IMPLICIT METHODS

The difference scheme obtained in explicit method can be modified by replacing the space derivative by its central difference difference approximation at the points $(mh, (n+1)k)$, (mh, nk) and $(mh, (n-1)k)$ as a weighted sum, i.e.,

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{m,n} = c^2\left[\theta\left(\frac{\partial^2 u}{\partial x^2}\right)_{m,n+1} + (1-2\theta)\left(\frac{\partial^2 u}{\partial x^2}\right)_{m,n} + \theta\left(\frac{\partial^2 u}{\partial x^2}\right)_{m,n-1}\right]$$

$$0 \le \theta \le 1$$

$$\Rightarrow \frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{k^2} = c^2\left[\theta\left[\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right]\right.$$

$$\left. + (1-2\theta)\left[\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2}\right] + \theta\left[\frac{u_{m+1}^{n-1} - 2u_m^{n-1} + u_{m-1}^{n-1}}{h^2}\right]\right]$$

$$\Rightarrow \boxed{\begin{aligned} -r^2\theta\, u_{m-1}^{n+1} + (1+2r^2\theta)\, u_m^{n+1} - r^2\theta\, u_{m+1}^{n+1} &= \left[2 - 2(1-2\theta)r^2\right] u_m^n + (1-2\theta)r^2 u_{m+1}^n \\ &+ (1-2\theta)r^2 u_{m-1}^n + \theta r^2 u_{m-1}^{n-1} - (1+2\theta r^2)\, u_m^{n-1} \\ &+ r^2\theta\, u_{m+1}^{n-1} \end{aligned}} \quad \text{—(1)}$$

Recall:

$\Delta f(x) = f(x+h) - f(x)$ forward diff. operator

$\nabla f(x) = f(x) - f(x-h)$ backward diff. operator

$\delta f(x) = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$ central diff. operator.

Similarly,

$$\delta^2 f(x) = \delta(\delta f(x)) = f(x+h) - 2f(x) + f(x-h) \quad \ldots$$

Rewriting (1) as:

$$\left(u_m^{n+1} - 2u_m^n + u_m^{n-1}\right) = r^2\theta\left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right)$$

$$+ (1-2\theta)r^2\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right) + \theta r^2\left(u_{m-1}^{n-1} - 2u_m^{n-1} + u_{m+1}^{n-1}\right)$$

Using notion δ, we can rewrite the above scheme as:

$$\boxed{\delta_t^2 u_m^n = r^2\delta_x^2\left[\theta u_m^{n+1} + (1-2\theta)u_m^n + \theta u_m^{n-1}\right]} \quad (2)$$

$$\delta_t^2 u_m^n = u_m^{n+1} - 2u_m^n + u_m^{n-1}$$

$$\delta_x^2 u_m^n = u_{m+1}^n - 2u_m^n + u_{m-1}^n$$

(2) may rewritten as

$$\delta_t^2 u_m^n = r^2\delta_x^2\left[u_m^n + \theta\delta_t^2 u_m^n\right]$$

or $\boxed{\left(1-\theta r^2\delta_x^2\right)\delta_t^2 u_m^n = r^2\delta_x^2 u_m^n}$

For $\theta = \frac{1}{4}$ this scheme is known as von-Neumann scheme:

$$\boxed{\left(1-\frac{1}{4}r^2\delta_x^2\right)\delta_t^2 u_m^n = r^2\delta_x^2 u_m^n}$$

Example: Find the solution at the FIRST time step of

$$u_{tt} = u_{xx} \qquad 0 < x < 1$$

subject to

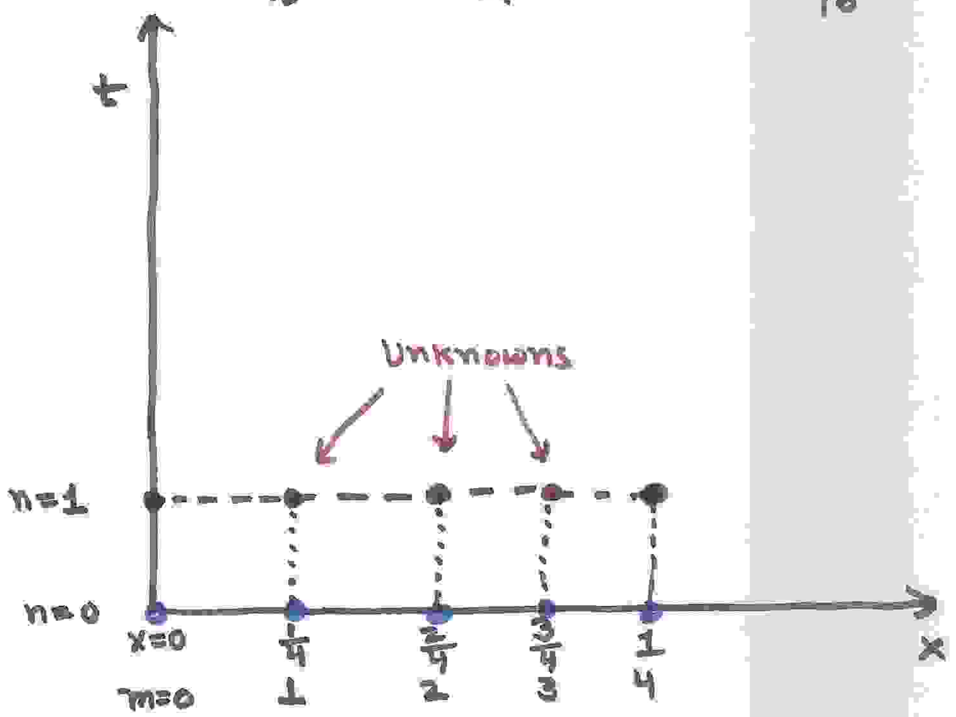
$$\left.\begin{aligned} u(x,0) &= \sin \pi x \\ u_t(x,0) &= 0 \end{aligned}\right\} \quad 0 \le x \le 1$$

$$u(0,t) = u(1,t) = 0, \; t > 0$$

by implicit scheme with $\theta = \frac{1}{2}$. Take $h = \frac{1}{4}$ & $r = \frac{3}{4}$.

