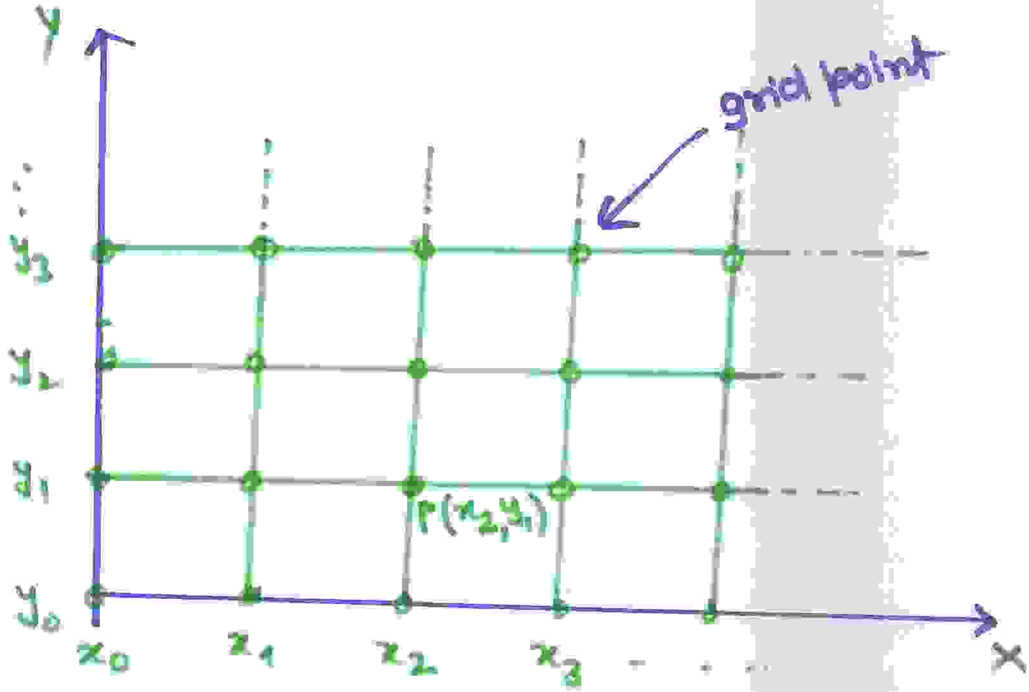
# Finite difference approximations to partial derivatives

Let the xy plane be divided into a set of equal rectangles of sides $\Delta x = h$ and $\Delta y = k$ by drawing rectangles of sides $\Delta x = h$ and $\Delta y = k$ by drawing the equally spaced grid lines parallel to the co-ordinate axes, defined by

$$x_m = mh, \quad m = 0, 1, 2, \dots$$
$$y_n = nk, \quad n = 0, 1, 2, \dots$$




The approximate value of $u$ at a grid point $P(x_m, y_n)$ is denoted by $u_m^n$ i.e.,

$$u_m^n \fallingdotseq u(x_m, y_n) = u(mh, nk).$$

To this end, we define

$$u_x(x_m, y_n) \fallingdotseq \frac{u_{m+1}^n - u_m^n}{h} + O(h) \quad \text{(forward difference)}$$

$$u_x(x_m, y_n) \simeq \frac{u_m^n - u_{m-1}^n}{h} + O(h) \quad \begin{pmatrix}\text{backward} \\ \text{difference}\end{pmatrix}$$

$$= \frac{u_{m+1}^n - u_{m-1}^n}{2h} + O(h^2) \quad \begin{pmatrix}\text{central} \\ \text{difference}\end{pmatrix}$$

Similarly,

$$u_y(x_m, y_n) \simeq \frac{u_m^{n+1} - u_m^n}{k} + O(k) \quad \text{forward}$$

$$\simeq \frac{u_m^n - u_m^{n-1}}{k} + O(k) \quad \text{backward}$$

$$\simeq \frac{u_m^{n+1} - u_m^{n-1}}{2k} + O(k^2) \quad \text{central}$$

&

$$u_{xx}(x_m, y_n) \simeq \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2} + O(h^2)$$

