DIFFERENTIAL EQUATIONS:

An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Examples:

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^3 = e^t \qquad \text{— (i)}$$

$$y(y^2+1)\,dx + x(y^2-1)\,dy \qquad \text{— (ii)}$$

$$\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2 \qquad \text{— (iii)}$$

Mathematical classifications:

ORDINARY DIFF. EQUATIONS: → Involves derivatives w.r.t single independent variables

PARTIAL DIFF. EQUATIONS: → Involves partial derivatives (more than one independent variables

ORDER OF A DIFFERENTIAL EQUATION: → The order of the highest order derivative involved

DEGREE OF A DIFFERENTIAL EQUATION: → The degree of the highest order derivative involved.

i) ODE, order - 4 degree 1

ii) ODE, order - 1 degree 1

iii) PDE, order - 3 degree 2.

LINEAR AND NONLINEAR DIFFERENTIAL EQUATION:

A differential equation is called <u>linear</u> if

i) every dependent variable and every derivative occur in the first degree only, and

ii) no products of dependent variables and/or derivatives occur.

If not linear than it is called nonlinear.

Note: Every linear equation is of first degree, but every first degree equation may not be linear.

$$\frac{d^2y}{dx^2} + y \cdot \frac{dy}{dx} + y = 0 \qquad \text{1st degree but nonlinear}$$

SOLUTION OF A DIFFERENTIAL EQUATION:

Any relation between the dependent and independent variables which satisfies the differential equation is called a solution or integral of the differential equation.

Ex. $y = \frac{A}{x} + B$ is a solution of

$$y'' + \left(\frac{2}{x}\right) y' = 0$$

Check! $\quad y' = -\frac{A}{x^2} \Rightarrow y'' = \frac{2A}{x^3}$

Subst. in the equation: $\underline{0 = 0}$.

Note: It should be noted that a solution of a differential equation does not involve the derivatives of the dep. variable w.r.t the indep. variable or variables.

Family of curves: An n-parameter family of curves is a set of relations of the form

$$\{(x,y) : f(x,y,c_1,c_2,\cdots c_n)=0\}$$

Example:

i) set of concentric circles

$x^2+y^2=c$ → one parameter family if c takes non-negative real values

ii) Set of circles:

$(x-c_1)^2+(y-c_2)^2=c_3$ → three parameters family if c_1, c_2 takes all real values and c_3 takes all non-negative real values.

Note: Solution of a differential equation is a family of curves.

Formation of differential equations from a given n-parameters family of curves:

From a given family of curves containg n arbitrary constants, we can obtain an nth order differential equation whose solution is the given family:

- Differentiate the given equation n times to get n additional equations containing those arbitrary constants.
- Eliminate n arbitrary constants from the $(n+1)$ equations.
- Obtain a differential equation of the nth order.

Ex: Obtain the differential equation satisfied by

$$xy = ae^{x} + be^{-x} + x^2$$

where a & b are arbitrary constants.

Sol: Given family of curves:

$$xy = ae^{x} + be^{-x} + x^2 \quad \text{—(1)}$$

Differentiating w.r.t x, we get

$$xy' + y = ae^{x} - be^{-x} + 2x$$

Differentiating again:

$$xy'' + 2y' = ae^{x} + be^{-x} + 2$$

Using (1) we get

$$xy'' + 2y' = xy - x^2 + 2$$

which is the desired differential equation.

Remark: Observe that the number of arbitrary constants in a solution of a differential equation depends upon the order of the differential equation. It is evident from the above example that a general solution (defined later) of an nth order differential equation will contain n arbitrary constants.

General, particular, and Singular Solution

Let $F(x, y, y', y'', \cdots y^{(n)}) = 0$ be an nth order ordinary differential equation.

i) General solution: solution containing n-independent arbitrary constants.

ii) Particular solution: Solution by giving particular values to one or more of the n-independent constants.

iii) Singular solution: cannot be obtained by any choice of independent arbitrary constant.

