LINE INTEGRAL:

Complex definite integrals are called line integrals:

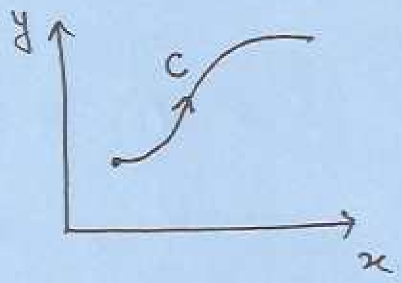
$$\int_C f(z)\, dz$$

- The integrand $f(z)$ is integrated over a given curve C in the complex plane.

- C is called the path of integration

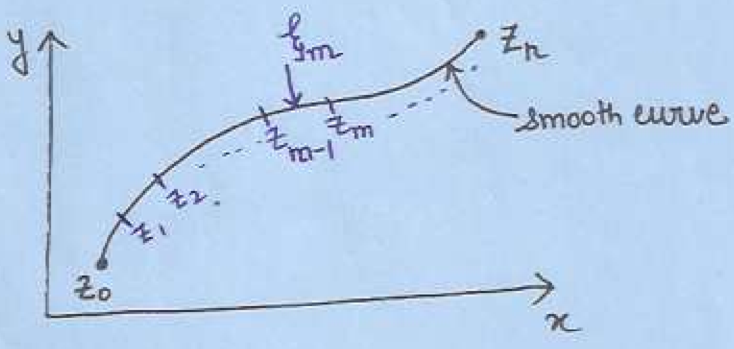
- C may be represented parametrically as

$$z(t) = x(t) + i\,y(t) \qquad a \le t \le b$$



The sense of increasing t is called the positive sense of C.

Definition



$$\lim_{n\to\infty} \sum_{m=1}^{n} f(\xi_m)\underbrace{(z_m - z_{m-1})}_{\Delta z_m} = \int_C f(z)\, dz$$

If C is a closed path then the line integral is denoted by

$$\oint_C f(z)\, dz$$

Basic properties of integration:

1. Linearity

$$\int_C [k_1 f_1(z) + k_2 f_2(z)]\, dz = k_1 \int_C f_1(z)\, dz + k_2 \int_C f_2(z)\, dz$$

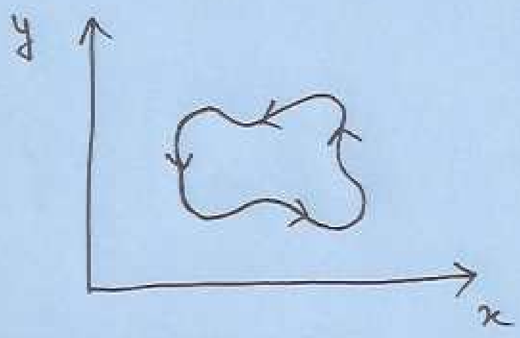
2. $$\int_{z_0}^{z} f(z)\, dz = -\int_{z}^{z_0} f(z)\, dz$$

3. $$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz \; ; \quad C = C_1 + C_2$$

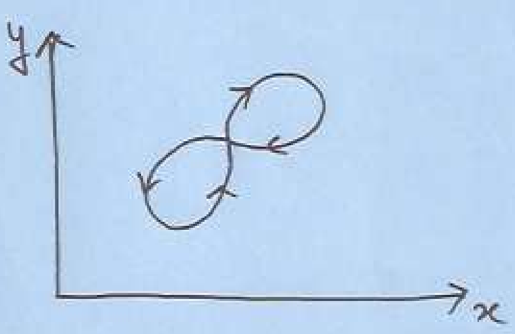
4. Suppose $f(z)$ is integrable along a curve C having finite length L and suppose there exists a positive number M such that $|f(z)| \leq M$ on C, then

$$\left| \int_C f(z)\, dz \right| \leq ML.$$

SIMPLE CLOSED CURVE: A closed curve that does not intersect (or touch) itself anywhere is called a simple closed curve.



