ T_2 \neq T_2 \circ T_1$ in general.

$$T_1^2(x, y) = T_1\left(T_1(x, y)\right) = T_1(y, x) = (x, y)$$

$$T_1^2(x, y) = I(x, y) = (x, y).$$

$I \to$ identity mapping

$$T_2^2(x, y) = (0, 0).$$

Inverse Linear Transformation (T^{-1})

Suppose, $T: V \to W$ & $T': W \to V$

If $T \circ T' = T' \circ T = I$

then T' is the inverse mapping / transformation of T.

$T' = T^{-1}$.

Theorem. If an inverse mapping T^{-1} of T exists, then

(i) it is linear i.e. $T^{-1}(c_1\omega_1 + c_2\omega_2) = c_1 T^{-1}(\omega_1) + c_2 T^{-1}(\omega_2)$

(ii) it is unique. i.e. if $T \circ T_1 = I = T_1 \circ T$ & $T \circ T_2 = I = T_2 \circ T$

then, $T_1 = T_2 = T^{-1}$

Definition. If T^{-1} exists, then T is said to be non-singular.

Theorem. If $\ker\{T\} = \{\underset{\sim}{0}\}$ then T^{-1} exists.

Example. Verify whether $T: \mathbb{R}^3 \to \mathbb{R}^3$ is invertible or not where $T(x_1, x_2, x_3) = (2x_1, 4x_1 - x_2, 2x_1 + 3x_2 - x_3)$.

If inverse exists, find T^{-1}.

Soln. $\ker\{T\} = \{(x_1, x_2, x_3) : T(x_1, x_2, x_3) = (0,0,0)\}$.

$\therefore (2x_1, 4x_1 - x_2, 2x_1 + 3x_2 - x_3) = (0, 0, 0)$

$$\therefore \begin{aligned} 2x_1 &= 0 \\ 4x_1 - x_2 &= 0 \\ 2x_1 + 3x_2 - x_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

$\therefore \ker\{T\} = \{\underset{\sim}{0}\}$

$\therefore T^{-1}$ exists.

$$T(x_1, x_2, x_3) = (y_1, y_2, y_3)$$

$$\therefore T^{-1}(y_1, y_2, y_3) = (x_1, x_2, x_3).$$

$2x_1 = y_1$	$4x_1 - x_2 = y_2$	$2x_1 + 3x_2 - x_3 = y_3$
$\Rightarrow x_1 = \frac{y_1}{2}$	$2y_1 - x_2 = y_2$	$y_1 + 3(2y_1 - y_2) - x_3 = y_3$
	$\Rightarrow x_2 = 2y_1 - y_2$	$x_3 = 7y_1 - 3y_2 - y_3$

$$\therefore T^{-1}(y_1, y_2, y_3) = \left(\frac{y_1}{2}, 2y_1 - y_2, 7y_1 - 3y_2 - y_3\right).$$

Matrix Representation of a linear mapping:

let $T: V \to W$, where $V \to$ vector space of dim n.
$W \to$ vector space of dim m

Then their corresponds a matrix of order $m \times n$.

let $\{e_1, e_2, \dots, e_n\}$ be some basis of V,
Let $\{f_1, f_2, \dots, f_n\}$ be some basis of W.

let $v \in V$, $w \in W$

$$v = c_1 e_1 + c_2 e_2 + \dots + c_n e_n$$
$$w = d_1 f_1 + d_2 f_2 + \dots + d_n f_n$$

$$T(e_1) = a_{11} f_1 + a_{21} f_2 + \dots + a_{m1} f_m$$
$$T(e_2) = a_{12} f_1 + a_{22} f_2 + \dots + a_{m2} f_m$$
$$\vdots$$
$$T(e_n) = a_{1n} f_1 + a_{2n} f_2 + \dots + a_{mn} f_m$$

The matrix involved here is.

$$B = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & & a_{mn} \end{pmatrix}_{n \times m}$$

$A = B^T$ is called the matrix representation (or matrix) of T w.r. to the ordered bases $(e_1, e_2, \ldots, e_n)$ of V & $(f_1, f_2, \ldots, f_m)$ of W. We write $A = [T]_e^f$.

<u>Example</u>: Let (e_1, e_2, e_3), (f_1, f_2) be ordered bases of the real vector spaces V & W respectively. A linear transformation $T: V \to W$ maps the basis vectors as $T(e_1) = f_1 + f_2$, $T(e_2) = 3f_1 - f_2$, $T(e_3) = f_1 + 3f_2$. Find the matrix of T with respect to the ordered bases.

(i) (e_1, e_2, e_3) of V & (f_2, f_1) of W

(ii) (e_1+e_2, e_2, e_3) of V & (f_1, f_2+f_1) of W

<u>Soln.</u> (i) $T(e_1) = f_2 + f_1$

$T(e_2) = -f_2 + 3f_1$

$T(e_3) = 3f_2 + f_1$

$$[T]_{(e_1, e_2, e_3)}^{(f_1, f_2)} = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

(ii) $T(e_1+e_2) = Te_1 + Te_2 = (f_1 + f_2) + 3f_1 - f_2 = 4f_1 + 0.(f_1+f_2)$.

