MTH 202 : Probability and Statistics

Homework 7 18th February, 2017

(1) **(Exercise from Homework-6)** Let Y be uniformly distributed on $(0,1)$. Find a function φ such that $\varphi(Y)$ has the gamma density $\Gamma(\frac{1}{2},\frac{1}{2})$. [Ref : Exercise-45, Hoel, Port, Stone, Page-138]

Solution : From the discussion in section 5.4 (Page-131, Hoel, Port, Stone) we have if Y has uniform distribution on $(0,1)$, then $\Phi^{-1}(Y)$ has normal distribution Φ with parameters $(0,1)$. Next from Example-12 (Page-128, Hoel, Port, Stone) we have that $Z = (\Phi^{-1}(Y))^2$ has gamma distribution with parameters $(\frac{1}{2},\frac{1}{2})$. Since the image of the function Φ is contained in the open interval $(-1,1)$, we see that φ can be taken as the function :

$$\varphi(x) = [\Phi^{-1}(x)]^2 \qquad (-1 < x < 1)$$

Exercises of Homework-7

(2) Let Y be uniformly distributed on $(0,1)$. Find a function φ such that $X = \varphi(Y)$ has the density f given by :

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{elsewhere} \end{cases}$$

[Ref : Exercise-44, Hoel, Port, Stone, Page-138]

Solution : The distribution function F of X is as follows

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^2 & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x > 1 \end{cases}$$

Now this means $F \equiv 0$ on the left of $I = (0,1)$ and $F \equiv 1$ on the right of I. Since F is strictly increasing on I, its inverse $g(y) = \sqrt{y}$ is well defined on $I = (0,1)$.
Consider the function $\varphi : (0,\infty) \longrightarrow \mathbb{R}$ given by $\varphi(y) = \sqrt{y}$. We will verify that while Y has uniform distribution on $(0,1)$, the random variable $X = \varphi(Y)$ has the required density.

Recall that the density f_1 of Y is given by

$$f_1(y) = \begin{cases} 1 & \text{if } 0 < y < 1, \\ 0 & \text{elsewhere} \end{cases}$$

Setting up with $I = (0,1)$ in Theorem-1, Page-119, Hoel, Port, Stone, we have $x = \sqrt{y}$, i.e. $y = x^2$ and $\frac{dy}{dx} = 2x$. Using the theorem we clearly have the density g_1 of $X = \sqrt{Y}$ is as given as f except at the point $x = 1$ (i.e. $g_1(1) = 0$ according to the Theorem used), which can be modified to define f.

(3) Let X be a integer valued random variable having distribution function F, and let Y be uniformly distributed on $(0,1)$. Define the integer valued random variable Z in terms of Y by

$$Z = m \qquad \text{if } F(m-1) < Y \leq F(m)$$

for an integer m. Show that Z has the same density as X.
[Ref : Exercise-50, Hoel, Port, Stone, Page-138]

Solution : Recall that the distribution of Y is given by

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0, \\ y & \text{if } 0 \leq y \leq 1, \\ 1 & \text{if } y > 1 \end{cases}$$

Now since the distribution function F has image contained in $[0,1]$ we have $F(m-1), F(m) \in [0,1]$. Hence using the definition of F_Y we have,

$$P(Y \leq F(m-1)) = F_Y(F(m-1)) = F(m-1)$$

Similarly, $P(Y \leq F(m)) = F(m)$. Now, first notice that X is a discrete random variable from its description given. Using the definition of the new (discrete) random variable Z we have,

$P(Z = m) = P(F(m-1) < Y \leq F(m))$
$= P(Y \leq F(m)) - P(Y \leq F(m-1)) = F(m) - F(m-1)$
$= P(m-1 < X \leq m) = P(X = m)$

since X is integer valued. Hence, X and Z has the same density function.

