Integer Programming

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Integer Linear Programming

General Model:

$$\max / \min : z = \sum_{j=1}^{n} c_j x_j \tag{1.1}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j (\leq = \geq) b_i, \qquad i = 1, 2, \ldots, m \tag{1.2}$$

$$x_j = 0, 1, 2, 3, ... \qquad j = 1, 2, ..., n \tag{1.3}$$

where c_j, a_{ij} and b_i are integers.

Integer Linear Programming(cont.)

Methods:

(i) Cutting Plane method
(ii) Branch and Bound method

Applications:

- Transportation Problem,
- Assignment Problem,
- Job-Shop Scheduling Problem,
- Man power Planning,
- Production Planning,
- Transhipment Problem.

Integer Linear Programming(cont.)

Methods:

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(ii) Branch and Bound method

Applications:

- Transportation Problem,
- Assignment Problem,
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- Man power Planning,
- Production Planning,
- Transhipment Problem.

Cutting Plane method

Cutting Plane method of Gomory (1958)

At first an Integer Programming Problem is solved as a regular LPP by dropping the integral condition. If the optimal solution (x^*) happens to be integer, terminate the process.
Otherwise, the secondary constrained will be added that will force the solution toward the integer solution. These constraints can be developed as follows:

Cutting Plane method(cont.)

Let the optimal Tableau for the LPP be given by:

c_B	BV \ NBV	w_1	w_2	$\cdots$	w_j	$\cdots$	w_n	X_B
$*$	x_1	α_{11}	α_{12}	$\cdots$	α_{1j}	$\cdots$	α_{1n}	β_1
$*$	x_2	α_{21}	α_{22}	$\cdots$	α_{2j}	$\cdots$	α_{2n}	β_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$*$	x_i	α_{i1}	α_{i2}	$\cdots$	α_{ij}	$\cdots$	α_{in}	β_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$*$	x_m	α_{m1}	α_{m2}	$\cdots$	α_{mj}	$\cdots$	α_{mn}	β_m
		$\overline{z_1 - c_1}$	$\overline{z_2 - c_2}$	$\cdots$	$\overline{z_j - c_j}$	$\cdots$	$\overline{z_n - c_n}$	$\overline{z}$

where $\overline{z_j - c_j} \geq 0, \quad j = 1, 2, ..., n.$
x_i $(i = 1, 2, ..., m)$ is the i-th basic variable
w_j $(j = 1, 2, ..., n)$ is the j-th non-basic variable

Cutting Plane method(cont.)

Let x_i be non-integer. Its value β_i has the largest fractional part. i.e.,

$$x_i + \sum_{j=1}^{n} \alpha_{ij} w_j = \beta_i, \quad \beta_i > 0$$

where $\beta_i = [\beta_i] + f_i,\ 0 < f_i < 1$ and $\alpha_{ij} = [\alpha_{ij}] + f_{ij},\ 0 \leq f_{ij} < 1$
Example: $[k] \leq k$: *greatest integer function*
$[2\frac{1}{2}] = 2 \Rightarrow \frac{1}{2} + 2 = 2\frac{1}{2}$
$[-3\frac{1}{2}] = -4 \Rightarrow \frac{1}{2} - 4 = -3\frac{1}{2}$
$[5] = 5$
$[-5] = -5$ *etc*

Cutting Plane method(cont.)

Now,

$$x_i + \sum_{j=1}^{n}([\alpha_{ij}] + f_{ij})w_j = [\beta_i] + f_i$$

$$\Rightarrow \qquad x_i - [\beta_i] + \sum_{j=1}^{n}[\alpha_{ij}]w_j = f_i - \sum_{j=1}^{n} f_{ij}w_j$$

For all $x_i,\ w_j$ LHS is an integer. Hence RHS is an integer.
But $f_i - \sum_{j=1}^{n} f_{ij}w_j \leq f_i < 1$ an integer.

Cutting Plane method(cont.)

Therefore,

$$
\begin{aligned}
& f_i - \sum_{j=1}^{n} f_{ij} w_j \leq 0 \\
\Rightarrow \quad & \sum_{j=1}^{n} f_{ij} w_j \geq f_i \\
\Rightarrow \quad & -\sum_{j=1}^{n} f_{ij} w_j \leq -f_i \\
\Rightarrow \quad & -\sum_{j=1}^{n} f_{ij} w_j + s_i = -f_i
\end{aligned}
$$

This fractional cut may be added to the last simplex Tableau. The problem may be solved by Dual Simplex method. This procedure is repeated till we find an integer solution.

