MA10002 Mathematics-II : Tutorial Sheet - 6

1. Determine if each of the following integrals converge or diverge. If the integral converges determine its value.
   (i) $\int_0^\infty (1+2x)\, e^{-x} dx$ (ii) $\int_{-\infty}^1 \sqrt{6-x}\, dx$ (iii) $\int_{-\infty}^\infty \frac{6x^3}{(x^4+1)^2} dx$.

2. Examine the convergence or divergence of the following integrals. If the integral converges determine its value.
   (i) $\int_{-5}^1 \frac{1}{10+2x} dx$ (ii) $\int_1^2 \frac{4x}{\sqrt[3]{x^2-4}} dx$ (iii) $\int_0^4 \frac{x}{x^2-9} dx$ (iv) $\int_0^1 \log t\, dt$ (v) $\int_{-2}^3 \frac{dx}{x-1}$.

3. Test the integral $\int_0^3 \frac{1}{x^2-3x+2} dx$ for its convergence.

4. Discuss the convergence of the following integrals.
   (i) $\int_1^\infty \frac{1}{x^3+1} dx$ (ii) $\int_6^\infty \frac{x^2+1}{x^3(\cos^2 x+1)} dx$ (iii) $\int_2^\infty \frac{1}{\log x} dx$ (iv) $\int_0^\infty e^{-x^2} dx$.

5. Test the integral $\int_1^\infty \frac{x-1}{x^4+2x^2} dx$, if it is convergent or divergent.

6. Test the convergence or divergence of the integral $\int_1^\infty \frac{x \tan^{-1} x}{\sqrt{4+x^3}} dx$.

7. Examine the convergence or divergence of the following integrals.
   (i) $\int_0^{\frac{\pi}{2}} \frac{\cos^m x}{x^n} dx, n < 1$ (ii) $\int_1^{\frac{\pi}{2}} \frac{\tan x}{x^{3/2}} dx$.

8. Determine if the following integrals converge or diverge.
   (i) $\int_2^5 \frac{x-1}{\sqrt{x}(x-2)} dx$ (ii) $\int_1^2 \frac{\sqrt{x}}{\ln x} dx$.

9. Show that the integral $\int_0^1 \frac{1}{(1+x)(2+x)\sqrt{x(1-x)}} dx$ is convergent.

10. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$, if it is convergent.

11. Show that $\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, $k^2 < 1$ is convergent.

12. Discuss the convergence of the integral $\int_1^\infty f(x)\, dx$, where the function $f(x)$ is given by as follows:

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \text{ is rational number} \\ -\frac{1}{x^2} & \text{if } x \text{ is irrational number} \end{cases}$$

13. Prove that $\int_1^\infty e^{-x} x^{m-1}\, dx$ is convergent for $m > 0$.

14. Show that $\int_1^\infty \sin x \log(\sin x) dx$ converges and find its value.

15. Find the value of the integrals $\int_0^{\frac{\pi}{2}} \log(\sin x) dx$ and $\int_0^{\frac{\pi}{2}} \log(\cos x) dx$ by discussing their convergence.

16. Show that the integral $\int\limits_{-1}^{1} \frac{\sin x}{x} dx$ is a proper integral.

17. Show that $\int\limits_{1}^{\infty} \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x} dx = \frac{\pi}{2} \log(\frac{a}{b}), 0 < b < a$.

18. Let $f(x, t) = (2x + t^3)^2$ then

    (i) find $\int\limits_{0}^{1} f(x, t)\ dx$

    (ii) Prove that $\frac{d}{dt} \int\limits_{0}^{1} f(x, t)\ dx = \int\limits_{0}^{1} \frac{\partial}{\partial t} f(x, t)\ dx$

19. i) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,t) = \begin{cases} \frac{\sin xt}{t} & \text{if } t \neq 0 \\ x & \text{if } t = 0 \end{cases}$$

    Find F', where $F(x) = \int\limits_{0}^{\frac{\pi}{2}} f(x, t)\ dt$.

    ii) Given $f : x \to \int\limits_{0}^{x^2} \tan^{-1} \frac{t}{x}\ dt$, find f'.

20. For any real numbers x and t, let

$$f(x,t) = \begin{cases} \frac{xt^3}{(x^2+t^2)^2} & \text{if } x \neq 0, t \neq 0 \\ 0 & \text{if } x = 0, t = 0 \end{cases}$$

    and $F(t) = \int\limits_{0}^{1} f(x, t)\ dx$. Is $\frac{d}{dt} \int\limits_{0}^{1} f(x, t)\ dx = \int\limits_{0}^{1} \frac{\partial}{\partial t} f(x, t)\ dx$? Give the justification.

21. Find the value of the integral $\int\limits_{0}^{\infty} \frac{e^{-bx} \sin ax}{x} dx$, where $a > 0, b > 0$ are fixed, and hence deduce the value of the integral $\int\limits_{0}^{\infty} \frac{\sin ax}{x} dx$.

22. Find the value of the following integrals

    i) $\int\limits_{0}^{\infty} \frac{e^{-bx}(1 - \cos ax)}{x} dx, b > 0$

    ii) $\int\limits_{0}^{\frac{\pi}{2}} \log(1 - x^2 \sin^2 \theta)\ d\theta, |x| < 1$

    iii) $\int\limits_{0}^{\infty} \frac{e^{-px} \cos qx - e^{-ax} \cos bx}{x}\ dx$

    iv) $\int\limits_{0}^{\infty} e^{-x^2} \cos 2ax\ dx$

Improper Integrals (Assignment) (6)

(1) (i) $\int_0^{\infty} (1+2x)e^{-x}\,dx$

$= \lim_{t\to\infty} \int_0^t (1+2x)e^{-x}\,dx$

Now, $\int_0^t (1+2x)e^{-x}\,dx = -(1+2x)e^{-x} + 2\int e^{-x}\,dx$

$= -(1+2x)e^{-x} - 2e^{-x} + C$

$= -(3+2x)e^{-x} + C$

So, $\int_0^{\infty} (1+2x)e^{-x}\,dx = \lim_{t\to\infty} \left[-(3+2x)e^{-x}\right]\Big|_0^t$

$= \lim_{t\to\infty} \left(3 - (3+2t)e^{-t}\right)$

$= 3 - \lim_{t\to\infty} \frac{3+2t}{e^t}$

$= 3 - \left[0 + 2\lim_{t\to\infty} \frac{t}{e^t}\right]$ ($\frac{\infty}{\infty}$ form L'Hospital rule)

$= 3 - \left[2\lim_{t\to\infty} \frac{1}{e^t}\right]$

$= 3 - 0 = 3$

$\therefore$ The integral converges and its value is 3.

(ii) $\int_{-\infty}^{1} \sqrt{6-x}\,dx = \lim_{t\to-\infty} \int_t^1 \sqrt{6-x}\,dx$

Now $\int \sqrt{6-x}\,dx = -\frac{2}{3}(6-x)^{3/2} + C$

So, $\int_{-\infty}^{1} \sqrt{6-x}\, dx = \lim_{t \to -\infty} \left[-\frac{2}{3} (6-x)^{3/2} \right] \Big|_{t}^{1}$

$$= \lim_{t \to -\infty} \left[-\frac{2}{3} (5)^{3/2} + \frac{2}{3} (6-t)^{3/2} \right]$$

$$= -\frac{2}{3} (5)^{3/2} + \frac{2}{3} \lim_{t \to -\infty} (6-t)^{3/2}$$

$$= \infty .$$

Thus, the integral diverges.

