Date 18/07/2017

Lecture 2

Exponential order

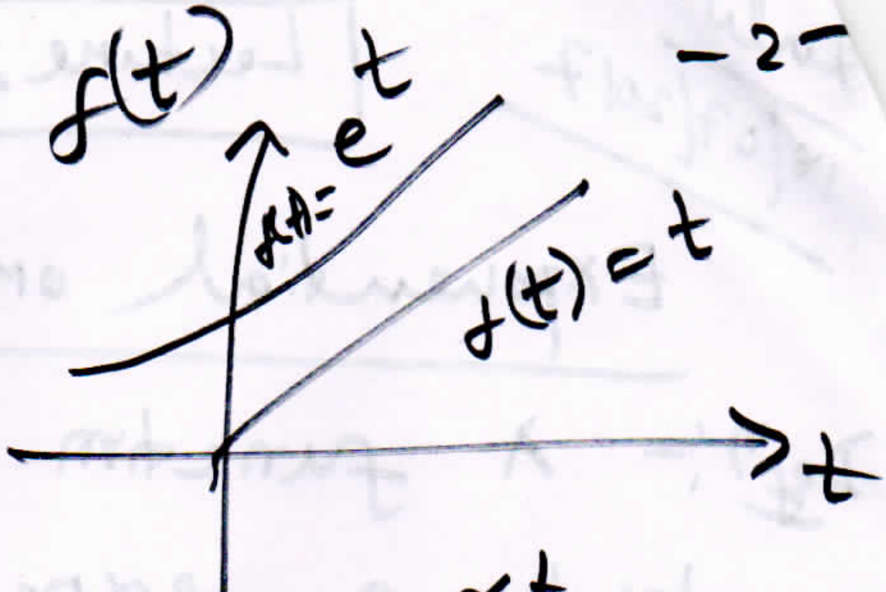
Defⁿ:- A function $f(t)$ is said to be of exponential order α, if $\exists$ constants α & $M > 0$ such that

$$|f(t)| \le M e^{\alpha t}, \; t \geq 0$$

Geometrically, this condⁿ implies that the graph of $f(t)$, $t > 0$ does not grow faster than the graph of the exponential fⁿ

