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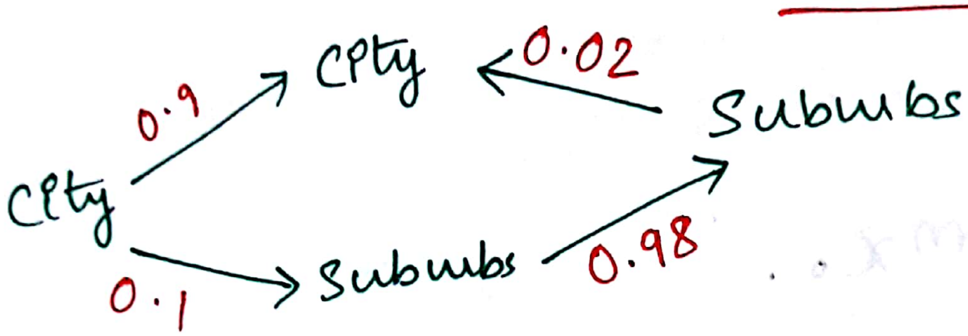
DIAGONALIZABILITY AND APPLICATIONS

30th October

1) Matrix limits
2) Matrix functions (Matrix polynomials | exponentials)

MARKOV CHAINS



$$\begin{array}{ccc} & \text{Currently living in the city} & \text{Currently living in the suburb} \\ \text{Living in the city next year} & 0.9 & 0.02 \\ \text{Living in the suburb next year} & 0.1 & 0.98 \end{array}$$

2×2

Transition State Matrix

① Column sums should be 1

② No entry should be negative.

$$P = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} \begin{matrix} \leftarrow \text{Proportion of city dwellers} \\ \leftarrow \text{Proportion of suburb dwellers} \end{matrix}$$

After 1 year

$$AP = \begin{pmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{pmatrix} \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.636 \\ 0.364 \end{pmatrix}$$

After 2 years

$$A(AP) = A^2 P = \begin{pmatrix} 0.5796 \\ 0.4203 \end{pmatrix}$$

$$\lim_{m \to \infty} A^m P \ ??$$

Is A diagonalizable?

$$Q = \begin{pmatrix} 1/6 & -1/6 \\ 5/6 & 1/6 \end{pmatrix} ; \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.88 \end{pmatrix}$$

$$Q^{-1} A Q = D$$

$$\therefore L = \lim_{m \to \infty} A^m = \lim_{m \to \infty} Q D^m Q^{-1}$$

$$= Q \left[\lim_{m \to \infty} D^m \right] Q^{-1}$$

See $\lim_{m \to \infty} D^m = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

x^2

$$L = \begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix} \rightarrow$$ stationary distribution of Stochastic process.

$$\therefore \lim_{m\to\infty} A^m P = LP = \begin{pmatrix} 1/6 \\ 5/6 \end{pmatrix}$$

Eventually, $1/6$th of the population will stay in the city and rest in the suburbs.

Can you always have an existential limit

<u>Definition</u>: Let $A \in M_{n\times n}(\mathbb{C})$. For, $1 \leq i, j \leq n$ define $\rho_i(A)$ to be the sum of absolute value of entries in the ith row of A and define $\gamma_j(A)$ to be the sum of absolute value of entries in the jth column.

$$\rho_i(A) = \sum_{j=1}^{n} |A_{ij}| \quad \text{for } i = 1, 2, \dots n$$

$$\gamma_j(A) = \sum_{i=1}^{n} |A_{ij}| \quad \text{for } j = 1, 2, \dots n.$$

Gerschgoren Disk. C_i:

