Lecture - 4 : Friday - 3-5 p.m.
15.1.16

Ex. Find the solution space of $W = \{(x, y, z, u, v) : Ax = 0\}$ where

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 3 & 0 & 3 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 3 & 5 & 0 & 6 & 2 \\ 2 & 3 & 2 & 5 & 2 \end{bmatrix}$$

Hence find the basis & dimension of the solution space of W.

Soln

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 3 & 0 & 3 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 3 & 5 & 0 & 6 & 2 \\ 2 & 3 & 2 & 5 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 2 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Equivalent system is given by

$$\begin{aligned} x + 2y \quad + 3u + v &= 0 \\ y \quad + 3u + v &= 0 \\ 2z + 2u + v &= 0 \end{aligned}$$

$v = t,\ u = s;\quad z = -\frac{1}{2}(2u+v) = -\frac{1}{2}(2s+t)$

$y = -3u - v = -3s - t$

$x = -2y - 3u - v = 3s + t$

$$(x, y, z, u, v) = \left(3s+t, -3s-t, -\tfrac{1}{2}(2s+t), s, t\right)$$

$$= s\underbrace{(3, -3, -1, 1, 0)}_{e_1} + t\underbrace{(1, -1, -\tfrac{1}{2}, 0, 1)}_{e_2}$$

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 3 & -3 & -1 & 1 & 0 \\ 1 & -1 & -\frac{1}{2} & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & -3 & -1 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & -3 \end{pmatrix}$$

$\therefore$ e_1, e_2 are linearly independent.

Also e_1, e_2 span the solution space or W.

$\therefore$ dim of the solution space $= 2$.

<u>Theorem.</u> The dimension of the solution space W of the homogeneous system of linear equations $AX = 0$ is '$n-r$' where 'n' is the number of unknowns & 'r' is the rank of the coefficient matrix.

Coordinates

$$v = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$$

$(a_1, a_2, \ldots, a_n)$ is the coordinate vector of v relative to the basis $\{e_1, e_2, \ldots, e_n\}$

$$P_2(t) = \{at^2 + bt + c : a, b, c \in \mathbb{R}\}.$$

$e = \{e_1 = 1,\ e_2 = t-1,\ e_3 = (t-1)^2\}$ form a basis

Let $v = 2t^2 - 5t + 6$

$$2t^2 - 5t + 6 = c_1 \times 1 + c_2(t-1) + c_3(t-1)^2$$

$$c_1 = 3,\quad c_2 = -1,\quad z = 2$$

$$[v]_e = (3, -1, 2).$$

$$[v]_{e = \{1, t, t^2\}} = (6, -5, 2)$$

Ex 1. If u, v, w be linearly independent vectors, show that $u+v$, $u-v$, $u-2v+w$ are also linearly independent.

Ex 2. Let W be the subspace of $\mathbb{R}^4$ generated by the vectors $(1, -2, 5, -3)$, $(2, 3, 1, -4)$, $(3, 8, -3, -5)$. Find a basis & dim of W. Extend the basis of W to for a basis of the whole space.

Hint.
$$\begin{pmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 5 & 3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows $(1, -2, 5, 3)$ & $(0, 7, -9, 2)$ of the echelon matrix form a basis of the row space i.e. of W. $\dim W = 2$

Add e_3, e_4

Linear Transformation (Mapping)

Let V & W be two vector spaces over the same field F. Then the mapping / transformation $T: V \to W$ is said to be linear if.

1) $T(v_1 + v_2) = T(v_1) + T(v_2)$, $\forall\, v_1, v_2 \in V$
2) $T(cv) = c\,T(v)$ $\forall\, c \in F, v \in V$

1 & 2 can be merged as $T(c_1 v_1 + c_2 v_2) = c_1 T(v_1) + c_2 T(v_2)$.

Theorem. $T: V \to W$ is a linear transformation if

1. $T(\underset{\sim}{0}_V) = \underset{\sim}{0}_W$
2. $T(C_1 v_1 + C_2 v_2) = C_1 T(v_1) + C_2 T(v_2)$.

Ex 1. Check whether $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$$T(x_1, x_2, x_3) = (x_1 + 1, x_2 + 1, x_3 + 1)$$
is a linear transformation or not.

Soln. $T(0,0,0) = (1,1,1) \neq (0,0,0)$

Thus T is not a linear transformation.

Ex 2. $T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x^2$ is a linear transformation or not.

Soln. $T(0) = 0$

$T(x+y) = (x+y)^2$

$T(x) + T(y) = x^2 + y^2$

$\therefore T(x+y) \neq T(x) + T(y)$.

