## Tutorial Problems set-I

**Note:** All these problems can be solved using the results of Chapter-I.

[**0.0.1**] ***Exercise*** Check that each of the following sets are vector space with respect to usual addition and scalar multiplication.
(i) The set of all real sequences over the field $\mathbb{F} = \mathbb{R}$.
(ii) The set of all bounded real sequences over the filed $\mathbb{R}$.
(iii) The set of all convergent real sequences over the field $\mathbb{R}$.
(iv) $\{(a_n) \mid a_n \in \mathbb{R},\, a_n \to 0\}$ over the field $\mathbb{R}$.
(v) The set of all **eventually** 0 sequences over the field $\mathbb{R}$. We call $(x_n)$ eventually 0 if $\exists k$ s.t. $x_n = 0$ for all $n \geq k$.
(vi) $\mathbb{P}(x)= \{p(x) \mid p(x)$ is a real polynomial in $x\}$ over the field $\mathbb{R}$.
(vii) $\mathbb{P}_5(x)= \{p(x) \in \mathbb{R}[x] \mid$ degree of $p(x) \leq 5\}$ over the field $\mathbb{R}$.
(viii) $\{A_{n\times n} \mid a_{ij} \in \mathbb{R},\, A$ upper triangular$\}$ over the field $\mathbb{R}$.

[**0.0.2**] ***Exercise*** Consider $\mathbb{P}_n(x)$ and $\mathbb{P}(x)$ over $\mathbb{R}$. Check that each of the following sets is subspace or not.
(i) $\{P(x) \in \mathbb{P}_3(x) \ \mid \ P(x) = ax + b, a, b \in \mathbb{R}\}$.
(ii) $\{P(x) \in \mathbb{P} \ \mid \ P(0) = 0\}$.
(iii) $\{P(x) \in \mathbb{P} \ \mid \ P(0) = 1\}$.
(iv) $\{P(x) \in \mathbb{P} \ \mid \ P(-x) = P(x)\}$.
(v) $\{P(x) \in \mathbb{P} \ \mid \ P(-x) = -P(x)\}$.

[**0.0.3**] ***Exercise*** Fix $A \in \mathcal{M}_n(\mathbb{R})$. Let $\mathbb{U} = \{B \in \mathcal{M}_n(\mathbb{R}) : AB = BA\}$.
a) Show that $\mathbb{U}$ is a subspace of $\mathcal{M}_n(\mathbb{R})$.
b) Let $\mathbb{W} = \{a_0 I + a_1 A + \cdots a_n A^m | m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, a_i \in \mathbb{R}\}$. Show that $\mathbb{W}$ is a subspace of $\mathbb{U}$.

[**0.0.4**] ***Exercise*** Find basis and dimension for each of the following vector spaces.
(i) $\mathbb{M}_n(\mathbb{C})$ over $\mathbb{R}$.
(ii) $\mathbb{H}_n(\mathbb{C})$, $n \times n$ Hermitian matrices, over $\mathbb{R}$.
(iii) $\mathbb{S}_n(\mathbb{C})$, $n \times n$ Skew-Hermitian matrices, over $\mathbb{R}$.

[**0.0.5**] ***Exercise*** Check whether the following vector space is finite dimensional or infinite dimensional.
(i) The set of all real sequences over the field $\mathbb{F} = \mathbb{R}$.
(ii) The set of all bounded real sequences over the field $\mathbb{R}$.
(iii) The set of all convergent real sequences over the field $\mathbb{R}$.
(iv) $\{(a_n) \mid a_n \in \mathbb{R},\, a_n \to 0\}$ over the field $\mathbb{R}$.
(v) The set of all **eventually** 0 sequences over the field $\mathbb{R}$.
(vi) We call $(x_n)$ eventually 0 if $\exists k$ s.t. $x_n = 0$ for all $n \geq k$.
(vii) $\mathbb{P}(x)= \{p(x) \mid p(x)$ is a real polynomial in $x\}$ over the field $\mathbb{R}$.
(viii) $\mathbb{P}_5(x)= \{p(x) \in \mathbb{R}[x] \mid$ degree of $p(x) \leq 5\}$ over the field $\mathbb{R}$.

