Indian Institute of Technology Kharagpur
Department of Mathematics
Course: Linear Algebra
Autumn Semester 2018
Problem Set 1

July 6, 2018

Notation:
V is a vector space over an arbitrary filed $\mathbb{F}$.
$\mathbb{R}$ denotes the field of real numbers.
$M_{m\times n}(\mathbb{F})$ denotes the vector space of all matrices of size $m \times n$ with entries from the field $\mathbb{F}$.
$\mathcal{C}(\mathbb{R}, \mathbb{R})$ denotes the real vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
$P_n(\mathbb{R})$ denotes the real vector space of all polynomials upto degree n.
$P(\mathbb{R})$ denotes the real vector space of all polynomials

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1. In $M_{3\times 2}(\mathbb{R})$ prove that the set

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$$

is linearly dependent.

2. Let S $= \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ be a subset of vector space over $\mathbb{F}^3$.

   (a) Prove that S is linearly independent if $\mathbb{F} = \mathbb{R}$.

   (b) Prove that S is linearly dependent if $\mathbb{F} = \mathbb{F}_2 = \{0, 1\}$, the binary field.

3. Show that the set $\{1, x, x^2, \ldots\}$ is linearly independent set in $P(\mathbb{R})$, the real vector space of all polynomials.

4. Let u and v be two non-zero distinct vectors in a vector space V over field $\mathbb{F}$. Then u and v are linearly dependent if and only if u is a linear multiple of v.

5. Give an example of three linearly dependent vectors in $\mathbb{R}^3$ where none of the three is a mutiple of another.

6. Let $u_1, u_2, \ldots, u_n$ be linearly independent vectors in a vector space V over finite field $\mathbb{F}_2 = \{0, 1\}$. Is the set $\text{span}\{u_1, u_2, \ldots, u_n\}$ finite? If yes, how many elements are there?

7. Let $f, g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ be the functions defined as $f = e^{rt}$ and $g = e^{st}$ for some real constants s and r such that $r \neq s$. Prove that the functions f and g are linearly independent in $\mathcal{C}(\mathbb{R}, \mathbb{R})$.

8. Let $S_{n\times n}(\mathbb{R}) \subset M_{n\times n}(\mathbb{R})$ be a subset of all symmetric matrices.

   (a) Prove that the set $S_{n\times n}(\mathbb{R})$ is a subspace of $M_{n\times n}(\mathbb{R})$.

   (b) Write down the basis of $M_{n\times n}(\mathbb{R})$ and hence find the dimension of $M_{n\times n}(\mathbb{R})$.

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9. Given three different bases for $\mathbb{F}^2$ and $M_{2\times 2}(\mathbb{F})$.

10. Let u and v be non-zero vectors in $\mathbb{R}^2$. Then prove u and v are independent if and only if $u + v$ and $u - v$ are independent.

11. Prove that the set of the all skew-symmetric matrices in $M_{n\times n}(\mathbb{R})$ is a finite dimensional subspace. Write down a basis and hence calculate the dimension of this subspace.

12. Prove that the set of the all lower triangular matrices in $M_{n\times n}(\mathbb{R})$ is a finite dimensional subspace. Write down a basis and hence calculate the dimension of this subspace.

13. Prove that of W_1 and W_2 are finite dimensional vector space over a field $\mathbb{F}$, then $W_1 + W_2$ is finite dimensional. Also

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$$

14. For a fixed $a \in \mathbb{R}$, determine the dimension of the subspace of $P_n(\mathbb{R})$ defined as $\{f \in P_n(\mathbb{R}) \mid f(a) = 0\}$.

15. Let $V = M_{2\times 2}(\mathbb{R})$, $W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$, and $W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$. Prove that W_1 and W_2 are subspaces of V and compute the dimensions of W_1, W_2 and $W_1 + W_2$.

16. Prove that if $\{A_1, A_2, \ldots, A_k\}$ is a linearly independent set of $M_{n\times n}(\mathbb{R})$, then $\{A_1^T, A_2^T, \ldots, A_k^T\}$ is also linearly independent.

17. Let S be a set of nonzero polynomials in $P(\mathbb{R})$ such that no two polynomials are of same degree. Prove that the set is linearly independent. Is the converse true? In other words, can a set with more than one polynomial of same degree be linearly independent?