INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

Date: .... FN / AN **Time:** 3 hrs **Full Marks:** 50 **No. of Students:** 1375
End Autumn Semester 2011-12 Department: **Mathematics** Subject No.: **MA10001**
1^{st} Yr. B.Tech (H)/ B.Arch (H) / M.Sc. Subject Name: **Mathematics I**

Instructions: Answer all parts of a question at one place.
Start answering a new question from a new page.

1. (a) Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). If $\exists\ c \in (a,b)$ such that $f'(c)=0$, does it imply $f(a)=f(b)$?. Justify your answer.

   (b) Find the values of p and q such that $\lim\limits_{x\to 0}\dfrac{x(1-p\cos x)+q\sin x}{x^3}=\dfrac{1}{3}$.

   (c) Find the radius of curvature for the curve $x^3+y^3=3axy$ at the point $\left(\dfrac{3a}{2},\dfrac{3a}{2}\right)$.

   (d) Express $f(x)=3x^3-4x^2+5x-1$ in powers of $(x-3)$ using Taylor series expansion.

(2+2+2+2)

2. (a) If $e^u = x^3+y^3+z^3-3xyz$ then find $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}$.

   (b) For a homogeneous function $z(x,y)$ of degree n, prove that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=nz$.

   If $z=\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$, for $x\neq 0, y\neq 0$, then evaluate $\dfrac{1}{y}\dfrac{\partial z}{\partial x}+\dfrac{1}{x}\dfrac{\partial z}{\partial y}$.

   (c) Obtain the expression for $\dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial y^2}$ in terms of $\psi(r,\theta)$ using the relation $x=r\cos\theta,\ y=r\sin\theta$. **(3+3+3)**

3. (a) Show that the function $f(z)=\begin{cases}\dfrac{(\bar{z})^2}{z}, & z\neq 0\\ 0, & z=0\end{cases}$

   is continuous and the Cauchy-Riemann equations are satisfied at origin. Does $f'(0)$ exist? Justify your answer.

   (b) Show that the function $u=2x(3-y)$ is harmonic. Find the conjugate harmonic function v and express $u+iv$ as an analytic function of z.

(c) If $f(z)$ is an analytic function of z, show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

(d) Find $\lim\limits_{z\to 0}\dfrac{\operatorname{Im}\left(z^2\right)}{|z|^2}$ if exists. **(3+2+2+1)**

4. (a) Does the integral $\int\left(x^2 - iy^2\right)dz$ depend upon the path $y = 2x^2$ from $1+i$ to $2+8i$? Justify your answer and evaluate the integral.

(b) Using Cauchy Integral Formula find the value $\int_C \dfrac{z+3i}{\left(z^2 - iz + 2\right)^3}\,dz$ where C is $|z-1-2i| = 2$ which oriented in the anti-clockwise direction.

(c) Evaluate $\int_C \dfrac{z^2-5z+3}{(z+1)(z-2)^2}\,dz$, where C is the circle $|z-2| = 1$ in the anticlockwise direction. **(2+3+3)**

5. (a) Solve the differential equation $(y\log x - 1)\,y\,dx = x\,dy$.

(b) For the differential equation $x\,dy - y\,dx = \left(x^2+y^2\right)dx$ determine that particular solution $y(x)$ for which $y = \dfrac{\pi}{2}$ when $x = \dfrac{\pi}{2}$.

(c) Find the general solution of $\dfrac{d^5y}{dx^5} - \dfrac{dy}{dx} = 12e^x + 8\sin x - 2x$.

(d) Find the Wronskian of $f(x) = x^2$ and $g(x) = e^{-x^2}$. **(2+2+3+1)**

6. (a) Find the general solution of $x^2y'' - 3xy' + y = \dfrac{\log x\,\sin(\log x)}{x}$.

(b) Using the method of variation of parameters, find the particular solution of the differential equation $y'' - y = \dfrac{2}{1+e^x}$.

(c) Solve the simultaneous system of differential equations

$\dfrac{dx}{dt} + 2y + \sin t = 0, \dfrac{dy}{dt} - 2x - \cos t = 0$ subject to the conditions

$x = 1,\ y = 1$ at $t = 0$. **(3+3+3)**

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