INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR

Date———FN/AN 3 Hrs. Full Marks: 50 No. of Students 66

End Spring Semester 2011-2012 Deptt: MATHEMATICS Sub No: MA 50002/ MA 51002

——Yr. B.Tech.(H)/B.Arch.(H)/M.Sc. Sub. Name: Measure Theory & Integration

Instruction: Answer all questions, which are of equal values

1. (a) Show that if f is absolutely continuous, then f has a derivative almost everywhere.

   (b) Let g be a nonnegative measurable function on $[0,1]$. Then show that $\ln \int g(t)\,dt \geq \int \ln(g(t))\,dt$, whenever the right hand side is defined.

2. (a) Let ϕ have a second derivative at each point of (a,b). Show that ϕ is convex iff $\phi''(x) \geq 0$ $\forall x \in (a,b)$.

   (b) Let $f(x) = |x|$. Find D^+f, D_+f, D^-f, and D_-f at $x = 0$.

3. (a) Define f on $[0,1]$ by $f(0) = 0$, $f(x) = x^p \sin(\frac{1}{x})$ for $x > 0$, where $p \geq 2$. Show that f is of bounded variation on $[0,1]$.

   (b) State and prove Holder's inequality.

4. (a) State and prove Jordan decomposition theorem.

   (b) Let $\nu(E) = \int_E xe^{-x^2}\,dx$. Find the positive, negative and null sets with respect to ν.

5. (a) Show that if $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}, (x,y) \neq (0,0)$, then $\int_0^1 \int_0^1 f(x,y)\,dydx$ does not exist but $\int_0^1 dx \int_0^1 f(x,y)\,dy = \frac{\pi}{4}$ and $\int_0^1 dy \int_0^1 f(x,y)\,dx = -\frac{\pi}{4}$. Does this contradict Fubini's theorem? Justify your result.

   (b) Let μ and ν be complete measures. Show that $\mu \times \nu$ need not be complete.

6. (a) Let $f \in L(0,a)$ and $g(x) = \int_x^a \frac{f(t)}{t}\,dt$ $(0 < x \leq a)$. Then show that $g \in L(0,a)$ and $\int_0^a g\,dx = \int_0^a f\,dx$.

   (b) If f is integrable on $[a,b]$ and $\int_a^x f(t)\,dt = 0$ $\forall x \in [a,b]$, show that $f(t) = 0$ a.e. in $[a,b]$.

7. (a) Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and for almost all x in E we have $f(x) = \lim_{n\to\infty} f_n(x)$. Then show that $\int_E f = \lim_{n\to\infty} \int_E f_n$.

   (b) If f is an integrable function on $(-\infty,\infty)$, show that $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\cos nx\,dx = 0$.

8. (a) Show that if $\{x : f(x) = \alpha\}$ is measurable, then f is not necessarily measurable.

   (b) Show that if f is measurable function and $f = g$ a.e., then g is measurable.

9. (a) Show that every Borel set is measurable.

   (b) Let f be absolutely continuous on $[a,b]$, then show that $T_a^b(f) = \int_a^b |f'|$.

10. (a) Show that $||f+g||_\infty \leq ||f||_\infty + ||g||_\infty$.

    (b) Show that we may have strict inequality in Fatou's lemma.