Example: Using shooting method, solve the Boundary value problem

$$y'' = 2yy' \quad 0 < x < 1$$

$$y(0) = 0.5 \qquad y(1) = 1.$$

Use the two stage Runge-Kutta method

$$K_1 = \frac{h^2}{2} f(x_j, u_j, u_j')$$

$$K_2 = \frac{h^2}{2} f\left(x_j + \frac{2}{3}h,\ u_j + \frac{2}{3}h u_j' + \frac{2}{3}K_1,\ u_j' + \frac{4}{3h}K_1\right)$$

$$u_{j+1} = u_j + h u_j' + \frac{1}{2}(K_1 + K_2)$$

$$u_{j+1}' = u_j' + \frac{1}{2h}(K_1 + 3K_2)$$

with $h = 0.25$ to solve the corresponding IVP.

Use Newton's method assuming the starting values of the slope at $x=0$ as $s^{(0)} = u'(0) = 0.3$. Perform two iterations and compare numerical results with the exact solution $y(x) = \frac{1}{2-x}$.

Solution: We consider the following two problems for the application of NR method:

(I) $u'' = 2uu'$

$u(0) = 0.5 \qquad u'(0) = 0.3 = s^{(0)}$

(II) $v'' = 2u'v + 2uv' = 2(u'v + uv')$

$v(0) = 0 \ ; \qquad v'(0) = 1$

Discretization:

$$\left.\begin{aligned}
u_{j+1} &= u_j + h u_j' + \tfrac{1}{2}(k_1 + k_2) \\
u_{j+1}' &= u_j' + \tfrac{1}{2h}(k_1 + 3k_2) \\
&\text{where} \\
k_1 &= \tfrac{h^2}{2}\cdot 2 u_j u_j' = h^2 u_j u_j' \\
k_2 &= h^2 \left(u_j + \tfrac{2}{3} h u_j' + \tfrac{2}{3} k_1\right)\left(u_j' + \tfrac{4}{3h} k_1\right)
\end{aligned}\right\} \text{(I)}$$

$$\left.\begin{aligned}
v_{j+1} &= v_j + h v_j' + \tfrac{1}{2}(k_1^* + k_2^*) \\
v_{j+1}' &= v_j' + \tfrac{1}{2h}(k_1^* + 3k_2^*) \\
k_1^* &= h^2 (u_j' v_j + u_j v_j') \\
k_2^* &= h^2 \left[u_j' \left(v_j + \tfrac{2}{3} h v_j' + \tfrac{2}{3} k_1^*\right) + u_j \left(v_j' + \tfrac{4}{3h} k_1^*\right)\right]
\end{aligned}\right\} \text{(II)}$$

Newton iteration

$$g(s^{(i)}) = u_4 - 1;$$

$$g'(s^{(i)}) = v_4$$

$$s^{(i+1)} = s^{(i)} - \frac{g(s^{(i)})}{g'(s^{(i)})};$$

x = 0.00 u = 0.5000 du = 0.3000 v = 0.0000 dv = 1.0000

x = 0.25 u = 0.5858 du = 0.3918 v = 0.2856 dv = 1.3023

x = 0.50 u = 0.7005 du = 0.5364 v = 0.6743 dv = 1.8410

x = 0.75 u = 0.8631 du = 0.7833 v = 1.2561 dv = 2.8869

x = 1.00 u = 1.1122 du = 1.2543 v = 2.2417 dv = 5.1836

$$g(s) = 1.1122 - 1 = 0.1123 \qquad g'(s) = 2.2417$$

$$s^{(1)} = s^{(0)} - \frac{g(s^{(0)})}{g'(s^{(0)})} = 0.3 - \frac{0.1123}{2.2417} = 0.2499$$

x = 0.00 u = 0.5000 du = 0.2499 v = 0.0000 dv = 1.0000

x = 0.25 u = 0.5714 du = 0.3254 v = 0.2853 dv = 1.2988

x = 0.50 u = 0.6662 du = 0.4406 v = 0.6697 dv = 1.8066

x = 0.75 u = 0.7982 du = 0.6291 v = 1.2304 dv = 2.7407

x = 1.00 u = 0.9941 du = 0.9678 v = 2.1368 dv = 4.6538

$$g(s) = 0.9941 - 1 = -0.0059$$

$$g'(s) = 2.1368$$

$$s^{(2)} = s^{(1)} - \frac{g(s^{(1)})}{g'(s^{(1)})} = 0.2499 + \frac{0.0059}{2.1368} = +0.2527$$

x = 0.00 u = 0.5000 exact 0.5

x = 0.25 u = 0.5722 exact 0.5714

x = 0.50 u = 0.6681 exact 0.6667

x = 0.75 u = 0.8017 exact 0.8000

x = 1.00 u = 1.0004 exact 1

Ex: Solve the following nonlinear boundary value problem using shooting method:

$$y'' = \frac{3}{2} y^2 \qquad 0 < x < 1$$

$$y(0) = 1 \qquad y(1) = 4.$$

Use fourth order Runge-Kutta method to solve the initial value problems and the Newton-Raphson method (1 iter) for iteration using the initial guess $s^{(0)} = 0.9$ and $h = 0.25$.

