

1.12 Lattice Paths and Catalan Numbers

Consider a lattice of integer lines in the plane. The set $S = \{(m, n) : m, n = 0, 1, 2, \dots\}$ are said to be the points of the lattice and the lines joining these points are called the edges of the lattice. Now, let us fix two points in this lattice, say (m_1, n_1) and m_2, n_2 , with $m_2 \geq m_1$ and $n_2 \geq n_1$. Then we define an INCREASING/LATTICE PATH from (m_1, n_1) to (m_2, n_2) to be a subset $\{e_1, e_2, \dots, e_k\}$ of S such that

1. either $e_1 = (m_1, n_1 + 1)$ or $e_1 = (m_1 + 1, n_1)$;
2. either $e_k = (m_2, n_2 - 1)$ or $e_k = (m_2 - 1, n_2)$; and
3. if we represent the tuple $e_i = (a_i, b_i)$, for $1 \leq i \leq k$, then for $2 \leq j \leq k$,
 - (a) either $a_j = a_{j-1}$ and $b_j = b_{j-1} + 1$
 - (b) or $b_j = b_{j-1}$ and $a_j = a_{j-1} + 1$.

That is, the movement on the lattice is either to the RIGHT or UP (see Figure 1.2). Now, let us look at some of the questions related to this topic.

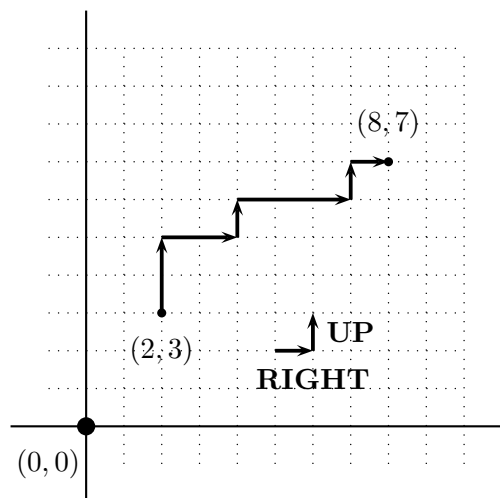


Figure 1.2: A lattice with a lattice path from $(2,3)$ to $(8,7)$

Example 1.12.1. 1. Determine the number of lattice paths from $(0,0)$ to (m,n) .

Solution: Note that at each stage, the coordinates of the lattice path increases, by exactly one positive value, either in the X -coordinate or in the Y -coordinate. Therefore, to reach (m,n) from $(0,0)$, the total increase in the X -direction is exactly m and in the Y -direction is exactly n . That is, each lattice path is a sequence of length $m+n$, consisting of m R 's (movement along X -axis/ $RIGHT$) and n U 's (movement along the Y -axis/ UP). So, we need to find m places for the R 's among the $m+n$ places (R and U together). Thus, the required answer is $\binom{m+n}{m}$.

2. Use the method of lattice paths to prove the following result on Binomial Coefficients:

$$\sum_{\ell=0}^m \binom{n+\ell}{\ell} = \binom{n+m+1}{m}.$$

Solution: Observe that the right hand side corresponds to the number of lattice paths from $(0,0)$ to $(m, n+1)$, whereas the left hand side corresponds to the number of lattice paths from $(0,0)$ to (ℓ, n) , where $0 \leq \ell \leq m$.

Now, fix $\ell, 0 \leq \ell \leq m$. Then to each lattice path from $(0,0)$ to (ℓ, n) , say P , we adjoin the path $Q = U \underbrace{RR \cdots R}_{m-\ell \text{ times}}$. Then the path $P \cup Q$, corresponding to a lattice path from $(0,0)$ to (ℓ, n) and from (ℓ, n) to $(\ell, n+1)$ and finally from $(\ell, n+1)$ to $(m, n+1)$, gives a lattice path from $(0,0)$ to $(m, n+1)$. These lattice paths, as we vary ℓ , for $0 \leq \ell \leq m$, are all distinct and hence the result follows.

Determine the number of lattice paths from $(0,0)$ to (n,n) that do not go above the line $Y = X$ (see Figure 1.3).

Solution: The first move from $(0,0)$ is R (corresponding to moving to the point $(1,0)$) as we are not allowed to go above the line $Y = X$. So, in principle, all our lattice paths are from $(1,0)$ to (n,n) with the condition that these paths do not cross the line $Y = X$.

Using Example 1.12.1.1, the total number of lattice paths from $(1,0)$ to (n,n) is $\binom{2n-1}{n}$. So, we need to subtract from $\binom{2n-1}{n}$ a number, say N_0 , where N_0 equals the number of lattice paths from $(1,0)$ to (n,n) that **cross** the line $Y = X$.

