# Boundary Value problems

Consider a two points boundary value problem

$$y'' = f(x, y, y'), \quad x \in (a, b) \qquad \text{——— (1)}$$

with one of the three boundary conditions

a) B.C. of the first kind (DIRICHLET B.C.)

$$y(a) = \gamma_1, \quad y(b) = \gamma_2$$

b) B.C. of the second kind (NEUMANN B.C.)

$$y'(a) = \gamma_1, \quad y'(b) = \gamma_2$$

c) B.C. of the third kind (ROBIN B.C.)

$$a_0 y(a) - a_1 y'(a) = \gamma_1$$
$$b_0 y(b) + b_1 y'(b) = \gamma_2$$

If all terms in (1) involve only the dependent variable $y$ & $y'$, then the differential equation is called homogeneous, otherwise non-homogeneous.

Similarly, the BCs are homogeneous if $\gamma_1$ & $\gamma_2 = 0$ otherwise they are non homogeneous.

REMARK: A homogeneous boundary value problem, that is, a homogeneous differential equation along with homogeneous BCs, always possesses a trivial solution

$$y(x) = 0.$$

# NUMERICA METHODS FOR SOLVING BVPs

i) SHOOTING METHODS — IVP Method

ii) DIFFERENCE METHODS — DIFFERENCE equation

## SHOOTING METHOD

Consider the BVP:

$$y'' = f(x, y, y') \qquad y(a) = \gamma_1, \; y(b) = \gamma_2$$

In order to solve the BVP using IVP Methods, we need to define the following initial values at $x = a$:

$$y(a) = \gamma_1, \qquad y'(a) = s$$

where s is unknown.

The question is: can we find the value of s for which the solution of the resulting IVP is identical to the solution of BVP?

OR

for what values of s, the IVP satisfies $y(b) = \gamma_2$.

The idea is to pick a value of s, then use the IVP method to march over to $x = b$ and see whether $y(b) = \gamma_2$. If not, then adjust the value of s and use the IVP method again and see how much close $y(b)$ is to $\gamma_2$. This is continued until $|y(b) - \gamma_2|$ is sufficiently small.

Again the question arises: how to adjust $s$ so that $y(b)$ ends up close to $r_2$? To address this, set

$$g(s) = y(s,b) - r_2$$

where $y(s,b)$ is the solution of IVP corresponding to the parameter $s$.

The function $g(s)$ enables us to express the question of getting $y(b)$ close to $r_2$ in terms of finding the value of $s$ such that $g = 0$.

Hence we can use something such as the secant or Newton's method to improve the value of $s$.

For example, to use the secant method we need to specify two values for $s$, say $s_1$ & $s_2$. In this case, the subsequent values for $s$ are determined using the secant method

$$s_{j+1} = s_j - \frac{g(s_j)}{g(s_j) - g(s_{j-1})} \times (s_j - s_{j-1}), \quad j = 2, 3, \ldots$$

This procedure for finding $s$ works whether the BVP is linear or non-linear. However it is possible to simplify the procedure a bit for linear problems.

If $y_1$ & $y_2$ are two solutions of a linear differential equation than their linear combination that is $(c_1 y_1 + c_2 y_2)$ will be the solution of the linear diff. equation.

# Setting Initial Conditions from Boundary Conditions

i) From the first kind BCs ( $y(a) = r_1$ , $y(b) = r_2$ )

$\boxed{y(a) = r_1}$ & $\boxed{y'(a) = s}$.

ii) From the second kind BCs ( $y'(a) = r_1$ , $y'(b) = r_2$ )

$\boxed{y'(a) = r_1}$ & $\boxed{y(a) = s}$ .

iii) BCs of the third kind: $a_0 y(a) - a_1 y'(a) = r_1$

$b_0 y(b) + b_1 y'(b) = r_2$

Here we guess $y(a)$ or $y'(a)$.

Let us guess $\boxed{y'(a) = s}$ then:

$$\boxed{y(a) = \frac{a_1 s + r_1}{a_0}}$$

## Shooting Method for a linear second order problem:

$$y'' + p(x) y' + q(x) y = r(x) \qquad a < x < b \qquad \text{—— (1)}$$

$$y(a) = r_1 \qquad \text{—— (1a)}$$

$$y(b) = r_2 \qquad \text{—— (1b)}$$

In order to use an initial value integrator for (1), we need to set $y(a)$ & $y'(a)$. By (1a) we have $y(a) = r_1$.

Let us therefore guess a "shooting angle" $s_1$; i.e.,

$$y'(a) = s_1 \qquad \text{—— (1c)}$$

we take $s_1 = 0$ in practice.

The IVP (1, 1a, 1c) can be solved yielding a solution, say $u(x)$.

Not that, in general, $u \not\equiv y$, because $u(b) \neq \gamma_2$.

Let us then guess another value

$$y'(a) = s_2 \qquad \text{——} \quad (1d)$$

we take $s_2 = 1$ in practice.

We call $v(x)$, the solution of the IVP (1, 1a, 1d).

Again $v(b) \neq \gamma_2$, so $y \not\equiv v$.

Linearity of the problem implies that

$$y(x) = \theta\, u(x) + (1-\theta)\, v(x), \quad 0 < x < b$$

satisfies the equation (1).

Assuming that $u(b) \neq v(b)$, we can define $\theta$ by

$$\gamma_2 = y(b) = \theta\, u(b) + (1-\theta)\, v(b)$$

$$\Rightarrow \boxed{\theta = \frac{\gamma_2 - v(b)}{u(b) - v(b)}}$$

Summary: we need to solve:

$$u'' + p(x)\, u' + q(x)\, u = r(x) \qquad v'' + p(x)\, v' + q(x)\, v = r(x)$$

$$u(a) = \gamma_1 \qquad v(a) = \gamma_1$$

$$u'(a) = 0 \qquad v'(a) = 1$$

$$y(x) = \theta\, u(x) + (1-\theta)\, v(x); \quad \theta = \frac{\gamma_2 - v(b)}{u(b) - v(b)}$$