Example: a) $y = (x+c)^2$ is the general solution of

$$\left(\frac{dy}{dx}\right)^2 - 4y = 0 \quad -①$$

b) $y = x^2$ is a particular solution of ① $(c=0)$

c) $y = 0$ is a singular solution.

Ex: Consider $yy' - x(y')^2 = 1$

General solution: $y = cx + \frac{1}{c}$

Particular solution: $y = x + 1 \quad (c=1)$

Singular solution: $y^2 = 4x$.

Explicit & Implicit solutions:

Explicit : $y = y(x)$

Implicit : $F(x, y) = 0$

Example: $y'' + k^2 y = 0$

solution: $y = C_1 \cos kx + C_2 \sin kx$

↳ explicit solution

Example: $x + 3yy' = 0$

Solution: $x^2 + 3y^2 = C$

↳ Implicit solution

Equation of first order and first degree :

We shall consider two standard forms of differential equation:

i) $\frac{dy}{dx} = f(x, y)$

ii) $M(x, y)\, dx + N(x, y)\, dy = 0$.

Solution methods:

- **Separation of variables:** If a differential equation can be written in the form

$$f_1(y)\,\frac{dy}{dx} = f_2(x) \qquad \text{— ①}$$

then we say variables are separable in the given differential equation.

Solution of ①:

$$\int f_1(y)\,dy = \int f_2(x)\,dx + C \quad \text{(how?)}$$

Example: $\dfrac{dy}{dx} = e^{x-2y} + x^2 e^{-2y}$

$$\Rightarrow\ e^{2y}\frac{dy}{dx} = e^x + x^2$$

Integrating both side:

$$\frac{e^{2y}}{2} = e^x + \frac{x^3}{3} + c_1$$

or $\boxed{e^{2y} = 2e^x + \frac{2}{3}x^3 + C}$

Equation reducible to separation of variables:

Consider $\frac{dy}{dx} = f(ax+by+c)$ —①

OR $\frac{dy}{dx} = f(ax+by)$

Subst. $ax+by+c = v$ OR $ax+by = v$

$\Rightarrow a + b\cdot\frac{dy}{dx} = \frac{dv}{dx}$

Then (1) reduces to

$$\frac{1}{b}\left[\frac{dv}{dx} - a\right] = f(v)$$

$$\Rightarrow \frac{dv}{dx} = bf(v) + a$$

$$\Rightarrow \int \frac{dv}{bf(v)+a} = \int dx$$

Example: $\frac{dy}{dx} = \sec(x+y)$

Sol: Let $x+y = v \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Then the diff. eq. becomes:

$$\frac{dv}{dx} = \sec v + 1 \qquad \text{(separable form)}$$

$$= \frac{1+\cos v}{\cos v} = \frac{2\cos^2\frac{v}{2}}{2\cos^2\frac{v}{2} - 1}$$

$$\Rightarrow \int \left[1 - \frac{1}{2}\sec^2\left(\frac{v}{2}\right)\right] dv = \int dx$$

$$\Rightarrow v - \tan\left(\frac{v}{2}\right) = x + c$$

Subst. $v = x+y$: $\boxed{y - \tan\left(\frac{x+y}{2}\right) = c}$

Homogeneous equations: A differential equation of first order and first degree is said to be homog. if it is of the form or can be put in the form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad — (1)$$

<u>Solution:</u> $\frac{y}{x} = v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$(1) \Rightarrow v + x\frac{dv}{dx} = f(v)$$

$$\Rightarrow x\frac{dv}{dx} = f(v) - v \quad \text{(separable form)}$$

$$\Rightarrow \int \frac{dv}{f(v)-v} = \int \frac{dx}{x} + c.$$

Example: $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$

<u>Sol:</u> $\frac{dy}{dx} = -\frac{x^3 + 3xy^2}{y^3 + 3x^2y} = -\frac{1 + 3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3 + 3\left(\frac{y}{x}\right)}$

