Vector Space
Lecture - 2

Friday
8.1.16.

Let V be a non-empty set and F be a field (Here $R \to$ Set of all real no.s). Then V is said to be a vector space over the field F (and denoted as $V(F)$), with respect to two operations - vector addition '+' & scalar multiplication $(\cdot)$, if the elements of V satisfy the following properties —

(A1). Suppose $v_1, v_2 \in V$, then $v_1 + v_2 \in V$.
(V is closed w.r. to '+')

(A2). For all $v_1, v_2 \in V$, $v_1 + v_2 = v_2 + v_1$.
(V is commutative w.r. to '+')

(A3). For all $v_1, v_2, v_3 \in V$, $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$
(V is associative w.r. to '+')

(A4). For all $v \in V$, $\exists$ an identity element $\underset{\sim}{0}$

$$v + \underset{\sim}{0} = v$$

(A5). For all $v \in V$, $\exists$ an inverse element $(-v) \in V$ such that $v + (-v) = \underset{\sim}{0}$

Note - $-v \to$ additive inverse (depends on v)
$\underset{\sim}{0} \to$ " identity (Unique for V)

M1. For all $v \in V$ & $k \in R$, $kv \in V$

V is closed under scalar multiplication.

M2. For all $v \in V$, $k_1, k_2 \in R$, $(k_1 + k_2)v = k_1 v + k_2 v$.

M3. For all $v_1, v_2 \in V$, $k \in R$, $k(v_1 + v_2) = k v_1 + k v_2$.

M4. For all $v \in V$, $k_1, k_2 \in R$, $(k_1 k_2)v = k_1(k_2 v)$.

M5. For all $v \in V$, $\exists$ '1' $\in R$: $1.v = v$.

'1' $\longrightarrow$ multiplicative identity.

Examples of vector space

$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$.

$k(x_1, x_2) = (kx_1, kx_2)$.

A1. If $v_1 = (x_1, x_2)$, $v_2 = (y_1, y_2)$,

then $v_1 + v_2 = (x_1 + y_1, x_2 + y_2) \in \mathbb{R}^2$

$\therefore \mathbb{R}^2$ is closed w.r.to '+'.

A2. $v_1 = (x_1, x_2)$, $v_2 = (y_1, y_2) \in \mathbb{R}^2$.

$$\begin{aligned} v_1 + v_2 &= (x_1 + y_1, x_2 + y_2) \\ &= (y_1 + x_1, y_2 + x_2) \\ &= (y_1, y_2) + (x_1, x_2) = v_2 + v_1 \end{aligned}$$

(A3) $v_1 = (x_1, x_2)$, $v_2 = (y_1, y_2)$, $v_3 = (z_1, z_2)$

$v_1 + v_2 = (x_1 + y_1, x_2 + y_2)$

$(v_1 + v_2) + v_3 = (x_1 + y_1 + z_1, x_2 + y_2 + z_2)$

$(v_2 + v_3) = (y_1 + z_1, y_2 + z_2)$

$v_1 + (v_2 + v_3) = (x_1 + y_1 + z_1, x_2 + y_2 + z_2)$

(A4). Let $v = (x, y) \in \mathbb{R}^2$.

Then $v + \underset{\sim}{0} = (x, y) + (0, 0) = (x + 0, y + 0)$
$= (x, y) \in \mathbb{R}^2$

Also $\underset{\sim}{0} = (0, 0) \in \mathbb{R}^2$

$\therefore (0, 0) \longrightarrow$ identity element.

(A5). Additive inverse.

Note, $\forall (x, y) \in \mathbb{R}^2$, $\exists (-x, -y) \in \mathbb{R}^2$

$: (x, y) + (-x, -y) = (x + (-x), y + (-y))$.
$= (0, 0)$

$\therefore (-x, -y)$ is the additive inverse w.r.to (x, y).

(M1). For any $k \in \mathbb{R}$, $v \in \mathbb{R}^2$

$kv = k(x, y) = (kx, ky) \in \mathbb{R}^2$

V closed w.r.to scalar multiplication.