$$C_i = \{ z \in \mathbb{C} \mid |z - A_{ii}| < r_i \}$$

where $r_i = \rho_i(A) - |A_{ii}|$

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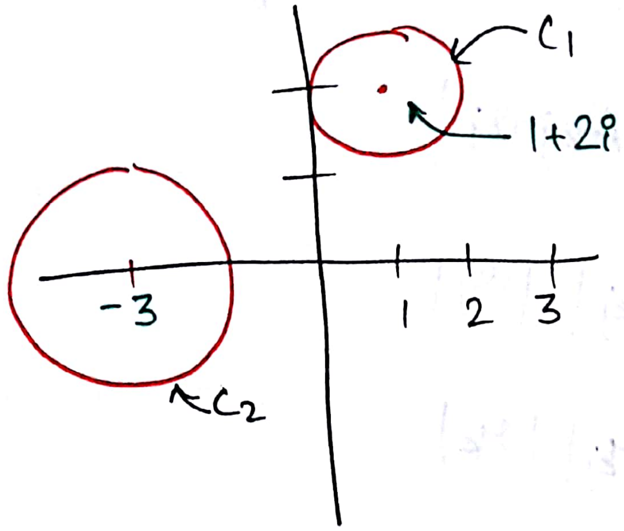
Example

$$A = \begin{bmatrix} 1+2i & 1 \\ 2i & -3 \end{bmatrix}$$



Theorem Gerschgorin's disk theorem

Let $A \in M_{n\times n}(\mathbb{C})$ Then every eigenvalue of A is contained in a Gerschgorin's disk.

Proof: Let λ be an eigenvalue of A and the corresponding eigenvector is

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = V$$

$$Av = \lambda v$$

$$\sum_{j=1}^{n} A_{ij} v_j = \lambda v_i \qquad i = 1, 2, \ldots n$$

Let v_k be the coordinate of v having the largest absolute value.

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We will show $\lambda \in C_k$.

$$|\lambda v_k - A_{kk} v_k|$$
$$= \left|\sum_{j=1}^{n} A_{kj} v_j - A_{kk} v_k\right|$$
$$= \left|\sum_{j\neq k} A_{kj} v_j\right|$$
$$\leq \sum_{j\neq k} |A_{kj}|\,|v_j|$$
$$\leq \sum_{j\neq v} |A_{kj}|\,|v_k|$$
$$= |v_k| \sum_{j\neq k} |A_{kj}|$$
$$= |v_k|\, r_k .$$

Thus,
$$|v_k|\,|\lambda - A_{kk}| \leq |v_k|\, r_k$$
$$\Rightarrow |\lambda - A_{kk}| \leq r_k .$$

Corollary: Let λ be an eigenvalue of $A \in M_{n\times n}(\mathbb{C})$. Then $|\lambda| < \rho(A)$

Corollary: Let λ be an eigenvalue of $A \in M_{n\times n}(\mathbb{C})$. Then $|\lambda| \leq \min(\rho(A), \gamma(A))$

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Invariant Subspaces and Cayley Hamilton Theorem

DEFINITION: Let T be a linear operator on a vector space V. A subspace W of V is called a T invariant subspace of V if $T(W) \subseteq W$, that is, if $v \in W$, $T(v) \in W$.

Examples

V be a vector space. $T: V \to V$.

1. $\{0\}$
2. V
3. $R(T)$
4. $N(T)$
5. E_λ for any eigenvalue of T.

Examples:

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+b \\ b+c \\ 0 \end{pmatrix}$$

$$W = \left\{ \begin{pmatrix} t \\ s \\ 0 \end{pmatrix} : t, s \in \mathbb{R} \right\}$$

↳ T invariant subspace.

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$\frac{}{1-x^2}$ linear operator on $\lambda = 2k$

Let T be a linear operator on a vector space V. Let x be a non zero vector

$x \neq 0, \; x \in V.$

$W = \text{span}\{x, T(x), T^2(x), \ldots\}$

$w \in W$

$w = \sum_{i=0}^{k} a_i T^i(x)$

T-cyclic subspace of V generated by x.

EX W: cyclic subspace of V generated by x is the smallest T-invariant subspace containing x

Ex

$T: R^3 \longrightarrow R^3$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -b+c \\ a+c \\ 3c \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in R^3$$

~~W = span~~

$W = \text{span}\{e_1, T(e_1), T^2(e_1) \ldots\}$

$$T(e_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$T(T(e_1)) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -e_1$$

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$W = \text{span}\{e_1, e_2\}$

Theorem: Let T be a linear operator on a finite dimensional vector space V. Let W be a T invariant subspace of V. Then the characteristic polynomial of T_W divides the characteristic polynomial of T.

Q) What is T_W?

$T : V \longrightarrow V$

W : T-invariant subspace of V.

$T(W) \subseteq W$.

