Exact Differential Equations

If M and N are functions of x and y, the equation $Mdx + Ndy = 0$ is called exact when there exists a function $f(x,y)$ such that

$$d(f(x,y)) = Mdx + Ndy$$

or

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

Theorem: The necessary and sufficient condition for the differential equation

$$Mdx + Ndy = 0$$

to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{— (1)}$$

Proof: The condition is necessary '$\Rightarrow$'

Let the equation be exact, then

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

Equating coefficients of dx & dy, we get:

$$M = \frac{\partial f}{\partial x} \qquad N = \frac{\partial f}{\partial y}$$

Assuming f to be continuous upto 2nd order partial derivatives, we obtain

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the equation is exact then M & N satisfy (1).

Now we show that the condition ① is sufficient.

We assume the $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and show that the equation

$M\,dx + N\,dy$ is exact.

That means we find a function $f(x,y)$ such that

$$df = M\,dx + N\,dy.$$

Let $g(x,y) = \int M\,dx$ be the partial integral of M such that

$$\frac{\partial g}{\partial x} = M.$$

We first prove that $\left(N - \frac{\partial g}{\partial y}\right)$ is a function of y only.

Consider $\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x \partial y}$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y \partial x} \qquad \left(\text{assuming } \frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Now consider:

$$f = g(x,y) + \int \left(N - \frac{\partial g}{\partial y}\right) dy. \text{ and then}$$

$$df = dg + d\left(\int \left(N - \frac{\partial g}{\partial y}\right) dy\right) = \frac{\partial g}{\partial x}\cdot dx + \frac{\partial g}{\partial y}\,dy + \overbrace{\frac{\partial}{\partial x}\left(\underbrace{\int\left(N - \frac{\partial g}{\partial y}\right) dy}_{\text{function of } y}\right)}^{=0} dx$$

$$+ \frac{\partial}{\partial y}\left(\int\left(N - \frac{\partial g}{\partial y}\right) dy\right) dy$$

$$= \frac{\partial g}{\partial x}\,dx + \cancel{\frac{\partial g}{\partial y}\,dy} + N\,dy - \cancel{\frac{\partial g}{\partial y}\,dy}$$

$$= M\,dx + N\,dy.$$

$\Rightarrow$ The given differential equation is exact.

Remark: The solution of an exact differential equation $Mdx + Ndy = 0$ can be written as

$$f = c$$

i.e.,

$$\int_{(y\text{ const.})} M\,dx + \int \underbrace{\left(N - \frac{\partial g}{\partial y}\right)}_{\text{function of } y \text{ alone}} dy = c$$

OR

$$\int_{(y\text{ const})} M\,dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c.$$

Example: Solve $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

Sol: $M = x^2 - 4xy - 2y^2 \qquad N = y^2 - 4xy - 2x^2$

$$\frac{\partial M}{\partial y} = -4x - 4y = \frac{\partial N}{\partial x} \Rightarrow \text{the equation is exact.}$$

Hence, there exists a function $f(x,y)$ such that

$$d(f(x,y)) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M\,dx + N\,dy$$

$$\Rightarrow \frac{\partial f}{\partial x} = x^2 - 4xy - 2y^2 \quad \& \quad \frac{\partial f}{\partial y} = y^2 - 4xy - 2x^2$$

Int. of 1st w.r.t. x $\Rightarrow$ $f = \frac{x^3}{3} - 2x^2y - 2xy^2 + c_1(y)$

On differentiation w.r.t y:

$$\frac{\partial f}{\partial y} = -2x^2 - 4xy + c_1'(y) = \underbrace{y^2 - 4xy - 2x^2}_{}$$

from above.

$$\Rightarrow c_1'(y) = y^2 \Rightarrow c_1(y) = \frac{y^3}{3} + c_2$$

Hence: $f = c_3 \Rightarrow \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} + c_2 = c_3$

$$\Rightarrow \boxed{x^3 - 6xy(x+y) + y^3 = c}$$

Example: Show that the differential equation

$$(3xy+y^2)\,dx + (x^2+xy)\,dy = 0$$

is not exact and hence it cannot be solved by the method discussed above.

Sol: Check: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$3x+2y \neq 2x+y$$

So the given equation is not exact.

However, if we proceed with the method given above, we get

$$\frac{\partial f}{\partial x} = 3xy+y^2 \qquad \frac{\partial f}{\partial y} = x^2+xy$$

$$\Rightarrow f = \frac{3}{2}x^2y + y^2x + f_1(y)$$

$$\frac{\partial f}{\partial y} = \frac{3}{2}x^2 + 2y\,x + f_1'(y) = x^2+xy$$

$$\Rightarrow f_1'(y) = \underbrace{-\frac{x^2}{2} - xy}_{}$$

depends on x & y. (Not possible to solve)

Thus, there is no $f(x,y)$ exists and hence it can not be solved in this way.