HOMOGENEOUS FUNCTION

We say an expression in (x, y) is homogeneous of order n, if it can be expressed as

$$x^n f\left(\frac{y}{x}\right)$$

Examples:

i) $f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$

$$= \underbrace{x^n \left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \dots + a_n\left(\frac{y}{x}\right)^n\right]}_{=: \, g\left(\frac{y}{x}\right)}$$

$\Rightarrow f(x,y)$ is a homo. func. of order n.

ii) $f(x,y) = \dfrac{\sqrt{y} + \sqrt{x}}{y + x} = \dfrac{\sqrt{x}}{x}\left[\dfrac{\sqrt{\left(\frac{y}{x}\right)} + 1}{\frac{y}{x} + 1}\right]$

$$= x^{-\frac{1}{2}} g(y/x)$$

$\Rightarrow f(x,y)$ is a homo. func. of order $-\frac{1}{2}$.

ALTERNATIVE DEF. A function $f(x,y)$ is said to be homogeneous of degree n if it satisfies

$$f(tx, ty) = t^n f(x,y).$$

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS:

If $z = f(x,y)$ be a homogeneous function of x & y of order n, then

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \qquad \forall\, x, y \in D.$$

D: Domain of the function f.

PROOF: Given $z = f(x,y) = x^n g\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = nx^{n-1} g\left(\frac{y}{x}\right) + x^n \cdot \left(-\frac{y}{x^2}\right) g'\left(\frac{y}{x}\right)$$

$$= nx^{n-1} g\left(\frac{y}{x}\right) - x^{n-2} \cdot y\, g'\left(\frac{y}{x}\right) \quad \text{--- } (1)$$

$$\frac{\partial z}{\partial y} = x^n \cdot g'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \quad \text{--- } (2)$$

from (1) and (2)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n g\left(\frac{y}{x}\right) - yx^{n-1} g'\left(\frac{y}{x}\right) + yx^{n-1} g'\left(\frac{y}{x}\right)$$

$$= nz$$

□

Theorem: If $z = f(x,y)$ is a homogeneous function of x & y of degree n. Then

$$\boxed{x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1) z}$$

Example: If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, $x \neq y$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2x$$

Sol: Let $z = \tan u = \dfrac{x^3+y^3}{x-y} = x^2\left[\dfrac{1+(y/x)^3}{1-y/x}\right]$

Clearly z is a homogeneous of degree 2.

$$\Rightarrow \quad x\cdot\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$$

Subst. $z=\tan u$ gives,

$$x\cdot \sec^2 u\cdot\frac{\partial u}{\partial x}+y\cdot\sec^2 u\cdot\frac{\partial u}{\partial y}=2\cdot\tan u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2.\sin u\cdot\cos u.$$

$$=\sin 2u.$$

<u>Ex</u>: If $u=ze^{ax+by}$ where z is a homogeneous function in x & y of degree n.

Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(ax+by+n)u$

<u>Sol</u>: Since z is a homogeneous function of degree n,

we have $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$ (Euler's theorem)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\left[\frac{\partial z}{\partial x}\cdot e^{ax+by}+z\cdot e^{ax+by}\cdot a\right]+y\left[\frac{\partial z}{\partial y}e^{ax+by}+z\cdot e^{ax+by}\cdot b\right]$$

$$=e^{ax+by}\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]+z\cdot\left[axe^{ax+by}+bye^{ax+by}\right]$$

$$=\left(nz+azx+bzy\right)e^{ax+by}$$

$$=(n+ax+by)\,u:$$

Ex. Let $z = xy\, f\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ where f & g are continuous and 2 times differentiable functions. Then, evaluate

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$$

Sol: Let $z = u_1 + u_2$ where $u_1 = \underbrace{xy\, f\left(\frac{y}{x}\right)}_{\text{homo fun. of deg. 2}}$ & $u_2 = \underbrace{g\left(\frac{y}{x}\right)}_{\text{homo. func. of deg. 0}}$

Applying Euler's theorem on u_1 & u_2 we get

$$x^2 \frac{\partial^2 u_1}{\partial x^2} + 2xy \frac{\partial^2 u_1}{\partial x \partial y} + y^2 \frac{\partial^2 u_1}{\partial y^2} = 2(2-1)\cdot u_1 \quad \text{—} (1)$$

& $$x^2 \frac{\partial^2 u_2}{\partial x^2} + 2xy \frac{\partial^2 u_2}{\partial x \partial y} + y^2 \frac{\partial^2 u_2}{\partial y^2} = 0 \quad \text{—} (2)$$

Adding (1) & (2):

$$x^2 \cdot \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 2 \cdot u_1$$

$$= 2 \cdot xy\, f\left(\frac{y}{x}\right)$$

Ex. If $z = y + f\left(\frac{x}{y}\right)$ where f is cont. & differentiable function.

Find the value of $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$.

Sol. Let $z = u_1 + u_2$ where $u_1 = y$ & $u_2 = f\left(\frac{x}{y}\right)$

Now. $x \frac{\partial u_1}{\partial x} + y \frac{\partial u_1}{\partial y} = y$ (Either by Euler's theorem or direct result

& $x \frac{\partial u_2}{\partial x} + y \frac{\partial u_2}{\partial y} = 0$

Adding the above two, we get: $\boxed{x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = y}$