No queries will be entertained during examination

Indian Institute of Technology, Kharagpur

Date ………FN/AN, Time: 2 hrs, Full Marks 30, Deptt : Mathematics

No. of students 60 Year 2017 Mid Semester Examination

Sub. No.: MA31007 Sub. Name: Mathematical Methods M. Sc./ M. Tech (Dual)

ATTEMPT ALL QUESTIONS. EACH QUESTION CARRIES SIX MARKS

1. (a) Define ordinary point, regular singular point and irregular singular points of second order ordinary differential equation.

   Odinary:- Point x_0 which is not a pole of P(x) and Q(x)
   Regular Singular:- Point x_0 is a pole of p(x) and Q(x) of order less than 1 and 2 resp
   Irregular Singular:- Neither of above two

   (b) What are the regular singular point(s) in the finite domain of the following ODE:

   $$(1-x^2)y'' - xy' + 4y = 0$$

   Obtain series solution of this ODE around $x = 0$.

   x=-1 and +1 are the regular singular points
   x=0 Regular Point

2. (a) Solve the Legendre equation of order n around $x = 0$. Show that Legendre polynomial $P_n(x)$ is one solution when n is positive integer or zero.

   (b) Establish Rodrigue's formula for $P_n(x)$.

3. (a) What is geodesics of a Riemannian space ?
   (b) Define Christoffel symbols of the first and second kind.
   (c) Define the line element and metric tensor in N-dimensional space.
   (d) Determine the metric tensor and the conjugate metric tensor in (i) cylindrical and (ii) spherical coordinates.

4. (a) Check whether $x = \infty$ is a regular singular point of the hypergeometric equation

$$x(1-x)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha\beta y = 0$$

It is a regular singular point

(b) Find two linearly independent solutions around $x = 0$, where $1 - \gamma \neq$ integer or zero.

5. (a) Solve the hypergeometric equation around $x = \infty$ and write down the solutions in terms of hypergeometric functions.

(b) Prove the following orthogonal property of Legendre polynomials

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0, \qquad m \neq n$$

$$= \frac{2}{2n+1}, \quad m = n$$

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MM (Mid Sem 2017)

Solution pages

1. a) Consider a 2nd order linear ODE

$$y'' + p(x)\, y' + q(x)\, y = 0 \quad (1)$$

A point $x = x_0$ is ordinary pt of (1) if it is not a pole of $p(x)$ and $q(x)$, otherwise it is singular point.

A singular point $x = x_0$ is called regular singular pt. if it is a pole of $p(x)$ of order $\not> 1$ and is a pole of $q(x)$ of order $\not> 2$, otherwise it is irregular singular point. (1)

b) $(1-x^2)\, y'' - x y' + 4y = 0$ (1)

Regular Singular points are at $x = \pm 1$. (1/2)

Sol^n around $x=0$: $y = \sum_{r=0}^{\infty} a_r x^r$ (2)

Substituting (2) into (1), (1/2)

$$\sum_{r=2}^{\infty} a_r r(r-1)\left[x^{r-2} - x^r\right] - \sum_{r=1}^{\infty} r a_r x^r + 4\sum_{r=0}^{\infty} a_r x^r = 0$$

[$r-2 = r'$ (first term), $r-1 = r''$ (2nd term)]

$$\Rightarrow \sum_{r'=0}^{\infty} a_{r'+2}(r'+2)(r'+1)\left[x^{r'} - x^{r'+2}\right] - \sum_{r''=0}^{\infty} (r''+1)\, a_{r''+1}\, x^{r''+1} + 4\sum_{r=0}^{\infty} a_r x^r = 0$$

[Dropping primes]

$$\Rightarrow \sum_{r=0}^{\infty}\left[(r+1)(r+2)\, a_{r+2} + 4a_r\right]x^r - \sum_{r=0}^{\infty}(r+1)(r+2)\, a_{r+2}\, x^{r+2} - \sum_{r=0}^{\infty}(r+1)\, a_{r+1}\, x^{r+1} = 0$$

x^{r+2}: $\Rightarrow a_{r+4} = \dfrac{r}{r+3}\, a_{r+2}, \quad r = 0, 1, \cdots$ (1/2)

x^0: $a_2 = -2a_0$, x^1: $a_3 = -\frac{1}{2} a_1$,

$a_4 = 0 \Rightarrow a_4 = a_6 = \cdots = 0$; $a_5 = -\frac{1}{8} a_1, \cdots$ (1)