$T(e_2) = 3f_1 - f_2 = 4f_1 + (-1)(f_1 + f_2)$

$T(e_3) = f_1 + 3f_2 = -2f_1 + 3(f_1 + f_2)$

$$[T]_{\{e_1+e_2, e_2, e_3\}}^{\{f_1, f_1+f_2\}} = \begin{bmatrix} 4 & 4 & -2 \\ 0 & -1 & 3 \end{bmatrix}$$

Example. The matrix of Linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ relative to $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ & $\{(1,0), (1,1)\}$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find the matrix of T relative to the ordered bases $\{(1,1,0), (1,0,1), (0,1,1)\}$ of $\mathbb{R}^3$ & $\{(1,1), (0,1)\}$ of $\mathbb{R}^2$

Soln $T(0,1,1) = 1(1,0) + 2(1,1) = (3,2)$

$T(1,0,1) = 2(1,0) + 1(1,1) = (3,1)$

$T(1,1,0) = 4(1,0) + 0.(1,1) = (4,0)$.

Let $(x,y,z) \in \mathbb{R}^3$

$\therefore (x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0) = (c_2+c_3, c_3+c_1, c_1+c_2)$.

$T(x,y,z) = c_1(3,2) + c_2(3,1) + c_3(4,0)$

$= (3c_1 + 3c_2 + 4c_3, 2c_1 + c_2)$.

Since $\left.\begin{aligned} c_2 + c_3 &= x \\ c_3 + c_1 &= y \\ c_1 + c_2 &= z \end{aligned}\right\} \Rightarrow c_1 + c_2 + c_3 = \frac{x+y+z}{2}$

$\therefore c_1 = \frac{y+z-x}{2}$; $c_2 = \frac{z+x-y}{2}$, $c_3 = \frac{x+y-z}{2}$

$\therefore T(x,y,z) = \left(2x+2y+z, \frac{-x+y+3z}{2}\right)$

Next. Ans: $\begin{pmatrix} 4 & 3 & 3 \\ -4 & -2 & -1 \end{pmatrix}$.

Eigen Values, Eigen Vectors of a square matrix

Let A be an $n \times n$ matrix. A real / complex no. λ is said to be an eigen value of A, if there exists a non zero vector $\underset{\sim}{v}$ such that $A_{n\times n} \underset{\sim}{v}_{n\times 1} = \lambda \underset{\sim}{v}$.

Note1. Corresponding to a particular eigen value, there may exist many eigenvectors.

Note2. If A be an $n \times n$ matrix, then the no. of eigen values is exactly equal to n, taking multiplicity into consideration.

Note3. Collection of all the eigen values of A is called spectrum of A. Magnitude of the largest eigen value is called the spectral radius of A i.e. if $-10, 3, -1$ are the eigen values of A, then $|-10| = 10$ is the spectral radius of A.

How to find eigen values & eigen vectors?

$$A\underset{\sim}{v} = \lambda \underset{\sim}{v} \quad ; \underset{\sim}{v} \neq \underset{\sim}{0}.$$

$$(A - \lambda I_n)\underset{\sim}{v} = \underset{\sim}{0}$$

$$\because \underset{\sim}{v} \neq \underset{\sim}{0} \quad \therefore \; |A - \lambda I_n| = 0 \longrightarrow (1)$$

$$(\lambda I_n - A)\underset{\sim}{v} = 0 \Rightarrow |\lambda I_n - A| = 0 \longrightarrow (1').$$

(1) or (1') is called the characteristic equation of the matrix A. It is an n^{th} degree polynomial equation if A is of order n.

$|A - \lambda I_n|$ or $|\lambda I_n - A| \rightarrow$ characteristic polynomial of A.

Solving the charactoristic eavuation $|\lambda I_n - A| = 0$, we get the eigen values of A.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \lambda I - A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix}$$

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = 0.$$

$\Rightarrow \lambda = 0, 0 \Rightarrow 0$ is the only eigen value of this matrix.

Example. Find the eigen values of the matrix.

$$A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$|\lambda I_3 - A| = \begin{vmatrix} \lambda-3 & 2 & 0 \\ 2 & \lambda-3 & 0 \\ 0 & 0 & \lambda-5 \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1, 5, 5$$

$$A\underset{\sim}{v} = \lambda \underset{\sim}{v}$$

$$\Rightarrow (\lambda I_n - A)\underset{\sim}{v} = 0$$

$$\begin{bmatrix} \lambda-3 & 2 & 0 \\ 2 & \lambda-3 & 0 \\ 0 & 0 & \lambda-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (1)$$

To find eigen vector corresponding to $\lambda = 1$, put $\lambda = 1$, in eavuation (1). Then,

$$\begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} -2x_1 + 2x_2 &= 0 \\ 2x_1 - 2x_2 &= 0 \\ -4x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} -2x_1 + 2x_2 = 0 &\Rightarrow x_1 = x_2 \\ x_3 &= 0 \end{aligned}$$

Let $x_1 = a$, $x_2 = a$, $x_3 = 0$

$\therefore (x_1, x_2, x_3) = (a, a, 0)$; $a \neq 0$; $a \in \mathbb{R} \setminus \{0\}$

$E_{\lambda=1}$ = eigen space of $\lambda = 1 = \{(a, a, 0);\ a \in \mathbb{R}\}$

$(x_1, x_2, x_3) = a(1, 1, 0)$.

