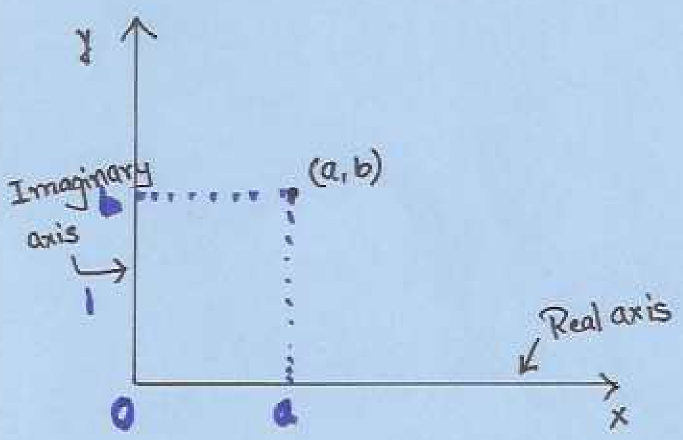
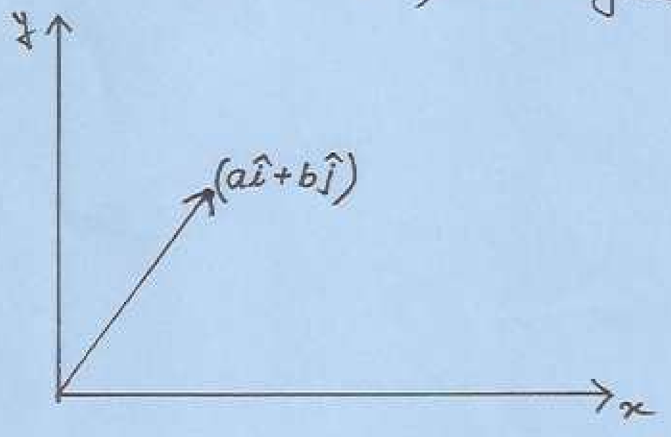
COMPLEX ANALYSIS

COMPLEX NUMBERS: $x+iy$ OR $x+yi$

x & y are real numbers & $i^2=-1$, i: imaginary uni



A point in the plane



Vector in a plane

ARITHMETIC OF COMPLEX NUMBERS:

- Equality: $a+ib=c+id$ exactly when $a=c$ & $b=d$
- Addition: $(a+ib)+(c+id) = (a+c)+i(b+d)$
- Multiplication (as first order polynomial multiplication)

$$(a+bi)(c+di) = ac+adi+bci+bdi^2$$
$$= (ac-bd)+i(ad+bc)$$

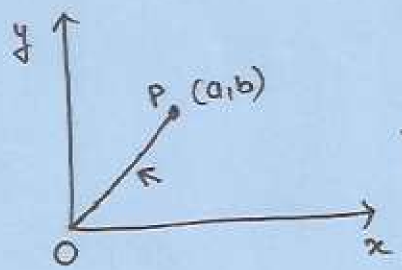
COMPLEX CONJUGATE:

$$Z = x+iy$$

Conjugate of $Z = \bar{Z} = x-iy$

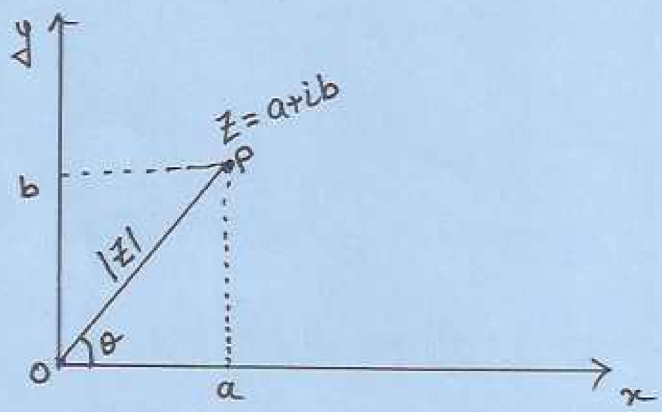
MAGNITUDE: Magnitude of $a+ib$ is denoted by $|a+ib|$ and is defined as

$$|a+ib| = \sqrt{a^2+b^2}$$



Distance $OP = \sqrt{a^2+b^2} = |a+ib|$

POLAR FORM OF COMPLEX NUMBERS:



If z has polar coordinate (r, θ) then

$$r = |z|$$

The angle θ (OP makes with the positive x-axis) is called the argument of z.

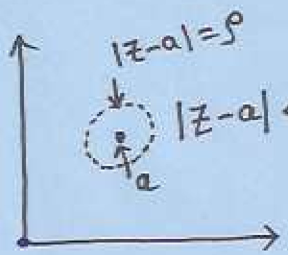
So, $\quad a = r\cos\theta \quad b = r\sin\theta$

Then,
$$z = a+ib = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

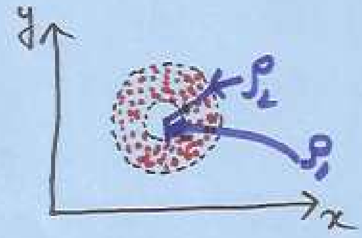
PROPERTIES:

- $|z_1 z_2| = |z_1||z_2|$
- $z\bar{z} = |z|^2$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $|z| = |\bar{z}|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $z^n = r^n(\cos n\theta + i \sin n\theta)$

