1(a) Apply mean value theorem to prove that $\left(1-\frac{1}{x}\right)^{x}, x>0$ is an increasing function.

1(b) Is Rolle's theorem applicable to the function

$$f(x)=\begin{cases}1-x^2, & x\leq 0\\ \cos x, & x>0\end{cases}$$

in $\left[-1,\frac{\pi}{2}\right]$? Justify your answer with proper arguments. If Rolle's theorem is applicable, then find $c\in\left(-1,\frac{\pi}{2}\right)$ such that $f'(c)=0$.

[2+2]

2(a) Expand the function $f(x,y) = x^y$ in powers of $(x-1)$ and $(y-1)$ upto terms including 2nd order (without remainder). Use the result to compute the approximate value of $(1.1)^{1.02}$.

2(b) Find $\lim_{x\to 0}\left(\frac{1}{2x} - \frac{1}{x(1+e^x)}\right)$.

[2+2]

3(a) Find the intervals for which the function $f(x) = \ln(x^2 + 1)$ is convex upwards (concave downwards) and convex downwards (concave upwards). Further, find the point of inflexion(s).

3(b) Using $\epsilon - \delta$ approach, show that $\lim_{(x,y)\to(1,1)} 2x^2 + 3y = 5$.

[2+2]

4(a) Find the radius of curvature of the curve $x = 6t^2 - 3t^4,\ y = 8t^3$ at an arbitrary point t. Evaluate its maximum value over $t \in [0, 1]$.

4(b) Find the asymptote(s) of the curve $x^3 - x^2y + axy + a^3 = 0$.

[2+2]

5(a) Is the function

$$f(x,y)=\begin{cases}\dfrac{2x^3+3y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

differentiable at the origin? Justify your answer.

5(b) Is the partial derivative with respect to x of the function

$$f(x,y)=\begin{cases}\dfrac{x^3y}{x^6+y^2}, & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

continuous at the origin? Justify your answer.

[3+2]

6(a) Let

$$u = \sin^{-1}\left(\frac{x^{\frac{5}{2}} - y^{\frac{5}{2}}}{x^2 + y^2}\right), (x, y) \neq (0, 0)$$

Find the value of $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy}$ as a function of u.

6(b) If $z = f(u, v), u = x + 4y,\ v = -x - 4y$ and f has continuous first and second order partial derivatives, then find the relation between z_{xx} and z_{yy}.

[2+2]

7(a) Using the Lagrange multiplier method, find the minimum value of the function

$$f(x, y) = x^2 + y^2$$

subject to

$$x^2 - xy + y^2 = 48.$$

7(b) If the curves $f(x, y) = 0$ and $\varphi(x, y) = 0$ touch each other at the point P, then evaluate

$$\frac{\partial f}{\partial x}\frac{\partial \varphi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial x}$$

at the point P.

[3+2]