A minor of a matrix A is the determinant of some smaller square matrix, cut-down from A by removing one or more its rows or columns. Minors obtained by removing just one row and one column from square matrices are called first minors.

Let: $A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix} \rightarrow$ Some minors of A are.

$$\begin{vmatrix} 1 & 2 & 1 \\ 9 & 5 & 2 \\ 7 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 2 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 0 & 4 \end{vmatrix}, \quad \begin{vmatrix} 2 & 2 \\ 0 & 4 \end{vmatrix} \text{ etc.}$$

$B = \begin{pmatrix} 2 & 3 & 2 & 1 \\ 8 & 4 & 1 & 0 \\ 6 & 1 & 1 & 5 \\ 3 & -1 & 5 & 6 \end{pmatrix} \rightarrow$ First minors of A are.

$$\begin{vmatrix} 4 & 1 & 0 \\ 1 & 1 & 5 \\ -1 & 5 & 6 \end{vmatrix}, \quad \begin{vmatrix} 3 & 2 & 1 \\ 1 & 1 & 5 \\ -1 & 5 & 6 \end{vmatrix} \text{ etc.}$$

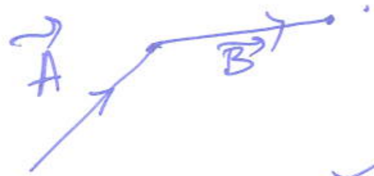
Det B is computed with the help of 1st minors.

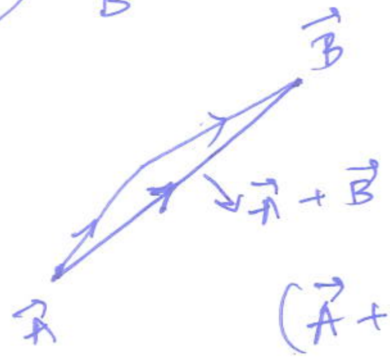
<u>Rank A</u>: The rank of a matrix A is 'r' if and only if A has some $r \times r$ submatrix with a nonzero determinant. and all square submatrics of larger size have determinant zero.

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix} \qquad \begin{vmatrix} 1 & 2 & 1 \\ 9 & 5 & 2 \\ 7 & 1 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 \\ 9 & 5 \end{vmatrix} \neq 0.$$

$\therefore$ rank $A = 2$.

Vector Space.





$\underset{\sim}{v}$

$\vec{A}+\vec{B}$ is another vector.

$\vec{B}+\vec{A} = \vec{A}+\vec{B}$.

$(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})$ (check geometri-cally)

$\vec{A}+\vec{O} = \vec{A}$.

$\vec{A}+(-\vec{A}) = \vec{O}$.

Def —

V ~~be~~ a non-empty set over a field F.
(we will always use F = R (real no. line))

Suppose two operations called vector addition '+' and scalar multiplication '·', are defined on V.
(scalars → elements of field F. In our case, scalars are real nos.)

Then V is said to be a vector space, if the elements of V satisfy the following properties

A1. If $\underset{\sim}{v}_1 \in V$, $\underset{\sim}{v}_2 \in V$, then $\underset{\sim}{v}_1 + \underset{\sim}{v}_2 \in V$.
(V is closed under vector addition)

A2. If $\underset{\sim}{v}_1, \underset{\sim}{v}_2 \in V$, then $\underset{\sim}{v}_1 + \underset{\sim}{v}_2 = \underset{\sim}{v}_2 + \underset{\sim}{v}_1$.
(V is commutative w.r.to '+').

A3. If $\underset{\sim}{v}_1, \underset{\sim}{v}_2, \underset{\sim}{v}_3 \in V$, $\underset{\sim}{v}_1 + (\underset{\sim}{v}_2 + \underset{\sim}{v}_3) = (\underset{\sim}{v}_1 + \underset{\sim}{v}_2) + \underset{\sim}{v}_3$.
(~~also~~ V is associative w.r.to '+')

A4. There exists ($\exists$) an identity element $\underset{\sim}{0} \in V$ (called zero vector) ~~ostead~~ such that

$$\underset{\sim}{v} + \underset{\sim}{0} = \underset{\sim}{v} \quad \underset{\text{(for all)}}{\forall} \ \underset{\sim}{v} \in V .$$

(note: the $\underset{\sim}{0}$ is unique for a particular $\underset{\sim}{V}$).

A5. $\exists$ an inverse element $(-\underset{\sim}{v}) \in V$ correspon-ding to each $\underset{\sim}{v} \in V$, such that

$$\underset{\sim}{v} + (-\underset{\sim}{v}) = \underset{\sim}{0} .$$

(note: inverse ~~of~~ $\underset{\sim}{v}$ is not unique, it depends on $\underset{\sim}{v}$).

