Simple Leibnitz rule

consider $I = \int_c^d f(x,y)\, dy$

let

1) $f(x,y)$ be continuous in the rectangle $R: \{(x,y): a \leq x \leq b,\ c \leq y \leq d\}$

2) $f_x(x,y)$ exists in R & is continuous in R,

If $g(x) = \int_c^d f(x,y)\, dy.$

then $\dfrac{dg(x)}{dx} = \int_c^d \dfrac{\partial f(x,y)}{\partial x}\, dy.$

Generalized Leibnitz rule

consider $I = \int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy.$

let

1) $f(x,y)$ be continuous in $R: \{(x,y): a \leq x \leq b,\ c \leq y \leq d,\}$

2) $f_x(x,y)$ exists in R & continuous in R.

3) $\alpha(x), \beta(x)$ are differentiable for $a \leq x \leq b.$

if $g(x) = \int_{\alpha(x)}^{\beta(x)} f(x,y)\, dy.$

$$\frac{dg(x)}{dx} = \int_{\alpha(x)}^{\beta(x)} \frac{\partial f(x,y)}{\partial x}\, dy + \beta'(x) f(x, \beta(x)) - \alpha'(x) f(x, \alpha(x))$$

Application of Leibnitz rule -

$$I(a) = \int_0^{\pi} \frac{\log(1+a\cos x)}{\cos x}\, dx.$$

$$\frac{dI}{da} = \int_0^{\pi} \frac{1}{(1+a\cos x)} \cdot \frac{\cos x}{\cos x}\, dx.$$

$$= \int_0^{\pi} \frac{dx}{(1+a\cos x)}.$$

$$\tan\frac{x}{2} = z.$$

$$\frac{1}{2}\sec^2\frac{x}{2}\, dx = dz$$

$$dx = \frac{2\,dz}{1+z^2}$$

$$\cos x = 2\cos^2\frac{x}{2} - 1.$$

$$= \frac{2}{\sec^2\frac{x}{2}} - 1$$

$$= \frac{2}{1+z^2} - 1.$$

$$\frac{dI}{da} = \frac{\pi}{\sqrt{1-a^2}}$$

$$I(a) = \int \frac{\pi}{\sqrt{1-a^2}}\, da + c$$

$$I(a) = \pi \sin^{-1} a + c$$

Note $I(0) = 0$.

$$\therefore\ 0 = \pi \sin^{-1} 0 + c \Rightarrow c = 0$$

$$I(a) = \pi \sin^{-1} a.$$

Ex
$$I(\alpha) = \int_0^{\infty} e^{-x}\left(\frac{1-\cos\alpha x}{x}\right) dx.$$

$$\frac{dI}{d\alpha} = \int_0^{\infty} \frac{\partial}{\partial\alpha}\left\{ e^{-x}\left(\frac{1-\cos\alpha x}{x}\right)\right\} dx.$$

$$= \int_0^{\infty} \frac{e^{-x}}{x}\, x\sin\alpha x\, dx$$

$$= \int_0^{\infty} e^{-x}\sin\alpha x\, dx = \frac{\alpha}{\alpha^2+1}$$

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$$\therefore\ I(\alpha) = \int \frac{\alpha \, d\alpha}{\alpha^2+1} + c.$$

$$I(\alpha) = \frac{1}{2} \ln |\alpha^2+1| + c.$$

$$I(0) = 0 \Rightarrow 0 = \frac{1}{2} \ln 1 + c$$

$$I(\alpha) = \frac{1}{2} \ln |\alpha^2+1|.$$

Ex $$J(a,b) = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx$$

$$= \int_0^\infty \frac{e^{-ax}}{x} dx - \int_0^\infty \frac{e^{-bx}}{x} dx.$$

$$= I(a) - I(b)$$

$$I(\alpha) = \int_0^\infty \frac{e^{-\alpha x}}{x} dx.$$

$$I(a) = -\ln a + c_1, \quad I(b) = -\ln b + c_2$$

$$J(a,b) = I(a) - I(b)$$

$$= -\ln a + c_1 + \ln b - c_2$$

$$= -\ln \frac{a}{b} + (c_1 - c_2)$$

$$J(a,a) = 0.$$

$$0 = J(a,a) = \underbrace{\ln \frac{a}{a}}_{=0} + (c_1 - c_2)$$

$$\Rightarrow c_1 - c_2 = 0.$$

So, $$J(a,b) = -\ln \frac{a}{b} = \ln \frac{b}{a}.$$

$$f(x,t) = \begin{cases} \dfrac{xt^3}{(x^2+t^2)^2} ; & x \neq 0,\ t \neq 0. \\ 0 ; & x = 0,\ t = 0. \end{cases}$$