Ex 3. Let P be a vector space of all polynomials. The mapping $D: P \to P$ is defined by $\frac{d}{dx} p(x)$; $p(x) \in P$. Show that D is a linear map.

Soln.

1. $\frac{d \cdot 0}{dx} = 0$

2. $\frac{d}{dx}\left(c_1 p_1(x) + c_2 p_2(x) \right) = c_1 \frac{d}{dx} p_1(x) + c_2 \frac{d}{dx} p_2(x)$

Kernel & Image of a Linear Transformation

Let $T: V \to W$ be a linear transformation.

Kernel of $T = \text{Ker}\{T\}$

$= \{v \in V : T(v) = \underset{\sim}{0}_W\}$

= set of all vectors $v \in V$ which are mapped to zero vector of W.

Theorem. $\text{Ker}\{T\}$ is a subspace of V.

Theorem. Let $T: V \to W$ such that $\text{Ker}\{T\} = \{\underset{\sim}{0}_v\}$. Then the images of a linearly independent set of vectors in V are linearly independent in W.

Example. $T: \mathbb{R}^3 \to \mathbb{R}^4$ given by

$T(x_1, x_2, x_3) = (x_2 + x_3,\ x_3 + x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3)$

Find kernel (T). Verify that Te_1, Te_2, Te_3 are linearly independent, where $\{e_1, e_2, e_3\}$ is the natural basis of $\mathbb{R}^3$.

Soln. $\operatorname{Ker}(T) = \{(x_1, x_2, x_3) : T(x_1, x_2, x_3) = \underset{\sim}{0}_W\}$

$$\Rightarrow \left.\begin{array}{r} x_2 + x_3 = 0 \\ x_3 + x_1 = 0 \\ x_1 + x_2 = 0 \\ x_1 + x_2 + x_3 = 0 \end{array}\right\} \Rightarrow x_1 = 0,\ x_2 = 0,\ x_3 = 0$$

$\therefore \operatorname{Ker}(T) = \{(0,0,0)\} = \{\underset{\sim}{0}_V\}$.

$\{e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)\}$ forms a basis of $\mathbb{R}^3$.

$Te_1 = (0,1,1,1)$

$Te_2 = (1,0,1,1)$

$Te_3 = (1,1,0,1)$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

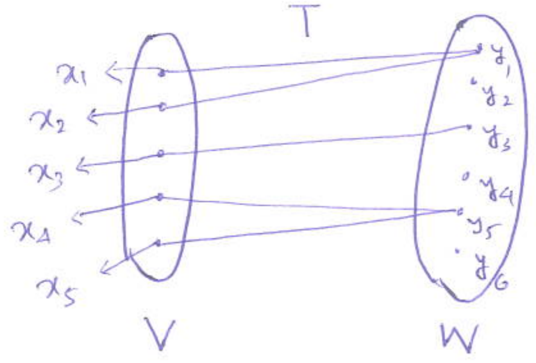
$$\longrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & -1 \end{pmatrix}$$

$\because$ there are 3 non-zero rows, the vectors Te_1, Te_2, Te_3 are linearly independent.

Image of T.

Let V & W be vector space over a field F.

$$\mathrm{Im}\, T = \{ w \in W : T(v) = w \,;\, v \in V \}$$



$\mathrm{Im}\, T = \{ y_1, y_3, y_5 \} \subset W$

Theorem. $\mathrm{Im}(T)$ is a subspace of W.

Definitions.

1. dimension of Ker T = nullity of T
2. dimension of Im T = rank of T.

Theorem. Rank T + nullity T = dim V

Theorem. Let V & W be vector spaces over a field F. Let $T : V \to W$ be a linear transformation & $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. Then the vectors $T(v_1), T(v_2), \ldots, T(v_n)$ span / generate Im T.

Ex.1. Determine the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ of $\mathbb{R}^3$ to $(1,1,1)$, $(1,1,1)$, $(1,1,1)$. Verify that rank T + nullity T $= \dim \mathbb{R}^3$.