Sol:

$$h = \frac{1}{4}, \quad \frac{k}{h} = r = \frac{3}{4} \Rightarrow k = \frac{3}{16}$$



IC: $u_m^0 = \sin \frac{\pi m}{4}$ & $u_m^{-1} = u_m^1$, $m = 0, 1, 2, 3, 4$

BCs: $u_m^n = 0$ for $m = 0$ & 4.

Implicit scheme for $\theta = \frac{1}{2}$.

$$\left(1 - \frac{1}{2} r^2 \delta_x^2\right) \delta_t^2 u_m^n = r^2 \delta_x^2 u_m^n$$

$$\Rightarrow \left(1-\frac{1}{2}\frac{9}{16}\delta_x^2\right)\left(u_m^{n-1}-2u_m^{n}+u_m^{n+1}\right) = \frac{9}{16}\left(u_{m-1}^{n}-2u_m^{n}+u_{m+1}^{n}\right)$$

$$\Rightarrow \left(u_m^{n-1}-2u_m^{n}+u_m^{n+1}\right) - \frac{9}{32}\left(u_{m-1}^{n-1}-2u_m^{n-1}+u_{m+1}^{n-1}\right)$$

$$+\cancel{\frac{9}{16}\left(u_{m-1}^{n}-2u_m^{n}+u_{m+1}^{n}\right)} - \frac{9}{32}\left(u_{m-1}^{n+1}-2u_m^{n+1}+u_{m+1}^{n+1}\right)$$

$$= \cancel{\frac{9}{16}\left(u_{m-1}^{n}-2u_m^{n}+u_{m+1}^{n}\right)}$$

$$\Rightarrow -\frac{9}{32}u_{m-1}^{n+1} + \frac{25}{16}u_m^{n+1} - \frac{9}{32}u_{m+1}^{n+1}$$

$$= 2u_m^{n} + \frac{9}{32}u_{m-1}^{n-1} - \frac{25}{16}u_m^{n-1} + \frac{9}{32}u_{m+1}^{n-1}$$

$m = 1, 2, 3.$

For $n=0$: (using $u_m^{-1}=u_m^{1}$)

$$-\frac{9}{16}u_{m-1}^{1} + \frac{25}{8}u_m^{1} - \frac{9}{16}u_{m+1}^{1} = 2u_m^{0}$$

For $m = 1,2,3$, we have the system

$$\begin{bmatrix} \frac{25}{8} & -\frac{9}{16} & 0 \\ -9/16 & 25/8 & -9/16 \\ 0 & -9/16 & 25/8 \end{bmatrix}\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} 2u_1^0 \\ 2u_2^0 \\ 2u_3^0 \end{bmatrix}$$

solving the above system we get:

$$u_1^1 = u_3^1 = 0.60709$$

$$u_2^1 = 0.85855$$

Von Neumann Stability Analysis (Fourier series stability analysis)

Fourier series in complex form:

Let $f(x)$ is a periodic function over period $2l$ defined in $[-l, l]$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Using Euler formula

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

we obtain

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left\{e^{i\frac{n\pi x}{l}} + e^{-\frac{in\pi x}{l}}\right\} + \frac{b_n}{2i}\left\{e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}}\right\}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)e^{\frac{in\pi x}{l}} + \frac{1}{2}(a_n + ib_n)e^{-\frac{in\pi x}{l}}\right]$$

Denoting $\quad c_0 = \frac{a_0}{2} \quad c_n = \frac{1}{2}(a_n - ib_n)$

$$c_{-n} = \frac{1}{2}(a_n + ib_n)$$

We get

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{\frac{in\pi x}{l}} + c_{-n} e^{-\frac{in\pi x}{l}} \right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

where

$$c_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} dx$$

$$n = 0, \pm 1, \pm 2, \dots$$

Stability analysis (Boundedness of numerical solution)

Consider the explicit method for solving the heat equation

$$u_j^{n+1} = (1-2\lambda) u_j^n + \lambda (u_{j-1}^n + u_{j+1}^n) \quad \text{——} \quad (1)$$

The exact solution of (1) for a single step can be expressed as

$$u_j^{n+1} = G u_j^n$$

where G, called the amplification factor, is in general a complex constant.

The solution of the FDS at time $T = N \Delta t$ is then

$$u_j^N = G^N u_j^0$$

For u_j^N to remain bounded, we must have

$$|G| \leq 1$$

Stability analysis thus reduces to the determination of the single step exact solution of the finite difference equation (1), i.e., the amplification factor G, and an investigation of the conditions necessary to ensure that $|G| \leq 1$.

From equation (1) it is seen that u_j^{n+1} depends not only on u_j^n but also on u_{j-1}^n and u_{j+1}^n. Consequently u_{j-1}^n and u_{j+1}^n must be related to u_j^n so that equation (1) can be solved for G. It is accomplished by expressing $u(x, t^n) = F(x)$ in a complex Fourier series.

The complex Fourier series of $F(x)$ is given as

$$u(x, t^n) = F(x) = \sum_{m=-\infty}^{\infty} A_n e^{i K_m x}$$

Where the wave number K_m is defined as

$$K_m = \frac{m\pi}{l}.$$