Basic Discrete Mathematics

Logic, Set Theory, & Probability

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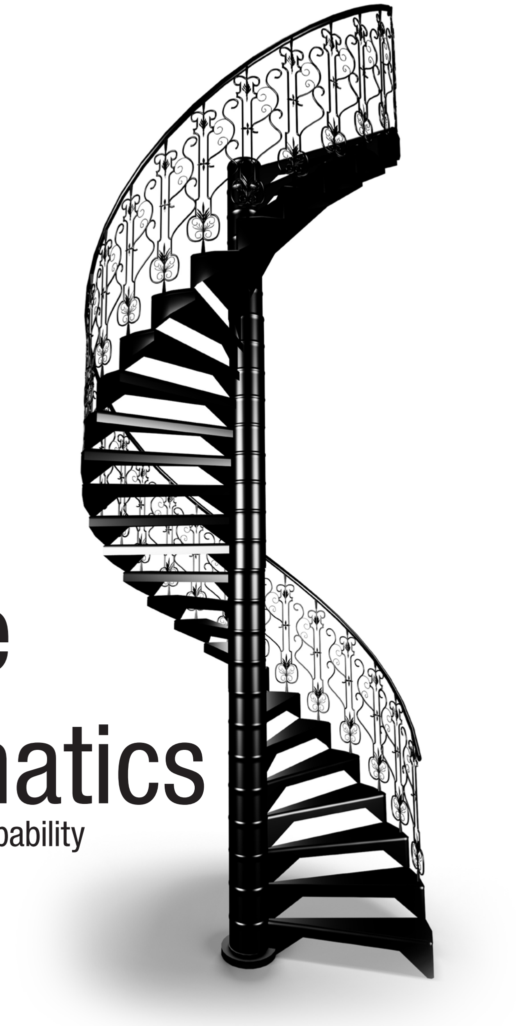

Basic Discrete Mathematics

Logic, Set Theory, & Probability

Richard Kohar
Royal Military College of Canada

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Dedicated to

James D. Stewart (1941–2014)

To the scholar who I only knew
through his high school textbooks,
but showed me the beauty of mathematics,
and inspired me to become a mathematician and a writer.

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Preface

> A textbook must be exceptionally bad if it is not more intelligible than the majority of notes made by students . . . the proper function of lectures is not to give a student all the information he [or she] needs, but to rouse his [or her] enthusiasm so that he [or she] will gather knowledge himself [or herself], perhaps under difficulties.
>
> —J. J. Thomson (1856–1940)

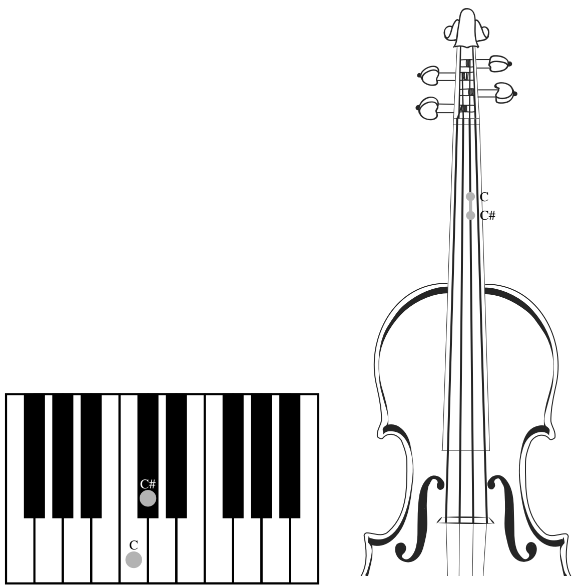
To the student

What kinds of problems can discrete mathematics solve? It can answer the following:

- What is the chance that you have HIV given that you have a positive test? (see p. 375)
- How do casinos and insurance companies make money in the long term? (see p. 404)
- How long do you expect to live if you are currently 20 years old? (see p. 351)
- How can you show whether a politician's argument is valid or invalid? (see p. 67)
- How can you show that you and another person in the city of London have the same number of strands of hair on your heads? (see p. 138)
- What is your chance of winning the lottery? (see p. 336)
- What is the maximum number of passwords you will have to attempt before you break into someone's computer account? (see p. 158)
- How many paths can you take from your home to school? (see p. 178)
- How can you quickly calculate $1 + 2 + 3 + \cdots + 100$? (see p. 222)

In this book, you will learn how to solve all of these types of problems and many more. But, what exactly is discrete mathematics?

