Mas

MA 31005: Real Analysis

End Semester Examination (Autumn 2015)

Time: 3 Hours, Full Marks: 50, Number of students =61

Answer **all** the problems. Numbers at the right hand side after each question denote marks.

(1) Prove that if f is continuous at $x = a$, then $|f|$ is continuous at $x = a$. Prove or disprove whether the converse of the aforementioned statement is true or not. [5]

(2) Show that the function $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$ but it is uniformly continuous on (a, ∞) where $a > 0$. [5]

(3) Let $f : [0, 1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Prove that there exist a point $c \in [0, 1/2]$ such that $f(c) = f(c + 1/2)$. [5]

(4) Suppose f is differentiable on $[a, b]$, with $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. [5]

(5) Let there be a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t. Prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and $x_{n+1} = f(x_n)$ for $n = 1, 2, 3, \cdots$. [5]

(6) If a function f is continuous on $[0, 1]$, prove that

$$\lim_{n\to\infty} \int_0^1 \frac{nf(x)}{1 + n^2x^2}\, dx = \frac{\pi f(0)}{2}. \qquad [5]$$

(7) Determine whether the integral $\int_0^{\pi/4} \frac{1}{\sqrt{\tan x}}$ converges or diverges. Find the Cauchy Principal value of the integral $\int_{-1}^{4} \frac{dx}{(x-1)^3}$ [3+2]

(8) If f is a continuous function on $[a, b]$ and $\int_a^b f(x)g(x)\, dx = 0$ for every continuous function g on $[a, b]$, then prove that $f \equiv 0$ on $[a, b]$. Show the details of your work. [5]

(9) Prove that $f_n(x) = \left\{\frac{x}{nx+1}\right\}$ converges pointwise for $x \in [0, \infty)$ and uniformly for $x \in [0, k]$ where k is a positive number. [5]

(10) Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2} \qquad [5]$$

converges uniformly in every bounded interval but not absolutely for any x.