Lesson 2

Numerical Solutions of IVP

Let us consider the following IVP and discuss its existence and uniqueness of solution:

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0 \quad t \in [t_0, b] \tag{2.1}$$

Existence and Uniqueness of the Solution of IVP

The IVP (2.1) admits a unique solution $y(t)$ if $f(t,y)$ is uniformly Lipschitz continuous, that is,

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2| \text{ for any } t \in [t_0, b] \text{ and any } y_1 \text{ and } y_2.$$

Here L is called Lipschitz constant.

Finding exact solution of a practical problem is hardly possible, therefore numerical solutions are required.

Numerical Solutions of IVP

The numerical methods produce approximate values of $y(t)$ at certain points along the t coordinate called grids or mesh points. In case of equally spaced grid points we have

$$t_{i+1} = t_i + h, \quad i = 0, 1, \ldots, N-1; \quad t_N = b$$

where h is called the step size.

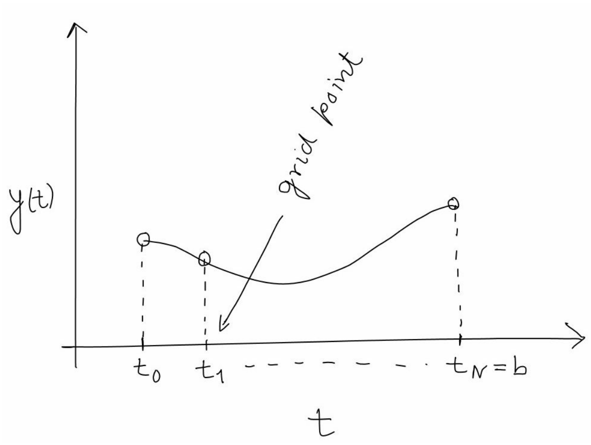


Figure 2.1: Grid points

We shall use the following notation for the the approximation of solution of IVPs

$$u_n \approx y(t_n) =: y_n.$$

Single or Multi Step Methods: If the method advances the solution from one grid point to the next grid point using only data at the single grid point, that is, u_{n+1} depends only on u_n, it is called one-step or single step method otherwise it is called multistep method.

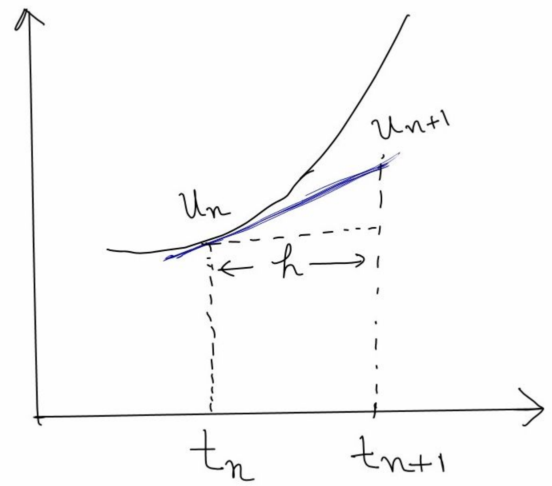
Explicit and Implicit Methods: A method is called *explicit* method if u_{n+1} can be computed directly in terms of the previous values u_k, $k \leq n$, *implicit* if u_{n+1} depends implicitly on itself through f.

Single Step Method

A single step method can be written as

$$u_{n+1} = u_n + h\phi(t_n, u_n, f_n, h)$$

where ϕ is called an increment function.



$$\text{slope} = \frac{u_{n+1} - u_n}{h} \Rightarrow u_{n+1} = u_n + h \times \text{slope}$$

Figure 2.2: Increment function

Consistency, Stability and Convergence:

Consider a single step Method:

$$u_{n+1} = u_n + h\,\Phi(t_n, u_n, f_n, h)$$

Consider:

$$y_{n+1} = y_n + h\,\Phi(t_n, y_n, f(t_n, y_n), h) + \tau_{n+1}$$

consistency error

If we define

$$u^*_{n+1} = y_n + h\,\Phi(t_n, y_n, f(t_n, y_n), h)$$

Then

$$\boxed{\tau_{n+1} = y_{n+1} - u^*_{n+1}}$$

Define the error at node t_{n+1} as

$$e_{n+1} = y_{n+1} - u_{n+1}$$

$$= (y_{n+1} - u^*_{n+1}) + (u^*_{n+1} - u_{n+1})$$