2.6 Application to Recurrence Relation

This section contains the applications of formal power series to solving recurrence relations. Let us try to understand it using the following examples.

Example 2.6.1. *1. Determine a formula for the numbers $a(n)$'s, where $a(n)$'s satisfy the recurrence relation $a(n) = 3\,a(n-1) + 2n$, for $n \geq 1$ with $a(0) = 1$.*

Solution: *Define $A(x) = \sum\limits_{n\geq 0} a(n)x^n$. Then using Example 2.5.1.1, one has*

$$\begin{aligned} A(x) &= \sum_{n\geq 0} a(n)x^n = a_0 + \sum_{n\geq 1} a(n)x^n = 1 + \sum_{n\geq 1} \left(3\,a(n-1) + 2n\right) x^n \\ &= 3x \sum_{n\geq 1} a(n-1)x^{n-1} + 2\sum_{n\geq 1} nx^n + 1 = 3xA(x) + 2\frac{x}{(1-x)^2} + 1. \end{aligned}$$

So, $A(x) = \dfrac{1+x^2}{(1-3x)(1-x)^2} = \dfrac{5}{2(1-3x)} - \dfrac{1}{2(1-x)} - \dfrac{1}{(1-x)^2}$. *Thus,*

$$a(n) = [x^n]A(x) = \frac{5}{2}3^n - \frac{1}{2} - (n+1) = \frac{5\cdot 3^n - 1}{2} - (n+1).$$

2. Determine a generating function for the numbers $f(n)$ that satisfy the recurrence relation

$$f(n) \quad = \quad f(n-1) + f(n-2), \quad \text{for} \quad n \geq 2 \qquad \text{with} \;\; f(0) = 1 \;\; \text{and} \; f(1) = 1.$$

Hence or otherwise find a formula for the numbers $f(n)$.

Solution: *Define $F(x) = \sum\limits_{n\geq 0} f(n)x^n$. Then one has*

$$\begin{aligned} F(x) &= \sum_{n\geq 0} f(n)x^n = 1 + x\sum_{n\geq 2} \left(f(n-1) + f(n-2)\right) x^n \\ &= 1 + x + x\sum_{n\geq 2} f(n-1)x^{n-1} + x^2 \sum_{n\geq 2} f(n-2)x^{n-2} = 1 + xF(x) + x^2F(x). \end{aligned}$$

Therefore, $F(x) = \dfrac{1}{1-x-x^2}$. *Let* $\alpha = \dfrac{1+\sqrt{5}}{2}$ *and* $\beta = \dfrac{1-\sqrt{5}}{2}$. *Then it can be checked that* $(1-\alpha x)(1-\beta x) = 1 - x - x^2$ *and*

$$F(x) = \frac{1}{\sqrt{5}}\left(\frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x}\right) = \frac{1}{\sqrt{5}}\left(\sum_{n\geq 0} \alpha^{n+1}x^n - \sum_{n\geq 0} \beta^{n+1}x^n\right).$$

Therefore,

$$f(n) = [x^n]F(x) = \frac{1}{\sqrt{5}}\sum_{n\geq 0}\left(\alpha^{n+1} - \beta^{n+1}\right).$$

As $\beta < 0$ and $|\beta| < 1$, we observe that $f(n) \approx \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1}$.

Remark 2.6.2. *The numbers $f(n)$, for $n \geq 0$ are called* FIBONACCI NUMBERS. *It is related with the following problem:* Suppose a couple bought a pair of rabbits (each one year old) in the year 2001. If a pair of rabbits starts giving birth to a pair of rabbits as soon as they grow 2 years old, determine the number of rabbits the couple will have in the year 2025.

3. *Determine a formula for the numbers $a(n)$'s, where $a(n)$'s satisfy the recurrence relation $a(n) = 3\, a(n-1) + 4a(n-2)$, for $n \geq 2$ with $a(0) = 1$ and $a(1) = c$, a constant.*

Solution: *Define* $A(x) = \sum\limits_{n \geq 0} a(n)x^n$. *Then*

$$\begin{aligned} A(x) &= \sum_{n\geq 0} a(n)x^n = a_0 + a_1 x + \sum_{n \geq 2} a(n)x^n = 1 + cx + \sum_{n\geq 2} (3\, a(n-1) + 4\, a(n-2))\, x^n \\ &= 1 + cx + 3x \sum_{n\geq 2} a(n-1)x^{n-1} + 4x^2 \sum_{n \geq 2} a_{n-2} x^{n-2} \\ &= 1 + cx + 3x(A(x) - a_0) + 4x^2 A(x). \end{aligned}$$

So, $A(x) = \dfrac{1 + (c-3)x}{(1 - 3x - 4x^2)} = \dfrac{1+(c-3)x}{(1+x)(1-4x)}$.