Subst. $\frac{y}{x} = v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{1+3v^2}{v^3+3v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^4 - 6v^2 - 1}{v^3 + 3v}$$

$$\Rightarrow -\int \frac{4(v^3+3v)}{v^4+6v^2+1} \cdot dv = \int 4\cdot\frac{dx}{x}$$

$$\Rightarrow -\ln(v^4 + 6v^2 + 1) = 4\ln x + \ln c \qquad (x>0)$$

$$\Rightarrow x^4 c(v^4 + 6v^2 + 1) = 0$$

$$\Rightarrow c x^4\left(\frac{y^4}{x^4} + 6\frac{y^2}{x^2} + 1\right) = 0$$

$$\Rightarrow \boxed{c(y^4 + 6y^2x^2 + x^4) = 0}$$

Equation reducible to homogeneous form:

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \quad , \text{ where } \frac{a}{a'} \neq \frac{b}{b'} \ (*) \qquad \text{—} ①$$

Take $\left.\begin{array}{l} x = X+h \\ y = Y+K \end{array}\right\} \text{ — } ②$

Where X & Y are new variables and h & K are constants to be so chosen that the resulting equation in X and Y becomes homogeneous

$② \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dY}{dX}$

$$\left[\begin{array}{l} y(x) = Y(X) + K \\ \text{or } Y(X) = y(x) + K \\ \dfrac{dY}{dX} = \dfrac{d}{dX}\left[y(x)+K\right] = \dfrac{dy}{dx}\cdot\underbrace{\dfrac{dx}{dX}}_{=1} \end{array}\right]$$

$① \Rightarrow$

$$\frac{dY}{dX} = \frac{aX+bY+ah+bK+c}{a'X+b'Y+a'h+b'K+c'} \qquad \text{—} ③$$

In order to make ③ homog. choose h and K such that

$\left.\begin{array}{l} ah+bK+c = 0 \\ a'h+b'K+c' = 0 \end{array}\right\}$ and (alway possible because $ab'-a'b \neq 0$

Getting h & K we have $X = x-h$ & $Y = y-K$

$$\Rightarrow \frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y} = \frac{a+b\left(\frac{Y}{X}\right)}{a'+b'\left(\frac{Y}{X}\right)} \qquad \text{homogeneous in } X \text{ \& } Y$$

(*) In case $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{1}{\lambda} \Rightarrow a' = \lambda a$ & $b' = \lambda b$

Subst. $\dfrac{dy}{dx} = \dfrac{ax+by+c}{\lambda(ax+by)+c'} = f(ax+by)$ (Can be solved by subst $ax+by = u$)

Ex: $\dfrac{dy}{dx} = \dfrac{x+2y-3}{2x+y-3}$ —①

Sol. Take $x = X+h$ & $y = Y+K$ so that $\dfrac{dy}{dx} = \dfrac{dY}{dX}$

$$\frac{dY}{dX} = \frac{X+2Y+(h+2K-3)}{2X+Y+(2h+K-3)} \quad \text{—②}$$

Choose h, K so that $\left.\begin{array}{l} h+2K-3=0 \\ 2h+K-3=0 \end{array}\right\} \Rightarrow h=1$ & $K=1$.

So from ① $X = x-1$ $Y = y-1$

② $\Rightarrow$ $\dfrac{dY}{dX} = \dfrac{X+2Y}{2X+Y}$

Take $Y = vX \Rightarrow \dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$$X\frac{dv}{dX} = \left(\frac{1+2v}{2+v}\right) - v = \frac{1-v^2}{2+v}$$

$$\Rightarrow \frac{dX}{X} = \left[\frac{1}{2}\left(\frac{1}{1+v}\right) + \frac{3}{2}\left(\frac{1}{1-v}\right)\right] dv$$

Integrating:

$$\Rightarrow \ln X + \ln C = \frac{1}{2}\left[\ln(1+v) - 3\ln(1-v)\right]$$

$$\Rightarrow 2\ln(XC) = \ln\left(\frac{1+v}{(1-v)^3}\right) \Rightarrow X^2C^2 = \frac{1+v}{(1-v)^3}$$

sub: $v = \dfrac{y-1}{x-1}$

$$\boxed{C^2(x-y)^3 = x+y-2}$$