2.5 Formal Power Series Continued ...

Before proceeding further, let us look at some important examples, where the above results can be used.

Example 2.5.1. *Using the generalized binomial theorem (see Theorem 1.14.1), recall that, for a positive integer r, the formal power series*

$$\sum_{n\geq 0}\binom{n+r-1}{n}x^n \text{ equals } \frac{1}{(1-x)^r}. \tag{2.1}$$

In particular, for $r=1$, the formal power series $\sum\limits_{n\geq 0} x^n$ equals $\frac{1}{1-x}$.

1. *Determine a closed form expression for $\sum\limits_{n\geq 0} nx^n \in \mathcal{P}(x)$.*

   Solution: *As $\frac{1}{1-x} = \sum\limits_{n\geq 0} x^n$, one has $\frac{1}{(1-x)^2} = D\left(\frac{1}{1-x}\right) = D\left(\sum\limits_{n\geq 0} x^n\right) = \sum\limits_{n\geq 0} nx^{n-1}$. Thus, the closed form expression is $\frac{x}{(1-x)^2}$.*

   This can also be computed as follows:
   Let $S = \sum\limits_{n\geq 0} nx^n = x + 2x^2 + 3x^3 + \cdots$. Then $xS = x^2 + 2x^3 + 3x^4 + \cdots$. Hence, $(1-x)S = x + x^2 + x^3 + \cdots = x(1+x+x^2+\cdots) = \frac{x}{1-x}$. Thus, $S = \frac{x}{(1-x)^2}$.

2. *Let $f(x) = \sum\limits_{n\geq 0} a_n x^n \in \mathcal{P}(x)$. Determine $\sum\limits_{k=0}^{n} a_k$.*

   Solution: *Recall that the Cauchy product of $f(x) = \sum\limits_{n\geq 0} a_n x^n$ and $g(x) = \sum\limits_{n\geq 0} b_n x^n$ equals $\sum\limits_{n\geq 0} c_n x^n$, where $c_n = \sum\limits_{k=0}^{n} a_k b_{n-k}$, for $n \geq 0$. Therefore, to get $c_n = \sum\limits_{k=0}^{n} a_k$, one needs $b_k = 1$, for all $k \geq 0$. That is, $\sum\limits_{k=0}^{n} a_k = c_n = [x^n]\left(f(x)\cdot\frac{1}{1-x}\right)$.*

   Hence, the Cauchy product helps us in computing the sum of the first N coefficients of a formal power series, for any $N \geq 1$.

3. *Determine the sum of the squares of the first N positive integers.*

   Solution: *Using Equation (2.1), observe that $k = [x^{k-1}]\left(\frac{1}{(1-x)^2}\right)$. Therefore, using Example 2.5.1.2, one has*

   $$\sum_{k=1}^{N} k = [x^{N-1}]\left(\frac{1}{(1-x)^2}\cdot\frac{1}{1-x}\right) = [x^{N-1}]\,\frac{1}{(1-x)^3} = \binom{N-1+3-1}{N-1} = \frac{N(N+1)}{2}.$$

4. *Determine a closed form expression for $\sum\limits_{k=1}^{N} k^2$.*

   Solution: *Using Example 2.5.1.1, observe that $\sum\limits_{k\geq 0} kx^k = \frac{x}{(1-x)^2}$. Therefore, using the* *differentiation operator, one obtains*

$$\sum_{k\geq 0} k^2 x^k = x\left(\sum_{k\geq 0} k^2 x^{k-1}\right) = xD\left(\frac{x}{(1-x)^2}\right) = \frac{x(1+x)}{(1-x)^3}. \tag{2.2}$$

Thus, by Example 2.5.1.2

$$\begin{aligned}\sum_{k=1}^{N} k^2 &= [x^N]\left(\frac{x(1+x)}{(1-x)^3}\cdot\frac{1}{1-x}\right) = [x^{N-1}]\left(\frac{1}{(1-x)^4}\right) + [x^{N-2}]\left(\frac{1}{(1-x)^4}\right)\\ &= \binom{N-1+4-1}{N-1} + \binom{N-2+4-1}{N-2}\\ &= \frac{N(N+1)(2N+1)}{6}.\end{aligned}$$

5. *Determine a closed form expression for* $\sum\limits_{k=1}^{N} k^3$.