simple closed curve

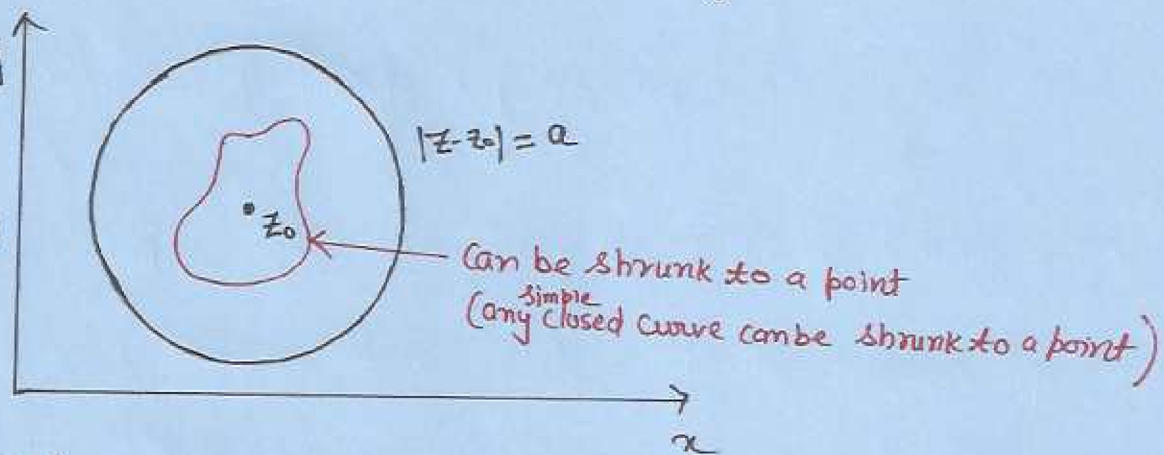


NOT a simple closed curve

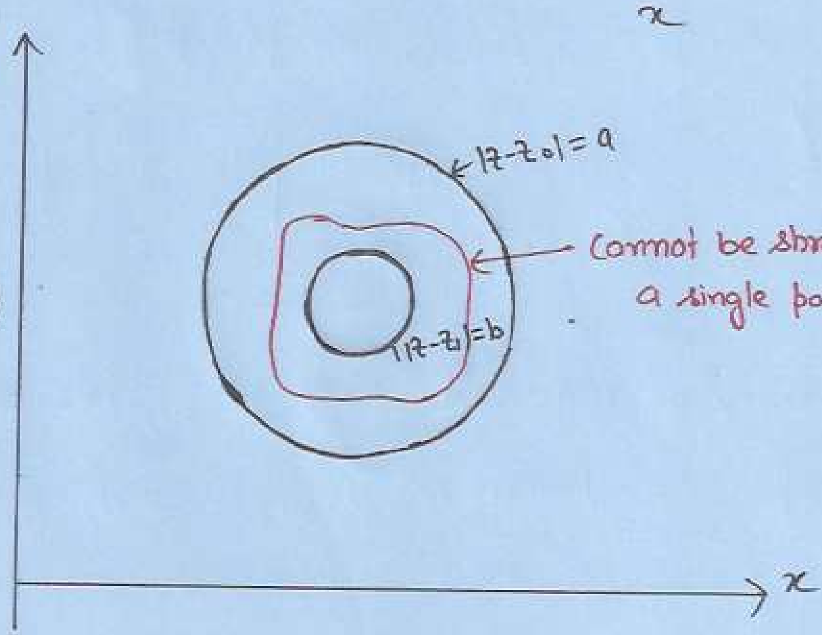
SIMPLY AND MULTIPLY CONNECTED DOMAINS

A domain D is called simply-connected if <u>any</u> simple closed curve which lies in D can be shrunk to a point without leaving D. A region which is not simply connected is called multiply-connected.

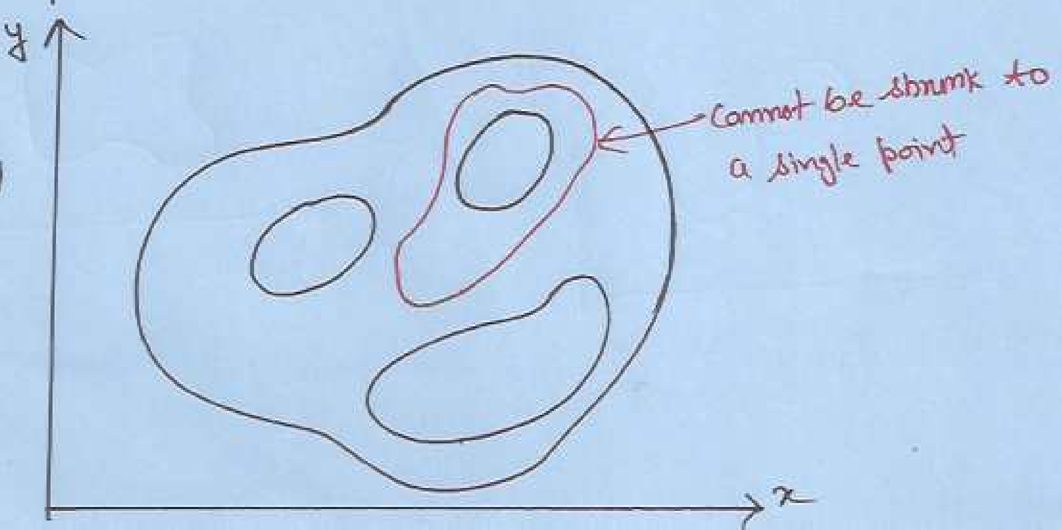
Simply Connected domain



multiply connected domain



multiply connected domain



EVALUATION OF LINE INTEGRALS:

