2.8 Application to Recurrence Relation Continued ...

Example 2.8.1. *Determine the number of ways of arranging n pairs of parentheses (left and right) such that at any stage the number of right parentheses is always less than or equal to the number of left parentheses.*

Solution: *Recall that this number equals C_n, the n^{th} Catalan number (see Page 39). Let us obtain a recurrence relation for these numbers and use it to get a formula for C_n's.*

Let P_n denote the arrangements of those n pairs of parentheses that satisfy "at any stage, the number of left parentheses is always greater than or equal to the number of right parentheses". Then $|P_n| = C_n$, for all $n \geq 1$. Also, let Q_n denote those elements of P_n for which, "at the $2k$-th stage, for $k < n$, the number of left parentheses is strictly greater than the number of right parentheses".

We now claim that $|Q_1| = 1$ and $|Q_n| = |P_{n-1}|$, for $n \geq 2$.
Clearly $|Q_1| = 1$. Note that, for $n \geq 2$, any element of Q_n, necessarily starts with two left parentheses and ends with two right parentheses. So, if we remove the first left parenthesis and the last right parenthesis from each element of Q_n then one obtains an element of P_{n-1}. In a similar way, if we add one left parenthesis at the beginning and a right parenthesis at the end of an element of P_{n-1}, one obtains an element of Q_n. Hence, there is one-to-one correspondence between the set Q_n and P_{n-1}. Thus, $|Q_n| = |P_{n-1}| = C_{n-1}$.

Let $n \geq 2$ and consider an element of P_n. Then, for some k, $1 \leq k \leq n$, the first k pairs of parentheses will have the property that the number of left parentheses is strictly greater than the number of right parentheses, for $1 \leq \ell < k$, i.e., they form an element of Q_k and the remaining $(n-k)$ pairs of parentheses will form an element of P_{n-k}. Hence, if $|P_0| = |Q_1| = 1$, one has $C_n = |P_n| = \sum\limits_{k=1}^{n} |Q_k||P_{n-k}| = \sum\limits_{k=1}^{n} |P_{k-1}||P_{n-k}| = \sum\limits_{k=1}^{n} C_{k-1}C_{n-k}$, for $n \geq 2$. Now, define $C(x) = \sum\limits_{n\geq 0} C_n x^n$. Then

$$\begin{aligned}
C(x) &= \sum_{n\geq 0} C_n x^n = 1 + \sum_{n\geq 1} C_n x^n = 1 + \sum_{n\geq 1}\left(\sum_{k=1}^{n} C_{k-1}C_{n-k}\right) x^n \\
&= 1 + x\left(\sum_{k\geq 1} C_{k-1}x^{k-1} \sum_{n\geq k} C_{n-k}x^{n-k}\right) = 1 + x\left(C(x)\sum_{k\geq 1} C_{k-1}x^{k-1}\right) \\
&= 1 + x\left(C(x)\right)^2.
\end{aligned}$$

Thus, $x\,C(x)^2 - C(x) + 1 = 0$ and $C(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$. Therefore, using $C_0 = 1$, one obtains

$C(x) = \dfrac{1-\sqrt{1-4x}}{2x}$. *Hence,*

$$\begin{aligned}
C_n &= [x^n]C(x) = \frac{1}{2}\cdot[x^{n+1}]\left(1-\sqrt{1-4x}\right) \\
&= -\frac{1}{2}\cdot\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n\right)}{(n+1)!}(-4)^{n+1} \\
&= 2(-4)^n\cdot\frac{1\cdot(-1)\cdot(-3)\cdot(-5)\cdots(1-2n)}{2^{n+1}(n+1)!} = 2^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(n+1)!} \\
&= \frac{1}{n+1}\binom{2n}{n}, \quad \text{the } n^{th} \quad \text{Catalan Number.}
\end{aligned}$$

The ideas learnt in the previous sections will be used to get closed form expressions for sums arising out of binomial coefficients. To do so, recall the list of formal power series that appear on Page 58.

Example 2.8.2. 1. *Find a closed form expression for the numbers* $a(n) = \sum\limits_{k\ge 0}\binom{k}{n-k}$.

