1.6 Functions

Definition 1.6.1 (Function). 1. *Let A and B be two sets. Then a function $f : A \longrightarrow B$ is a rule that assigns to each element of A exactly one element of B.*

2. *The set A is called the* domain *of the function f.*

3. *The set B is called the* co-domain *of the function f.*

The readers should carefully read the following important remark before proceeding further.

Remark 1.6.2. 1. *If $A = \emptyset$, then by convention, one assumes that there is a function, called the empty function, from A to B.*

2. *If $B = \emptyset$, then it can be easily observed that there is no function from A to B.*

3. *Some books use the word "map" in place of "function". So, both the words may be used interchangeably throughout the notes.*

4. *Throughout these notes, whenever the phrase "let $f : A \longrightarrow B$ be a function" is used, it will be assumed that both A and B are non-empty sets.*

Example 1.6.3. 1. *Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $C = \{3, 4\}$. Then verify that the examples given below are indeed functions.*

   (a) $f : A \longrightarrow B$, *defined by* $f(a) = 3, f(b) = 3$ *and* $f(c) = 3$.

   (b) $f : A \longrightarrow B$, *defined by* $f(a) = 3, f(b) = 2$ *and* $f(c) = 2$.

   (c) $f : A \longrightarrow B$, *defined by* $f(a) = 3, f(b) = 1$ *and* $f(c) = 2$.

   (d) $f : A \longrightarrow C$, *defined by* $f(a) = 3, f(b) = 3$ *and* $f(c) = 3$.

   (e) $f : C \longrightarrow A$, *defined by* $f(3) = a, f(4) = c$.

2. *Verify that the following examples give functions, $f : \mathbb{Z} \longrightarrow \mathbb{Z}$.*

   (a) $f(x) = 1$, *if x is even and* $f(x) = 5$, *if x is odd.*

   (b) $f(x) = -1$, *for all* $x \in \mathbb{Z}$.

   (c) $f(x) = x \pmod{10}$, *for all* $x \in \mathbb{Z}$.

   (d) $f(x) = 1$, *if* $x > 0$, $f(0) = 0$ *and* $f(x) = 1$, *if* $x < 0$.

Definition 1.6.4. *Let $f : A \longrightarrow B$ be a function. Then,*

1. *for each $x \in A$, the element $f(x) \in B$ is called the* image *of x under f.*

2. *the* range/image *of A under f equals $f(A) = \{f(a) : a \in A\}$.*

3. *the function f is said to be* one-to-one *if "for any two distinct elements $a_1, a_2 \in A$, $f(a_1) \neq f(a_2)$".*

4. *the function f is said to be* onto *if "for every element $b \in B$ there exists an element $a \in A$, such that $f(a) = b$".*

5. *for any function $g : B \longrightarrow C$, the composition $g \circ f : A \longrightarrow C$ is a function defined by $(g \circ f)(a) = g\big(f(a)\big)$, for every $a \in A$.*

Example 1.6.5. 1. *Let $f : \mathbb{N} \longrightarrow \mathbb{Z}$ be defined by $f(x) = \begin{cases} \frac{-x}{2}, & \text{if } x \text{ is even,} \\ \frac{x+1}{2}, & \text{if } x \text{ is odd.} \end{cases}$ Then prove that f is one-one. Is f onto?*

Solution: *Let us use the contrapostive argument to prove that f is one-one. Let if possible $f(x) = f(y)$, for some $x, y \in \mathbb{N}$. Using the definition, one sees that x and y are either both odd or both even. So, let us assume that both x and y are even. In this case, $\frac{-x}{2} = \frac{-y}{2}$ and hence $x = y$. A similar argument holds, in case both x and y are odd.*

Claim: *f is onto.*

Let $x \in \mathbb{Z}$ with $x \geq 1$. Then $2x - 1 \in \mathbb{N}$ and $f(2x-1) = \frac{(2x-1)+1}{2} = x$. If $x \in \mathbb{Z}$ and $x \leq 0$, then $-2x \in \mathbb{N}$ and $f(-2x) = \frac{-(-2x)}{2} = x$. Hence, f is indeed onto.

