LECTURE - 10

11.02.2016

Linear algebra

Properties of eigen-values

1. If λ is an e-value of A, then $c\lambda$ is an e-value of CA
2. If λ is an e-value of A, λ^m is an e-value of A^m
3. $\lambda + c$ is an e-value of the matrix $A_{n\times n} + cI_n$.

Theorem - Zero is an e-value of A iff A is singular.

4. λ^{-1} is an e-value of A^{-1}.
5. Sum of e-values of A = trace of A = Sum of diagonal elements of A.
6. Product of eigenvalues of A = det A.

Ex If 2 is an e-value of A, then what is the corresponding eigen value of $3A^3 + 5I$?

Ans - $3.2^3 + 5$ (Check it)

Numerical Analysis

1. NR method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \; ; \quad n = 0, 1, 2, 3, \ldots$$

To find a root of $f(x) = 0$.

Assumption - α is a simple root of $f(x) = 0$.

Suppose, now α is a root of multiplicity k.

Case - I - Multiplicity k is known.

Then NR formula will be -

$$x_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)}$$

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Case - II

Multiplicity is not known.

$$h(x) = \frac{f(x)}{f'(x)}$$

Suppose α be a root of $f(x)=0$ of multiplicity m.

$$f(x) = (x-\alpha)^m g(x); \quad g(\alpha) \neq 0.$$

$$\frac{f(x)}{f'(x)} = \frac{(x-\alpha)^m g(x)}{m(x-\alpha)^{m-1} g(x) + (x-\alpha)^m g'(x)}.$$

or, $$h(x) = \frac{(x-\alpha) g(x)}{m g(x) + (x-\alpha) g'(x)} = (x-\alpha) G(x);$$

where $G(\alpha) \neq 0$.

since α is a simple root of $h(x)=0$,
then, the NR formula for $h(x)$ is,

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}; \quad n = 0, 1, 2, 3$$

Now, $h = \frac{f}{f'}$; $h' = \frac{f'^2 - ff''}{f'^2}$

So,

$$x_{n+1} = x_n - \left(\frac{ff'}{f'^2 - ff''}\right)_{x = x_n}; \quad n = 0, 1, 2, 3, \ldots$$

Order of convergence of FP method is 1.

" NR " " 2.

" bisection " " 1.

Finite Difference

$X:\ 0.2 \quad 0.5 \quad 0.8 \quad 0.11 \qquad h = 0.3$

$Y:\ 0.7 \quad -0.9 \quad 0.5 \quad 0.13$

$X:\ x \quad x+h \quad x+2h \quad x+3h \ \dots \ x+nh$

$Y:\ f(x) \quad f(x+h) \quad f(x+2h) \quad f(x+3h) \dots \ f(x+nh)$

- Forward Difference operator $\rightarrow \Delta f(x) = f(x+h) - f(x)$
- Backward Difference operator $\rightarrow \nabla f(x) = f(x) - f(x-h)$

$$\Delta^2 f(x) = \Delta\left(\Delta f(x)\right) = \Delta\left(f(x+h) - f(x)\right)$$
$$= \Delta f(x+h) - \Delta f(x)$$
$$= f(x+2h) - 2f(x+h) + f(x)$$

$$\Delta^2 f(x_0) = \Delta\left(\Delta f(x_0)\right) = f(x_0+2h) - 2f(x_0+h) + f(x_0)$$
$$= y_2 - 2y_1 + y_0$$

$\Delta^r f(x_k) = ?$

- Shift operator $E \rightarrow E f(x) = f(x+h)$

$$\Delta f(x) = f(x+h) - f(x) = E f(x) - I f(x) = (E - I) f(x)$$

$$\boxed{\Delta = E - I}$$

$$\Delta^r y_k = (E-I)^r y_k = \sum_{j=0}^{r} \binom{r}{j} E^{r-j} (-1)^j y_k$$
$$= \sum_{j=0}^{r} \binom{r}{j} (-1)^j E^{r-j} y_k .$$

$$\Delta^r y_k = \sum_{j=0}^{r} \binom{r}{j} (-1)^j y_{k+r-j}$$

Properities of Δ

(i) $\Delta c = 0$ (ii) $\Delta c f(x) = c \Delta f(x)$

(iii) $\Delta (f \pm g) = \Delta f \pm \Delta g$.

