1.14 Generalizations

1. Let n, k be non-negative integers with $0 \le k \le n$. Then in Lemma 1.8.1, "the number of ways of choosing a subset of size k from a set consisting of n elements" was denoted by the binomial coefficients, $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. Since, for each $k, 0 \le k \le n$, $(n-k)!$ divides $n!$, let us think of $\binom{n}{k} = \dfrac{n \cdot (n-1) \cdots (n-k+1)}{k!}$. With this understanding, the numbers $\binom{n}{k}$ can be generalized. That is, in the generalized form, for any $n \in \mathbb{C}$ and for any non-negative integer k, one has

$$\binom{n}{k} = \begin{cases} 0, & \text{if } \ k < 0 \\ 0, & \text{if } \ n = 0, n \neq k \\ 1, & \text{if } \ n = k \\ \dfrac{n \cdot (n-1) \cdots (n-k+1)}{k!}, & \text{otherwise.} \end{cases} \tag{1.1}$$

With the notations as above, one has the following theorem which is popularly known as the generalized binomial theorem. We state it without proof.

Theorem 1.14.1. *Let n be any real number. Then*

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots .$$

In particular, $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ *and if* $a, b \in \mathbb{R}$ *with* $|a| < |b|$, *then*

$$(a+b)^n = b^n \left(1 + \frac{a}{b}\right)^n = b^n \sum_{r \ge 0} \binom{n}{r} \left(\frac{a}{b}\right)^r = \sum_{r \ge 0} \binom{n}{r} a^r b^{n-r}.$$

Let us now understand Theorem 1.14.1 through the following examples.

(a) Let $n = \dfrac{1}{2}$. In this case, for $k \ge 1$, Equation (1.1) gives

$$\binom{\frac{1}{2}}{k} = \frac{\frac{1}{2} \cdot \left(\frac{1}{2} - 1\right) \cdots \left(\frac{1}{2} - k + 1\right)}{k!} = \frac{1 \cdot (-1) \cdot (-3) \cdots (3-2k)}{2^k k!} = \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}(k-1)!k!}.$$

Thus,

$$(1+x)^{1/2} = \sum_{k \ge 0} \binom{\frac{1}{2}}{k} x^k = 1 + \frac{1}{2}x + \frac{-1}{2^3}x^2 + \frac{1}{2^4}x^3 + \sum_{k \ge 4} \frac{(-1)^{k-1}(2k-2)!}{2^{2k-1}(k-1)!k!} x^k.$$

This can also be obtained using the Taylor series expansion of $f(x) = (1+x)^{1/2}$ around $x = 0$. Recall that the Taylor series expansion of $f(x)$ around $x = 0$ equals $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \sum\limits_{k \ge 3} \frac{f^{(k)}(0)}{k!}x^k$, where $f(0) = 1, f'(0) = \frac{1}{2}, f''(0) = \frac{-1}{2^2}$ and in general $f^{(k)}(0) = \frac{1}{2} \cdot (\frac{1}{2} - 1) \cdots (\frac{1}{2} - k + 1)$, for $k \ge 3$.

(b) Let $n = -r$, where r is a positive integer. Then, for $k \geq 1$, Equation (1.1) gives

$$\binom{-r}{k} = \frac{-r \cdot (-r-1) \cdots (-r-k+1)}{k!} = (-1)^k \binom{r+k-1}{k}.$$

Thus,

$$(1+x)^n = \frac{1}{(1+x)^r} = 1 - rx + \binom{r+1}{2} x^2 + \sum_{k \geq 3} \binom{r+k-1}{k} (-x)^k.$$

The readers are advised to get the above expression using the Taylor series expansion of $(1+x)^n$ around $x = 0$.

2. Let $n, m \in \mathbb{N}$. Recall the identity $n^m = \sum\limits_{k=0}^{m} \binom{n}{k} k! S(m,k) = \sum\limits_{k=0}^{n} \binom{n}{k} k! S(m,k)$ that appeared in Lemma 1.9.5 (see Equation (1.2)). We note that for a fixed positive integer m, the above identity is same as the matrix product $X = AY$, where

$$X = \begin{bmatrix} 0^m \\ 1^m \\ 2^m \\ 3^m \\ \vdots \\ n^m \end{bmatrix}, \quad A = \begin{bmatrix} \binom{0}{0} & 0 & 0 & 0 & \cdots & 0 \\ \binom{1}{0} & \binom{1}{1} & 0 & 0 & \cdots & 0 \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & 0 & \cdots & 0 \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n} \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0! S(m,0) \\ 1! S(m,1) \\ 2! S(m,2) \\ \vdots \\ n! S(m,n) \end{bmatrix}.$$

