Von Neumann Stability Analysis (Fourier series stability analysis)

Fourier series in complex form:

Let $f(x)$ is a periodic function over period $2l$ defined in $[-l, l]$ then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Using Euler formula

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

we obtain

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2}\left\{e^{\frac{in\pi x}{l}} + e^{-\frac{in\pi x}{l}}\right\} + \frac{b_n}{2i}\left\{e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}}\right\}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)e^{\frac{in\pi x}{l}} + \frac{1}{2}(a_n + ib_n)e^{-\frac{in\pi x}{l}}\right]$$

Denoting $\quad c_0 = \frac{a_0}{2} \quad c_n = \frac{1}{2}(a_n - ib_n)$

$$c_{-n} = \frac{1}{2}(a_n + ib_n)$$

We get

$$f(x) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{\frac{in\pi x}{\ell}} + c_{-n} e^{-\frac{in\pi x}{\ell}} \right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{\ell}}$$

where

$$c_n = \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x) e^{-\frac{in\pi x}{\ell}} dx$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Stability analysis (Boundedness of numerical solution)

Consider the explicit method for solving the heat equation

$$u_j^{n+1} = (1-2\lambda) u_j^n + \lambda (u_{j-1}^n + u_{j+1}^n) \quad \text{—— (1)}$$

The exact solution of (1) for a single step can be expressed as

$$u_j^{n+1} = G\, u_j^n$$

where G, called the amplification factor, is in general a complex constant.

The solution of the FDS at time $T = N\Delta t$ is then

$$u_j^N = G^N u_j^0$$

For u_j^N to remain bounded, we must have

$$|G| \leq 1$$

Stability analysis thus reduces to the determination of the single step exact solution of the finite difference equation (1), i.e., the amplification factor G, and an investigation of the conditions necessary to ensure that $|G| \leq 1$.

From equation (1) it is seen that u_j^{n+1} depends not only on u_j^n but also on u_{j-1}^n and u_{j+1}^n. Consequently u_{j-1}^n and u_{j+1}^n must be related to u_j^n so that equation (1) can be solved for G. It is accomplished by expressing $u(x,t^n) = F(x)$ in a complex Fourier series.

The complex Fourier series of $F(x)$ is given as

$$u(x,t^n) = F(x) = \sum_{m=-\infty}^{\infty} A_n e^{i k_m x}$$

where the wave number k_m is defined as

$$k_m = \frac{m\pi}{l}.$$

For simplicity, let us examine the behavior of the solution by taking a single term of the series:

$$u_j^n = u(x_j, t^n) = A_m e^{i k_m x_j}$$

(one can also work with full sum)

then

$$u_{j+1}^n = A_m e^{i k_m (x_j + \Delta x)}$$

$$= A_m e^{i k_m x_j} \cdot e^{i k_m \Delta x}$$

$$= u_j^n e^{i\theta}$$

taking $k_m \Delta x = \theta$

$\theta \in [0, 2\pi]$

Note that $e^{i k_m \Delta x}$ represents sine and cosine functions, which have a period of 2π. Therefore $\theta \in [0, 2\pi]$ will covers all possible values of the $e^{i k_m \Delta x}$.

Similarly,

$$u_{j-1}^n = u_j^n e^{-i\theta}$$

&

$$u_{j\pm 1}^{n+1} = u_j^{n+1} e^{\pm i\theta}.$$

Working steps for stability

1. Substitute the complex components for $u_{j\pm1}^{n}$ & $u_{j\pm1}^{n+1}$ into the finite diff. equation, i.e.,

$$u_{j\pm1}^{n} = u_j^{n}\, e^{\pm i\theta}$$

$$u_{j\pm1}^{n+1} = u_j^{n+1}\, e^{\pm i\theta}$$

2. express $e^{\pm i\theta}$ in terms of $\sin\theta$ and $\cos\theta$, i.e.

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta,$$

and determine the amplification factor, G.

3. Analyse G (i.e., $|G|<1$) to determine the stability criteria of the finite difference equation.

Example 1. Explicit method for solving heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Consider the explicit method

$$\frac{u_m^{n+1} - u_m^n}{k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$

Substituting $u_{m\pm1}^n = u_m^n e^{\pm i\theta}$ & $u_{m\pm1}^{n+1} = u_m^{n+1} e^{\pm i\theta}$

we obtain:

$$u_m^{n+1} = u_m^n + \lambda\left[u_m^n e^{-i\theta} - 2u_m^n + u_m^n e^{i\theta}\right]$$

$$= \left[1 + \lambda(e^{i\theta} + e^{-i\theta}) - 2\lambda\right] u_m^n$$

$$= \left[1 + 2\lambda\cos\theta - 2\lambda\right] u_m^n$$

This implies that the amplification factor is

$$G = 1 + 2\lambda\cos\theta - 2\lambda$$

$$= 1 + 2\lambda(\cos\theta - 1)$$

For stability we require

$$|G| \le 1$$

$$\Rightarrow \left|1 + 2\lambda(\cos\theta - 1)\right| \le 1$$

$$\Rightarrow -1 \le \underbrace{1 + 2\lambda(\cos\theta - 1) \le 1}_{\text{always true}}$$

The upper inequality is always true because $\lambda > 0$ and $(\cos\theta - 1)$ ranges from -2 to 0.

