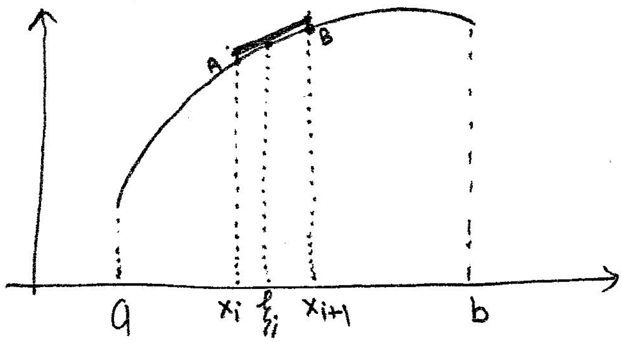
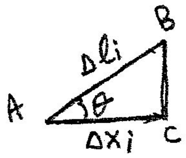
Computing the Area of a surface:

ONE dimension case: computation of curve length



length of the curve

$$L = \lim_{n\to\infty} \sum_{i=1}^{n-1} \Delta l_i$$



$$\tan\theta = f'(\xi_i)$$

From triangle: $\cos\theta = \dfrac{\Delta x_i}{\Delta l_i}$

$$\Rightarrow \Delta l_i = \Delta x_i \sec\theta \quad \text{—— (1)}$$

Also. $\cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}} \Rightarrow \sec\theta = \sqrt{1+f'(\xi_i)^2}$

$\Rightarrow$ from (1): $\Delta l_i = \Delta x_i \sqrt{1+f'(\xi_i)^2}$

So, $L = \lim_{n\to\infty} \sum_{i=1}^{n-1} \sqrt{1+f'(\xi_i)^2} \cdot \Delta x_i$

$$\boxed{L = \int_a^b \sqrt{1+f'^2}\, dx}$$

In two dimension we will take tangent plane instead of tangent line and similar to One dimensional case we get surface area:

$$S = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy.$$

if the equation is given in the form

$x = \mu(y, z)$ or in the form $y = \chi(x, z)$

then

$$S = \iint_{\tilde{D}} \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dy\, dz$$

$$S = \iint_{\breve{D}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dx\, dz :$$

where $\tilde{D}$ & $\breve{D}$ are the domain in the yz and xz plane in which the given surface is projected.

<u>Example:</u> Compute the surface area of the sphere

$$x^2 + y^2 + z^2 = R^2$$

<u>Solution:</u> equation of the surface

$$z = \sqrt{R^2 - x^2 - y^2}.$$

In this case : $\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}$

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$$

Domain of integration : $x^2 + y^2 \le R^2$.

$$S = 2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dy\,dx$$

$$\rightarrow \frac{R}{\sqrt{R^2-x^2-y^2}}$$

transformation to polar coordinate :

$$S = 2\int_{0}^{2\pi}\int_{0}^{R} \frac{R}{\sqrt{R^2-r^2}}\, r\, dr\, d\theta$$

$$= 2R \int_0^{2\pi} \left[-\sqrt{R^2-r^2}\right]_0^R d\theta$$

$$= 2R \int_0^{2\pi} R\, d\theta$$

$$= 2R^2\, 2\pi$$

$$= 4\pi R^2.$$

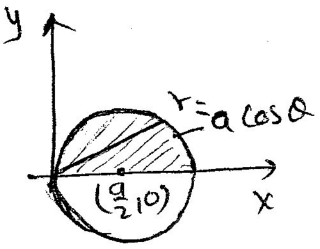
Ans.

Q: Find the area of that part of the sphere $x^2+y^2+z^2=a^2$ which is cut off by the cylinder $x^2+y^2=ax$.

Sol::

$$x^2+y^2=ax \Rightarrow \left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2.$$

$$z=\sqrt{a^2-x^2-y^2}$$



$$S = 2\cdot 2\cdot \int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$$

$$= 4\cdot a\cdot \int_0^{\pi/2} \left[-\sqrt{a^2-r^2}\right]_0^{a\cos\theta} d\theta$$

$$= 4a\cdot \int_0^{\pi/2} \left[-a\sin\theta + a\right] d\theta$$

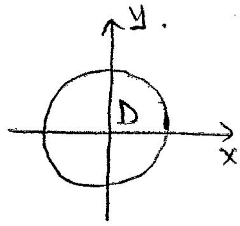
$$= 4a\cdot \left[\{a\cos\theta\}_0^{\pi/2} + a\{\theta\}_0^{\pi/2}\right]$$

$$= 4a\cdot\left[-a + a\pi/2\right]$$

$$= 2a^2(\pi-2)$$

Ans

Example! Determine the surface area of the part of $z = xy$ that lies in the cyl: $x^2+y^2=1$.



Solution: $f(x,y) = z = xy$.

$f_x = y \qquad f_y = x$.

S = Surface area: $= \iint_D \sqrt{1+x^2+y^2} \cdot dA$.

In polar coordinate.

$$S = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\sqrt{1+r^2} \cdot dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\left[\frac{2}{3}\cdot(1+r^2)^{3/2}\right]_0^1 \cdot d\theta$$

$$= \frac{2\pi}{3}\cdot\left(2\sqrt{2}-1\right)$$

ANS.