Example

Problem-1

$$\max : Z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \leq 3$$
$$4x_1 + x_2 \leq 8$$
$$x_1, x_2 = 0, 1, 2, \ldots.$$

Initial Simplex Tableau

c_B	BV \ NBV	x_1	x_2	X_B
0	x_3	1	1	3
0	x_4	4	1	8
		-5	-4	0

Example

Problem-1

$$\max : Z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \leq 3$$
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Initial Simplex Tableau

c_B	BV \ NBV	x_1	x_2	X_B
0	x_3	1	1	3
0	x_4	4	1	8
		-5	-4	0

Example(cont.)

Simplex Tableau (cont.)

		0	4	
c_B	BV \ NBV	x_4	x_2	X_B
0	x_3	$-\frac{1}{4}$	$\frac{3}{4}$	1
5	x_1	$\frac{1}{4}$	$\frac{1}{4}$	2
		$\frac{5}{4}$	$-\frac{11}{4}$	10

Simplex Tableau (cont.)

		0	0	
c_B	BV \ NBV	x_4	x_3	X_B
4	x_2	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
5	x_1	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$
		$\frac{1}{3}$	$\frac{11}{3}$	$\frac{41}{3}$

Example(cont.)

Simplex Tableau (cont.)

		0	4	
c_B	BV \ NBV	x_4	x_2	X_B
0	x_3	$-\frac{1}{4}$	$\frac{3}{4}$	1
5	x_1	$\frac{1}{4}$	$\frac{1}{4}$	2
		$\frac{5}{4}$	$-\frac{11}{4}$	10

Simplex Tableau (cont.)

		0	0	
c_B	BV \ NBV	x_4	x_3	X_B
4	x_2	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
5	x_1	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$
		$\frac{1}{3}$	$\frac{11}{3}$	$\frac{41}{3}$

Example(cont.)

Second constraint has the largest fractional part. Hence it is selected.

$$x_1 + \frac{1}{3}x_4 - \frac{1}{3}x_3 = \frac{5}{3}$$

Using the cutting plane method we establish the constraint as follows:

$$\frac{2}{3}x_3 + \frac{1}{3}x_4 \geq \frac{2}{3}$$

$$\Rightarrow \quad -\frac{2}{3}x_3 - \frac{1}{3}x_4 \leq -\frac{2}{3}$$

Simplex Tableau

c_B	BV \ NBV	x_4	x_3	X_B
4	x_2	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
5	x_1	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$
0	s_1	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
		$\frac{1}{3}$	$\frac{11}{3}$	$\frac{41}{3}$

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Simplex Tableau

c_B	BV \ NBV	x_4	x_3	X_B
4	x_2	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
5	x_1	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$
0	s_1	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
		$\frac{1}{3}$	$\frac{11}{3}$	$\frac{41}{3}$

Example(cont.)

Append the cutting plane to the last simplex tableau and apply Dual Simplex method.

c_B	BV \ NBV	s_1	x_3	X_B
4	x_2	-1	2	2
5	x_1	1	-1	1
0	x_4	3	2	2
		1	3	13

Optimal Solution:

$x_1^* = 1, \; x_2^* = 2, \; Z^* = 13$

Example(cont.)

Append the cutting plane to the last simplex tableau and apply Dual Simplex method.

c_B	BV \ NBV	s_1	x_3	X_B
4	x_2	-1	2	2
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Branch and Bound Method (1960)

General Model

$$\max / \min : Z = \sum_{j=1}^{n} c_j x_j \tag{1.4}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j (\leq = \geq) b_i, \qquad i = 1, 2, \ldots, m \tag{1.5}$$

$$x_j = 0, 1, 2, 3, ... \qquad j = 1, 2, ..., n \tag{1.6}$$

where the cost coefficients c_j, technological coefficients a_{ij} and target value b_i are integers for $i = 1, 2, ..., m; j = 1, 2, ..., n$.

Branch and Bound Method (cont.)

- Solve the Integer Programming problem by graphical method (2D)/ Simplex method by dropping integer restrictions.
- Let x_j be an integer variable whose optimal value x_j^* is fractional. Then the range

$$[x_j^*] < x_j < [x_j^*] + 1$$

can not include any solution which is integer.

- A feasible integral value of x_j must satisfy either

$$x_j \geq [x_j^*] + 1 \tag{1.7}$$

or

$$x_j \leq [x_j^*] \tag{1.8}$$

Branch and Bound Method (cont.)

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Branch and Bound Method (cont.)