(4) Suppose the times it takes two students to solve a problem are independently and exponentially distributed with parameter λ. Find the probability that the first student will take least twice as long as the second student to solve the problem.
[Ref : Exercise-6, Hoel, Port, Stone, Page-169]

Solution : Let X and Y denote the random variables representing the time taken by the first and the second students to solve the problem. X and Y are independent and each have the exponential density with parameter λ, which is same as the gamma density $\Gamma(1, \lambda)$. We wish to compute $P(X \geq 2Y) = P(X/Y \geq 2)$.
Now using the description of the density of X/Y in Theorem-3, Page-152, Hoel, Port, Stone, we have

$$P(X/Y \geq 2) = \int_2^\infty \frac{\Gamma(2)}{(\Gamma(1))^2} \frac{dz}{(z+1)^2} = \int_3^\infty \frac{dt}{t^2} = 1/3$$

(5) Let X and Y be continuous random variables having the joint density f given by :

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} & \text{if } 0 \leq x \leq y, \\ 0 & \text{elsewhere} \end{cases}$$

Find the marginal densities of X and Y. Find the joint distribution function of X and Y. [Ref : Exercise-7, Hoel, Port, Stone, Page-169]

Solution : Using the equations derived section-6.1, Page-141, Hoel, Port, Stone, we have for $x > 0$,

$$f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y} dy = \lambda e^{-\lambda x}$$

Hence the marginal density of X is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$$

which is exponential with parameter λ. On the other hand

$$f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y} dx = \lambda^2 y e^{-\lambda y}$$

Hence the marginal density of Y is given by

$$f_Y(y) = \begin{cases} \lambda^2 y e^{-\lambda y} & \text{if } y > 0, \\ 0 & \text{if } y \leq 0 \end{cases}$$

which represent gamma density $\Gamma(2, \lambda)$. Next to find the joint distribution we consider the following cases :

Case I : $x > y$

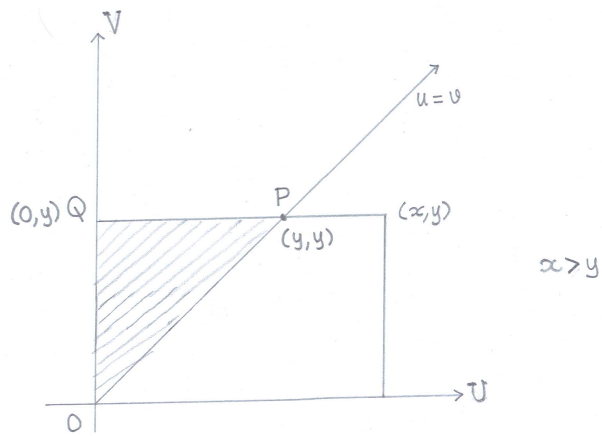

FIGURE 1

$$F(x,y) = \int_0^y \left(\int_0^u f(u,v)du \right) dv = \int_0^y \left(\int_0^v \lambda^2 e^{-\lambda v} du \right) dv$$

$$= \lambda^2 \int_0^y v e^{-\lambda v} dv = (\lambda^2) \frac{v e^{-\lambda v}}{-\lambda} \Big|_{v=0}^{y} - (\lambda^2) \int_0^y \frac{e^{-\lambda v}}{-\lambda} dv$$

$$= -\lambda y e^{-\lambda y} + (\lambda) \frac{e^{-\lambda v}}{-\lambda} \Big|_{v=0}^{y} = 1 - e^{-\lambda y}(1 + \lambda y)$$

Case II : $x \leq y$

$$F(x,y) = \int_0^x \left(\int_0^v f(u,v)du \right) dv + \int_x^y \left(\int_0^x f(u,v)du \right) dv$$

$$= \int_0^x \lambda^2 v e^{-\lambda v} dv + \int_x^y \lambda^2 x e^{-\lambda v} dv$$

$$= 1 - e^{-\lambda x}(1 + \lambda x) - \lambda x e^{-\lambda y} + \lambda x e^{-\lambda x} = 1 - e^{-\lambda x} - \lambda x e^{-\lambda y}$$

Hence the distribution function is given by

$$F_{X,Y}(x,y) = \begin{cases} 1 - e^{-\lambda y}(1 + \lambda y) & \text{if } 0 \leq y < x, \\ 1 - e^{-\lambda x} - \lambda x e^{-\lambda y} & \text{if } 0 \leq x \leq y, \\ 0 & \text{elsewhere} \end{cases}$$