Discrete mathematics is the study of discrete objects rather than continuous objects. An intuitive way of seeing the distinction between the discrete and continuous brings us into the realm of music. For example, a piano can play discrete notes: you can play a C or a C-sharp, but no notes in between. A violin, however, can play any sound between C and C-sharp: the finger can be placed anywhere between C and C-sharp on the continuous string.



(a) A piano is discrete

(b) A violin can play a continuous range of notes

Fig. 0.1 Music can be discrete or continuous

In contrast with real numbers (which are continuous), discrete mathematics studies objects like integers. Integers are easy to count one by one—they are **countable**—while the real numbers are not. For example, there are five integer

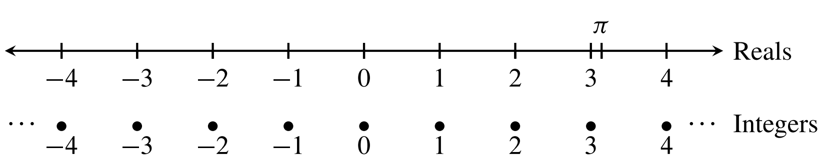


Fig. 0.2 The integers have gaps between them, while the reals do not. We can count all of the integers between 0 and 4, but we cannot count all the reals between 0 and 4.

numbers between 0 and 4 (inclusively), while we are unable to count the number of real numbers between 0 and 4. To see why, consider how close we can get to the number π without being equal to π.[1] You could say that a number close to π is 3.1, but we can still get closer with 3.14, or 3.1415, or 3.141 592 653 589 7 and so on and so forth, but we will never get to the true value. The real numbers have no gaps between them—they are continuous. Unlike the integers that have gaps between the numbers, we can keep repeating this process forever, and hence, we cannot count all of the number of real numbers between 0 and 4. (You could say, they're uncountable.) But what about numbers you can express as fractions—the rational numbers—can you count all of them too?[2]

> And it's time for our mathematics curriculum to change from analogue to digital, from the more classical, continuous mathematics, to the more modern, discrete mathematics—the mathematics of uncertainty, of randomness, of data—that being probability and statistics.
>
> —Arthur T. Benjamin (1961–)
>
> TED Talk given in February 2009

You may have struggled in the past with mathematics courses in high school, but consider this course a fresh start. This book is designed to stretch the understanding of mathematics that you achieved in high school, and to show you that mathematics indeed requires some creativity when solving problems. We will see that even some truly great mathematicians were either stumped or had to resort to ground-breaking new ways of thinking about a problem. John Waters (1946–) once said, "A poor person to me can have a big bank balance, but is stupid by choice—uncurious, judgmental, isolated, and unavailable to change."[3] In short, if by your own choice you are not open to these new ideas, I can't force them upon you; just put the book down now. If you're still reading, discuss the text with your classmates, get help from the instructor as soon as you need it, and promise yourself never to procrastinate and to complete your homework. With proper time management and some effort, this could very well be a class you enjoy.

Here are some general tips to help you use this book:

(a) *Read the book before the lecture*. Ask the instructor which section will be covered the following day, and read that section before the lecture. This will give you an overview of the next lecture's material, and you will be ready with questions if there is material that you didn't understand.

[1] The Greek letter π is pronounced as pie. If you think back to high school or elementary school, π is the mathematical constant that is equal to a circle's circumference divided by its diameter regardless of the circle's size (see Appendix B.2). Its decimal number representation never ends, and it never settles on any repeating pattern.

