Recapitulation (Single step Methods)

$$y' = f(t, y), \quad y(t_0) = y_0$$

Explicit method:

$$u_{n+1} = u_n + h\,\phi(t_n, u_n, f_n, h)$$

i) Taylor's series method of order p:

$$u_{n+1} = u_n + h u_n' + \frac{h^2}{\lfloor 2} u_n'' + \cdots\cdots + \frac{h^p}{\lfloor p} u_n^{(p)}$$

$$u_n' = f(t_n, u_n) \quad \cdots\cdots$$

Error: $\left|\frac{h^{p+1}}{\lfloor p+1} y^{(p+1)}(\xi)\right| < \varepsilon$

ii) Euler method:

$$u_{n+1} = u_n + h f(t_n, u_n)$$

iii) Runge-Kutta method of second order

a)

$$K_1 = f(t_j, u_j)$$

$$K_2 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} K_1\right)$$

$$u_{j+1} = u_j + h K_2$$

modified Euler-Cauchy method

b)

$$K_1 = f(t_j, u_j)$$

$$K_2 = f(t_j + h, u_j + h K_1)$$

$$u_{j+1} = u_j + h\left(\frac{K_1 + K_2}{2}\right)$$

Euler-Cauchy method

IV) Fourth order Runge-Kutta Method:

$$k_1 = f(t_j, u_j)$$

$$k_2 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} k_2\right)$$

$$k_4 = f(t_j + h, u_j + h k_3)$$

$$u_{j+1} = u_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Implicit methods:

i) Backward Euler Method

$$u_{n+1} = u_n + h f(t_{n+1}, u_{n+1})$$

ii) Second order Runge-Kutta method:

$$u_{j+1} = u_j + k_1 h$$

$$k_1 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} k_1\right)$$

Single step methods for solving higher order differential equations / system of first order differential equation.

Consistency + Stability = Convergence

Consistency Error:

$$\tau_{n+1} = y_{n+1} - y_n - h\,\Phi(t_n, y_n, f(t_n, y_n), h)$$

Order of a method:

$$\left| \frac{1}{h} \tau_{n+1} \right| = O(h^p)$$

Stability:

A single step method when applied to $y' = \lambda y$ leads to a first order difference equation

$$u_{j+1} = E(\lambda h) u_j$$

Def: We call a single step method

- absolute stable: if $|E(\lambda h)| < 1$, $\lambda < 0$ or $Re(\lambda) < 0$
- relatively stable: if $|E(\lambda h)| < e^{\lambda h}$, $\lambda > 0$