Problem Set - 13

AUTUMN 2016

ANSWER/HINTS

MATHEMATICS-I (MA10001)

1. (a) $\dfrac{1}{z-2} = -\dfrac{1}{2}\left[1+\dfrac{z}{2}+\dfrac{z^2}{4}+\cdots\right]$, radius of convergence is 2.

   (b) $\dfrac{1}{z-2} = -\dfrac{1}{(2+2i)}\left[1+\dfrac{z+2i}{2+2i}+\left(\dfrac{z+2i}{2+2i}\right)^2+\cdots\right]$, radius of convergence is $2\sqrt{2}$.

   (c) $\dfrac{1}{z-2} = \dfrac{1}{i-2}\left[1-\dfrac{z-i}{i-2}+\left(\dfrac{z-i}{i-2}\right)^2+\cdots\right]$, radius of convergence is $\sqrt{5}$.

   (d) $\dfrac{1}{z-2} = -\dfrac{1}{3}\left[1+\dfrac{z+1}{3}+\dfrac{(z+1)^2}{3}+\cdots\right]$, radius of convergence is 3.

2. (a) $\dfrac{1}{z^2-3z+2} = -\displaystyle\sum_{n=-1}^{\infty}(z-1)^n, \quad 0<|z-1|<1.$

   (b) $\dfrac{1}{z^2-3z+2} = -\displaystyle\sum_{n=-1}^{\infty}(-1)^{n+1}(z-2)^n, \quad 0<|z-2|<1.$

   (c) For $z=0$, consider three different domains $|z|<1, 1<|z|<2$ and $|z|>2$.

      i. $\dfrac{1}{z^2-3z+2} = \displaystyle\sum_{n=0}^{\infty}(1-\frac{1}{2^{n+1}})z^n, \quad |z|<1.$

      ii. $\dfrac{1}{z^2-3z+2} = -\displaystyle\sum_{n=-\infty}^{-1}z^n - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n, \quad 1<|z|<2.$

      iii. $\dfrac{1}{z^2-3z+2} = \displaystyle\sum_{n=-\infty}^{-1}\left(\frac{1}{2^{n+1}}-1\right)z^n, \quad |z|>2.$

3. (a) $\ln\left(\dfrac{1+z}{1-z}\right) = \displaystyle\sum_{n=0}^{\infty}\frac{2z^{2n+1}}{2n+1}$, radius of convergence 1.

   (b) $\sinh z = \displaystyle\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}$, radius of convergence ∞.

4. $-1,-2$ are the singular points whose distance from 1 is 2 and 3 resp. Now consider three domains $|z-1|<2,\ 2<|z-1|<3,\ |z-1|>3$.

   (a) $\dfrac{1}{(z+1)(z+2)^2} = \displaystyle\sum_{n=0}^{\infty}(-1)^n[2^{-n-1}-(n+4)3^{-n-2}](z-1)^n, \quad |z-1|<2.$

   (b) $\dfrac{1}{(z+1)(z+2)^2} = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{(z-1)^{(n+1)}} - \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n(n+4)\left(\frac{z-1}{3}\right)^n, \quad 2<|z-1|<3.$

(c) $\dfrac{1}{(z+1)(z+2)^2} = \sum_{n=0}^{\infty}(-1)^n[\dfrac{2^n-3^n}{(z-1)^n} - \dfrac{3^n(n+1)}{(z-1)^{n+2}}], \quad |z-1|>3.$

5. Use Taylor's theorem.

   (a) $\dfrac{1}{2+e^z} = \dfrac{1}{3}(1-\dfrac{z}{3}-\dfrac{z^2}{18})+\cdots$

   (b) $e^{z\cos z} = 1+z+\dfrac{z^2}{2}+\cdots$

6. (a) $z=i$ is the only singular point of the function in the given domain, which is a simple pole, so only find the residue at that point to have principal part.
   Ans. $-\dfrac{i}{4(z-i)}$

   (b) Expand $\sin z$ in Taylor's series about $z=0$. Then the given function has Laurent's series expansion about $z=0$ in the domain $|z|>0$.
   Ans. $\dfrac{1}{z^3}-\dfrac{1}{6z}$

   (c) $z=0$ is an essential singularity of the given function.
   Ans. $-\left[\dfrac{3}{z}+\dfrac{1}{3!}\dfrac{1}{z^2}+\dfrac{3}{3!}\dfrac{1}{z^3}+\dfrac{1}{5!}\dfrac{1}{z^4}+\dfrac{3}{5!}\dfrac{1}{z^5}+\cdots\right]$

   (d) $z=0$ is a removable singular point, hence no principal part.

7. $z=0$ is a simple pole, use the Taylor's series expansion of e^z or otherwise use the integral formula for coefficients in Laurent's series expansion of $f(z)$.
   $\dfrac{1}{e^z-1} = \dfrac{1}{z}-\dfrac{1}{2}+\dfrac{z}{12}+\cdots, \quad 0<|z|<2\pi$

8. $e^{\frac{z}{z-2}} = e\left[1+\dfrac{2}{z-2}+\dfrac{2}{(z-2)^2}+\dfrac{4}{3(z-2)^3}+\cdots\right]$, region of convergence $|z-2|>0$.

9. (a) $z=\frac{1}{(2n+1)\pi i}, n\in\mathbb{Z}$ are also simple pole of $f(z)$.
   $z=0$ is a non-isolated essential singularity.
   $z=\infty$ is an removable singularity of $f(z)$.

   (b) $z_n=\frac{1}{n\pi}, n\in\mathbb{Z}$ are simple poles of $f(z)$.
   $z=0$ is a non-isolated essential singularity of $f(z)$.
   $f(z)$ has simple pole at $z=\infty$.

   (c) $f(z)$ has essential singularity at $z=0$.
   $f(z)$ has essential singularity at $z=\infty$.

   (d) $f(z)$ has removable singularity at $z=0$.
   Show that $\lim_{z\to\infty} f(z)$ does not exist, hence $f(z)$ has essential singularity at $z=\infty$.

10. (a) $z=\frac{i}{2n\pi}$ are simple poles of $f(z)$.
    $z=0$ is a non-isolated essential singularity.
    $z=\infty$ is a pole of order 2 for the function $f(z)$.

    (b) $z=0$ is an essential singularity of $f(z)$.
    $z=\infty$ is an essential singularity of $f(z)$.

(c) $z = 0$ is removable singularity.
$z = 2n\pi i, n \in \mathbb{Z}$ are simple poles of $f(z)$.
$z = \infty$ is a non-isolated essential singularity.

(d) $z = 0$ is a pole of order 3 for $f(z)$.
$z = \infty$ is an essential singularity of $f(z)$.

11. (a) $z = \infty$ is a isolated essential singularity of $f(z)$.

(b) $z = \infty$ is a isolated essential singularity of $f(z)$.

(c) $z = \infty$ is a isolated essential singularity of $f(z)$.

(d) $z = \infty$ is a non-isolated essential singularity of $f(z)$.

12. $f(z) = \dfrac{(z - z_0)e^{\frac{1}{(z-z_2)}}}{(z - z_0)(z - z_1)^k}$.

13. Order of the pole is (a) 4, (b) 9.