(iv) $\Delta^r p_n(x) = \begin{cases} \text{a pol. of degree } n-r & \text{if } n > r \\ 0 & \text{if } n < r \\ \text{a constant} & \text{if } n = r. \end{cases}$

$p_n(x) \downarrow$ pol. of degree n.

Thm $\Delta^n p_n(x) = n!\, a_n h^n$

where $p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$

proof $\rightarrow$ Exercise

[Hint- Use mathematical induction]

$\Delta x^2 = (x+h)^2 - x^2$

Example of a finite difference table

x	2	5	8	11	14
$f(x)$	3	-9	0	8	13

$x_r = x_0 + rh$

$\Delta f(x_0) = f(x_0+h) - f(x_0)$

$\Delta^r f(x_k) = \Delta^r y_k = \sum_{j=0}^{r} \binom{r}{j} y_{k+r-j} (-1)^j.$

$\nabla f(x) = f(x) - f(x-h) = (I - E^{-1}) f(x) = E^{-1}(E - I) f(x)$

$= E^{-1} \Delta f(x)$

$E^{-r} = f(x - rh)$, $\Delta f(x) = f(x+h) - f(x) = (E - I) f(x)$

$\Delta = E \nabla$

$\nabla^r f(x_k) = (E^{-1}\Delta)^r y_k = E^r \Delta^r y_k = \Delta^r E^{-r} y_k = \Delta^r y_{k-r}$

$\Delta^r f(x_k) = (E\nabla)^r y_k = \nabla^r E^r y_k = \nabla^r y_{k+r}.$

$\nabla^r y_k = \Delta^r y_{k-r}$ and $\Delta^r y_k = \nabla^r y_{k+r}$

Interpolation

x	x_0 (2)	x_1 (4)	x_2 (5) $\cdots$	x_n (10)
y	y_0 (6)	y_1 (8)	y_2 (13) $\cdots$	y_n (100)

Find y at $x = 4.5$

i.e. y at a pt. which is not listed in the given data.

$\longrightarrow$ Interpolation.

Find 1.5 or 10.2

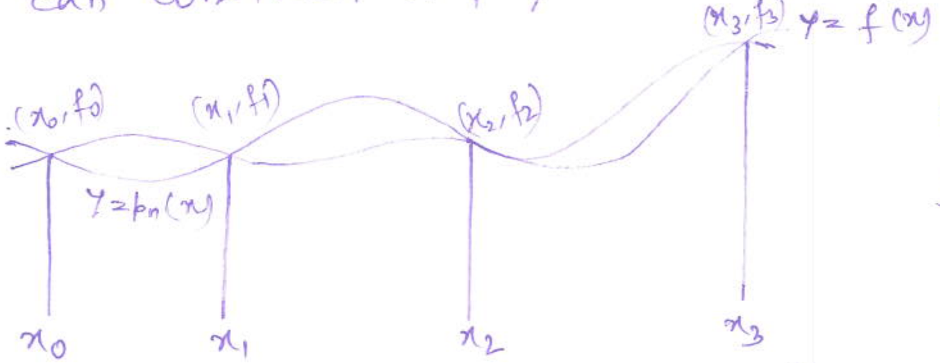
$\longrightarrow$ Extrapolation.

Interpolating polynomials

Given $(n+1)$ pairs of data

$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$;

We can construct a polynomial of degree atmost n.



$p_n(x_j) = f(x_j)$

$j = 0, 1, 2, \ldots, n$

$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

a_j's $(j = 0, 1, \ldots, n)$ are determined from

$p_n(x_j) = f(x_j)$; $j = 0, 1, 2, \ldots, n$.