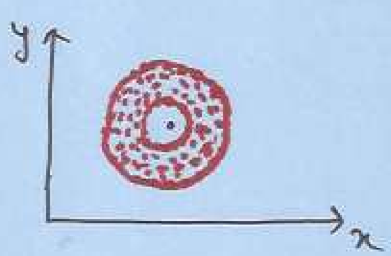
NEIGHBOURHOOD:



OPEN ANNULUS: $\rho_1 < |z-a| < \rho_2$



CLOSED ANNULUS: $\rho_1 \leq |z-a| \leq \rho_2$



COMPLEX FUNCTION:

$$f : D \to R$$

D & R are some sets of complex numbers

$$w = f(z) = f(x+iy) = u(x,y) + i v(x,y)$$

LIMIT: Let $f(z)$ be defined and single valued in a neighboured of $z = z_0$. Let l be a complex number

then,

$$\lim_{z \to z_0} f(z) = l$$

if and only if

for given $\varepsilon > 0$, there exists a positive number δ such that

$$|f(z) - l| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

We call l the limit of $f(z)$ as z approahes z_0.

OR

$\lim_{z \to z_0} f(z) = l$ if the difference in absolute value between $f(z)$ and l can be made arbitrarily small by choosing z close enough to z_0.

Ex. Find $\lim_{z \to i} \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$.

$$= \lim_{z \to i} 3z^3 - (2-3i)z^2 + (5-2i)z + 5i$$

$$= -3i + (2-3i) + (5-2i)i + 5i = 4i + 4$$

Prove using ε-δ approach $\lim_{z \to i} \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} = 4 + 4i$

CONTINUITY:

$f(z)$ is continuous at $z = z_0$ if

1. $\lim_{z \to z_0} f(z) = l$, i.e. the limit $\lim_{z \to z_0} f(z)$ exists

2. $f(z)$ is defined at z_0 i.e. $f(z_0)$ exists

3. $l = f(z_0)$.

OR:

$f(z)$ is said to be continuous at $z = z_0$ if for any $\varepsilon > 0$ we can find $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$.

NOTE: If $\lim_{z \to z_0} f(z)$ exists but is not equal to $f(z_0)$, we call z_0 removable discontinuity since by redefining $f(z_0)$ to be the same as $\lim_{z \to z_0} f(z)$ the function becomes continuous.

<u>Ex</u>: $f(z) = z^2$ is continuous at $z = z_0$ as

$$\lim_{z \to z_0} z^2 = z_0^2 = f(z_0).$$

- Show that $\lim_{z \to z_0} z^2 = z_0^2$ using δ-ε approach.

To show: $|f(z) - f(z_0)| = |z^2 - z_0^2| < \varepsilon$ whenever $|z - z_0| < \delta$

If we take $\delta \leq 1$, then $|z - z_0| < \delta$ implies:

$$|z^2 - z_0^2| = |z - z_0||z + z_0| < \delta\, |z - z_0 + 2z_0| < \delta\left[|z - z_0| + |2z_0|\right]$$
$$< \delta\left[1 + |2z_0|\right]$$

Take δ smaller of 1 & $\varepsilon/(1 + 2|z_0|)$ then, $|z^2 - z_0^2| < \varepsilon$ whenever $|z - z_0| < \delta$

Ex. Discuss the continuity of the function

$$f(z) = \begin{cases} \dfrac{3z^4-2z^3+8z^2-2z+5}{z-i} & z \neq i \\ 0 & z = i \end{cases}$$

Sol: We have already seen that

$$\lim_{z\to i} f(z) = 4i+4 \neq f(i)$$

Hence the function is not continuous at $z=i$. However, $z=i$ is a removable discontinuity since redefining the function as $f(z) = 4+4i$ at $z=i$, it becomes continuous.

DERIVATIVE OF A COMPLEX FUNCTION:

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} \quad \text{—(1)}$$

provided the limit exists independent of the path in which $\Delta z \to 0$.

(1) can also be written as

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

Ex: Find the derivative of $f(z) = z^2$.

Sol.

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{(z+\Delta z)^2-z^2}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\cancel{z^2}+\Delta z^2+2z\Delta z-\cancel{z^2}}{\Delta z} = \lim_{\Delta z\to 0}(2z+\Delta z)$$

$$= 2z.$$

Ex. $f(z) = \bar{z}$ is not differentiable at any z.

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

If Δz approaches to 0 along the real axis: then

$$\frac{\overline{\Delta z}}{\Delta z} = 1 \quad \text{as} \quad \overline{\Delta z} = \Delta z$$

But if Δz approaches to 0 along the imaginary axis then

$\Delta z = ik$ for some real k; and

$$\frac{\overline{\Delta z}}{\Delta z} = \frac{-ik}{ik} = -1$$

$\Rightarrow \lim\limits_{\Delta z \to 0} \dfrac{\overline{\Delta z}}{\Delta z}$ does not exist, so f has no derivatives at any point

Th. Let f be differentiable at z_0, then f is continuous at z_0.

Proof: Consider

$$f(z_0+\Delta z) - f(z_0) = \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \cdot \Delta z \;; \; \Delta z \neq 0$$

Now,

$$\lim_{\Delta z \to 0} \left(f(z_0+\Delta z) - f(z_0) \right) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \cdot \lim_{\Delta z \to 0} \Delta z$$

$$= f'(z_0) \cdot 0$$

$$= 0$$

Thus $\lim\limits_{\Delta z \to 0} f(z_0+\Delta z) = f(z_0)$

Proved.