From the lower limit, we get

$$-1 \le 1 + 2\lambda(\cos\theta - 1)$$

$$\Rightarrow -2 \le 2\lambda(\cos\theta - 1)$$

$$\Rightarrow \lambda \le \frac{1}{1-\cos\theta} \Rightarrow \lambda \le \frac{1}{2}$$

because the largest value of $(1-\cos\theta)$ is 2.

Hence the explicit scheme is conditionally stable with the condition:

$$\frac{k}{h^2} \le \frac{1}{2}$$

$$\Rightarrow \boxed{k \le \frac{1}{2}h^2}$$

Ex: Stability of Richardson (Leapfrog) Method.

Equation $u_t = u_{xx}$

method:

$$\frac{u_m^{n+1} - u_m^{n-1}}{2K} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$

OR

$$u_m^{n+1} = u_m^{n-1} + 2\lambda\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right)$$

Substitute $u_{m\pm1}^n = u_m^n e^{\pm i\theta}$

& $u_m^{n+1} = G u_m^n \Rightarrow u_m^n = G u_m^{n-1}$

$$\Rightarrow u_m^{n-1} = \frac{1}{G} u_m^n$$

we get:

$$u_m^{n+1} = \frac{1}{G} u_m^n + 2\lambda\left[u_m^n e^{-i\theta} - 2u_m^n + u_m^n e^{i\theta}\right]$$

$$= \left[\frac{1}{G} + 2\lambda\left\{e^{-i\theta} + e^{i\theta} - 2\right\}\right] u_m^n$$

$$= \left[\frac{1}{G} + 2\lambda\left(2\cos\theta - 2\right)\right] u_m^n$$

$$\Rightarrow u_m^{n+1} = \left[\frac{1}{G} + 4\lambda\left(\cos\theta - 1\right)\right] u_m^n$$

amplification factor:

$$G = \left[\frac{1}{G} + 4\lambda\left(\cos\theta - 1\right)\right]$$

$$\Rightarrow G^2 - 4\lambda(\cos\theta - 1)G - 1 = 0$$

$$\Rightarrow G_{1,2} = \frac{4\lambda(\cos\theta - 1) \pm \sqrt{16\lambda^2(\cos\theta-1)^2 - 4 \times -1}}{2}$$

$$= \left(2\lambda(\cos\theta - 1) \pm \sqrt{4\lambda^2(\cos\theta - 1)^2 + 1}\right)$$

Consider.

$$|G_2| = \left|2\lambda(\cos\theta - 1) - \sqrt{1 + 4\lambda^2(\cos\theta - 1)^2}\right|$$

$$= \left|2\lambda(1 - \cos\theta) + \sqrt{1 + 4\lambda^2(\cos\theta - 1)^2}\right| > 1$$

$\Rightarrow$ The leapfrog method is unconditionally unstable.

Ex: Laasonen method: (Implicit)

$$\frac{u_m^{n+1} - u_m^n}{k} = \frac{u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}}{h^2}$$

$$\Rightarrow u_m^{n+1} = u_m^n + \lambda\left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right)$$

$$\Rightarrow u_m^{n+1} = u_m^n + \lambda\left[u_m^{n+1}e^{-i\theta} - 2u_m^{n+1} + u_m^{n+1}e^{i\theta}\right]$$

$$\Rightarrow u_m^{n+1} = u_m^n + \lambda\left[2u_m^{n+1}\cos\theta - 2u_m^{n+1}\right]$$

$$= u_m^n + 2\lambda\left[\cos\theta - 1\right]u_m^{n+1}$$

$$\Rightarrow \quad u_m^{n+1}\left[1 + 2\lambda(1-\cos\theta)\right] = u_m^n$$

$$\Rightarrow \quad u_m^{n+1} = \frac{1}{1+2\lambda(1-\cos\theta)}\, u_m^n$$

amplification factor

$$G = \frac{1}{\underbrace{1+2\lambda(1-\cos\theta)}_{\geq 1}} \leq 1$$

Hence the method is unconditionally stable.