(iii) $\int_{-\infty}^{\infty} \frac{6x^3}{(x^4+1)^2}\, dx$

Since we have infinities in both the limits we'll need to split up the integral. We shall use $x=0$ as the split point. splitting up the integral gives

$$\int_{-\infty}^{\infty} \frac{6x^3}{(x^4+1)^2}\, dx = \int_{-\infty}^{0} \frac{6x^3}{(x^4+1)^2}\, dx + \int_{0}^{\infty} \frac{6x^3}{(x^4+1)^2}\, dx$$

lim

So, now we can eliminate the infinities as

$$\int_{-\infty}^{\infty} \frac{6x^3}{(x^4+1)^2}\, dx = \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{6x^3}{(x^4+1)^2}\, dx$$

$$+ \lim_{t_2 \to \infty} \int_{0}^{t_2} \frac{6x^3}{(x^4+1)^2}\, dx$$

Now, $\int \frac{6x^3}{(x^4+1)^2}\, dx = -\frac{3}{2} \frac{1}{x^4+1} + C$

(B)

So, $\int_{-\infty}^{\infty} \frac{6x^3}{(x^4+1)^2}\,dx = \lim_{t_1\to-\infty}\left(-\frac{3}{2}\cdot\frac{1}{x^4+1}\right)\Big|_{t_1}^{0}$

$$+ \lim_{t_2\to\infty}\left(-\frac{3}{2}\cdot\frac{1}{x^4+1}\right)\Big|_{0}^{t_2}$$

$$= \lim_{t_1\to-\infty}\left(-\frac{3}{2}+\frac{3}{2}\cdot\frac{1}{t_1^4+1}\right) + \lim_{t_2\to\infty}\left(-\frac{3}{2}\cdot\frac{1}{t_2^4+1}+\frac{3}{2}\right)$$

$$= -\frac{3}{2}+\frac{3}{2} = 0$$

Thus, the integral converges and its value is 0.

(2) (i) $\int_{-5}^{1} \frac{1}{10+2x}\,dx$

There is a discontinuity in the integrand at $x=-5$. We'll need to eliminate the discontinuity first as follows

$$\int_{-5}^{1} \frac{1}{10+2x}\,dx = \lim_{t\to5^+}\int_{t}^{1}\frac{1}{10+2x}\,dx$$

Now, $\int \frac{1}{10+2x}\,dx = \frac{1}{2}\ln|10+2x| + C$

Thus, $\int_{-5}^{1}\frac{1}{10+2x}\,dx = \lim_{t\to5^+}\left[\frac{1}{2}\ln|10+2x|\right]\Big|_{t}^{1}$

$$= \lim_{t\to5^+}\left(\frac{1}{2}\ln|12| - \frac{1}{2}\ln|10+2t|\right)$$

$$= \frac{1}{2}\ln|12| + \infty = \infty.$$

Thus, the integral diverges

(ii) $\int_1^2 \frac{4x}{\sqrt[3]{x^2-4}}\,dx$

There is an infinite discontinuity in the integrand at $x = 2$

We'll eliminate the discontinuity as follows

$$\int_1^2 \frac{4x}{\sqrt[3]{x^2-4}}\,dx = \lim_{t \to 2^-} \int_1^t \frac{4x}{\sqrt[3]{x^2-4}}\,dx$$

$$\int \frac{4x}{\sqrt[3]{x^2-4}}\,dx = 3(x^2-4)^{2/3} + c$$

Thus, $\int_1^2 \frac{4x}{\sqrt[3]{x^2-4}}\,dx = \lim_{t \to 2^-} \left[3(t^2-4)^{2/3} - 3(-3)^{2/3} \right]$

$$= -3(-3)^{2/3} = (-3)^{5/3}$$

Thus, the integral converges and its value is $(-3)^{5/3}$.

(iii) $\int_0^4 \frac{x}{x^2-9}\,dx$

There is an infinite discontinuity in the integrand at $x = 3$, which is inside the interval $[0,4]$

We'll need to break up the integral at $x = 3$

$$\Rightarrow \int_0^4 \frac{x}{x^2-9}\,dx = \int_0^3 \frac{x}{x^2-9}\,dx + \int_3^4 \frac{x}{x^2-9}\,dx \quad \text{--- } (*)$$

$$= \lim_{t \to 3^-} \int_0^t \frac{x}{x^2-9}\,dx + \lim_{s \to 3^+} \int_s^4 \frac{x}{x^2-9}\,dx$$

Now, $\int \frac{x}{x^2-9}\,dx = \frac{1}{2} \ln|x^2-9| + c$

(i)

So, $$\int_0^4 \frac{x}{x^2-9}\,dx = \lim_{t\to 3^-}\left(\frac{1}{2}\ln|x^2-9|\right)\Big|_0^t + \lim_{s\to 3^+}\left(\frac{1}{2}\ln|x^2-9|\right)\Big|_s^4$$

$$= \lim_{t\to 3^-}\left(\frac{1}{2}\ln|t^2-9| - \frac{1}{2}\ln(9)\right) + \lim_{s\to 3^+}\left(\frac{1}{2}\ln(7) - \frac{1}{2}\ln|s^2-9|\right)$$

$$= \left[-\infty - \frac{1}{2}\ln(9)\right] + \left[\frac{1}{2}\ln(7) + \infty\right]$$

Thus, we see that

$$\int_0^3 \frac{x}{x^2-9}\,dx = -\infty.$$

$$\& \quad \int_3^4 \frac{x}{x^2-9}\,dx = \infty.$$

Sol^{ns} of Q. 2 (iv), (v) – on sheet (13)

That is, each of these integral is divergent which means that we can not break up the integral as we did in (*)

This means that the integral diverges.

<u>Note</u> :- Point to remember is we can only break up an integral (like we did in step *) provided that both the new integrals are convergent. If it turns out that even one of them is divergent, it will turn out that we couldn't have done this and the original integral will be divergent.

(3) $\displaystyle\int_0^3 \frac{1}{x^2-3x+2}\,dx$.

The integrand has infinite discontinuities at $x=1$ and $x=2$, both of which lie inside the interval $[0,3]$

So, we'll split up the integral as follows.

$$\int_0^3 \frac{1}{x^2-3x+2}\,dx = \int_0^1 \frac{1}{x^2-3x+2}\,dx + \int_1^2 \frac{1}{x^2-3x+2}\,dx + \int_2^3 \frac{1}{x^2-3x+2}\,dx$$

$$= \int_0^1 \frac{1}{(x-1)(x-2)}\,dx + \int_1^2 \frac{1}{(x-1)(x-2)}\,dx + \int_2^3 \frac{1}{(x-1)(x-2)}\,dx$$

Now, the integral $\displaystyle\int_1^2 \frac{1}{(x-1)(x-2)}\,dx$ has infinite discontinuity at both the end points $x=1$ and $x=2$.

