1.13 Catalan Numbers Continued ...

Remark 1.13.1. *The book titled "enumerative combinatorics" by Stanley [10] gives a comprehensive list of places in combinatorics where Catalan numbers appear. A few of them are mentioned here for the inquisitive mind. The equivalence between these problems can be better understood after the chapter on recurrence relations.*

1. *Suppose in an election two candidates A and B get exactly n votes. Then C_n equals the number of ways of counting the votes such that candidate A is always ahead of candidate B. For example,*

$$C_3 = 5 = |\{AAABBB, AABABB, AABBAB, ABAABB, ABABAB\}|.$$

2. *Suppose, we need to multiply $n+1$ given numbers, say $a_1, a_2, \ldots, a_{n+1}$. Then the different ways of multiplying these numbers, without changing the order of the elements, equals C_n. For example,*

$$C_3 = 5 = |\{(((a_1a_2)a_3)a_4), ((a_1a_2)(a_3a_4)), ((a_1(a_2a_3))a_4), (a_1((a_2a_3)a_4)), (a_1(a_2(a_3a_4)))\}|.$$

3. *C_n also equals the number of ways that a convex $(n+2)$-gon cab be sub-divided into triangles by its diagonals so that no two diagonals intersect inside the $(n+2)$-gon. For example, for a pentagon, the different ways are*

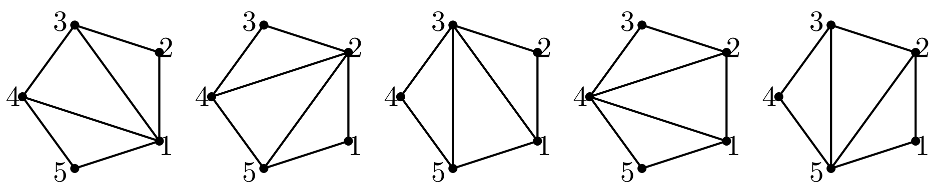

Figure 1.4: Different divisions of pentagon

4. *C_n is also equal to the number of full binary trees on $2n+1$ vertices, where recall that a full binary tree is a rooted binary tree in which every node either has exactly two offsprings or has no offspring.*

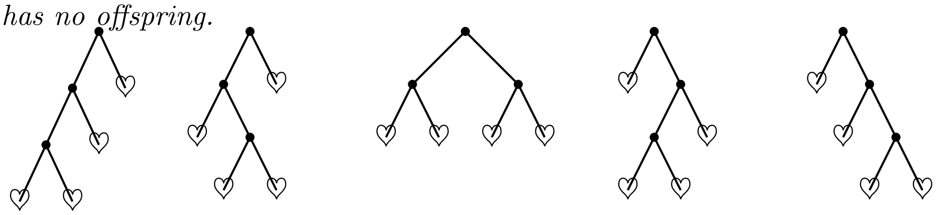

Figure 1.5: Full binary trees on 7 vertices (or 4 leaves)

5. *C_n is also equal to the number of Dyck paths from $(0,0)$ to $(2n,0)$, where recall that a Dyck path is a movement on an integer lattice in which each step is either in the North*

East or in the South East direction (so the only movement from $(0,0)$ *is either to* $(1,1)$ *or to* $(1,-1)$*).*

6. C_n *also equals the number of* n *nonintersecting chords joining* $2n$ *points on the circumference of a circle.*

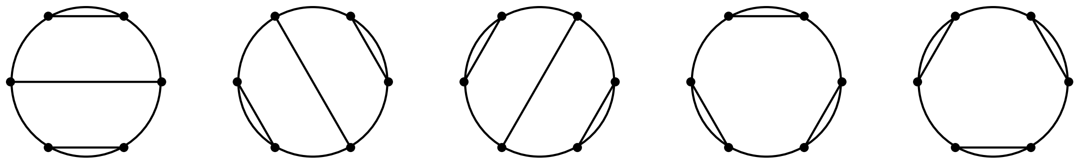

Figure 1.6: Non-intersecting chords using 6 points on the circle

7. C_n *also equals the number of integer sequences that satisfy* $1 \le a_1 \le a_2 \le \cdots \le a_n$ *and* $a_i \le i$*, for all* $i, 1 \le i \le n$.

The following article has been taken from [2].

Let A_n denote the set of all lattice paths from $(0,0)$ to (n,n) and let $B_n \subset A_n$ denote the set of all lattice paths from $(0,0)$ to (n,n) that does not go above the line $Y = X$. Then, the numerical values of $|A_n|$ and $|B_n|$ imply that $(n+1)\cdot|B_n| = |A_n|$. The question arises:

can we find a partition of the set A_n into $(n+1)$-parts, say $S_0, S_1, S_2, \ldots, S_n$ such that $S_0 = B_n$ and $|S_i| = |S_0|$, for $1 \le i \le n$?

The answer is in affirmative. Observe that any path from $(0,0)$ to (n,n) has n right moves. So, the path is specified as soon as we know the successive right moves $R_1, R_2, \ldots, R_n$, where R_i equals ℓ if and only if R_i lies on the line $Y = \ell$. For example, in Figure 1.3, $R_1 = 0,\ R_2 = 0,\ R_3 = 1,\ R_4 = 1, \ldots$. These R_i's satisfy

$$0 \le R_1 \le R_2 \le \cdots \le R_n \le n. \tag{1.1}$$

That is, any element of A_n can be represented by an ordered n-tuple $(R_1, R_2, \ldots, R_n)$ satisfying Equation (1.1). Conversely, it can be easily verified that any ordered n-tuple $(R_1, R_2, \ldots, R_n)$ satisfying Equation (1.1) corresponds to a lattice path from $(0,0)$ to (n,n). Note that, using Remark 1.13.1.7, among the above n-tuples, the tuples that satisfy $R_i \le i-1$, for $1 \le i \le n$ are elements of B_n, and vice-versa. Now, we use the tuples that represent the elements of B_n to get $n+1$ maps, $f_0, f_1, \ldots, f_n$, in such a way that $f_j(B_n)$ and $f_k(B_n)$ are disjoint, for $0 \le j \ne k \le n$, and $A_n = \bigcup\limits_{k=0}^{n} f_k(B_n)$. In particular, for a fixed $k, 0 \le k \le n$, the map $f_k : B_n \longrightarrow A_n$ is defined by

$$f_k\big((R_1, R_2, \ldots, R_n)\big) = (R_{i_1} \oplus_{n+1} k, R_{i_2} \oplus_{n+1} k, \ldots, R_{i_n} \oplus_{n+1} k),$$

where $\oplus_{n+1}$ denotes addition modulo $n+1$ and $i_1, i_2, \ldots, i_n$ is a rearrangement of the numbers $1, 2, \ldots, n$ such that $0 \le R_{i_1} \oplus_{n+1} k \le R_{i_2} \oplus_{n+1} k \le \cdots \le R_{i_n} \oplus_{n+1} k$. The readers are advised to prove the following exercise as they give the required partition of the set A_n.