REVIEW (COMPLEX ANALYSIS)

COMPLEX FUCNCTION:

$f: D \to R$ $D, R \to$ set of complex numbers

$$\omega = f(z) = f(x+iy) = u(x,y) + i v(x,y)$$

Not that $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$

$v(x,y) : \mathbb{R}^2 \to \mathbb{R}$

LIMIT & CONTINUITY

$$\lim_{z \to z_0} f(z) = f(z_0)$$

DIFFERENTIABILITY

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

ANALYTIC FUNCTIONS:

A function is said to be analytic at a point z_0 if $\exists$ a neighbourhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

CAUCHY RIEMANN EQUATIONS (C-R Equations)

NECESSARY CONDITIONS FOR $f'(z)$ to exist at a point z:

C-R equations hold at z, i.e.,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{at } z$$

SUFFICIENT CONDITIONS for the existence of $f'(z)$:

- u_x, u_y, v_x, v_y are continuous at z.
- u_x, v_x, u_y, v_y exist at z and a neighbourhood about z
- The C-R equations hold at z.

A necessary condition that $f(z) = u + iv$ be analytic in a domain D is that u & v satisfy C-R equations in D.

Moreover, if the partial derivatives in C-R equations are continuous in D then the C-R equations are sufficient for analyticity of f in D.

Th: If $f(z) = u + iv$ is analytic in a domain D then u & v satisfy Laplace's equation, i.e.

$$u_{xx} + u_{yy} = 0 \quad \& \quad v_{xx} + v_{yy} = 0.$$

- ## CONSTRUCTION OF ANALYTIC FUCTION

Let u be harmonic on a domain D then for some v, $u + iv$ defines an analytic function for $z = x + iy$ in D.

Example: Determine the analytic function $w = u + iv$ if

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2$$

Sol: $\dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \qquad \dfrac{\partial u}{\partial y} = -6xy - 6y$

C-R equation: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Rightarrow v = \int (3x^2 - 3y^2 + 6x)\,dy + C(x)$

$$= 3x^2y - y^3 + 6xy + C(x)$$

$$\Rightarrow \frac{\partial v}{\partial x} = 6xy + 6y + C'(x) = -(-6xy - 6y)$$

$$\Rightarrow C'(x) = 0 \Rightarrow C(x) = C.$$

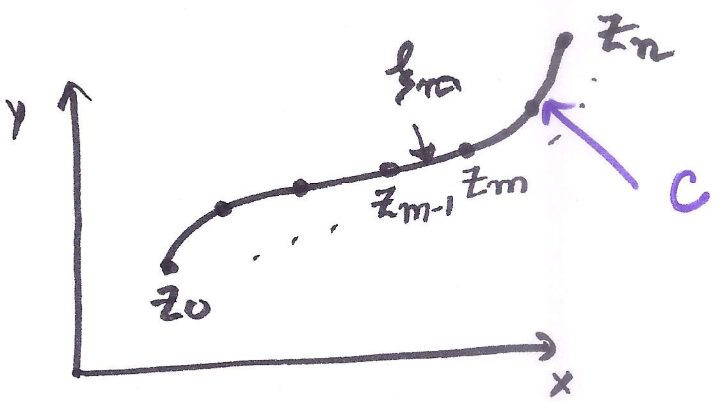
$$\Rightarrow v = 3x^2y - y^3 + 6xy + C.$$

$$w = x^3 - 3xy^2 + 3x^2 - 3y^2 + i(3x^2y - y^3 + 6xy + C)$$

$$= (x + iy)^3 + 3(x + iy)^2 + C$$

$$\boxed{w = z^3 + 3z^2 + C}$$

Line Integral:



$$\lim_{n\to\infty} \sum_{m=1}^{n} f(\xi_m)(z_m - z_{m-1}) = \int_C f(z)\,dz$$

$f(z)$ is integrated over C.

Important properties:

1. $\displaystyle\int_C \left(k_1 f_1(z) + k_2 f_2(z)\right) dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz$

2. $\displaystyle\int_{z_0}^{z} f(z)\,dz = -\int_{z}^{z_0} f(z)\,dz$

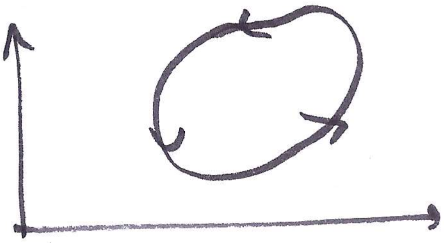
3. $\displaystyle\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz \; ; \; C = C_1 + C_2$

4. $\displaystyle\left|\int_C f(z)\,dz\right| \leq M \cdot L$ (ML-Inequality)

$M \to$ maximum of $|f(z)|$ over C.