So, we take any pt. say $x=c$ inside the limits of integration at which $f(x)$ is defined. We also find that $f(x)<0$ when $1<x<2$. We write $g(x)=-f(x)$ so that $g(x)>0$ when $1<x<2$. Therefore, we can write

$$\int_1^2 \frac{1}{(x-1)(x-2)}\,dx = -\int_1^c \frac{1}{(x-1)(2-x)}\,dx - \int_c^2 \frac{1}{(x-1)(2-x)}\,dx$$

Thus, $\displaystyle\int_0^3 \frac{1}{x^2-3x+2}\,dx$

$$= \lim_{t_1\to 0} \int_0^{1-t_1} \frac{1}{(x-1)(x-2)}\,dx - \lim_{t_2\to 0} \int_{1+t_2}^{c} \frac{1}{(x-1)(2-x)}\,dx$$

$$- \lim_{t_3\to 0} \int_c^{2-t_3} \frac{1}{(x-1)(2-x)}\,dx + \lim_{t_4\to 0} \int_{2+t_4}^{3} \frac{1}{(x-1)(x-2)}\,dx$$

$$= \lim_{t_1\to 0}\left[\ln\left(\frac{t_1+1}{t_1}\right) - \ln 2\right] - \lim_{t_2\to 0}\left[\ln\left(\frac{c-1}{2-c}\right) - \ln\left(\frac{t_2}{1-t_2}\right)\right] + \lim_{t_3\to 0}$$

(D)

$$- \lim_{t_3 \to 0} \left[\ln\left(\frac{1-t_3}{t_3}\right) - \ln\left(\frac{c-1}{2-c}\right) \right]$$
$$+ \lim_{t_4 \to 0} \left[\ln\left(\frac{1}{2}\right) - \ln\left(\frac{t_4}{t_4+1}\right) \right]$$

The limits do not exist, the improper integral diverges.

(4) (i) $\displaystyle\int_1^\infty \frac{1}{x^3+1}\,dx$

For $x > 1$ we have, $x^3 < x^3+1$

or $x^3+1 > x^3$

$$\Rightarrow \frac{1}{x^3+1} < \frac{1}{x^3}$$

Now, we know that $\displaystyle\int_1^\infty \frac{dx}{x^3}$ converges so, by comparison test $\displaystyle\int_1^\infty \frac{dx}{x^3+1}$ must also converge.

(ii) $\displaystyle\int_6^\infty \frac{x^2+1}{x^3(\cos^2 x+1)}\,dx$

For $x > 6$, $x^2+1 > x^2$

$\therefore$ we have $\displaystyle\frac{x^2+1}{x^3(\cos^2 x+1)} > \frac{x^2}{x^3(\cos^2 x+1)} = \frac{1}{x(\cos^2 x+1)}$ — (1)

Now, $0 \le \cos^2 x \le 1$ so we'll have

$\cos^2 x + 1 < 1+1 = 2$

$$\therefore \frac{1}{x(\cos^2 x+1)} > \frac{1}{2x} \quad \text{— (2)}$$

from (1) & (2) we get

$$\frac{x^2+1}{x^3(\cos^2 x+1)} > \frac{1}{2x}.$$

we know that $\int_6^\infty \frac{1}{2x}\,dx = \frac{1}{2}\int_6^\infty \frac{1}{x}\,dx$ diverges

Thus, by comparison test, the given integral diverges.

Sol^{ns} of Q. 4 (iii) (iv) → on sheet (14)

(5) $$\int_1^\infty \frac{x-1}{x^4+2x^2}\,dx$$

For $x>1$, $x-1<x$

$$\Rightarrow \quad \frac{x-1}{x^4+2x^2} < \frac{x}{x^4+2x^2} = \frac{1}{x^3+2x} \quad \text{---} (1)$$

Again, for $x>1$ $\quad x^3+2x > x^3$

$$\therefore \quad \frac{1}{x^3+2x} < \frac{1}{x^3} \quad \text{---} (2)$$

from (1) & (2) we have

$$\frac{x-1}{x^4+2x^2} < \frac{1}{x^3}$$

and we know that $\int_1^\infty \frac{1}{x^3}\,dx$ converges.

Thus, by comparison test

$$\int_1^\infty \frac{x-1}{x^4+2x^2}\,dx \quad \text{converges}$$

(6) $\int_1^\infty \frac{x\tan^{-1}x}{\sqrt{4+x^3}}\,dx$

Let $f(x) = \frac{x\tan^{-1}x}{\sqrt{4+x^3}}$ and $g(x) = \frac{1}{\sqrt{x}}$.

Now, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\tan^{-1}x}{\sqrt{1+4x^{-3}}} = \frac{\pi}{2}$.

Thus, by Limit comparison ~~Limit~~ test, the integrals $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ converge or diverge together.

Now, $\int_1^\infty g(x)dx$ is divergent.

$\therefore$, $\int_1^\infty f(x)dx$ is also divergent.

(7) (i) $\int_0^{\pi/2} \frac{\cos^m x}{x^n}\,dx$, $n<1$

Let $f(x) = \frac{\cos^m x}{x^n}$

$x=0$ is the pt of infinite discontinuity of $f(x)$

Now, $\frac{\cos^m x}{x^n} < \frac{1}{x^n}$, $0<x<\pi/2$

Now, $\int_0^{\pi/2} \frac{1}{x^n}\,dx$ converges for $n<1$

Thus, by comparison test

$\int_0^{\pi/2} \frac{\cos^m x}{x^n}\,dx$ is convergent for $n<1$

(ii) $\displaystyle\int_1^{\pi/2} \frac{\tan x}{x^{3/2}}\, dx$

$x = \pi/2$ is a point of infinite discontinuity.

Now, when $x \approx \pi/2$

$$\frac{\tan x}{x^{3/2}} \sim \frac{\tan x}{(\pi/2)^{3/2}} \sim \frac{\sec x}{(\pi/2)^{3/2}} = \frac{1}{(\pi/2)^{3/2} \cos x}$$

Approximate $\cos x$ when $x \approx \pi/2$ using taylor polynomials

$\cancel{\cos x \sim \cos(x - \pi/2)}$

we'll have $\cos x \sim -(x - \pi/2) = \frac{\pi}{2} - x$.

Thus, $$\frac{\tan x}{x^{3/2}} \sim \frac{1}{(\pi/2)^{3/2} \left(\frac{\pi}{2} - x\right)}$$

Now, the improper integral $\displaystyle\int_1^{\pi/2} \frac{1}{(\pi/2)^{3/2}\left(\frac{\pi}{2} - x\right)}$ is divergent

Therefore, the integral $\displaystyle\int_1^{\pi/2} \frac{\tan x}{x^{3/2}}$ is also divergent

by comparison test.

(F)

(8) (i) $\displaystyle\int_2^5 \frac{x-1}{\sqrt{x}\,(x-2)}\,dx$ 2 is ~~a~~ the only pt of infinite discontinuity.

Let $f(x) = \dfrac{x-1}{\sqrt{x}\,(x-2)}$ & $g(x) = \dfrac{1}{x-2}$

Then, $\displaystyle\lim_{x\to 2^+} \frac{f(x)}{g(x)} = \lim_{x\to 2^+} \frac{(x-1)\cancel{(x-2)}}{\sqrt{x}\,\cancel{(x-2)}} = \frac{1}{\sqrt{2}}$

Thus, by Limit Comparison test $\displaystyle\int_2^5 f(x)\,dx$ and $\displaystyle\int_2^5 g(x)\,dx$ converge or diverge together.

Now, $\displaystyle\int_2^5 \frac{dx}{x-2} = \int_0^3 \frac{dy}{y}$ putting $y = x-2$, $dy = dx$.

