TOTAL DIFFERENTIAL AND DIFFERENTIABILITY

ONE VARIABLE:

Def. 1: We call a function $y = f(x)$ differentiable at a point (x, y) if

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

exists. The value of the above limit is called the derivative of f at x.

Def. 2: The function $y = f(x)$ is said to be differentiable at the point (x, y) if, at this point

$$\Delta y = f(x+\Delta x) - f(x) = a\,\Delta x + \varepsilon \cdot \Delta x$$

where a is independent of Δx and $\lim_{\Delta x \to 0} \varepsilon = 0$.
The value of a is the derivative of f at x.

REMARK: Not that Def 1 & 2 are equivalent as

$$f(x+\Delta x) - f(x) = a\,\Delta x + \varepsilon\,\Delta x$$

$$\Leftrightarrow \quad \frac{f(x+\Delta x) - f(x)}{\Delta x} = a + \varepsilon$$

$$\Leftrightarrow \quad \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(a)}{\Delta x} = a \qquad (\varepsilon \to 0 \text{ as } \Delta x \to 0)$$

Def 1 is more practical for verifying differentiability of a function.

DIFFERENTIAL: The differential of the dependent variable y, written as dy, is defined to be

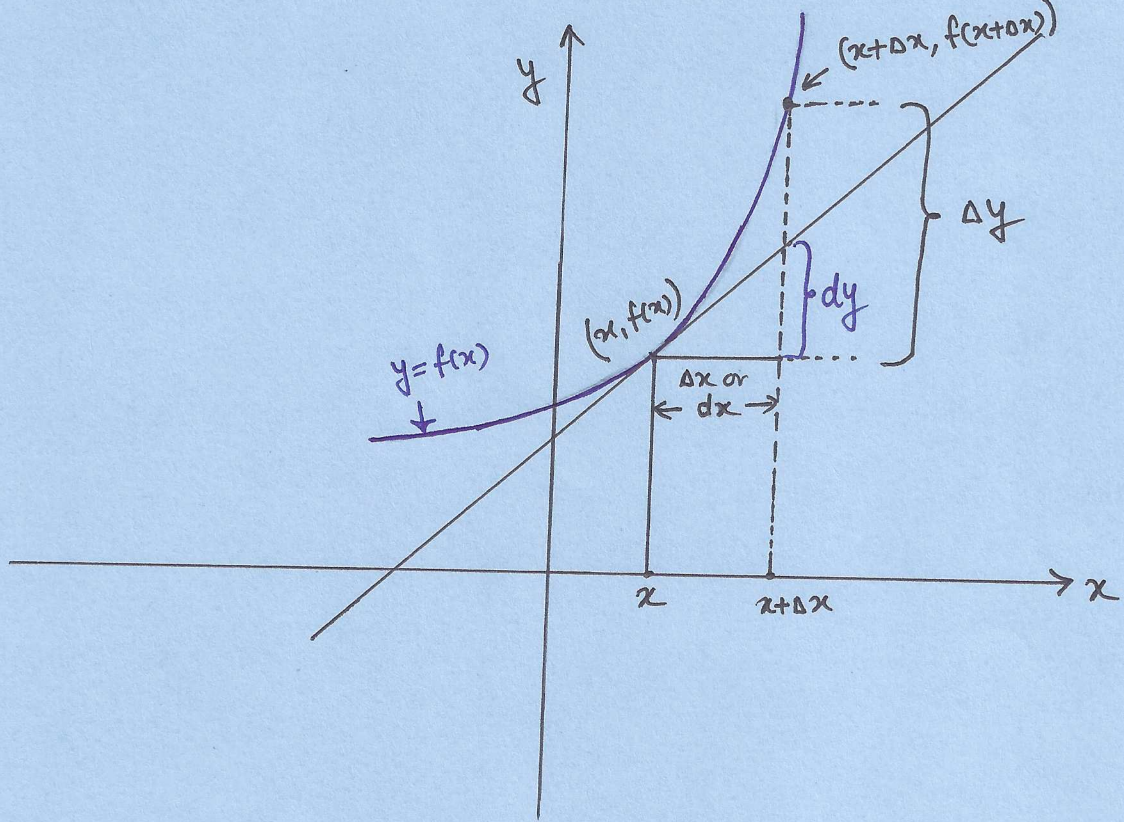
$$dy = f'(x)\,\Delta x, \qquad \text{where } y = f(x)$$

$$\text{or } dy = f'(x)\,dx$$

$$\text{or } df = f'(x)\,dx$$

Differential of the independent variable x, written as dx, is same as Δx. one can also observed this by taking $y = x$ and using the above definition of differential as

$$dx = (x)' \Delta x \Rightarrow dx = \Delta x$$



Note that Δx (or dx) is an increment while dy is total differential.

Δy is the change in y due to change in x by Δx or dx

Also Note that $$\Delta y = \underbrace{f'(x)\, \Delta x}_{\text{linear part}} + \varepsilon\, \Delta x$$

So the differential is a linear function of the increment Δx.

TWO VARIABLE :

The function $z = f(x, y)$ is said to be differentiable at the point (x, y) if, at this point

$$\Delta z = a\,\Delta x + b\,\Delta y + \epsilon_1 \cdot \Delta x + \epsilon_2 \cdot \Delta y$$

where a and b are independent of Δx, Δy and ϵ_1 and ϵ_2 are functions of Δx and Δy such that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_1 = 0 \quad \text{and} \quad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_2 = 0$$

The linear function of Δx and Δy $a\,\Delta x + b\,\Delta y$ is called the total differential of z at the point (x, y) and is denoted by dz

$$\begin{aligned} dz &= a\,\Delta x + b\,\Delta y \\ &= a\,dx + b\,dy \end{aligned}$$

If Δx and Δy are sufficiently small, dz gives a close approximation to Δz.

EXAMPLE: Show that $z = x^2 + xy + xy^2$ is differentiable and write down its total differential.

SOLUTION:

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= (x+\Delta x)^2 + (x+\Delta x)(y+\Delta y) + (x+\Delta x)(y+\Delta y)^2 - x^2 - xy - xy^2$$

$$= \Delta x\,(2x + y + y^2) + \Delta y\,(x + 2x$$

$$+ (\Delta x + \Delta y\,(1+2y))\,\Delta x$$

$$+ (x.\Delta y + \Delta x\,\Delta y)\,\Delta y$$

hence the function is differentiabl

Total differential

$$dz = (2x + y + y^2)\,dx + (x + 2xy)\,dy .$$

NECESSARY CONDITION FOR DIFFERENTIABILITY

THEOREM: If $z = f(x, y)$ is differentiabl then $f(x, y)$ is continuous and has partial derivatives with respe to x and y at the point (x, y) and that

$$a = f_x(x, y) = \frac{\partial z}{\partial x} , \quad b = f_y(x, y) = \frac{\partial z}{\partial y}$$

PROOF: Let f be differentiable, the