3.8 The Cycle Index Polynomial

Let G be a group acting on a set X. Then as mentioned at the end of the previous section, we need to understand the cycle decomposition of each $g \in G$ as product of disjoint cycles. Redfield and Polya observed that elements of G with the same cyclic decomposition made the same contribution to the sets of *fixed points*. They defined the notion of cycle index polynomial to keep track of the cycle decomposition of the elements of G. Let us start with a few definitions and examples to better understand the use of cycle decomposition of an element of a permutation group.

Definition 3.8.1. *A permutation $\sigma \in \mathcal{S}_n$ is said to have the cycle structure $1^{\ell_1}\, 2^{\ell_2} \cdots n^{\ell_n}$, if the cycle representation of σ has ℓ_i cycles of length i, for $1 \le i \le n$. Observe that $\sum\limits_{i=1}^{t} i \cdot \ell_i = n$.*

Example 3.8.2. 1. *Let e be the identity element of $\mathcal{S}_n$. Then $e = (1)\,(2)\,\cdots(n)$ and hence the cycle structure of e, as an element of $\mathcal{S}_n$ equals 1^n.*

2. *Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 3 & 6 & 7 & 10 & 14 & 1 & 2 & 13 & 15 & 4 & 11 & 5 & 8 & 12 & 9 \end{pmatrix}$. Then it can be easily verified that in the cycle notation, $\sigma = (1\ 3\ 7\ 2\ 6)\ (4\ 10)\ (5\ 14\ 12)\ (8\ 13)\ (9\ 15)\ (11)$. Thus, the cycle structure of σ is $1^1 2^3 3^1 5^1$.*

3. *Consider the group G of symmetries of the tetrahedron (see Example 3.2.1.2a). Then the elements of G have the following cycle structure:*

1^4 *for exactly* 1 *element corresponding to the identity element*;

$1^1 3^1$ *for exactly* 8 *elements corresponding to* 3 *cycles*;

2^2 *for exactly* 3 *elements corresponding to* $(12)(34), (13)(24) \& (14)(23)$.

Definition 3.8.3. *Let G be a permutation group on n symbols. For a fixed $g \in G$, let $\ell_k(g)$ denote the number of cycles of length $k, 1 \le k \le n$, in the cycle representation of g. Then the cycle index polynomial of G, as a permutation group on n symbols, is a polynomial in n variables $z_1, z_2, \ldots, z_n$ given by*

$$P_G(z_1, z_2, \ldots, z_n) = \frac{1}{|G|}\left(\sum_{g \in G} z_1^{\ell_1(g)}\, z_2^{\ell_2(g)} \cdots z_n^{\ell_n(g)}\right).$$

Before, we look at a few examples, note that for each fixed $g \in G$, the condition that g has exactly $\ell_k(g)$ cycles of length $k, 1 \le k \le n$, implies that each term $z_1^{\ell_1(g)}\, z_2^{\ell_2(g)} \cdots z_n^{\ell_n(g)}$ in the summation satisfies $1 \cdot \ell_1(g) + 2 \cdot \ell_2(g) + \cdots + n \cdot \ell_n(g) = n$.

Example 3.8.4. 1. *Let G be the dihedral group D_4 (see Example 3.2.1.2). Then*

$e = (1)(2)(3)(4) \longrightarrow z_1^4,\ r = (1234) \longrightarrow z_4,\ r^3 = (1432) \longrightarrow z_4,\ r^2 = (13)(24) \longrightarrow z_2^2,$
$f = (14)(23) \longrightarrow z_2^2,\ rf = (1)(3)(24) \longrightarrow z_1^2 z_2,\ r^2 f = (12)(34) \longrightarrow z_2^2,\ r^3 f = (13)(2)(4) \longrightarrow z_1^2 z_2.$

Thus, $P_G(z_1, z_2, z_3, z_4) = \frac{1}{8}\left(z_1^4 + 2z_4 + 3z_2^2 + 2z_1^2 z_2\right).$

2. *Let G be the dihedral group D_5 (see Example 3.2.1.1c). Then* $P_G(z_1, z_2, z_3, z_4, z_5) = \frac{1}{10}\left(z_1^5 + 4z_5 + 5z_1 z_2^2\right).$

3. *Verify that the cycle index polynomial of the symmetries of a cube induced on the set of vertices equals*
$$P_G(z_1, z_2, \ldots, z_8) = \frac{1}{24}\left(z_1^8 + 6z_4^2 + 9z_2^4 + 8z_1^2 z_3^2\right).$$

3.8.1 Applications

Let S be an object (a geometrical figure) and let X be the finite set of points of S. Also, let C be a finite set (say, of colors). Consider the set Ω that denotes the set of all functions from X to C. Observe that an element of Ω gives a color pattern on the object S. Let G be a subgroup of the group of symmetries of the object S. Hence, G acts on the elements of X. Let us denote this action by $\star$. So, $g \star x \in X$, for all $x \in X$.

