1.10 Indistinguishable Balls and Distinguishable Boxes

Example 1.10.1. *Consider the following three problems which have the same solution.*

1. *Determine the number of distinct strings that can be formed using* 3 A*'s and* 6 B*'s.*

2. *Determine the number of solutions to the equation* $x_1+x_2+x_3+x_4=6$*, where each* $x_i \in \mathbb{Z}$ *and* $0 \le x_i \le 6$.

3. *Determine the number of ways of placing* 6 *indistinguishable balls into* 4 *distinguishable boxes.*

   Solution: *The solution is based on the understanding that all the three problems correspond to forming strings using* $+$*'s (or* $|$*'s) and* 1*'s (or balls) in place of* A*'a and* B*'s?*

$BBABBBABA$	$11+111+1+=2+3+1+0$	●●|●●●|●|
$ABBBBBAAB$	$+11111++1=0+5+0+1$	|●●●●●||●
$ABBBABABB$	$+111+1+11=0+3+1+2$	|●●●|●|●●

Figure 1.1: Understanding the three problems

Note that the 3 A*'s are indistinguishable among themselves and the same holds for* 6 B*'s. Thus, we need to find* 3 *places, from the* $9=3+6$ *places, for the* A*'s. Hence, the answer is* $\binom{9}{3}$*. The answer will remain the same as we just need to replace* A*'s with* $+$*'s (or* $|$*'s) and* B*'s with* 1*'s (or balls) in any string of* 3 A*'s and* 6 B*'s. See Figure 1.1 or note that four numbers can be added using* 3 $+$*'s or four adjacent boxes can be created by putting* 3 *vertical lines or* $|$*'s.*

We now generalize this example to a general case.

Lemma 1.10.2. *Determine the number of*

1. *solutions to the equation* $x_1+x_2+\cdots+x_n=m$*, where each* $x_i \in \mathbb{Z}$ *and* $0 \le x_i \le m$.

2. *ways to put* m **indistinguishable** *balls into* n **distinguishable** *boxes.*

Proof: Note that the number m or the m balls can be replaced with m 1's or m $\bigstar$'s. Once this is done, using the idea in Example 1.10.1.2, we see that it is enough to find the number of distinct strings formed using $n-1$ $+$'s (or $|$'s) and m 1's (or m $\bigstar$'s). Then the indistinguishability of the $+$'s (or $|$'s) and placing them among the 1's (or $\bigstar$'s), we get

$$\binom{n-1+m}{m} = \binom{n-1+m}{n-1} = \binom{m+(n-1)}{m}.$$

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Remark 1.10.3. *Observe that the problems in Lemma 1.10.2 is same as " Determine the number of non-decreasing sequences of length m using the numbers $1, 2, \ldots, n$". Hint: Since we are looking at a non-decreasing sequences, we note that the sequence is determined if we know the number of times a particular number has appeared in the sequence. So, let x_i, for $1 \leq i \leq n$, denote the number of times the number i has appeared in the sequence.*