MA20104 Probability and Statistics

Problem Set 4

1. Let a point be chosen uniformly over the interval $[0,a]$. Let X denote the distance of the point chosen from the origin. Find the distribution function of X. Also, find the distribution function of $Y = \min(X, a/2)$.

2. Let the point (u,v) be chosen uniformly from the square $0 < u < 1$, $0 < v < 1$. Let X be the random variable that assigns to the point (u,v) the number $u+v$. Find the distribution function of X.

3. Let X be a continuous random variable having density f given by
$$f(x) = \frac{1}{2}e^{-|x|}, \qquad -\infty < x < \infty.$$
Find $P(1 \le |X| \le 2)$.

4. Let F be the distribution function defined by
$$F(x) = \frac{1}{2} + \frac{x}{2(|x|+1)}, \qquad \infty < x < \infty.$$
Find a density function f for F.

5. Let X be a uniformly distributed random variable on $(0,1)$. Find the density of $Y = X^{\frac{1}{\beta}}$, where $\beta \neq 0$.

6. Let X be a positive continuous random variable having density f. Find a formula for the density of $Y = \frac{1}{X+1}$.

7. Let X be a random variable, g a density function with respect to integration, and ϕ a differentiable strictly increasing function on $(-\infty, +\infty)$. Suppose that
$$P(X \le x) = \int_{-\infty}^{\phi(x)} g(z)dz$$
Show that the random variable $Y = \phi(X)$ has density g.

8. Let X be a random variable that is uniformly distributed on (a,b). Find a linear function ϕ such that $Y = \phi(X)$ is uniformly distributed on $(0,1)$.

9. Let $g(x) = x(1-x)^2$, $0 \le x \le 1$, and $g(x) = 0$ elsewhere. How should g be normalized to make it a density?

10. Let X have the normal density $n(O, \sigma^2)$. Find the density of $Y = |X|$.

11. Let X have the normal density $n(\mu, \sigma^2)$. Find the density of $Y = e^X$. This density is called a *lognormal* density.

12. Suppose a very large number of identical radioactive particles have decay times which are exponentially distributed with some parameter λ. If one half of the particles decay during the first second, how long will it take for 75% of the particles to decay?

13. Show that if $\alpha > 1$, the gamma density has a maximum at $(\alpha - 1)/\lambda$.

14. Let X have the gamma density $\Gamma(\alpha, \lambda)$. Find the density of $Y = \sqrt{X}$.

As additional problems you may try the following problems from the fourth edition of the book "Probability and Statistics in Engineering" by the authors William Hines, Douglas Montgomery, David Goldsman, Connie M. Borror. The list of the problems is as follows:
Chapter 6: 6-1, 6-2, 6-3, 6-6, 6-8, 6-10, 6-11, 6-12, 6-14, 6-16, 6-20, 6-23, 6-25, 6-26
Chapter 7: 7-1, 7-2, 7-3, 7-4, 7-5, 7-6, 7-7, 7-8, 7-9, 7-10, 7-12, 7-15, 7-16, 7-18