One can also obtain an action of G on Ω, denoted $\circledast$, by the following rule:
Fix an element $x \in X$. Then, for each $\phi \in \Omega$ and $g \in G$, $g \circledast \phi$ is an element of Ω, a function from X to C. Hence, one defines
$$(g \circledast \phi)(x) = \phi(g^{-1} \star x), \text{ for all } \phi \in \Omega \text{ and } x \in X.$$

We claim that $\circledast$ indeed defines a group action on the set Ω. To do so, note that for each $h, g \in G$ and $\phi \in \Omega$, the definition of the action on X and Ω gives
$$\begin{aligned} \left(h \circledast (g \circledast \phi)\right)(x) &= (g \circledast \phi)(h^{-1} \star x) = \phi\left(g^{-1} \star (h^{-1} \star x)\right) = \phi\left(g^{-1}h^{-1} \star x\right) \\ &= \phi\left((hg)^{-1} \star x\right) = (hg \circledast \phi)(x). \end{aligned}$$

Since, $\left(h \circledast (g \circledast \phi)\right)(x) = (hg \circledast \phi)(x)$, for all $x \in X$, one has $h \circledast (g \circledast \phi) = hg \circledast \phi$, for each $h, g \in G$ and $\phi \in \Omega$. Hence, the proof of the claim is complete. Now, using the above notations, we have the following theorem.

Theorem 3.8.5 (Polya-Redfield Theorem)**.** *Let C, S, X and Ω be as defined above. Also, let G be a subgroup of the group of permutations of the object S. Then the number of distinct color patterns (distinct elements of Ω), distinct up to the action of G, is given by*
$$P_G(|C|, |C|, \ldots, |C|).$$

Proof. Let $|X| = n$. Then observe that G is a subgroup of $\mathcal{S}_n$. So, each $g \in G$ can be written as a product of disjoint cycles. Also, by Burnside's Lemma 3.7.3, N, the number of distinct color patterns (distinct orbits under the action of G), equals $\frac{1}{|G|}\sum\limits_{g \in G} |F_g|$, where
$$F_g = \{\phi \in \Omega \ : g \circledast \phi = \phi\} = \{\phi \in \Omega \ : \ (g \circledast \phi)(x) = \phi(x), \ \text{ for all } \ x \in X\}.$$

We claim that "$g \in G$ fixes a color pattern (or an element of Ω) if and only if ϕ colors the elements in a given cycle of g with the same color".

Suppose that $g \circledast \phi = \phi$. That is, $(g \circledast \phi)(x) = \phi(x)$, for all $x \in X$. So, using the definition, one has $\phi(g^{-1} \star x) = \phi(x)$, for all $x \in X$. In particular, for a fixed $x_0 \in X$, one also has

$$\phi(x_0) = \phi(g \star x_0) = \phi\big(g^2 \star x_0\big) = \cdots .$$

Note that, for each fixed $x_0 \in X$ and $g \in G$, the permutation $(x_0,\ g \star x_0,\ g^2 \star x_0, \ldots)$ corresponds to a cycle of g. Therefore, if g fixes a color pattern ϕ, *i.e.*, $g \circledast \phi = \phi$, then ϕ assigns the same color to each element of any cycle of g.

Conversely, fix an element $g \in G$ and let ϕ be a color pattern (a function) that has the property that every point in a given cycle of g is colored with the same color. That is, $\phi(x) = \phi(g \star x)$, for each $x \in X$. Or equivalently, $\phi(x) = \phi\big(g^{-1} \star x\big) = (g \circledast \phi)(x)$, for all $x \in X$. Hence, by definition, $g \circledast \phi = \phi$. Thus, g fixes the color pattern ϕ. Hence, the proof of the claim is complete.

