AD

End Semester Examination (Spring 2017)
Subject Number: MA51002, Subject Name: Measure Theory and Integration
Department: Mathematics, Full Marks: 50, Duration: 3 Hrs.

Answer **all** the problems. Numbers at the right hand side after each question denote marks. No clarification will be entertained during the examination.

(1) Suppose f is a non-negative measurable function on $\mathbb{R}$. Prove that there exists an increasing sequence of non-negative simple functions that converges pointwise to f. [5]

(2) State and prove the Borel-Cantelli Lemma. [2+3]

(3) "Every function is nearly continuous" - Justify this statement in the sense of Littlewood. [5]

(4) State and prove a continuous parameter version of the Dominated convergence theorem. [2+3]

(5) Show that

$$\lim_{n\to\infty}\int_a^\infty \frac{n^2 x e^{-n^2x^2}}{1+x^2}\,dx = 0$$

if $a > 0$. What happens to the value of the above integral if $a = 0$? [3+2]

(6) Show that if f is measurable, then the set $\{x : f(x) = \alpha\}$ is also measurable where $\alpha \in [-\infty, +\infty]$. Prove that the set of points on which a sequence of measurable functions $\{f\}_n$ converges is measurable. [2+3]

(7) Let $f_n(x) = \frac{n^{3/2}x}{1+n^2x^2}$ for $x \in [0, 1]$.
  (i) Show that $f_n(x) \to 0$ for all $x \in [0, 1]$
  (ii) Show that the sequence $\{f_n\}$ is not uniformly bounded
  (iii) Explain why the conditions of the Dominated Convergence Theorem are satisfied and make a conclusion concerning the limit of $\int f_n$ [1+2+2]

(8) Is $C[a, b]$–the space of all continuous functions, a complete metric space in L_1-metric? Justify. Prove that $L_1[a, b]$ is the completion of the space of all Riemann integrable functions in [a,b]. [2+3]

(9) Let f be a bounded measurable function on (a, b). Show that [5]

$$\lim_{n\to\infty}\int_a^b f(x)e^{inx} = 0$$

(10) State Fubini's theorem. Prove that the condition $f \in L_1(X \times Y)$ is necessary in the hypothesis of the theorem. [2+3]