Indian Institute of Technology Kharagpur
Departments: MA, MF, CS and EC.
MA41002 / MA30002 Modern Algebra
Spring End Semester Examination, 2016 No. of Students: 85
Full Marks: 50, Time: 3 Hrs.

INSTRUCTION: Answer all the questions. Each question carries equal marks.

1. Let $R = M_2(\mathbb{R})$, the ring of all 2-by-2 matrices with real co-efficients.

   (a) Define a subset $S \subseteq R$ by

   $$S = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

   Verify that S is a subring of R, and that $S^{\times} = S - \{0\}$.

   (b) Define $\phi : \mathbb{C} \to M_2(\mathbb{R})$ by $\phi(a + bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Prove that ϕ is a ring isomorphism of $\mathbb{C}$ onto the subring S defined in part (a).

   (2+3 = 5 marks)

2. (a) For $n \geq 3$, show that S_n is a non-abelian group.

   (b) If G is a finite group of prime order p, then show that G is cyclic.

   (3+2 = 5 marks)

3. (a) Let G be a finite group. Then show that every element $a \in G$ has finite order. If G is an infinite group, can we have elements of finite order?

   (b) Let R be a commutative ring with identity. Show that if $u \in R$ is a unit, then u is not a zero divisor. Therefore, any field is necessarily an integral domain.

   (3+2 = 5 marks)

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4. (a) If $\phi : G_1 \to G_2$ is an isomorphism, then show that $o(\phi(a)) = o(a), \ \forall\, a \in G_1$.

   (b) Find the cycle decomposition and order of the following permutation
   $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$.

(3 + 2 = 5 marks)

5. (a) Show that disjoint cycles commute. That is, if $\sigma, \tau \in S_n$ are disjoint cycles, then $\sigma\tau = \tau\sigma$.

   (b) Let G be a group of order 49. Show that G must have a subgroup of order 7.

(3+2 = 5 marks)

6. (a) In the following statement, either give an example which has the given property, or explain why no such example exists:
   " A ring R and a ring homomorphism $\phi : \mathbb{Q} \to R$ such that $\ker \phi = \mathbb{Z}$."

   (b) Let G be a group with N a normal subgroup of G, and define a function $\pi : G \to G/N$ by $\pi(g) = Ng$, for all $g \in G$. Prove that π is a homomorphism, and that $\ker \pi = N$.

(2+3 = 5 marks)

7. (a) Show that a permutation $\sigma \in S_n$ cannot be both even and odd.

   (b) Prove that if G is cyclic, then G/H is also cyclic.

(3+2 = 5 marks)

8. State and prove the First Isomorphism Theorem for groups.

(5 marks)

9. (a) Determine all abelian groups of order 600, upto isomorphism.

   (b) Define a ring homomorphism with an example.

(3+2 = 5 marks)

10. (a) Give definitions for Field extension and splitting field? What is the splitting field of $f(x) = x^4 - 5x^2 + 6$?

    (b) Let R be a finite integral domain with identity $1 \in R$. Show that R is actually a field.

(2+3 = 5 marks)

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