$\displaystyle\int_0^3 \frac{dy}{y}$ is divergent $\Rightarrow \displaystyle\int_2^5 g(x)\,dx$ is divergent

Thus, by Limit Comparison test $\displaystyle\int_2^5 \frac{x-1}{\sqrt{x}(x-2)}\,dx$ is divergent.

(ii) $\displaystyle\int_1^2 \frac{\sqrt{x}}{\ln x}\,dx$.

Let $f(x) = \dfrac{\sqrt{x}}{\ln x} \geq 0$, $1 < x \leq 2$.

$x = 1$ is the only pt. of infinite discontinuity

Let $g(x) = \dfrac{1}{x \ln x}$

Then, $\displaystyle\lim_{x\to 1^+} \frac{f(x)}{g(x)} = \lim_{x\to 1^+} \frac{\sqrt{x}}{\cancel{\ln x}} \times x \cancel{\ln x}$

$= \underline{\underline{1}}$

Thus, by Limit Comparison test both the integrals $\int_1^2 f(x)\,dx$ and $\int_1^2 g(x)\,dx$ Converge or diverge together.

Now, $\int_1^2 g(x)\,dx = \int_1^2 \frac{dx}{x \ln x} = \lim_{\epsilon \to 0} \int_{1+\epsilon}^2 \frac{dx}{x \ln x}$

$$= \lim_{\epsilon \to 0} \ln(\ln x)\Big]_{1+\epsilon}^{2}$$

$$= \lim_{\epsilon \to 0} \left[\ln(\ln 2) - \ln(\ln(1+\epsilon)) \right] \longrightarrow \infty$$

so, $\int_1^2 g(x)\,dx$ diverges hence the integral $\int_1^2 f(x)\,dx$ diverges.

(9) $$\int_0^1 \frac{1}{(1+x)(2+x)\sqrt{x(1-x)}}\,dx$$

Let $f(x) = \dfrac{1}{(x+1)(x+2)\sqrt{x(1-x)}}$

Then, 0 and 1 are the only points of discontinuity of f. Also, $f(x) > 0 \quad \forall\ x \in (0,1)$

Let us now examine the convergence of the improper integral $\int_0^{1/2} f(x)\,dx$ and $\int_{1/2}^1 f(x)\,dx$

Convergence of $\int_0^{1/2} f(x)\,dx$ at 0.

Let $g(x) = \frac{1}{\sqrt{x}}$ $\quad \lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \sqrt{x} \cdot \frac{1}{(1+x)(2+x)\sqrt{x}\sqrt{1-x}}$

$$= \frac{1}{2}.$$

Since $\int_0^{1/2} g(x)\,dx$ is cgt so by Limit Comparison test $\int_0^{1/2} f(x)\,dx$ converges —(i)

(r)

Convergence of $\int_{1/2}^{1} f(x)\,dx$ at 1.

Let $g(x) = \frac{1}{\sqrt{1-x}}$.

Then, $\lim_{x\to 1^-} \frac{f(x)}{g(x)} = \lim_{x\to 1^-} \cancel{\sqrt{1-x}} \cdot \frac{1}{(1+x)(2+x)\sqrt{x}\,\cancel{\sqrt{1-x}}}$

$$= \frac{1}{6}$$

Now, $\int_{1/2}^{1} g(x)\,dx$ is cgt so by Limit comparison test

$\int_{1/2}^{1} f(x)\,dx$ is Convergent ——— (ii)

from (i) & (ii) we get that $\int_0^1 \frac{dx}{(x+1)(x+2)\sqrt{x(1-x)}}$ is Convergent.

(16) $\int_0^{\pi/2} \log\cos x\,dx$ is an improper integral. The integrand is continuous and therefore integration on $[0, \pi/2 - \epsilon]$ $\forall\, \epsilon$ with $0 < \epsilon < \pi/2$

Let $\psi(\epsilon) = \int_0^{\pi/2-\epsilon} \log\cos x\,dx$

Then, $\psi(\epsilon) = \int_\epsilon^{\pi/2} \log\sin y\,dy \quad [x = \pi/2 - y]$

$$= \int_\epsilon^{\pi/2} \log\sin x\,dx$$

Since, $\int_0^{\pi/2} \log\sin x\,dx$ is convergent, $\lim_{\epsilon\to 0} \int_\epsilon^{\pi/2} \log\sin x\,dx$ is finite i.e. $\lim_{\epsilon\to 0} \psi(\epsilon)$ is finite and this proves that $\int_0^{\pi/2} \log\cos x\,dx$ is Convergent

Let $I = \int_0^{\pi/2} \log \cos x \, dx$

Then, $I = \lim_{\epsilon \to 0} \varphi(\epsilon) = \lim_{\epsilon \to 0} \int_\epsilon^{\pi/2} \log \sin x \, dx$

$= \int_0^{\pi/2} \log \sin x \, dx$

$= \frac{\pi}{2} \log \frac{1}{2}$

$I = \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$

$= \int_0^{\pi/2} \log \cos x \, dx$

$\Rightarrow \quad 2I = \int_0^{\pi/2} \log\left(\frac{1}{2} \sin 2x\right) dx = \int_0^{\pi/2} \log\left(\frac{1}{2}\right) dx$

$+ \int_0^{\pi/2} \log \sin 2x \, dx = \frac{\pi}{2} \log\left(\frac{1}{2}\right) + \int_0^{\pi/2} \log \sin 2x \, dx$

$(2x = z)$

$\int_0^{\pi/2} \log \sin 2x \, dx = \frac{1}{2} \int_0^{\pi} \log \sin x \, dx \Rightarrow$

$\Rightarrow 2I = \frac{\pi}{2} \log\left(\frac{1}{2}\right) + I$

$= \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \sin x \, dx$

$\Rightarrow \quad 2I = \frac{\pi}{2} \log\left(\frac{1}{2}\right) + I \Rightarrow \quad I = \frac{\pi}{2} \log \frac{1}{2}$

(10) Q: (10) Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$, if it Converges.

Solⁿ: Since the integrand becomes infinite as $x \to 1$

We evaluate $\int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(1-\epsilon)$

As $\epsilon \to 0+$, $\sin^{-1}(1-\epsilon) \to \sin^{-1} 1 = \frac{\pi}{2}$, Hence

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$$

Q: (11) Show that $\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-K^2x^2)}}$, $K^2 < 1$ is Convergent

Solⁿ: Only singularity is at $x = 1$

$$\lim_{x \to 1^-} (1-x)^{1/2} \frac{1}{\sqrt{(1-x^2)(1-K^2x^2)}} = \frac{1}{\sqrt{2(1-K^2)}}$$

By μ-test $(\mu = 1/2 < 1)$, $\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-K^2x^2)}}$ is Convergent

Q: Prove that the improper integration $\int_0^1 \frac{1}{x^{3/2}} \sin(1/x)\, dx$ is Convergent.

Not in syllabus

$$\left| \frac{\sin(1/x)}{x^{3/2}} \right| \le \frac{1}{x^{3/2}}$$

Now $\int_0^1 \frac{dx}{x^{3/2}}$ Convergence $\Rightarrow \int_0^1 \frac{\sin(1/x)\, dx}{x^{3/2}}$ Convergent

☐ Using Dirichlet's test :

$$\int_0^1 \frac{1}{x^{3/2}} \sin(1/x)\, dx = \int_1^\infty \frac{\sin t}{\sqrt{t}}\, dt \quad \left(\text{Change } x \text{ to } 1/t\right)$$

Take $f(t) = \frac{1}{\sqrt{t}}$ $\phi(t) = \sin t$

Now we see that $f(t)$ is monotone decreasing for $t \geq 1$, $f(t) \to 0$ as $t \to \infty$, Also $f(t)$ is bounded.

