MA 51002: Measure Theory and Integration
Mid Semester Examination (Spring 2017)
Time: 2 Hours, Full Marks: 30, Number of students = 80.

Answer **all** the problems. Numbers at the right hand side after each question denote marks. No clarification will be entertained during the examination.

(1) For any set A, we let $\mathcal{P}(A)$ be the collection of all subsets of A. Let B^A be the set of all functions mapping A into the set $B = \{0, 1\}$. Show that $|B^A| = |\mathcal{P}(A)|$. [3]

(2) Prove that the Cantor one-third set is a perfect and totally disconnected set of measure zero. Is it countable? Justify. [1+1+2+1]

(3) Define a σ-algebra and measure. Prove that if E_1 is measurable and E_2 differs from E_1 (symmetric difference, denoted by $E_1 \Delta E_2$) by a set of measure zero, then E_2 is also measurable. [2+2]

(4) Prove or disprove: Every set is measurable. [5]

(5) Suppose E is a measurable subset of $\mathbb{R}$ with $m(E) < \infty$. Prove that for every $\epsilon > 0$, there exists a finite union $F = \cup_{j=1}^{N} Q_j$ of closed intervals such that $m(E \Delta F) \leq \epsilon$. [4]

(6) Define a Borel set. Is Borel set measurable? Justify. [1+1]

(7) Prove that [3 +3 +1]
(a) every continuous function in the closed interval $[a, b]$ is Riemann integrable.
(b) Popcorn function/ Thomae's function $f : [0, 1] \to [0, 1]$ is Riemann integrable where

$$f(x) = \begin{cases} 1/q, & \text{if } x = p/q \in (0, 1] \cap \mathbb{Q} \\ 0, & \text{if } x \in [0, 1] \cap \mathbb{Q}^c \text{ or } x = 0 \end{cases}$$

(c) Prove or disprove that composition of two Riemann integrable function is again Riemann integrable.