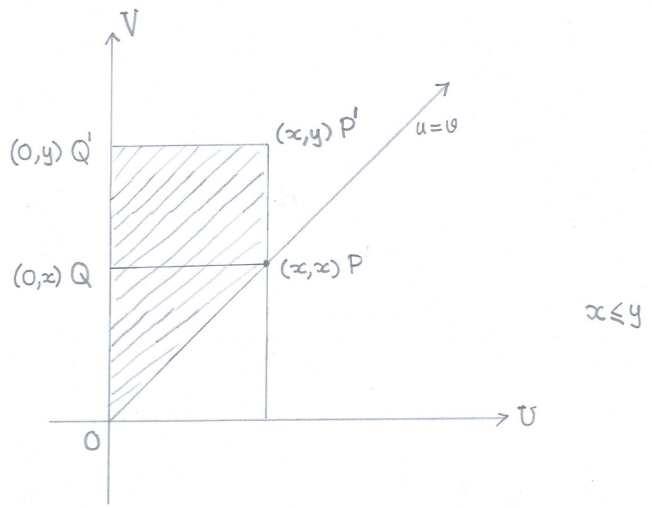

FIGURE 2

(6) Let $f(x,y) = ce^{-(x^2-xy+4y^2)/2}$ for $(x,y) \in \mathbb{R}^2$. How should c be chosen to make f a density? Find the marginal densities of f. [Ref : Exercise-9, Hoel, Port, Stone, Page-169]

Solution : Ignoring the sign, the factor on the exponent is

$$\frac{1}{2}(x^2 - xy + 4y^2) = \frac{1}{2}[(x - \frac{y}{2})^2 + \frac{15}{4}y^2]$$

From the definition of density,

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy$$

$$= c\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x-\frac{y}{2})^2+\frac{15}{4}y^2]}dx)dy = c\int_{-\infty}^{\infty} e^{-\frac{15}{8}y^2}(\int_{-\infty}^{\infty} e^{-t^2/2}dt)dy$$

(with a change of variable $(x - y/2) = t$ in the inner integral)

$$= c\sqrt{2\pi}\int_{-\infty}^{\infty} e^{-\frac{15}{8}y^2}dy = c\sqrt{2\pi}\frac{2}{\sqrt{15}}\int_{-\infty}^{\infty} e^{-u^2/2}du = \frac{4\pi c}{\sqrt{15}}$$

(with another change of variable $\frac{\sqrt{15}}{2}y = u$). Hence we require $c = \frac{\sqrt{15}}{4\pi}$

(7) Let X and Y be independent continuous random variables having the indicated marginal densities. Find the density of $Z = X + Y$.

(a) X and Y are exponentially distributed with parameters λ_1 and λ_2, where $\lambda_1 \neq \lambda_2$.

(b) X is uniform on $(0,1)$, and Y is exponentially distributed with parameter λ. [Ref : Exercise-11, Hoel, Port, Stone, Page-169-170]

Solution : (a) Recall that the marginal densities of X and Y are given by

$$f_X(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$f_Y(y) = \begin{cases} \lambda_2 e^{-\lambda_2 y} & \text{if } y > 0, \\ 0 & \text{if } y \leq 0 \end{cases}$$

Using the convolution formula (14), Page-145, Hoel, Port, Stone, for $z > 0$ we have,

$$f_{X+Y}(z) = \int_0^z f(x, z-x)dx = \int_0^z f_X(x) f_Y(z-x)dx$$

$$= \int_0^z \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 (z-x)} dx = \lambda_1 \lambda_2 e^{-\lambda_2 z} \int_0^z e^{-(\lambda_1 - \lambda_2)x} dx$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 z} \frac{e^{-(\lambda_1 - \lambda_2)x}}{-(\lambda_1 - \lambda_2)} \Bigg|_{x=0}^{z} = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} [e^{-\lambda_2 z} - e^{-\lambda_1 z}]$$

Hence the density is given by

$$f_{X+Y}(z) = \begin{cases} \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} [e^{-\lambda_2 z} - e^{-\lambda_1 z}] & \text{if } z > 0, \\ 0 & \text{if } z \leq 0 \end{cases}$$

(b) The marginal densities are given by

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & \text{if } y > 0, \\ 0 & \text{if } y \leq 0 \end{cases}$$