- By imposing the constraints (2.7) and (2.8) to the original LPP we find two mutually exclusive LP Problems.
- Hence the problem is branched into two subproblems. The optimal value of the objective function Z for the non-integer case be Z^*.
- For the integer case (discrete case) the optimal value Z^* is $\leq [Z^*]$. It is a bound (upper bound) for the objective function.
- This procedure is to be repeated a number of times to find an all integer solution x^*

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Example-1

P:

$$\max : Z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \leq 3$$
$$4x_1 + x_2 \leq 8$$
$$x_1, x_2 = 0, 1, 2, ...$$

P_1 :

$$\max : Z = 5x_1 + 4x_2$$

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Example-1(continued)

Initial Simplex Tableau:

	c_N	5	4	
c_B	B \ NB	x_1	x_2	X_B
0	x_3	1	1	3
0	x_4	4	1	8
		-5	-4	0

Optimal (Final) Simplex Tableau:

	c_N	0	0	
c_B	B \ NB	x_4	x_3	X_B
4	x_2	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
5	x_1	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$
		$\frac{1}{3}$	$\frac{11}{3}$	$\frac{41}{3}$

Optimal Solution:

$x_1^* = \frac{5}{3} = 1\frac{2}{3}$
$x_2^* = \frac{4}{3} = 1\frac{1}{3}$
$Z^* = \frac{41}{3} = 13\frac{2}{3}$
$[Z^*] = 13$

Example-1(continued)

Initial Simplex Tableau:

	c_N	5	4	
c_B	NB / B	x_1	x_2	X_B
0	x_3	1	1	3
0	x_4	4	1	8
		-5	-4	0

Optimal (Final) Simplex Tableau:

	c_N	0	0	
c_B	NB / B	x_4	x_3	X_B
4	x_2	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
5	x_1	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$
		$\frac{1}{3}$	$\frac{11}{3}$	$\frac{41}{3}$

Optimal Solution:

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$Z^* = \frac{41}{3} = 13\frac{2}{3}$

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Example-1(continued)

Initial Simplex Tableau:

	c_N	5	4	
c_B	B \ NB	x_1	x_2	X_B
0	x_3	1	1	3
0	x_4	4	1	8
		-5	-4	0

Optimal (Final) Simplex Tableau:

	c_N	0	0	
c_B	B \ NB	x_4	x_3	X_B
4	x_2	$-\frac{1}{3}$	$\frac{4}{3}$	$\frac{4}{3}$
5	x_1	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$
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Optimal Solution:

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$x_2^* = \frac{4}{3} = 1\frac{1}{3}$
$Z^* = \frac{41}{3} = 13\frac{2}{3}$
$[Z^*] = 13$

Example-1(continued)

Select x_1, then we have $x_1^* = \frac{5}{3} \Rightarrow [\frac{5}{3}] < x_1 < [\frac{5}{3}] + 1 \Rightarrow 1 < x_1 < 2$.
There is no integer solution in $1 < x_1 < 2$. Hence either $x_1 \leq 1$ *or* $x_1 \geq 2$.

P_2 :

$$\max : Z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \leq 3$$
$$4x_1 + x_2 \leq 8$$
$$x_1 \leq 1$$
$$x_1, x_2 \geq 0$$

$x_1^* = 1, \; x_2^* = 2, \; Z^* = 13$

P_3 :

$$\max : Z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \leq 3$$
$$4x_1 + x_2 \leq 8$$
$$x_1 \geq 2$$
$$x_1, x_2 \geq 0$$

$x_1^* = 2, \; x_2^* = 0, \; Z^* = 10$

Example-1(continued)

Select x_1, then we have $x_1^* = \frac{5}{3} \Rightarrow [\frac{5}{3}] < x_1 < [\frac{5}{3}] + 1 \Rightarrow 1 < x_1 < 2.$
There is no integer solution in $1 < x_1 < 2$. Hence either $x_1 \leq 1$ *or* $x_1 \geq 2$.

P_2 :

$$\max : Z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \leq 3$$
$$4x_1 + x_2 \leq 8$$
$$x_1 \leq 1$$
$$x_1, x_2 \geq 0$$

$x_1^* = 1,\ x_2^* = 2,\ Z^* = 13$

P_3 :

$$\max : Z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \leq 3$$
$$4x_1 + x_2 \leq 8$$
$$x_1 \geq 2$$
$$x_1, x_2 \geq 0$$

$x_1^* = 2,\ x_2^* = 0,\ Z^* = 10$

Example-1(continued)

Select x_1, then we have $x_1^* = \frac{5}{3} \Rightarrow [\frac{5}{3}] < x_1 < [\frac{5}{3}] + 1 \Rightarrow 1 < x_1 < 2$. There is no integer solution in $1 < x_1 < 2$. Hence either $x_1 \leq 1$ *or* $x_1 \geq 2$.

