3.2 Example of Groups

Example 3.2.1. *1. Symmetries of regular n-gons in plane.*

(a) *Suppose a unit square is placed at the coordinates* $(0,0,0),(1,0,0),(0,1,0)$ *and* $(1,1,0)$. *Our aim is to move the square in space such that the position of the vertices may change but they are still placed at the above mentioned coordinates. The question arises, what are the possible ways in which this can be done? It can be easily verified that the possible configurations are as follows (see Figure 3.1):*

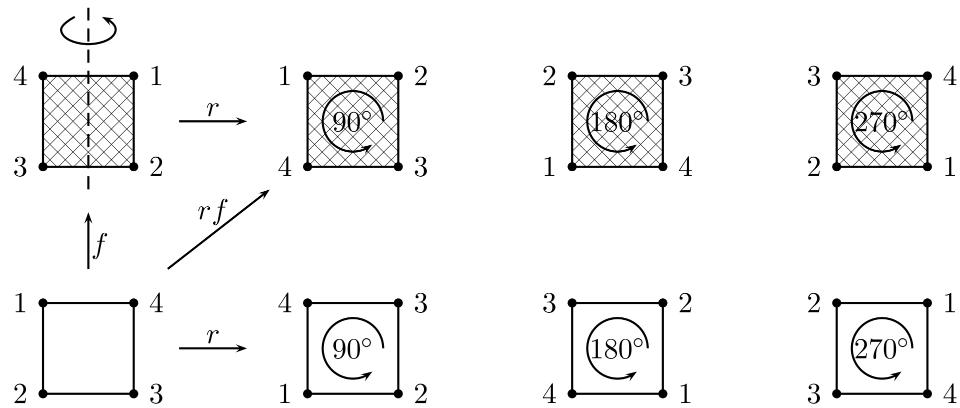


Figure 3.1: Symmetries of a square.

Let r denote the counter-clockwise rotation of the square by $90°$ *and f denote the flipping of the square along the vertical axis passing through the midpoint of opposite horizontal edges (see Figure 3.1). Then note that the possible configurations correspond to the set*

$$G = \{e, r, r^2, r^3, f, rf, r^2f, r^3f\} \text{ with relations } r^4 = e = f^2 \text{ and } fr^3 = rf. \quad (3.1)$$

Using (3.1), observe that $(rf)^2 = (rf)(rf) = r(fr)f = r(r^3f)f = r^4f^2 = e$. *Similarly, it can be checked that* $(r^2f)^2 = (r^3f)^2 = e$. *That is, all the terms f, rf, r^2f and r^3f are flips. The group G is generally denoted by D_4 and is called the* **Dihedral group** *or the* **symmetries of a square**. *This group can also be represented as follows:*

$$\{e, (1234), (13)(24), (1432), (14)(23), (24), (12)(34), (13)\}.$$

Exercise: *Relate the two representations of the group D_4.*

(b) *In the same way, one can define the symmetries of an equilateral triangle (see Figure 3.2). This group is denoted by D_3 and is represented as*

$$D_3 = \{e, r, r^2, f, rf, r^2f\} \text{ with relations } r^3 = e = f^2 \text{ and } fr^2 = rf, \quad (3.2)$$

where r is a counter-clockwise rotation by $120^\circ = \dfrac{2\pi}{3}$ and f is a flip. Using Figure 3.2, one can check that the group D_3 can also be represented by

$$D_3 = \{e, (ABC), (ACB), (BC), (CA), (AB)\}.$$

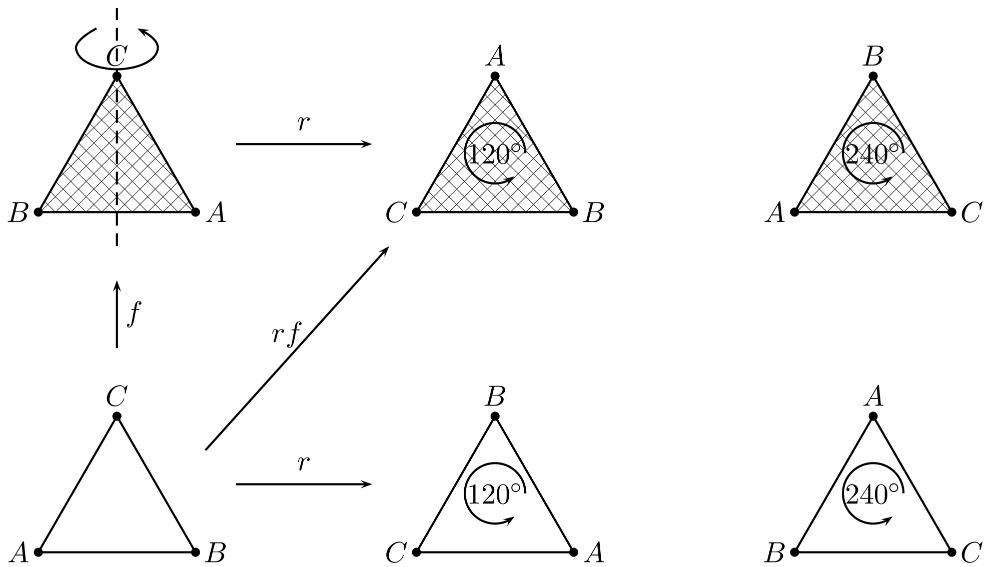


Figure 3.2: Symmetries of an Equilateral Triangle.