(M2). $k_1, k_2 \in \mathbb{R}^*$, $v = (x, y) \in \mathbb{R}^2$

$(k_1 + k_2)(x, y) = ((k_1 + k_2)x, (k_1 + k_2)y)$.

$k_1(x, y) = (k_1 x, k_1 y)$, $k_2(x, y) = (k_2 x, k_2 y)$

P.T.O

$$k_1(x,y) + k_2(x,y) = (k_1x + k_2x,\ k_1y + k_2y)$$
$$= ((k_1+k_2)x,\ (k_1+k_2)y)$$

M3) $k \in \mathbb{R}$, $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$

$$k(v_1 + v_2) = k((x_1, y_1) + (x_2, y_2))$$
$$= k(x_1 + x_2,\ y_1 + y_2)$$
$$= (k(x_1 + x_2),\ k(y_1 + y_2))$$

$$kv_1 = k(x_1, y_1) = (kx_1, ky_1)$$
$$kv_2 = k(x_2, y_2) = (kx_2, ky_2)$$

$$kv_1 + kv_2 = (kx_1 + kx_2,\ ky_1 + ky_2)$$
$$= (k(x_1 + x_2),\ k(y_1 + y_2))$$

M4) $k_1, k_2 \in \mathbb{R}$, $v \in \mathbb{R}^2$, $v = (x,y)$

$$(k_1 \cdot k_2)(x,y) = (k_1k_2x,\ k_1k_2y)$$
$$k_1(k_2(x,y)) = k_1(k_2x,\ k_2y)$$
$$= (k_1(k_2x),\ k_1(k_2y)) = (k_1k_2x,\ k_1k_2y)$$

M5) $v \in (x,y) \in \mathbb{R}^2$, $1 \in \mathbb{R}$

Then $1 \cdot (x,y) = (1 \cdot x,\ 1 \cdot y) = (x,y)$

2nd Example

Define the operation

$$(x_1, y_1) + (x_2, y_2) = (x_1^2 + x_2, y_1 + y_2)$$

$$k(x, y) = (kx, ky)$$

$\underset{\sim}{v}_1 = (x_1, y_1)$, $\underset{\sim}{v}_2 = (x_2, y_2)$

$$\underset{\sim}{v}_1 + \underset{\sim}{v}_2 = (x_1^2 + x_2, \cancel{y}_1 + y_2)$$

$$\underset{\sim}{v}_2 + \underset{\sim}{v}_1 = (x_2^2 + x_1, y_1 + y_2)$$

$$\Longrightarrow \underset{\sim}{v}_1 + \underset{\sim}{v}_2 \neq \underset{\sim}{v}_2 + \underset{\sim}{v}_1$$

$$(x_1, y_1) + (x_2, y_2) \neq (x_2, y_2) + (x_1, y_1)$$

$\therefore$ w.r.to the given operation, $\mathbb{R}^2$ is not a Vector space.

3rd Example

Check whether the set $\{sum\}$ is a vector space w.r.to the operation $sum + sum = sum$ and $k(sum) = sum$, k a real no.

A1. V is closed under above operation

A2. Operation is commutative

A3. " " associative

A4. Additive identity is sum itself

A5. " inverse is also sum itself.

M1. $k(sum) = sum \in V$

M2. $(k_1 + k_2)\, sum = sum$, again $k_1\, sum + k_2\, sum = sum + sum = sum$

M3. $k(sum + sum) = k\, sum = sum$.

M4. $(k_1 k_2)\, sum = sum$ and M5. $k \cdot sum = sum$.
So any scalar can be multiplicative identity.

Exercise

Set of all $m \times n$ matrices is a vector space over $\mathbb{R}$, w.r.to usual matrix addition and multiplication by a scalar.

Exercise

Set of all polynomials –

$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, $a_i \in \mathbb{R}$.

form a vector space.

Exercise

let V be a set containing all functions $f : X \to R$; (X is a non-empty set).

Then V is a vector space, if we define

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$. $\forall f_1, f_2 \in V$

$(k f_1)(x) = k f_1(x)$, $k \in R$.