Solution: *Define* $A(x) = \sum\limits_{n\ge 0} a(n)x^n$. *Then*

$$\begin{aligned}
A(x) &= \sum_{n\ge 0} a(n)x^n = \sum_{n\ge 0}\left(\sum_{k\ge 0}\binom{k}{n-k}\right)x^n = \sum_{k\ge 0}\left(\sum_{n\ge 0}\binom{k}{n-k}x^n\right) \\
&= \sum_{k\ge 0}x^k\left(\sum_{n\ge k}\binom{k}{n-k}x^{n-k}\right) = \sum_{k\ge 0}x^k(1+x)^k = \sum_{k\ge 0}\big(x(1+x)\big)^k = \frac{1}{1-x(1+x)}.
\end{aligned}$$

Therefore, Example 2.6.1.2 implies $a(n) = [x^n]A(x) = [x^n]\dfrac{1}{1-x(1+x)} = F_n$, *the n-th Fibonacci number.*

2. *Find a closed form expression for the polynomials* $a(n,x) = \sum\limits_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}(-1)^k x^{n-2k}$.

Solution: *Define* $A(x,y) = \sum\limits_{n\ge 0} a(n,x)y^n$. *Then*

$$\begin{aligned}
A(x,y) &= \sum_{n\ge 0} a(n,x)y^n = \sum_{n\ge 0}\left(\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}(-1)^k x^{n-2k}\right)y^n \\
&= \sum_{k\ge 0}(-1)^k y^{2k}\left(\sum_{n\ge 2k}\binom{n-k}{k}(xy)^{n-2k}\right) \\
&= \sum_{k\ge 0}(-1)^k y^{2k}(xy)^{-k}\left(\sum_{t\ge k}\binom{t}{k}(xy)^t\right) \\
&= \sum_{k\ge 0}(-y^2)^k(xy)^{-k}\frac{(xy)^k}{(1-xy)^{k+1}} = \frac{1}{1-xy}\cdot\sum_{k\ge 0}\left(\frac{-y^2}{1-xy}\right)^k \\
&= \frac{1}{1-xy}\cdot\frac{1}{1-\frac{-y^2}{1-xy}} = \frac{1}{1-xy+y^2} = \frac{1}{(1-\alpha y)(1-\beta y)},
\end{aligned}$$

where $\alpha = \dfrac{x+\sqrt{x^2-4}}{2}$ *and* $\beta = \dfrac{x-\sqrt{x^2-4}}{2}$. *Thus,*

$$\begin{aligned} a(n,x) &= [y^n]A(x,y) = [y^n]\frac{1}{1-xy+y^2} = [y^n]\ \frac{1}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha y} - \frac{\beta}{1-\beta y}\right) \\ &= \frac{1}{\alpha-\beta}\left(\alpha^{n+1}-\beta^{n+1}\right) \\ &= \frac{1}{\sqrt{x^2-4}}\left(\left(\frac{x+\sqrt{x^2-4}}{2}\right)^{n+1} - \left(\frac{x-\sqrt{x^2-4}}{2}\right)^{n+1}\right). \end{aligned}$$

Since α *and* β *are the roots of* $y^2-xy+1=0$, $\alpha^2=\alpha x-1$ *and* $\beta^2=\beta x-1$. *Therefore, verify that the* $a(n,x)$*'s satisfy the recurrence relation* $a(n,x) = xa(n-1,x)-a(n-2,x)$, *for* $n \geq 2$, *with initial conditions* $a(0,x)=1$ *and* $a(1,x)=x$.

Let $A=(a_{ij})$ *be an* $n\times n$ *matrix, with* $a_{ij}=1$, *whenever* $|i-j|=1$ *and* 0, *otherwise. Then* A *is an adjacency matrix of a tree* T *on* n *vertices, say* $1,2,\ldots,n$ *with the vertex* i *being adjacent to* $i+1$, *for* $1\leq i\leq n-1$. *It can be verified that if* $a(n,x)=\det(xI_n-A)$, *the characteristic polynomial of* A, *then* $a(n,x)$*'s satisfy the above recurrence relation. The polynomials* $a(n,2x)$*'s are also known as* CHEBYSHEV'S *polynomial of second kind.*