Department of Mathematics
IIT Kharagpur
MA20103 Partial Differential Equations
Autumn 2015 - End-Examination, Time: 3 hrs.; Max. Marks: 50, Number of students: 460

Carefully read the following instructions:

- Please answer all parts of a specific question at one place.
- No queries will be entertained during the examination.
- There are 2 pages in this question paper.
- Answer all the questions. Please follow the notations: z or u: dependent variables; x, y, t: independent variables; for $h = u$ or z, $\frac{\partial h}{\partial x} = h_x = p$; $\frac{\partial h}{\partial y} = h_y = q$; $D \equiv \frac{\partial}{\partial x}$; $D' \equiv \frac{\partial}{\partial y}$
- For questions 2(D), 3(A) and 3(C), show the following details systematically. (i) separation process, boundary conditions, (ii). solutions of the corresponding ODEs, (iii) explicitly mention the eigenvalues and the corresponding eigenfunctions, (iii) precise form of the series solution, (iv) process to determine the coefficients of the series solution, and (v). final solution.

Questions

1. (A). Find the solution of $2xyp + (x^2 + y^2)q = y^3 z$ passing through the curve $x_0(s) = s, y_0(s) = 1 + s, z_0(s) = e^{s^2/4}$.

   [5 marks]

   (B). Find the one parameter family of common solutions of the PDEs: $p^2 + q^2 = 4$ and $(p^2 + q^2)y = qz$ and hence obtain the corresponding singular solution.

   [4 marks]

   (C). Find a complete integral of $p^2 x + q^2 y = \frac{z}{4}$.

   [5 marks]

   (D). Given the second order PDE $xu_{xx} + (y + 1)u_{yy} + yu_x + xu_y + 2xy = 0$, identify the regions in which the PDE is parabolic, hyperbolic and elliptic.

   [3 marks]

2. (A). Obtain a first order PDE by eliminating the arbitrary function f from $z = e^y f(x + y)$ and classify the PDE thus obtained.

   [2 marks]

   (B). Find the complementary function of the second order PDE $(D^2 + DD' - 2D'^2)z = e^{x+y}$. Also find a particular integral and verify that the particular integral satisfies the given PDE.

   [3 marks]

   (C). Classify the second order PDE and reduce it to canonical form. Hence find the general solution of the given PDE

$$3z_{xx} + 10z_{xy} + 3z_{yy} = 0$$

   [5 marks] p.t.o

(D). Solve the one dimensional diffusion equation $u_t = u_{xx}$ by the method of separation of variables subject to the boundary conditions $u(0,t) = 0; u(1,t) = 0$ for all $t > 0$ and the initial condition

$$u(x,0) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \leq x \leq 1 \end{cases}.$$

Further $u(x,t)$ remains finite as $t \to \infty$. [6 marks]

3. (A). A finite string is fixed at its both ends $x = 0$ and $x = 1$. For a given initial velocity and displacement, the displacement $u(x,t)$ satisfy the following initial boundary value problem, $u_{tt} = u_{xx}$ with the boundary conditions $u(0,t) = 0; u(1,t) = 0$ and the initial conditions $u(x,0) = \frac{3}{2}\sin 3\pi x; u_t(x,0) = 5\sin 7\pi x - 10\sin 20\pi x$. Use separation of variables and derive the corresponding general solution for the displacement $u(x,t)$. Further, determine any arbitrary constants involved by satisfying initial conditions given. Do not use Fourier series expansion and integration to determine the coefficients. You must force the initial conditions and determine coefficients via direct comparison.

[5 marks]

(B). Given the canonical form $u_{\xi\eta} = 0$ in (ξ, η) plane where the characteristics are $\xi = x + 2t$ and $\eta = x - 2t$, write down the corresponding general solution. Hence, obtain a particular solution subject to the following data that is prescribed on these characteristics

$$u(x,t) = x, \quad \text{on} \quad x + 2t = 0$$

$$u(x,t) = x^2, \quad \text{on} \quad x - 2t = 0$$

and $u(0,0) = 0$. [3 marks]

(C). A scalar potential $u(x,y)$ on a rectangular metal sheet $D = \{(x,y) : 0 < x < a, 0 < y < b\}$ satisfy the boundary value problem (BVP)

$$u_{xx} + u_{yy} = 0, \quad \text{on } D; u_x(0,y) = 0; u_x(a,y) = 0; u(x,0) = 0; u_y(x,b) = f(x).$$

Use separation of variables to obtain a series solution of the above BVP. [5 marks]

(D). A scalar potential $u(r,\theta)$ on a circular disk $r \leq 1$ is defined by

$$u(r,\theta) = \sum_{n=1}^{\infty} \left(A_n r^n + B_n r^{-n}\right) \cos n\theta$$

where A_n and B_n are unknown coefficients to be determined. Find A_n and B_n (i). for the interior problem subject to the Neumann data $\frac{\partial u}{\partial r} = 2\cos 2015\theta$ on $r = 1$ and regular at $r = 0$, and (ii). for the exterior problem subject to the Dirichlet data $u(r,\theta) = 0$ on $r = 1$ and $u(r,\theta) \to 3r^2\cos 2\theta$ as $r \to \infty$. [4 marks]

Best of Luck