2.9 Applications to Generating Functions Continued ...

Let us now substitute different values for x to obtain different expressions and then use them to get binomial identities.

1. Let $x = z + \dfrac{1}{z}$. Then $\sqrt{x^2-4} = z - \dfrac{1}{z}$ and we obtain $a(n, z+\frac{1}{z}) = \dfrac{z^{2n+2}-1}{(z^2-1)z^n}$. Hence, we have $\sum\limits_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}(-1)^k \left(z+\frac{1}{z}\right)^{n-2k} = \dfrac{z^{2n+2}-1}{(z^2-1)z^n}$. Or equivalently,
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}(-1)^k \left(z^2+1\right)^{n-2k} z^{2k} = \frac{z^{2n+2}-1}{z^2-1}.$$
2. Writing x in place of z^2, we obtain the following identity.
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}(-1)^k (x+1)^{n-2k} x^k = \frac{x^{n+1}-1}{x-1} = \sum_{k=0}^{n} x^k. \tag{2.1}$$
3. Hence, equating the coefficient of x^m in (2.1) gives the identity
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \binom{n-k}{k}\binom{n-2k}{m-k} = \begin{cases} 1, & \text{if } 0 \le m \le n; \\ 0, & \text{otherwise.} \end{cases}$$
4. Substituting $x = 1$ in (2.1) gives $\sum\limits_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k \binom{n-k}{k} 2^{n-2k} = n+1$.

Example 2.9.1. *Determine the generating function for the numbers* $a(n, y) = \sum\limits_{k \ge 0} \binom{n+k}{2k} y^k$.

Solution: *Define* $A(x, y) = \sum\limits_{n \ge 0} a(n, y) x^n$. *Then*
$$\begin{aligned} A(x,y) &= \sum_{n\ge 0}\left(\sum_{k\ge 0}\binom{n+k}{2k}y^k\right)x^n = \sum_{k\ge 0}\left(\frac{y}{x}\right)^k \sum_{n\ge k}\binom{n+k}{2k}x^{n+k} \\ &= \sum_{k\ge 0}\left(\frac{y}{x}\right)^k \frac{x^{2k}}{(1-x)^{2k+1}} = \frac{1}{1-x}\sum_{k\ge 0}\left(\frac{yx}{(1-x)^2}\right)^k \\ &= \frac{1-x}{(1-x)^2 - xy}. \end{aligned}$$

1. *Verify that if we substituting* $y = -2$ *then*
$$\sum_{k\ge 0}\binom{n+k}{2k}(-2)^k = [x^n]\ A(x,-2) = [x^n]\ \frac{1-x}{(1+x^2)} = (-1)^{\lceil n/2 \rceil}.$$
2. *Verify that if we substituting* $y = -4$ *then*
$$\sum_{k\ge 0}\binom{n+k}{2k}(-4)^k = [x^n]\ A(x,-4) = [x^n]\ \frac{1-x}{(1+x)^2} = (-1)^n(2n+1).$$

3. *Let* $f(n) = \sum_{k \geq 0} \binom{n+k}{2k} 2^{n-k}$ *and let* $F(z) = \sum_{n \geq 0} f(n) z^n$. *Then verify that*

$$F(z) = A(2z, \frac{1}{2}) = \frac{1-2z}{(1-z)(1-4z)} = \frac{2}{3} \cdot \frac{1}{1-4z} + \frac{1}{3} \cdot \frac{1}{1-z}.$$

Hence, $f(n) = [z^n]F(z) = \dfrac{2 \cdot 4^n}{3} + \dfrac{1}{3} = \dfrac{2^{2n+1}+1}{3}$.

Notes: Most of the ideas for this chapter have come from book [11].