Multistep Method

The general multistep method or K-step method can be written as:

$$u_{j+1} = a_1 u_j + a_2 u_{j-1} + \cdots\cdots + a_K u_{j-K+1}$$
$$+ h\left(b_0 u'_{j+1} + b_1 u'_j + b_2 u'_{j-1} + \cdots\cdots + b_K u'_{j-K+1}\right)$$

OR $\qquad j = K-1, K, \ldots\ldots$

$$\boxed{u_{j+1} = \sum_{i=1}^{K} a_i u_{j-i+1} + h \sum_{i=0}^{K} b_i u'_{j-i+1}}$$

Define a shift operator.

$$E f(x_i) = f(x_{i+1})$$
$$E^2 f(x_i) = f(x_{i+2})$$
$$\vdots$$
$$E^K f(x_i) = f(x_i+K)$$

OR in discrete form

$$E^K f_i = f_{i+K}$$

With the shift operator the multistep method can be rewritten as

$$E^K u_{j-K+1} = a_1 E^{K-1} u_{j-K+1} + a_2 E^{K-2} u_{j-K+1} + \cdots\cdots + a_K u_{j-K+1}$$
$$+ h\left(b_0 E^K u'_{j-K+1} + b_1 E^{K-1} u'_{j-K+1} + \cdots\cdots + b_K u'_{j-K+1}\right)$$

OR

$$\left(E^K - a_1 E^{K-1} - a_2 E^{K-2} + \cdots\cdots - a_K\right) u_{j-K+1}$$

$$- h\left(b_0 E^K + b_1 E^{K-1} + \cdots\cdots + b_K\right) u'_{j-K+1} = 0$$

$$\Rightarrow \quad \rho(E)\, u_{j-K+1} - h\, \sigma(E)\, u'_{j-K+1} = 0$$

wher ρ & σ are polynomials defined by

$$\rho(\xi) = \left(\xi^K - a_1 \xi^{K-1} - a_2 \xi^{K-2} - \cdots\cdots - a_K\right)$$

and

$$\sigma(\xi) = \left(b_0 \xi^K + b_1 \xi^{K-1} + b_2 \xi^{K-2} + \cdots\cdots + b_K\right)$$

Note that, if $b_0 = 0$, the method is called an explicit or predictor method. When $b_0 \neq 0$, it is called an implicit method.

Example: Midpoint Method:

$$u_{n+1} = u_{n-1} + 2h f_n$$

where u_0, u_1 is to be known before applying the above two-step explicit method.

The local truncation error will be given as:

$$\tau_{j+1} = y(t_{j+1}) - \sum_{i=1}^{k} a_i\, y(t_{j-i+1}) - h\sum_{i=0}^{k} b_i\, y'(t_{j-i+1})$$

Further simplifications using Taylor's series expansion of $y(t_{j+1})$, $y(t_{j-i+1})$ & $y'(t_{j-i+1})$ give:

$$\tau_{j+1} = y(t_j) + h\,y'(t_j) + \frac{h^2}{\lfloor\underline{2}}\,y''(t_j) + \cdots + \frac{h^p}{\lfloor\underline{p}}\,y^{(p)}(t_j) + \frac{h^{p+1}}{\lfloor\underline{p+1}}\,y^{(p+1)}(t_j) + O(h^{p+2})$$

$$- \sum_{i=1}^{k} a_i \Big[y(t_j) + \underbrace{(t_{j-i+1} - t_j)}_{\begin{subarray}{l} = t_0 + (j-i+1)h - t_0 - jh \\ = (1-i)h \end{subarray}}\, y'(t_j) + \frac{(1-i)^2 h^2}{\lfloor\underline{2}}\, y''(t_j)$$

$$+ \cdots + \frac{(1-i)^p}{\lfloor\underline{p}}\, h^p\, y^{(p)}(t_j) + \frac{(1-i)^{p+1}}{\lfloor\underline{p+1}}\, h^{p+1}\, y^{(p+1)}(t_j) + O(h^{p+2}) \Big]$$

$$- h\sum_{i=0}^{k} b_i \Big[y'(t_j) + (1-i)h\, y''(t_j) + \cdots + \frac{(1-i)^{p-1}}{\lfloor\underline{p-1}}\, h^{p-1}\, y^{(p)}(t_j)$$

$$+ \frac{(1-i)^p}{\lfloor\underline{p}}\, h^p\, y^{(p+1)}(t_j) + O(h^{p+\cancel{2}}) \Big]$$

This can be rewritten in the following form

$$\tau_{j+1} = c_0\, y(t_j) + c_1 h\, y'(t_j) + c_2 h^2 y''(t_j) + \cdots + c_p h^p\, y^{(p)}(t_j) + T_{p+1}$$

Where

$$C_0 = 1 - \sum_{i=1}^{k} a_i$$

$$C_q = \frac{1}{q!}\left[1 - \sum_{i=1}^{k} a_i (1-i)^q\right] - \frac{1}{(q-1)!} \sum_{i=0}^{k} b_i (1-i)^{q-1}$$

$$q = 1, 2, \ldots, p+1 .$$

$$T_{p+1} = C_{p+1} h^{p+1} y^{(p+1)}(t_j) + O(h^{p+2})$$

Definition: 1. The linear multistep method is said to be consistent if it has order $p \geq 1$.

