DERIVATIVE (GEOMETRICAL INTERPRETATION)

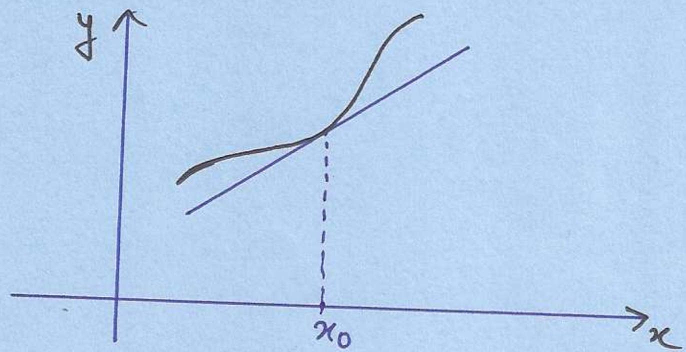
ONE VARIABLE:

$$f(x) - f(x_0) = (x - x_0)A + \varepsilon_1 (x - x_0)$$

where $\varepsilon_1 \to 0$ as $x \to x_0$

OR $$f(x) = \underbrace{f(x_0) + (x - x_0)A}_{\text{linear function, say } \Phi(x)} + \varepsilon_1 (x - x_0)$$

$\Phi(x) = f(x_0) + (x - x_0)A \to$ tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$.



GENERAL DEF.

A function $y = f(x)$ (or $z = f(x,y)$) is differentiable at the point P if it can be approximated in the neighbourhood of this point by a linear function.

Two variables:

$$f(x,y) = \underbrace{f(x_0,y_0) + (x - x_0) f_x(x_0,y_0) + (y - y_0) f_y(x_0,y_0)}_{\text{linear function, say } \psi(x,y)} + \varepsilon_1 (x - x_0) + \varepsilon_2 (y - y_0)$$

$$\psi(x,y) = f(x_0,y_0) + (x - x_0) f_x(x_0,y_0) + (y - y_0) f_y(x_0,y_0)$$

$\hookrightarrow$ tangent plane.

TAYLOR'S THEOREM FOR A FUNCTION OF TWO VARIABLES

Let a function $f(x,y)$ be defined in some domain D in $\mathbb{R}^2$ and have continuous partial derivatives up to $(n+1)$th order in some neighbourhood of a point $P(x_0,y_0)$ in D. Then,

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0,y_0) + \frac{1}{\lfloor 2}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(x_0,$$

$$+ \cdots\cdots + \frac{1}{\lfloor n}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0,y_0) + R_n$$

where the remainder is given by

$$R_n = \frac{1}{\lfloor (n+1)}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k), \quad 0<\theta<1.$$

Proof: For simplicity, $n=2$.

Let $x = x_0+th$, $y = y_0+tk$ where the parameter $t\in[0,1]$.

Define $\Phi(t) = f(x_0+th, y_0+tk)$

Using chain rule:

$$\Phi'(t) = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0+th, y_0+tk)$$

$$\Phi''(t) = h\left\{\frac{\partial^2 f}{\partial x^2}\cdot h + \frac{\partial^2 f}{\partial y\partial x}k\right\} + k\left\{\frac{\partial^2 f}{\partial x\partial y}h + \frac{\partial^2 f}{\partial y^2}k\right\}$$

$$= h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\partial y} + k^2\frac{\partial^2 f}{\partial y^2}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0+th, y_0+tk)$$

$$\Phi'''(t) = h^2\left\{\frac{\partial^3 f}{\partial x^3}h + \frac{\partial^3 f}{\partial y\,\partial x^2}k\right\} + 2hk\left\{\frac{\partial^3 f}{\partial x^2\,\partial y}h + \frac{\partial^3 f}{\partial x\,\partial y^2}k\right\}$$

$$+ k^2\left\{\frac{\partial^3 f}{\partial x\,\partial y^2}h + \frac{\partial^3 f}{\partial y^3}k\right\}$$

$$= h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2\,\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\,\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0+th,\, y_0+tk)$$

Using Taylor's theorem for a function ($\Phi(t)$) of one variable about the point 0 as:

$$\Phi(t) = \Phi(0) + t\,\Phi'(0) + \frac{t^2}{\underline{|2}}\Phi''(0) + \frac{t^3}{\underline{|3}}\Phi'''(\theta t)$$

$$0 < \theta < 1$$

For $t = 1$:

$$\Phi(1) = \Phi(0) + \Phi'(0) + \frac{1}{2}\Phi''(0) + \frac{t^3}{\underline{|3}}\Phi'''(\theta)$$

$$\Rightarrow\quad f(x_0+h,\, y_0+k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x_0, y_0)$$

$$+ \frac{1}{\underline{|2}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \frac{1}{\underline{|3}}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0+\theta h,\, y_0+\theta k)$$

where $0 < \theta < 1$.

