2.4 Formal Power Series

In this chapter, we will first try to develop the theory of generating functions by getting closed form expressions for some known recurrence relations. These ideas will be used later to get some binomial identities.

To do so, we first recall from Page 41 that for all $n \in \mathbb{Q}$ and $k \in \mathbb{Z}$, $k \geq 0$, the binomial coefficients, $\binom{n}{k}$, are well defined, using the idea that $\binom{n}{k} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$. We now start with the definition of "formal power series" over $\mathbb{Q}$ and study its properties in some detail.

Definition 2.4.1 (Formal power series)**.** *An algebraic expression of the form $f(x) = \sum\limits_{n\geq 0} a_n x^n$, where $a_n \in \mathbb{Q}$ for all $n \geq 0$, is called a formal power series in the indeterminate x over $\mathbb{Q}$.*

The set of all formal power series in the indeterminate x, with coefficients from $\mathbb{Q}$ will be denoted by $\mathcal{P}(x)$.

Remark 2.4.2.

1. *Given a sequence of numbers $\{a_n \in \mathbb{Q} : n = 0, 1, 2, \ldots\}$, one associates two formal power series, namely, $\sum\limits_{n\geq 0} a_n x^n$ and $\sum\limits_{n\geq 0} a_n \dfrac{x^n}{n!}$. The expression $\sum\limits_{n\geq 0} a_n x^n$ is called the* generating function *and the expression $\sum\limits_{n\geq 0} a_n \dfrac{x^n}{n!}$ is called the* exponential generating function*, for the numbers $\{a_n : n \geq 0\}$.*

2. *Let $f(x) = \sum\limits_{n\geq 0} a_n x^n$ be a formal power series. Then the coefficient of x^n, for $n \geq 0$, in $f(x)$ is denoted by $[x^n]f(x)$. That is, $a_0 = [x^0]f(x)$ and $a_n = [x^n]f(x)$, for $n \geq 1$.*

3. *One thinks of $\sum\limits_{n\geq 0} a_n x^n$ as an algebraic expression. In general, one is interested only in computing the coefficient of certain power of x and not in evaluating them for any value of x. But, if at all there is a need to evaluate it at a point, say x_0, then one needs to determine its "radius of convergence" and then evaluate it if x_0 lies within that radius.*

We need the following definition to proceed further.

Definition 2.4.3 (Equality of two formal power series)**.** *Two elements $f(x) = \sum\limits_{n\geq 0} a_n x^n$ and $g(x) = \sum\limits_{n\geq 0} b_n x^n$ of $\mathcal{P}(x)$ are said to be equal if $a_n = b_n$, for all $n \geq 0$.*

We are now ready to define the algebraic rules:

Definition 2.4.4. *Let $f(x) = \sum\limits_{n\geq 0} a_n x^n, g(x) = \sum\limits_{n\geq 0} b_n x^n \in \mathcal{P}(x)$. Then their*

1. *sum/addition is defined by*

$$f(x) + g(x) = \sum_{n\geq 0} a_n x^n + \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0} (a_n + b_n) x^n.$$

2. product is defined by

$$f(x)\cdot g(x) = \left(\sum_{n\ge 0} a_n x^n\right)\cdot\left(\sum_{n\ge 0} b_n x^n\right) = \sum_{n\ge 0} c_n x^n, \quad \textit{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad \textit{for } n \ge 0.$$

This product is also called the Cauchy *product.*

Remark 2.4.5. *1. In case of exponential power series, the product of* $f(x) = \sum_{n\ge 0} a_n \frac{x^n}{n!}$ *and* $g(x) = \sum_{n\ge 0} b_n \frac{x^n}{n!}$ *equals* $\sum_{n\ge 0} d_n \frac{x^n}{n!}$, *where* $d_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$, *for* $n \ge 0$.

2. Note that the expression e^{e^x-1} *is a well defined formal power series as the definition* $e^y = \sum_{n\ge 0} \frac{y^n}{n!}$ *implies that* $e^{e^x-1} = \sum_{n\ge 0} \frac{(e^x-1)^n}{n!}$ *and hence*

$$[x^m]e^{e^x-1} = [x^m]\sum_{n\ge 0} \frac{(e^x-1)^n}{n!} = \sum_{n=0}^{m} [x^m]\,\frac{(e^x-1)^n}{n!}. \tag{2.1}$$

That is, for each $m \ge 0$, $[x^m]e^{e^x-1}$ *is a sum of a finite number of real numbers. Where as the expression* e^{e^x} *is not a formal power series as the computation of* $[x^m]e^{e^x}$, *for all* $m \ge 0$, *will indeed require an infinite sum.*

Thus, under the algebraic operations defined above, it can be checked that the set $\mathcal{P}(x)$ forms a *Commutative Ring* with identity, where the identity element is given by the formal power series $f(x) = 1$. In this ring, the element $f(x) = \sum_{n\ge 0} a_n x^n$ is said to have a *reciprocal* if there exists another element $g(x) = \sum_{n\ge 0} b_n x^n \in \mathcal{P}(x)$ such that $f(x)\cdot g(x) = 1$. So, the question arises, under what conditions on the coefficients of $f(x)$, can we find $g(x) \in \mathcal{P}(x)$ such that $f(x)g(x) = 1$. The answer to this question is given in the following proposition.