Ex: Stability of Crank-Nicolson Method

method:
$$\frac{u_m^{n+1} - u_m^n}{k} = \frac{1}{2}\left[\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} + \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right]$$

$$\Rightarrow u_m^{n+1} = u_m^n + \frac{\lambda}{2}\left[u_{m+1}^n - 2u_m^n + u_{m-1}^n + u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}\right]$$

$\Rightarrow$

$$u_m^{n+1} = u_m^n + \frac{\lambda}{2}\left[u_m^n e^{i\theta} - 2u_m^n + u_m^n e^{-i\theta} + u_m^{n+1} e^{i\theta} - 2u_m^{n+1} + u_m^{n+1} e^{-i\theta}\right]$$

$$\Rightarrow u_m^{n+1} = u_m^n + \frac{\lambda}{2}\left[2u_m^n \cos\theta - 2u_m^n + 2u_m^{n+1}\cos\theta - 2u_m^{n+1}\right]$$

$$\Rightarrow u_m^{n+1}\left[1 - \lambda(\cos\theta - 1)\right] = \left[1 + \lambda(\cos\theta - 1)\right] u_m^n$$

$$\Rightarrow u_m^{n+1} = \frac{1 + \lambda(\cos\theta - 1)}{1 - \lambda(\cos\theta - 1)}\, u_m^n$$

amplification factor:

$$G = \frac{1 - \lambda(1 - \cos\theta)}{1 + \lambda(1 - \cos\theta)}$$

$\Rightarrow |G| \leq 1$ for all λ.

$\Rightarrow$ The method is unconditionally stable.

Ex: Hyperbolic Equation (explicit method)

Equation: $u_{tt} = c^2 u_{xx}$

method: $u_m^{n+1} = r^2 u_{m-1}^n + 2(1-r^2) u_m^n + r^2 u_{m+1}^n - u_m^{n-1}$

Substituting $u_{m\pm1}^n = u_m^n e^{\pm i\theta}$ & $u_m^n = G u_m^{n-1}$

$$\Rightarrow \quad u_m^{n+1} = r^2 u_m^n e^{-i\theta} + 2(1-r^2) u_m^n + r^2 u_m^n e^{i\theta} - \frac{1}{G} u_m^n$$

$$u_m^{n+1} = 2r^2 u_m^n \cos\theta + 2(1-r^2) u_m^n - \frac{1}{G} u_m^n$$

$$\Rightarrow u_m^{n+1} = \left[2r^2(\cos\theta - 1) + 2 - \frac{1}{G}\right] u_m^n$$

amplification factor: $G = 2r^2(\cos\theta - 1) + 2 - \frac{1}{G}$

$$\Rightarrow \quad G^2 - \left(2 - 2r^2(1-\cos\theta)\right) G + 1 = 0$$

$$\Rightarrow G^2 - \left[2 - 2r^2\, 2\sin^2\frac{\theta}{2}\right] G + 1 = 0$$

$$\Rightarrow G^2 - \left[2 - 4r^2 \sin^2 \phi\right] G + 1 = 0 \qquad \text{where } \phi = \theta/2$$

$$G_{1,2} = \frac{(2 - 4r^2\sin^2\phi) \pm \sqrt{(2-4r^2\sin^2\phi)^2 - 4}}{2}$$

$$= (1 - 2r^2\sin^2\phi) \pm \sqrt{(1-2r^2\sin^2\phi)^2 - 1}$$

Case I: If $|1-2r^2\sin^2\Phi| > 1$

In this case $|G_1| > 1$ or $|G_2| > 1$ and the scheme is unstable

case II: If $|1-2r^2\sin^2\Phi| < 1$

then $G_{1,2}$ are complex pair whose magnitude is

$$|G_{1,2}| = \sqrt{\cancel{(1-2r^2\sin^2\Phi)^2} - \cancel{(1-2r^2\sin^2\phi)^2} + 1}$$

$$= 1$$

Hence the scheme is stable.

Case III: $|1-2r^2\sin^2\phi| = 1$

then $G_{1,2} = 1$

again the scheme is stable.

Hence the scheme is stable for

$$-1 \le \underbrace{1-2r^2\sin^2\phi \le 1}_{\text{always true}}$$

The first inequality gives:

$$-1 \leq 1 - 2r^2 \sin^2 \Phi$$

$$\Rightarrow -2 \leq -2r^2 \sin^2 \Phi$$

$$\Rightarrow r^2 \sin^2 \phi \leq 1$$

$$\Rightarrow \qquad r^2 \leq \frac{1}{\sin^2 \phi}$$

$$\Rightarrow r^2 \leq 1 \Rightarrow \boxed{r \leq 1}$$