[2] Yes. See Theorem 4.2.9 on p. 126, but you might have to go back to the beginning of the Chap. 4 to understand why this is true. So if you thought we were just limited to integers, don't worry.

[3] From John Waters' 2015 Commencement Speech at Rhode Island School of Design on 30 May 2015.

(b) *Keep the book open during the lecture*. I found it helpful to keep the book open during a lecture. If I didn't feel comfortable asking my question, I could quickly skim through the text to find my answer. This could easily answer questions such as what a symbol means, or what a particular word means. Instructors sometimes gave hints in class about content of forthcoming examinations, so I would make a note in the relevant section of the textbook, and I could pay particular attention when studying and reviewing.

(c) *Read the book after lecture*. If you didn't read the textbook before the lecture, you should read the textbook after. I believe, just as did J. J. Thomson, that a good lecture should be motivating enough for the student to read the textbook and delve deeper into the details. During this second exposure to the material, I recommend that when you come up on an example, cover up the solution, and try to work it out yourself. Give it a zealous all-out effort. When you are finished, you can check to see if you got the right answer; if you did not, go back over your work and see where you made a mistake. If you get stuck, I mean really stuck, then take a peak and see if you can continue. By making mistakes, and correcting yourself, you will improve and gain an understanding of the material.

(d) *Read with a pencil/pen*. You solve mathematics problems with a pencil/pen, why not read mathematics with a pencil/pen as well? If there is something you don't understand, try figuring it out on a scrap of paper. If it's important, put your condensation of ideas in the margin. Don't be afraid to mark up the book if you own it—you can shed your high school mentality of not "defacing" your book. If you haven't noticed, the book is in black and white; feel free to add some color yourself.[4] By marking and adding your thoughts and drawings to the book, it helps you place mental index markers in the book, so you can find material quickly as you flip back and forth through the book. You should not be discouraged if you find your reading rate considerably slower than normal.

(e) *Skip ahead*. A book is linear: one page follows another. This does not mean that it has to be read in a linear way! If I have written about something and you don't understand the reason for me to present the idea, try skipping ahead and reading the next section. You may see why I had to go through some difficult material[5] to get to present an exciting topic or application. Read that topic or application, and then go back. You'll now understand the motivation for presenting the difficult material.

[4]Perhaps use a red pen to highlight the boxes of theorems or important equations. Try coloring the portrait of George Boole on p. 2 or Georg Cantor on p. 131.

[5]What one person finds boring, another may find exciting.

> Mathematics is logical to be sure; each conclusion is drawn from previously derived statements. Yet the whole of it, the real piece of art, is not linear; worse than that its perception should be instantaneous.
>
> —Artin (1953, p. 475)

(f) *Don't skip ahead*. There may be a hard example that you are trying to work through the solution on your own. Don't give up, and persevere! This is the hard part of learning. You will be spending a lot of time working on this subject, so do not worry if you feel you are slow.

> Being a mathematician is a bit like being a manic depressive: you spend your life alternating between giddy elation and black despair. You will have difficulty being objective about your own work: before a problem is solved, it seems to be mightily important; after it is solved, the whole matter seems trivial and you wonder how you could have spent so much time on it.
>
> —Krantz (1997, p. 78)

One way to overcome this is to celebrate the small successes. If you see a friend who is in your class walking down the hallway, tell your friend that you solved that problem you were struggling with. Your friend may be interested in how you solved the problem; your friend might have been stuck on the same problem too.

(g) *Study quietly by yourself*. We plug directly ourselves into social media feeds, music, videos, and the list goes on and on. However, to become good or even great at something, you need to learn to turn off technology (at least while studying). Take your phone, switch it off, and put it in a drawer. Even better, just *throw it out the window*. Why? Ericsson *et al.* (1993) found that to acquire expert performance as a chess grandmaster, a violinist, or a gymnast, required a substantial amount of deliberate practice. Best conducted in solitude, deliberate practice allows for intense focus, which allows you to identify gaps in your knowledge or skills, strive to fill in those gaps, monitor your performance, and correct accordingly (Cain, 2012, p. 81). Distractions hinder deliberate practice by causing us to lose focus, and when you are interrupted, it takes about 25 minutes to cycle back to the original task (Mark *et al.*, 2005). Two of the reasons I included the solutions at the back of the book are

(i) you can avoid the temptation of searching for solutions online if you become stuck and need a hint (and hence, you avoid being distracted by a text message or email alerts from others), and

(ii) you can extend your deliberate practice time longer as it does not depend on the availability of external resources, such as a professor or a tutor.

