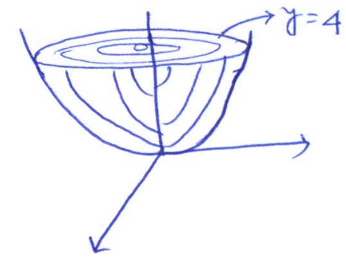
Triple Integral.

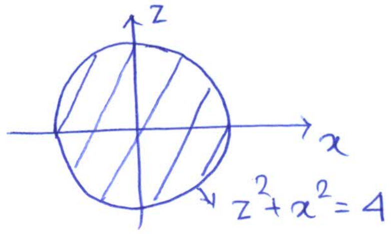
Ex. Compute

$$\iiint \sqrt{x^2+z^2}\ dx\, dy\, dz$$

R = portion of the paraboloid

$y = z^2 + x^2$ for $y \leq 4$.





$$= \iint_{z^2+x^2\leq 4} \int_{y=z^2+x^2}^{4} \sqrt{x^2+z^2}\ dx\, dy\, dz$$

$$= \iint_{x^2+z^2\leq 4} \sqrt{x^2+z^2}\ \left(4 - \overline{z^2+x^2}\right) dx\, dz$$

$x = r\cos\theta \quad y = r\sin\theta$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} r\,(4-r^2)\ \underbrace{|J|}_{=r}\, dr\, d\theta \qquad |J| = r$$

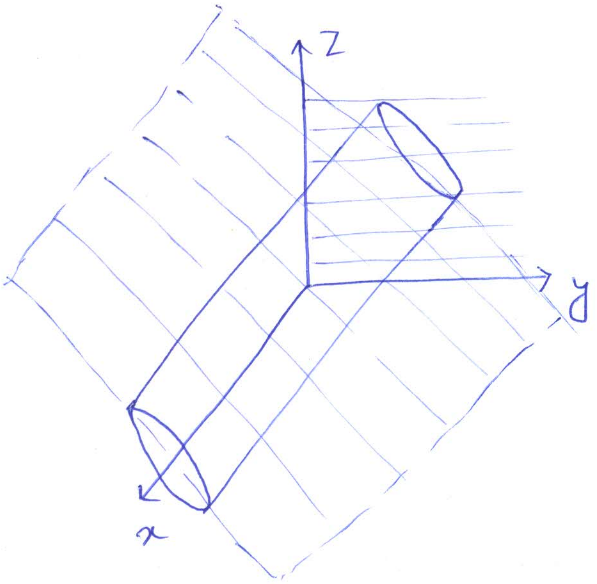
$$= 2\pi \int_0^2 (4r^2 - r^4)\, dr = 2\pi\left(\frac{4}{3}\times 8 - \frac{2^5}{5}\right)$$

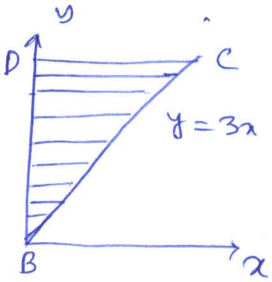
$$= \frac{\pi \times 2^6}{15} \times 2.$$

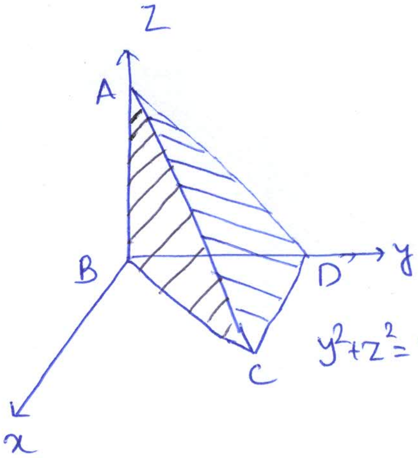
2. $\iiint z \, dx \, dy \, dz.$

R = region bounded by :

