Another Class of Numerical Methods:

Given $\sigma(\xi) = \xi^k$. Find $\rho(\xi)$ so that the resulting linear multistep method is implicit.

looking for $\rho(\xi)$ of degree k.

For $k=2$:
$$\sigma(\xi) = \xi^2 = (\xi-1)^2 + 2(\xi-1) + 1$$

$$(\ln \xi)\,\sigma(\xi) = \left[(\xi-1) - \tfrac{1}{2}(\xi-1)^2 + \cdots\right]\left[(\xi-1)^2 + 2(\xi-1) + 1\right]$$

$$= (\xi-1) + \left(2 - \tfrac{1}{2}\right)(\xi-1)^2 + O(\xi-1)^3$$

$$= \tfrac{3}{2}\xi^2 - 2\xi + \tfrac{1}{2} + O(\xi-1)^3$$

The numerical method is given by

$$\rho(E)\,u_{j-1} - h\,\sigma(E)\,u'_{j-1} = 0$$

$$\Rightarrow \left[\tfrac{3}{2}E^2 - 2E + \tfrac{1}{2}\right]u_{j-1} - h\,E^2\,u'_{j-1} = 0$$

$$\Rightarrow \tfrac{3}{2}u_{j+1} - 2u_j + \tfrac{1}{2}u_{j-1} = h\,u'_{j+1}$$

ORDER OF THE METHOD = 2.

Ex. Given $\sigma(\xi) = (23\xi^2 - 16\xi + 5)/12$.

Find out $\rho(\xi)$ and write down an explicit linear multi-step method.

→ Adams-Bashforth method of order 3.

Ex. Derive a fourth order method of the form

$$u_{n+1} = a u_{n-2} + h(b u_n' + c u_{n-1}' + d u_{n-2}' + e u_{n-3}')$$

for the solution of $y' = f(x,y)$.

Sol: The local truncated error of the method is given by

$$\tau_{n+1} = y(x_{n+1}) - a y(x_{n-2}) - h[b y'(x_n) + c y'(x_{n-1}) + d y'(x_{n-2}) + e y'(x_{n-3})]$$

$$= y(x_n) + h y'(x_n) + \frac{h^2}{\underline{|2}} y''(x_n) + \frac{h^3}{\underline{|3}} y'''(x_n) + \frac{h^4}{\underline{|4}} y^{(iv)}(x_n) + O(h^5)$$

$$- a\left[y(x_n) - 2h y'(x_n) + \frac{4h^2}{\underline{|2}} y''(x_n) + \cdots\cdots + O(h^5)\right]$$

$$- h\left[b\{y'(x_n)\} + c\{y'(x_n) - h y''(x_n) + \frac{h^2}{\underline{|2}} y'''(x_n) - \cdots\}\right.$$

$$\left. + d\{\cdots\cdots\} + e\{\cdots\cdots\}\right]$$

To determine a, b, c, d, e, we have

$$\left.\begin{aligned} &1 - a = 0 \\ &1 + 2a - (b + c + d + e) = 0 \\ &\tfrac{1}{2}(1 - 4a) + (c + 2d + 3e) = 0 \\ &\tfrac{1}{6}(1 + 8a) - \tfrac{1}{2}(c + 4d + 9e) = 0 \\ &\tfrac{1}{24}(1 - 16a) + \tfrac{1}{6}(c + 8d + 27e) = 0 \end{aligned}\right\} \quad \begin{aligned} a &= 1 \\ b &= \tfrac{21}{8} \\ c &= -9/8 \\ d &= 15/8 \\ e &= -3/8 \end{aligned}$$

The method can be written as

$$u_{n+1} = u_{n-2} + \frac{h}{8}(21 u_n' - 9 u_{n-1}' + 15 u_{n-2}' - 3 u_{n-3}')$$

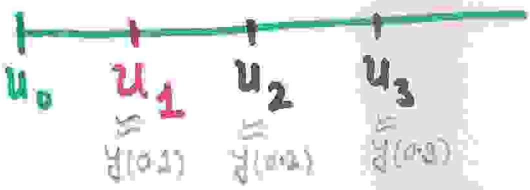
Ex. Find the solution at $x = 0.3$ for the differential equation

$$y' = x - y^2 \qquad y(0) = 1,$$

by the Adams-Bashforth Method of order two with $h = 0.1$. Determine the starting values using a second order Runge-Kutta method.