(a) *If $c = 4$ then* $A(x) = \dfrac{1}{1-4x}$ *and hence* $a_n = [x^n]\, A(x) = 4^n$.

(b) *If $c \neq 4$ then* $A(x) = \dfrac{1+c}{5} \cdot \dfrac{1}{1-4x} + \dfrac{4-c}{5} \cdot \dfrac{1}{1+x}$ *and hence*
$a_n = [x^n]\, A(x) = \dfrac{(1+c)\, 4^n}{5} + \dfrac{(-1)^n(4-c)}{5}$.

4. *Determine a sequence, $\{a(n) \in \mathbb{R} : n \geq 0\}$, such that $a_0 = 1$ and $\sum\limits_{k=0}^{n} a(k)a(n-k) = \binom{n+2}{2}$, for all $n \geq 1$.*

Solution: *Define* $A(x) = \sum\limits_{n\geq 0} a(n)x^n$. *Then, using the Cauchy product, one has*

$$A(x)^2 = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a(k)a(n-k) \right) x^n = \sum_{n \geq 0} \binom{n+2}{2} x^n = \frac{1}{(1-x)^3}.$$

Hence, $A(x) = \frac{1}{(1-x)^{3/2}}$ *and thus* $a(n) = (-1)^n \binom{-3/2}{n} = \dfrac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{2^n\, n!}$, *for all $n \geq 1$.*

5. *Determine a generating function for the numbers $f(n,m)$, $n, m \in \mathbb{Z}, n, m \geq 0$ that satisfy*

$$\begin{aligned} f(n,m) &= f(n-1,m) + f(n-1,m-1),\ (n,m) \neq (0,0) \qquad \text{with} \qquad (2.1) \\ & \quad f(n,0) = 1, \ \text{ for all } n \geq 0 \ \text{ and } f(0,m) = 0, \ \text{ for all } m > 0. \end{aligned}$$

Hence or otherwise, find a formula for the numbers $f(n,m)$.

Solution: *Note that in the above recurrence relation, the value of m need not be $\leq n$.*

METHOD 1: *Define* $F_n(x) = \sum_{m \geq 0} f(n,m)x^m$. *Then, for* $n \geq 1$, *Equation (2.1) gives*

$$\begin{aligned} F_n(x) &= \sum_{m\geq 0} f(n,m)x^m = \sum_{m \geq 0} \left(f(n-1,m) + f(n-1,m-1)\right) x^m \\ &= \sum_{m \geq 0} f(n-1,m)x^m + \sum_{m\geq 0} f(n-1,m-1)x^m \\ &= F_{n-1}(x) + xF_{n-1}(x) = (1+x)F_{n-1}(x) = \cdots = (1+x)^n F_0(x). \end{aligned}$$

Now, using the initial conditions, $F_0(x) = 1$ *and hence* $F_n(x) = (1+x)^n$. *Thus,*

$$f(n,m) = [x^m](1+x)^n = \binom{n}{m} \quad \text{if} \quad 0 \leq m \leq n \quad \text{and} \quad f(n,m) = 0, \quad \text{for} \quad m > n.$$

METHOD 2: *Define* $G_m(y) = \sum_{n\geq 0} f(n,m)y^n$. *Then, for* $m \geq 1$, *Equation (2.1) gives*

$$\begin{aligned} G_m(y) &= \sum_{n\geq 0} f(n,m)y^n = \sum_{n\geq 0} \left(f(n-1,m) + f(n-1,m-1)\right) y^n \\ &= \sum_{n\geq 0} f(n-1,m)y^n + \sum_{n \geq 0} f(n-1,m-1)y^n \\ &= yG_m(y) + yG_{m-1}(y). \end{aligned}$$

Therefore, $G_m(y) = \dfrac{y}{1-y} G_{m-1}(y)$. *Now, using initial conditions,* $G_0(y) = \dfrac{1}{1-y}$ *and hence* $G_m(y) = \dfrac{y^m}{(1-y)^{m+1}}$. *Thus,* $f(n,m) = [y^n]\dfrac{y^m}{(1-y)^{m+1}} = [y^{n-m}]\dfrac{1}{(1-y)^{m+1}} = \binom{n}{m}$, *whenever* $0 \leq m \leq n$ *and* $f(n,m) = 0$, *for* $m > n$.