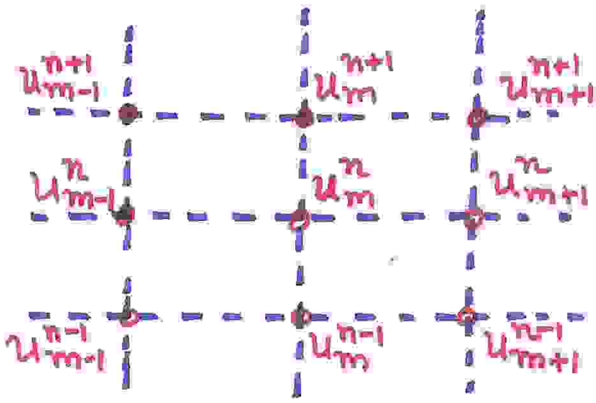
$$u_{yy}(x_m, y_n) \simeq \frac{u_m^{n-1} - 2u_m^n + u_m^{n+1}}{k^2} + O(k^2)$$

# Parabolic Partial differential equation

We consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad a \le x \le b, \quad 0 \le t \le T \qquad \text{--- (1)}$$

Let us denote $u_m^n$ the approximation of $u$ at $(x_m, t_n)$.




The possible approximations of the equation (1) are:

2(i): $$\frac{u_m^{n+1} - u_m^n}{k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Schmidt method (explicit) (two level) (cond. stable) $O(k+h^2)$

2(ii) $$\frac{u_m^{n} - u_m^{n-1}}{k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Laasonen method (Implicit) (two level) uncond. stable $O(k+h^2)$

2(iii) $$\frac{u_m^{n+1} - u_m^{n-1}}{2k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Richardson (Leapfrog) method (explicit) (three level) uncond. ~~stable~~ unstable $O(k^2+h^2)$.

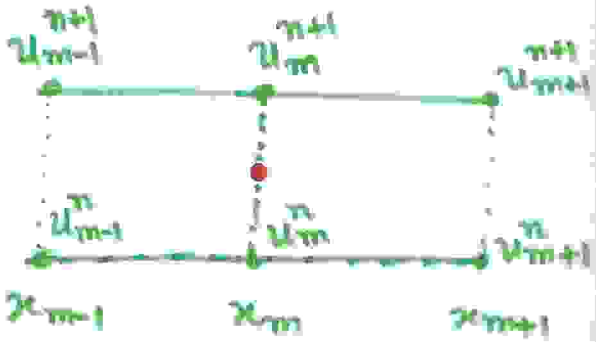
**Dufort Frankel method:** modification to 2 iii):
$u_m^n$ is replaced by the average of $u_m$ at time level $(n+1)$ and $(n-1)$; i.e.,

$$u_m^n = \frac{u_m^{n+1} + u_m^{n-1}}{2}.$$ Then the method 2(iii) becomes

2(iv) $$\frac{u_m^{n+1} - u_m^{n-1}}{2K} = \frac{u_{m-1}^{n} - (u_m^{n+1} + u_m^{n-1}) + u_{m+1}^{n}}{h^2}.$$

Now the method becomes unconditionally stable. However the method is not consistent. ORDER of the method is 0.

## CRANK-NICOLSON METHOD:




Crank & Nicolson proposed approximating the partial derivatives at the point $(x_m, t_n + \frac{K}{2})$ or $(m, n+\frac{1}{2})$ as.

$$u_t\Big|_{(m,n+\frac{1}{2})} \simeq \frac{u_m^{n+1} - u_m^{n}}{K}$$

and

$$u_{xx}\Big|_{(m,n+\frac{1}{2})} \simeq \frac{1}{2}\left[u_{xx}\Big|_{(m,n)} + u_{xx}\Big|_{(m,n+1)}\right]$$

$$\simeq \frac{1}{2}\left[\frac{u_{m+1}^{n} - 2u_m^{n} + u_{m-1}^{n}}{h^2} + \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right]$$

Finally, the scheme becomes:

2(v):

$$\frac{u_m^{n+1}-u_m^n}{k} = \frac{1}{2}\left[\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} + \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right]$$

ORDER: $O(k^2+h^2)$

STABILITY: UNCOND. STABLE.