Solution: *Using Equation (2.2), observe that* $\sum\limits_{k\geq 0} k^2x^k = \dfrac{x(1+x)}{(1-x)^3}$. *So,*

$$\sum_{k\geq 0} k^3 x^k = x\left(\sum_{k\geq 0} k^3 x^{k-1}\right) = xD\left(\frac{x(1+x)}{(1-x)^3}\right) = \frac{x(1+4x+x^2)}{(1-x)^4}.$$

Thus, by Example 2.5.1.2

$$\begin{aligned}\sum_{k=1}^{N} k^3 &= [x^N]\left(\frac{x(1+4x+x^2}{(1-x)^4)}\cdot\frac{1}{1-x}\right)\\ &= [x^{N-1}]\left(\frac{1}{(1-x)^5}\right) + [x^{N-2}]\left(\frac{4}{(1-x)^5}\right) + [x^{N-3}]\left(\frac{1}{(1-x)^5}\right)\\ &= \binom{N-1+5-1}{N-1} + 4\binom{N-2+5-1}{N-2} + \binom{N-3+5-1}{N-3}\\ &= \left(\frac{N(N+1)}{2}\right)^2.\end{aligned}$$

Hence, we observe that we can inductively use this technique to get a closed form expression for $\sum\limits_{k=1}^{N} k^r$, *for any positive integer* r.

6. *Determine a closed form expression for* $\sum\limits_{n\geq 0} \dfrac{n^2+n+6}{n!}$.

Solution: *As we need to compute the infinite sum, Cauchy product cannot be used. Also, one needs to find a convergent series, which when evaluated at some* x_0, *gives the required expression. Therefore, recall that the series* $e^x = \sum\limits_{n\geq 0} \dfrac{x^n}{n!}$, *converges for all* $x \in \mathbb{R}$ *and evaluating* e^x *at* $x = 1$, *gives* $\sum\limits_{n\geq 0} \dfrac{1}{n!} = e$. *Similarly,* $\dfrac{n}{n!} = [x^n]\,(xD(e^x)) = [x^n]\,(xe^x)$ *and*

$\dfrac{n^2}{n!} = [x^n]\,(xD(xDe^x)) = [x^n]\left((x+x^2)e^x\right)$. *Thus,*

$$\sum_{n\geq 0} \frac{n^2+n+6}{n!} = (x+x^2)e^x + xe^x + 6e^x\Big|_{x=1} = 9e.$$

7. *Let n and r be two fixed positive integers. Then determine the number of non-negative integer solutions to the system $x_1 + x_2 + \cdots + x_n = r$.*

   Solution: *Recall that this number equals $\binom{r+n-1}{r}$ (see Lemma 1.10.2).*

   In this chapter, we can think of the problem as follows: the above system can be interpreted as coming from the monomial x^r, where $r = x_1+x_2+\cdots+x_n$. That is, the problem reduces to finding the coefficients of y^{x_k} of a formal power series, for non-negative integers x_k's. Now, recall that the terms y^{x_k} appear with a coefficient 1 in the expression $\dfrac{1}{1-y} = \sum\limits_{i\geq 0} y^i$. Hence, the question reduces to computing

$$[y^r]\left(\frac{1}{(1-y)(1-y)\cdots(1-y)}\right) = [y^r]\,\frac{1}{(1-y)^n} = \binom{r+n-1}{r}.$$

We now look at some examples that may require the use of the package "MATHEMATICA" or "MAPLE" or ... to obtain the exact answer. So, in the examples given below, one is interested in getting only a formal power series whose coefficient gives the required result even though it may be difficult to compute the coefficient.

Example 2.5.2. 1. *For fixed positive integers n and r, determine the number of non-negative integer solutions to the system $x_1 + 2x_2 + \cdots + nx_n = r$.*

   Solution: *The ideas from Example 2.5.1.7 imply that one needs to consider the formal power series $\sum\limits_{i\geq 0} x^{ki} = \dfrac{1}{1-x^k}$. Hence, need to compute*

$$[x^r]\,\frac{1}{(1-x)(1-x^2)\cdots(1-x^n)}.$$

2. *Determine the number of solutions to the system $x_1 + x_2 + \cdots + x_5 = n$, where x_i's are non-negative integer, $x_1 \geq 4$, $x_4 \leq 10$ and x_r is a multiple of r, whenever $r \neq 1, 4$.*