2. The linear multistep method is said to be of order p if $C_0 = C_1 = \cdots = C_p = 0$ & $C_{p+1} \neq 0$

Hence for a consistent method C_0 & C_1 must be ZERO.

Ex: For a consistent method, show that

$$\rho'(1) = a_1 + 2a_2 + \cdots + k a_k$$

Sol:
$$\rho(\xi) = \xi^k - a_1 \xi^{k-1} - a_2 \xi^{k-2} - \cdots - a_k$$

$$\rho'(\xi) = k\xi^{k-1} - a_1 (k-1)\xi^{k-2} - a_2 (k-2)\xi^{k-3} - \cdots - a_{k-1}$$

Then

$$\rho'(1) = k - a_1(k-1) - a_2(k-2) - \cdots - a_{k-1}$$

$$= k - a_1(k-1) - a_2(k-2) - \cdots - a_{k-1}(k-(k-1))$$

$$= k(1 - a_1 - a_2 - \cdots - a_{k-1} - a_k + a_k)$$
$$+ a_1 + 2a_2 + \cdots + (k-1)a_{k-1}$$

We know that for a consistent method c_0 must be 0, that means $1 - \sum_{i=1}^{k} a_i = 0$

Then

$$\rho'(1) = k(0 + a_k) + a_1 + 2a_2 + \cdots + (k-1)a_{k-1}$$

$$= a_1 + 2a_2 + \cdots + k a_k$$

$\square$.

Ex: Prove that if a method is consistent then

$$\rho(1) = 0 \quad \&$$

$$\rho'(1) = \sigma(1)$$

<u>Proof</u>: For a consistent method:

$$C_0 = 0 \quad \& \quad C_1 = 0.$$

$$C_0 = 0 \Rightarrow 1 - (a_1 + a_2 + \cdots + a_K) = 0$$

$$\Rightarrow \rho(1) = 0$$

$$C_1 = 0 \Rightarrow [1 + a_2 + 2a_3 + \cdots + (K-1)a_K]$$

$$- [b_0 + b_1 + \cdots + b_K] = 0$$

$$\Rightarrow 1 + [a_1 + 2a_2 + 3a_3 + \cdots + Ka_K]$$

$$- [a_1 + a_2 + \cdots + a_K] - [b_0 + b_1 + \cdots + b_K] = 0$$

$$\Rightarrow 1 + \rho'(1) - 1 - \sigma(1) = 0$$

$$\Rightarrow \rho'(1) = \sigma(1)$$

Therefor a method is consistent if

$$\rho(1) = 0 \quad \text{and}$$

$$\rho'(1) = \sigma(1).$$

Definition: The multistep method

$$\rho(E)\, u_{j-k+1} - h\,\sigma(E)\, u'_{j-k+1} = 0$$

is said to satisfy the root condition if all roots of the equation $\rho(\xi) = 0$ are contained within the unit circle centered at the origin of the complex plane, otherwise, if they fall on its boundary, they must be simple roots of ρ. Equivalentty

let r_j be the roots of $\rho(\xi)$ then

$$\begin{cases} |r_j| \leq 1 \qquad j = 1, 2, \cdots k. \\ \text{furthermore, for those } j \text{ such that } |r_j| = 1 \text{ then } \rho'(r_j) \neq 0 \end{cases}$$

Remark: For a consistent method, the root condition is equivonlent to zero-stability. More stronger versions of stability will be considered later.

Theorem: The linear multistep method is convergent iff the method is consistent and satisfies the root condition.

Ex: Show that the method

$$u_{j+1} - 3u_j + 2u_{j-1} = \frac{h}{2}\left(f_j - 3f_{j-1}\right)$$

is not convergent.

Sol: The given multistep method can be written in the following form:

$$(E^2 - 3E + 2)u_{j-1} - h\left(\frac{E}{2} - \frac{3}{2}\right)f_{j-1} = 0$$

$$\Rightarrow \quad \rho(\xi) = \xi^2 - 3\xi + 2$$

$$\sigma(\xi) = \frac{1}{2}(\xi - 3)$$

Consistency Check: $\rho(1) = 1 - 3 + 2 = 0$ ✓

$$\sigma(1) = \frac{1}{2}(1-3) = -1$$

$$\rho'(\xi) = 2\xi - 3$$

$$\rho'(1) = -1$$

hence $\rho(1) = 0$ & $\rho'(1) = \sigma(1)$.

The method is consistent.

Root condition:

$$\rho(\xi) = \xi^2 - 3\xi + 2 = 0$$

$$\Rightarrow \xi^2 - 2\xi - \xi + 2 = 0$$

$$\Rightarrow (\xi - 2)(\xi - 1) = 0$$

$$\Rightarrow \xi = 1, \boxed{2.}$$

Hence the root condition is not satisfied.

The given method is not convergent.