(I) First Method: (Restricted to analytical function)

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have:

$$\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0)$$

(II) Second Method (general)

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \leq t \leq b$. Let $f(z)$ be a continuous function on C, then

$$\int_C f(z)\, dz = \int_a^b f(z(t))\, \dot{z}(t)\, dt$$

Example: Find $\oint_C (z-z_0)^m dz$, m is an integer and C is the Circle of radius ρ and center at z_0.

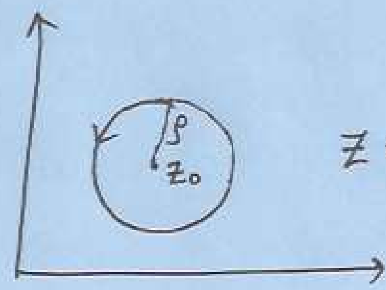
Sol:

Case I: $m \geq 0$ then $(z-z_0)^m$ is analytic

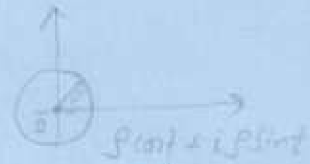
then $\oint_C (z-z_0)^m dz = 0$ from (I).

Case II: $m = -1$ i.e. $f(z) = \frac{1}{(z-z_0)}$, then the function (integrand) is not analytic in C and therefor method (I) is not applicable.

Not that C is a circle of radius ρ and center z_0



$$z = z_0 + \rho(\cos t + i\sin t) = z_0 + \rho e^{it}$$
$$0 \le t \le 2\pi$$



$$\oint_C \frac{1}{(z-z_0)}\,dz = \int_0^{2\pi} [\rho e^{it}]^{-1} \rho i\, e^{it}\,dt \quad \text{(using Method II)}$$

$$= \int_0^{2\pi} \frac{1}{\rho} \cdot e^{-it} \rho i\, e^{it}\,dt$$

$$= 2\pi i$$

Case III: $m \le -2$

$$\oint (z-z_0)^m\,dz = \int_0^{2\pi} [\rho e^{it}]^m \rho i\, e^{it}\,dt$$

$$= i\rho^{m+1} \int_0^{2\pi} e^{it(m+1)}\,dt$$

$$= i\rho^{m+1} \cdot \left.\frac{e^{it(m+1)}}{i(m+1)}\right|_0^{2\pi} \quad (m \ne -1)$$

$$= \rho^{m+1} \cdot \frac{1}{(m+1)} \cdot \left[e^{i\,2(m+1)\pi} - 1\right]$$

$$= \frac{\rho^{m+1}}{m+1}\,[1-1] = 0$$

Hence

$$\oint (z-z_0)^m\,dz = \begin{cases} 2\pi i, & m = -1 \\ 0, & m \ne -1,\ m \text{ is integer} \end{cases}$$

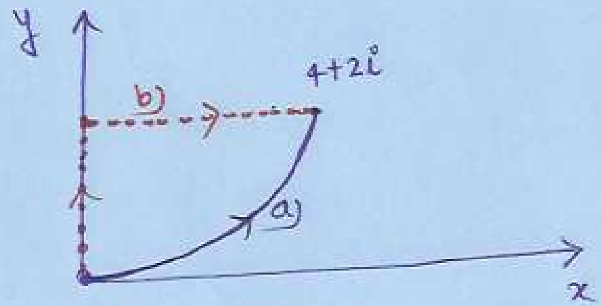
REMARK: A complex line integral depends not only on the end points of the path but in general also on the path itself.

Example: Evaluate: $\int_C \bar{z}\, dz$ from $z=0$ to $z=4+2i$

along the curve C given by

a) $z = t^2 + it$

b) the line from $z=0$ to $z=2i$ and then the line from $z=2i$ to $z=4+2i$.