$$f(t) = M e^{\alpha t}, \; \alpha > 0$$

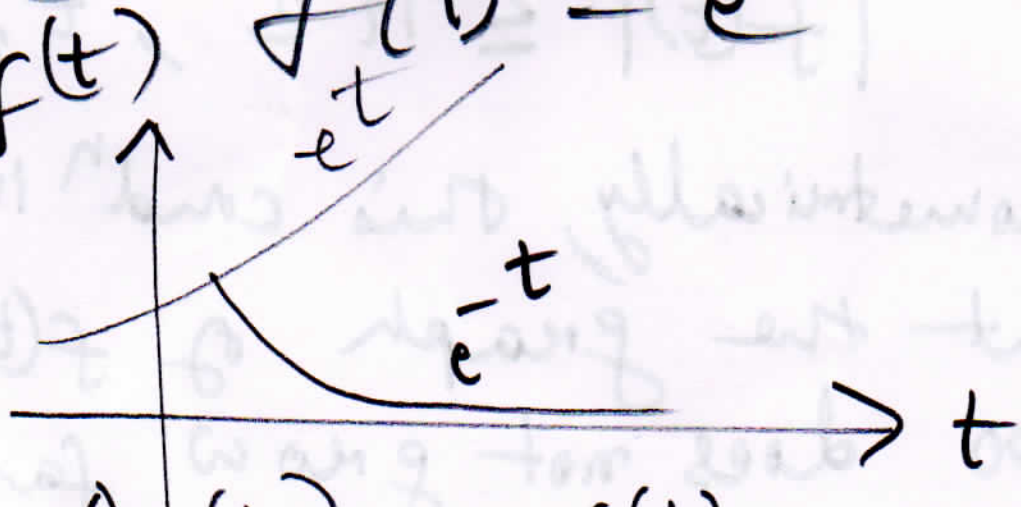
① $f(t)=t$



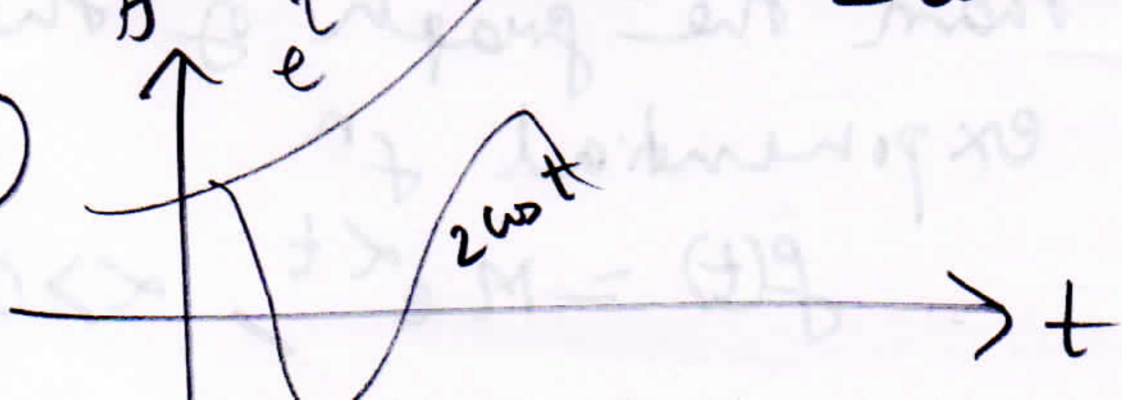
e^{t^2}, e^{t^3}

$|t| \le M \cdot e^{\alpha t}$?

② $f(t) = e^{-t}$

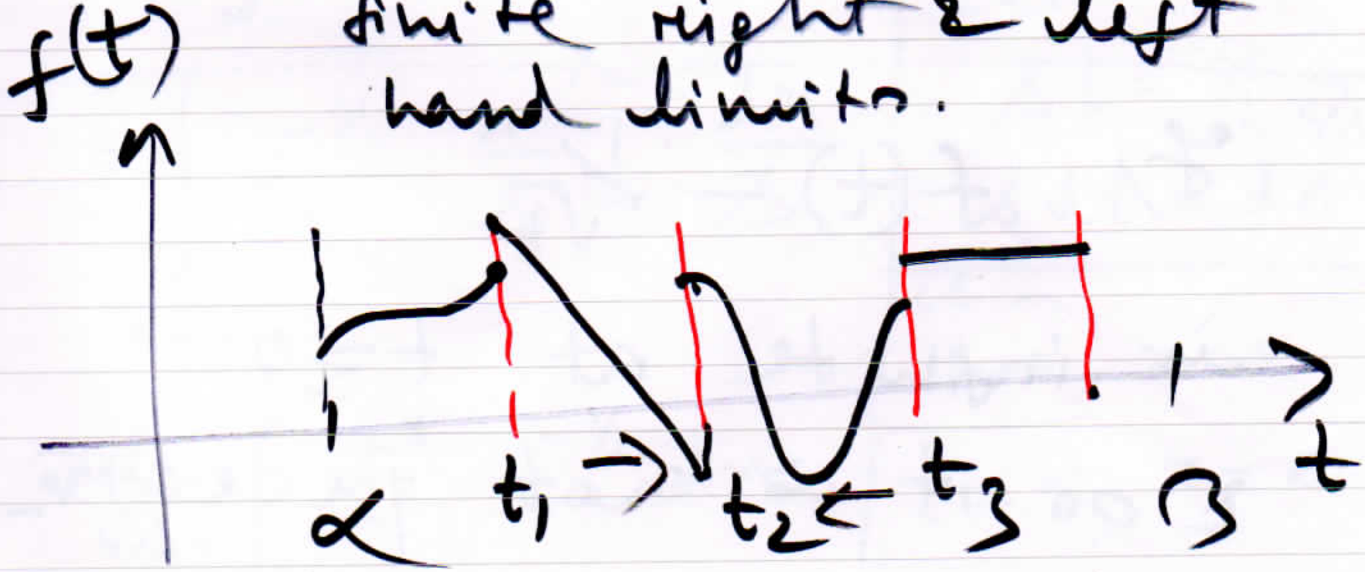


③ $f(t) = 2\cos t$



Sectional or Piece-wise Continuity

Def^n:- A function $f(t)$ is called piecewise or sectional continuous in an interval $\alpha \le t \le \beta$, if the interval can be subdivided into a finite no. of intervals in each of which the f^n is continuous & has finite right & left hand limits.



This function $f(t)$ has discontinuities at t_1, t_2 & t_3.

Note that the right & left hand limits at t_2 are

are given by

$$\lim_{\epsilon \to 0} f(t_2+\epsilon) = f(t_2+0) = f(t_2+)$$

& $$\lim_{\epsilon \to 0} f(t_2-\epsilon) = f(t_2-0) = f(t_2-)$$

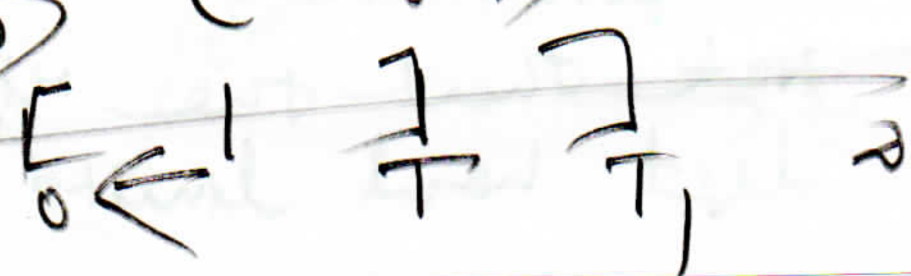
for $\epsilon > 0$

Eg, $f(t) = 1/\sqrt{t}$

is infinite at $t=0$

& so it is not piecewise continuous in the range $[0, \infty)$ (why?)

