1. Groups, Subgroups

(1) Let $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} | a \neq 0, a \in \mathbb{R} \right\}$. Is it a group under multiplication?

(2) Let $G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} | |a| + |b| \neq 0, a, b \in \mathbb{R} \right\}$. Is it a group under multiplication?

(3) Put $G = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Show that the nonzero elements in G form a group denoted bu $G^{\times}$ under multiplication.

(4) Show that the set of all transformations of the type $z \mapsto \frac{az+b}{cz+d}$, $ad - bc \neq 0$ of the complex numbers in itself, is a group for the operation of composite transformations. (This group is called Mobius transformation group.)

(5) Find the order of the group $GL_n(\mathbb{F}_p)$ where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field of integers modulo a prime number p.

(6) Show that every $\sigma \in S_n$ can be expressed as a product of disjoint cycles.

(7) Every $\sigma \in S_n$ can be expressed as a product of transpositions.

(8) The set A_n of all even permutations forms a subgroup of S_n of order $n!/2$.

(9) List all the elements of order 2 in S_4. How many elements of S_n have order 2?

(10) Write elements as permutation in S_6 of the dihedral group symmetries of a regular hexagon inscribed in a unit circle with one vertex on the x-axis.

(11) Let $|x|$ denote order of an element x in a group G. Show that $|x| = |x^{-1}| = |gxg^{-1}|$ for any $g \in G$. Deduce that $|ab| = |ba|$ for any $a, b \in G$.

(12) Prove that if $x^2 = 1$ for all $x \in G$, then G is abelian.

(13) Show that G is abelian group if and only if $(ab)^2 = a^2b^2$.

(14) Prove that any finite group of even order contains an element of order 2.

(15) Let F be a field. The Heisenburg group $H(F)$ is defined to be the multiplicative group:

$$H(F) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in F \right\}$$

(a) Find formulas for products and inverses of elements in $H(F)$.

(b) Show that $H(F)$ is a nonabelian group.

(c) Prove that every nonidentity element of $H(\mathbb{R})$ has infinite order.

(d) Let F be a finite field with q elements. Show that $|H(F)| = q^3$.

(e) Find orders of elements of $H(\mathbb{F}_2)$.

(16) Let G be an abelian group. Prove that the set $t(G) = \{g \in G : |g| < \infty\}$ is a subgroup of G, (called the torsion subgroup of G). Give an example to show that $t(G)$ is not a subgroup when G is not abelian.

(17) Let H and K be subgroups of a group G. Then $HK = \{hk | h \in H, k \in K\}$ is a subgroup of G if and only if $HK = KH$.

(18) Give example of a group G and two subgroups H and K such that HK is not a subgroup.

(19) Show that a group can not be the union of two proper subgroups.

(20) Find all subgroups of S_3 and D_4.
(21) Let H be a subgroup of G and $a \in G$. Show that $a \in H$ if and only if $aH = H$.
(22) Show that every subgroup of a cyclic group is cyclic.
(23) Let $G =< a >$ and $|a| = n$. Find $|a^r|$ where $r \in [n]$. Find all $r \in [n]$ such that $G =< a^r >$.
(24) Let G be a group and $x, y \in G$ have finite orders m and n respectively. Prove that $|xy|$ divides $[m, n]$ if $xy = yx$. Give an example of x and y so that $|xy| < [m, n]$. What can you say if $xy \neq yx$.
(25) Give an example of a group which is not cyclic group but every proper subgroup of which is cyclic.
(26) Show that a cyclic group with just one generator has at most two elements.
(27) Prove that an infinite cyclic group has exactly two generators.