$\Phi(t)$ is bounded on $[1, B]$ for $B > 1$ and

$$\left| \int_1^B \sin t \, dt \right| \leq 2 \quad \text{for } B > 1$$

Hence by Dirichlets test $\int_0^1 \frac{1}{x^{3/2}} \sin(1/x)\, dx$ is Convergent.

Q: Show that the impropor integral $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$ is Convergent if $a \geq 0$.

Sol^n: If $a = 0$, then the integral reduces to

$\int_0^\infty \frac{\sin x}{x}$ and it is Convergent by Dirichlets test, Since, if $f(x) = 1/x$ and $\Phi(x) = \sin x$, then $f(x)$ is monotone decreasing for $x > 0$ and $f(x) \to 0$ as $x \to \infty$, also $\Phi(x)$ is bounded in any interval $[0+, B]$, $B > 0$.

Let $a > 0$ and $\phi(x) = e^{-ax}$ $x \geq 0$

Then $\phi'(x) = -a e^{-ax} < 0 \quad \forall x \geq 0$

Therefore, ϕ is a bounded monotone function on $[0, \infty)$.

And $\int_0^\infty \frac{\sin x}{x}$ is convergent, by Dirichlet's test

By Abel's test, $\int_0^\infty \phi(x) \frac{\sin x}{x}$ is convergent

$\Rightarrow \int_0^\infty e^{-ax} \frac{\sin x}{x}$ is convergent for $a \geq 0$.

Q: Show that $\int_a^\infty \frac{\cos x}{\log x} dx$ is convergent for $a > 1$

Sol^n: Let $f(x) = (\log x)^{-1}$ and $\phi(x) = \cos x$

Then $f(x) \to 0$ as $x \to \infty$ and $f(x)$ is monotone decreasing, $\phi(x)$ is bounded in $[a, A]$, $A > a$

Hence by Dirichlet's test $\int_a^\infty \frac{\cos x}{\log x} dx$ is convergent for $a > 1$

Q: Test the convergence of (i) $\int_0^\infty \sin x^2 dx$ (ii) $\int_0^\infty \cos x^2 dx$

Sol^n: See that $\int_0^1 \sin x^2 dx$ is a proper integral.

Hence, let us discuss the convergence of $\int_1^\infty \sin x^2 dx$

$$\int_1^\infty \sin x^2 = \int_1^\infty \frac{1}{2x} 2x \sin x^2 dx$$

Now, let $f(x) = \frac{1}{2x}$ and $\phi(x) = 2x \sin x^2$

Thus, By Dirichlets test $\int_0^\infty \sin x^2 dx$ is convergent

Q : 12 Discuss the Convergence of the integral

$$\int_1^{\infty} f(x)\,dx, \text{ where } f(x) = \begin{cases} \frac{1}{x^2} & x \text{ is rational} \\ -\frac{1}{x^2} & x \text{ is irrational} \end{cases}$$

Soln: $\int_1^{\infty} |f(x)|\,dx = \int_1^{\infty} \frac{1}{x^2}\,dx$ is Convergent

Now, every absolutely Convergent integral is Convergent

Therefore, the given integral is Convergent.

~~Q~~ : Show that $\int_1^{\infty} \frac{\sin x \log x}{x}$ is Convergent

Soln: Let $f(x) = \sin x$ $\phi(x) = \frac{\log x}{x}$

Now $\left| \int_1^{X} \sin x\,dx \right|$ is bounded for $X \geq 1$

ϕ is monotone decreasing, $\phi \to 0$ as $x \to \infty$

Hence $\int_1^{\infty} \frac{\sin x \log x}{x}$ is Convergent.

Q : 13 Prove that $\int_0^{\infty} e^{-x} x^{m-1}$ is Convergent for $m > 0$

Soln: Let $f(x) = e^{-x} x^{m-1} = \frac{e^{-x}}{x^{1-m}}$

The integrand f has infinite discontinuity at 0 if $m < 1$. So we have to examine Convergence at 0 and ∞ both.

Putting $\int_0^{\infty} e^{-x} x^{m-1}\,dx = \int_0^1 e^{-x} x^{m-1} + \int_1^{\infty} x^{m-1} e^{-x}$.

Convergence at 0, $m<1$.

Let $g(x) = \frac{1}{x^{1-m}}$ So that $\frac{f(x)}{g(x)} = e^{-x} \to 1$ as $x \to 0$

Also $\int_0^1 g\,dx = \int_0^1 \frac{dx}{x^{1-m}}$ converges $\Leftrightarrow m > 0$

Hence $\int_0^1 x^{m-1} e^{-x} dx$ Converges $\Leftrightarrow m > 0$

Convergence at ∞

Let $g(x) = \frac{1}{x^2}$, So that $\frac{f(x)}{g(x)} = \frac{x^{m+1}}{e^x} \to 0$ as $x \to \infty$ $\forall m$

As $\int_1^\infty \frac{dx}{x^2}$ Converges, therefore $\int_1^\infty e^{-x} x^{m-1}$ also Converges $\forall m$.

Q : 14 Show that $\int_0^{\frac{\pi}{2}} \sin x \log \sin x$ Converges and find its value.

Soln: The only Singularity is at $x=0$. Now

$$\int_\epsilon^{\pi/2} (\log \sin x) \sin x \, dx$$

$$= \left[-\cos x \log \sin x\right]_\epsilon^{\pi/2} - \int_\epsilon^{\frac{\pi}{2}} (\sin x - \operatorname{cosec} x)\, dx$$

$$= \cos\epsilon \log \sin\epsilon - \cos\epsilon - \log \tan\frac{\epsilon}{2}$$

Now $\lim_{\epsilon \to 0^+} \left(\cos\epsilon \log \sin\epsilon - \cos\epsilon - \log\tan\frac{\epsilon}{2}\right)$

$$= \lim_{\epsilon \to 0+} \left\{ (\cos\epsilon - 1) \log \sin\frac{\epsilon}{2} + \cos\epsilon \log 2\cos\epsilon/2 + \log \cos\frac{\epsilon}{2} - \cos\epsilon \right\}$$

$$= \lim_{\epsilon \to 0+} \frac{\log \sin\epsilon/2}{-\frac{1}{2}\operatorname{cosec}^2\epsilon/2} + \lim_{\epsilon \to 0+} \cos\epsilon \log 2\cos\epsilon/2 + \log\cos\epsilon/2 - \cos\epsilon$$

$$= \lim_{\epsilon \to 0} \frac{1}{2}\sin^2\frac{\epsilon}{2} + \lim_{\epsilon \to 0} \cos\epsilon \log 2\cos\frac{\epsilon}{2} + \log\cos\frac{\epsilon}{2} - \cos\epsilon$$

$$= \log 2 - 1.$$

Q : 15 Find the value of the integral $\int_0^{\pi/2} \log \sin x \, dx$, and $\int_0^{\pi/2} \log(\cos x)\, dx$ by discussing the convergence.

