# MA10002 Mathematics-II : Tutorial Sheet - 6

1. Determine if each of the following integrals converge or diverge. If the integral converges determine its value.
   (i) $\int_0^\infty (1+2x)\, e^{-x} dx$ (ii) $\int_{-\infty}^1 \sqrt{6-x}\, dx$ (iii) $\int_{-\infty}^\infty \frac{6x^3}{(x^4+1)^2} dx$.

2. Examine the convergence or divergence of the following integrals. If the integral converges determine its value.
   (i) $\int_{-5}^1 \frac{1}{10+2x} dx$ (ii) $\int_1^2 \frac{4x}{\sqrt[3]{x^2-4}} dx$ (iii) $\int_0^4 \frac{x}{x^2-9} dx$ (iv) $\int_0^1 \log t\, dt$ (v) $\int_{-2}^3 \frac{dx}{x-1}$.

3. Test the integral $\int_0^3 \frac{1}{x^2-3x+2} dx$ for its convergence.

4. Discuss the convergence of the following integrals.
   (i) $\int_1^\infty \frac{1}{x^3+1} dx$ (ii) $\int_6^\infty \frac{x^2+1}{x^3(\cos^2 x+1)} dx$ (iii) $\int_2^\infty \frac{1}{\log x} dx$ (iv) $\int_0^\infty e^{-x^2} dx$.

5. Test the integral $\int_1^\infty \frac{x-1}{x^4+2x^2} dx$, if it is convergent or divergent.

6. Test the convergence or divergence of the integral $\int_1^\infty \frac{x \tan^{-1} x}{\sqrt{4+x^3}} dx$.

7. Examine the convergence or divergence of the following integrals.
   (i) $\int_0^{\frac{\pi}{2}} \frac{\cos^m x}{x^n} dx,\ n<1$ (ii) $\int_1^{\frac{\pi}{2}} \frac{\tan x}{x^{3/2}} dx$.

8. Determine if the following integrals converge or diverge.
   (i) $\int_2^5 \frac{x-1}{\sqrt{x}(x-2)} dx$ (ii) $\int_1^2 \frac{\sqrt{x}}{\ln x} dx$.

9. Show that the integral $\int_0^1 \frac{1}{(1+x)(2+x)\sqrt{x(1-x)}} dx$ is convergent.

10. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$, if it is convergent.

11. Show that $\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},\ k^2<1$ is convergent.

12. Discuss the convergence of the integral $\int_1^\infty f(x)\, dx$ , where the function $f(x)$ is given by as follows:

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \text{ is rational number} \\ -\frac{1}{x^2} & \text{if } x \text{ is irrational number} \end{cases}$$

13. Prove that $\int_1^\infty e^{-x} x^{m-1}\, dx$ is convergent for $m>0$.

14. Show that $\int_1^\infty \sin x \log(\sin x) dx$ converges and find its value.

15. Find the value of the integrals $\int_0^{\frac{\pi}{2}} \log(\sin x) dx$ and $\int_0^{\frac{\pi}{2}} \log(\cos x) dx$ by discussing their convergence.

16. Show that the integral $\int\limits_{-1}^{1} \frac{\sin x}{x} dx$ is a proper integral.

17. Show that $\int\limits_{1}^{\infty} \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x} dx = \frac{\pi}{2} \log(\frac{a}{b}), 0 < b < a.$

18. Let $f(x,t) = (2x + t^3)^2$ then

    (i) find $\int\limits_{0}^{1} f(x,t)\, dx$

    (ii) Prove that $\frac{d}{dt} \int\limits_{0}^{1} f(x,t)\, dx = \int\limits_{0}^{1} \frac{\partial}{\partial t} f(x,t)\, dx$

19. i) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,t) = \begin{cases} \frac{\sin xt}{t} & \text{if } t \neq 0 \\ x & \text{if } t = 0 \end{cases}$$

    Find $F'$, where $F(x) = \int\limits_{0}^{\frac{\pi}{2}} f(x,t)\, dt$.

    ii) Given $f : x \to \int\limits_{0}^{x^2} \tan^{-1} \frac{t}{x}\, dt$, find $f'$.

20. For any real numbers $x$ and $t$, let

$$f(x,t) = \begin{cases} \frac{xt^3}{(x^2+t^2)^2} & \text{if } x \neq 0, t \neq 0 \\ 0 & \text{if } x = 0, t = 0 \end{cases}$$

    and $F(t) = \int\limits_{0}^{1} f(x,t)\, dx$. Is $\frac{d}{dt} \int\limits_{0}^{1} f(x,t)\, dx = \int\limits_{0}^{1} \frac{\partial}{\partial t} f(x,t)\, dx$? Give the justification.

21. Find the value of the integral $\int\limits_{0}^{\infty} \frac{e^{-bx} \sin ax}{x} dx$, where $a > 0, b > 0$ are fixed, and hence deduce the value of the integral $\int\limits_{0}^{\infty} \frac{\sin ax}{x} dx$.

22. Find the value of the following integrals

    i) $\int\limits_{0}^{\infty} \frac{e^{-bx}(1 - \cos ax)}{x} dx, b > 0$

    ii) $\int\limits_{0}^{\frac{\pi}{2}} \log(1 - x^2 \sin^2 \theta)\, d\theta, |x| < 1$

    iii) $\int\limits_{0}^{\infty} \frac{e^{-px} \cos qx - e^{-ax} \cos bx}{x}\, dx$

    iv) $\int\limits_{0}^{\infty} e^{-x^2} \cos 2ax\, dx$