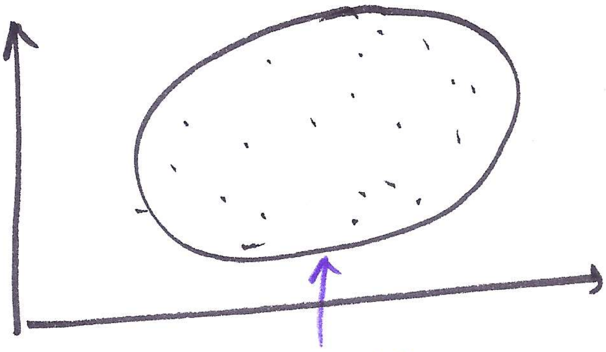
$L \to$ length of the curve C.

Simple closed curve:



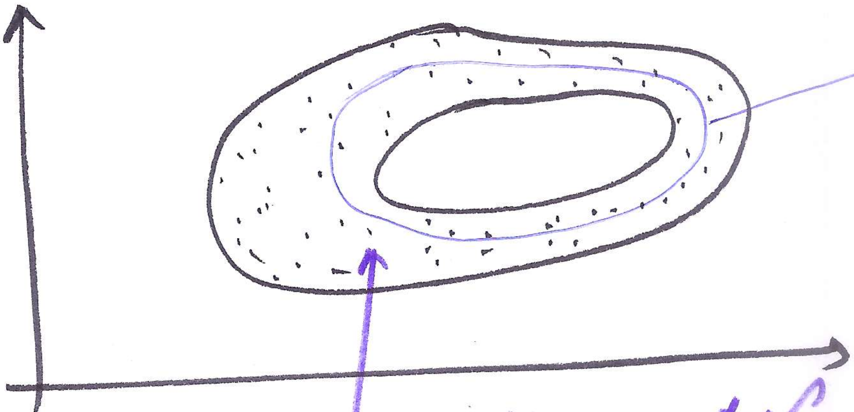
does not intersect or touch itself.

Simply and Multiply connected domains



Simply connected domain

If any simple closed curve can be shrunk to a point without leaving the domain D.



cannot be shrunk to a point without leaving D.

multiply connected domain

Evaluation of Line Integral:

(I) Method restricted to analytical function:

Let $f(z)$ be analytic in a simply connected domain D, then

$$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0)$$

z_0 & z_1 are two points in D.

(II) General approach:

Let the curve C be represented by

$$z = z(t), \quad a \leq t \leq b.$$

then

$$\int_C f(z)\, dz = \int_a^b f(z(t)) \cdot \dot{z}(t)\, dt$$

$\uparrow$ a continuous function on C.

Usually we write $\int_C f(z)dz = \int_{z_0}^{z_1} f(z)dz$ if the integral is path indep. i.e. $f(z)$ is analytic.

Ex: Find the value of the integral

$$\int_0^{1+i} (x - y + ix^2)\, dz$$

i) along the straight line $z=0$ to $z=1+i$

ii) along the real axis $z=0$ to $z=1$ and then $z=1$ to $1+i$.

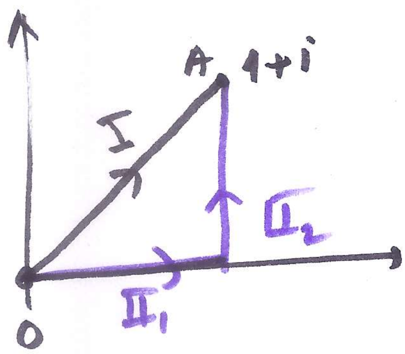
Sol: Is the integrand analytic?

$$u = x-y \qquad v = x^2$$

$$u_x = 1 \qquad u_y = -1$$

$$v_x = 2x \qquad v_y = 0$$

C-R equation are not satisfied at any point and therefore the function $x-y+ix^2$ is not analytic anywhere.



