Useful Inequality

Exercise

Prove that

$$(1+x)^{\frac{p}{q}} \leq 1+\frac{p}{q}x,$$

where p and q be positive integer integers such that $p \leq q$.

Proof:

Let p and q be positive integer integers such that $p \leq q$.

The geometric mean of q numbers

$$1, 1, \dots, 1, \underbrace{(1+x), (1+x), \dots, (1+x)}_{p \text{ times}}$$

is $(1+x)^{\frac{p}{q}}$. While their arithmetic mean is $1+\frac{p}{q}x$. As geometric mean is always less that the arithmetic mean, we get

$$(1+x)^{\frac{p}{q}} \leq 1+\frac{p}{q}x$$

Hence proved. $\square$

A more general result is known as Bernoulli inequality. It is read as

$$(1+x)^s \leq 1+sx \quad \text{for} \quad x > -1 \quad \text{and} \quad 0 \leq s \leq 1.$$

For s > 1, the inequality reverses.

Hint: Use the above inequality to prove

$$\ln(1+x) \leq \frac{x}{\sqrt{(1+x)}}, \text{for} \quad x > 0$$