$x \geq 0, \; z \geq 0, \; y \geq 3x \;\; \& \;\; y^2+z^2 \leq 9$







$y^2+z^2=9 \cap z=0$

$\Rightarrow y = 3$

ABC, ACD, ABD, BCD

AC, AD $\rightarrow$ intersection of cylinder with $y=3x$, $x=0$

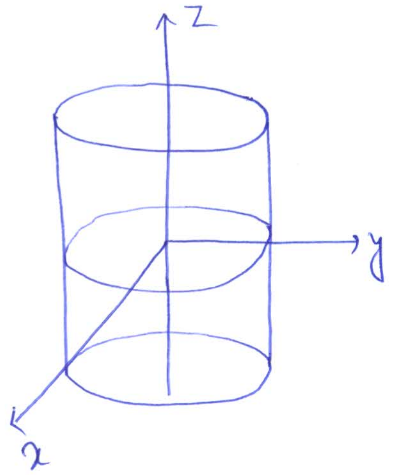
$$I = \iint_{D_{xy}} \int_{z=0}^{\sqrt{9-y^2}} z \, dx\, dy\, dz = \int_{y=0}^{3} \int_{x=0}^{y/3} \frac{z^2}{2} \Big|_0^{\sqrt{9-y^2}} dx\, dy$$

$$= \int_{y=0}^{3} \int_{x=0}^{y/3} \frac{9-y^2}{2} \, dx\, dy = \int_{y=0}^{3} \frac{y(9-y^2)}{6} \, dy = \frac{27}{8}.$$

Change in variables.

Cylindrical polar $(x, y, z) \to (r, \theta, z)$; $f(x^2+y^2)$ / $f(y^2+z^2)$ / $f(x^2+z^2)$

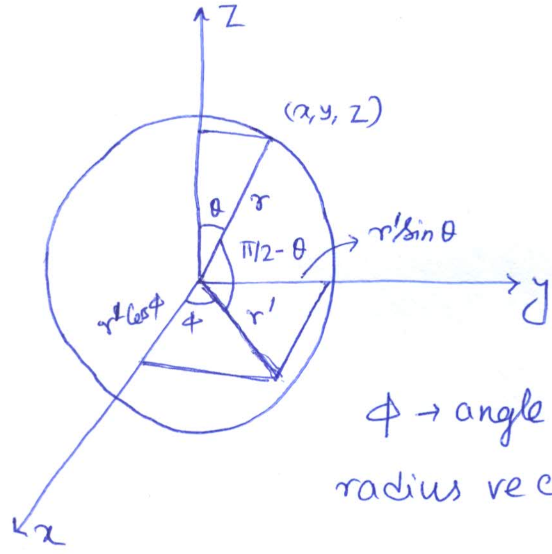
spherical polar. $(x, y, z) \to (r, \theta, \phi)$; $f(x^2+y^2+z^2)$



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \text{cylindrical polar coordinates.}$$

$$\iiint_{R_{xyz}} f(x, y, z)\, dx\, dy\, dz = \iiint_{R'_{r\theta z}} F(r, \theta, z)\, |J|\, dr\, d\theta\, dz$$

$$J_{\substack{cyl\\polar}} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$



$Z = r \cos\theta$

$\phi \rightarrow$ angle between projection of radius vector with x-axis.

$$Z = r\cos\theta \;,\; x = r'\cos\phi \;,\; y = r'\sin\phi$$

$$r' = r\cos\left(\tfrac{\pi}{2} - \theta\right) = r\sin\theta$$

$$\boxed{\begin{gathered} \therefore\; Z = r\cos\theta \;,\; x = r\sin\theta\cos\phi \;,\; y = r\sin\theta\sin\phi \\ -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2} \;,\; 0 \le \phi \le 2\pi \end{gathered}}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\cos\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

$$\iiint_{R_{xyz}} f(x,y,z)\,dx\,dy\,dz = \iiint_{R'_{r\theta\phi}} F(r,\theta,\phi)\, \overset{|J|}{\overset{\shortparallel}{r^2\sin\theta}}\, dr\,d\theta\,d\phi$$

1. Evaluate

$$\iiint (10-x^2-y^2-z^2)\,dx\,dy\,dz \quad \nearrow f(x^2+y^2+z^2).$$

R = a sphere of radius 3

Soln. $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

$\therefore x^2+y^2+z^2 = r^2 \qquad 0 \le r \le 3, \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \ 0 \le \phi \le 2\pi$

$$I = \int_{\phi=0}^{2\pi}\int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{3} (10-r^2)\, r^2 |\sin\theta|\, dr\, d\theta\, d\phi$$

$|J| = r^2|\sin\theta|$ $\qquad \because -\frac{\pi}{2} \le \theta \le 0 \Rightarrow |J| = -r^2\sin\theta$

