MAXIMA AND MINIMA OF A FUNCTION

DEF:

1. A function $z = f(x,y)$ has a maximum (or a minimum) at the point (x_0, y_0) if at every point in a neighbourhood of (x_0, y_0) the function assumes a smaller value (or a larger value) than at the point itself. Such a maximum or minimum is ofter called relative (or local) maximum or minimum respectively.

2. For a given closed and bounded domain, a function may also attain its greatest value (or least value) on the boundary of the domain.

   The smallest and the largest values attained by a function over the entire domain including the boundary are called the absolute (or global) minimum and absolute (or global) maximum, respectively.

3. The point (x_0, y_0) is called critical point (or stationary point) of $f(x,y)$ if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.

4. A critical point where the function has no minimum or maximum is called a saddle point.

5. Minimum and maximum values together are called extreme values.

Theorem (Necessary conditions for a function to have extremum)

Let $f(x,y)$ be continuous and have first order partial derivatives at a point $P(a,b)$. Then necessary conditions for the existence of an extreme value of it at the point P are

$$f_x(a,b) = 0 \quad \& \quad f_y(a,b) = 0 .$$

OR

If the point (a,b) is a relative extrema of the function $f(x,y)$ then (a,b) is also a critical point of $f(x,y)$.

Proof: Let $(a+h, b+k)$ be a point in the neighbourhood of the point $P(a,b)$. Then P will be a point of maximum if

$$\Delta f = f(a+h, b+k) - f(a,b) \leq 0 \text{ for all sufficiently small } h \,\&\, k$$

and a point of minimum if

$$\Delta f = f(a+h, b+k) - f(a,b) \geq 0 \text{ for all sufficiently small } h \,\&$$

Taylor's series expansion about the point (a,b):

$$f(a+h, b+k) = f(a,b) + \left(hf_x + kf_y\right)_{(a,b)} + \frac{1}{2}\left(hf_x + kf_y\right)^2_{(a,b)} + \cdots$$

For sufficiently small h & k, we can neglect second and higher order terms, to set

$$\Delta f \simeq h f_x(a,b) + k f_y(a,b)$$

The sign of Δf depends on the sign of $h f_x(a,b) + k f_y(a,b)$.

Letting $h \to 0$ we find that Δf changes sign with k,

i.e., assuming $f_y(a,b) > 0$:

for $k > 0$; $\Delta f > 0$

for $k < 0$; $\Delta f < 0$

Therefore the function cannot have an extremum unless $f_y = 0$

Similarly, letting $k \to 0$, we find that the function f cannot have an extremum unless $f_x = 0$.

Therefore the necessary conditions for the existence of an extremum at the point (a,b) is that

$f_x(a,b) = 0$ & $f_y(a,b) = 0$.

$\square$.

SUFFICIENT CONDITIONS FOR A FUNCTION TO HAVE MINIMA/MAXIMA

For simplicity, we set

$$r = f_{xx}(a,b), \quad s = f_{xy}(a,b), \quad t = f_{yy}(a,b)$$

Let a function $f(x,y)$ be continuous and have first and second order partial derivatives at a point $P(a,b)$. If (a,b) is a critical point, then the point P is a point of

i) local maximum if $rt - s^2 > 0$ and $r < 0$ (r<0)

ii) local minimum if $rt - s^2 > 0$ and $r > 0$ (r>0)

iii) saddle point if $rt - s^2 < 0$

iv) may be a local minimum, local maximum or a saddle point if $rt - s^2 = 0$.

<u>Proof</u>: consider $\Delta f = f(a+h, b+k) - f(a,b)$

Note that $(a+h, b+k)$ is a point in the neighbourhood of (a,b)

By Taylor's series expansion

$$\Delta f = \left(h f_x + k f_y\right)_{(a,b)} + \frac{1}{2}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{(a,b)} + \cdots$$

As (a,b) is a critical point, meaning $f_x|_{(a,b)} = f_y|_{(a,b)} = 0$

$$\Rightarrow \Delta f = \frac{1}{2}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{(a,b)} + R$$

$$= \frac{1}{2}\left[h^2 r + 2hks + k^2 t\right] + R$$

$$= \frac{1}{2r}\left[h^2 r^2 + 2hkrs + k^2 rt\right] + R \quad (\text{Assuming } r \neq 0)$$

$$= \frac{1}{2r}\left[(hr+ks)^2 - k^2s^2 + k^2rt\right] + R$$

$$= \frac{1}{2r}\left[(hr+ks)^2 + k^2(rt-s^2)\right] + R \quad \left(\text{OR } \frac{1}{2t}\left[(hs+kt)^2 + h^2(rt-s^2)\right] + \dots\right)$$

same conclusions follow if $r=0$ & $t\neq 0$

Since $(hr+ks)^2$, the sufficient condition for the expression $\left[(hr+ks)^2 + k^2(rt-s^2)\right]$ to be positive is that

$$rt - s^2 > 0$$

$\Rightarrow$ If $rt-s^2>0$, then

i) $\Delta f > 0$ if $r>0$

ii) $\Delta f < 0$ if $r<0$

$\Rightarrow$ The point (a,b) is a point of $\begin{cases} \text{minimum if } (rt-s^2)>0 \ \& \ r>0 \\ \text{maximum if } (rt-s^2)>0 \ \& \ r<0 \end{cases}$

iii) If $rt-s^2<0$, then the sign of Δf depends on h & k.

For example,

let $k\to 0$ & $h\neq 0$ $\Rightarrow$ $\Delta f>0$ if $r>0$

and if $k\neq 0$ & we choose h such that $hr+ks=0$

$\Rightarrow \Delta f<0$ for $r>0$

Hence no maximum/minimum of f can occur at $P(a,b)$.

$\Rightarrow P(a,b)$ is a saddle point

(iv) If $rt-s^2=0$, then

$$\Delta f = \frac{1}{2r}\left[(hr+ks)^2\right] + R$$

If we take h & k such that $hr=-ks$ i.e, $\frac{h}{k} = -\left(\frac{s}{r}\right)$, then the whole second order terms of right hand side will vanish.

Therefore for these points in the neighbourhood we have to consider third order terms in the remainder. Other than these points we have

$\Delta f > 0$ for $r>0$ and

$\Delta f < 0$ for $r<0$.

Thus the conclusion will depend on the higher order terms.

$\Rightarrow$ A FURTHER INVESTIGATION IS REQUIRED.

WORKING RULES:

1) FIND CRITICAL POINTS OR STATIONARY POINTS $f_x=0$ & $f_y=0$.

2) FOR EACH CRITICAL POINT, EVALUATE

$$r=f_{xx}, \quad s=f_{xy}, \quad t=f_{yy}$$

3) IDENTIFICATION:

i) If $rt-s^2>0$ & $r<0$ $\rightarrow$ Maximum

ii) If $rt-s^2>0$ & $r>0$ $\rightarrow$ Minimum

iii) If $rt-s^2<0$ $\rightarrow$ Saddle point

iv) If $rt-s^2=0$ $\rightarrow$ Doubtful, needs further investigation