Show that $\dfrac{d}{dt}\displaystyle\int_0^1 f(x,t)\,dx \neq \int_0^1 \frac{\partial f}{\partial t}(x,t)\,dx.$

Justify the answer.

$$\int_0^1 f(x,t)\,dx = \int_0^1 \frac{xt^3}{(x^2+t^2)^2}\,dx.$$

$$= -\frac{1}{2}\left[\frac{t^3}{x^2+t^2}\right]_0^1 = \frac{t}{2(1+t^2)}$$

$$\frac{d}{dt}\int_0^1 f(x,t)\,dx = \frac{1}{2}\,\frac{1+t^2-2t^2}{(1+t^2)^2} = \frac{1}{2}\,\frac{1-t^2}{(1+t^2)^2}.$$

Try to complete $\quad = \displaystyle\int_0$

Ex

$$I(\alpha^2) = \int_0^{\alpha^2}\left(\tan^{-1}\frac{y}{\alpha^2}\right)dy \Rightarrow I(p) = \int_0^p \tan^{-1}\frac{y}{p}\,dy$$

taking $\alpha^2 = p$

$$I'(p) = \int_0^p \frac{1}{1+\frac{y^2}{p^2}} \times \left(-\frac{y}{p^2}\right)dy + 1\cdot\tan^{-1}\frac{p}{p} - 0.$$

$$= -\int_0^p \frac{y\,dy}{p^2+y^2} + \frac{\pi}{4}$$

$$\Rightarrow I'(p) = -\frac{1}{2}\ln 2 + \frac{\pi}{4} \Rightarrow I(p) = \left(\frac{1}{2}\ln \tfrac{1}{2} + \pi/4\right)p + C.$$

Now, $I(0) = 0$, $\Rightarrow I(\alpha^2) = \left(\frac{1}{2}\ln \tfrac{1}{2} + \pi/4\right)\alpha^2.$

Beta - Gamma function

A. **Gamma function** -

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\, dx$$

B. **Beta function** -

$$\int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\, dx$$

$$= \int_0^{a(>0)} e^{-x} x^{\alpha-1}\, dx + \int_a^\infty e^{-x} x^{\alpha-1}\, dx = I_1 + I_2$$

$I_1 = \int_0^1 e^{-x} x^{\alpha-1}\, dx$; I_1 is proper if $\alpha \geq 1$.

let $\alpha < 1$, let $1-\alpha = m > 0$.

$I_1 = \int_0^1 \frac{e^{-x}}{x^m}\, dx$. $\quad f(x) = \frac{e^{-x}}{x^m}$; choose $g(x) = \frac{1}{x^m}$

Then, $\frac{f(x)}{g(x)} = e^{-x} \longrightarrow 1$ as $x \longrightarrow 0$

Now, $\int_0^1 \frac{dx}{x^m}$ $\rightarrow$ converges when $m < 1$
$\quad\hookrightarrow$ diverges when $m \geq 1$

Thus $\int_0^1 \frac{e^{-x}}{x^m}\, dx$ $\rightarrow$ converges when $m < 1$
$\quad\hookrightarrow$ diverges when $m \geq 1$.

$m = 1-\alpha$
$1-\alpha < 1$
$\Rightarrow \alpha > 0$
$1-\alpha \geq 1$
$\Rightarrow \alpha \leq 0$

$\Rightarrow \int_0^1 e^{-x} x^{\alpha-1}\, dx$ converges when $\alpha > 0$
diverges when $\alpha \leq 0$.

Thus, the I_1 is defined for $\alpha > 0$.
For $0 < \alpha < 1$, it is a convergent improper integral
For $\alpha \geq 1$, it is a proper integral.

$$I_2 = \int_a^\infty e^{-x} x^{\alpha-1}\, dx = \int_a^\infty \frac{e^{-x}}{x^{1-\alpha}}\, dx.$$

Now for comparison test let us consider $g(x) = \frac{1}{x^2}$.

then, $\frac{f}{g} = \frac{x^{\alpha+1}}{e^x} \to 0$ as $x \to \infty$ $\forall \alpha$

& $\int_a^\infty g(x)\, dx = \int_a^\infty \frac{dx}{x^2}$ converges.

