Fluid Mechanics

GAURAV SURYAWANSHI

Basic Concepts

Heading	Definition
Viscous Fluid	A fluid is called a Viscous fluid if it has both Normal And shear Force, if non-viscous it doesn't have a shear force value
Viscosity	Viscosity of a fluid is the property by which it exhibits resistance to the alteration of the form when upper plate moves in x dirn with certain velocity but the lower plate is stationary
Laminar Flow	A flow in which each particle traverses a definite curve, and the curve traced by any two particles never intersect.
Steady Flow	A flow in which we have that the properties such as temperature and pressure independent of time is called steady flow
Langrage Approach	We compare position p_0 at time t_0 with position p_1 at time t_1 i.e $$V = f(V_0, t, t_0)$$ $$\frac{dV}{dt} = \sum_i \left(\frac{dx_i}{dt}\,\hat{x}_i\right)$$
Euler	We fix a point in space and look how fluid is behaving at that point $$V, \frac{dV}{dt}, \frac{d^2V}{dt^2}$$

Important Theorems

Heading	Definition
Stokes Law	Let S be an open surface bounded by a closed curve C then $$\int_C \vec{F} d\vec{r} = \int\int\int_S (\nabla \times \mathrm{F})\,\hat{n}\,ds$$
Gauss Divergence Theorem	Let S be a closed surface bounding a Volume V, and let $\hat{n}$ be a unit vector perpendicular to the surface of V then $$\int\int_S \vec{F}.\hat{n}\,dS = \int\int\int_V \vec{\nabla}.\vec{F}\,dV$$
Green's theorem	Let ϕ and φ be continuous differentiable function $$\int_C \phi \times \frac{d\varphi}{dt} - \varphi \times \frac{d\phi}{dt}\,ds = \int_s \phi\,\nabla^2\varphi - \varphi\nabla^2\phi\,dx\,dy$$

Important Formulae's

Heading	Definition's	Formulae's
Thermal Conductivity	*Here $\mathrm{q_n}$ is the conductive heat per unit area, k is the coefficient*	$q_n = -\frac{k\partial T}{\partial n}$

Viscosity		$$\tau = u \times \frac{du}{dy}$$ Where u is constant of proportionality
Density, Velocity, Div		$\rho, \vec{q}, \vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$
Material Derivative		$$\frac{Df}{Dt} = \left(\frac{\partial}{\partial t} + \vec{\nabla}.\vec{q}\right) f$$ Where $\frac{Df}{Dt}$ is material derivative and ∂ is local derivative
Streamline Equation	*Continuous Line of flow such that the tangent drawn at any point of streamline coincides with the dirn of motion of the fluid*	$$d\vec{s} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}\,dz$$ $$\vec{q} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$ $$d\vec{s} \times \vec{q} = 0$$ $$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$ Last equation is streamline at Point P
Path lines		$$\vec{r} = \int \vec{q}.dt$$
Velocity and Acceleration of fluid particle		$\vec{q} = vel = \frac{d\vec{r}}{dt}$ $\vec{A} = accn = \frac{d\vec{q}}{dt} = \frac{\partial\vec{q}}{\partial t} + \vec{q}.(\vec{\nabla}.\vec{q})$
Equation of Continuity	Normal Form LaGrange's form	$\frac{d\rho}{dt} + \vec{\nabla}.(\rho\vec{q}) = 0$ $\rho_0 = \rho.J$ where J is Jacobian
Velocity potential (Scalar)		$\vec{q} = -\vec{\nabla}.\phi$
Rotational Flow Vector wz for 2-D flow		$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ Where v is velocity in y dirn and u in x
Circulation	*Scalar integral indicating measure of rotation along a closed curve Flow across closed curve is called circulation around that curve*	$\Gamma = \oint_c \vec{q}.d\vec{r} = \int_s \vec{w}.\hat{n}\,ds$ And this integral is >0 for anti-clockwise. W is the voriticity
Vorticity	*Pseudo vector field describing spinning of fluid around a point*	$$\vec{w} = \vec{\nabla} \times \vec{q}$$ $$\vec{w} = w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}$$
Vortex Line	*A vortex Line is a curve drawn such that the tangent drawn at any point*	$$\vec{w} \times \vec{r} = \vec{0}$$ $$\frac{dx}{w_1} = \frac{dy}{w_2} = \frac{dz}{w_3}$$
Streak Lines	*A streak line is defined to be locus of different particles passing through the same point*	Let at any time t, r be the fixed point $$\vec{r} = f(\vec{r}_0, t)$$ $$\vec{r_0} = F(\vec{r}, t)$$ Hence, the streak line at time t' would be $$\vec{r} = G(\vec{r}_0, t')$$ $$\vec{r} = G(F, t')$$
Linear Strain Rule		$Strain_x = \frac{\partial u_x}{\partial x}$ $Strain_y = \frac{\partial u_y}{\partial x}$ $Strain_i = \partial u_i/\partial x_i\ \forall i$ $Total\ Strain = \sum \frac{\partial u_i}{\partial x_i} = \frac{\partial u_\beta}{\partial x_\beta}$ Where β is dummy sum convention
Shear Stress or deformation of fluid element bcoz of Strains		$$\epsilon_{ij} = e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ ϵ_{ii} denotes principle stress
Change in Density		$\frac{1}{\rho} \times \frac{D\rho}{Dt} = \sum \frac{\partial u_i}{\partial x_i}$