Thm Interpolating polynomials with a set of n ~~points~~ pairs of data points are unique.

Newton's forward difference interpolating polynomial

x	x_0	x_1	x_2 ...	x_n
y	y_0	y_1	y_2	y_n

$x_j - x_{j-1} = h$

$j = 1, 2, \ldots, n$.

We need $p_n(x_j) = f(x_j)$, $j = 0, 1, \ldots, n$.

Let $p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0) + a_2(x - x_0)(x - x_1)$
$+ a_3(x - x_0)(x - x_1)(x - x_2) + \cdots + a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1})$

We have $p_n(x_j) = y_j$; $y = 0, 1, 2, \ldots, n$.

put $x = x_0$ in (1) : $y_0 = a_0$

put $x = x_1$ in (2) : $y_1 = a_0 + a_1(x - x_0)$

$$a_1 = \frac{y_1 - a_0}{x_1 - x_0} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}.$$

put $x = x_2$ in (1) : $y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$

or, $y_2 = y_0 + \frac{y_1 - y_0}{h} \cdot 2h + a_2 \cdot 2h \times h$

or $2h^2 a_2 = y_2 - y_0 - 2y_1 + 2y_0$

$= y_2 - 2y_1 + y_0$

$\therefore a_2 = \frac{\Delta^2 y_0}{2!\, h^2}$

In this way $a_3 = \frac{\Delta^3 y_0}{3!\, h^3}, \ldots$

i.e. $a_n = \frac{\Delta^n y_0}{n!\, h^n}$.

$\therefore p_n(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\, h^2}(x - x_0)(x - x_1)$

$+ \frac{\Delta^3 y_0}{3!\, h^3}(x - x_0)(x - x_1)(x - x_2) + \cdots$

$+ \frac{\Delta^n y_0}{n!\, h^n}(x - x_0)(x - x_1) \cdots (x - x_{n-1})$

taking $x = x_0 + kh \Rightarrow k = \frac{x - x_0}{h}$

$x - x_1 = x - x_0 + x_0 - x_1 = (x - x_0) - (x_1 - x_0)$

$= hk - h = (k-1)h$

$p_n(x) = P_n(k) = y_0 + \frac{\Delta y_0}{\cancel{h}} \times k\cancel{h} + \frac{\Delta^2 y_0}{2!\, \cancel{h^2}} kh \times (k-1)h$

$+ \cdots$

$= y_0 + \Delta y_0\, k + \frac{k(k-1)}{2!} \Delta^2 y_0 + k(k-1)(k-2) \frac{\Delta^3 y_0}{3!}$

$+ \cdots + \frac{\Delta^n y_0}{n!} k(k-1)(k-2) \cdots (k-n+1)$

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Newton's backward difference interpolating polynomial

Let us take the polynomial in the form.

$$(1)\begin{cases} q_n(x) = b_0 + b_1(x-x_n) + b_2(x-x_n)(x-x_{n-1}) \\ \quad + b_3(x-x_n)(x-x_{n-1})(x-x_{n-2}) \\ \quad + \dots + b_n(x-x_n)(x-x_{n-1})\dots(x-x_1) \end{cases}$$

To find b_n's from $q_n(x_j) = f(x_j) = y_j$; $j = 0, 1, \dots, n$

put $x = x_n$ in (1):

$$y_n = b_0$$

put $x = x_{n-1}$ in (1):

$$y_{n-1} = b_0 + b_1(x_{n-1} - x_n)$$

$$b_1 = \frac{y_{n-1} - y_n}{x_{n-1} - x_n} = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} = \frac{\nabla y_n}{h}$$

put $x = x_{n-2}$ in (1):

$$y_{n-2} = b_0 + b_1(x_{n-2} - x_n) + b_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

or, $$y_{n-2} = y_n - \frac{y_n - y_{n-1}}{h} \cdot 2h + b_2\, 2h^2$$

$$\Rightarrow \quad 2h^2 b_2 = \nabla^2 y_n$$

$$\Rightarrow \quad b_2 = \frac{\nabla^2 y_n}{2!\, h^2}$$

Similarly ~~by~~ $b_n = \dfrac{\nabla^n y_n}{n!\, h^n}$.