$$\lim_{\epsilon \to 0} f(t+\epsilon) = \infty$$



but its transform exists.

$$\Gamma(1/2) = \sqrt{\pi}.$$

$$\mathcal{L}(1/\sqrt{t}) = \mathcal{L}(t^{-1/2})$$

$$= \int_0^\infty e^{-st} \cdot t^{-1/2}\, dt$$

$$= \int_{x=0}^{\infty} e^{-x} \cdot \frac{x^{-1/2}}{s^{-1/2}} \cdot \frac{dx}{s}$$

Let $st = x \Rightarrow t = x/s$

$\therefore s\,dt = dx$

when $t=0$, $x=0$

$t=\infty$, $x=\infty$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} \cdot x^{-1/2}\, dx$$

$$= \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} \cdot x^{(1/2 - 1)}\, dx$$

$$\Gamma(\alpha) = \int_0^\infty e^{-x} \cdot x^{\alpha - 1}\, dx$$

$$= \frac{\Gamma(1/2)}{\sqrt{s}} = \sqrt{\pi/s}, \quad (s>0)$$

Note:- Excluded f^n are those that have singularities such as

$\ln x$ or, $\frac{1}{(x-1)}$

& f^{ns} that have growth rate more rapid than exponential (eg, e^{t^2}, e^{t^3} etc.)

Functions that have a finite no. of finite discontinuities are also included in this list.

eg, $f(x) = \begin{cases} 1, & 2n < x < 2n+1 \\ 0, & 2n+1 < x < 2n+2 \end{cases}$
$n = 0, 1, 2, \ldots$

However, the function

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

is excluded because although all the discontinuities are finite, there are infinitely many of them.

Integral Transforms

A class of transformations, which are called integral transforms are defined by

$$T\{f(t)\} = \int_{-\infty}^{\infty} \boxed{k(s,t)}\, f(t)\, dt = F(s)$$

kernel $k(s,t)$

If we choose

$$k(s,t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-st} & \text{for } t \geq 0 \end{cases}$$

is called Laplace transform

Application O.D.eq^n

x

P.D eq^n $(x, y, \ldots)$

$T\{f(t)\} \longrightarrow f(t) = T^{-1}$

Uniqueness of Laplace Transform (L.T)

Let $f(t)$ & $g(t)$ be two functions such that $F(s) = G(s)$, $\forall\ s > k$.

Then $f(t) = g(t)$ at all t where both the f^ns f & g are continuous.

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Th-2 / (Linearity)

If $f_1(t)$ & $f_2(t)$ are two functions whose Laplace Transform exists, then

$$\mathcal{L}\{a f_1(t) + b f_2(t)\} = a\,\mathcal{L}\{f_1(t)\} + b\,\mathcal{L}\{f_2(t)\}$$

where a & b are arbitrary const.

Sol^n:- $\mathcal{L}\{a f_1(t) + b f_2(t)\}$

$$= \int_0^\infty (a f_1 + b f_2) e^{-st}\, dt$$

$$= a \underbrace{\int_0^\infty f_1 e^{-st}\, dt} + b \int_0^\infty f_2 e^{-st}\, dt$$

$$= a\,\mathcal{L}\{f_1(t)\} + b\,\mathcal{L}\{f_2(t)\}$$

where we have assumed that

$$|f_1| \le M_1 e^{\alpha_1 t}, \quad |f_2| \le M_2 e^{\alpha_2 t}$$

so that

$$|a f_1 + b f_2| \le |a||f_1| + |b||f_2|$$

where $\alpha_3 = \max\{\alpha_1, \alpha_2\}$ $\quad \le (|a| M_1 + |b| M_2) e^{\alpha_3 t}$

Ex/ Hyperbolic f^ns.

Let $f(t) = \cosh at$

$$= \frac{1}{2}\left(e^{at} + e^{-at}\right),$$

Find $\mathcal{L}(f)$

$$= \frac{1}{2}\mathcal{L}\left(e^{at}\right) + \frac{1}{2}\mathcal{L}\left(e^{-at}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(s-a)} + \frac{1}{(s+a)}\right] \text{ [how?]}$$

$$= \frac{s}{s^2-a^2} \quad \left(\text{when } s > a \geq 0\right)$$

Slly, $\mathcal{L}(\sinh at) = \frac{1}{2}\mathcal{L}\left(e^{at}\right) - \frac{1}{2}\mathcal{L}\left(e^{-at}\right)$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right)$$

$$= \frac{a}{s^2-a^2}, \quad (s > a \geq 0)$$

H.W.
Ex

Find $\mathcal{L}(e^{at}), t \geq 0$

$a = i\omega$ that is a is a complex no.

$i = \sqrt{-1}$

Q) what about $\tan t$, $\cot t$, $\csc t$, $\sec t$