Proposition 2.4.6. *Let* $f(x) = \sum_{n\ge 0} a_n x^n \in \mathcal{P}(x)$. *Then there exists* $g(x) \in \mathcal{P}(x)$ *satisfying* $f(x)\cdot g(x) = 1$ *if and only if* $a_0 \ne 0$.

Proof. Let $g(x) = \sum_{n\ge 0} b_n x^n \in \mathcal{P}(x)$. Then, by the definition of Cauchy product, $f(x)g(x) = \sum_{n\ge 0} c_n x^n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, for all $n \ge 0$. Therefore, using the definition of equality of two power series, we see that $f(x)g(x) = 1$ if and only if $c_0 = 1$ and $c_n = 0$, for all $n \ge 1$.

Therefore, if $a_0 = 0$ then $c_0 = 0$ and hence the Cauchy product $f(x)g(x)$ can never equal 1. However, if $a_0 \ne 0$, then the coefficients b_n's can be recursively obtained as follows:

$b_0 = \dfrac{1}{a_0}$ as $1 = c_0 = a_0 b_0$.

$b_1 = \dfrac{-1}{a_0}\cdot(a_1 b_0)$ as $0 = c_1 = a_0 b_1 + a_1 b_0$.

$b_2 = \dfrac{-1}{a_0} \cdot (a_2 b_0 + a_1 b_1)$ as $0 = c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$. And in general, if we have already computed the values of b_k, for $k \leq r$, then

$$b_{r+1} = \frac{-1}{a_0} \cdot (a_{r+1} b_0 + a_r b_1 + \cdots + a_1 b_r) \text{ as } 0 = c_{r+1} = a_{r+1} b_0 + a_r b_1 + \cdots + a_1 b_r + a_0 b_{r+1}.$$

■

Note that if the coefficients a_n's come from $\mathcal{R}$, a commutative ring with unity, then one needs b_0 to be an invertible element of $\mathcal{R}$ for Proposition 2.4.6 to hold true. Let us now look at the composition of two formal power series. Recall that, if $f(x) = \sum\limits_{n \geq 0} a_n x^n, g(x) = \sum\limits_{n \geq 0} b_n x^n \in \mathcal{P}(x)$ then the composition $(f \circ g)(x) = f(g(x)) = \sum\limits_{n \geq 0} a_n (g(x))^n = \sum\limits_{n \geq 0} a_n (\sum\limits_{m \geq 0} b_m x^m)^n$ may not be defined (just to compute the constant term of the composition, one may have to look at an infinite sum). For example, let $f(x) = e^x$ and $g(x) = x + 1$. Note that $g(0) = 1 \neq 0$. Here, $(f \circ g)(x) = f(g(x)) = f(x+1) = e^{x+1}$. So, as function $f \circ g$ is well defined, but there is no formal procedure to write e^{x+1} as $\sum\limits_{k \geq 0} a_k x^k \in \mathcal{P}(x)$ (*i.e.*, with $a_k \in \mathbb{Q}$) and hence e^{x+1} is not a formal power series over $\mathbb{Q}$.

The next result gives the condition under which the composition $(f \circ g)(x)$ is well defined.

Proposition 2.4.7. *Let $f(x) = \sum\limits_{n \geq 0} a_n x^n$ and $g(x) = \sum\limits_{n \geq 0} b_n x^n$ be two formal power series. Then the composition $(f \circ g)(x)$ is well defined if either f is a polynomial or $b_0 = 0$.*

Moreover, suppose that $a_0 = 0$. Then, there exists $g(x) = \sum\limits_{n \geq 0} b_n x^n$, with $b_0 = 0$, such that $(f \circ g)(x) = x$. Furthermore, $(g \circ f)(x)$ is well defined and $(g \circ f)(x) = x$.

Proof. Let $(f \circ g)(x) = f(g(x)) = \sum\limits_{n \geq 0} c_n x^n$ and suppose that either f is a polynomial or $b_0 = 0$. Then to compute $c_k = [x^k] \ (f \circ g)(x)$, for $k \geq 0$, one just needs to consider the terms $a_0 + a_1 g(x) + a_2 (g(x))^2 + \cdots + a_k (g(x))^k$. Hence, each c_k is a real number and $(f \circ g)(x)$ is well defined. This completes the proof of the first portion. The proof of the other part is left to the readers. ■

We now define the formal differentiation of elements of $\mathcal{P}(x)$ and state a result without proof.

Definition 2.4.8 (Differentiation)**.** *Let $f(x) = \sum\limits_{n \geq 0} a_n x^n \in \mathcal{P}(x)$. Then the formal differentiation of $f(x)$, denoted $Df(x)$, is defined by*

$$Df(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \cdots = \sum_{n \geq 1} n a_n x^{n-1}.$$

Proposition 2.4.9. *Let $f(x) = \sum\limits_{n \geq 0} a_n x^n \in \mathcal{P}(x)$. Then $f(x) = a_0$, a constant, whenever $Df(x) = 0$. Also, $f(x) = a_0 e^x$ whenever $Df(x) = f(x)$.*