P_2 :

$$\max : Z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 3 \\ 4x_1 + x_2 &\leq 8 \\ x_1 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$x_1^* = 1,\ x_2^* = 2,\ Z^* = 13$

P_3 :

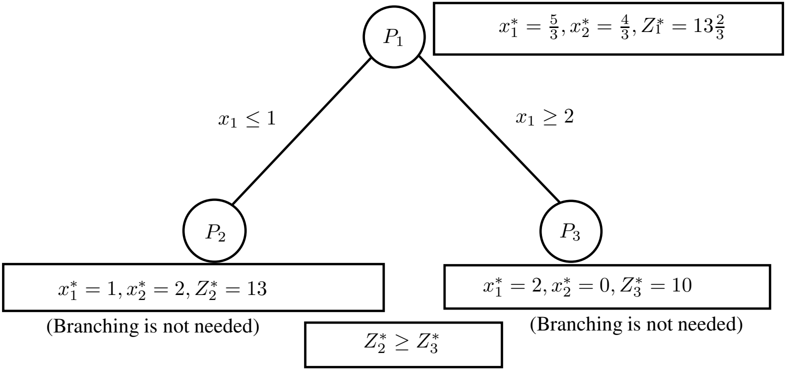
$$\max : Z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 3 \\ 4x_1 + x_2 &\leq 8 \\ x_1 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$x_1^* = 2,\ x_2^* = 0,\ Z^* = 10$

Example-1(continued)



Example-2

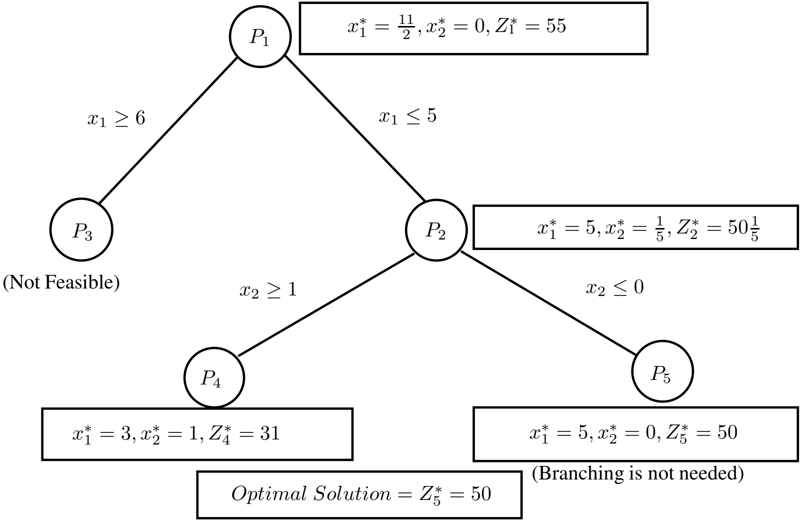
Example-2

$$\max : Z = 10x_1 + x_2$$

subject to

$$2x_1 + 5x_2 \leq 11$$
$$x_1, x_2 = 0, 1, 2, ...$$

Example-2(cont.)



Example-3

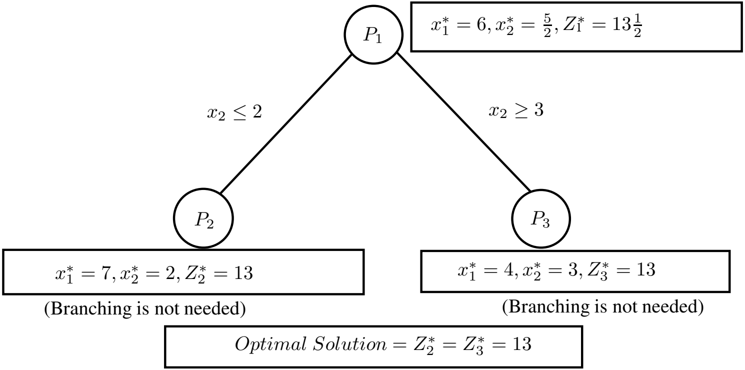
Example-3

$$\max : Z = x_1 + 3x_2$$

subject to

$$x_1 + 2x_2 \leq 11$$
$$x_1 + 4x_2 \leq 16$$
$$x_1, x_2 = 0, 1, 2, \ldots.$$

Example-3(cont.)



Some of the integer programming problem have alternative optimal solution also.

References

- H.A. Taha, Operations Reaserch, 2006.
- Ravindran, Philips and Solberg, Operations Reaserch, 2007.

Thank You