# Runge-Kutta Methods

Runge-Kutta methods use weighted average of slopes instead of a single slope.

A general Runge-Kutta method is defined as

$$u_{j+1} = u_j + h\,[\text{weighted average of slopes on the given interval}]$$

Consider $n$ slopes in $[t_j, t_{j+1}]$:

$$k_1 = f(t_j + c_1 h,\; u_j + h a_{11} k_1 + h a_{12} k_2 + \cdots + h a_{1n} k_n)$$

$$k_2 = f(t_j + c_2 h,\; u_j + h a_{21} k_1 + h a_{22} k_2 + \cdots + h a_{2n} k_n)$$

$$\vdots$$

$$k_n = f(t_j + c_n h,\; u_j + h a_{n1} k_1 + h a_{n2} k_2 + \cdots + h a_{nn} k_n)$$

The method will be given as

$$u_{j+1} = u_j + h\,[w_1 k_1 + w_2 k_2 + \cdots + w_n k_n]$$

This is called $n$-stage fully implicit Runge-Kutta Method.

To formulate a Runge-Kutta Method we need:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$[c_1, c_2, \ldots, c_n]$$

$$[w_1, w_2, \ldots, w_n]$$

## Semi-implicit methods:

The upper triangular part of $A$ is zero.

$$k_1 = f(t_j + c_1 h,\ u_j + h\, a_{11} k_1)$$

$$k_2 = f(t_j + c_2 h,\ u_j + h\, a_{21} k_1 + h\, a_{22} k_2)$$

$$\vdots$$

$$k_n = f(t_j + c_n h,\ u_j + h\, a_{n1} k_1 + \cdots + h\, a_{nn} k_n)$$

$$u_{j+1} = u_j + h\,[w_1 k_1 + w_2 k_2 + \cdots + w_n k_n]$$

## Explicit method:

The upper triangular part including diagonal is zero.

$$k_1 = f(t_j + c_1 h,\ u_j)$$

$$k_2 = f(t_j + c_2 h,\ u_j + h\, a_{21} k_1)$$

$$\vdots$$

$$k_n = f(t_j + c_n h,\ u_j + h\, a_{n1} k_1 + \cdots + h\, a_{n,n-1} k_{n-1})$$

$$u_{j+1} = u_j + h\,[w_1 k_1 + w_2 k_2 + \cdots + w_n k_n]$$

# Explicit Runge-Kutta Methods (DERIVATION)

Consider the Runge-Kutta Method with two slopes:

$$K_1 = f(t_j, u_j)$$

$$K_2 = f(t_j + c_2 h,\ u_j + h a_{21} K_1)$$

$$u_{j+1} = u_j + h[\omega_1 K_1 + \omega_2 K_2] \quad \text{--- (1)}$$

where the parameters $c_2, a_{21}, \omega_1, \omega_2$ will be determined so that the error $u_{j+1} - y(t_{j+1})$ becomes small.

First, we write the Taylor's series of the solution

$$y(t_{j+1}) = y(t_j) + h\, y'(t_j) + \frac{h^2}{\underline{|2}}\, y''(t_j) + \frac{h^3}{\underline{|3}}\, y'''(t_j) + \cdots\cdots \quad \text{--- (2)}$$

where

$$y' = f(t, y)$$

$$y'' = f_t + f f_y$$

$$y''' = f_{tt} + 2 f f_{ty} + f_{yy} f^2 + f_y (f_t + f f_y)$$

Expand $K_2$ about $(t_j, u_j)$:

$$K_2 = f(t_j, u_j) + c_2 h f_t + h a_{21} f_j f_y + \frac{1}{\underline{|2}} h^2 \big( c_2^2 f_{tt} + 2 c_2 a_{21} f_j f_{ty} + a_{21}^2 f_j^2 f_{yy} \big) + \cdots$$

Substituting $K_1$ and $K_2$ in (1)

$$u_{j+1} = u_j + h\Big[\omega_1 f_j + \omega_2 \Big\{ f_j + h(c_2 f_t + a_{21} f_j f_y) + \frac{h^2}{2}\big(c^2 f_{tt} + 2 c_2 a_{21} f_j f_{ty} + a_{21}^2 f_j^2 f_{yy}\big)\Big\}\Big]$$

$\Rightarrow$

$$u_{j+1} = u_j + (\omega_1+\omega_2) f_j h + (\omega_2 c_2 f_t + \omega_2 a_{21} f_j f_y) h^2 + + \frac{h^3}{2}\omega_2\left(c_2^2 f_{tt} + 2c_2 a_{21} f_j f_{ty} + a_{21}^2 f_j^2 f_{yy}\right) + \cdots \quad \text{—③}$$

Comparing ② & ③

$$\omega_1 + \omega_2 = 1$$

$$\omega_2 c_2 = \frac{1}{2}$$

$$\omega_2 a_{21} = \frac{1}{2}$$

If $c_2$ is chosen arbitoarily then

$$\omega_2 = \frac{1}{2c_2}\ ; \qquad a_{21} = c_2$$

$$\omega_1 = 1 - \frac{1}{2c_2} \qquad 0 \le c_2 \le 1$$

Now ③ becomes:

$$u_{j+1} = u_j + \underbrace{h f_j + \frac{h^2}{2}(f_t + f_j f_y)}_{h\Phi(t_j, y_j, h)} + \frac{h^3}{4} c_2 \left(f_{tt} + 2 f_j f_{ty} + f_j^2 f_{yy}\right) + \cdots$$

TRUNCATION ERROR: $\tau_{j+1} = y(t_{j+1}) - y(t_j) - h\Phi(t_j, y(t_j), h)$

$$\Rightarrow \tau_{j+1} = \frac{h^3}{6}\left(f_{tt} + 2ff_{ty} + f_{yy}f^2 + f_y(f_t + ff_y)\right)_{t=t_j} - \frac{h^3}{4}c_2\left(f_{tt} + 2f(t_j)f_{ty} + f(t_j)^2 f_{yy}\right) + \cdots$$

$$= h^3\left[\left(\frac{1}{6} - \frac{c_2}{4}\right)\left(f_{tt} + 2ff_{ty} + f^2 f_{yy}\right)_{t=t_j} + \frac{1}{6} f_y (f_t + ff_y)_{t=t_j}\right] + \cdots$$

ORDER OF THE METHOD = 2

## Special Cases:

**1:** $c_2 = \frac{1}{2}$ ; $w_2 = 1$ $w_1 = 0$ $a_{21} = \frac{1}{2}$

Method:

$$k_1 = f(t_j, u_j)$$

$$k_2 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} k_1\right)$$

$$u_{j+1} = u_j + h k_2$$

Coeff. in Table form:

$$\begin{array}{c|cc} c_2 & a_{21} & \\ \hline & w_1 & w_2 \end{array} \qquad \begin{array}{c|cc} \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$$

This method is called <u>Modified Euler-Cauchy</u> method.

**2:** $c_2 = 1$ $a_{21} = 1$ $w_1 = \frac{1}{2}$ $w_2 = \frac{1}{2}$

Method:

$$k_1 = f(t_j, u_j)$$

$$k_2 = f(t_j + h, u_j + h k_1)$$

$$u_{j+1} = u_j + h\left(\frac{k_1 + k_2}{2}\right)$$

In Table form:

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

This method is called as Euler-Cauchy method.