Euler Equation of Motion		$\frac{D\vec{q}}{Dt} = \vec{F}_{ext,per\ unit\ mass} - \frac{1}{\rho}(\nabla.\rho)$
Lamb Hydrodynamics Equation		$\frac{\partial q}{\partial t} + \frac{1}{2} \times \vec{\nabla}.\vec{q}^2 - \vec{q} \times \vec{w} = \vec{F}_{ext} - \frac{1}{\rho}(\nabla.\rho)$
Generic Functions of stress and strain		Strain $= \frac{\delta x}{x}$ Stress= $\frac{F}{A}$ Shear= $\frac{d\theta}{dt}$ where θ is the angle
Properties of incompressible Liquids		$\vec{\nabla}.\vec{q} = 0$ $\vec{\nabla}^2\phi = 0$
Properties of Irrotational Fluids		$\vec{\nabla} \times \vec{q} = \vec{0}$ Curl of Q $\omega_z = \vec{0}$
If force is conservative, we can use force field		$\vec{F} = -\vec{\nabla}.\phi$ Where ϕ is a force field

Stress and Strain

Heading	**Definition's**	**Formulae's**
Cauchy stress tensor	*2nd order tensor defining a stress at a point inside the material*	Let Stress Vector be $\vec{T_n}$ and $\hat{n}$ be the unit vector in direction of T then $\vec{T_n} = \vec{\sigma}.\hat{n} = \sigma_{ij}n_i$
Moment of Force(torque)	*It is the rotational equivalent of linear force and is equivalent to magnitude of Force and the perpendicular distance between force and axis*	$\tau = \lvert\vec{r}\rvert\lvert\vec{F}\rvert \sin\theta$ Where θ is the angle between r and F $\vec{\tau} = \vec{r} \times \vec{F}$ $\vec{\tau} = \frac{d\vec{L}}{dt}$
Angular Momentum		$\vec{L} = \vec{r} \times \vec{p}$ *Where* $\vec{p} = m\vec{v}$ $\rightarrow \vec{L} = m\vec{r} \times \vec{v}$
Moment of Inertia (analogy mass)	*Defined to be as angular momentum by angular Velocity*	$I = \frac{L}{\omega}$ Where L is the angular momentum
Stress Tensor and vector		$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$ $T^n = \hat{n}.\vec{\sigma}$ $T_i = \sigma_{ij}n_j$
Levi-Civia System		$\epsilon_{i1,i2,\ldots,in} = (-1)^p\epsilon_{1234\ldots n}$ Where LHS has distinct and in ordered fashion
Cauchy First Law of Motion		$\frac{\partial(\sigma_{ji})}{\partial j} + F_i = 0$
Principal Stress		$\sigma = \begin{bmatrix} \sigma'_{11} & 0 & 0 \\ 0 & \sigma'_{22} & 0 \\ 0 & 0 & \sigma'_{33} \end{bmatrix}$ where $\sigma'_{11} = \max(\lambda_1, \lambda_2, \lambda_3)$ $\sigma'_{33} = \min(\lambda_1, \lambda_2, \lambda_3)$ $\sigma'_{22} = \mathrm{I}_1 - \sigma'_{11} - \sigma'_{33}$ Where

		$\det \sigma_{ij} - \lambda\delta_{ij} = 0$ $-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0$ $I_1 = \sigma_{kk}$ $I_2 = \frac{1}{2}(\sigma_{ii}\sigma_{kk} - \sigma_{ik}\sigma_{ki})$ $I_3 = \det(\sigma)$
Stress Derivative Tensor	S_{ij} *is derivative stress tensor which tends to distort the body.* $\pi\delta_{ij}$ *tends to change its volume.*	$\sigma_{ij} = \pi\delta_{ij} + S_{ij}$ $\pi = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$
Bernoulli Equation	*Here* $\vec{F}_{ext} = -\vec{\nabla}.V$ $\vec{q} = \text{velocity}$ $\phi = \text{Velocity potential}$ $P = \text{Pressure}$ $\rho = \text{Density}$	$\varphi(t) = -\frac{d\phi}{dt} + \frac{\vec{q}^2}{2} + \int \frac{dP}{\rho} + V$
Bernoulli Equation with no velocity potential & conservative force		$\frac{\vec{q}^2}{2} + V + \int \frac{dP}{\rho} = Const$