Therefore, we observe that for a fixed $g \in G$, a cycle of g can be given a color independent of another cycle of g. Also, the number of distinct colors equals $|C|$. Hence, using the principle of basic counting (see Item 2 on Page 25), for a fixed $g \in G$, $|F_g| = |C|^{\ell_1(g)} \cdot |C|^{\ell_2(g)} \cdots |C|^{\ell_n(g)}$, where for each k, $1 \le k \le n$, $\ell_k(g)$ denotes the number of cycles of g of length k. Thus,

$$N = \frac{1}{|G|} \sum_{g \in G} |F_g| = \frac{1}{|G|} \sum_{g \in G} |C|^{\ell_1(g)} \cdot |C|^{\ell_2(g)} \cdots |C|^{\ell_n(g)} = P_G(|C|, |C|, \ldots, |C|). \qquad \blacksquare$$

We now give a few examples to indicate the importance of Theorem 3.8.5.

Example 3.8.6. 1. *Determine the number of distinct color patterns, when the vertices of a pentagon is colored with* 3 *colors.*

Solution: *It can be easily observed that the group* D_5, *the group of symmetries of a pentagon, acts on the color patterns. Now, verify that*

$$P_{D_5}(z_1, z_2, \ldots, z_5) = \frac{1}{|D_5|}(z_1^5 + 4z_5 + 5z_1 z_2^2) = \frac{z_1^5 + 4z_5 + 5z_1 z_2^2}{10}.$$

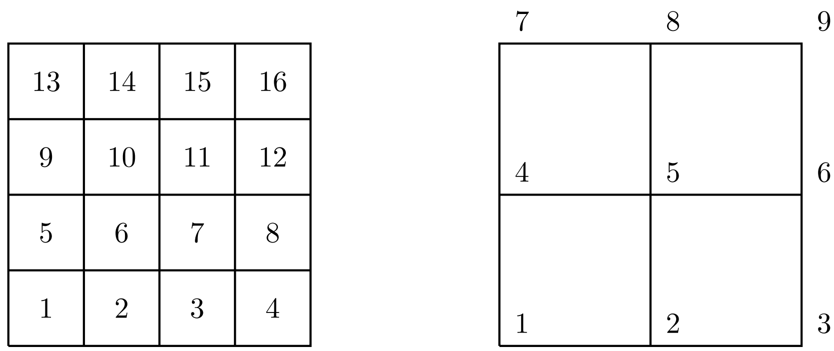
Thus, by Theorem 3.8.5, the required number equals $N = \dfrac{1}{10}(3^5 + 4 \cdot 3 + 5 \cdot 3 \cdot 3^2) = 39.$

2. *Suppose we are given beads of* 3 *different colors and that there are at least* 6 *beads of each color. Determine the distinct necklace patterns that are possible using the* 6 *beads.*

Solution: *Since we are forming a necklace using* 6 *beads, the group* D_6 *acts on the* 6 *beads of the necklace. Also, the cycle index polynomial of* D_6 *equals* $P_{D_6}(z_1, z_2, \ldots, z_5, z_6) = \dfrac{1}{|D_6|}(z_1^6 + 2z_6 + 2z_3^2 + z_2^3 + 3z_2^3 + 3z_1^2 z_2^2)$. *Hence, by Theorem 3.8.5, the number of distinct necklace patterns equals* $\dfrac{1}{12}(3^6 + 2 \cdot 3 + 2 \cdot 3^2 + 4 \cdot 3^3 + 3 \cdot 3^2 \cdot 3^2) = 92.$

3. *Consider the 2×2 square given in Figure 3.7. Determine the number of distinct color patterns, when the vertices of the given figure are colored with two colors.*

Solution: *Observe that D_4 is the group of symmetries of the 2×2 square and it needs to act on 9 vertices. So, we need to write the elements of D_4 as a subgroup of $\mathcal{S}_9$. Hence, the cycle index polynomial is given by* $P_{D_4}(z_1, \ldots, z_9) = \dfrac{z_1^9 + 2z_1z_4^2 + z_1z_2^4 + 4z_1^3z_2^3}{8}$ *and the number of distinct color patterns equals* 102.



The 4×4 Square The 2×2 Square

Figure 3.7: Faces and Vertices of Squares

4. *Determine the number of distinct color patterns when the edges of a cube are colored with 2 colors.*

Solution: *Using the group of symmetries of the cube given on Page 77, the cycle index polynomial corresponding to the faces equals* $P_G(z_1, \ldots, z_{12}) = \dfrac{z_1^{12} + 6z_4^3 + 3z_2^6 + 8z_3^4 + 6z_1^2z_2^5}{24}$.
Thus, the required number is 218.