$\qquad 0 \le \theta \le \frac{\pi}{2} \Rightarrow |J| = r^2\sin\theta$

$$\therefore I = \int_{\phi=0}^{2\pi}\int_{\theta=-\pi/2}^{0}\int_{r=0}^{3} -(10-r^2)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$+ \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{3} (10-r^2)\, r^2\sin\theta\, dr\, d\theta\, d\phi.$$

$$= \frac{36 \times 23}{15}\pi.$$

2. $$\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} \frac{r^2\sin\theta\, dr\, d\theta\, d\phi}{\sqrt{1-r^2}} = \frac{\pi^3}{8}.$$

R = portion of the sphere $x^2+y^2+z^2=1$ in the first octant

3. $$\iiint z\overbrace{(x^2+y^2)}^{f(x^2+y^2)}\,dx\,dy\,dz$$

R = portion of the cylinder $x^2+y^2=1$

between the planes $z=2$ & $z=3$

<u>Soln.</u> $x = r\cos\theta$; $y = r\sin\theta$; $z = z$.

$$I = \int_{z=2}^{3}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} z\, r^2 \underset{r}{\overset{|J|}{\shortparallel}}\, dr\, d\theta\, dz$$

$$= 2\pi \left[\frac{r^4}{4}\right]_0^1 \left[\frac{z^2}{2}\right]_2^3 = \frac{\pi}{2}\left[\frac{3^2-2^2}{2}\right] = \frac{5\pi}{4}$$

4. Using cylindrical polar coordinates evaluate

$$\iiint z\,dx\,dy\,dz$$

R = upper part of the sphere $x^2+y^2+z^2=a^2$

<u>Soln,</u> $x = r\cos\theta$; $y = r\sin\theta$; $z = z$

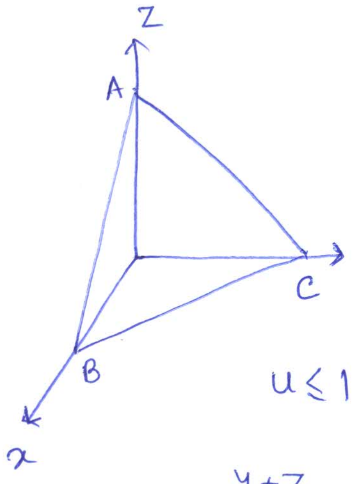
$$I = \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{\sqrt{a^2-r^2}} z\, r\, dr\, d\theta\, dz \quad = \frac{\pi}{4}a^4$$

5. $\iiint\limits_R \left[\dfrac{1-x-y-z}{xyz}\right]^{1/2} dx\,dy\,dz.$

$R \to$ region bounded by the planes

$x \geq 0, y \geq 0, z \geq 0$ & $x+y+z \leq 1$

Soln.



$$x+y+z=u$$

$$y+z=uv$$

$$z=uvw$$

$$u \geq 0, \; v \geq 0, \; w \geq 0$$

$$u \leq 1 \quad \because x+y+z \leq 1.$$

$$v = \frac{y+z}{x+y+z} \leq 1$$

$$w = \frac{z}{y+z} \leq 1$$

$$0 \leq u \leq 1, \quad 0 \leq v \leq 1, \quad 0 \leq w \leq 1.$$

$$z = uvw$$

$$\therefore y = uv - z$$
$$= uv - uvw$$
$$= uv(1-w).$$

$$x = u - y - z$$
$$= u - uv = u(1-v)$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & u(1-w) & vw \\ -u & u & uw \\ 0 & -uv & uv \end{vmatrix} = u^2 v$$

$$I = \iiint \frac{(1-x-y-z)^{1/2}}{x^{1/2}\, y^{1/2}\, z^{1/2}}\, dx\, dy\, dz$$

$$= \int_{w=0}^{1} \int_{v=0}^{1} \int_{u=0}^{1} \frac{(1-u)^{1/2}\ \overset{|J|}{\overset{\shortparallel}{u^2 \cancel{v}}}}{u^{1/2}(1-v)^{1/2}\, u^{1/2} \cancel{v^{1/2}}\, (1-w)^{1/2}\, u^{1/2} \cancel{v^{1/2}}\, w^{1/2}}\, du\, dv\, dw$$

$$= \int_{w=0}^{1} \int_{v=0}^{1} \int_{u=0}^{1} u^{1/2}(1-u)^{1/2}\, (1-v)^{-1/2}\, w^{-1/2}(1-w)^{-1/2}\, du\, dv\, dw$$