Using the convolution formula again, while $0 < z \leq 1$ we have

$$f_{X+Y}(z) = \int_0^z f(x, z-x)dx = \int_0^z f_X(x) f_Y(z-x)dx$$

$$= \lambda e^{-\lambda z} \int_0^z e^{\lambda x} dx = 1 - e^{-\lambda z}$$

In case $z > 1$ we have

$$f_{X+Y}(z) = \int_0^z f(x, z-x)dx = \int_0^1 f_X(x) f_Y(z-x)dx$$

$$= \lambda e^{-\lambda z} \int_0^1 e^{\lambda x} dx = e^{-\lambda z}(e^{\lambda} - 1)$$

Hence the density function is given by

$$f_{X+Y}(z) = \begin{cases} 0 & \text{if } z \le 0, \\ 1 - e^{-\lambda z} & \text{if } 0 < z \le 1, \\ e^{-\lambda z}(e^{\lambda} - 1) & \text{if } 1 < z < \infty \end{cases}$$

(8) Let a point be chosen randomly in the plane in such a manner that its x and y coordinates are independently distributed according to the normal density $n(0, \sigma^2)$. Find the density function for the random variable R denoting the distance from the point to the origin. (**Note :** This density occurs in electrical engineering and is known there as a *Rayleigh* density.)
[Ref : Exercise-17, Hoel, Port, Stone, Page-170]

Solution : From the discussions of section-5.3.3, Page-129, Hoel, Port, Stone, we have that X^2 and Y^2 are independently distributed as $\Gamma(\frac{1}{2}, \frac{1}{2\sigma^2})$. Now using Theorem-1, Page-148, Hoel, Port, Stone, we see that X^2+Y^2 has the gamma density $\Gamma(1, \frac{1}{2\sigma^2})$.

Now using Exercise-43, Page-138, Hoel, Port, Stone (see the last but one solved exercise in "More solved problems from Chapter-5") we have the density of $R = \sqrt{X^2 + Y^2}$ is given by

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} & \text{if } r > 0, \\ 0 & \text{if } r \le 0 \end{cases}$$

(9) Let X and Y be independent random variables each having the normal density $n(0, \sigma^2)$. Show that both of Y/X and $Y/|X|$ have the Cauchy density. [Ref : Exercise-19, Hoel, Port, Stone, Page-170]

Solution : We first compute the density of $|X|$. For $x < 0$, clearly $P(|X| \le x) = 0$ and for $x \ge 0$, we have $F_{|X|}(x)$

$$= P(|X| \le x) = P(-x \le X \le x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-x}^{x} e^{-t^2/2\sigma^2} dt$$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_0^x e^{-t^2/2\sigma^2} dt = -\frac{1}{2} + \frac{2}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2\sigma^2} dt$$

(since $F_X(0) = 1/2$ using the symmetry of the density of X). Now differentiating w.r.t. x we have,

$$f_{|X|}(x) = \frac{2}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

Hence the density of $|X|$ is given by

$$f_{|X|}(x) = \begin{cases} \frac{2}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}$$

To compute the density of $Y/|X|$ we need to redo the steps in section-6.2.2. In this case the random variable $|X|$ takes only non-negative values and hence its relevant to consider the set

$$A_z = \{(x, y) \ : \ x > 0, \text{ and } y \leq xz\}$$

Consequently, $F_{Y/|X|}(z)$

$$= \int_0^\infty (\int_{-\infty}^{xz} f_{|X|,Y}(x, y)dy)dx = \int_0^\infty (\int_{-\infty}^{z} x f_{|X|,Y}(x, xv)dv)dx$$

using the change of variable $y = xv$ and hence $dy = xdv$ to the inner integral. Now interchanging the order of integration we have

$$F_{Y/|X|}(z) = \int_{-\infty}^{z} (\int_{-\infty}^{\infty} x f_{|X|,Y}(x, xv)dx)dv$$

Now differentiating w.r.t z we see that the density of $Y/|X|$ is given by

$$f_{Y/|X|}(z) = \int_{-\infty}^{\infty} x f_{|X|,Y}(x, xz)dx \qquad (z \in \mathbb{R})$$

Next since $|X|$ and Y are independent (using the fact that X and Y are independent) we have that

$$f_{Y/|X|}(z) = \int_{-\infty}^{\infty} x f_{|X|}(x) f_Y(xz)dx = \frac{1}{\pi\sigma^2} \int_0^\infty e^{-\frac{x^2}{2\sigma^2}(1+z^2)} x dx$$