$\square$

Ex.: Find the quadratic Taylor's polynomial approximation to the function $f(x,y) = \frac{x-y}{x+y}$ about the point $(1,1)$.

Sol: $$f_x = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\Rightarrow f_x(1,1) = \frac{1}{2}$$

$$f_y = \frac{-(x+y)-(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

$$\Rightarrow f_y(1,1) = -\frac{1}{2}.$$

$$f_{xx} = \frac{-4y}{(x+y)^3} \Rightarrow f_{xx}(1,1) = -\frac{1}{2}$$

$$f_{yy} = \frac{4x}{(x+y)^3} \Rightarrow f_{yy}(1,1) = \frac{1}{2}$$

$$f_{xy} = \frac{2x-2y}{(x+y)^3} \Rightarrow f_{xy}(1,1) = 0$$

$$P_2(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$
$$+ \frac{1}{2} f_{xx}(1,1)(x-1)^2 + f_{xy}(1,1)(x-1)(y-1) + \frac{1}{2} f_{yy}(1,1)(y-1)^2$$

$$= \frac{1}{2}(x-1) - \frac{1}{2}(y-1) - \frac{1}{4}(x-1)^2 + \frac{1}{4}(y-1)^2$$

Q: Let $f(x,y) = x^2 + xy + y^2$ be linearly approximated by the Taylor's polynomial about the point $(1,1)$. Find out the maximum error in this approximation at a point in the square $|x-1| \leq 0.1$, $|y-1| \leq 0.1$.

Sol:

$$f(x,y) = x^2 + xy + y^2$$

$f_x = 2x + y$ $\quad$ $f_{xx} = 2$ $\quad$ $f_{xy} = 1$

$f_y = 2y + x$ $\quad$ $f_{yy} = 2$

Remainder R_1 :

$$R_1 = \frac{1}{2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0 + \theta h, y_0 + \theta k)$$

$$= \frac{1}{2}\left((x-1)^2 f_{xx} + 2(x-1)(y-1) f_{xy} + f_{yy}(y-1)^2\right)$$

$$= \frac{1}{2}\left((x-1)\cdot 2 + 2\cdot(x-1)(y-1) + 2\cdot(y-1)\right]$$

$$= (x-1)^2 + (x-1)(y-1) + (y-1)^2$$

Maximum error :

$$R_1 = (0.1)^2 + (0.1)^2 + (0.1)^2$$

$$= 3 * 0.01$$

$$= 0.03$$

Ex. Obtain Taylor's formula $(n=2)$ for $f(x,y) = \cos(x+y)$ at $(0,0)$

Sol: $$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{\underline{|2}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \frac{1}{\underline{|3}}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(\theta x, \theta y)$$

$0<\theta<1.$

- $f(0,0) = 1$

- First order derivatives: $f_x = -\sin(x+y) \Rightarrow f_x(0,0) = 0$

  $f_y = -\sin(x+y) \Rightarrow f_y(0,0) = 0$

- Second order derivatives: $f_{xx} = f_{yy} = f_{xy} = -\cos(x+y)$

  At $(0,0)$, $f_{xx} = f_{yy} = f_{xy} = -1$

- Third order derivatives:

  $$f_{xxx} = f_{yyy} = f_{xxy} = f_{yyx}\Big|_{(\theta x, \theta y)} = \sin(\theta x + \theta y)$$

Taylor's Theorem:

$$f(x,y) = 1 + 0 - \frac{1}{\underline{|2}}\left(x^2 + 2xy + y^2\right) + \frac{1}{\underline{|3}}\left(x^3 + 3x^2y + 3xy^2 + y^3\right) \sin(\theta x + \theta y)$$

$$= 1 - \frac{1}{\underline{|2}}(x+y)^2 + \frac{1}{\underline{|3}}(x+y)^3 \sin(\theta x + \theta y).$$

□