$$= \left(\int_0^1 u^{1/2}(1-u)^{1/2}\, du \right) \left(\int_0^1 (1-v)^{-1/2}\, dv \right) \left(\int_0^1 w^{-1/2}(1-w)^{-1/2}\, dw \right)$$

$$= B\left(\tfrac{3}{2}, \tfrac{3}{2}\right) B\left(1, \tfrac{1}{2}\right) B\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$= \frac{\Gamma(\frac{3}{2})\cancel{\Gamma(\frac{3}{2})}}{\Gamma(3)} \ \frac{\Gamma(1)\Gamma(\frac{1}{2})}{\cancel{\Gamma(\frac{3}{2})}} \ \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}$$

$$= \frac{\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{2\,\Gamma(2)} = \frac{\pi^2}{4}.$$

Application of multiple integrals.

$$\iint_{D_{xy}} dx\,dy = \text{area of } D_{xy} \text{ (plane region)}$$

$$\iiint_R dx\,dy\,dz = \text{volume of } R$$

R is bounded by the surfaces $z = f_2(x,y)$ (on top) & by $z = f_1(x,y)$ on the bottom.

$$\text{Volume of } R = \iint_{D_{xy}} f_2(x,y)\,dx\,dy - \iint_{D_{xy}} f_1(x,y)\,dx\,dy$$

Ex. Find the volume enclosed by the paraboloid $y = z^2 + x^2$ & the planes $y = -1$, $y + z = 4$.

Soln

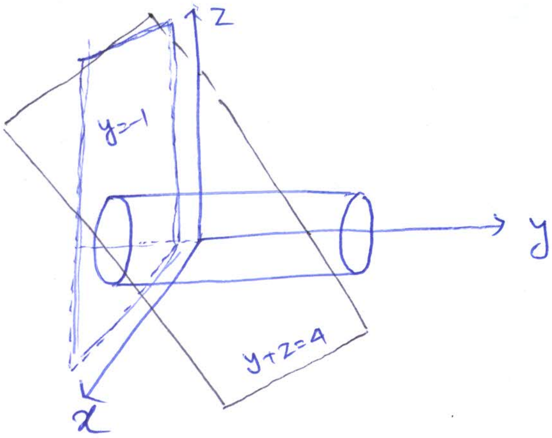
R is bounded by $y = f_2(z,x) = 4 - z$ & by $y = f_1(z,x) = -1$

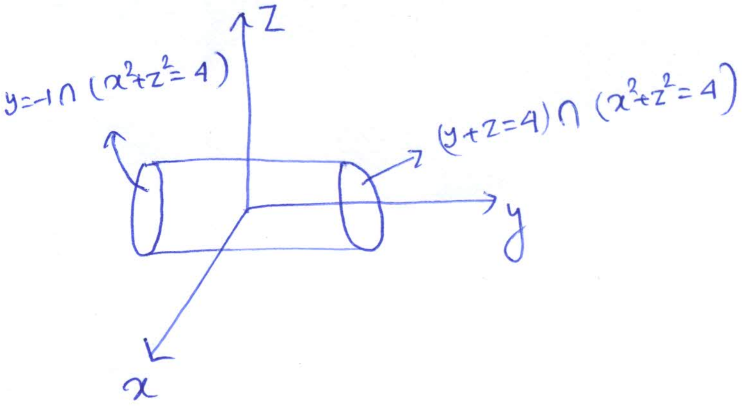
$$V = \iint_{D_{zx}} f_2(z,x)\,dz\,dx - \iint_{D_{zx}} f_1(z,x)\,dz\,dx$$

$$= \iint_{x^2+y^2 \leq 4} (4-z)\,dz\,dx - \iint_{x^2+y^2 \leq 4} (-1)\,dz\,dx$$

Ex. Find the volume enclosed by the cylinder $x^2+z^2=4$ & the planes $y=-1$, $y+z=4$.

Soln.