Soln let $(x, y, z) \in \mathbb{R}^3$ then

$$(x, y, z) = c_1 (0,1,1) + c_2 (1,0,1) + c_3 (1,1,0)$$

since $\{ (0,1,1), (1,0,1), (1,1,0) \}$ is a basis of $\mathbb{R}^3$.

$$T(x,y,z) = T(c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0))$$
$$= c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$$
$$= c_1(1,1,1) + c_2(1,1,1) + c_3(1,1,1)$$
$$= (c_1+c_2+c_3,\ c_1+c_2+c_3,\ c_1+c_2+c_3)$$

$$(x,y,z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$$
$$= (c_2+c_3,\ c_1+c_3,\ c_1+c_2).$$

$$\therefore 2(c_1+c_2+c_3) = x+y+z$$
$$\Rightarrow c_1+c_2+c_3 = \frac{x+y+z}{2}$$

$$\therefore T(x,y,z) = \left(\frac{x+y+z}{2}, \frac{x+y+z}{2}, \frac{x+y+z}{2}\right).$$

$$\text{Ker } T = \{(x,y,z) : T(x,y,z) = (0,0,0)\}$$

$$\therefore x+y+z=0$$
$$z=c,\ y=b,\ x=-b-c$$
$$\therefore (x,y,z) = (-b-c, b, c) = b\underbrace{(-1,1,0)}_{f_1} + c\underbrace{(-1,0,1)}_{f_2}$$

Also $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\therefore$ f_1, f_2 are linearly independent.

Also $\{f_1, f_2\}$ spans Ker T.

$\therefore$ $\{f_1, f_2\}$ forms a basis for Ker T

$\therefore$ nullity of T = dim of Ker $T = 2$.

Since $\{\underbrace{(0,1,1)}_{e_1}, \underbrace{(1,0,1)}_{e_2}, \underbrace{(1,1,0)}_{e_3}\}$ forms a basis for $\mathbb{R}^3$. So $\{T(e_1), T(e_2), T(e_3)\}$ span Im T.

$T(e_1) = (1,1,1)$, $T(e_2) = (1,1,1)$, $T(e_3) = (1,1,1)$

$\therefore$ $(1,1,1)$ spans $\operatorname{Im} T$.

$(1,1,1)$ being a single vector is linearly independent

So $\{(1,1,1)\}$ forms a basis for $\operatorname{Im} T$.

$\therefore$ dimension of $\operatorname{Im} T$ = Rank of $T = 1$.

$$\therefore \quad \text{Rank}(T) + \text{Nullity}(T) = 1+2$$
$$= 3 = \dim \mathbb{R}^3.$$

<u>Ex 2</u>. Determine the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ of $\mathbb{R}^3$ to the vectors $(2,0,0)$, $(0,2,0)$, $(0,0,2)$ respectively. Find $\operatorname{Ker} T$ & $\operatorname{Im} T$. Verify that rank T + nullity T = $\dim \mathbb{R}^3$

<u>Soln.</u>

$$(x,y,z) = c_1 \underbrace{(0,1,1)}_{e_1} + c_2 \underbrace{(1,0,1)}_{e_2} + c_3 \underbrace{(1,1,0)}_{e_3}$$

$$T(x,y,z) = c_1(2,0,0) + c_2(0,2,0) + c_3(0,0,2)$$
$$= 2c_1 + 2c_2 + c_3.$$

Also

$$\left.\begin{array}{l} c_2 + c_3 = x \\ c_1 + c_3 = y \\ c_1 + c_2 = z \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = \dfrac{y+z-x}{2} \\ c_2 = \dfrac{x-y+z}{2} \\ c_3 = \dfrac{x+y-z}{2} \end{array}$$

$$\therefore \quad T(x,y,z) = (y+z-x,\ z+x-y,\ x+y-z).$$

$$\operatorname{Ker} T = \{(x,y,z) : T(x,y,z) = 0\}$$

$$\therefore \begin{array}{l} -x+y+z=0 \\ x-y+z=0 \\ x+y-z=0 \end{array} \Rightarrow \begin{array}{r} -x+y+z=0 \\ z=0 \\ y=0. \end{array}$$

$\therefore$ $\operatorname{Ker} T = \{0,0,0\}$. $\therefore$ Nullity $T = 0$.

$\because \{e_1, e_2, e_3\}$ forms a basis for $\mathbb{R}^3 (= V)$.

$\therefore \{Te_1, Te_2, Te_3\} = \{\underbrace{(2,0,0)}_{f_1}, \underbrace{(0,2,0)}_{f_2}, \underbrace{(0,0,2)}_{f_3}\}$

spans $\operatorname{Im} T$.

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

$\Rightarrow \{f_1, f_2, f_3\}$ are linearly independent

$\Rightarrow \{f_1, f_2, f_3\}$ forms a basis for $\operatorname{Im} T$.

$\therefore$ rank $T = 3$

$\therefore$ nullity $T +$ rank $T = 3$.