Solⁿ: Let $f(x) = \log \sin x$, $x \in \left(0, \frac{\pi}{2}\right]$. 0 is a point of infinite discontinuity of f. $f(x) > 0 \ \forall x \in \left(0, \frac{\pi}{2}\right]$

We have $\lim_{x \to 0+} \sqrt{x}\,(\log x) = 0$ $\lim_{x \to 0+} \sqrt{x} \log \frac{\sin x}{x} = 0$

Therefore, $\lim_{x \to 0+} \sqrt{x}\left[\log x + \log \frac{\sin x}{x}\right] = 0$

or $\lim_{x \to 0+} \sqrt{x} \log(\sin x) = 0$

Let $g(x) = \frac{1}{\sqrt{x}}$ $x \in \left(0, \frac{\pi}{2}\right]$, then $g(x) > 0 \ \forall x \in \left(0, \frac{\pi}{2}\right]$

$\lim_{x \to 0+} \frac{f(x)}{g(x)} = 0$ and $\int_0^{\frac{\pi}{2}} g(x)\,dx$ is convergent.

By Comparison test, $\int_0^{\pi/2} \log \sin x \, dx$ is convergent.

$I = \int_0^{\pi/2} \log \sin x \, dx$. Let $\Phi(\epsilon) = \int_\epsilon^{\pi/2} \log \sin x \, dx \quad 0 < \epsilon < \frac{\pi}{2}$

Then, $I = \lim_{\epsilon \to 0} \Phi(\epsilon)$

$$\Phi(\epsilon) = \int_\epsilon^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2 - \epsilon} \log \cos y \, dy \quad \left(x = \frac{\pi}{2} - y\right)$$

$$2\Phi(\epsilon) = \int_\epsilon^{\pi/2} \log \sin x \, dx + \int_0^{\pi/2-\epsilon} \log \cos x \, dx$$

$$= \int_\epsilon^{\pi/2-\epsilon} [\log \sin x + \log \cos x] \, dx + \int_{\pi/2-\epsilon}^{\pi/2} \log \sin x \, dx + \int_0^\epsilon \log \cos x \, dx$$

$$= \int_\epsilon^{\pi/2-\epsilon} \log\left(\frac{\sin 2x}{2}\right) dx + 2\int_0^\epsilon \log \cos x \, dx$$

Therefore, $2I = \lim_{\epsilon \to 0} 2\Phi(\epsilon)$

$$= \lim_{\epsilon \to 0} \left[\int_\epsilon^{\pi/2-\epsilon} \log \frac{\sin 2x}{2} \, dx + 2\int_0^\epsilon \log \cos x \, dx\right]$$

$$= \lim_{\epsilon \to 0} \left[\int_\epsilon^{\pi/2-\epsilon} \log \sin 2x \, dx + 2\int_0^\epsilon \log \cos x \, dx - \int_\epsilon^{\frac{\pi}{2}-\epsilon} \log 2 \, dx\right]$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{2}\int_{2\epsilon}^{\pi-2\epsilon} \log \sin u \, du - \left(\frac{\pi}{2} - 2\epsilon\right) \log 2 + 2\int_0^\epsilon \log \cos x \, dx\right]$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{2}\int_{2\epsilon}^{\pi/2} \log \sin u \, du + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi-2\epsilon} \log \sin u \, du - \left(\frac{\pi}{2} - 2\epsilon\right) \log 2 + 2\int_0^\epsilon \log \cos x \, dx\right]$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{2}\int_0^{\pi/2} \log \sin u \, du + \frac{1}{2}\int_{2\epsilon}^{\pi/2} \log \sin t \, dt - \left(\frac{\pi}{2} - 2\epsilon\right) \log 2 + 2\int_0^\epsilon \log \cos x \, dx\right]$$

$$= \lim_{\epsilon \to 0} \left[\int_{2\epsilon}^{\pi/2} \log \sin x \, dx - \left(\frac{\pi}{2} - 2\epsilon\right) \log 2 + 2 \int_0^{\epsilon} \log \cos x \, dx \right]$$

$$= \lim_{\epsilon \to 0} \left[\Phi(2\epsilon) - \left(\frac{\pi}{2} - 2\epsilon\right) \log 2 + 2 \int_0^{\epsilon} \log \cos x \, dx \right] \quad \text{---} \; *$$

Let $f(\epsilon) = \int^{\epsilon} \log \cos x \, dx \quad 0 \le \epsilon \le \frac{\pi}{2}$. Then f is a continuous function on $[0, \frac{\pi}{4}]$, since $\log$ $\log \cos x$ is integrable on $[0, \frac{\pi}{4}]$.

Therefore, $\lim_{\epsilon \to 0} f(\epsilon) = f(0) = 0$

$\lim_{\epsilon \to 0} \phi(2\epsilon) = \lim_{\epsilon \to 0} \phi(\epsilon) = I$ and $\lim_{\epsilon \to 0} \left[\frac{\pi}{2} - 2\epsilon\right] = \frac{\pi}{2}$

From $*$, $\quad 2I = I - \frac{\pi}{2} \log 2 \Rightarrow I = \frac{\pi}{2} \log \frac{1}{2}$. [For $\log(\cos x)dx$ see ques. $*$

Q: 17 Show that $\int_0^{\infty} \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x} dx = \frac{\pi}{2} \log(a/b)$

$0 < b < a$

Let $\phi(x) = \tan^{-1} x, \quad x \geq 0$. Then ϕ is continuous on $[0, \infty)$

$\lim_{x \to 0+} \Phi(x) = \Phi(0) = 0 \quad \lim_{x \to \infty} \Phi(x) = \frac{\pi}{2}$

Therefore $\int_0^{\infty} \frac{\Phi(ax) - \Phi(bx)}{x} dx = \left[0 - \frac{\pi}{2}\right] \log(b/a)$

$$= \frac{\pi}{2} \log(a/b).$$

(12)

Q: (18) Find the value of $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ if it ever converges.

Solⁿ: By definition,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

Now, $$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{b \to -\infty} \left[\tan^{-1} x\right]_b^0$$

$$= 0 - \lim_{b \to -\infty} \tan^{-1} b = \frac{\pi}{2}$$

Similarly, $$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$$

Hence, $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

Q: (19) Show that $\int_0^{\infty} \frac{q \sin px - p \sin qx}{x^2}\,dx = pq \log(q/p)$

$0 < q < p$

Let $f(x) = \frac{\sin x}{x}$, $x > 0$.

Then f is continuous on $(0, \infty)$

$\lim_{x \to 0+} f(x) = 1$ $\quad \lim_{x \to \infty} f(x) = 0$

Therefore, $$\int_0^{\infty} \frac{f(px) - f(qx)}{x}\,dx = \cancel{\left[\left\{f(px) - f(qx)\right\}\log x\right]_0^{\infty} - \int_0^{\infty} \frac{p f'(px) - q f'(qx)}{x}\,dx}$$

$$= (1-0)\log(q/p)$$ [Apply the thm]

$$\Rightarrow \int_0^\infty \frac{\frac{\sin px}{px} - \frac{\sin qx}{qx}}{x} = \log(q/p)$$

$$\Rightarrow \int_0^\infty \frac{q \sin px - p \sin qx}{x^2} = pq \log(q/p)$$

Q: 20 Prove that $\int_0^{\frac{\pi}{2}} \log \cos x \, dx$ is Convergent and find the value.

