Elliptic Partial Differential Equation

Let us consider the two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{--- (1)}$$

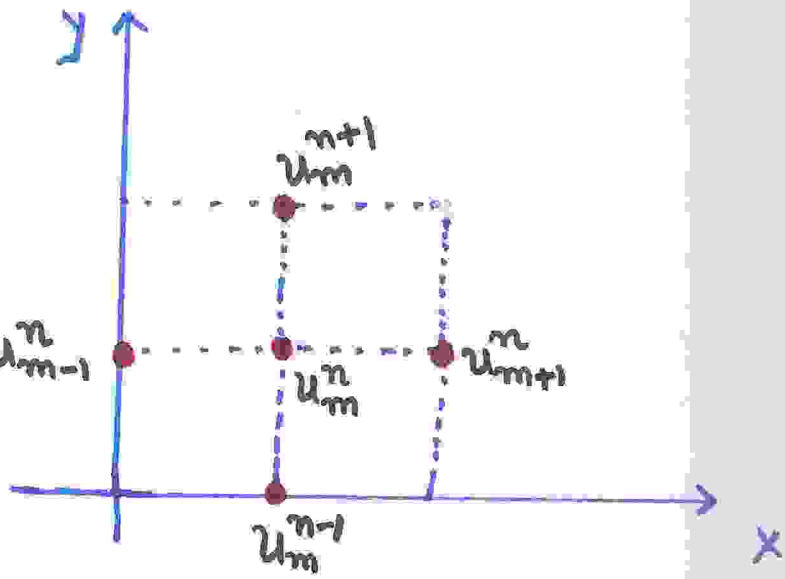
with a rectangular domain.

Using the central difference approximation to both the space and derivatives, the finite difference approximation of the above equation is given by

$$\frac{u_{m-1}^{n} - 2u_m^n + u_{m+1}^{n}}{h^2} + \frac{u_m^{n-1} - 2u_m^n + u_m^{n+1}}{k^2} = 0 \qquad \text{--- (2)}$$

If the grid points are uniform in both directions then it becomes

$$u_m^n = \frac{1}{4}\left[u_{m-1}^{n} + u_{m+1}^{n} + u_m^{n-1} + u_m^{n+1}\right] \qquad \text{--- (3)}$$



This shows that the values of u at the point (m, n) is the average of its values at the four neighbours.
This formula is know as Standard five point formula.

Remark: An equation of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

is called Poisson's equation.

Its finite difference approximation is given by

$$u_m^n = \frac{1}{4}\left[u_{m-1}^n + u_{m+1}^n + u_m^{n-1} + u_m^{n+1} - h^2 f(x_m, y_n)\right] \quad -(4)$$

Iterative Methods:

Let $u_m^{n(r)}$ denote the rth iterative value of u_m^n.

Jacobi Method:

$$u_m^{n(r+1)} = \frac{1}{4}\left[u_{m-1}^{n(r)} + u_{m+1}^{n(r)} + u_m^{(n-1)(r)} + u_m^{(n+1)(r)}\right] - (5)$$

Gauss-Seidel's Method:

In this method, the most recently computed values as soon as they are available are used and the values of u along each row are computed systematically from left to right.
The iterative formula takes the following form:

$$u_m^{n(r+1)} = \frac{1}{4}\left[u_{m-1}^{n(r+1)} + u_{m+1}^{n(r)} + u_m^{n-1(r+1)} + u_m^{n+1(r)}\right]$$

The rate of convergence of this method is twice as fast as the Jacobi's method. This method is also known as Liebmann's method.

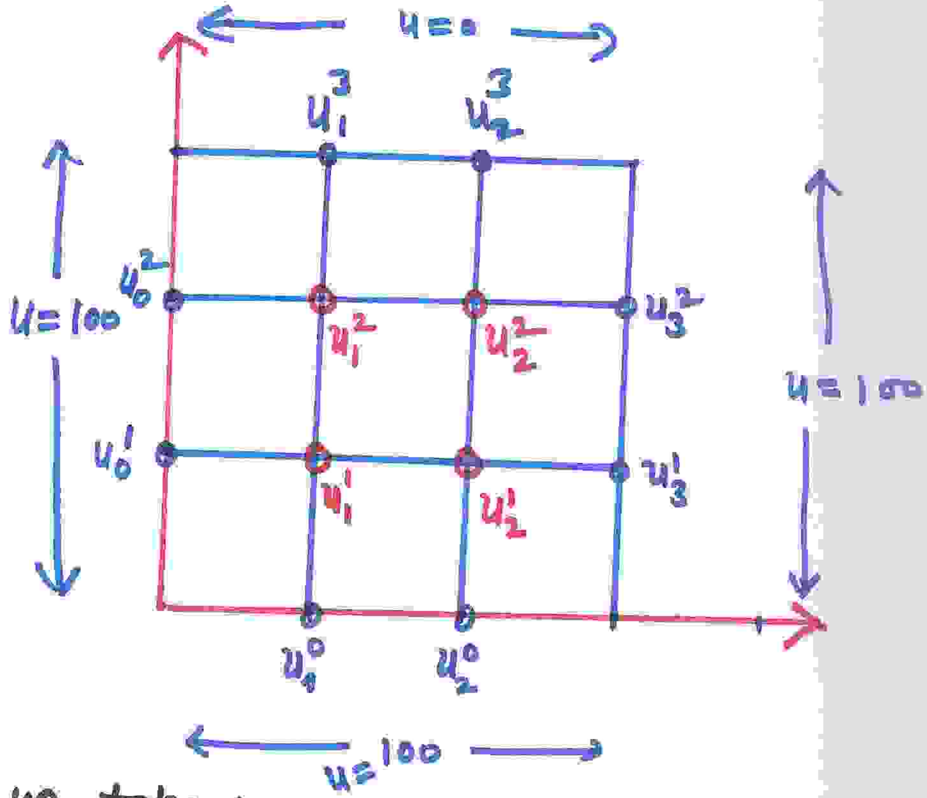
Problem: Solve the following Dirichlet problem:

$$u_{xx} + u_{yy} = 0$$

$$u(x,0) = 100$$
$$u(x,12) = 0$$
$$u(0,y) = 100$$
$$u(12,y) = 100$$

Take $h = k = 4$ and use Gauss-Seidel iteration method to solve the system of linear equations.

Solution:



Let us take initial guess as

$$u_1^{1(0)} = u_2^{1(0)} = u_1^{2(0)} = u_2^{2(0)} = 100.$$

Then,

$$u_1^{1(1)} = \frac{1}{4}\left[u_0^1 + u_2^{1(0)} + u_1^0 + u_1^{2(0)}\right]$$

$$= \frac{1}{4}\left[100 + 100 + 100 + 100\right] = 100$$

$$u_2^{1(1)} = \frac{1}{4}\left[u_1^{1(1)} + u_3^1 + u_2^0 + u_2^{2(0)}\right]$$

$$= \frac{1}{4}\left[100 + 100 + 100 + 100\right] = 100$$

$$u_1^{2(1)} = \frac{1}{4}\left[u_0^2 + u_2^{2(0)} + u_1^{1(1)} + u_1^3\right]$$

$$= \frac{1}{4}\left[100 + 100 + 100 + 0\right] = 75$$

$$u_2^{2(1)} = \frac{1}{4}\left[u_1^{2(1)} + u_3^2 + u_2^{1(1)} + u_2^3\right]$$

$$= \frac{1}{4}\left[75 + 100 + 100 + 0\right] = 68.75$$

<u>NEXT ITERATION:</u>

$$u_1^{1(2)} = \frac{1}{4}\left[u_0^1 + u_2^{1(1)} + u_1^0 + u_1^{2(1)}\right] = 93.75$$

$$u_2^{1(2)} = \frac{1}{4}\left[u_1^{1(2)} + u_3^1 + u_2^0 + u_2^{2(1)}\right] = 90.625$$

$$u_1^{2(2)} = \frac{1}{4}\left[u_0^2 + u_2^{2(1)} + u_1^{1(2)} + u_1^3\right] = 65.625$$

$$u_2^{2(2)} = \frac{1}{4}\left[u_1^{2(2)} + u_3^2 + u_2^{1(2)} + u_2^3\right] = 64.0625$$

Remark: Use of symmetry:

Problem is symmetric about $x = 6$, i.e.,

$$u_1^1 = u_2^1 \quad \& \quad u_1^2 = u_2^2$$

$$\Rightarrow \quad u_1^1 = \frac{1}{4}\left[u_0^1 + u_1^1 + u_1^0 + u_1^2\right]$$

$$\Rightarrow \quad 3u_1^1 = 200 + u_1^2 \qquad \text{———} \quad (*)$$

and

$$u_1^2 = \frac{1}{4}\left[u_0^2 + u_2^2 + u_1^1 + u_1^3\right]$$

$$= \frac{1}{4}\left[100 + u_1^2 + u_1^1 + 0\right]$$

$$\Rightarrow \quad 3u_1^2 = 100 + u_1^1 \qquad \text{———} \quad (**)$$

Setup Gauss Seidel iterations:

$$u_1^{1(r)} = \frac{1}{3}\left[200 + u_1^{2(r-1)}\right]$$

$$\& \quad u_1^{2(r)} = \frac{1}{3}\left(100 + u_1^{1(r)}\right)$$

$r = 0, 1, 2 \ldots$

$\Rightarrow$ From $(*)$ & $(**)$, we get

$$u_1^2 = u_2^2 = 62.5$$

$$u_1^1 = u_2^1 = 87.5$$

Problem: Solve the Poisson's Equation

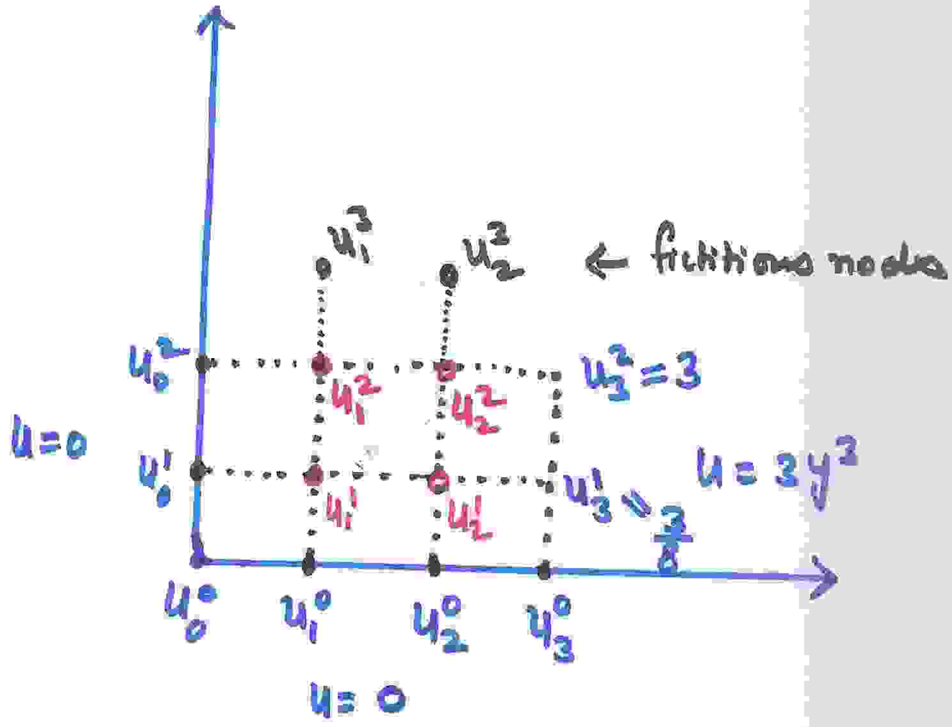
$$u_{xx} + u_{yy} = 12xy$$

BCs:

$$u(x,0) = 0, \quad 0<x<1.5$$
$$u(0,y) = 0, \quad 0 \le y \le 1$$
$$u(1.5,y) = 3y^3, \quad 0 \le y \le 1$$
$$\left.\frac{\partial u}{\partial y}\right|_{(x,1)} = 6x, \quad 0<x<1.5$$

Take $h = k = 0.5$.

Sol:



At u_1^1:

$$u_1^1 = \frac{1}{4}\left[0 + u_2^1 + 0 + u_1^2 - 12\left(\tfrac{1}{2}\right)^2 \tfrac{1}{2} \times \tfrac{1}{2}\right]$$

$$\Rightarrow \quad -4u_1^1 + u_2^1 + u_1^2 = \frac{3}{4} \qquad \text{—— (1)}$$

At u_2^1:

$$u_2^1 = \frac{1}{4}\left[u_1^1 + \frac{3}{8} + 0 + u_2^2 - 12\,\frac{1}{4} \times 1 \times \frac{1}{2}\right]$$

$$\Rightarrow 4u_2^1 = u_1^1 + u_2^2 - \frac{9}{8}$$

$$\Rightarrow -4u_2^1 + u_1^1 + u_2^2 = \frac{9}{8} \quad \text{—— (2)}$$

At: u_1^2:

$$u_1^2 = \frac{1}{4}\left[0 + u_2^2 + u_1^1 + u_1^3 - 12 \cdot \frac{1}{4} \times \frac{1}{2} \times 1\right]$$

$$\Rightarrow 4u_1^2 = u_2^2 + u_1^3 + u_1^1 - \frac{3}{2}$$

$$\Rightarrow -4u_1^2 + u_2^2 + u_1^3 + u_1^1 = 3/2 \quad \text{—— (3)}$$

At u_2^2: $u_2^2 = \frac{1}{4}\left[u_1^2 + 3 + u_2^1 + u_2^3 - 12 \cdot \frac{1}{4} \times 1 \times 1\right]$

$$\Rightarrow -4u_2^2 + u_1^2 + u_2^1 + u_2^3 = 0 \quad \text{—— (4)}$$

Use of Neumann BC.

$$\frac{\partial u_1^2}{\partial y} = 6 \times \frac{1}{2} \Rightarrow \frac{\partial u_1^2}{\partial y} = 3 \Rightarrow \frac{u_1^3 - u_1^1}{1} = 3$$

$$\Rightarrow u_1^3 = 3 + u_1^1$$

& $\frac{\partial u_2^2}{\partial y} = 6 \times 1 = 6 \Rightarrow \frac{u_2^3 - u_2^1}{1} = 6$

$$\Rightarrow u_2^3 = 6 + u_2^1$$

Substituting u_1^3 & u_2^3 in (3) and (4) we obtain

$$-4u_1^2 + u_2^2 + 2u_1^1 = -\frac{3}{2} \quad \text{--- (5)}$$

& $$-4u_2^2 + u_1^2 + 2u_2^1 = -6 \quad \text{--- (6)}$$

In matrix form:

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 2 & 0 & -4 & 1 \\ 0 & 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 9/8 \\ -3/2 \\ -6 \end{bmatrix}$$

Solving these equations using Gauss-elimination, we get

$$u_1^1 = 0.0769$$

$$u_2^1 = 0.1910$$

$$u_1^2 = 0.8665$$

$$u_2^2 = 1.8121.$$