Finally making the change of variable

$$\frac{x^2}{2\sigma^2}(1 + z^2) = u \Longrightarrow xdx = \frac{\sigma^2}{1 + z^2} du$$

in the above integral we have

$$f_{Y/|X|}(z) = \frac{1}{\pi\sigma^2} . \frac{\sigma^2}{1 + z^2} \int_0^\infty e^{-u} du = \frac{1}{\pi} . \frac{1}{1 + z^2} \ (z \in \mathbb{R})$$

which represents the Cauchy density. On the other hand, to compute the density of Y/X we can immediately apply the formula (22) (Page-151, Hoel, Port, Stone). Thus, for $z \in \mathbb{R}$ we have,

$$f_{Y/X}(z) = \int_{-\infty}^{\infty} |x| f(x, xz) dx = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(xz) dx$$

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} |x| e^{-\frac{x^2}{2\sigma^2}(1+z^2)} dx = \frac{1}{\pi\sigma^2} \int_0^{\infty} e^{-\frac{x^2}{2\sigma^2}(1+z^2)} x dx$$

which is the same integral we obtained in the expression of $f_{Y/|X|}(z)$. This proves that both of Y/X and $Y/|X|$ have the Cauchy density.

(10) Let X and Y be independent random variables having respective gamma densities $\Gamma(\alpha_1, \lambda)$ and $\Gamma(\alpha_2, \lambda)$. Find the density of $Z = X/(X+Y)$ [Ref : Exercise-22, Hoel, Port, Stone, Page-170]

Solution : Write $Z = \frac{X}{X+Y} = \frac{1}{1+Y/X}$. Now, for $z > 0$ (as both of X and Y are non-negative valued)
$F_Z(z) = P(Z \leq z) = P(1 + Y/X \geq 1/z)$
$= 1 - P(Y/X < (1/z) - 1) = 1 - F_{Y/X}((1/z) - 1)$
Differentiating w.r.t. z we have

$$f_Z(z) = \frac{1}{z^2} f_{Y/X}((1/z) - 1)$$

Next using Theorem-3, Page-152, Hoel, Port, Stone, we have if $(1/z) - 1 > 0$ (and also $z > 0$) then,

$$f_Z(z) = \frac{1}{z^2} \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{((1/z) - 1)^{\alpha_2 - 1}}{((1/z) - 1 + 1)^{\alpha_1 + \alpha_2}}$$

i.e. while $0 < z < 1$ we have

$$f_Z(z) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (1 - z)^{\alpha_2 - 1} z^{\alpha_1 - 1}$$

and $f_Z(z) = 0$ otherwise. Using formulation (20), Page-148, Hoel, Port, Stone, we find this is same as the Beta density with parameters α_1 and α_2.

(11) Let X_1, X_2, X_3 denote the three components of the velocity of a molecule of gas. Suppose that X_1, X_2, X_3 are independent and each has the normal density $n(0, \sigma^2)$. In physics the magnitude of the velocity $Y = (X_1^2 + X_2^2 + X_3^2)^{1/2}$ is said to have a *Maxwell* distribution. Find f_Y. [Ref : Exercise-28, Hoel, Port, Stone, Page-171]

Solution : Recall again from section-5.3.3, Page-128, Hoel, Port, Stone, that the densities of X_1^2, X_2^2 and X_3^2 are $\Gamma(1/2, 1/2\sigma^2)$. Hence using Theorem-1, Page-148, Hoel, Port, Stone, the random variable $X_1^2+X_2^2+X_3^2$ has the gamma density $\Gamma(3/2, 1/2\sigma^2)$. Finally using Exercise-43, Page-138, Hoel, Port, Stone (solved in "More solved problems from Chapter-5") the density of Y is given by

$$f_Y(y) = \begin{cases} \frac{\sqrt{2}}{\sigma^3\sqrt{\pi}} y^2 e^{-y^2/2\sigma^2} & \text{if } y > 0, \\ 0 & \text{if } y \leq 0 \end{cases}$$

(12) Let $X_1, \ldots, X_n$ be independent random variables having a common normal density. Show that there are constants A_n and B_n such that

$$\frac{X_1 + \ldots + X_n - A_n}{B_n}$$

has the same density as X_1. [Ref : Exercise-29, Hoel, Port, Stone, Page-171]