$\therefore \int_a^\infty e^{-x} x^{\alpha-1}\, dx$ converges for all α.

Thus, $\Gamma(\alpha) = I_1 + I_2$ converges for $\alpha > 0$.

Convergence of Beta function

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\, dt.$$

$$= \int_0^{a(>0)} t^{m-1}(1-t)^{n-1}\, dt + \int_a^1 t^{m-1}(1-t)^{n-1}\, dt = I_1 + I_2$$

$I_1 = \int_0^a t^{m-1}(1-t)^{n-1}\, dt$: I_1 is proper if $m \geq 1$.

$m < 1$, say $1-m = p > 0$.

$I_1 = \int_0^a \frac{(1-t)^{n-1}}{t^p}\, dt$, $\frac{f(t)}{g(t)} = (1-t)^{n-1}$, taking $g(t) = \frac{1}{t^p}$

~~Now~~, $\lim_{t \to 0} \frac{f(t)}{g(t)} = 1$

Now, $\int_0^a \frac{dt}{t^p}$ converges if $p < 1$, diverges if $p \geq 1$.

$\therefore I_1 = \int_0^a \frac{(1-t)^{n-1}}{t^p}\, dt \to$ converges if $p < 1$.

$\hookrightarrow$ diverges if $p \geq 1$.

P.T.O

Which implies

$$I_1 = \int_0^a t^{m-1}(1-t)^{n-1}\,dt \rightarrow \text{converges if } m>0$$
$$\hookrightarrow \text{diverges if } m \le 0$$

Now, $$I_2 = \int_a^1 t^{m-1}(1-t)^{n-1}\,dt$$

taking $g(t) = (1-t)^{n-1}$ we get $\frac{f(t)}{g(t)} = t^{m-1}$

again $$\int_a^1 g(t)\,dt = \int_a^1 \frac{dt}{(1-t)^{1-n}}$$ $\rightarrow$ converges if $1-n<1 \Rightarrow n>0$

$\hookrightarrow$ diverges if $1-n \ge 1 \Rightarrow n \le 0$

$\therefore$ $I_1 + I_2$ converges for $m>0$ and $n>0$.

Different forms of Beta function

1. Show that $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$.

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt.$$

put $t = \sin^2\theta$

$dt = 2\sin\theta\cos\theta\, d\theta$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \; 2\sin\theta\cos\theta\, d\theta.$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.$$

2. Show that $B(m,n) = B(n,m)$.

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2m-1}\phi \sin^{2n-1}\phi\, d\phi, \text{ taking } \theta = \frac{\pi}{2}-\phi$$

$$= B(n,m)$$

3. Show that $B(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt$.

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\, dt$$

put $t = \frac{1}{1+u}$, $dt = -\frac{1}{(1+u)^2}\, du$.

$$\therefore B(m,n) = \int_\infty^0 \frac{1}{(1+u)^{m-1}} \frac{u^{n-1}}{(1+u)^{n-1}} \left[\frac{-1}{(1+u)^2}\right] du$$

$$= \int_0^\infty \frac{u^{n-1}}{(1+u)^{m+n}}\, du.$$

Now, since $B(m,n) = B(n,m)$

then $B(n,m) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{n+m}}\, dt$

Ex If $\int_2^5 (x-2)^{m-1}(5-x)^{m-1}\, dx = c_0\, B(m,n)$

find c_0.

Hint - put $x = 2\cos^2\theta + 5\sin^2\theta$.

In general, if in $\int_a^b (x-a)^{m-1}(b-x)^{n-1}\, dx$ we put

$x = a\cos^2\theta + b\sin^2\theta$, we get $I = (b-a)^{m+n-1} B(m,n)$.

Exercise. Show that $\int_a^b \frac{x\, dx}{(x-a)^{1/3}(b-x)^{2/3}} = \frac{2\pi}{3\sqrt{3}}(a+2b)$

Ex $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx$, use $\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt = B(m,n)$

$= B(9,15) - B(15,9) = 0$

Ex prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(b+cx)^{m+n}}\,dx$

$$= \cancel{b}\,(b+c)^{-m}\, b^{-n} \times B(m,n).$$

Hint. put $y = \frac{(b+c)x}{b+cx}$

Exercise -

Prove that -

$$\int_{-1}^{1} (1+x)^{m-1}(1-x)^{n-1}\,dx = 2^{m+n-1} B(m,n).$$