Indian Institute of Technology Kharagpur

Date:........ FN/AN, Time: 2 Hrs, Full Marks: 30, Deptt: Mathematics
No. of Students: 77, Mid Autumn Semester Examination: 2012–2013,
Sub. No. MA31005, Sub. Name: Real Analysis, 2^{nd} and 3^{rd} year B.Tech/M.Sc.

Instructions:
(i) Answer all questions.
(ii) All the parts of the same question should be done at one place.

Question 1. [4+1]

a) Let (X, d) be a metric space. Define

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that ρ is a metric on X. Are ρ and d equivalent metrics on X? Justify your answer.

b) Is the set of complex numbers $\mathbb{C}$ countable? Justify your answer.

Question 2. [2+2+1]

a) Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$ be a sequence of real numbers. Test the convergence of the sequence $\{S_{10^n-1}\}$.

b) Prove or disprove that "countable union of countable sets is countable."

c) Prove or disprove that "in any metric space, arbitrary union of closed sets is closed."

Question 3. [3+2]

a) Let X be a metric space and $A \subseteq X$. The show that $\overline{A} = A \cup D(A)$.

b) Let X be a metric space and $A, B \subseteq X$. Prove or disprove that $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Question 4. [3+2]

a) Prove that "any compact subset of a metric space is closed". Does the converse of this statement true? Justify your answer.

b) In $[0, 1]$ with usual metric find $B(\frac{1}{2}, 1)$ and $B(\frac{1}{4}, \frac{1}{4})$.

Question 5. [3+2]

a) The Fibonacci numbers $x_1, x_2, \ldots,$ are defined recursively by

$$x_1 = 1, \ x_2 = 2, \quad \text{and} \ \ x_{n+1} = x_n + x_{n-1} \quad \text{for } n \geq 2.$$

Show that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists, and evaluate the limit.

b) Evaluate $\lim_{n\to\infty} \cos\frac{\pi}{2^2} \cos\frac{\pi}{2^3} \cdots \cos\frac{\pi}{2^n}.$

Question 6. [2+2+1]

a) Evaluate

$$\lim_{n\to\infty} \left(\frac{(n+1)(n+2)\cdots(n+n)}{n^n}\right)^{\frac{1}{n}}.$$

b) Prove that the sequence

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

is convergent.

c) Test the convergence of the sequence

$$b_n = \frac{n!}{n^n}.$$

***************** THE END *****************