$(1,1,0)$ being a single vector is itself linear independent.
So $(1,1,0)$ forms a basis for $E_{\lambda=1}$, $\therefore$ dimension of $E_{\lambda=1}$ is 1.

<u>Definition.</u> Dimension of the eigen space corresponding to an eigen value λ is called the geometric multiplicity of λ & is denoted by g_λ. So, here $g_{\lambda=1} = 1$.

<u>Definition.</u> Multiplicity of an eigen value λ as a root of the characteristic earuation is called the algebraic multiplicity of λ & is denoted by a_λ.

Here $a_\lambda = 1$.

<u>Theorem.</u> $a_\lambda \geq g_\lambda$, for a particular eigen value λ

$a_\lambda - g_\lambda$ is called the defect of λ.

To find eigen vectors corresponding to $\lambda = 5$, put $\lambda = 5$ in (1) & solve the homogeneous system

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\Rightarrow 2x_1 + 2x_2 = 0$ $\quad$ Let $x_2 = b$,

x_3 is arbitrary $\quad$ $x_3 = c$.

$(x_1, x_2, x_3) = (-b, b, c) = b(-1,1,0) + c(0,0,1)$

$E_{\lambda=5} = \{(-b, b, c) ; b, c \in \mathbb{R}\}$.

Any $(x_1, x_2, x_3) \in E_{\lambda=5}$ is a linear combination of $(-1,1,0), (0,0,1)$. These vectors are also linearly independent, since they form nonzero rows of an echelon matrix.

$\therefore \{(-1,1,0), (0,0,1)\}$ is a basis for $E_{\lambda=5}$.

$\therefore$ dim of $E_{\lambda=5} = 2$.

$\therefore g_{\lambda=5} = 2$

$a_{\lambda=5} = 2$. $\quad$ $\therefore$ defect of $\lambda = 5 = 0$.

<u>Example</u> Consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$(\lambda I - A)\underset{\sim}{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \lambda = 0, 0$.

algebraic multiplicity of $\lambda = 0$ is 2.

$A\underset{\sim}{v} = \lambda \underset{\sim}{v} \Rightarrow (\lambda I_n - A)\underset{\sim}{v} = \underset{\sim}{0}.$

$$\Rightarrow \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \because \lambda = 0.$$

$x_2 = 0$

$x_1 =$ arbitrary $= a$ (say)

$\therefore (x_1, x_2) = (a, 0) = a(1, 0).$

$E_{\lambda=0} = \{(a,0) : a \in \mathbb{R}\}.$

Dimension of $E_{\lambda=0} = 1 = g_{\lambda=0}.$

Here $a_{\lambda=0} = 2$, $g_{\lambda=0} = 1.$

$\therefore$ defect of $\lambda = 0 = 2 - 1 = 1.$

Excercise 1. Consider a matrix A given by

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Obtain eigen values, eigen vectors of A. Also find the algebraic & geometric multiplicity of eigen values and defect of each eigen values.

Ans. $\lambda = 5, -3, -3$

$E_{\lambda=5} = [1, 2, -1]^T.$

$E_{\lambda=-3} = \{(-2a+3b, a, b) : a, b \in \mathbb{R}\}.$

Excercise 2. A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3,\ x_2 + 4x_3,\ x_1 - x_2 + 3x_3),$$
$$(x_1, x_2, x_3) \in \mathbb{R}^3.$$

Find the matrix of T relative to the ordered bases $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ & $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$

Ans. matrix of T = $\begin{bmatrix} \frac{7}{2} & \frac{7}{2} & -1 \\ -\frac{3}{2} & \frac{1}{2} & 1 \\ \frac{3}{2} & \frac{1}{2} & 2 \end{bmatrix}$

Excercise 3. The matrix of a linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ w.r to the order basis $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ is given by

$$\begin{pmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix}$$

Find T.

Ans. $T(x, y, z) = (-x + y + 3z,\ x + y + z,\ x - 3y + 5z)$.

Excercise 4. A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T(x, y, z) = (x - y,\ x + 2y,\ y + 3z),\ (x, y, z) \in \mathbb{R}^3.$$

Find T^{-1}.

Ans. $T^{-1}(x, y, z) = \left(\frac{2}{3}x + \frac{1}{3}y,\ -\frac{1}{3}x + \frac{1}{3}y,\ \frac{1}{9}x - \frac{1}{9}y + \frac{1}{3}z\right),$
$(x, y, z) \in \mathbb{R}^3.$