Solution : Suppose each of X_i has the common normal density $n(\mu, \sigma^2)$ $(1 \leq i \leq n)$. Then using Theorem-2, Page-149, Hoel, Port, Stone, the density of $S = X_1 + \ldots + X_n$ is $n(n\mu, n\sigma^2)$. Using the discussion in section-5.3.1, Page-124, Hoel, Port, Stone, the density of $Z = (S - n\mu)/\sigma\sqrt{n}$ is $n(0, 1)$. Hence $\mu + \sigma Z$ has density $n(\mu, \sigma^2)$ (which is same as X_1). Now,

$$\mu + \sigma Z = \frac{X_1 + \ldots + X_n - (n\mu - \sqrt{n}\mu)}{\sqrt{n}}$$

i.e. we may regard $A_n = n\mu - \sqrt{n}\mu$ and $B_n = \sqrt{n}$.

(13) Let X_1, X_2, X_3 be independent random variables each uniformly distributed on $(0, 1)$. Find the density of the random variable $Y = X_1 + X_2 + X_3$. Find $P(X_1 + X_2 + X_3 \leq 2)$. [Ref : Exercise-30, Hoel, Port, Stone, Page-171]

Solution : We shall make use of the density computed in Example-4, Page-147, Hoel, Port, Stone. Since X_2, X_3 are uniformly distributed on $(0, 1)$, the density of $X_2 + X_3$ is given by

$$f_{X_2+X_3}(z) = \begin{cases} z & \text{if } 0 \leq z \leq 1, \\ 2 - z & \text{if } 1 < z \leq 2, \\ 0 & \text{elsewhere} \end{cases}$$

Setting $W = X_1 + X_2 + X_3$, we have the density $f_W(w) = 0$ if $w < 0$ since X_1, X_2, X_3 are non-negative valued. Using the convolution formula (16), Page-146, Hoel, Port, Stone, we have for $w \geq 0$,

$$f_W(w) = \int_0^w f_{X_1}(t) f_{X_2+X_3}(w-t)dt$$

Now we consider the three different intervals. If $0 \leq w < 1$ we have

$$f_W(w) = \int_0^w t dt = w^2/2$$

Next if $1 \leq w < 2$ we have

$$f_W(w) = \int_0^1 f_{X_1}(t) f_{X_2+X_3}(w-t)dt$$

$$= \int_0^{w-1} f_{X_2+X_3}(w-t)dt + \int_{w-1}^1 f_{X_2+X_3}(w-t)dt$$

Now notice that for the first integral $1 \leq w - t \leq w < 2$ and for the second integral we have $0 \leq w - t \leq 1$. Hence

$$f_W(w) = \int_0^{w-1} (2-(w-t))dt + \int_{w-1}^1 (w-t)dt$$

$$= ((2-w)t + t^2/2)\Big|_0^{w-1} + (wt - t^2/2)\Big|_{w-1}^1 = -w^2 + 3w - 3/2$$

While $2 \leq w < 3$ we have

$$f_W(w) = \int_0^1 f_{X_1}(t) f_{X_2+X_3}(w-t)dt$$

$$= \int_0^{w-2} f_{X_2+X_3}(w-t)dt + \int_{w-2}^1 f_{X_2+X_3}(w-t)dt$$

We again notice that in the first integral $w-t \geq 2$ which makes it 0, while in the second one $1 \leq w-1 \leq w-t \leq 2$, and hence

$$f_W(w) = \int_{w-2}^1 (2-(w-t))dt = ((2-w)t + t^2/2)\Big|_{w-2}^1$$

$= w^2/2 - 3w + 9/2$. The last case $w \geq 3$ and $0 \leq t \leq 1$ make $f_W(w) = 0$ since $w - t > 2$.

Hence the density of Z is given by

$$f_W(w) = \begin{cases} w^2/2 & \text{if } 0 \leq w < 1, \\ -w^2 + 3w - 3/2 & \text{if } 1 \leq w < 2, \\ w^2/2 - 3w + 9/2 & \text{if } 2 \leq w \leq 3, \\ 0 & \text{elsewhere} \end{cases}$$

For the last part, we compute

$$P(W \leq 2) = \int_0^1 \frac{w^2}{2} dw + \int_1^2 (-w^2 + 3w - 3/2) dw$$

$$= w^3/6 \Big|_0^1 + (-w^3/3 + 3/2w^2 - 3/2w) \Big|_1^2 = 5/6$$