LINEAR ALGEBRA

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Assignment -1

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Q.1) (c) Uncountably infinite:

Consider $f_{r_1} = e^{r_1 x} - e^{r_1}$

$f_{r_2} = e^{r_2 x} - e^{r_2}$

$\forall r_1 \neq r_2 \rightarrow f_{r_1}(x), f_{r_2}(x)$ are linearly independent.

hence in Basis, f_{r_i} must be present $\underline{\forall r_i \in R}$

but R is uncountably infinite.

Q.E.D

Q.2] (c) No inner product.

Check parallelogram law.

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Consider $n=3$

$x = \{3,3,3\}$, $y = \{1,2,2\}$

LHS = 29, RHS = 26

hence doesn't hold.

Q.3) Since $\dim(V) = 2$, atleast one basis $\{v_1, v_2\}$ exists

note: $v_1 + v_2 = v_1 - v_2$, as $v_2 + v_2 = 0$

$B_1 = \{v_1, v_2\}$, $B_2 = \{v_1, v_1+v_2\}$, $B_3 = \{v_2, v_1+v_2\}$

any other is repetition.

$\boxed{(c)}$ Exactly 3 bases

Q.4) After some calculation, (trivial)

$$W = LS\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}\right)$$

Since this set is LI, it's cardinality is $\dim(W) = 3$

$\boxed{\text{(c) Dimension of W is 3}}$

Q.5) to make v_1, v_2, v_3 LI

we must enforce:

if $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$

We know:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & a & 5 \\ 0 & 0 & 2 \end{bmatrix}}_{A} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Since $\det(A) = 2a$, if $a \neq 0$

We can get unique sol^n & show u_1, u_2, u_3 are LI,

Conversely, if $a=0$, then $\frac{v_2}{5} = \frac{v_3}{2}$, hence v_2, v_3 would be LD,

hence if LI, $a \neq 0$, (Contrapositive)

if $a \neq 0$ they are LI. QED

Ans: $\boxed{[b]\ \text{LI iff } a \neq 0}$

Q.6)

Observation: W is subspace & finite dimensional.

$\therefore \dim(W^\perp) = n^2 - \dim(W)$

$= n^2 - \left(n - 1 + \frac{n(n-1)}{2}\right)$

$= \frac{n^2 - n + 2}{2}$

None of the above

Q.7) $\langle x+y, x+y \rangle = \langle x, y \rangle + \langle x, x \rangle + \langle y, y \rangle + \langle y, x \rangle$

$\therefore \langle x, y \rangle + \langle y, x \rangle = 0$

if $K = C$, $\not\Rightarrow \langle x, y \rangle = 0$ (take standard inner product to disprove)

if $K = R \Rightarrow \langle x, y \rangle + \overline{\langle x, y \rangle} = 0$

$= \langle x, y \rangle = 0$

$\boxed{\text{ans: } |x+y|^2 = |x|^2 + |y|^2 \Rightarrow x \text{ is orthogonal to } y \text{ if } K=R \quad (c)}$

Q.8) By property of Tutorial 1,

if $n \geq 2$, $\dim(W) = n-1 \geq \frac{n}{2}$

$\boxed{\text{b) W has 2 virtually disjoint components}}$

Q.9] a) If $\phi^\top = V$, $\phi = \{0\}$ is true

, since if $\phi \neq \{0\}$, i.e ϕ has non zero element v_1, then $v_1 \in \phi^\top$, but $\langle v_1, v_1 \rangle > 0$, hence v_1 cant exist

b) Easily false with many counter examples

c) let $V = R^2$, and standard inner product

let $\phi = LS(e_1)$, $\phi^\top$ is a non trivial subspace

$\boxed{\text{ANSWER} \rightarrow (a) \rightarrow \text{If } \phi^\top = V, \phi = \{0\}}$

Q.10) a) is false, since $U^\top = \{0\}$

b) is true, proof: assume $g(x)$ exists $\neq 0$

Construct 'f_x'

$$f_x(t) = \begin{cases} g(t), & -1 \leq t \leq -\epsilon \\ g(t), & \epsilon \leq t \leq 1 \\ \frac{-g(\epsilon)}{\epsilon} t, & -\epsilon < t < 0 \\ \frac{g(\epsilon)}{\epsilon} t, & 0 \leq t < \epsilon \end{cases}$$

$\boxed{\text{for any } g(x) \neq 0}$ $\exists\, \epsilon$ s.t $\int_{-1}^{-1} f_x(t) g(t)\, dt > 0$ if $g(t) \neq 0$

Clearly f_x is continuous, $f_x(0) = 0$, $\therefore f_x \in U$

so $g(x) = 0$

c) is false since (b) is true

Answer: $\left[(B) \rightarrow U^\top = \{0\}\right]$