$\therefore\ y_1(x) = 1 - 2x^2$, (1/2)

$y_2(x) = x - \dfrac{x^3}{2} - \dfrac{x^5}{8} + \cdots$ (1/2)

2. a) $(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad , n \in \mathbb{R}$

Sol^n around $x=0$: $y = \sum_{r=0}^{\infty} a_r x^r$

$$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)(x^{r-2} - x^r) - 2\sum_{r=1}^{\infty} a_r r x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$$

$[r' = r-2 \, , \, r'' = r-1]$ and dropping prime,

$$\Rightarrow \sum a_{r+2}(r+1)(r+2)(x^r - x^{r+2}) - 2\sum a_{r+1}(r+1)x^{r+1} + n(n+1)\sum a_r x^r = 0$$

x^r: $\quad a_{r+2} = \frac{(r-n)(r+n+1)}{(r+1)(r+2)} a_r \, , \; r = 0, 1, 2, \dots$ 1/2

$$\therefore \frac{a_2}{a_0} = -\frac{n(n+1)}{1.2} \, , \; \frac{a_4}{a_2} = -\frac{(n-2)(n-3)}{3.4} \, , \; \frac{a_6}{a_4} = -\frac{(n-4)(n+5)}{5.6} \, , \dots$$

and,

$$\frac{a_3}{a_1} = -\frac{(n-1)(n+2)}{2.3} \, , \; \frac{a_5}{a_3} = -\frac{(n-3)(n+4)}{4.5} \, , \; \frac{a_7}{a_5} = -\frac{(n-5)(n+6)}{6.7} \, , \dots$$ 1

$$\therefore y = c_0\left[1 + \frac{c_2}{c_0}x^2 + \frac{c_4}{c_0}x^4 + \frac{c_6}{c_0}x^6 + \dots\right] + c_1\left[x + \frac{c_3}{c_1}x^3 + \frac{c_5}{c_1}x^5 + \frac{c_7}{c_5}x^7 + \dots\right]$$

$$= c_0 y_1(x) + c_1 y_2(x)$$

$$y_1(x) = 1 - \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n-2)(n+3)}{1.2.3.4}x^4 - \frac{n(n+1)(n-2)(n+3)(n-4)(n+5)}{1.2.3.4.5.6}x^6 + \dots$$ 1

$$y_2(x) = x - \frac{(n-1)(n+2)}{2.3}x^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{2.3.4.5}x^5 - \frac{(n-1)(n+2)(n-3)(n+4)(n-5)(n+6)}{2.3.4.5.6.7}x^7 + \dots$$ 1

n even pos. integer $\Rightarrow y_1(x)$ becomes a polynomial of degree n.

n odd pos. integer $\Rightarrow y_2(x)$ becomes a polynomial of degree n.

This polynomial is Legendre polynomial. 1/2

2. b) $$P_n(x) = \sum_{r=0}^{p} \frac{(-1)^r (2n-2r)!}{2^n\, r!\, (n-r)!\, (n-2r)!}\, x^{n-2r}$$

where $p = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

$$\frac{d^n}{dx^n}\left(x^{2n-2r}\right) = (2n-2r)(2n-2r-1)\cdots\{(2n-2r)-n+1\}\, x^{n-2r}$$

$$= \frac{(2n-2r)!}{(n-2r)!}\, x^{n-2r}$$ 1

$$\therefore P_n(x) = \frac{d^n}{dx^n} \sum_{r=0}^{p} \frac{(-1)^r}{2^n\, r!\,(n-r)!}\, x^{2n-2r}$$

$$= \frac{d^n}{dx^n} \sum_{r=0}^{n} \frac{(-1)^n}{2^n\, r!\, (n-r)!}\, (x^2)^{n-r}$$ 1

$$= \frac{1}{2^n\, n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

3. a) $$s = \int_{t_1}^{t_2} \sqrt{g_{pq} \frac{dx^p}{dt} \frac{dx^q}{dt}}\, dt$$ $\Rightarrow$ distance s between t_1 and t_2 on a curve $x^r = x^r(t)$ in a Riemannian space.
That curve in the space which makes the distance a minimum is called a geodesic of the space. 1

b) The symbols
$$[pq, r] = \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^r}\right)$$ 1
$$\left\{ {s \atop pq} \right\} = g^{sr}\, [pq, r]$$

c) In N-dimensional space with coordinates $(x^1, x^2, \dots, x^N)$, the line element ds is defined by quadratic form, called the metric form $ds^2 = g_{pq}\, dx^p dx^q$ 1