TRUNCATION ERROR OF THE METHOD: 2(i)

$$\text{T.E.} = \frac{u(x_m, t_n+k) - u(x_m,t_n)}{k} - \frac{1}{h^2}\Big[u(x_m-h,t_n) - 2u(x_m,t_n) + u(x_m+h,t_n)\Big]$$

$$= \frac{u(x_m,t_n) + k\,u_t(x_m,t_n) + \frac{k^2}{\underline{2}}u_{tt}(x_m,t_n) + \cdots - u(x_m,t_n)}{k}$$

$$-\frac{1}{h^2}\Big[u(x_m,t_n) - h\,u_x(x_m,t_n) + \frac{h^2}{2}u_{xx}(x_m,t_n) - \frac{h^3}{2}u_{xxx} + \frac{h^4}{\underline{4}}\cdots$$
$$-2u(x_m,t_n)$$
$$+u(x_m,t_n) + h\,u_x(x_m,t_n) + \frac{h^2}{2}u_{xx}(x_m,t_n) + \frac{h^4}{4}u_{xxx} + \frac{h^4}{\underline{4}}\cdots\Big]$$

$$= \underline{u_t} + \frac{k}{2}u_{tt} + \cdots - \underline{u_{xx}} - \frac{h^2}{12}u_{xxxx} + \cdots$$

$$= \frac{k}{2}u_{tt} + \cdots - \frac{h^2}{12}u_{xxxx} + \cdots$$

$$= \left(\frac{k}{2} - \frac{h^2}{12}\right)u_{xxxx} + O(k^2) + O(h^3) = O(k) + O(h^2)$$
$$= O(k+h^2)$$

The method is said to be second order accurate in space and 1st order in time.

The methods can be rewritten in simplified form:

2(i): $$u_m^{n+1} = (1-2\lambda)\,u_m^n + \lambda\left(u_{m-1}^n + u_{m+1}^n\right)$$

$\lambda$ is called mesh ratio parameter.

$$\lambda = \frac{k}{h^2}.$$

2(ii): $$-\lambda u_{m-1}^n + (1+2\lambda)\,u_m^n - \lambda u_{m+1}^n = u_m^{n-1}$$

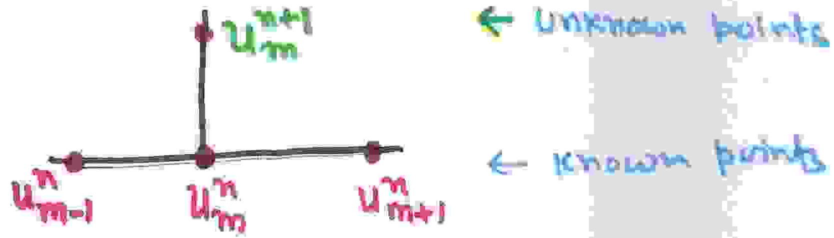
OR

$$-\lambda u_{m-1}^{n+1} + (1+2\lambda)\,u_m^{n+1} - \lambda u_{m+1}^{n+1} = u_m^n$$

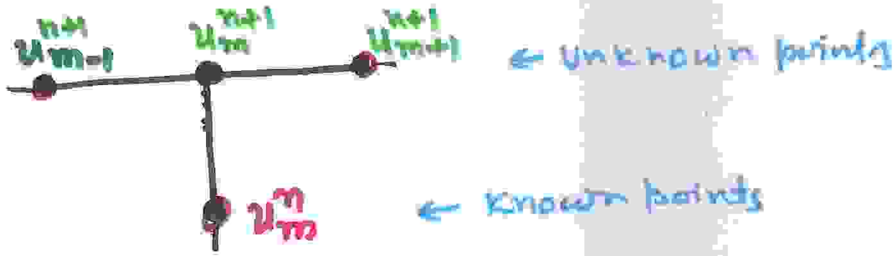
2(v): $$-\lambda u_{m-1}^{n+1} + (2+2\lambda)\,u_m^{n+1} - \lambda u_{m+1}^{n+1} = \lambda u_{m-1}^n + (2-2\lambda)\,u_m^n + \lambda u_{m+1}^n$$

SCHEMATIC DIAGRAM (STENCIL)

2(i):




2(ii)




2(v)