M1. V is closed w.r.to scalar multiplication. if $k \in R(F)$, $\underset{\sim}{v} \in V$, then $k\underset{\sim}{v} \in V$.

M2. $k_1, k_2 \in F$, $\underset{\sim}{v} \in V$, then $(k_1+k_2)\underset{\sim}{v} = k_1\underset{\sim}{v} + k_2\underset{\sim}{v}$

M3. $k \in F$, $\underset{\sim}{v_1}, \underset{\sim}{v_2} \in V$, then $k(\underset{\sim}{v_1}+\underset{\sim}{v_2}) = k\underset{\sim}{v_1} + k\underset{\sim}{v_2}$.

M4. $k_1, k_2 \in F$, $\underset{\sim}{v} \in V$, then $k_1(k_2\underset{\sim}{v}) = (k_1k_2)\underset{\sim}{v}$

M5. Existence of multiplicative identity. $\forall v \in V$, $\exists$ '1' (multiplicative identity) $\in F$, such that $1.\underset{\sim}{v} = \underset{\sim}{v}$.

(You must try to prove — left as exercise)

Examples. 1. $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n)\}$ is a vector space with respect to the vector addition '+' and scalar multiplication '$\cdot$' defined as,

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$$

and,

$$k(x_1, x_2, \dots, x_n) = (kx_1, kx_2, \dots, kx_n).$$

2. Let V_{mn} = set of all $m \times n$ matrices. Then V_{mn} is also a vector space w.r.to usual matrix addition and multiplication of matrices by a scalar.

3. Let $P(x) = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n;\ a_i \in R,\ n \in N\}$ be the set of all polynomials. Then $P(x)$ is a vector space w.r.to usual addition of two polynomials and multiplication by a scalar.

4. $V_f \to$ set of all functions defined on real line.

Then V_f is a vector space w.r.to the ~~scalar~~ vector addition & scalar multiplication defined as,

$$(f+g)(x) = f(x) + g(x)$$

$$(kf)(x) = k \cdot f(x);\ k \to \text{scalar}.$$

To show

$\mathbb{R}^2$ is a vector space w.r.to '+', scalar mult. (·) defined as

$(x_1, \cancel{x_2}) + (\cancel{y_1}, y_2) = (x_1 + y_1, x_2 + y_2)$.

and $k(x_1, x_2) = (kx_1, kx_2)$; $k \in R$.

A1. Let $\underset{\sim}{v_1} = (x_1, x_2)$, $\underset{\sim}{v_2} = (y_1, y_2) \in \mathbb{R}^2$

$\underset{\sim}{v_1} + \underset{\sim}{v_2} = (x_1 + y_1, x_2 + y_2) \in \mathbb{R}^2$.

$\therefore$ ~~(+) is~~ $\mathbb{R}^2$ is closed w.r.to vector addition '+'.

A2. Let $\underset{\sim}{v_1} = (x_1, x_2)$, $\underset{\sim}{v_2} = (y_1, y_2) \in \mathbb{R}^2$.

$\underset{\sim}{v_1} + \underset{\sim}{v_2} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

$\underset{\sim}{v_2} + \underset{\sim}{v_1} = (y_1, y_2) + (x_1, x_2) = (y_1 + x_1, y_2 + x_2)$

$= (x_1 + y_1, x_2 + y_2)$

$= \underset{\sim}{v_1} + \underset{\sim}{v_2}$.

$\therefore$ $\mathbb{R}^2$ is commutative w.r.to '+'.

A3. Let $\underset{\sim}{v_1} = (x_1, x_2)$, $\underset{\sim}{v_2} = (y_1, y_2)$, $\underset{\sim}{v_3} = (z_1, z_2) \in \mathbb{R}^2$.

$(\underset{\sim}{v_1} + \underset{\sim}{v_2}) + \underset{\sim}{v_3} = (x_1 + y_1, x_2 + y_2) + (z_1, z_2)$

$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2)$.

$= (x_1 + y_1 + z_1, x_2 + y_2 + z_2)$

$\underset{\sim}{v_1} + (\underset{\sim}{v_2} + \underset{\sim}{v_3}) = (x_1, x_2) + (y_1 + z_1, y_2 + z_2)$

$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2))$

$= (x_1 + y_1 + z_1, x_2 + y_2 + z_2)$

$\therefore (\underset{\sim}{v_1} + \underset{\sim}{v_2}) + \underset{\sim}{v_3} = \underset{\sim}{v_1} + (\underset{\sim}{v_2} + \underset{\sim}{v_3})$

$\mathbb{R}^2$ is associative w.r.to '+'.

A4. Let $\underset{\sim}{v} = (x_1, x_2) \in \mathbb{R}^2$.

