INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR

Date ________________FN/AN, Time : 2 Hrs., Full Marks _30_, Deptt.________Mathematics_____

No. of Students ___83___, Mid Autumn Semester Examination __________2012-13___________

Sub. No. ______MA41007__________ Sub. Name ___Functional Analysis__________________

__4th_ Yr. (Integ. M.Sc./ Ist yr. M.Sc.-Ph.D./ 4th yr CS)

Instruction : Attempt All questions. Each question carries 5 marks.

1. Establish the following Inequalities :

(a) For $p \geq 1$, $(\sum_{k=1}^{n}|a_k|)^p \leq n^{p-1}\sum_{k=1}^{n}|a_k|^p$

(b) Let $p \geq 1$, $a_1, a_2, \ldots, a_n \geq 0$ and $b_1, b_2, \ldots, b_n \geq 0$, then

$$\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p}$$

2(a). Discuss whether $d(x,y) = |\lim_n (x_n - y_n)|$ is a metric on the set of all convergent sequences.

(b) Let $(X_1, d_1), (X_2, d_2)$ be metric spaces and let $x = (x_1, x_2), y = (y_1, y_2)$ be in the product set $X = X_1 \times X_2$. Define $d(x,y) = max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$.
Is (X,d) a metric space ? Justify your answer.

3(a). Prove that the space $C = C(R)$ convergent sequences of real numbers, with $d(x,y) = \sup_i |x_i - y_i|$; $x = (x_i)$, $y = (y_i)$ in C is complete.

(b) Prove that every convergent sequence in a metric space is a Cauchy Sequence but converse need not be true in general. Justify your answer.

4(a). If d is a metric on a vector space $X \neq \{0\}$ which is obtained from a norm, and $\bar{d}$ is defined by $\bar{d}(x,x) = 0$, $\bar{d}(x,y) = d(x,y) + 1$, $(x \neq y)$. Show that $\bar{d}$ can not be obtained from a norm.

(b) Prove that every finite dimensional Normed space is complete.

5(a) Let X *and* Y be metric spaces and $T: X \rightarrow Y$ a continuous mapping. Prove that the image of a compact subset M of X under T is compact.

(b) Let X *and* Y be metric spaces, X compact and $T: X \rightarrow Y$ bijective and continuous. Show that T is a homeomorphism.

6(a) Show that an Integral operator $T = C[0,1] \rightarrow C[0,1]$ by $y = Tx$ where $y(t) = \int_0^1 k(t,\tau)x(\tau)d\tau$ is a bounded linear operator under maximum norm where $k(t,\tau)$, kernel of T, is assumed to be continuous on $[0,1] \times [0,1]$.

(b) Show that the range $R(T)$ of a bounded linear operator $T = (X, \|.\|) \rightarrow (Y, \|.\|)$ need not be closed.