ASSIGNMENT – 6

Numerical Solutions of Ordinary and Partial Differential Equations

1. Find the Local truncation error and order of the Crank Nicolson method that is used to solve the one dimensional heat equation.

2. Discuss the consistency of Dufort-Frankel method used to solve $u_t = u_{xx}$.

3. Derive the Crank-Nicolson method. Use it to solve the parabolic partial differential equation

   $u_t = u_{xx}$, $\quad x \in (0,1), t \in (0, \infty)$

   with initial condition $u(x,0) = 2x$, boundary conditions $u_x(0,t) = 0$ and $u_x(1,t) = 1$. Use the central difference approximation for the boundary conditions. Take $h = k = 0.5$. Mention the value of $u(0.5, 0.5)$.

4. Using the Crank-Nicolson method with $h = \frac{1}{2}$ and the mesh ratio parameter $r = \frac{1}{3}$

   find the solution of $\quad u_t = u_{xx}$ with

   Initial condition $\quad u(x,0) = \cos\frac{\pi x}{2}, \quad -1 \le x \le 1, t = 0;$

   boundary conditions $\quad u(-1,t) = u(1,t) = 0, \quad t > 0$

   at the first time step (i.e. $t = k$).

5. Use the Crank-Nicolson method and the central difference for the boundary condition to

   solve the B.V.P. $\quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 1,$

   $$u(x,0) = 2, 0 \le x \le 1,$$

   $$u(0,t) = 2, t \ge 0,$$

   $$\frac{\partial u}{\partial t}(1,t) = -u(1,t), t \ge 0,$$

   With step length $h = 1/3$ and $\lambda = 1/3$. Integrate upto two time steps.

6. Use the explicit method to solve the wave equation

$u_{tt} = u_{xx}, \; 0 < x < 1, t > 0$

with boundary and initial conditions

$u(0,t) = -\sin t, \; u(1,t) = \sin(1-t), \; u(x,0) = \sin x, \quad u_t(x,0) = -\cos(x).$

Take step length along x-axis and t-axis as 1/5 and 1 respectively. Find solution for $t = 2.$

7. Using standard 5-point formula, derive the system of algebraic equations at the nodal points for the elliptic equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2, \quad -1 < x < 1, -1 < y < 1,$ $u = 2$ at $x = -1$ & $x = 1$; $u = 1$ at $y = -1$ & $y = 1$. Take $h = k = 1/2$. Setup the Gauss-Seidel iteration for the system of equations.

8. Use the explicit method

$u_m^{n+1} = 2(1 - p^2)u_m^n + p^2(u_{m-1}^n + u_{m+1}^n) - u_m^{n-1}$

to find the solution of the below pde at the second time step $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ with $u(x,0) = \frac{1}{10}x^2, \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < 1$

and $\frac{\partial u}{\partial x}(0,t) = \frac{1}{5}t, u(1,t) = \frac{1}{10}(1+t)^2, \quad t > 0$.

Use $h = 1/2$, $k = 0.1$; $x \in [0,1]$ and use central difference approximation for the derivatives in the initial and boundary conditions.

9. Use the implicit scheme

$\delta_t^2 u_m^n = r^2 \delta_x^2 [\theta u_m^{n+1} + (1 - 2\theta)u_m^n + \theta u_m^{n-1}]$

with $\theta = 1/2$ and other symbols have their usual meanings, to solve the hyperbolic equation

$$u_{tt} = u_{xx}$$

with initial conditions $u(x,0) = \sin x$ and $u_t(x,0) = -\frac{1}{5}\cos x$

And the boundary conditions $u(0,t) = -\sin(\frac{t}{5})$ and $u(1,t) = \sin(1 - \frac{t}{5})$.

Take $h = k = 0.25$. Solve for the first time level .

10. Use the explicit method to solve the wave equation $u_{tt} = \frac{1}{25} u_{xx}$, $0 < x < 1, t > 0$ with boundary and initial conditions $u(0,t) = -\sin(t/5)$, $u(1,t) = \sin(1 - t/5)$,

$u(x,0) = \sin(x)$, $u_t(x,0) = -\frac{1}{5}\cos(x)$. Take step length along x-axis and t-axis as 1/5 and 1 respectively. Find solution for $t = 2$.

11. Using standard 5-point formula, derive the system of algebraic equations at the nodal points for the elliptic equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8xy$, $-1 < x < 1, -1 < y < 1$,

$u = 2$ at $x = -1$ & $x = 1$, $u = 1$ at $y = -1$ & $y = 1$, with $h = k = 1/2$.

Setup the Gauss-Seidel iteration for the system of equations.

12. The torsion of an elastic beam of square cross section requires the solution of the BVP

$u_{xx} + u_{yy} + 2 = 0$, $(x, y) \in (-1,1) \times (-1,1)$

with $u = 0$ on the boundary of the square. First write the discretization scheme using a step length $h = k = 0.5$. Now use symmetry of the problem to reduce the number of unknowns. Solve the equation by a direct method to find $u(0,0)$.

13. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in $0 \le x, y \le 1$ with $u(x, y) = e^{3x} \cos 3y$ on the boundary using the standard 5-point formula with $h = k = \frac{1}{3}$. Use Gauss-Seidel iteration to solve the system of equations .

*****END*****