(I): Along OA: $\underline{y=x}$

$$z = x+ix = (1+i)x$$

$$\int_0^1 (ix^2)\cdot(1+i)\,dx = (1+i)\,i\,\frac{1}{3} = \underline{\underline{\frac{1}{3}(i-1)}}$$

(II) $$\int_{II} (x-y+ix^2)dz = \int_{II_1} (x-y+ix^2)dz + \int_{II_2} (x-y+ix^2)\,dz$$

II_1: $z=x$ $\qquad$ II_2: $z = iy+1$

$$= \int_0^1 (x+ix^2)\,dx + \int_0^1 (1-y+i)\,i\,dy$$

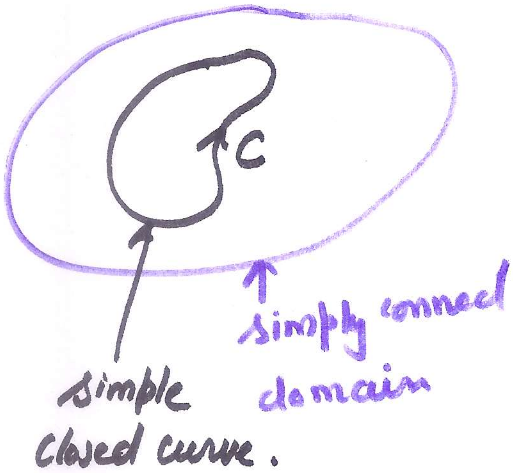
$$= \left(\frac{1}{2}+\frac{i}{3}\right) + (1+i)i - \frac{1}{2}i$$

$$= \underline{\underline{-\frac{1}{2}+\frac{5}{6}i}}$$

CAUCHY THEOREM (CAUCHY INTEGRAL THEOREM)
(CAUCHY-GOURSAT THEOREM)

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,dz = 0.$$



OR:

Cauchy theorem states that

$$\oint_C f(z)\,dz = 0 \text{ if } f \text{ is}$$

analytic on the curve and on all points enclosed by the curve.

if f is analytic on C & all the point enclosed by C then

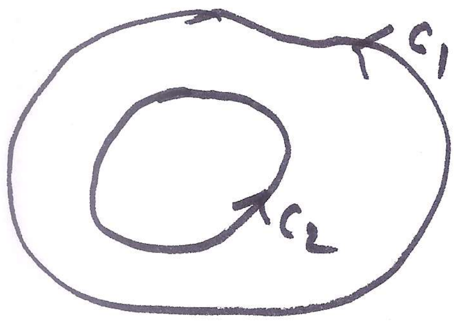
$$\oint_C f(z)\,dz = 0$$

CONSEQUENCES OF CAUCHY'S THEOREM:

Let $f(z)$ be analytic in a domain D bounded by two simple closed curve C_1 & C_2 and also on C_1 & C_2, then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

where C_1 & C_2 both are traversed counter clockwise direction.



The theorem states that the integral of f has the same value over both paths when one can be deformed into other, moving only over points at which the function is analytic. This means that we can replace C_1 with another path C_2 that may be more convenient to use in evaluating the integral.

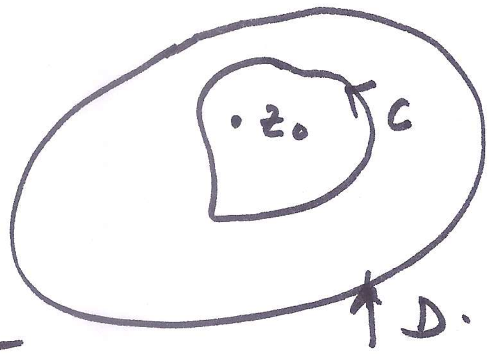
CAUCHY INTEGRAL FORMULA

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0 we have

$$\oint_C \frac{f(z)}{(z-z_0)}\, dz = 2\pi i\, f(z_0)$$

OR

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)}\, dz$$



Ex. $\oint_C \frac{\tan z}{(z^2-1)}\, dz \quad C: |z| = 3/2$

Singularities of $\frac{\tan z}{z^2-1}$: $z = 1, z = -1, z = \pm\pi/2, \pm\frac{3\pi}{2}, \ldots$

Not that the points $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ doesnot lies inside C, ~~then~~ hence $\tan z$ is analytic inside $|z| = 3/2$.

$$\Rightarrow \oint_C \frac{\tan z}{(z-1)(z+1)}\, dz = \oint_C \frac{\tan z}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right] dz$$

$$= \frac{1}{2}\left[\oint_C \frac{\tan z}{z-1}\, dz - \oint_C \frac{\tan z}{z+1}\, dz\right]$$

$$= \frac{1}{2}\cdot\left[2\pi i \tan(1) - 2\pi i \tan(-1)\right] = 2\pi i \tan(1)$$