Hence, if we know the inverse of the matrix A, we can write $Y = A^{-1} X$. Check that

$$A^{-1} = \begin{bmatrix} \binom{0}{0} & 0 & 0 & 0 & \cdots & 0 \\ -\binom{1}{0} & \binom{1}{1} & 0 & 0 & \cdots & 0 \\ \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} & 0 & \cdots & 0 \\ -\binom{3}{0} & \binom{3}{1} & -\binom{3}{2} & \binom{3}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^n \binom{n}{0} & (-1)^{n-1} \binom{n}{1} & (-1)^{n-2} \binom{n}{2} & (-1)^{n-3} \binom{n}{3} & \cdots & \binom{n}{n} \end{bmatrix}.$$

This gives us a way to calculate the Stirling numbers of second kind as a function of binomial coefficients. That is, verify that

$$S(m,n) = \frac{1}{n!} \sum_{k \geq 0} (-1)^k \binom{n}{k} (n-k)^m. \tag{1.2}$$

The above ideas imply that for all non-negative integers n the identity

$$a(n) = \sum_{k \geq 0} \binom{n}{k} b(k) \text{ holds if and only if } b(n) = \sum_{k \geq 0} (-1)^k \binom{n}{k} a(k) \text{ holds.}$$

3. We end this chapter with an example which has a history (see [5]) of being solved by many mathematicians such as Montmort, N. Bernoulli and de Moivre. We present the idea that was proposed by Euler.

Example 1.14.2. *On a rainy day, n students leave their umbrellas (which are indistinguishable) outside their examination room. Determine the number of ways of collecting the umbrellas so that no student collects the correct umbrella when they finish the examination? This problem is generally known by the* DERANGEMENT PROBLEM.

Solution: Let the students be numbered $1, 2, \ldots, n$ and suppose that the i^{th} student has the umbrella numbered i, $1 \le i \le n$. So, we are interested in the number of permutations of the set $\{1, 2, \ldots, n\}$ such that the number i is not at the i^{th} position, for $1 \le i \le n$. Let D_n represent the number of derangements. Then it can be checked that $D_2 = 1$ and $D_3 = 2$. They correspond the permutations 21 for $n = 2$ and $231,\ 312$ for $n = 3$. We will try to find a relationship of D_n with D_i, for $1 \le i \le n-1$.

Let us have a close look at the required permutations. We note that n should not be placed at the n^{th} position. So, n has to appear some where between 1 and $n-1$. That is, for some $i,\ 1 \le i \le n-1$

(a) n appears at the i^{th} position and i appears at the n^{th} position, or

(b) n appears at the i^{th} position and i does not appear at the n^{th} position.

CASE (a): Fix $i, 1 \le i \le n-1$. Then the position of n and i is fixed and the remaining numbers j, for $j \ne i, n$ should not appear at the j^{th} place. As $j \ne i, n$, the problem reduces to the number of derangements of $n-2$ numbers which by our notation equals D_{n-2}. As i can be any one of the integers $1, 2, \ldots, n-1$, the number of derangements corresponding to the first case equals $(n-1)D_{n-2}$.

CASE (b): Fix $i, 1 \le i \le n-1$. Then the position of n is at the i^{th} place but i is not placed at the n^{th} position. So, in this case, the problem reduces to placing the numbers $1, 2, \ldots, n-1$ at the places $1, 2, \ldots, i-1, i+1, \ldots, n$ such that the number i is not to be placed at the n^{th} position and for $j \ne i$, j is not placed at the j^{th} position. Let us rename the positions as $a_1, a_2, \ldots, a_{n-1}$, where $a_i = n$ and $a_j = j$ for $j \ne i$.

Then, with this renaming, the problem reduces to placing the numbers $1, 2, \ldots, n-1$ at places $a_1, a_2, \ldots, a_{n-1}$ such that the number j, for $1 \le j \le n-1$, is not placed at a_j'th position. This corresponds to the derangement of $n-1$ numbers that this by our notation equals D_{n-1}. Thus, in this case the number of derangements equals $(n-1)D_{n-1}$.

Hence, $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$. Or equivalently, $D_2 = 1$ and $D_1 = 0$ imply $D_n - nD_{n-1} = -\big(D_{n-1} - (n-1)D_{n-2}\big) = (-1)^2\big(D_{n-2} - (n-2)D_{n-3}\big) = \cdots = (-1)^n.$

Therefore,

$$\begin{aligned}
D_n &= nD_{n-1} + (-1)^n = n\big((n-1)D_{n-2} + (-1)^{n-1}\big) + (-1)^n \\
&= n(n-1)D_{n-2} + n(-1)^{n-1} + (-1)^n \\
&\vdots \\
&= n(n-1)\cdots 4 \cdot 3\ D_2 + n(n-1)\cdots 4(-1)^3 + \cdots + n(-1)^{n-1} + (-1)^n \\
&= n!\left(1 + \frac{-1}{1!} + \frac{(-1)^2}{2!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right).
\end{aligned}$$

Or, in other words $\lim\limits_{n \longrightarrow \infty} \dfrac{D_n}{n!} = \dfrac{1}{e}$.

Notes: Most of the ideas for this chapter have come from the books [4], [7] and [8].