4.3 Matrices related with Graphs

Let X be a graph on n vertices and let us fix a labeling of the vertices of X. Then the *adjacency matrix* of the graph X, on n vertices, denoted $A(X) = [a_{ij}]$ (or A), is an $n \times n$ matrix with $a_{ij} = 1$, if the i-th vertex is adjacent to the j-th vertex and 0, otherwise. Note that another labeling of the vertices of X gives rise to another matrix B such that $B = S^{-1}AS$, for some permutation matrix S (for a permutation matrix $S^t = S^{-1}$). Hence, we talk of the adjacency matrix of a graph X and we do not worry about the labeling of the vertices of X.

Clearly, A is a real symmetric matrix. Hence, A has n real eigenvalues, A is diagonalizable, and the eigenvectors can be chosen to form an orthonormal basis of $\mathbb{R}^n$. The eigenvalues, eigenvectors, the minimal polynomial and the characteristic polynomial of a graph X are defined to be that of its adjacency matrix.

The definition of the adjacency matrix of a graph indicates that there is one to one correspondence between graphs and its adjacency matrix. For the graph X, drawn in Figure 4.10, the adjacency matrix of X equals

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

For example, it can be easily shown that a graph X is disconnected if and only if there exists a permutation matrix P such that $A(X) = \begin{bmatrix} A_{11} & \mathbf{0} \\ \mathbf{0} & A_{22} \end{bmatrix}$, for some matrices A_{11} and A_{22}. Recall that a symmetric permutation of the rows and columns of $A(X)$ corresponds to another labeling of the graph X.

We now state and prove a few results related to the adjacency matrix of a graph.

Theorem 4.3.1. *Let A be the adjacency matrix of a graph $X = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$. Then, for each positive integer k, the (i, j)-th entry of A^k gives the number of walks of length k from the vertex v_i to the vertex v_j.*

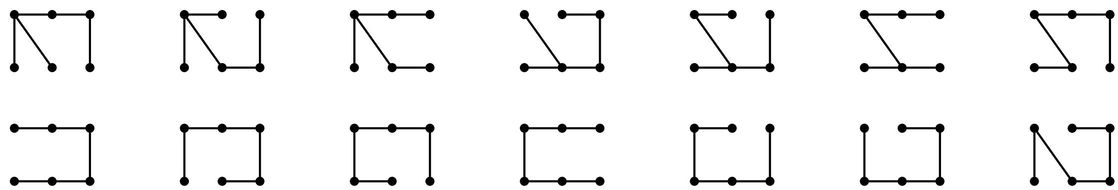
Proof. We prove the result by induction of k. The result is clearly true for $k = 1$. So, let the result be true for k and let us consider $(A^{k+1})_{ij}$, for $1 \le i \le j \le n$. By definition,

$$(A^{k+1})_{ij} = \sum_{\ell=1}^{n} (A^k)_{i\ell} a_{\ell j} = \sum_{v_\ell \sim j} (A^k)_{i\ell}.$$

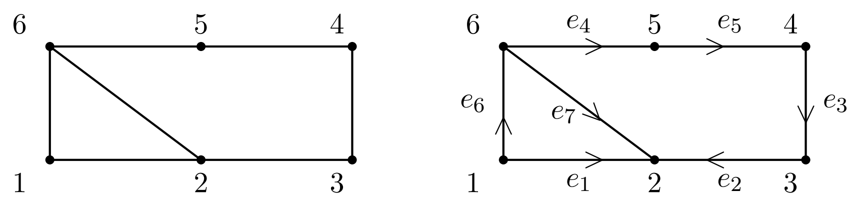
By induction hypothesis, $(A^k)_{i\ell}$ gives the number of walks of length k from the vertex v_i to the vertex v_ℓ and then a walk of length from v_ℓ to v_j. Thus, the required result follows. ■

One now immediately has the following corollary. We skip the proof as it is a direct application of Theorem 4.3.1.

Corollary 4.3.2. *Let A be the adjacency matrix of a graph X on n vertices. Then $(A^2)_{i,j}$, for $i \neq j$, gives the number of paths of length 2 from the vertex v_i to the vertex v_j. Also, $(A^2)_{ii}$, for $1 \leq i \leq n$, gives* $\deg(v_i)$.



The labeled spanning subgraphs of X (labeling not shown)



The graph X An orientation of X

Figure 4.10: A graph on 6 vertices and its spanning trees

One also defines the $\{-1, 0, 1\}$-incidence (or the edge incidence) matrix of a graph $X = (V, E)$, denoted $Q(X)$ or in short Q, as follows:
Give an orientation to each edge of X and let $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. Then, the matrix $Q = (q_{ij})$ is defined to be an $n \times m$ matrix that has its rows and columns indexed by the elements of V and E, respectively with

$$q_{ij} = \begin{cases} 1 & e_j \text{ originates at the vertex } v_i, \\ -1 & e_j \text{ terminates at the vertex } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix QQ^t, denoted $L(X)$ or in short L, is called the Laplacian matrix of X. It can be easily verified that $L = D - A$, where D is the diagonal matrix with the i-th diagonal entry being $\deg(v_i)$. Consider the oriented graph X in Figure 4.10. Based on this orientation, the edge incidence matrix and the Laplacian matrix of X, respectively equal

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \text{ and } L = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & -1 & 0 & 0 & -1 & 3 \end{bmatrix}.$$

Note that depending on the orientation of the graph X, the incidence matrices may differ, but there is no change in its Laplacian matrix. Hence, there is again a one to one correspondence between the Laplacian matrix and the corresponding graph.

Let X be a graph on n vertices and let L be its Laplacian matrix. Then note that if we add the 2-nd, 3-rd and so on till n-th row of L to the first row of L then the first row of L equals the zero vector. Hence, $\det(L) = 0$. A similar statement is true for the columns as L is a symmetric matrix. The above argument also implies that 0 is an eigenvalue of L with corresponding eigenvector $\mathbf{e} = (1, 1, \ldots, 1)^t$. Also, $L = QQ^t$ and hence L is a positive semi-definite matrix. That is, all the eigenvalues of L are non-negative and there is a non-zero vector $\mathbf{x}_0$ such that $\mathbf{x}_0^t L \mathbf{x}_0 = 0$. Using a result in matrix theory, it can be easily shown that the graph X is disconnected if and only if the multiplicity of the eigenvalue 0 is at least 2.