Prerequisite for Linear Algebra

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For this course, we have to have ideas about partial ordered set(poset), maximal and minimal elements of poset, lower bound and upper bound, Zorn's lemma and fields.

I suggest you to please recall the definitions of vector space, subspace, linear combination, linear span, linearly independent set, linearly dependent set, basis and dimensions.

Definition 0.1. [**Relation**] Let X and Y be two nonempty sets. A relation R from X to Y is a subset of $X \times Y$, i.e., it is a collection of certain ordered pairs. We write xRy to mean $(x, y) \in R \subseteq X \times Y$.

Example 0.1.

1. Let X be any nonempty set and consider the set $\mathcal{P}(X)$. Define a relation R on $\mathcal{P}(X)$ by $R = \{(S, T) \in \mathcal{P}(X) \times \mathcal{P}(X) \;\; : \;\; S \subseteq T\}$.
2. Let $A = \{a, b, c, d\}$. Some relations R on A are:
   (a) $R = A \times A$.
   (b) $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (b, c)\}$.
   (c) $R = \{(a, a), (b, b), (c, c)\}$.

Definition 0.2. Let A be a nonempty set. Then, a relation R on A is said to be

1. **reflexive:** if for each $a \in A$, $(a, a) \in R$.
2. **symmetric:** if for each pair of elements $a, b \in A$, $(a, b) \in R$ implies $(b, a) \in R$.
3. **transitive:** if for each triple of elements $a, b, c \in A$, $(a, b), (b, c) \in R$ imply $(a, c) \in R$.
4. **antisymmetric** if for each pair of elements $a, b \in A$, $(a, b) \in R$ and $(b, a) \in R$ imply $a = b$.

Definition 0.3. Let X be a nonempty set. A relation R on X is called partially order relation if R is reflexive, anti-symmetric and transitive.

Example 0.2. 1. Let $X = \mathbb{R}$ and xRy if and only if x is less than or equal to y. Then R is a partially order relation.

2. Let $X = \mathbb{N}$ and xRy if and only if x divides y. Then R is a partial order relation.

3. Let X be a non-empty set. Let R is a relation on $\mathcal{P}(x)$ defined by ARB if and only if $A \subseteq B$. Then R is a partial order relation on $\mathcal{P}(X)$.

A partial order relation is denoted by ' $\leq'$.

Let ' $\leq'$ be a partial order relation on X. Then $(x, \leq)$ is called partially ordered set.

Definition 0.4. Let $(x, \leq)$ be a partially ordered set. Let $x, y \in X$. Then x and y are comparable if $x \leq y$ or $y \leq x$.

Remark: Let $(x, \leq)$ be a partially ordered set. Let $x, y \in X$. If $x \leq y$, then we say x is less than equal to y or y is getter than equal to x.

Definition 0.5. Let $(X, \leq)$ be a partially ordered set and $A \subseteq X$. Then A is called **totally ordered set** if any two elements in X are comparable. A **chain** in X is a totally ordered subset.

Definition 0.6. Let $(X, \leq)$ be a partially ordered set. Let $A \subseteq X$. An element $a \in A$ is called **maximal element** of A if a is not smaller than any other element of A. That is there is no $b \in A - \{a\}$ such that $a \leq b$.

Let $(X, \leq)$ be a partially ordered set. Let $A \subseteq X$. An element $a \in A$ is called **minimal element** of A if a is not smaller than any other element of A. That is there is no $b \in A - \{a\}$ such that $a \leq b$.

Remarks:

1. Maximal (minimal) element of a set if it exists must be an element of that set.

2. Maximal (minimal) element of a set may not be unique.

Example 0.3. 1. Let $X = \{1, 2, 3\}$. $R = \{(1, 1), (2, 2), (3, 3)\}$. You can easily check that R is a partial order relation on X. Let $A = \{1, 2\}$. Then $1, 2$ and 3 are the maximal elements of A.

2. Let $(\mathbb{R}, \leq)$ be a poset where the relation $\leq$ is usual less than or equal on $\mathbb{R}$. Let $A = (0, 1)$. Then A does not have any maximal element.

3. Let $X = \{1, 2, 3\}$. $R = \{(1, 1), (2, 2), (3, 3)\}$. You can easily check that R is a partial order relation on X. Let $A = \{1, 2\}$. Then $1, 2$ and 3 are the minimal elements of A.

4. Let $(\mathbb{R}, \leq)$ be a poset where the relation $\leq$ is usual less than or equal on $\mathbb{R}$. Let $A = (0, 1)$. Then A does not have minimal element.

Definition 0.7. Let $(X, \leq)$ be a partially ordered set. Let $A \subseteq X$ be a totally ordered subset. An element $a \in X$ is called **upper bound** of A if $b \leq a$ for all $b \in A$.

Let $(X, \leq)$ be a partially ordered set. Let $A \subseteq X$ be a totally ordered subset. An element $a \in X$ is called **lower bound** of A if $b \leq a$ for all $b \in A$.

Example 0.4. 1. Let $(\mathbb{R}, \leq)$ be a poset where the relation $\leq$ is usual less than or equal on $\mathbb{R}$. Let $A = (0, 1)$. Then $[1, \infty)$ is the set of all upper bounds of A.

2. Let $(\mathbb{N}, \leq)$ be a poset where the relation $x \leq y$ iff x divides y. Let $A = \{2, 4, 6, 8, \ldots\}$. Then 2 and 1 are the lower bounds of A.

Zorn's Lemma: Let $(X, \leq)$ be a partially ordered set. Let every chain have upper bound. Then X has a maximal element.

Definition 0.8. [Group] A non-empty set G is said to form a **group** with respect to binary operation $\circ$, if

1. G is closed under $\circ$,
2. $\circ$ is associative,
3. there exists an element e in G such that $e \circ a = a \circ e = a$ for all in G,
4. for each element a in G , there exists an element a' in G such that $a' \circ a = a \circ a' = e$.

The group is denoted by the symbol $(G, \circ)$.

Definition 0.9. [Field] A nonempty set R is said to be a **field** if in R there are defined two binary operations, denoted by $+$ and $\cdot$, respectively, such that

1. $(R, +)$ is abelian group, the identity element is denoted by zero.
2. $(R - \{0\}, \cdot)$ is abelian group, the identity element is denoted 1.
3. $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in R$