$T(W)$: restriction of T on W.

$T_W : W \longrightarrow W$.

Proof: Let $\gamma_1 = \{v_1, \ldots, v_k\}$ be an ordered basis of W.

$\gamma_2 = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ extended ordered basis of V.

Let $B_1 = [T_W]_{\gamma_1}$

$A = [T]_{\gamma_2}$

So, it is clear, that

$$A = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}$$

$$\det(A - tI) = \det\left[\begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix} - t[I]\right]$$

$\det(A-tI) = \det(B_1 - tI)\det(B_3 - tI)$ ...uple

Characteristic polynomial of T. Characteristic polynomial of T_W.

Theorem Let T be a linear operator in a finite dimensional vector space V. Let W denote T-cyclic subspace of V generated by non zero vector $v \in V$ and let $k = \dim(W)$. Then.

(a) $\{v, T(v), T^2(v) \ldots\ldots, T^{k-1}(v)\}$ is a basis of W.

(b) If $a_0 v + a_1(T(v)) + \ldots + a_{k-1}T^{k-1}(v) + T^k(v) = 0$ then the characteristic polynomial of T_W is $f(t) = (-1)^k(a_0 + a_1 t + \ldots + a_{k-1}t^{k-1} + t^k)$

Proof: Let $v \neq 0$. $\{v\}$ is L.I.

$B = \{v, T(v), \ldots\ldots, T^{j-1}(v)\}$

a linearly independent set where j is the largest positive integer.

Let $Z = \mathrm{span}(B)$

Also, $T^j(v) \in Z$.

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b) $W = \text{span}\{v, T(v), \dots T^{(k-1)}(v)\}$

$T^k(v) = -a_0 v - a_1 T(v) + \dots - a_{k-1} T^{k-1}(v)$.

$$[T_W]_B = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & 0 & -a_1 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & & & \\ & & & 1 & -a_{k-1} \end{bmatrix}$$

$\det([T_W]_B - tI_k) = (-1)^k (a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k)$

Controllable canonical form.

The Cayley Hamilton Theorem

Let T be a linear operator on a finite dimensional vector space V. Let $f(t)$ be the characteristic polynomial of T. Then $f(T) = 0$, the zero transformation. That is, T "satisfies", its characteristic polynomial.

Proof: We will show that

$$f(T)(v) = 0 \quad \forall v \in V.$$

Let $v \neq 0$,

W is a T-cyclic subspace generated by v.

Let (dimension of W) $\dim(W) = k$.

$\exists$ scalars $a_0, a_1, \dots, a_{k-1}$ such that

$a_0 v + a_1 T(v) + a_{k-1} T^{k-1}(v) + T^k(v) = 0$.

This means that

$g(t) = (-1)^k (a_0 + a_1 t + a_2 t^2 \dots + a_{k-1} t^{k-1} + t^k)$

is the char. Polynomial of T_W.

$\lambda =$

$$g(T) = (-1)^k (a_0 I + a_1 T + \ldots + a_{k-1} T^{k-1} + T^k)(v).$$

$$= 0.$$

Since $g(t) | f(t)$ → characteristic polynomial of T.

$f(T)(v) = 0 \quad \forall v \in V$

$\Rightarrow$ $f(T)$ is a zero linear operator

$\Rightarrow$ T satisfies its characteristic polynomial

Corrollary: Let A be an $n \times n$ matrix with $f(T)$ as its characteristic polynomial then $f(A) = 0$

Ex 1

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$f(t) = t^2 - 1$

C.H. $\Rightarrow f(A) = 0$

$A^2 - I = 0$

$\Rightarrow A^2 = I$

$\Rightarrow \underline{A = A^{-1}}$

Ex

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$f(t) = (t-1)^2$$

$$f(t) = t^2 - 2t + 1$$

$$\Rightarrow \quad f(A) = 0$$

$$\Rightarrow \quad A^2 - 2A + I = 0$$

$$\Rightarrow \quad \boxed{A^2 = 2A - I} \qquad (*)$$

$$A^3 = 2A^2 - A = 4A - 2I - A$$

$$= 3A - 2I$$

$$\boxed{A^{-1} = 2I - A} \qquad (\text{If } A^{-1} \text{ exists})$$