2. *Let $f : \mathbb{N} \longrightarrow \mathbb{Z}$ and $g : \mathbb{Z} \longrightarrow \mathbb{Z}$ be defined by, $f(x) = 2x$ and $g(x) = \begin{cases} 0, & \text{if } x \text{ is odd,} \\ x/2, & \text{if } x \text{ is even,} \end{cases}$ respectively. Then prove that the functions f and $g \circ f$ are one-one but g is not one-one.*

Solution: *By definiton, it is clear that f is indeed one-one and g is not one-one. But*

$$g \circ f(x) = g(f(x)) = g(2x) = \frac{2x}{2} = x,$$

for all $x \in \mathbb{N}$. Hence, $g \circ f : \mathbb{N} \longrightarrow \mathbb{Z}$ is also one-one.

The next theorem gives some result related with composition of functions.

Theorem 1.6.6 (Properties of Functions)**.** *Consider the functions $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$.*

1. *Then $(h \circ g) \circ f = h \circ (g \circ f)$ (*associativity holds).

2. If f and g are one-to-one then the function $g \circ f$ is also one-to-one.

3. If f and g are onto then the function $g \circ f$ is also onto.

Proof. First note that $g \circ f : A \longrightarrow C$ and both $(h \circ g) \circ f$, $h \circ (g \circ f)$ are functions from A to D.

Proof of Part 1: The first part is direct, as for each $a \in A$,

$$\left((h \circ g) \circ f\right)(a) = (h \circ g)\left(f(a)\right) = h\left(g\left(f(a)\right)\right) = h\left((g \circ f)(a)\right) = \left(h \circ (g \circ f)\right)(a).$$

Proof of Part 2: Need to show that "whenever $(g \circ f)(a_1) = (g \circ f)(a_2)$, for some $a_1, a_2 \in A$ then $a_1 = a_2$".

So, let us assume that $g(f(a_1)) = (g \circ f)(a_1) = (g \circ f)(a_2) = g(f(a_2))$, for some $a_1, a_2 \in A$. As g is one-one, the assumption gives $f(a_1) = f(a_2)$. But f is also one-one and hence $a_1 = a_2$.

Proof of Part 3: To show that "given any $c \in C$, there exists $a \in A$ such that $(g \circ f)(a) = c$".

As g is onto, for the given $c \in C$, there exists $b \in B$ such that $g(b) = c$. But f is also given to be onto. Hence, for the b obtained in previous step, there exists $a \in A$ such that $f(a) = b$. Hence, we see that $c = g(b) = g(f(a)) = (g \circ f)(a)$. ■

Definition 1.6.7 (Identity Function)**.** *Fix a set A and let $e_A : A \longrightarrow A$ be defined by $e_A(a) = a$, for all $a \in A$. Then the function e_A is called the identity function or map on A.*

The subscript A in Definition 1.6.7 will be removed, whenever there is no chance of confusion about the domain of the function.

Theorem 1.6.8 (Properties of Identity Function)**.** *Fix two non-empty sets A and B and let $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be any two functions. Also, let $e : A \longrightarrow A$ be the identity map defined above. Then*

1. *e is a one-one and onto map.*
2. *the map $f \circ e = f$.*
3. *the map $e \circ g = g$.*

Proof. Proof of Part 1: Since $e(a) = a$, for all $a \in A$, it is clear that e is one-one and onto.

Proof of Part 2: BY definition, $(f \circ e)(a) = f(e(a)) = f(a)$, for all $a \in A$. Hence, $f \circ e = f$.