(c) *For a regular pentagon, it can be verified that the group of symmetries of a regular pentagon is given by $G = \{e, r, r^2, r^3, r^4, f, rf, r^2f, r^3f, r^4f\}$ with $r^5 = 2 = f^2$ and $rf = fr^4$, where r denotes a counter-clockwise rotation through an angle of $72^\circ = \dfrac{2\pi}{5}$ and f is a flip along a line that passes through a vertex and the midpoint of the opposite edge. Or equivalently, if we label the vertices of a regular pentagon, counter-clockwise, with the numbers $1, 2, 3, 4$ and 5 then*

$$\begin{aligned} G &= \{e, (1,2,3,4,5), (1,3,5,2,4), (1,4,2,5,3), (1,5,4,3,2), (2,5)(3,4), \\ &\qquad (1,3)(4,5), (1,5)(2,4), (1,2)(3,5), (1,4)(2,3)\}. \end{aligned}$$

(d) *In general, one can define symmetries of a regular n-gon. This group is denoted by D_n,has $2n$ elements and is represented as*

$$\{e, r, r^2, \ldots, r^{n-1}, f, rf, \ldots, r^{n-1}f\} \;\; \textit{with} \;\; r^n = e = f^2 \;\; \textit{and} \;\; fr^{n-1} = rf. \tag{3.3}$$

Here the symbol r stands for a counter-clockwise rotation through an angle of $\dfrac{2\pi}{n}$ and f stands for a vertical flip.

2. *Symmetries of regular platonic solids.*

(a) *Recall from geometry that a tetrahedron is a 3-dimensional regular object that is composed of 4-equilateral triangles such that any three triangles meet at a vertex (see*

Figure 3.1). Observe that a tetrahedron has 6 *edges,* 4 *vertices and* 4 *faces. If we denote the vertices of the tetrahedron with numbers* $1, 2, 3$ *and* 4*, then the symmetries of the tetrahedron can be represented with the help of the group,*

$$G = \{e, (234), (243), (124), (142), (123), (132), (134), (143), (12)(34), (13)(24), (14)(23)\},$$

where, for distinct numbers i, j, k *and* ℓ*, the element* (ijk) *is formed by a rotation of* $120°$ *along the line that passes through the vertex* ℓ *and the centroid of the equilateral triangle with vertices* i, j *and* k*. Similarly, the group element* $(ij)(k\ell)$ *is formed by a rotation of* $180°$ *along the line that passes through mid-points of the edges* (ij) *and* $(k\ell)$*.*

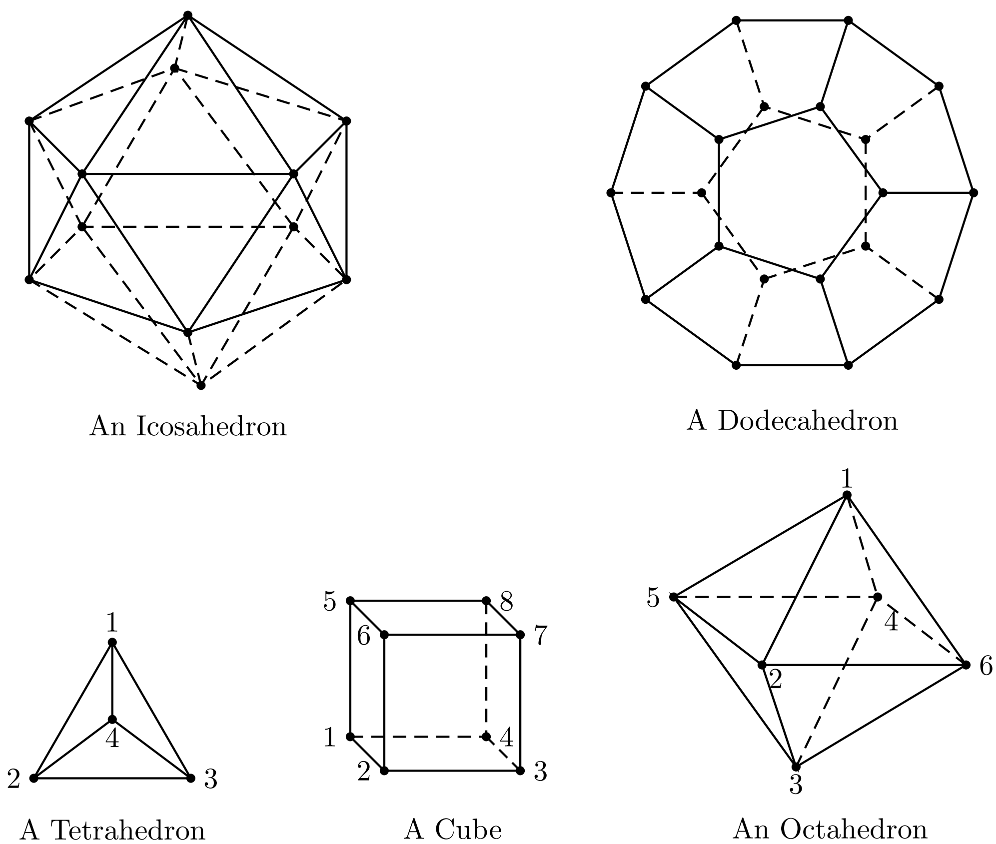


Figure 3.3: Regular Platonic solids.

