LINEAR ALGEBRA-2

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Assignment -2

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Q.1] Claim: Every linear transformation f can be written as

$$f(A) = \sum_{i=1, j=1}^{n} a_{ij} c_{ij}, \text{ for some fixed } c_{ij} \in R$$

(here f is LT from $M_{n\times n} \rightarrow R$)

Proof:

Consider 2 bases, one of $M_{n\times n}$, other of R

$M_{n\times n}$ basis $= \{ e_{ij}, 1 \leq i,j \leq n, i,j \in N \}$

R basis $= \{1\}$

Now, $A = \sum e_{ij} a_{ij}$, where a_{ij} denotes the term of matrix at i^{th} row, j^{th} column

$\therefore T(A) = \sum T(e_{ij} a_{ij})$

$= \sum a_{ij} T(e_{ij})$

$= \sum a_{ij} c_{ij}$, $c_{ij} \in R$, $c_{ij} = T(e_{ij})$

hence proved.

Claim: $f(AB) = f(BA)$ implies

$c_{ij} = \delta(i,j) K$, $K \in R$

δ is Kronecker Delta function.

Proof: $AB = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ik} b_{kj}$

$BA = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} b_{ik} a_{kj}$

$T(AB) = T(BA) \Rightarrow \sum\sum\sum c_{ij} a_{ik} b_{kj} = \sum\sum\sum c_{ij} b_{ik} a_{kj}$

☆ This is an identity in a_{ij}, b_{ij}, since it is true $\forall$ matrices $\forall$ $M_{n\times n}(R)$

Put $A = e_{11}$, $B = e_{1i}$ $\forall i \in N (i \leq n)$

this will give $c_{12} = c_{13} = c_{14} \cdots = c_{1n} = 0$

Putting $A = e_{j1}$, $B = e_{ji}$, $\forall i,j \in N (i,j \leq n)$ will give $c_{ij} = 0$, $i \neq j$

let us make some observations.

Coefficient of $a_{1n} b_{n1}$ in LHS is c_{11},

Coefficient of $a_{1n} b_{n1}$ in RHS is c_{nn}

$\boxed{\text{So } c_{11} = c_{nn}}$ (since this is an identity)

In particular, coefficient of $a_{in} b_{ni}$ in LHS is c_{ii}, & RHS $= c_{nn}$

$$\boxed{\text{So } c_{ii} = c_{nn} \; \forall i \in \{1,2,3 \ldots n\}}$$

(again, since if $c_{ii} \neq c_{nn}$, this would imply $\forall A,B \in M_{n\times n}(R)$ $a_{in} b_{ni} = 0$ which is absurd)

One can easily check that rest of the terms of c_{ii} either cancel, or give equations of the form

$c_{ii} = c_{jj}$ $(i \neq j)$

Hence $c_{ij} = K\delta(i,j)$, $K \in R$

So $T(A) = \sum\sum K\delta(i,j) a_{ij}$

$= K \operatorname{trace}(A)$

So set of all linear transformations are

$$\boxed{T: T(A) = K \operatorname{trace}(A), \text{ for some } K \in R.}$$