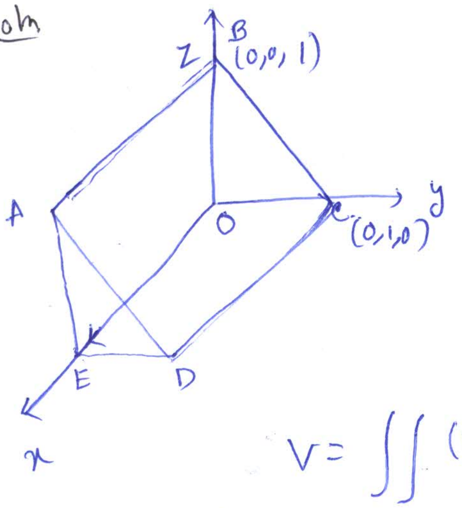


$$V = \iint_{x^2+z^2\le 4} \int_{y=-1}^{4-z} dx\,dy\,dz = \iint_{x^2+z^2\le 4} (5-z)\,dz\,dx$$

$$= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} (5 - r\sin\theta)\, r\, dr\, d\theta.$$

Ex. Find the volume of the portion of the solid in the first octant bounded by the planes $x=2$ & $y+z=1$.

Soln



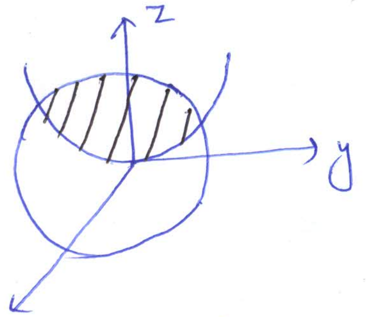
$$V = \int_{x=0}^{2} \int_{z=0}^{1} \int_{y=0}^{1-z} dx\,dy\,dz = 1 \text{ unit}^3$$

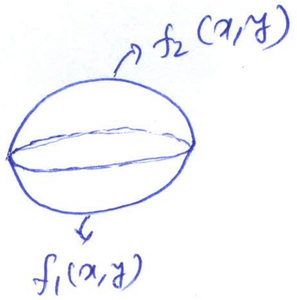
$$V = \int_{y=0}^{1} \int_{x=0}^{2} \int_{z=0}^{1-y} dx\,dy\,dz = 1 \text{ unit}^3$$

$$V = \iint_{f_2(z,x)} (1-z)\,dx\,dz - \iint_{f_1(z,x)} 0\,dx\,dz$$

<u>Ex.</u> Volume of solid bounded by $x^2+y^2+z^2=6$ & the paraboloid $x^2+y^2=z$.

<u>Soln.</u>





$$x^2+y^2+z^2=6 \quad ; \quad x^2+y^2=z$$

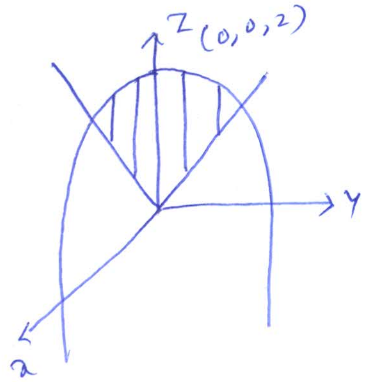
$$\Rightarrow \quad z+z^2=6$$
$$\Rightarrow \quad z^2+z-6=0$$
$$\Rightarrow (z+3)(z-2)=0$$
$$\Rightarrow z=-\cancel{3},\ 2 \qquad \therefore z=2 \Rightarrow \underline{x^2+y^2=2}$$

$$V=\iint\limits_{x^2+y^2\le 2}\int_{z=x^2+y^2}^{\sqrt{6-x^2-y^2}} dx\,dy\,dz=\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2}}\left(\sqrt{6-r^2}-r^2\right)r\,dr\,d\theta$$

$$=\frac{6\sqrt{6}-11}{3} \text{ cubic units.}$$

Ex. Find the volume of the solid bounded by the paraboloid $z = 2 - x^2 - y^2$ & the conic surface $z = \sqrt{x^2+y^2}$.

Soln.



$$z = 2 - x^2 - y^2$$

$$z = \sqrt{x^2+y^2}$$

$$\Rightarrow z = 2 - z^2$$

$$\Rightarrow z^2 + z - 2 = 0$$

$$\Rightarrow (z+2)(z-1) = 0$$

$$\Rightarrow z = -\cancel{2}, 1$$

$$\therefore z = 1 \Rightarrow x^2 + y^2 = 1.$$

$$V = \iint\limits_{x^2+y^2 \leq 1} \int_{z=\sqrt{x^2+y^2}}^{2-x^2-y^2} dx\,dy\,dz$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \{(2-r^2) - r\} r\,dr\,d\theta.$$