Functional Analysis
Problem Sheet 1

August 22, 2019

1 Metric Spaces

1. Determine whether d is a metric on the set $\mathbb{R}$ of all real numbers in the following cases.
   (a) $d(x, y) = (x - y)^2$ for all $x, y \in \mathbb{R}$.
   (b) $d(x, y) = \sqrt{|x - y|}$ for all $x, y \in \mathbb{R}$.

2. If A is the subspace of l^∞ consisting of all sequences of zeros and ones, what is the induced metric on A?

3. Let X be the set of all ordered triples of zeros and ones. Show that X consists of eight elements and a metric d on X is defined by

$$d(x, y) = \text{number of places where } x \text{ and } y \text{ have different entries}$$

   for all $x, y \in X$.

4. (a) Find a sequence which converges to 0, but is not in any space l^p for $1 \leq p < \infty$.
   (b) Find a sequence which is in l^p for all $p > 1$ but not in l^1.

5. Let (X, d) be a metric space. Define

$$D(A, B) = \inf_{a \in A, b \in B} d(a, b)$$

   for all $A, B \subseteq X$. Show that D is not a metric on the power set of X.

6. Show that the image of an open set under a continuous mapping need not be open.

7. Show that a metric space (X, d) is separable if and only if X has a countable subset Y such that for every $\epsilon > 0$ and every $x \in X$, there is a $y \in Y$ such that $d(x, y) < \epsilon$.

8. (a) Let (X, d) be a metric space. Suppose that there exists an uncountable subset Z of X such that the topology on Z generated by the induced metric is the discrete topology. Show that (X, d) is not separable.
   (b) Show that for $a, b \in \mathbb{R}$ with $a < b$, the metric space $B[a, b]$ is not separable.

9. Let (x_n) be a sequence in X and let $x \in X$. Show that if every subsequence of (x_n) has a subsequence that converges to x, then (x_n) converges to x.

10. Show that a Cauchy sequence which has a convergent subsequence is convergent.

11. If d_1 and d_2 are metrics on the same set X and there are positive numbers a and b such that for all $x, y \in X$,
$$ad_1(x, y) \leq d_2(x, , y) \leq bd_1(x, y),$$
then prove the following.
    (a) A subset A of X is open in (X, d_1) if and only if it is open in (X, d_2).
    (b) A sequence (x_n) in X is Cauchy in (X, d_1) if and only if it is Cauchy in (X, d_2).
12. Determine whether $(\mathbb{N}, d)$ is a complete metric space in the following cases.
    (a) $d(m, n) = |m - n|$ for all $m, n \in \mathbb{N}$.
    (b) $d(m, n) = |\frac{1}{m} - \frac{1}{n}|$ for all $m, n \in \mathbb{N}$.
13. Show that $C[0, 1]$ and $C[a, b]$ are isometric.
14. Let d be a metric on a nonempty set X and $\overline{d} = \frac{d}{1+d}$. Show that (X, d) is complete if and only if $(X, \overline{d})$ is complete.
15. Let $\tilde{X}$ be the set of all Cauchy sequences in a metric space (X, d).
    (a) Show that $(d(x_n, y_n))$ converges for $(x_n), (y_n) \in \tilde{X}$.
    (b) Let $(x_n), (y_n) \in \tilde{X}$ such that $(d(x_n, y_n))$ converges to 0 and let $z \in X$. Show that (x_n) converges to z if and only (y_n) converges to z.
    (c) Define a relation $\sim$ on $\tilde{X}$ such that for $(x_n), (y_n) \in \tilde{X}$, $(x_n) \sim (y_n)$ if and only if $(d(x_n, y_n))$ converges to 0. Show that $\sim$ is an equivalence relation.

2 Normed spaces, Banach spaces

1. (a) Let $(X, \|\cdot\|)$ be a normed space. Show that if $X \neq \{0\}$, then $\{\|x\| : x \in X\} = [0, \infty)$.
    (b) Let d be a metric on a vector space $X \neq \{0\}$ and define
$$\overline{d}(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ d(x, y) + 1 & \text{if} \quad x \neq y \end{cases}$$
    for all $x, y \in X$. Show that $\overline{d}$ is a metric on X that cannot be obtained from any norm on X.
2. Show that a subset A of a normed space is bounded if and only if there exists $m > 0$ such that $\|x\| \leq m$ for all $a \in A$.
3. (a) Let X be a normed space and let $Y \neq \{0\}$ be a subspace of X. Show that Y is unbounded.
    (b) Show that there exists no norm on a vector space X which induces the discrete metric on X.
4. Let Y be a subspace of a vector space X. For $x \in X$, let $x + Y = \{x + y : y \in Y\}$. Show that for any $a, b \in X$, either $a + Y = b + Y$ or $a + Y \cap b + Y = \emptyset$
5. For a subspace Y of a vector space X, let $X/Y = \{x + Y : y \in Y\}$. Show that under the algebraic operations defined by
$$(a + Y) + (b + Y) = a + b + Y$$
$$\alpha(a + Y) = \alpha a + Y$$
for any $a, b \in X$ and scalar α, X/Y is a vector space.
6. Let c = set of all convergent sequences of real numbers, c_0 = set of all sequences of real numbers converging to 0 and c_{00} = set of all sequences of real numbers with only finitely many nonzero terms.

