FOURTH ORDER METHOD: (EXPLICIT)

$$u_{j+1} = u_j + h\left[w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4\right]$$

$$k_1 = f(t_j, u_j)$$

$$k_2 = f(t_j + c_2 h,\ u_j + h a_{21} k_1)$$

$$k_3 = f(t_j + c_3 h,\ u_j + h a_{31} k_1 + h a_{32} k_2)$$

$$k_4 = f(t_j + c_4 h,\ u_j + h a_{41} k_1 + h a_{42} k_2 + h a_{43} k_3)$$

Classical Runge-Kutta Method:

$$u_{j+1} = u_j + h \cdot \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = f(t_j, u_j)$$

$$k_2 = f\left(t_j + \frac{h}{2},\ u_j + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(t_j + \frac{h}{2},\ u_j + \frac{h}{2} k_2\right)$$

$$k_4 = f(t_j + h,\ u_j + h k_3)$$

Table form:

$$\begin{array}{c|cccc}
c_2 & a_{21} & & & \\
c_3 & a_{31} & a_{32} & & \\
c_4 & a_{41} & a_{42} & a_{43} & \\
\hline
 & w_1 & w_2 & w_3 & w_4
\end{array}$$

$$\begin{array}{c|cccc}
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6}
\end{array}$$

← Classical

$$\begin{array}{c|cccc}
\frac{1}{3} & \frac{1}{3} & & & \\
\frac{2}{3} & -\frac{1}{3} & 1 & & \\
1 & 1 & -1 & 1 & \\
\hline
 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8}
\end{array}$$

← Kutta

Minimum number of function evaluation versus order

ORDER	2	3	4	5	6	7	8	.....
MNFE	2	3	4	6	7	9	11	

Remark: The order of an s-stage explicit method (RK) can not be greater than s.

Also, there does not exist a s-stage method (explicit RK) with order s if $s \geq 5$.

Example: Apply Classical Runge-Kutta Method to get $y(0.3)$ with step size $h = 0.1$ for the problem

$$\frac{dy}{dt} = t + y, \quad y(0) = 1.$$

Sol: Classical Runge-Kutta Method

$$K_1 = f(t_j, u_j)$$

$$K_2 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} K_1\right)$$

$$K_3 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2} K_2\right)$$

$$K_4 = f(t_j + h, u_j + h K_3)$$

$$u_{j+1} = u_j + \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right).$$

$h = 0.1$

$u_0 = 1.$

u_1, u_2, u_3 ?

$\underline{j=0:}$

$$k_1 = f(0,1) = 0+1 = 1$$

$$k_2 = f\left(\frac{0.1}{2}, 1+\frac{0.1}{2}\right) = 1.1$$

$$k_3 = f\left(\frac{0.1}{2}, 1+\frac{0.1}{2}\times 1.1\right) = 1.105$$

$$k_4 = f(0.1, 1+0.1\times 1.105) = 1.2105$$

$$u_1 = 1 + \frac{0.1}{6}\left[1 + 2\times 1.1 + 2\times 1.105 + 1.2105\right]$$

$$= 1.110341667$$

$\underline{j=1:}$

$$k_1 = 1.210341667$$

$$k_2 = 1.320858750$$

$$k_3 = 1.326384604$$

$$k_4 = 1.442980127$$

$$u_2 = 1.242805142$$

$j=2:$

$$k_1 = 1.442805142$$

$$k_2 = 1.564945399$$

$$k_3 = 1.57105241195$$

$$k_4 = 1.69991038319$$

$$u_3 = 1.39971699 44$$

t	exact y	Numerical y
0.1	1.110341836	1.110341667
0.2	1.242805516	1.242805142
0.3	1.399717615	1.399716994

exact solution

$$y = -1-t+2e^t$$

Ex. Use the Runge-Kutta Method to approximate the particular solution at $x=1$ of the differential equation $y' = xy$ through $(0,1)$.

Sol: $h = 1$; $y(0) = 1$

$K_1 = f(0,1) = 0.$

$K_2 = f(0.5, 1) = 0.5 \times 1 = 0.5.$

$K_3 = f(0.5, 1 + \frac{1}{2} \times 0.5) = (0.5)(1.25) = 0.625$

$K_4 = f(1, 1 + 0.625) = (1)(1.625) = 1.625$

$$u_1 = u_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1.0}{6}(0 + 2 \times 0.5 + 2 \times 0.625 + 1.625)$$

$$= 1.646$$

Ex: Consider a scalar problem

$$y' = -y^2 \qquad y(1) = 1$$

The exact solution is $y(t) = \frac{1}{t}$. Compute the numerical solution at $t = 1.5$ using $h = 0.5$.

Sol:

$$k_1 = f(\underset{1}{\underset{\parallel}{t}}, \underset{1}{\underset{\parallel}{y}}) = -1$$

$$k_2 = f\left(1 + \frac{0.5}{2},\ 1 + \frac{0.5}{2} \times (-1)\right)$$

$$= -\left(\frac{1.5}{2}\right)^2 = -0.5625$$

$$k_3 = f\left(1 + \frac{0.5}{2},\ 1 + \frac{0.5}{2} \times (-0.5625)\right)$$

$$= -0.738525390625$$

$$k_4 = f(1 + 0.5,\ 1 + 0.5 \times -0.738525390625$$

$$= -0.397829547524452$$

$$u_1 = u_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{0.5}{6}(\ \ldots\ldots\ldots\ )$$

$$= 0.\underline{6666}766639268796$$

EXACT SOLUTION: $0.\underline{6666}666\ldots\ldots$