Lesson 3

Numerical Solutions of IVP

Test Problem

Let us consider the following IVP:

$$\frac{du}{dt} = f(t,u), \quad u(t_0) = u_0 \quad t \in [t_0, b] \tag{2.1}$$

The behavior of solution of IVP (2.1) in the neighborhood of any point $(\bar{t}, \bar{u})$ can be predicted by the linearized form of the differential equation.

The nonlinear function $f(t,u)$ can be linearized in the neighborhood of the point $(\bar{t}, \bar{u})$ by expending it into the Taylor series as

$$f(t,u) = f(\bar{t},\bar{u}) + (t-\bar{t})\frac{\partial f}{\partial t}(\bar{t},\bar{u}) + (u-\bar{u})\frac{\partial f}{\partial u}(\bar{t},\bar{u}) + \text{higher order terms}$$

Defining

$$\lambda = \frac{\partial f}{\partial u}(\bar{t},\bar{u}), \qquad \mu = \frac{\partial f}{\partial t}(\bar{t},\bar{u})$$
$$c = f(\bar{t},\bar{u}) - \bar{u}\lambda + (t-\bar{t})\mu$$

The differential equation can be written as

$$u' \approx \lambda u + c$$

Substituting $w = u + (c/\lambda) + (\mu/\lambda^2)$ in the above linearized differential equation, we get

$$w' - \mu/\lambda = \lambda\left[w - (c/\lambda) - (\mu/\lambda^2)\right] + c$$

This implies

$$w' = \lambda w$$

The exact solution of the test problem is

$$w(t) = ke^{\lambda t}$$

where the constant k can be evaluated by the given initial condition.

Order of a Method

Note that the consistency error is given as

$$\tau_{n+1} = y_{n+1} - y_n - h\phi(t_n, y_n, f(t_n, y_n), h)$$

The order of a method is the largest integer p such that

$$\left|\frac{1}{h}\tau_{n+1}\right| = \mathcal{O}(h^p)$$

The big O Notation

If a is some real number (typically 0), we write

$$f(x) = \mathcal{O}(g(x)) \text{ for } x \to a$$

if and only if there exist constants $d > 0$ and C such that

$$|f(x)| \leq C|g(x)| \text{ for all } x \text{ with } |x - a| < d.$$

For example, we write

$$e^x = 1 + x + \frac{x^2}{2} + \mathcal{O}(x^3) \text{ for } x \to 0$$

TAYLOR SERIES METHOD:

Let the solution $y(t)$ of the IVP

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0, \quad t \in [t_0, b]$$

exists uniquely such that $y(t) \in C^{(p+1)}[t_0, b]$.

Expand the solution $y(t)$ in a Taylor series about any point t_n

$$y(t) = y(t_n) + (t-t_n)\, y'(t_n) + \frac{(t-t_n)^2}{\lfloor 2} y''(t_n) + \cdots + \frac{1}{\lfloor p}(t-t_n)^p y^{(p)}(t_n)$$
$$+ \frac{(t-t_n)^{p+1}}{\lfloor p+1} y^{(p+1)}(\xi_n)$$

where $t \in [t_0, b]$; $t_n < \xi_n < t$.

Substituting $t = t_{n+1}$

$$y(t_{n+1}) = y(t_n) + (t_{n+1}-t_n)\, y'(t_n) + \frac{(t_{n+1}-t_n)^2}{\lfloor 2} y''(t_n) + \cdots$$
$$+ \frac{1}{\lfloor p}(t_{n+1}-t_n)^p y^{(p)}(t_n) + \frac{(t_{n+1}-t_n)^{p+1}}{\lfloor p+1} y^{(p+1)}(\xi_n)$$

Let $t_{n+1} - t_n = h$, then

$$y(t_{n+1}) = y(t_n) + \underbrace{h\, y'(t_n) + \frac{h^2}{\lfloor 2} y''(t_n) + \cdots + \frac{1}{\lfloor p} h^p y^{(p)}(t_n) + \frac{h^{p+1}}{\lfloor p+1} y^{(p+1)}(\xi_n)}_{=:\, h\,\Phi(t_n, y_n, f_n, h)}$$

Hence the numerical scheme to approximate $y(t_{n+1})$ is given as

$$u_{n+1} = u_n + h\,\Phi(t_n, u_n, f_n, h), \quad n = 0, 1, 2, \ldots N-1.$$

This is called Taylor's series method of order p.

For $p=1$:

$$\boxed{u_{n+1} = u_n + h\, f(t_n, u_n)} \qquad n = 0, 1, \ldots N-1.$$

is known as Euler method.

How to get $y'(t_n)$, $y''(t_n)$ ..... ?

Notice that

$$y' = f(t, y)$$

$$y'' = \frac{d}{dt}\left(f(t,y)\right) = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= f_t + f_y f.$$

&

$$y''' = \frac{d}{dt}\left(f_t\right) + \frac{d}{dt}\left(f_y f\right)$$

$$= f_{tt} + f_{ty} f + f_y\left(f_t + f_y f\right) + f\left(f_{yt} + f_{yy} f\right)$$

$$= f_{tt} + 2 f f_{ty} + f^2 f_{yy} + f_y\left(f_t + f f_y\right)$$

$$\vdots$$

The consistency or truncation error is given by

$$T_{n+1} = \frac{h^{p+1}}{\lfloor p+1} y^{(p+1)}(\xi_n)$$

The number of terms to be included in the Taylor series can be obtained for a given accuracy ε as

$$\left| \frac{h^{p+1}}{\lfloor p+1} y^{(p+1)}(\xi_n) \right| < \varepsilon$$

$$\Rightarrow \quad h^{p+1} \left| y^{(p+1)}(\xi_n) \right| < \varepsilon \lfloor p+1$$

Since ξ_n is unknown, we replace $|y^{(p+1)}(\xi_n)|$ by its maximum value in $[t_0, b]$, i.e.,

$$h^{p+1} \max_{t \in [t_0, b]} \left| y^{(p+1)}(t) \right| < \varepsilon \lfloor p+1$$

With this relation, for a given h, the number of term to be included in the Taylor's series can be obtained.

OR

if the number of terms are fixed, then h can be estimated for given accuracy.