Sol: The second order Adams-Bashforth method is

$$u_{j+1} = u_j + \frac{h}{2}\left(3u_j' - u_{j-1}'\right) \qquad j = 1, 2, \ldots$$



We need to find the value of u_1 in order to start the computation. The second order Runge-Kutta method

$$u_{j+1} = u_j + \frac{h}{2}(K_1 + K_2)$$

$$K_1 = f(x_j, u_j)$$

$$K_2 = f(x_j + h, u_j + hK_1)$$

$$K_1 = 0 - 1^2 = -1$$

$$K_2 = f(0.1, 1 - 0.1) = 0.1 - (0.9)^2$$

$$= -0.71$$

$$u_1 = 1 + \frac{0.1}{2}(-1 - 0.71) = 0.9145$$

Using AB method:

$$u_2 = 0.9145 + \frac{0.1}{2}\Big[3\underbrace{(0.1 - 0.9145^2)}_{=0.73631} - (0 - 1^2)\Big] = 0.85405$$

$$u_3 = u_2 + \frac{h}{2}\left[3u_2' - u_1'\right]$$

$$= 0.85405 + \frac{0.1}{2}\left[3(0.2 - 0.85405^2) + 0.73631\right]$$

$$= 0.81146$$

PROBLEM: For the initial value problem

$$y' = t^2 + u^2 \qquad y(1) = 2$$

find an estimate for $y(1.2)$ using the Adams-Moulton third order method with $h = 0.1$. Use Taylor's series method of order 3 in order to determine starting values.

Solution: Adams-Moulton method:

$$u_{j+1} = u_j + \frac{h}{12}\left[5u_{j+1}' + 8u_j' - u_{j-1}\right] \qquad j = 1$$

$u_0 = y(1)$ $\qquad u_1 = y(1.1)$ $\qquad u_2 = y(1.2)$

We need to get u_1 to apply Adams-Moulton Method:

Taylor series method:

$$u_1 = u_0 + h u_0' + \frac{h^2}{2!} u_0'' + \frac{h^3}{3!} u_0'''$$

$h = 0.1$, $u_0 = 2$

$u_0' = t_0^2 + u_0^2 = 1^2 + 4 = 5$

$$u_0'' = 2t_0 + 2u_0u_0' = 2\times1 + 2\times2\times5 = 22$$

$$u_0''' = 2 + 2u_0'^2 + 2u_0u_0'' = 2 + 2\times25 + 2\times2\times22$$
$$= 140$$

Therefore

$$u_1 = 2 + 0.1\times5 + \frac{0.1^2}{\lfloor 2}\times22 + \frac{0.1^3}{\lfloor 3}\times140$$
$$= 2.633333$$

Using Adams-Moulton method:

$$u_2 = 2.633333 + \frac{0.1}{12}\left[5(1.2^2 + u_2^2) + 8(1.1^2 + 2.633333^2) - 5\right]$$

$$\Rightarrow u_2 = 0.041667\,u_2^2 + 3.194629$$

Newton Raphson method:

$$F(u_2) = 0.041667\,u_2^2 - u_2 + 3.194629$$

$$F'(u_2) = 0.083334\,u_2 - 1$$

$$u_2^{(s+1)} = u_2^{(s)} - \frac{F(u_2^{(s)})}{F'(u_2^{(s)})} \qquad s = 0, 1, 2 \ldots$$

$$u_2^{(0)} = u_1 = 2.633333$$

$$\Rightarrow u_2^{(3)} = 3.794588$$

$$y(1.2) \approx 3.794588$$