1.11 Indistinguishable Balls in Indistinguishable Boxes

To study the number of onto functions $f : M \longrightarrow N$, we were lead to the study of "partition of a set consisting of m elements into n parts". In this section, we study the partition of a number m into n parts and look at a few problems in which this idea can be used.

Definition 1.11.1 (Partition of a number)**.** *A partition of a positive integer m into n parts, is a non-increasing sequence of positive numbers $x_1, x_2, \ldots, x_n$ such that $\sum\limits_{k=1}^{n} x_k = m$. The number of such partitions is denoted by $\Pi(m, n)$.*

Remark 1.11.2.

1. *For example, the distinct partitions of 7 into 4 parts are given by $4+1+1+1$, $3+2+1+1$, $2+2+2+1$. Hence, $\Pi(7,4) = 3$. Verify that $\Pi(7,2) = 3$ and $\Pi(7,3) = 4$.*
2. *For a fixed positive integer m, $\Pi(m)$ denotes the number of partitions of m. Hence, $\Pi(m) = \sum\limits_{k=1}^{m} \Pi(m,k)$. Verify that $\Pi(7) = 15$.*
3. *By convention, we let $\Pi(0,0) = 1$ and $\Pi(m,n) = 0$, whenever $n > m$.*

We are now ready to associate the study of partitions of a number with the following problems.

Example 1.11.3.

1. *Determine the number of ways of putting m* **indistinguishable** *balls into n* **indistinguishable** *boxes with the restriction that no box is empty?*

   Solution: *Since the balls are indistinguishable balls, the problem reduces to counting the number of balls in each box with the condition that no box is empty. As the boxes are also indistinguishable, they can be arranged in such a way that the number of balls inside them are in non-increasing order. Hence, we have the answer as $\Pi(m, n)$.*

2. *Determine the number of ways to put m* **indistinguishable** *balls into n* **indistinguishable** *boxes.*

   Solution: *The problem can be rephrased as follows: "suppose that each box already has 1 ball, i.e., initially, each of the n boxes are non-empty. Now let us determine the number of ways of putting m indistinguishable balls into the n indistinguishable boxes that are already non-empty." This new problem is same as "in how many ways can $m+n$* **indistinguishable** *balls be put into n* **indistinguishable** *boxes with the restriction that no box is empty". Therefore, the answer to our problem reduces to computing $\Pi(m+n, n)$.*

3. *Use the above idea to show that $\Pi(2m, m) = \Pi(m)$ for any positive integer m. Hint: $\Pi(2m, m)$ corresponds to "putting $2m$ indistinguishable balls into m indistinguishable boxes with the condition that no box empty", where as $\Pi(m)$ corresponds to "putting m balls into m indistinguishable boxes".*

Till now, we have been looking at problems that required arranging the objects into a row. That is, we differentiated between the arrangements $ABCD$ and $BCDA$. In this section, we briefly study the problem of arranging the objects into a circular fashion. That is, if we are arranging the four distinct chairs, named A, B, C and D, at a round table then the arrangements $ABCD$ and the arrangement $BCDA$ are the same. That is, the main problem that we come across circular arrangements as compared to problems in the previous sections is "there is no object that can truly be said to be placed at the number 1 position". The problems related with round table configurations will be dealt at length in Chapter 3.

So, to get distinct arrangements at a round table, we need to fix an object and assign it the number 1 position and study the distinct arrangement of the other $n-1$ objects relative to the object which has been assigned position 1. We will look at two examples to understand this idea.

Example 1.11.4. 1. *Determine the number of ways to sit* 8 *persons at a round table.*

Solution: METHOD 1: *Let us number the chairs as* $1, 2, \ldots, 8$. *Then, we can pick one of the person and ask him/her to sit on the chair that has been numbered* 1. *Then relative to this person, the other persons (*7 *of them) can be arranged in* $7!$ *ways. So, the total number of arrangements is* $7!$.

METHOD 2: *The total number of arrangements of* 8 *persons if they are to be seated in a row is* $8!$. *Since the cyclic arrangement* $P_1P_2\cdots P_8$ *is same as the arrangement* $P_8P_1P_2\cdots P_7$ *and so on, we need to divide the number* $8!$ *by* 8 *to get the actual number as* $7!$.

2. *Recall Example 1.9.7. Suppose we are now interested in making the* 4 *couples sit in a round table. Find the number of seating arrangements.*

Solution: *Note that using Example 1.11.4.1, the* 4 *cohesive units can be arranged in* $3!$ *ways. But we can still have the couples to sit either as "wife and husband" or "husband and wife'. Hence, the required answer is* $2^4 3!$.