LA *Assignment - I*

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1) $V = \{ f \in C[0,1] : f(1) = 0 \}$ over $\mathbb{R}$.

$C[0,1]$ is the set of all continuous functions from $[0,1]$ to $\mathbb{R}$.

Every continuous function on $[0,1]$ has a unique representation as an infinite linear combination of these functions. This expression is called the Fourier series of a function and the coefficients of that are Fourier coefficients. But $C[0,1]$ does not have a countable Hamel basis, even if it has a countable orthonormal basis

Let $f(x) = e^{b1}(e^x - 1)$, $g(x) = e^{b2}(e^x - 1)$

$b1, b2 \in \mathbb{R}$ $f(x), g(x) \in V$ & they are LI.

$\therefore$ Basis contains $b1, b2$ ($b1, b2 \in \mathbb{R}$) and $\mathbb{R}$ is uncountably infinite, so basis is also uncountably infinite.

Ans 1c) The cardinality of each basis of V is uncountable.

2) $||x|| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$ $\forall$ $x \in R^n$ is a norm on $\mathbb{R}^n$.

Let there be an inner product which induces above norm.

We know, ~~for any~~ every inner product induces a norm of form $||x|| = \langle x, x\rangle^{1/2}$

$$\Rightarrow ||x||^2 = \langle x, x\rangle$$

By parallelogram identity,

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$$

Let $x = (1, 0, \dots 0)_n$, $y = (0, 0, \dots, 1)_n$

Then, from given definition of norm,

$$||x+y||^2 = 1^2 = 1 . \quad ||x-y||^2 = 1^2 = 1$$

$$||x||^2 = 1 , \quad ||y|| = 1$$

$$\therefore 1 + 1 = 2.1 + 2.1$$

— contradiction

$\therefore$ 2 Ans) c) There is no inner product on $R^n$ which induces the above norm $||.||$.

3) $\dim V = 2$, $|F| = 2$

$\therefore$ F has only two elements, additive and multiplicative identity i.e. 0 and 1.

$\therefore$ There are only 3 possible basis:

i) $\{(0,1), (1,0)\}$

ii) $\{(1,1), (0,1)\}$

iii) $\{(1,1), (1,0)\}$

Ans) 3 c) V has exactly 3 three bases.

4) $V = M_{3\times 2}(\mathbb{R})$ . $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$W = \{ A \in V : Ax = 0 \}$

Let $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ $\therefore \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} a+b \\ c+d \\ e+f \end{bmatrix} = 0 \Rightarrow \begin{matrix} a = -b \\ c = -d \\ e = -f \end{matrix}$

$\therefore$ Standard basis of W:

$\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$

Ans): 4. c) Dimension of W is 3.

5) If $v_1, v_2, v_3$ are LI

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ has only trivial soln. $c_1 = c_2 = c_3 = 0$

$\Rightarrow c_1(u_1 + 2u_2 + 3u_3) + c_2(au_2 + 5u_3) + c_3 \cdot 2u_3 = 0$

$\Rightarrow (c_1)u_1 + (2c_1 + ac_2)u_2 + (3c_1 + 5c_2 + 2c_3)u_3 = 0$

$u_1, u_2, u_3$ are LI.

$\therefore c_1 = 0$

$\therefore 2c_1 + ac_2 = 0$

$3c_1 + 5c_2 + 2c_3 = 0$

This system of homogeneous linear equ^n's has trivial soln. iff $D \neq 0$ $(c_1 = c_2 = c_3 = 0)$

$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 2 & a & 0 \\ 3 & 5 & 2 \end{vmatrix} \neq 0$

$\Rightarrow 1(2a) \neq 0 \Rightarrow a \neq 0$

Ans) 5b) $v_1, v_2$ and $v_3$ are linearly independent iff $a \neq 0$.

6) $V = M_{n\times n}(\mathbb{R})$, $\dim(V) = n^2$ (no. of places to put 1 to form standard basis)

$\mathbb{W} = \{A \in V : A$ is an upper triangular and $\text{trace}(A) = 0\}$

$2x + n = n^2 \Rightarrow x = \frac{n^2-n}{2}$

$\dim \mathbb{W} = \frac{n^2-n}{2} + n-1 = \frac{n^2}{2} + \frac{n}{2} - 1$

(no. of places to put 1 above the principal diagonal, all places below diagonal will have 0)

(no. of place to put 1 in the diagonal, $\because$ trace $= 0$ as we can consider $a_{11}$ as $a_{11} = -(a_{22} + \dots + a_{nn})$)

$V = \mathbb{W} \oplus \mathbb{W}^\perp \Rightarrow \dim V = \dim W + \dim W^\perp$

$\Rightarrow \dim W^\perp = n^2 - \left(\frac{n^2}{2} + \frac{n}{2} - 1\right)$

$= \frac{n^2}{2} - \frac{n}{2} + 1 = \frac{n^2-n+2}{2}$

6 Ans) c) Dimensions of orthogonal complement of W is $\frac{n^2-n+2}{2}$.

7) $\|x\| = \sqrt{\langle x, x\rangle}$

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle$

Given, $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

$\Rightarrow \langle x,y\rangle = \langle y,x\rangle = 0$

$\Rightarrow \overline{\langle y,x\rangle} = \langle y,x\rangle = 0$ (conjugate symmetry of IP)

$\Rightarrow x$ is orthogonal to $y$ & $\langle x,y\rangle \in \mathbb{R}$

Ans) 7c) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 \Rightarrow x$ is orthogonal to $y$ if $\mathbb{K} = \mathbb{R}$

8) $W = \{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = 0\}$ subspace of $R^n$

W is non-trivial

$$\frac{\dim(\mathbb{R}^n)}{2} = \frac{n}{2}$$

$\dim(W) = n-1$ (taking $x_1$ ~~as de~~ $-(x_2 + \ldots + x_n)$)

$\therefore \dim(W) \geq \frac{\dim(\mathbb{R}^n)}{2}$ $\left[ n-1 \geq \frac{n}{2} \right.$ $\because n \geq 2 \left.\right]$

$\therefore$ Ans) 8 a) W has two virtually disjoint complements.

9) $\{\phi\}^{\perp} = \{u \in V \mid \langle u, v \rangle = 0 \ \forall \ v \in \{\phi\}\}$

$\because \{\phi\}$ is an empty set.

$\therefore \forall \ u \in V, \ \langle u, v \rangle = 0, \ v \in \{\phi\}$

$\therefore \{\phi\}^{\perp} = V$

Ans) 9a) $\{\phi\}^{\perp} = V$, $\{\phi\}$ is an empty set.

10) $C[-1, 1]$ is an infinite dimensional inner product space.

$\therefore C[-1, 1] \neq U \oplus U^{\perp}$ (equality holds only for finite-dimensional IPS)

$U = \{ f \in C[-1, 1] : f(0) = 0 \}$

Let $f(x) = x^2 g(x)$ $\quad (f(0) = 0)$

If $f(x)$ & $g(x)$ are orthogonal,

$\langle f, g \rangle = 0$

$$\Rightarrow \int_{-1}^{1} x^2 (g(x))^2 \, dx = 0$$

$x^2 \geq 0$ & $(g(x))^2 \geq 0 \;\forall\; x \in [-1, 1]$

$$\therefore \int_{-1}^{1} x^2 (g(x))^2 dx = 0 \text{ iff } g(x) = 0$$

$\therefore U^{\perp} = \{0\}$

Ans) 10) b) $U^{\perp} = \{0\}$