The quantities g_{pq} are called metric tensor.

3. d) $g_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$ (cylindrical)

$x^1 \equiv \rho,\ x^2 \equiv \phi,\ x^3 \equiv z$ 1½

$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$ (polar)

$g_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$ $x^1 \equiv r,\ x^2 \equiv \theta,\ x^3 \equiv \phi$ 1½

similarly, compute reciprocal. ½ marks for each reciprocal.

4. a) $x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0$

put $x = 1/\xi \Rightarrow y' = -\xi^2 dy/d\xi,\ y'' = \xi^2\left(\xi^2 \frac{d^2y}{d\xi^2} + 2\xi\frac{dy}{d\xi}\right)$

$\Rightarrow \xi^2(\xi-1)\frac{d^2y}{d\xi^2} + \xi[(2-\gamma)\xi + (\alpha+\beta-1)]\frac{dy}{d\xi} - \alpha\beta y = 0$ 2

$\therefore \xi = 0$ i.e. $x = \infty$ is regular singular point.

b) Sol^n around $x=0$ $y = \sum_{r=0}^{\infty} a_r x^{\rho+r},\ a_0 \neq 0$

$\Rightarrow \sum a_r(\rho+r)(\rho+r-1)[x^{\rho+r-1} - x^{\rho+r}] + \sum a_r(\rho+r)[\gamma x^{\rho+r-1} - (\alpha+\beta+1)x^{\rho+r}]$
$- \alpha\beta \sum a_r x^{\rho+r} = 0$

$\Rightarrow \sum a_r(\rho+r)(\rho+r-1+\gamma)x^{\rho+r-1} = \sum a_r(\rho+r+\alpha)(\rho+r+\beta)x^{\rho+r}$ 1

Indicial: $x^{\rho-1}$: $a_0\rho(\rho-1+\gamma) = 0 \Rightarrow \rho = 0,\ 1-\gamma$

Given $1-\gamma \neq$ integer or zero.

x^ρ: $a_1 = \frac{(\rho+\alpha)(\rho+\beta)}{(\rho+1)(\rho+\gamma)}a_0$, $x^{\rho+r}$: $a_{r+1} = \frac{(\rho+r+\alpha)(\rho+r+\beta)}{(\rho+r+1)(\rho+r+\gamma)}a_r$, $r = 0, 1, 2, \ldots$

$\rho = 0$: $a_{r+1} = \frac{(r+\alpha)(r+\beta)}{(r+1)(r+\gamma)}a_r$ (A) $\Rightarrow a_1 = \frac{\alpha\beta}{1.\gamma}a_0,\ a_2 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma.(\gamma+1)}a_0, \ldots$ 1

$y_1(x) \equiv F(\alpha,\beta;\gamma;x) = 1 + \frac{\alpha\beta}{\gamma.1}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1).1.2}x^2 + \ldots$ 1

$\rho = 1-\gamma$: $a_{r+1} = \frac{(r+1-\gamma+\alpha)(r+1-\gamma+\beta)}{(r+2-\gamma)(r+1)}a_r,\ r = 0, 1, 2, \ldots$ (B)

(B) is obtained from (A) by replacing $\alpha \to 1-\gamma+\alpha,\ \beta \to 1-\gamma+\beta,\ \gamma \to 2-\gamma$ 1

$\therefore y_2(x) \equiv x^{1-\gamma}F(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma, x)$

5, a) $x(1-x)\,y'' + [\gamma - (\alpha+\beta+1)x]\,y' - \alpha\beta y = 0$ (1)

