1. Product and Quotient Groups

(1) Let $G = H \times K$. Show that G is abelian if and only if both H and K are abelian.

(2) Is the symmetric group S_3 is a direct product of its proper subgroups?

(3) Prove that the product of two infinite cyclic group is not cyclic.

(4) Prove that if $G/\mathbb{Z}(G)$ is cyclic then G is abelian.

(5) Let a group G contain normal subgroups of order 3 and 5. Show that G has an element of order 15.

(6) Let G be a group of order ab where G has two subgroups H and K of order a and b respectively. Show that if $H \cap K = (1)$ then $G = HK$. Is G isomorphic to $H \times K$?

(7) Show that $H = \{A \in GL_n(\mathbb{R}) | detA > 0\}$ is a normal subgroup of $GL_n(\mathbb{R})$. Describe the quotient group.

(8) Let G be a group. Prove that $N = \langle x^{-1}y^{-1}xy | x, y \in G \rangle$ is a normal subgroup of G and G/N is abelian (N is called the commutator subgroup of G).

(9) Let M and N be normal subgroups of a group G such that $G = MN$. Prove that $G/(M \cap N) \cong G/M \times G/N$.

(10) Show that $Z(G \times H) = Z(G) \times Z(H)$.

(11) Show that any normal subgroup of G of order 2 is contained in the centre of G.

(12) Show that the multiplicative group of non zero complex numbers is isomorphic to the direct product of $(\mathbb{R}^{\times}_{>0}, .)$ and $(\mathbb{R}/\mathbb{Z}, +)$.