(h) *Read widely*. Mathematics is a wonderfully broad area of study. There is no single book that can contain all of its ideas, and this book is no different.

Another way to explore ideas that we will discuss is to read through the Bibliographic Remarks at the end of the chapters. You may find other books that you may want to read to further your study about a particular topic. A recent invention is the digital object identifier (DOI), which is a string of characters that uniquely identifies an electronic document. In the Bibliography, I have recorded the DOI for many of the articles and books listed. Unlike web addresses which change or break over time, DOIs remain fixed over the lifetime of the document. These DOIs will direct you to the article, and if you access them from the university, they are usually available free of charge to you.

Pólya's Approach to Problem Solving

This book has a lot of mathematical problems in it. Some of these problems you may never have seen before, and you'll have to solve them. The Hungarian mathematician George Pólya (1945) gave some strategies for tackling a mathematical problem, and we will use them extensively throughout the book. I summarize them below:

(a) *Understand the problem*. Read the problem, and determine what you are required to find. What are the unknowns? What is the given information? You might find that starting to *draw a diagram* will help (especially if you are a visual learner) to organize the information you have, and recognize the information that you need.

(b) *Devise a plan*. What is the relationship between the given to the unknown? If it is not immediately apparent, then try devising a plan based on the following:

   (i) *Draw a diagram*. This can help organize the data that is given in the problem. Our working memory is one of our limitations; we can hold about seven pieces of data.[6] By drawing a diagram, it allows us to extend and augment our memory. Sometimes, the solution will emerge or become obvious just from seeing a visual representation. See Example 5.3.1 on p. 147.

   (ii) *Divide into smaller problems*. If the problem is complex, break it up into small problems that are easier to solve. See Problem 7.1.1 on p. 277.

[6] Miller (1956) proposed that the limit for our working memory is 7 ± 2 "chunks" of information. A chunk could be collection of digits, letters, words, or objects. You can see that it is hard to remember the digits 2, 0, 2, 4, 5, 6, 1, 4, 1, 4. We can chunk them in the following way: 202 is the area code for Washington D.C., 456 is the successive numbers 4, 5, and 6, and finally two pairs of 14. Now you can easily remember this telephone number—which so happens to be the White House switchboard number: (202) 456-1414. We have the ability to keep more information in our working memory, but this requires us to find a way to chunk efficiently (to find a compact representation of the data in short-term memory by using knowledge from our long-term memory).

(iii) *Find something familiar.* You might recall a similar problem or example that you have already seen. You can see if you can apply a similar strategy to your current problem at hand. See Example 7.1.2 on p. 279, which is similar to Problem 7.1.1 on p. 277. You can also review the proofs of theorems for the key ideas that were used, and decide if these same ideas could be applied to your (specific) problem at hand. For example, Exercise 10.5.4 (b) on p. 464 requires an argument that is similar to the proof used to prove Theorem 10.5.4—we could try to adapt the proof of Theorem 10.5.4 to handle this specific situation.

(iv) *Find a pattern.* You might find a pattern that will continue, and you can generalize. For example, the Fibonacci sequence in Example 6.1.4 on p. 202.

(v) *Take cases.* We can divide the problem into several cases, and find a solution for each of those cases. See Example 3.4.6 on p. 101.

(vi) *Work backwards.* We assume the answer and work backwards until we arrive at the given data in the problem. For example, this is how we can prove Theorem 6.10.3 on p. 260. This can also be done with any of the exercises where you can look at the answer, and try to work backwards to get back to the original data in the exercise.

