Chapter 2

Advanced Counting and Generating Functions

2.1 Pigeonhole Principle

The pigeonhole principle states that *if there are $n+1$ pigeons and n holes (boxes), then there is at least one hole (box) that contains two or more pigeons.* It can be easily verified that the pigeonhole principle is equivalent to the following statements:

1. If m pigeons are put into m pigeonholes, there is an empty hole if and only if there's a hole with more than one pigeon.

2. If n pigeons are put into m pigeonholes, with $n > m$, then there is a hole with more than one pigeon.

3. For two finite sets A and B, there exists a one to one and onto function $f : A \longrightarrow B$ if and only if $|A| = |B|$.

Remark 2.1.1. *Recall that the expression $\lceil x \rceil$, called the* CEILING FUNCTION*, is the smallest integer ℓ, such that $\ell \geq x$ and the expression $\lfloor x \rfloor$, called the* FLOOR FUNCTION*, is the largest integer k, such that $k \leq x$.*

1. [**Generalized Pigeonhole Principle**] if there are n pigeons and m holes with $n > m$, then there is at least one hole that contains $\lceil \frac{n}{m} \rceil$ pigeons.

2. Dirichlet was the one who popularized this principle.

Example 2.1.2. *Let a be an irrational number. Then prove that there exist infinitely many rational numbers $s = \dfrac{p}{q}$, such that $|a - s| < \dfrac{1}{q^2}$.*

Proof. Let $N \in \mathbb{N}$. Without loss of generality, we assume that $a > 0$. By $\{\alpha\}$, we will denote the fractional part of α. That is, $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$.

Now, consider the fractional parts $\{0\}, \{a\}, \{2a\}, \ldots, \{Na\}$ of the first $(N+1)$ multiples of a and the N subintervals $[0, \frac{1}{N}), [\frac{1}{N}, \frac{2}{N}), \ldots, [\frac{N-1}{N}, 1)$ of $[0,1)$. Clearly $\{ka\}$, for k a positive integer, cannot be an integer as a is an irrational number. Thus, by the pigeonhole principle, two of the above fractional parts must fall into the same subinterval. That is, there exist integers u, v and w such that $u > v$ but

$$\{ua\} \in [\frac{w}{N}, \frac{w+1}{N}) \quad \text{and} \quad \{va\} \in [\frac{w}{N}, \frac{w+1}{N}).$$

Thus, $|\{ua\} - \{va\}| < \dfrac{1}{N}$ and $|\{ua\} - \{va\}| = |(u-v)a - (\lfloor ua \rfloor - \lfloor va \rfloor)|$. Now, let $q = u - v$ and $p = \lfloor ua \rfloor - \lfloor va \rfloor$. Then $p, q \in \mathbb{Z}, q \neq 0$ and $|qa - p| < \frac{1}{N}$. Dividing by q, we get

$$|a - \frac{p}{q}| < \frac{1}{Nq} \leq \frac{1}{q^2} \quad \text{as } \ 0 < q \leq N.$$

Therefore, we have found a rational number $\dfrac{p}{q}$ such that $|a - \dfrac{p}{q}| < \dfrac{1}{q^2}$. We will now show that the number of such pairs (p,q) is infinite.

On the contrary, assume that there are only a finite number of rational numbers, say $r_1, r_2, \ldots, r_M$ such that

$$r_i = \frac{p_i}{q_i}, \quad \text{for } i = 1, \ldots, M, \quad \text{and} \quad |a - r_i| < \frac{1}{q_i^2}.$$

Since a is an irrational number, none of the differences $|a - r_i|$, for $i = 1, 2, \ldots, M$, will be exactly 0. Therefore, there exists an integer Q such that

$$|a - r_i| > \frac{1}{Q}, \quad \text{for all} \quad i = 1, 2, \ldots, M.$$

We now, apply our earlier argument to this Q. The argument gives the existence of a fraction $r = \dfrac{p}{q}$ such that $|a - r| < \frac{1}{Qq} < \frac{1}{Q} < |a - r_i|$, for $1 \leq i \leq M$. Hence, $r \neq r_i$, for all $i = 1, 2, \ldots, M$. On the other hand, we also have, $|a - r| < \dfrac{1}{q^2}$ contradicting the assumption that the fractions r_i, for $i = 1, 2, \ldots, M$, were all the fractions with this property. ■