(a) Show that c is a closed subspace of l^∞.

(b) Show that c_0 is a closed subspace of c.

(c) Show that c_{00} is not closed in c, c_0 and l^p for any $1 \leq p \leq \infty$.

7. Show that if a normed space has a Schauder basis, it is separable.

8. (a) Let $\{x_n\}$ be a Cauchy sequence in X. Show that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that the sequence $\sum_{n=1}^{\infty} \|x_{n_k} - x_{n_{k+1}}\| < \infty$.

   (b) Show that a normed space X is complete if and only if every absolutely convergent series is convergent.

9. Let Y be a subspace of a normed space $(X, \|\cdot\|)$. Define $\|x+Y\|_0 = \inf\{\|x+y\| : y \in Y\}$ for all $x \in X$.

   (a) Show that $\|x+Y\|_0 = D(x,Y)$.

   (b) Show that Y is closed if and only if $\|\cdot\|_0$ is a norm on X/Y.

10. Show that equivalent norms on a vector space X induce the same topology on X.

11. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent norms on a vector space X. Show that $(X, \|\cdot\|_1)$ is complete if and only if $(X, \|\cdot\|_2)$ is complete.

12. (a) Let $x_1, \ldots, x_n$ be linearly independent elements of a normed space X and let $x_{n+1} \in X$. Show that $x_1, \ldots, x_{n+1}$ are linearly independent if and only if $D(x_{n+1}, span\{x_1, \ldots, x_n\}) > 0$.

    (b) Let Y be a finite-dimensional proper subspace of a normed space X. Show that there exists $x \in X$ such that $\|x\| = D(x,Y) = 1$.

    (c) Show that if X is an infinite-dimensional normed space, then $\{x \in X : \|x\| \leq 1\}$ is not compact.

13. Let X, Y be vector spaces, let $T : X \to Y$ be a linear operator and let $x_1, \ldots, x_n \in X$..

    (a) Show that the image of a subspace of X is a subspace of Y, and the inverse image of a subspace of Y is a subspace of X.

    (b) Show that if $Tx_1, \ldots, Tx_n$ are linearly independent, then $x_1, \ldots, x_n$ are linearly independent.

    (c) Show that if T is injective and $x_1, \ldots, x_n$ are linearly independent in X, then $Tx_1, \ldots, Tx_n$ are linearly independent.

14. Let X, Y be normed spaces and let $T : X \to Y$ be a linear operator. Show that T is bounded if and only if it maps bounded sets into bounded sets.

15. Let X, Y be normed spaces and let $T : X \to Y$ be a bounded linear operator. If $T \neq 0$, show that for $x \in X$ such that $\|x\| < 1$, we have $\|Tx\| < \|T\|$.

16. For $(x_1, x_2, x_3, \ldots) \in l^\infty$, let

$$T(x_1, x_2, x_3, \ldots) = \left(\frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right).$$

    (a) Show that $T : l^\infty \to l^\infty$ is an injective bounded linear operator.

    (b) Show that $\mathcal{R}(T)$ is not closed in l^∞.

    (c) Show that $T^{-1} : \mathcal{R}(T) \to l^\infty$ is not bounded.

17. Let T be a bounded lineat operator from a normed space X onto a normed space Y. If there is $b > 0$ such that $\|Tx\| \geq b\|x\|$ for all $x \in X$, show that $T^{-1} : Y \to X$ exists and is bounded.

18. Let $f \neq 0$ be a linear functional on a vector space X. Show that $\mathcal{R}(f)$ is the scalar field of X.

19. (a) Let $f \neq 0$ be a linear functional on a vector space X. Fix $x_0 \notin \mathcal{N}$. Show that for every $x \in X$, there exists a unique pair (α, y) such that α is a scalar and $y \in \mathcal{N}(f)$ and $x = \alpha x_0 + y$.

    (b) Let f_1, f_2 be nonzero linear functional on a vector space X. Show that $\mathcal{N}(f_1) = \mathcal{N}(f_2)$, if and only if f_1 and f_2 are linearly dependent.

20. Let $f \neq 0$ be a linear functional on a vector space X. Show that

$$\inf\{\|x\| : f(x) = 1\} = \frac{1}{\|f\|}.$$