(vii) *Indirect reasoning.* Consider building a pyramid. There would be two basic ways you can do this. One way is to build the pyramid *directly*: you place one stone on top of another up you reach to the top. The other way is to build the pyramid *indirectly*: you begin with a stone cube, and you chisel and remove the stone pieces until the stone that remains is in the shape of a pyramid. This can be applied to a problem. If you are unable to solve the question in a direct fashion, consider if there is an indirect way; can you start with something and chisel away until you have the desired answer. See Example 5.4.4 on p. 153. You will see that you can usually solve a problem both ways, but one type of reasoning will often be easier to use than the other. Another example of indirect reasoning is proof by contradiction whereby we assume that the conclusion is false, and after encountering a contradiction, then we conclude that the conclusion must be true. For example, we can prove that there are infinitely many prime numbers by this method (see the proof of Theorem 4.1.2 on p. 120).

(c) *Carry out the plan.* Write out a detailed solution to prove each stage is correct.

(d) *Look back*, and ask yourself if your answer or conclusion makes sense. For example, if it is a question about probability, does it make sense that an event occurring has more than 100% chance? (No.) If your answer does not make sense, go back and double check your work, or try a different strategy. Trying a different strategy or method is helpful.

> In order to convince ourselves of the presence or of the quality of an object, we like to see and touch it. And as we prefer perception through two different senses, so we prefer conviction by two different [methods].
>
> —Pólya (1945, p. 15)

If we find two equivalent answers to the problem by two different methods, we increase our confidence that we have the correct answer.

Another suggestion that we often hear in writing essays is to put it away, and look at it later. The same advice can be applied to mathematics. The advantage of completing your homework assignment early is that it gives you a day or two to clear your mind, and look it over later with fresh eyes. You may be amazed at the mistakes that you can catch!

But most important of all, before giving up, try something!

For the more adventurous student, the exercises that are marked with † are meant to be challenging for you.

To the instructor

This book started out as a collection of handwritten lecture notes for a course that I gave in the Fall of 2013 and 2014 at the Royal Military College of Canada to humanities majors. The College required that these students take a course in introductory logic, sets, and discrete probability. The text in use at the time had placed logic as an afterthought in the form of an appendix, and other available logic textbooks for the humanities had tiptoed around the use of mathematical symbols for abstract reasoning and lacked diagrams. What a travesty! Without pictures and diagrams, visual learners (like me) felt isolated. I took it upon myself to create my own set of lecture notes.

A new textbook on basic discrete mathematics for the humanities cannot perpetuate the iniquity of earlier books. This new generation of students, infused by the spirit of the Internet, question authority, require transparency, and demand to know *why*. They expect reasons for any statements or formula that is stated. In this book, we give students what they want in the form of proofs. The proofs provide the rationale behind a concept or a formula, and all of them are all at the level of understanding for a humanities college freshmen.

For example, I could have stated the binomial coefficient formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!},$$

which is *how* to calculate the number of r-element subsets from a total of n elements, and walked away. But, this would have left the student scratching his or her head.

"But, *why* is this formula like this? I demand to know *why* we are dividing by $(n-r)!$ and $r!$. *Why* are we not subtracting?"

The student will become enraged (and rightly so) when met with the authoritative, "Because I say so" or even worse, "That's the way it's been done in mathematics for the past 400 years." In this case, the calming remedy for the student is a combinatorial proof—a proof which counts objects in one way, then counts it in another way, and then concludes by saying that the two counting methods are equivalent. (See Example 5.7.4 on p. 163 for the combinatorial proof of the binomial coefficient formula.)

I caution instructors against skipping proofs. This act signals to the students that the material is not important; this is detrimental. For without the reasoning, the mathematical formula is apt to be forgotten the moment the students burn their notes after their final examination. By going through the proofs, the notes may be gone, but the knowledge remains.