Energy Equations

Heading	Definition's	Formulae's
Basic Statement	*The rate of change of total energy (kinetic, potential etc) on a given segment of a compressible inviscid fluid as it moves is equal to the total work done by the pressure on the boundary provided*	Let T=Kinetic Energy, P=Potential, I=internal, W=work done, P=pressure Ω=Potenital field $\frac{dT}{dt} + \frac{dP}{dt} + \frac{dI}{dt} = \int P(\vec{q}.\hat{n})ds = W$ where $T = \int_v \rho\|\vec{q}\|^2 dV$ $P = \int_v \rho\,\Omega\,\mathrm{dV}$ $E = \int PdV$ $I = \int_v E\rho dV$

2-D Flow

Heading	Definition's	Formulae's
Stream Function	*Stream function signifies that the difference of its values across two points represent the flow across any line joining the points (inform of flux)*	$\vec{q} = (u, v)$ is velocity vector. Let φ be the potential $u = \frac{\partial\psi}{\partial y}$ $v = -\frac{\partial\psi}{\partial x}$
Flux		*Flux flowing is equal to* $Total\ flux = \Phi_1 - \Phi_2$ *Flux across a small segment* $Flux_{\delta s} = \delta s \times normal_{vel}$ $Velocity_{normal} = \frac{\delta\varphi}{\delta s}$
Velocity in Polar Coordinates		$q_r = \quad = \frac{1}{r} \times \frac{d\psi}{d\theta} = \frac{d\phi}{dr}$ $q_\theta = \quad = -\frac{d\psi}{dr} = \frac{d\phi}{rd\theta}$

Complex Potential	*Complex potential is potential function + I * stream function*	$\omega = f(z) = \phi(x,y) + i\psi(x,y)$ $z = x + iy$ $\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}; \frac{\partial\varphi}{\partial x} = -\frac{\partial\psi}{\partial y}$
Spin Component of q		$\Omega = \frac{1}{2} Curl\, \vec{q} = \frac{1}{2}(\vec{\nabla} \times \vec{q})$
Velocity in complex plane		$\frac{d\omega}{dz} = -u + iv$
Strength Source and Sink	*A source of strength m in 2D is such that the flow across any small curve surrounding it is* $2\pi m$	$m = rq_r$ $\psi = -m\theta$ $\phi = -m \log r$ $\omega = -m \log z$
Dipole	*A combination of source and sink.* μ *is the strength of the doublet represented as* $m\delta s$, δs *is the distance*	$\omega = \frac{\mu}{z}$ *Doublets At point* z_1,z_2 $\omega = \frac{\mu_1}{z - z_1} + \frac{\mu_2}{z - z_2} + \cdots$
Image of a source	*In a fluid, if there exists a surface S where there is no flow, then the system of sources, sinks and doublets on opposite sides of S is known as the system with regards to that curve*	*i)* For St line *Direct image* *ii)* *For circle* *Source at inverse point and sink at centre*
Milne Thompson Theorem	*Let f(z) be the complex potential in a flow having no rigid boundaries such that there are no singularities inside circle |z|=a. Then if we introduce a solid cylinder at |z|=a, then the new complex potential would be*	$\omega = f(z) + f\left(\frac{z^2}{a}\right)$ $\|z\| \geq a$
Blasius Theorem	*In steady 2D Irrotational flow of an incompressible liquid with complex potential w=f(z), the pressure thrust of a cylinder of any shape is represented by Force(X,Y) and the moment M*	$X - iY = \frac{i\rho}{2}\int_c \left(\frac{d\omega}{dz}\right)^2 dz$ $M = real\left(-\frac{i\rho}{2}\int_c z\left(\frac{d\omega}{dz}\right)^2 dz\right)$ *Where C is contour.*
Flow	*Value of integral* $\int_a^b \mathrm{udx} + \mathrm{vdy} + \mathrm{wdz}$ *is called flow from a to b.*	
Kelvin Circulation Theorem	*When the external forces are conservative and derived from single valued potential function and density is a function of pressure only. Then the circulation in a particular closed circuit is constant all the time.*	
Green's Theorem	*Let two single valued continuous function* ϕ *and* ϕ'	$\int_v \nabla\phi.\nabla\phi' dV = \int_s \phi \times \frac{d\phi'}{dt} ds - \int_v \phi\nabla^2\phi' dV = \int \phi' \times \frac{d\phi}{dt} ds - \int_v \phi'\nabla^2\phi\, dV$
Kelvin Minimum Energy Theorem	*The Irrotational motion of a liquid occupying a simply connected region would have less kinetic*	

	energy than any other motion having same normal velocity at the boundary.	

