INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR
MID SEMESTER EXAMINATION

Date: 22-09-2017 AN
Autumn Semester: 2000-2001
Subject No.: MA31005
No. of Students: 75

Time : 2 hours

Full Marks : 60
Department : Mathematics
Subject Name : Real Analysis
Course: M.Sc. 3rd Year (Maths. & Computing)/Breadth

Instructions: Answer all questions. Give complete arguments in support of your answers. No queries will be entertained during the exam. No marks will be awarded if explanations for your answers are not given.

1. Prove that $\sqrt{5}$ is not a rational number. 4M

2. Let $\mathbb{N}$ be the set of natural numbers. Find the infimum and supremum of each of the following sets of real numbers: 6M

(a) $E_1 = \left\{ \left(\frac{2}{3}\right)^{2m} + \left(\frac{1}{4}\right)^{n} : m, n \in \mathbb{N} \right\}$

(b) $E_2 = \left\{ \frac{n}{n+1} \cos(n\pi) : n = 0, 1, \ldots, 6 \right\}$

(c) $E_3 = \left\{ \frac{(-1)^n}{n+1} \sin\left(\frac{n\pi}{2}\right) : n \in \mathbb{N} \right\}$

3. Assuming the metric space to be $\mathbb{R}$ (the set of real numbers), determine the set of limit points of sets $E_i, i = 1, 2, 3$ defined in Q. No. 2. 6M

4. Define a Dedekind cut. Define order relation and show that the set of all Dedekind cuts is an ordered set. 4M

5. Define additive inverse of a Dedekind cut α by $\beta = \{ p \in \mathbb{Q} : \exists\, r > 0 \text{ such that } -p-r \notin \alpha \}$. Prove that β is a cut. 4M

6. Prove that in any metric space X, a set $E \subset X$ is open if and only if its complement is closed. 4M

7. Give construction of Cantor's set. Show that it is compact. 4M

8. For each of the following sets determine whether they are (i) open (ii) closed (iii) bounded (iv) compact (v) perfect (vi) connected. Present final conclusions in the form of a table, Also sketch all the sets graphically. 16M

(a) Metric space : $\mathbb{R}$, Set $E=\{x\in\mathbb{R}:|x|+|x+1|<2\}$

(b) Metric space : $\mathbb{R}$, Set $F=\{x\in\mathbb{R}:|2x-3|\geq 5\}$

(c) Metric space : $\mathbb{R}^2$, Set $G=\{(x,y)\in\mathbb{R}^2:|x|+|y|\leq 1\}$

(d) Metric space : $\mathbb{R}^2$, Set $H=\{(x,y)\in\mathbb{R}^2:|x|-|y|>1\}$

9. Let $E=\{\alpha,\beta:\alpha$ and β are roots of the equation $ax^2+bx+c=0$, for all $a,b,c\in\mathbb{Q}\}$. Determine if the set E is countable or uncountable. Here $\mathbb{Q}$ denotes the set of all rational numbers. 4M

10. Define (i) convergence of a sequence (ii) Cauchy sequence. Prove that in any metric space every convergent sequence is a Cauchy sequence. 4M

11. Let $\{p_n\}$ be a sequence in a metric space X. Prove that $\{p_n\}$ converges to $p\in X$ if and only if every neighbourhood of p contains all but finitely many points of $\{p_n\}$. 4M