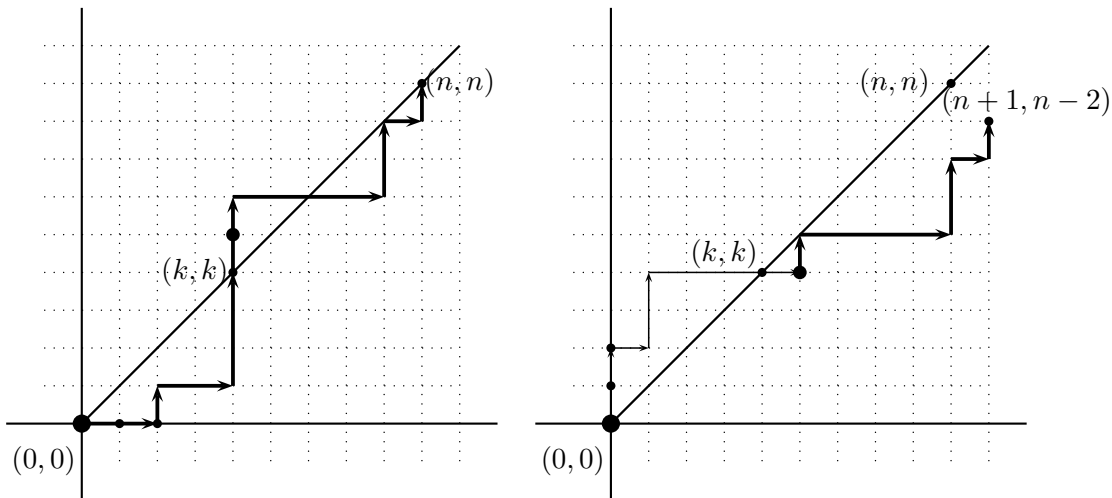


Figure 1.3: Lattice paths giving the mirror symmetry from $(0,0)$ to (k,k)

To compute the value of N_0 , we decompose each lattice path that crosses the line $Y = X$ into two sub-paths. Let P be a path from $(1,0)$ to (n,n) that crosses the line $Y = X$. Then this path crosses the line $Y = X$ for the first time at some point, say (k,k) , $1 \leq k \leq n-1$.

Claim: there exists a 1-1 correspondence between lattice paths from $(1, 0)$ to (n, n) that crosses the line $Y = X$ and the lattice paths from $(0, 1)$ to $(n + 1, n - 1)$.

Let $P = P_1 P_2 \dots P_{2k-1} P_{2k} P_{2k+1} \dots P_{2n-1}$ be the path from $(1, 0)$ to (n, n) that crosses the line $Y = X$. Then $P_1, P_2, \dots, P_{2k-2}$ consist of a sequence of $(k - 1)$ R 's and $(k - 1)$ U 's. Also, $P_{2k-1} = P_{2k} = U$ and the sub-path $P_{2k+1} P_{2k+2} \dots P_{2n-1}$ consist of a sequence that has $(n - k)$ R 's and $(n - k - 1)$ U 's. Also, for any $i, 1 \leq i \leq 2k - 2$, in the sub-path $P_1 P_2 \dots P_i$,

$$\# \text{ of } R\text{'s} = |\{j : P_j = R, 1 \leq j \leq i\}| \geq |\{\ell : P_\ell = U, 1 \leq \ell \leq i\}| = \# \text{ of } U\text{'s}. \quad (1.1)$$

Now, P is mapped to a path Q , such that $Q_i = \begin{cases} P_i, & \text{if } 2k + 1 \leq i \leq 2n - 1, \\ \{R, U\} \setminus P_i, & \text{if } 1 \leq i \leq 2k. \end{cases}$

Then we see that the path Q consists of exactly $(k - 1) + 2 + (n - k) = n + 1$ R 's and $(k - 1) + (n - k - 1) = n - 2$ U 's. Also, the condition that $Q_i = \{R, U\} \setminus P_i$, for $1 \leq i \leq 2k$, implies that the path Q starts from the point $(0, 1)$. Therefore, Q is a path that starts from $(0, 1)$ and consists of $(n + 1)$ R 's and $(n - 2)$ U 's and hence Q ends at the point $(n + 1, n - 1)$.

Also, if Q' is a path from $(0, 1)$ to $(n + 1, n - 1)$, then Q' consists of $(n + 1)$ R 's and $(n - 2)$ U 's. So, in any such sequence an instant occurs when the number of R 's exceeds the number of U 's by 2. Suppose this occurrence happens for the first time at the $(2k)^{\text{th}}$ instant, for some $k, 1 \leq k \leq n - 1$. Then there are $(k + 1)$ R 's and $(k - 1)$ U 's till the first $(2k)^{\text{th}}$ instant and $(n - k)$ R 's and $(n - 1 - k)$ U 's in the remaining part of the sequence. So, Q' can be replaced by a path P' , such that $(P')_i = \begin{cases} (Q')_i, & \text{if } 2k + 1 \leq i \leq 2n - 1, \\ \{R, U\} \setminus (Q')_i, & \text{if } 1 \leq i \leq 2k. \end{cases}$

It can be easily verified that P' is a lattice path from $(1, 0)$ to (n, n) that crosses the line $Y = X$. Thus, the proof of the claim is complete.

Hence, the number of lattice paths from $(1, 0)$ to (n, n) that crosses the line $Y = X$ equals the number of lattice paths from $(0, 1)$ to $(n + 1, n - 1)$. But, using Example 1.12.1.1, the number of lattice paths from $(0, 1)$ to $(n + 1, n - 1)$ equals $\binom{2n-1}{n+1}$. Hence, the number of lattice paths from $(0, 0)$ to (n, n) that does not go above the line $Y = X$ is

$$\binom{2n-1}{n} - \binom{2n-1}{n+1} = \frac{1}{n+1} \binom{2n}{n}.$$

This number is popularly known as the n^{th} CATALAN NUMBER, denoted C_n .