Proof of Part 3: The readers are advised to supply the proof. ■

Example 1.6.9. 1. *Let $f, g : \mathbb{N} \longrightarrow \mathbb{N}$ be defined by, $f(x) = 2x$ and $g(x) = \begin{cases} 0, & \text{if } x \text{ is odd,} \\ x/2, & \text{if } x \text{ is even.} \end{cases}$ Then verify that $g \circ f : \mathbb{N} \longrightarrow \mathbb{N}$ is the identity map, whereas $f \circ g$ maps even numbers to itself and maps odd numbers to 0.*

Definition 1.6.10 (Invertible Function)**.** *A function $f : A \longrightarrow B$ is said to be invertible if there exists a function $g : B \longrightarrow A$ such that the map*

1. *$g \circ f : A \longrightarrow A$ is the identity map on A, and*
2. *$f \circ g : B \longrightarrow B$ is the identity map on B.*

Let us now prove that if $f : A \longrightarrow B$ is an invertible map then the map $g : B \longrightarrow A$, defined above is unique.

Theorem 1.6.11. *Let $f : A \longrightarrow B$ be an invertible map. Then the map*

1. *g defined in Definition 1.6.10 is unique. The map g is generally denoted by f^{-1}.*

2. $\left(f^{-1}\right)^{-1} = f$.

Proof. The proof of the second part is left as an exercise for the readers. Let us now proceed with the proof of the first part.

Suppose $g, h : B \longrightarrow A$ are two maps satisfying the conditions in Definition 1.6.10. Therefore, $g \circ f = e_A = h \circ f$ and $f \circ g = e_B = f \circ h$. Hence, using associativity of functions, for each $b \in B$, one has

$$g(b) = g(e_B(b)) = g\left((f \circ h)(b)\right) = (g \circ f)\left(h(b)\right) = e_A(h(b)) = h(b).$$

Hence, the maps h and g are the same and thus the proof of the first part is over. ■

Theorem 1.6.12. *Let $f : A \longrightarrow B$ be a function. Then f is invertible if and only if f is one-one and onto.*

Proof. Let f be invertible. To show, f is one-one and onto.

Since, f is invertible, there exists the map $f^{-1} : B \longrightarrow A$ such that $f \circ f^{-1} = e_B$ and $f^{-1} \circ f = e_A$. So, now suppose that $f(a_1) = f(a_2)$, for some $a_1, a_2 \in A$. Then, using the map f^{-1}, we get

$$a_1 = e_A(a_1) = \left(f^{-1} \circ f\right)(a_1) = f^{-1}\left(f(a_1)\right) = f^{-1}\left(f(a_2)\right) = \left(f^{-1} \circ f\right)(a_2) = e_A(a_2) = a_2.$$

Thus, f is one-one. To prove onto, let $b \in B$. Then, by definition, $f^{-1}(b) \in A$ and $f\left(f^{-1}(b)\right) = \left(f \circ f^{-1}\right)(b) = e_B(b) = b$. Hence, f is onto as well.

Now, let us assume that f is one-one and onto. To show, f is invertible. Consider the map $f^{-1} : B \longrightarrow A$ defined by "$f^{-1}(b) = a$ whenever $f(a) = b$", for each $b \in B$. This map is well-defined as f is onto and onto (note that onto implies that for each $b \in B$, there exists $a \in A$ such that $f(a) = b$. Also, f is one-one implies that the element a obtained in the previous line is unique).

Now, it can be easily verified that $f \circ f^{-1} = e_B$ and $f^{-1} \circ f = e_A$ and hence f is indeed invertible. ■

We now state the following important theorem whose proof is beyond the scope of this book. The theorem is popularly known as the "Cantor-Bernstein-Schroeder theorem".

Definition 1.6.13 (Cantor-Bernstein-Schroeder Theorem)**.** *Let A and B be two sets. If there exist injective (one-one) functions $f : A \longrightarrow B$ (i.e., $|A| \leq |B|$) and $g : B \longrightarrow A$ (i.e., $|A| \geq |B|$), then there exists a bijective (one-one and onto) function $h : A \longrightarrow B$ (i.e., $|A| = |B|$).*