I must confess that when I was charged with the task of teaching to humanities majors at the College, I was nervous: I had grown up and spent many years learning and enjoying mathematics, and I wasn't oblivious to the fact that many students don't find it as enjoyable to say the least (especially the artsmen); these would be the kinds of students I would be teaching. To combat my nervousness and to ease the anxiety of many students, I made logic, sets, and probability more accessible by setting the topics in the context of history, and natural language, as well as in daily and professional life. Moreover, it became apparent to demonstrate the interplay between logic, sets, and probability as opposed to treating these isolated topics as separate islands of thought. By doing this, we can start to apply the concepts of associative learning from psychology to relate ideas of mathematics to one another and to concepts in life, thereby reinforcing this in the student's (long-term) memory.

The exercises are meant to help students gain skills in calculation, in problem solving, and in explaining concepts. Some of my favorite exercises require the student to give an explanation; *e.g.,* Exercise 8.3.10 (p. 318), or 8.5.1 (p. 326).

This book has many exercises for the students and solutions at the back of the book. I am sure there are many arguments against including solutions, but I believe it is to the student's advantage to include them. When I taught the course at the College, I ran an extra tutorial on Monday evenings for two or three hours. During this time, I did very little in terms of helping the students (I spent most of my time editing this book). I found that students were able to work quietly, and when they got stuck, they could ask me for help. Often, the questions were very minor, but could easily cause frustration and waste an hour of the student's time. (With this minimal effort, I received glowing reviews from the class for the extra help.) If we replace me with a set of solutions, the students can check the solutions when they get stuck. The only temptation that the student faces is not trying the problem and directly skipping to the solutions—they need to give it the ol' college try first.

The exercises that are marked with † are meant to be challenging for the student.

Another decision I made in this book was to avoid teaching technology to students; for example, teaching students to use a particular calculator, a particular software package, *etc*. There is such a plethora of technology, for me to pick a single tool would be a disadvantage. As soon as I would be done writing about a particular technology, this book would be out of date. So with this, you are free to select the tools you prefer, and not be hindered by nagging technology boxes found in many other books. I also hold the personal philosophy that everyone should be able to do a quick back-of-the-envelope calculation, so the book reflects this philosophy by emphasizing hand calculations; *sapiens omnia sua secum portat*.

Numbering scheme

The numbering scheme is designed to enable quick location of definitions, theorems, and examples, which are referred to in the text. Definitions, theorems, and examples are numbered sequentially within each Section (*e.g.,* Definition 3.1.2 is in Chapter 3, Section 1, and is the second highlighted item).

How the book was produced

This first edition was produced in LaTeX using MikTeX 2.9 with TeXnicCenter 2. I used the Times 10/12 typeface with mathematics typeset in MathTime Professional 2 produced by Michael Spivak of Publish or Perish, Inc. The section headings are set in Avant Garde. Some of the book design came from the Legrand Orange book originally created by Mathias Legrand. The line illustrations were drawn by me with Tikz developed by Till Tantau using TikzEdt 0.2.3.0 by Thomas Willwacher.

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Trojan Helmet by Javier Triana

For the license plate graphics, I used the *License Plate* typeface by Dave Hansen. Credits for certain illustrations appear in the figure caption.

Mistakes

The mistake that you found—yes, the one you just circled—has been keeping me up at night. I know they exist, but it is difficult to catch them all.

A book is never finished; it is simply abandoned. With this in mind, I will continue to revise for the next printing (or the next edition), and I am always happy to receive corrections and comments. There are many people who helped point out the blunders, flaws, fallacies, miscalculations, inaccuracies, and other miscellaneous grammatical and mathematical howlers, but the rest remain my own fault; no mortal is wise at all times. If you spot an error, please send me an email with the correction and the page number to `richard@math.kohar.ca`. I will be maintaining a list of errata on my website, `www.kohar.ca`.

Website

You can visit my website, `www.kohar.ca`, to obtain supplementary information.

Math isn't so bad

Thank you for choosing to use this book. I hope that you will enjoy reading it. It will teach you some useful and exciting mathematics; it will be a good reference book in the future, and perhaps it might convince you that mathematics isn't so bad after all.

Richard Kohar
Royal Military College of Canada
Kingston, Ontario, Canada
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