# Problem Set - 11

**AUTUMN 2016**

**MATHEMATICS-I (MA10001)** *October 24, 2016*

1. Find following limits (if exists)
   $(a)$ $\lim_{z\to 0} \frac{z^2}{|z|}$, $(b)$ $\lim_{z\to i}[x + \frac{i}{1-x}]e^{xy}$, $(c)$ $\lim_{z\to -i} \frac{z^4-1}{z+i}$, $(d)$ $\lim_{z\to\infty} \frac{z}{2-iz}$, $(e)$ $\lim_{z\to 0} \frac{z}{Rez}$,
   $(f)$ $\lim_{z\to 0} \frac{Im(z^2)}{|z|^2}$, $(g)$ $\lim_{z\to 0} \frac{Re(z^2)}{|z|^2}$, $(h)$ $\lim_{z\to 1} \frac{z^2+1}{z^2-3z+2}$, $(i)$ $\lim_{z\to 0} \frac{\overline{z}}{2z}$ $(j)$ $\lim_{z\to\infty} \frac{4+z^2}{(z-1)^2}$
   $(k)$ $\lim_{z\to 0} \frac{Im(z)}{z}$

2. Test the continuity of the following functions at $z = 0$, if $f(0) = 0$
   $(a)$ $f(z) = \frac{Re(z^3)}{|z|^2}$, $(b)$ $f(z) = e^{-\frac{1}{z^2}}$, $(c)$ $f(z) = \frac{Re(z^2)}{|z|}$, $(d)$ $f(z) = \frac{Rez}{1+|z|}$, $(e)$ $f(z) = \frac{(Rez-Imz)}{|z|^2}$,
   $(f)$ $f(z) = \frac{x^3y^5(x+iy)}{(x^4+y^4)}$, $(g)$ $f(z) = \frac{Imz}{|z|}$, $(h)$ $f(z) = \frac{(z+i)^2+1}{z}$.

3. Using $\epsilon - \delta$ method prove the following problems
   $(a)$ $\lim_{z\to i} \frac{3z^4-2z^3+8z^2-2z+5}{z-i} = 4 + 4i$
   $(b)$ $\lim_{z\to i}(z^2 + 2z) = 2i - 1$
   $(c)$ $\lim_{z\to -i} z^2 = -1$
   $(d)$ If $z_0$ and $w_0$ are points in the $z$-plane and $w$-plane respectively, then show that
   $(i)$ $\lim_{z\to z_0} f(z) = \infty$ iff $\lim_{z\to z_0} \frac{1}{f(z)} = 0$, $(ii)$ $\lim_{z\to\infty} f(z) = w_0$ iff $\lim_{z\to 0} f(\frac{1}{z}) = w_0$,
   $(e)$ $\lim_{z\to 2+i} z^2 = 3 + 4i$,
   $(f)$ $\lim_{z\to -i} \frac{1}{z} = i$,
   $(g)$ Let $f(x) = x^2$ for $x \geq 0$ and $f(x) = -x^2$ for $x < 0$. Show that $f'(x)$ exists and is continuous but $f''(x)$ does not exist at $x = 0$.

4. Using $\epsilon - \delta$ method show that $f(z) = z^2$ is continuous at $z = z_0$.

5. Show that if $f(z)$ is continuous at $z = a$ then $\overline{f(z)}$ is also continuous at $z = a$.

6. Using the definition find the derivative of $f(z) = z^3 - 2z$ at the point where $z = a$ and $z = -1$.

7. Show that $\frac{d}{dz}(z^2\overline{z})$ does not exist anywhere.

8. Show that $(a)$ $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}$, $(b)$ $\frac{\partial}{\partial y} = (\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}})i$, where $z = x + iy$ and $\overline{z} = x - iy$
   $(c)$ $\bigtriangledown \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \overline{z}}$, $(d)$ $\overline{\bigtriangledown} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial z}$

9. Prove that in polar form the C-R equations can be written as $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

10. Let $f(z) = u(x, y) + iv(x, y)$. If $u_1(x, y) = \frac{\partial u}{\partial x}$ and $u_2(x, y) = \frac{\partial u}{\partial y}$ prove that $f'(z) = u_1(z, 0) - iu_2(z, 0)$.

11. $(a)$ Prove that the function $u = 2x(1 - y)$ is harmonic.
    $(b)$ Find a function $v$ such that $f(z) = u + iv$ is analytic.
    $(c)$ Express $f(z)$ in terms of $z$.

12. Prove that $f(z) = |z|^4$ is differentiable at $z = 0$ but not analytic at $z = 0$.

13. Characterize and draw the set of points in the complex plane such that
(a) $|z|^2 + 3z + 3\overline{z} + 5 = 0$, (b) $|z - i| = |z + i|$.

14. At which points the following function is complex-differentiable?

$$f(z) = \begin{cases} \frac{\overline{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Does $f$ satisfy Cauchy-Reimann equation at the origin?

15. Let $f(z)$ be an analytic function on a connected open set $D$. If there are two constants $c_1$ and $c_2 \in \mathbb{C}$ not all zero such that $c_1 f(z) + c_2 \overline{f(z)} = 0$ for all $z \in D$, then $f(z)$ is constant on $D$.

16. Show that the function $\sin(\overline{z})$ is nowhere analytic on $\mathbb{C}$.

17. Let $u(x, y) = 4xy - x^3 + 3xy^2$, $(x, y) \in \mathbb{C}$. Find harmonic conjugate of $u$ such that $f = u + iv$ is an analytic function in $\mathbb{C}$. Also find explicitly $f$.

18. Show that if $f(z)$ is analytic and if $Re\ z =$ constant or $Im\ z =$ constant then $f(z)$ is constant.

19. Find an analytic function $f(z)$ such that $Re[f'(z)] = 3x^2 - 4y - 3y^2$ and $f(1 + i) = 0$.