[**0.0.6**] ***Exercise*** Write 4 nontrivial subspaces of $\mathbb{R}^4$.

[**0.0.7**] ***Exercise*** Show that $u_1, \cdots, u_k \in \mathbb{R}^n$ are linearly independent iff $Au_1, \cdots, Au_k$ are linearly independent for any invertible $A_n$.

[**0.0.8**] ***Exercise*** Show that $u_1, \cdots, u_k \in \mathbb{V}$ is linearly independent iff $\sum\limits_{i=1}^{k} a_{i1}u_i, \cdots, \sum\limits_{i=1}^{k} a_{ik}u_i$ are linearly independent for any invertible $A_{k\times k}$. Show that $\{u, v\}$ is linearly independent iff $\{u + v, u - v\}$ is linearly independent.

[**0.0.9**] ***Exercise*** Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $A$ and $B$ be two non-empty subsets of $\mathbb{V}$. Prove or disprove: $\text{LS}(A) \cap \text{LS}(B) \neq \{0\} \implies A \cap B \neq \phi$.

[**0.0.10**] ***Exercise*** Show that a vector space $\mathbb{V}$ over $\mathbb{F}$ has a unique basis if and only if either $\text{DIM}(\mathbb{V}) = 0$ or $\text{DIM}(\mathbb{V}) = 1$ and $|\mathbb{F}| = 2$.

[**0.0.11**] ***Exercise*** Let $\mathbb{V}$ be an $n$ dimensional vector space over $\mathbb{F}$ and let $\mathbb{F}$ has exactly $p$ elements. Then show that $|\mathbb{V}| = p^n$.

[**0.0.12**] ***Exercise*** Check whether vector space $\mathbb{R}$ (set of real numbers) over the field $\mathbb{Q}$ (set rational number) is infinite dimensional or finite dimensional.

[**0.0.13**] ***Exercise*** Let $S = \{ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} a \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \}$. Find the values of $a$ for which $\text{LS}(S) \neq \mathbb{R}^3$.

[**0.0.14**] ***Exercise*** Give 2 bases for the trace 0 real symmetric matrices of size $3 \times 3$. Extend these bases to bases of the real symmetric matrices of size $3 \times 3$. Extend these bases to bases of the real matrices of size $3 \times 3$.

[**0.0.15**] ***Exercise*** Consider $\mathbb{W} = \{v \in \mathbb{R}^6 | v_1 + v_2 + v_3 = 0, v_2 + v_3 + v_4 = 0, v_4 + v_5 + v_6 = 0\}$. Supply a basis for $\mathbb{W}$ and extend it to a basis of $\mathbb{R}^6$.

[**0.0.16**] ***Exercise*** For what values $\alpha$ are the vectors $(0, 1, \alpha), (\alpha, 1, 0)$ and $(1, \alpha, 1)$ in $\mathbb{R}^3$ linearly independent?

[**0.0.17**] ***Exercise*** If $S$ and $T$ are two subspaces of a vector spaces having a common complement set $W$, does it follow that $S = T$?

[**0.0.18**] ***Exercise*** In the vector space $\mathbb{R}^4$, find two different complements of the subspace $S = \{(x_1, x_2, x_3, x_4) : x_3 - x_4 = 0\}$

[**0.0.19**] ***Exercise*** Show that a non-trivial subspace $S$ of a finite dimensional vector space $\mathbb{V}$ has two virtually disjoint complements iff $\text{DIM}(S) \geq \frac{\text{DIM}(\mathbb{V})}{2}$.

[**0.0.20**] ***Exercise*** Find a complement of the subspace $\{(x_1, x_2, \ldots, x_n) : \sum\limits_{i=1}^{n} x_i = 0\}$ in $\mathbb{R}^n$.