$\xrightarrow{x = 1/\xi}$ $\xi^2(\xi-1)\dfrac{d^2y}{d\xi^2} + \xi[(2-\gamma)\xi + (\alpha+\beta-1)]\dfrac{dy}{d\xi} - \alpha\beta y = 0$

$\xrightarrow{y = \xi^{\alpha}\omega(\xi)}$ $\xi(1-\xi)\dfrac{d^2\omega}{d\xi^2} + [(1+\alpha-\beta) - \{\alpha + (\alpha-\gamma+1) + 1\}\xi]\dfrac{d\omega}{d\xi}$

$- \alpha(\alpha-\gamma+1)\,\omega = 0$ (2) 1

(2) is same as (1) with $\beta \to \alpha-\gamma+1$, $\gamma \to 1+\alpha-\beta$.

$\therefore$ Two solⁿs of (2) are

$\omega_1 = F(\alpha, \alpha-\gamma+1; 1+\alpha-\beta, \xi)$, $\omega_2 = \xi^{\beta-\alpha}F(\beta, 1+\beta-\gamma; 1+\beta-\alpha, \xi)$

$\therefore$ Two solⁿs of (1) around $x = \infty$ are

$y_5 = \dfrac{1}{x^{\alpha}} F(\alpha, \alpha-\gamma+1; 1+\alpha-\beta, \tfrac{1}{x})$, $y_6 = \dfrac{1}{x^{\beta}} F(\beta, 1+\beta-\gamma; 1+\beta-\alpha; \tfrac{1}{x})$ 2

b) $\dfrac{d}{dx}\left[(1-x^2)\dfrac{d}{dx}P_m(x)\right] + m(m+1)P_m = 0$ (1)

$\dfrac{d}{dx}\left[(1-x^2)\dfrac{d}{dx}P_n(x)\right] + n(n+1)P_n = 0$ (2)

(2) $\times P_m$ − (1) $\times P_n$, and integrate in $[-1, 1]$,

$$\int_{-1}^{1} P_m P_n\,dx = 0, \quad m \neq n$$ 1/2

To show, $\int_{-1}^{1} P_n^2\,dx = 2/(2n+1)$, $u_n = (x^2-1)^n$,

$u_n^{(m)} = \dfrac{d^m}{dx^m}(x^2-1)^n$

$P_n(x) = \dfrac{1}{2^n n!}u_n^{(n)}$, $\int_{-1}^{1} P_n^2\,dx = \dfrac{1}{2^{2n}(n!)^2}\int_{-1}^{1} u_n^{(n)} u_n^{(n)}\,dx$

2 1/2 $= -\dfrac{1}{2^{2n}(n!)^2}\int_{-1}^{1} u_n^{(n+1)} u_n^{(n-1)}\,dx$, since $u_n^{(n-1)} = 0$, at $x = \pm 1$

$$\therefore \int_{-1}^{1} u_n^{(n)} u_n^{(n)} dx = - \int_{-1}^{1} u_n^{(n+1)} u_n^{(n-1)} dx \quad \text{1/2}$$

Similarly,

$$\int_{-1}^{1} u_n^{(n+1)} u_n^{(n-1)} dx = - \int_{-1}^{1} u_n^{(n+2)} u_n^{(n-2)} dx$$

$$\int_{-1}^{1} u_n^{(n+n-1)} u_n^{(n-n+1)} dx = - \int_{-1}^{1} u_n^{(n+n)} u_n \, dx$$

Multiplying,

$$\int_{-1}^{1} u_n^{(n)} u_n^{(n)} dx = (-1)^n \int_{-1}^{1} u_n^{(2n)} u_n \, dx \quad 1$$

$$= (-1)^n \int_{-1}^{1} \frac{d^{2n}}{dx^{2n}} \{(x^2-1)^n\} (x^2-1)^n dx$$

$$= (2n)! \int_{-1}^{1} (1-x^2)^n dx$$

$$= 2(2n)! \int_{0}^{\pi/2} \cos^{2n+1}\theta \, d\theta \quad [x = \sin\theta]$$

$$= \frac{2(2n)! (2 \cdot 4 \cdots 2n)^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots 2n(2n+1)}$$

2

$$= \frac{2}{2n+1}$$