$\int_0^{\pi/2} \log \cos x \, dx$ is an improper integral

The integrand is Continuous on $[0, \frac{\pi}{2} - \epsilon]$ $\forall \epsilon \in (0, \frac{\pi}{2})$

Therefore it is integrable on $[0, \frac{\pi}{2} - \epsilon]$

Let $f(\epsilon) = \int_0^{\pi/2 - \epsilon} \log \cos x \, dx$

Then $f(\epsilon) = \int_0^{\frac{\pi}{2}} \log \sin y \, dy \quad (y = x - \frac{\pi}{2})$

Since $\int_0^{\frac{\pi}{2}} \log \sin y \, dy$ is Convergent. Hence $\int_0^{\pi/2} \log \cos x \, dx$ is Convergent

Let $I = \int_0^{\pi/2} \log \cos x \, dx$. Then $I = \lim_{\epsilon \to 0} f(\epsilon)$

$$= \lim_{\epsilon \to 0} \int_\epsilon^{\pi/2} \log \sin y \, dy = \int_0^{\pi/2} \log \sin y \, dy = \frac{\pi}{2} \log\left(\frac{1}{2}\right)$$

Q: 21 2 iv Find the value of the integral $\int_0^1 \log t\, dt$.

Sol^n: → The given integral is improper at $t=0$

Thus, $$\int_0^1 \log t\, dt = \lim_{b\to 0+} \int_b^1 \log t\, dt$$

$$= \lim_{b\to 0+} \Big[t\log t - t\Big]_b^1$$

$$= \lim_{b\to 0} \Big[(\log 1 - 1) - (b\log b - b)\Big]$$

$$= -1 - \lim_{b\to 0+} b\log b = -1 - \lim_{b\to 0+} \frac{\log b}{1/b} \quad \left[\frac{\infty}{\infty}\right]$$

$$= -1 - \lim_{b\to 0+} \frac{1}{b}(-b^2) \quad \text{[Applying L'Hospital]}$$

$$= -1$$

Q: 22 2 v Prove that the integral $\int_{-2}^{3} \frac{dx}{x-1}$ does not exist.

Sol^n: $$\int_{-2}^{3} \frac{dx}{x-1} = \int_{-2}^{1} \frac{dx}{x-1} + \int_{1}^{3} \frac{dx}{x-1}$$

$$= \lim_{b \to 1-} \int_{-2}^{b} \frac{dx}{x-1} + \lim_{d \to 1+} \int_{d}^{3} \frac{dx}{x-1}$$

$$= \lim_{b \to 1-} \left[\log|b-1| - \log 3 \right] + \lim_{d \to 1+} \left[\log 2 - \log|(d-1)| \right]$$

$$= \log\left(\tfrac{2}{3}\right) + \lim_{b \to 1-} \log|b-1| - \lim_{d \to 1+} \log|d-1|$$

$$= \cancel{\log} - \infty + \infty = \text{does not exist}$$

Q: ~~23~~ 16 Show that the integral $\int_{-1}^{1} \frac{\sin x}{x}\, dx$ is Proper integral.

Soln: $\to$ Let $f(x) = \frac{\sin x}{x}$, at $x = 0$ it takes the ~~intern~~ indeterminate form $[0/0]$. Now

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1,$$

Therefore, $f(x)$ is bounded on $[-1, 1]$ and can be continuously defined at $x = 0$ by assigning the value $f(0) = 1$

Q: 24 (A) (iv) Show that the integral $\int_0^{\infty} e^{-x^2} dx$ is Convergent (14)

Consider the Continuous function

$$f(x) = e^{-x^2} \text{ on } [0, \infty) \text{ and define}$$

$$g(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq 1 \\ e^{-x} & \text{for } 1 \leq x < \infty \end{cases}$$

Then, $0 < f(x) \leq g(x) \quad \forall x \in [0, \infty)$ and

$$\int_0^{\infty} g(x)\,dx = \int_0^1 e^{-x^2} dx + \int_1^{\infty} e^{-x} dx$$

$$= \int_0^1 e^{-x^2} dx + \left[-e^{-x}\right]_1^{\infty}$$

$$= (\text{finite value}) + \left[0 + e^{-1}\right] = \text{finite value}$$

Therefore,

$$\int_0^{\infty} f(x)\,dx = \int_0^{\infty} e^{-x^2} dx < \int_0^{\infty} g(x)\,dx < \infty$$

and so $\int_0^{\infty} f(x)\,dx$ is finite

Q: 25 (A) (iii) Prove that $\int_2^{\infty} \frac{1}{\log x} dx = \infty$, i.e. it diverges

On $[2, \infty)$ we have that $0 < 1/x < \frac{1}{\log x}$

Now $\int_2^\infty \frac{1}{x}\,dx = \left[\log x\right]_2^\infty = \infty$

Therefore, $\int_2^\infty \frac{dx}{\log x} = \infty$.

Q : 18 Let $f(x,t) = (2x+t^3)^2$ then find (i) $\int_0^1 f(x,t)\,dx$ 15

(ii) Prove that $\frac{d}{dt}\int_0^1 f(x,t)\,dx = \int_0^1 \frac{\partial}{\partial t} f(x,t)\,dx$

Soln: $\int_0^1 f(x,t)\,dx = \int_0^1 (2x+t^3)^2\,dx = \frac{4}{3} + 2t^3 + t^6$.

$$\frac{d}{dt}\int_0^1 f(x,t)\,dx = 6t^2 + 6t^5.$$

$$\int_0^1 \frac{\partial}{\partial t}(2x+t^3)^2\,dx = \int_0^1 2(2x+t^3)\,3t^2\,dx$$

$$= \left[6t^2x^2 + 6t^5x\right]_0^1 = 6t^2 + 6t^5.$$

Q : 19 (i) Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(x,t) = \begin{cases} \frac{\sin xt}{t} & t \neq 0 \\ x & t = 0 \end{cases}$

Find F' where $F(x) = \int_0^{\pi/2} f(x,t)\,dt$

Soln: We have $\lim_{t \to 0} \frac{\sin xt}{t} = x$

f is continuous on $\mathbb{D} = \left\{ (x,t) : -\infty < x < \infty,\ 0 \le t \le \pi/2 \right\}$.

and $\frac{\partial f}{\partial x} = \begin{cases} \cos xt & t \neq 0 \\ 1 & t = 0 \end{cases}$

Hence $\frac{\partial f}{\partial x}$ is continuous on $\mathbb{D}$.

By applying Leibniz's rule

$$F'(x) = \int_0^{\pi/2} \cos xt \, dt = \frac{\sin \pi/2 x}{x} \quad x \neq 0$$

and, $F'(0) = \frac{\pi}{2}$.

(11) Given $f : x \longrightarrow \int_0^{x^2} \tan^{-1}(t/x)\, dt$, find f'.

Soln: We get

$$\frac{\partial}{\partial x}\left(\tan^{-1}\frac{t}{x^2}\right) = -\frac{2tx}{t^2+x^4}$$

Using the general Leibniz rule, we get

$$f'(x) = (\tan^{-1} 1)\, 2x - \int_0^{x^2} \frac{2tx}{t^2+x^4}\, dt$$

Setting $t = x^2 u$

$$f'(x) = \frac{\pi x}{2} - x\int_0^1 \frac{2u\,du}{1+u^2} = x\left(\frac{\pi}{2} - \log 2\right).$$

Q: (20) For any real numbers x and t let

$$f(x,t) = \begin{cases} \dfrac{xt^3}{(x^2+t^2)^2} & x \neq 0,\ t \neq 0 \\ 0 & x = 0 \quad t = 0 \end{cases}$$

and $F(t) = \int_0^1 f(x,t)\, dx$.