(b) *Consider the Cube and the Octahedron given in Figure 3.3. It can be checked that the group of symmetries of the two figures has* 24 *elements. We give the group elements for the symmetries of the cube, when the vertices of the cube are labeled. The readers are required to compute the group elements for the symmetries of the octahedron. For the cube (see Figure 3.3), the group elements are*

i. *e, the identity element;*

ii. $3 \times 3 = 9$ *elements that are obtained by rotations along lines that pass through the center of opposite faces (*3 *pairs of opposite faces and each face is a square:*

corresponds to a rotation of 90°*). In terms of the vertices of the cube, the group elements are*

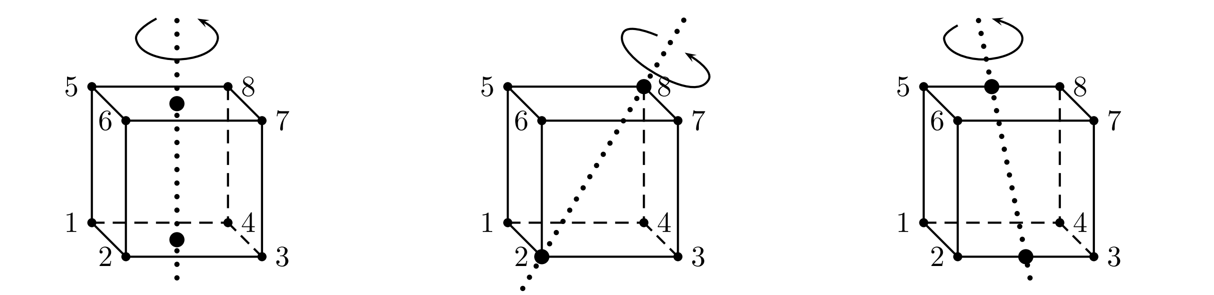
$$(1234)(5678), (13)(24)(57)(68), (1432)(5876), (1265)(3784), (16)(25)(38)(47),$$
$$(1562)(3487), (1485)(2376), (18)(45)(27)(36), (1584)(2673).$$

iii. $2 \times 4 = 8$ *elements that are obtained by rotations along lines that pass through opposite vertices (*4 *pairs of opposite vertices and each vertex is incident with* 3 *edges: correspond to a rotation of* 120°*). The group elements in terms of the vertices of the cube are*

$$(254)(368), (245)(386), (163)(457), (136)(475), (275)(138),$$
$$(257)(183), (168)(274), (186)(247).$$

iv. $1 \times 6 = 6$ *elements that are obtained by rotations along lines that pass through the midpoint of opposite edges (*6 *pairs of opposite edges: corresponds to a rotation of* 180°*). The corresponding elements in terms of the vertices of the cube are*

$$(14)(67)(28)(35), (23)(58)(17)(46), (15)(37)(28)(64), (26)(48)(17)(35),$$
$$(12)(78)(35)(46), (34)(56)(17)(28).$$



90° rotations along lines that pass through opposite faces. 120° rotations along lines that pass through opposite vertices. 180° rotations along lines that pass through opposite edges.

Figure 3.4: Understanding the group of symmetries of a cube.

(c) *Consider now the icosahedron and the dodecahedron (see Figure 3.3). Note that the icosahedron has* 12 *vertices,* 20 *faces and* 30 *edges and the dodecahedron has* 20 *vertices,* 12 *faces and* 30 *edges. It can be checked that the group of symmetries of the two figures has* 60 *elements. We give the idea of the group elements for the symmetries of the icosahedron. The readers are required to compute the group elements for the symmetries of the dodecahedron. For the icosahedron, one has*

i. e*, the identity element;*

ii. $2 \times 10 = 20$ *elements that are obtained by rotations along lines that pass through the center of opposite faces (*10 *pairs of opposite faces and each face is an equilateral triangle: corresponds to a rotation of* 120°*);*

iii. $6 \times 4 = 24$ *elements that are obtained by rotations along lines that pass through opposite vertices (*6 *pairs of opposite vertices and each vertex is incident with* 5 *edges: corresponds to a rotation of* 72°*);*

iv. $1 \times 15 = 15$ *elements that are obtained by rotations along lines that pass through the midpoint of opposite edges (*15 *pairs of opposite edges: corresponds to a rotation of* 180°*).*