   Solution: *The condition $x_1 \geq 4$ corresponds to looking at x^k, for $k \geq 4$, which corresponds to the formal power series $\sum\limits_{k\geq 4} x^k$. Similarly, $x_4 \leq 10$ gives the formal power series $\sum\limits_{k=0}^{10} x^k$ and the condition x_r is a multiple of r, for $r \neq 1, 4$, corresponds to $\sum\limits_{k\geq 0} x^{rk}$. So, we need to compute the coefficient of x^n in the product*

$$\left(\sum_{k\geq 4} x^k\right)\cdot\left(\sum_{k=0}^{10} x^k\right)\cdot\left(\sum_{k\geq 0} x^{2k}\right)\cdot\left(\sum_{k\geq 0} x^{3k}\right)\cdot\left(\sum_{k\geq 0} x^{5k}\right) = \frac{x^4(1-x^{11})}{(1-x)^2(1-x^2)(1-x^3)(1-x^5)}.$$

3. *Determine the number of ways in which* 100 *voters can cast their* 100 *votes for* 10 *candidates such that no candidate gets more than* 20 *votes.*

   Solution: *Note that we are assuming that the voters are identical. So, we need to solve the system in non-negative integers to the system* $x_1 + x_2 + \cdots + x_{10} = 100$, *with* $0 \le x_i \le 20$, *for* $1 \le i \le 10$. *So, we need to find the coefficient of* x^{100} *in*

$$\left(\sum_{k=1}^{20} x^k\right)^{10} = \frac{(1-x^{21})^{10}}{(1-x)^{10}} = \left(\sum_{i=0}^{10} (-1)^i \binom{10}{i} x^{21i}\right) \cdot \left(\sum_{j\ge 0} \binom{10+j-1}{j} x^j\right).$$

   Thus, the required answer equals $[x^{100}] \left(\sum\limits_{k=1}^{20} x^k\right)^{10} = \sum\limits_{i=0}^{4} (-1)^i \binom{10}{i} \cdot \binom{109-21i}{9}$.

Before moving to the applications of formal power series to the solution of recurrence relations, let us list a few well known power series. The readers are requested to get a proof for their satisfaction.

Table of Formal Power Series

e^x	$=$	$\sum\limits_{k\ge 0} \frac{x^k}{k!}$	$\log(1-x)$	$=$	$-\sum\limits_{k\ge 1} \frac{x^k}{k},\ \lvert x\rvert < 1$
$(1+x)^a$	$=$	$\sum\limits_{r\ge 0} \binom{a}{k} x^k,\ \lvert x\rvert < 1$	$\frac{1}{1-x}$	$=$	$\sum\limits_{k\ge 0} x^k,\ \lvert x\rvert < 1$
$\frac{1}{(1-x)^a}$	$=$	$\sum\limits_{k\ge 0} \binom{a+k-1}{k} x^k,\ \lvert x\rvert < 1$	$\frac{1}{\sqrt{1-4x}}$	$=$	$\sum\limits_{k\ge 0} \binom{2k}{k} x^k,\ \lvert x\rvert < \frac{1}{4}$
$x^{-r}(1+x)^n$	$=$	$\sum\limits_{k\ge -r} \binom{n}{r+k} x^k,\ \lvert x\rvert < 1$	$\frac{x^n}{(1-x)^{n+1}}$	$=$	$\sum\limits_{k\ge 0} \binom{k}{n} x^k,\ n \ge 0, \lvert x\rvert < 1$
$\sin(x)$	$=$	$\sum\limits_{r\ge 0} \frac{(-1)^r x^{2r+1}}{(2r+1)!}$	$\cos(x)$	$=$	$\sum\limits_{r\ge 0} \frac{(-1)^r x^{2r}}{(2r)!}$
$\sinh(x)$	$=$	$\sum\limits_{r\ge 0} \frac{x^{2r+1}}{(2r+1)!}$	$\cosh(x)$	$=$	$\sum\limits_{r\ge 0} \frac{x^{2r}}{(2r)!}$
		$\frac{1-\sqrt{1-4x}}{2x} = \sum\limits_{k\ge 0} \frac{1}{k+1}\binom{2k}{k} x^k,\ \lvert x\rvert < \frac{1}{4}$			