Sol. Note that $\bar{z}$ is not analytic and therefore we expect different integral values along different path.

a) Corresponding to $z=0$ & $z=4+2i$, we have $t=0$ & $t=2$ respectively. Then.

$$\int_C \bar{z}\, dz = \int_{t=0}^{2} \overline{(t^2+it)}\,(2t+i)\,dt$$

$$= \int_0^2 (t^2-it)(2t+i)\,dt$$

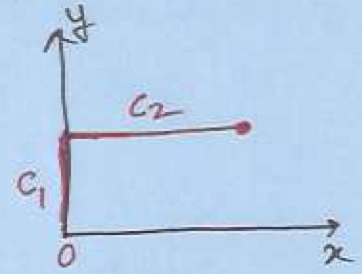
$$= \int_0^2 \left[2t^3 + it^2 - 2it^2 - i^2 t\right] dt$$

$$= \int_0^2 (2t^3 - it^2 + t)\,dt = \frac{2}{4}\cdot 16 - \frac{i}{3}\, 8 + \frac{1}{2}\cdot 4$$

$$= 10 - \frac{8}{3} i$$

b)

$$\int_C \bar{z}\,dz = \int_{C_1} \bar{z}\,dz + \int_{C_2} \bar{z}\,dz$$



C_1: $z = 2it \quad t \in [0,1]$

C_2: $z = 4t + 2i, \; t \in [0,1]$

$$= \int_0^1 -2it \cdot 2i\,dt + \int_0^1 (4t-2i)\,4\,dt$$

$$= 4\int_0^1 t\,dt + 8\int_0^1 (2t - i)\,dt$$

$$= 4\cdot\frac{1}{2} + 8\cdot\left[2\cdot\frac{1}{2} - i\right]$$

$$= 2 + 8\cdot(1-i)$$

$$= 10 - 8i$$

<u>OR</u>: PATH: $C: z = x + iy$;

Along C_1: $x = 0$, $y = 0$ to 2.

Along C_2: $y = 2$, $x = 0$ to 4.

$$\int_C \bar{z}\,dz = \int_0^2 \overline{iy}\; i\,dy + \int_0^4 \overline{(x+2i)}\,dx$$

$$= \int_0^2 y\,dy + \int_0^4 (x - 2i)\,dx$$

$$= \frac{1}{2}\cdot 4 + \frac{1}{2}\cdot 16 - 2i\cdot 4$$

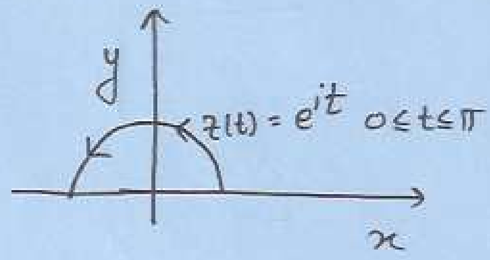
$$= 10 - 8i.$$

Ex. Evaluate $\int_C \bar{z}\, dz \qquad C: z = e^{it} \quad 0 \le t \le \pi$

Sol:

$$\int_C \bar{z}\, dz = \int_0^{\pi} e^{-it} \cdot e^{it} \cdot i\, dt$$

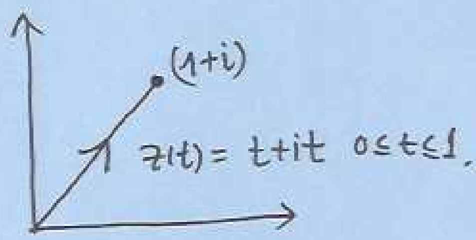
$$= \pi i$$



Ex. $\int_C z^2\, dz$ if $C: z = t + it \quad 0 \le t \le 1.$

Sol.

$$\int_0^1 (1+i)^2 t^2 \cdot (1+i)\, dt$$



$$= \int_0^1 2i\, t^2 (1+i)\, dt$$

$$= (2i-2)\,\frac{1}{3} = \frac{2}{3}(i-1).$$

Ex. Evaluate $\int_C z\, \mathrm{Re}(z)\, dz$ if $z(t) = t - it^2 \quad 0 \le t \le 2$

$$\int_C z\, \mathrm{Re}(z)\, dz = \int_0^2 (t - it^2)(t) \cdot (1 - 2it)\, dt$$

$$= \int_0^2 \left(t^2 - 2it^3 - it^3 - 2t^4\right) dt$$

$$= \int_0^2 \left(t^2 - 3it^3 - 2t^4\right) dt$$

$$= \frac{1}{3} \cdot 8 - \frac{3i}{4} \cdot 16 - \frac{2}{5}\, 32$$

$$= -\frac{152}{15} - 12i .$$