Department of Mathematics

Indian Institute of Technology, Kharagpur

Date:............ FN/AN Time: 3 Hrs Full Marks: 50 No. of Students: 650
End Autumn Semester: 2012, Deptt: AE/ME/CE/EX/NA/CH/MF/PH/EE/MA/IE/CS
Sub. No. MA 20102 Subject Name: Transform Calculus

Instructions:
(i) Answer all questions.
(ii) All the parts of the same question should be done at one place.
(iii) Start new question from new page.

Question 1 [2 + 3 + 3 + 4]

a) Consider a piecewise continuous function $f(t)$ having the only discontinuity at $t = a$, then find the Laplace transform of $f'(t)$.

b) Find the Laplace transform of $f(t) = \frac{1}{t}\,\delta\left(t - \frac{3}{2}\right)$.

c) Find the inverse Laplace transform $f(t)$ of $F(s) = \frac{e^{-s}}{s^2+4} + \frac{e^{-2s}}{s^2+4} + \frac{e^{-3s}}{(s+2)^2}$

d) Solve the following system of simultaneous differential equations using the Laplace transform technique

$$\frac{dx}{dt} - 2y = \cos(2t), \quad t > 0$$
$$\frac{dy}{dt} + 2x = \sin(2t), \quad t > 0$$

with $x(0) = 1$, $y(0) = 0$.

Question 2: Using notations $F\{f(t)\} = \hat{f}(\alpha)$ as Fourier transform, $F_c\{f(t)\} = \hat{f}_c(\alpha)$ as Fourier cosine transform, and $F_s\{f(t)\} = \hat{f}_s(\alpha)$ as Fourier sine transform [2 + 3 + 2 + 6]

a) Show that $F\{e^{-5|t|}\} = F_c\{e^{-5t}\}$.

b) Find $F_s\{e^{at}\}$ and use it to compute $F_s\{\frac{e^{at}}{t}\}$ for $a > 0$.

c) Calculate $F\{|t|e^{-a|t|}\}$.

d) Prove the Parseval's identity

$$\int_0^\infty |\hat{f}_s(\alpha)|^2 d\alpha = \int_0^\infty |\hat{f}_c(\alpha)|^2 d\alpha = \int_0^\infty |f(t)|^2 dt$$

Using these relations, evaluate $I_1 = \int_0^\infty \frac{1}{(\alpha^2+4)^2} d\alpha$ and $I_2 = \int_0^\infty \frac{\alpha^2}{(\alpha^2+9)^2} d\alpha$.

Question 3 [4 + 4 + 5]

a) Solve the wave equation using Fourier sine transform technique

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \, t > 0$$

subject to the initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$. Take u and $\frac{\partial u}{\partial x}$ both tend to zero as $x \to \infty$ and $u(0,t) = 0$.

b) Solve the diffusion equation using Laplace transform technique

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \, t > 0$$

subject to the initial condition $u(x,0) = 1 + \sin(\pi x)$ and boundary conditions $u(0,t) = 1$, $u(1,t) = 1$ for $t > 0$.

c) Using Fourier transform technique, solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \, y > 0$$

subject to $u(x,0) = f(x)$, $-\infty < x < \infty$ and $u(x,y)$ is bounded as $y \to \infty$. Assume u and $\frac{\partial u}{\partial x}$ vanish as $|x| \to \infty$.

Question 4 [7 + 3 + 2]

a) Let $f(x) = e^{2x}$ for $0 \le x \le 1$. Write down the half range Fourier *(i)* sine and *(ii)* cosine series of $f(x)$. Determine the sum of each series for $x = 0$ and $x = 1/2$.

b) Find the complex form of Fourier series of $f(x) = \cos(ax)$, $-\pi < x < \pi$, $0 < a < 1$.

c) Let $f : \mathbb{R} \to \mathbb{R}$ be a $\pi-$ periodic piecewise continuous function. If
$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2nx) + b_n \sin(2nx)]$ be the Fourier series of f on $[0, \pi]$ and
$\frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(nx) + A_n \sin(nx)]$ be the Fourier series of f on $[-\pi, \pi]$, then express the coefficients A_n and B_n in terms of the a_n and b_n.

***************** THE END *****************