Motion of Cylinders

Heading	Definition's	Formulae's
Stream function for pure rotational motion		$\Psi = vx - uy + \frac{\omega}{2}(x^2 + y^2) + C$ *where* ω *is the angular velocity*
Kinetic Energy		$KE = \frac{\rho}{2}\int_s \frac{\phi \partial \phi}{\partial n}\, ds$
When cylinder is moving in an infinite mass of liquid in x dirn		$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{dr} + \frac{1}{r^2}\frac{\partial^2\phi}{d\theta^2} = 0$ *Solution of the form* $\phi(r,\theta) = Ar\cos\theta + \frac{B}{r}\cos\theta$ $\omega = \frac{Ua^2}{z}$
Fluid flowing at vel=-U		$\omega = \frac{Ua^2}{z} + Uz$
Cylinder moving with velocity U		$\omega = \frac{Ua^2}{z - Ut}$
Complex potential due to circulation		$\omega = i\frac{k}{2\pi}\log z$
Streaming & Circulation around a fixed cylinder		$\omega = \frac{ik}{2\pi}\log z + \frac{Ua^2}{z} + Uz$

Aerofoils & Naiver Stokes Equation

Heading	Definition's	Formulae's
Conformal Mapping	*Preserves angle and side ratios*	$\zeta = f(z)$ $z = x + iy$ $\zeta = \xi + i\eta$ $f'(z) = \frac{d\zeta}{dz}$ *Area Ratio (ξ to z) equals to* $\lvert f'(z)\rvert^2$ *Velocity Ratio equals to 1/|f'(z)|* *Kinetic Energy Preserved* *Complex potential preserved*
Differentiation		$f'(z) = \frac{d\zeta}{dz} = \frac{d\xi}{dx} + i\frac{d\eta}{dx} = \frac{d\eta}{dy} - i\frac{d\xi}{dy}$
Joukowski's Transformation		$f(z) = \zeta = z + \frac{a^2}{z}$
Kutta- Joukowski Theorem	*When a cylinder of any shape is placed in uniform stream of speed U then the resultant thrust on the cylinder is KρU per unit length normal to the cylinder, where K is the circulation.*	$X - iY = -i\rho KUe^{-i\alpha}$ $\omega = Az + \frac{B}{z^2} + Uze^{-i\alpha} + i\frac{k\log z}{2\pi}$ $\alpha = dir^n$ *with the positive x axis, other variables have their usual meanings*

Flow past a circle		α =dirn with the -ive x axis $\omega = Ue^{i\alpha}(z - z_0) + \frac{Ub^2 e^{-i\alpha}}{z - z_0} + i\frac{k}{2\pi}\log z - z_0$
Flow past an aerofoil		$\frac{d\omega}{d\zeta} = \frac{\frac{d\omega}{dz}}{1 - \frac{a^2}{z^2}}$ $X - iY = -i\rho K e^{i\alpha} U$ $N = 2\pi\rho U^2[2bc\sin(\alpha + \beta)\cos\alpha + \gamma + a^2\sin 2\alpha]$
Naiver Stokes Equation		$\frac{\rho du}{dt} = \rho F_x - \frac{\partial P}{\partial x} + \left[2\mu \times \frac{\partial^2 u}{\partial x^2} - \frac{2\mu}{3}\frac{\partial(\vec{\nabla}.\vec{q})}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) + \mu\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right)\right]$
Constitutive law of Newtonian compressible flow		$\sigma_{ii} = 2\mu\frac{\partial q_i}{\partial i} - \frac{2\mu}{3}\vec{\nabla}.\vec{q} - P$ $\sigma_{ij} = \mu\left(\frac{\partial q_i}{\partial j} + \frac{\partial q_j}{\partial i}\right)$ *Where i, j= x,,y,z*
Naiver Stokes for incompressible fluid		$\frac{d\vec{q}}{dt} = \vec{F} - \frac{\vec{\nabla}p}{\rho} + \frac{\mu}{\rho}\vec{\nabla}^2\vec{q}$
Dissipation of energy	$\mathrm{TKE} = \frac{\rho}{2}\int_v q^2 dV$	$D = -\int_v \mu\lvert\Omega\rvert^2\, dV$ $\Omega = \vec{\nabla} \times \vec{q}$

Solving Differential Equations

Heading	Definition's	Formulae's
Generic Way to find PI		$\frac{X}{D - \alpha} = e^{\alpha x}\int X e^{-\alpha x} dx$

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