Note, $(x_1, x_2) + (0,0) = (x_1 + 0, x_2 + 0)$
$= (x_1, x_2)$

$\therefore \underset{\sim}{0} = (0,0)$ is the identity element of $\mathbb{R}^2$.

A5. Let $\underset{\sim}{v} = (x_1, x_2) \in \mathbb{R}^2$.

Now, $(x_1, x_2) + (-x_1, -x_2) = (x_1 + (-x_1), x_2 + (-x_2))$
$= (x_1 - x_1, x_2 - x_2) = (0,0)$

$\therefore (-x_1, -x_2)$ is an inverse element w.r.to (x_1, x_2)

i.e for every $\underset{\sim}{v} \in \mathbb{R}^2$, there exists $-\underset{\sim}{v} \in \mathbb{R}^2$, such that $\underset{\sim}{v} + (-\underset{\sim}{v}) = \underset{\sim}{0}$.

M1. Let $\underset{\sim}{v} = (x_1, x_2) \in \mathbb{R}^2$.

Then $k\underset{\sim}{v} = k(x_1, x_2) = (kx_1, kx_2) \in \mathbb{R}^2$.

M2. Let $\underset{\sim}{v} = (x_1, x_2) \in \mathbb{R}^2$.

$(k_1 + k_2)\underset{\sim}{v}$
$= k_1\underset{\sim}{v} + k_2\underset{\sim}{v}$

$(k_1 + k_2)\underset{\sim}{v} = (k_1 + k_2)(x_1, x_2)$
$= ((k_1 + k_2)x_1, (k_1 + k_2)x_2)$
$= (k_1x_1 + k_2x_1, k_1x_2 + k_2x_2) \longrightarrow (A)$

$k_1\underset{\sim}{v} = k_1(x_1, x_2) = (k_1x_1, k_1x_2)$
$k_2\underset{\sim}{v} = k_2(x_1, x_2) = (k_2x_1, k_2x_2)$

$\therefore k_1\underset{\sim}{v} + k_2\underset{\sim}{v} = (k_1x_1, k_1x_2) + (k_2x_1, k_2x_2)$
$= (k_1x_1 + k_2x_1, k_1x_2 + k_2x_2) \longrightarrow (B)$

From (A) & (B),
$(k_1 + k_2)\underset{\sim}{v} = k_1\underset{\sim}{v} + k_2\underset{\sim}{v}$.

M3. $k(\underset{\sim}{v}_1 + \underset{\sim}{v}_2) = k\underset{\sim}{v}_1 + k\underset{\sim}{v}_2$

$\underset{\sim}{v}_1 = (x_1, x_2), \quad \underset{\sim}{v}_2 = (y_1, y_2).$

$\underset{\sim}{v}_1 + \underset{\sim}{v}_2 = (x_1 + y_1, x_2 + y_2).$

$$k(\underset{\sim}{v}_1 + \underset{\sim}{v}_2) = k(x_1 + y_1, x_2 + y_2)$$
$$= (kx_1 + ky_1, kx_2 + ky_2) \longrightarrow (C)$$

$k\underset{\sim}{v}_1 = (kx_1, kx_2), \quad k\underset{\sim}{v}_2 = (ky_1, ky_2).$

$\therefore k\underset{\sim}{v}_1 + k\underset{\sim}{v}_2 = (kx_1 + ky_1, kx_2 + ky_2) \longrightarrow (D)$

from (C) & (D), $k(\underset{\sim}{v}_1 + \underset{\sim}{v}_2) = k\underset{\sim}{v}_1 + k\underset{\sim}{v}_2$.

M4. $(k_1 k_2)\underset{\sim}{v} = k_1(k_2\underset{\sim}{v})$. $\quad \underset{\sim}{v} = (x_1, x_2)$

$$k_1(k_2\underset{\sim}{v}) = k_1(k_2(x_1, x_2))$$
$$= k_1(k_2 x_1, k_2 x_2)$$
$$= (k_1(k_2 x_1), k_1(k_2 x_2))$$
$$= (k_1 k_2 x_1, k_1 k_2 x_2)$$

$$(k_1 k_2)\underset{\sim}{v} = (k_1 k_2)(x_1, x_2)$$
$$= (k_1 k_2 x_1, k_1 k_2 x_2).$$

$k_1(k_2\underset{\sim}{v}) = (k_1 k_2)\underset{\sim}{v}$

M5. $1.\underset{\sim}{v} = \underset{\sim}{v}$.

If $\underset{\sim}{v} = (x_1, x_2) \in \mathbb{R}^2$,

then $1.(x_1, x_2) = (x_1, x_2)$

$\therefore$ the number 1 is the multiplicative identity.

Note: 1. Additive identity is vector.

