System of equations:

Consider:

$$\frac{d\bar{y}}{dt} = \bar{f}(t, y_1, y_2, \dots, y_n)$$

$$\bar{y}(t_0) = \bar{\eta}$$

where:

$$\bar{y} = [y_1, y_2, \dots, y_n]^T$$

$$\bar{f} = [f_1, f_2, \dots, f_n]^T$$

$$\bar{\eta} = [\eta_1, \eta_2, \dots, \eta_n]^T.$$

Taylor's series method:

$$\bar{u}_{j+1} = \bar{u}_j + h\,\bar{u}'_j + \frac{h^2}{\lfloor 2}\bar{u}''_j + \dots + \frac{h^p}{\lfloor p}\bar{u}_j^{(p)}$$

$j = 0, 1, 2 \dots, N-1.$

In particular, the Euler method:

$$\bar{u}_{j+1} = \bar{u}_j + h\,\bar{f}_j \qquad j = 0, 1, 2, \dots, N-1$$

OR

$$\begin{bmatrix} u_{1,j+1} \\ \vdots \\ u_{n,j+1} \end{bmatrix} = \begin{bmatrix} u_{1,j} \\ \vdots \\ u_{n,j} \end{bmatrix} + h \begin{bmatrix} f_{1,j} \\ \vdots \\ f_{n,j} \end{bmatrix}$$

$$f_{ij} = f_i(t_j, u_{1,j}, \dots, u_{n,j})$$

Runge-Kutta Method of second order (Euler Cauchy)

$$\bar{u}_{j+1} = \bar{u}_j + \frac{h}{2}(\bar{K}_1 + \bar{K}_2)$$

$$K_{i1} = f_i(t_j, u_{1,j}, u_{2,j}, \dots, u_{n,j}) \quad i = 1,2,\dots,n.$$

$$K_{i2} = f_i(t_j + h, u_{1,j} + hK_{11}, u_{2,j} + hK_{21} + \dots, u_{nj} + hK_{n1})$$

Not that

$$\bar{u}_j = [u_{1,j}, u_{2,j}, \dots, u_{n,j}]^T$$

&

$$\bar{K}_1 = [K_{11}, K_{21}, \dots, K_{n1}]^T$$

$$\bar{K}_2 = [K_{12}, K_{22}, \dots, K_{n2}]^T$$

Similarly Runge-Kutta method of higher order can be formulated.

Example: compute an approximation to $u(1)$, $u'(1)$ and $u''(1)$ with the Taylor's series method of second order and step length $h = 1$ for the IVP

$$u''' + 2u'' + u' - u = \cos t \qquad 0 \le t \le 1$$

$$u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 2.$$

Sol: We can reduce the second order or higher order equations to an equivalent system of first order equations by setting

$v_1 = u \qquad v_2 = u' \qquad v_3 = u''$

System of equations:

$$v_1' = v_2$$

$$v_2' = v_3$$

$$v_3' = \cos t - 2v_3 - v_2 + v_1$$

$v_1(0) = 0$

$v_2(0) = 1$

$v_3(0) = 2$

Therefore the Taylor's series method gives:

$$\bar{u}_{j+1} = \bar{u}_j + h\bar{u}_j' + \frac{h^2}{2}\bar{u}_j''$$

$j = 0;$

$$\bar{u}_1 = \begin{bmatrix} v_{1,1} \\ v_{2,1} \\ v_{3,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 1 \cdot \begin{bmatrix} v_2(0) \\ v_3(0) \\ \cos(0) - 2v_3(0) - v_2(0) + v_1(0) \end{bmatrix}$$

$$+ \frac{1}{2}\begin{bmatrix} v_3(0) \\ \cos(0) - 2v_3(0) \dots \\ \sin(0) - 2v_3'(0) - v_3(0) + v_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 1 - 2\times2 - 1 + 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ -4 \\ 0 + 8 - 2 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3/2 \end{bmatrix}$$

Hence: $u(1) \simeq 2$

$u'(1) \simeq 1$

$u''(1) \simeq 3/2$

Ex: Use the Runge-Kutta Method to approximate the particular solution at $x=2$ of the differential equation

$$y'' = x + yy'$$

if $y' = 0$ and $y = 1$ when $x = 0$.

Sol:

let $y_1 = y$

$y_2 = y'$

then:

$$y_1' = y_2 \quad =: f_1(x, y_1, y_2)$$

$$y_2' = x + y_1 y_2 \quad =: f(x, y_1, y_2)$$

$$\bar{u}_1 = \bar{u}_0 + \frac{h}{6}\left[\bar{k}_1 + 2\bar{k}_2 + 2\bar{k}_3 + \bar{k}_4\right] \qquad h = 2$$

$$\bar{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \bar{k}_1 = \begin{bmatrix} y_2(0) \\ 0 + y_1(0)\, y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{k}_2 = \begin{bmatrix} f_1(1, 1 + 1\times 0, 0 + 1\times 0) \\ f_2(1, 1, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\bar{k}_3 = \begin{bmatrix} f_1(1, 1 + 1\times 0, 0 + 1\times 1) \\ f_2(1, 1, 1) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\bar{k}_4 = \begin{bmatrix} f_1(2, 1 + 2\times 1, 0 + 2\times 2) \\ f_2(2, 3, 4) \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$$

$$\bar{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{2}{6}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 14 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 6 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 20/3 \end{bmatrix}$$

$$\boxed{y(2) \simeq 3}$$