Is $\frac{d}{dt}\int_0^1 f(x,t)\,dx = \int_0^1 \frac{\partial}{\partial t} f(x,t)\,dx$? (16) Give the justification.

Soln: $F(0) = 0$, For $t \neq 0$

$$F(t) = \int_0^1 \frac{xt^3}{(x^2+t^2)^2}\,dx = \int_{t^2}^{1+t^2} \frac{t^3}{2z^2}\,dz \quad [z = x^2+t^2]$$

$$= -\frac{t^3}{2z}\Big|_{t^2}^{1+t^2} = -\frac{t^3}{2(1+t^2)} + \frac{t^3}{2t^2}$$

$$= \frac{t}{2(1+t^2)} \quad \forall t$$

$F(t)$ is differentiable and $F'(t) = \dfrac{1-t^2}{2(1+t^2)^2}$

Now $F'(0) = \frac{1}{2}$ and

$$\frac{\partial}{\partial t} f(x,t) = \begin{cases} \dfrac{xt^2(3x^2-t^2)}{(x^2+t^2)^3} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

In Particular, $\frac{\partial}{\partial t} f(x,t)\Big|_{t=0} = 0$. Hence

$\int_0^1 \frac{\partial}{\partial t} f(x,t)\,dx = 0$ at $t = 0$. But $F'(0) = 1/2$

Justification: $\frac{\partial}{\partial t} f(x,t)$ is not a continuous function of (x,t). If we let $(x,t) \to (0,0)$ along the line $x = t$, then on this line

$\frac{\partial}{\partial} f(x,t)$ has the value $1/4x$, which does not tend to 0 as $(x,t) \to (0,0)$

Q (2) Find the value of the integral $\int_0^\infty \frac{e^{-bx}\sin ax}{x}\,dx$ where $a>0$, $b>0$ are fixed, and hence deduce the value of the integral $\int_0^\infty \frac{\sin ax}{x}\,dx$

Sol^n: Let $F(a) = \int_0^\infty \frac{e^{-bx}\sin ax}{x}\,dx$

Then,
$$F'(a) = \int_0^\infty e^{-bx}\cos ax\,dx$$

Hence, $F'(a) = \frac{b}{b^2+a^2}$, therefore.

$$\int_0^\infty e^{-bx}\frac{\sin ax}{x}\,dx = \tan^{-1}(a/b) + c$$

Now, $F(0) = 0 \Rightarrow c = 0.$

$$\int_0^\infty e^{-bx}\frac{\sin ax}{x}\,dx = \tan^{-1}(a/b).$$

At this point we can set $b \to 0^+$ and take limits both side, we get

$$\int_0^\infty \frac{\sin ax}{x} = \frac{\pi}{2}. \quad \forall a>0$$

Q: 22 Find the value of the following integral:

(i) $\displaystyle\int_0^\infty e^{-bx}\,\frac{1-\cos ax}{x}\,dx$, $b>0$ is fixed.

Sol^n: Let $\displaystyle F(a)=\int_0^\infty e^{-bx}\,\frac{1-\cos ax}{x}\,dx$

The derivative is:

$$F'(a)=\int_0^\infty e^{-bx}\sin ax\,dx=\frac{a}{a^2+b^2}$$

$$\Rightarrow F(a)=\frac{1}{2}\log(a^2+b^2)+c$$

Setting $a=0$, we find $c=-\frac{1}{2}\log b^2$.

Thus, $\displaystyle\int_0^\infty e^{-bx}\,\frac{1-\cos ax}{x}\,dx=\frac{1}{2}\log\left(1+\frac{a^2}{b^2}\right)$.

(ii) $\displaystyle\int_0^{\pi/2}\log(1-x^2\sin^2\theta)\,d\theta$ for $|x|<1$

The function $\log(1-x^2\sin^2\theta)$ is well defined in the ~~integral~~ rectangle $[-1,1;\,0,\pi/2]$ and satisfies of the Leibnitz's rule

Let $\displaystyle F(x)=\int_0^{\pi/2}\log(1-x^2\sin^2\theta)\,d\theta$ $|x|<1$

By differentiating under the integral sign, w.r.t x, we get

$$\therefore\ F'(x) = \int_0^{\pi/2} \frac{-2x\sin^2\theta}{1-x^2\sin^2\theta}\,d\theta = \frac{2}{x}\int_0^{\pi/2} \frac{1-x^2\sin^2\theta-1}{1-x^2\sin^2\theta}\,d\theta$$

$$= \frac{\pi}{x} - \frac{2}{x}\int_0^{\pi/2} \frac{d\theta}{1-x^2\sin^2\theta}, \quad \text{put } \cot\theta = t$$

$$= \frac{\pi}{x} - \frac{2}{x}\int_0^{\infty} \frac{dt}{1+t^2-x^2}$$

$$= \frac{\pi}{x} - \frac{2}{x\sqrt{1-x^2}}\tan^{-1}\frac{t}{\sqrt{1-x^2}}\Bigg|_0^{\infty} = \frac{\pi}{x} - \frac{\pi}{x\sqrt{1-x^2}}$$

$$F(x) = \pi\log x - \pi\log\left\{\frac{1-\sqrt{1-x^2}}{x}\right\} + c$$

$$= \pi\log\left\{\frac{x^2\left(1+\sqrt{1-x^2}\right)}{1-(1-x^2)}\right\} + c$$

$$= \pi\log\left(1+\sqrt{1-x^2}\right) + c$$

But $F(0) = 0 \Rightarrow c = -\pi\log 2$

Hence, $$F(x) = \int_0^{\pi/2} \log\left(1-x^2\sin^2\theta\right)d\theta$$

$$= \pi\log\left(1+\sqrt{1-x^2}\right) - \pi\log 2.$$

(14) $$\int_0^{\infty} \frac{e^{-px}\cos qx - e^{-ax}\cos bx}{x}\,dx$$

Let $F(a,b) = \int_0^\infty \frac{e^{-px}\cos qx - e^{-ax}\cos bx}{x}\,dx$

By differentiating under integral sign w.r.t a, we get

$$F_a(a,b) = \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$$

Again, by differentiating under integral sign w.r.t b, We get

$$F_b(a,b) = \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

Hence, $F(a,b) = \log(a^2+b^2) + C$

Now, at $a = p$, $b = q$, $F(a,b) = 0$

$\Rightarrow \quad C = -\log(p^2+q^2).$

$$\Rightarrow \int_0^\infty \frac{e^{-px}\cos qx - e^{-ax}\cos bx}{x}\,dx = \log\left(\frac{a^2+b^2}{p^2+q^2}\right)$$

iv) $\int_0^\infty e^{-x^2}\cos 2ax\,dx$

Let $F(a) = \int_0^\infty e^{-x^2}\cos 2ax\,dx$

Here $F'(a) = -2\int_0^\infty xe^{-x^2}\sin 2ax\,dx$, and integration by parts leads to the following diff equation

$$F'(a) = 2aF$$

or

$$\frac{dF}{da} = -2aF$$

$$\Rightarrow \quad F = c\,e^{-a^2}$$

For $a = 0$, $F(0) = \frac{\sqrt{\pi}}{2}$. Therefore

$$F(a) = \frac{\sqrt{\pi}}{2} e^{-a^2}.$$