2. Multiplicative " is a scalar.

3. Elements of vector space are called vectors.

Prove or disprove.

Ex. $R^+ \to$ set of all real nos. is a vector space w.r.to usual addition of real no.s and multiplication by scalars (real no.s)

Prove that

Ex. R^+ is a vector space w.r.to the addition $x+y = xy$ & $k.x = x^k$.

Note $x+1 = x.1 = x$. $\therefore$ '1' is the additive identity.

$x + \left(\frac{1}{x}\right) = x.\frac{1}{x} = 1$. $\therefore$ $\frac{1}{x}$ is the additive inverse w.r.to x.

$1.x = x^1 = x$.

'1' is the multiplicative identity.

Exercise $V = \{moon\}$. Define '+' and (.) on V like moon + moon = moon.

and. $k(moon) = moon$, k is a real no.

Is it a vector space?

Subspace.

let V be a vector space & $W \subset V$ be non-empty.

Definition. W is said to be a sub-space of V if W is itself a vector space w.r.to the same vector addition & scalar multiplication defined on V.

Thm. W is a subspace of V if.

1) $\underset{\sim}{0} \in W$ ($\underset{\sim}{0}$ is the identity element of V)
2) $k\underset{\sim}{w} \in W \quad \forall\ \underset{\sim}{w} \in W,\ k \in R$.
3) $(\underset{\sim}{w_1} + \underset{\sim}{w_2}) \in W \quad \forall\ \underset{\sim}{w_1}, \underset{\sim}{w_2} \in W$.

Note. 2) & 3) can be combined as,

if $k_1, k_2 \in R$ & $\underset{\sim}{w_1}, \underset{\sim}{w_2} \in W$,

$$k_1 \underset{\sim}{w_1} + k_2 \underset{\sim}{w_2} \in W \quad (*)$$

(Most of the time it is enough to prove 1) & (*))

Ex 1. Show that $W = \{(a, b, 0);\ a, b \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

Sol. $\underset{\sim}{0} = (0, 0, 0) \in W$.

$w_1 = (a_1, b_1, 0), \quad w_2 = (a_2, b_2, 0) \in W$.

$\therefore k_1 w_1 = (k_1 a_1, k_1 b_1, 0), \ k_2 w_2 = (k_2 a_2, k_2 b_2, 0)$

$\therefore k_1 w_1 + k_2 w_2 = (k_1 a_1 + k_2 a_2,\ k_1 b_1 + k_2 b_2,\ 0) \in W$

$\therefore$ W is a subspace of $\mathbb{R}^3$.

Ex 2. ~~$W = \{(a+1, b, 0); a, b \in \mathbb{R}\}$~~

$W = \{(a, b, 1); a, b \in \mathbb{R}\}$.

$(0,0,0) \notin W$.

$\therefore$ W is not a subspace.

Ex 3. $V^{2\times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$ ~~$\in V^{2\times 2}$~~

$W \subset V$ such that $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \right\}$

$\underset{\sim}{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$.

Note

$\underset{\sim}{w}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W \quad \because \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$.

$\underset{\sim}{w}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in W \quad \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$.

$\underset{\sim}{w}_1 + \underset{\sim}{w}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$,

$\therefore \underset{\sim}{w}_1 + \underset{\sim}{w}_2 \notin W$.

Ex 4. (exercise). W be set of all symmetric matrices (2×2).

Is it a subspace of $V^{2\times 2}$.

Ex-5. $P(t)$ = ~~set~~ set of all polynomials is vector space.

Let $P_n(t) \rightarrow$ set of all polynomials of degree $\leq n$.

$P_5(t) \rightarrow$ set " " " " degree ≤ 5

$2+5t \in P_5(t)$, $3-4t+3t^2-5t^5 \in P_5(t)$

- - - -

Check. $P_n(t)$ is a subspace of $P(t)$.

$Q_n(t) \rightarrow$ set of all polynomials of degree $= n$.

then $Q_n(t)$ is not a subspace.

$Q_5(t) \Rightarrow$ set of all polynomial of degree $= 5$

$q_1(t) = 2+3t-5t^2+6t^3-4t^4+7t^5 \in Q_5(t)$

$q_2(t) = 5t-9t^2+6t^4-7t^5 \in Q_5(t)$

$q_1(t)+q_2(t) = 2+8t-14t^2+6t^3+2t^4 \notin Q_5(t)$

Exercise-1. $V(f)$ be the vector space of all continuous functions $f: R \rightarrow R$.

Show that 1) $W = \{f : f(6) = f(3)\}$ is a subspace of $V(f)$

2) $W = \{f : f(6) = f(3)+2\}$ is not a subspace

Exercise 2 Check whether $W = \{(a,b,0) : a \leq 0\}$ is a s.sp. of $\mathbb{R}^3$.