$$q_n(x) = y_n + \frac{\nabla y_n}{h}(x-x_n) + \frac{\nabla^2 y_n}{2!\, h^2}(x-x_n)(x-x_{n-1})$$
$$+ \frac{\nabla^3 y_n}{3!\, h^3}(x-x_n)(x-x_{n-1})(x-x_{n-2})$$
$$+ \dots + \frac{\nabla^n y_n}{n!\, h^n}(x-x_n)\dots(x-x_1)$$

P.T.O

let
$x = x_n + vh$; $x - x_n = vh$; $x - x_{n-1} = x - x_n + x_n - x_{n-1}$
$= vh + h$

$$q_n(x) = Q_n(x) = y_n + \frac{\nabla y_n}{h} \cdot vh + \frac{\nabla^2 y_n}{2!\,h^2} vh \cdot h(v+1) + \frac{\nabla^3 y_n}{3!\,h^3} vh\,(v+1)h\,(v+2)h + \cdots + \frac{\nabla^n y_n}{n!\,h^n} v \ldots (v+n-1)h^n$$

$$= y_n + \nabla y_n \cdot v + \frac{\nabla^2 y_n}{2!} v(v+1) + \cdots + \frac{\nabla^n y_n}{n!} v(v+1)\ldots(v+n-1).$$

Example

Using the following table and appropriate Newton's formula, find $f(0.5)$, $f(1.7)$, $f(3.8)$, $f(4.2)$

x	1	2	3	4
$f(x)$	5	9	14	20

$\Rightarrow$

Difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	$5 = y_0$			
		$4 = \Delta y_0 = \nabla y_1$		
2	$9 = y_1$		$1 = \Delta^2 y_0 = \nabla^2 y_2$	
		$5 = \Delta y_1 = \nabla y_2$		$0 = \Delta^3 y_0 = \nabla^3 y_3$
3	$14 = y_2$		$1 = \nabla^2 y_1 = \nabla^2 y_3$	
		$6 = \Delta y_2 = \nabla y_3$		
4	$20 = y_3$			

$\Delta^k y_r = \nabla^k y_{k+r}$

$P_n(k) = y_0 + \Delta y_0 \cdot k + \frac{\Delta^2 y_0}{2!} k(k-1) + \frac{\Delta^3 y_0}{3!} k(k-1)(k-2)$ $\Delta^3 y_0 = 0$

when $x = 0.5$, $x = x_0 + kh$, $h = 1$

$k = 0.5 - 1 = -0.5$

$\left.\begin{array}{l} P_n(-0.5) \\ P_n(0.5) \end{array}\right\} = 5 + 4(-0.5) + \frac{1}{2}(-0.5)(-1.5)$

$= 3.375$

Ans $f(0.5) \approx P_n(0.5)$
$= 3.375$

$P_n(1.7)$

$1.7 = 1 + k$

$\Rightarrow k = .7$

$P_n(1.7) = P_n(x = .7) = 7.695$

$f(3.8)$, $f(4.2)$

$Q_n(v) = y_3 + \nabla y_3 \cdot v + \frac{\nabla^2 y_3}{2!} v(v+1) + \frac{\nabla^3 y_3}{3!} v(v+1)(v+2)$

$= 20 + 6v + \frac{1}{2} v(v+1)$

$x = x_n + vh$

$x = 3.8$, $v = 3.8 - 4 = -0.2$

$x = 4.2$, $v = 4.2 - 4 = +0.2$

$f(3.8) \approx Q(3.8) = 18.72$, $f(4.2) = 21.32$

Ex Verify through the following table that the Newton's forward & backward interpolating polynomial are the same.

x	1	3	5	7
$f(x)$	3	5	7	10