Sol: We need to solve the following IVPs:

$$\left.\begin{array}{l} u'' = \frac{3}{2}u^2 \\ u(0) = 1 \qquad u'(0) = 0.9 \ (s^{(0)}) \end{array}\right\} -(\mathrm{I})$$

&

$$\left.\begin{array}{l} v'' = 3uv \\ v(0) = 0 \qquad v'(0) = 1 \end{array}\right\} -(\mathrm{II})$$

The two IVPs in system of equation form:

$$\begin{bmatrix} u \\ \bar{u} \end{bmatrix}' = \begin{bmatrix} \bar{u} \\ \frac{3}{2}u^2 \end{bmatrix} =: \begin{bmatrix} f_1(x, u, \bar{u}) \\ f_2(x, u, \bar{u}) \end{bmatrix} \qquad \left.\begin{array}{l} u(0) = 1 \\ \bar{u}(0) = 0.9 \end{array}\right\} (\mathrm{I})$$

&

$$\begin{bmatrix} v \\ \bar{v} \end{bmatrix}' = \begin{bmatrix} \bar{v} \\ 3uv \end{bmatrix} =: \begin{bmatrix} g_1(x, v, \bar{v}) \\ g_2(x, v, \bar{v}) \end{bmatrix} \qquad \left.\begin{array}{l} v(0) = 0 \\ \bar{v}(0) = 1 \end{array}\right\} (\mathrm{II})$$

Fourth order Runge-Kutta method for the problem (I)

$$\bar{K}_1 = \begin{bmatrix} K_1^{(1)} \\ K_1^{(2)} \end{bmatrix} = \begin{bmatrix} f_1(x_j, u_j, \bar{u}_j) \\ f_2(x_j, u_j, \bar{u}_j) \end{bmatrix} = \begin{bmatrix} \bar{u}_j \\ \frac{3}{2}u_j^2 \end{bmatrix}$$

$$\bar{K}_2 = \begin{bmatrix} K_2^{(1)} \\ K_2^{(2)} \end{bmatrix} = \begin{bmatrix} f_1(x_j + \frac{h}{2}, u_j + \frac{h}{2}K_1^{(1)}, \bar{u}_j + \frac{h}{2}K_1^{(2)}) \\ f_2(x_j + h/2, u_j + h/2\, K_1^{(1)}, \bar{u}_j + \frac{h}{2}K_1^{(2)}) \end{bmatrix} = \begin{bmatrix} \bar{u}_j + \frac{h}{2}K_1^{(2)} \\ \frac{3}{2}(u_j + \frac{h}{2}K_1^{(1)})^2 \end{bmatrix}$$

$$\bar{K}_3 = \begin{bmatrix} \bar{u}_j + \frac{h}{2}K_2^{(2)} \\ \frac{3}{2}(u_j + \frac{h}{2}K_2^{(1)})^2 \end{bmatrix} \qquad \bar{K}_4 = \begin{bmatrix} \bar{u}_j + hK_3^{(2)} \\ \frac{3}{2}(u_j + hK_3^{(1)})^2 \end{bmatrix}$$

$$\begin{bmatrix} u_{j+1} \\ \bar{u}_{j+1} \end{bmatrix} = \begin{bmatrix} u_j \\ \bar{u}_j \end{bmatrix} + \frac{h}{6}\left[\bar{K}_1 + 2\bar{K}_2 + 2\bar{K}_3 + K_4\right]$$

$j = 0, 1, 2, 3.$

Fourth order Runge-Kutta method for the problem (II):

$$\bar{K}_1 = \begin{bmatrix} \bar{v}_j \\ 3u_j v_j \end{bmatrix} \qquad \bar{K}_2 = \begin{bmatrix} \bar{v}_j + \frac{h}{2} K_1^{(2)} \\ 3u_j\left(v_j + \frac{h}{2} K_1^{(1)}\right) \end{bmatrix}$$

$$\bar{K}_3 = \begin{bmatrix} \bar{v}_j + \frac{h}{2} K_2^{(2)} \\ 3u_j\left(v_j + \frac{h}{2} K_2^{(1)}\right) \end{bmatrix} \qquad \bar{K}_4 = \begin{bmatrix} \bar{v}_j + h K_3^{(2)} \\ 3u_j\left(v_j + h K_3^{(1)}\right) \end{bmatrix}$$

$$\begin{bmatrix} v_{j+1} \\ \bar{v}_{j+1} \end{bmatrix} = \begin{bmatrix} v_j \\ \bar{v}_j \end{bmatrix} + \frac{h}{6}\left[\bar{K}_1 + 2\bar{K}_2 + 2\bar{K}_3 + \bar{K}_4\right]$$

$$g'(s^{(0)}) = v_4 \qquad g(s^{(0)}) = u_4 - 4;$$

$$s^{(1)} = s^{(0)} - \frac{g(s^{(0)})}{g'(s^{(0)})}$$

$u_1 = 1.2801$ $\quad u_2 = 1.7188$ $\quad u_3 = 2.4466$ $\quad u_4 = 3.7711$

$\bar{u}_1 = 1.3814$ $\quad \bar{u}_2 = 2.2114$ $\quad \bar{u}_3 = 3.8042$ $\quad \bar{u}_4 = 7.3195$

$v_1 = 0.2578$ $\quad v_2 = 0.5741$ $\quad v_3 = 1.0608$ $\quad v_4 = 1.9895$

$\bar{v}_1 = 1.0952$ $\quad \bar{v}_2 = 1.4867$ $\quad \bar{v}_3 = 2.5126$ $\quad \bar{v}_4 = 5.2062$

$s^{(1)} = 1.0150$

$\boxed{u_1 = 1.3099}$ $\quad \boxed{u_2 = 1.7862}$ $\quad \boxed{u_3 = 2.5754}$ $\quad \boxed{u_4 = 4.0283}$

$\bar{u}_1 = 1.5095$ $\quad \bar{u}_2 = 2.3942$ $\quad \bar{u}_3 = 4.1394$ $\quad \bar{u}_4 = 8.0981$

exact: $\boxed{1.3061}$ $\quad \boxed{1.7778}$ $\quad \boxed{2.5600}$ $\quad \boxed{4.0000}$