Lecture 5

Date 31/07/2017

Proof Th-9

(ie, Integration of the transform of a f^n $f(t)$ corresponds to the division of $f(t)$ by t)

Let $\boxed{g(t) = \dfrac{f(t)}{t}}$

so that $f(t) = t\, g(t)$.

We now use the property

$$\mathcal{L}\{t\, g(t)\} = -\frac{d}{ds}\mathcal{L}\{g(t)\}$$

We deduce that

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t\, g(t)\}$$

$$= -\frac{d}{ds}\mathcal{L}\left\{\frac{f(t)}{t}\right\}. \rightarrow (i)$$

Integrating both sides of (i) w.r.t s from s to ∞ gives

$$\int_s^\infty F(u)\,du = -\left[\mathcal{L}\left\{\frac{f(t)}{t}\right\}\right]_s^\infty$$

$$= +\,\mathcal{L}\left\{\frac{f(t)}{t}\right\}\Big|_s$$

$$= \mathcal{L}\left\{\frac{f(t)}{t}\right\}.$$

since $\mathcal{L}\left\{\frac{f(t)}{t}\right\} \to 0$ as $s \to \infty$.

$$Si(t) = \int_0^t \frac{\sin u}{u}\,du$$

Let $f(t) = \sin(t)$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du.$$

to find

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{du}{u^2+1}$$

$$= \left[\tan^{-1} u\right]_{u=s}^{\infty}$$

$$\left[\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}\right.$$

$$= \pi/2 - \tan^{-1}(s)$$

$$= \tan^{-1}\left(\frac{1}{s}\right) \quad (\underline{\underline{\text{how??}}})$$

We now use the result

$$\mathcal{L}\left(\int_0^t f(u)\,du\right) = \frac{F(s)}{s}$$

to deduce that

$$\mathcal{L}\left(\int_0^t \frac{\sin u}{u}\,du\right) = \mathcal{L}(\mathrm{Si}(t))$$

$$= \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$$

$$\left[\begin{array}{l} f(u) = \frac{\sin u}{u} \\ F(s) = \mathcal{L}\{f(u)\} \\ \quad = \tan^{-1}(1/s) \end{array}\right.$$

§/ Differential eqn, Initial Value Problems (I.V.P)

$$y'' + ay' + by = r(t),$$

$$y(0) = k_0, \quad y'(0) = k_1; \longrightarrow (i)$$

where a & b are constants.

Here, $r(t)$ is the input (driving force) applied to the mechanical system & $y(t)$ is the output (or response of the system).

In Laplace's method, we do three (main) steps:-

1st step

We transform (i) by means of

$$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)$$

& $$\mathcal{L}(f'') = s^2\mathcal{L}(f) - s\,f(0) - f'(0).$$

Writing $Y = \mathcal{L}\{y\}$ & $R = \mathcal{L}\{r\}$. this gives

$$\mathcal{L}\{y'' + a y' + b y\} = \mathcal{L}\{r(t)\}$$

$$\Rightarrow \mathcal{L}\{y''\} + a\,\mathcal{L}\{y'\} + b\,\mathcal{L}\{y\} = \mathcal{L}\{r(t)\}$$ [why?]

$$\Rightarrow \left[s^2\mathcal{L}\{y(t)\} - s\,y(0) - y'(0)\right] + a\left[\mathcal{L}\{y(t)\} - y(0)\right] + b\,\mathcal{L}\{y(t)\} = R(s)$$

$$\Rightarrow \left[s^2 Y - s\, y(0) - y'(0)\right]$$

$$+ a\left[s\, Y(s) - y(0)\right] + bY = R(s)$$

This is called the subsidiary equation.

Collecting Y — terms, we have

$$(s^2 + as + b)\,Y = (s+a)\, y(0) + y'(0) + R(s)$$

2nd step: We solve the subsidiary eq^n algebraically for Y

Division by (s^2+as+b) & use of the so-called transfer function

$$Q(s) = \frac{1}{s^2+as+b} \rightarrow (2)$$

gives the solution

$$Y(s) = [(s+a)\,y(0) + y'(0)]\,Q(s) + R(s)\,Q(s) \rightarrow (3)$$

If $\boxed{y(0) = y'(0) = 0}$

this is simply $Y = RQ$

and Q is the quotient

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

Q depends only on a & b

but neither on $r(t)$ nor on

the initial conditions.

3rd step :- we reduce (3)
(usually by partial fractions, as in calculus) to a sum of terms whose inverses can be found from the table, so that the solution

$$y(t) = \mathcal{L}^{-1}(Y(s))$$

of eq^n (1) is obtained.

Ex1/ Solve:

$$y'' - y = t,$$
$$y(0) = 1, \; y'(0) = 1$$

Solve:- 1st step:- $\mathcal{L}\{y'' - y\} = \mathcal{L}(t)$

$$\Rightarrow \mathcal{L}\{y''\} - \mathcal{L}\{y\} = 1/s^2$$

$$\Rightarrow \left[s^2 Y - s\,\underset{(=1)}{y(0)} - \underset{(=1)}{y'(0)}\right] - Y = 1/s^2$$

$$\boxed{\Rightarrow (s^2 - 1)\,Y = s + 1 + 1/s^2}$$

2nd step:- The transfer function is $Q = \dfrac{1}{(s^2-1)}$ & we get we get

$$Y = (s+1)\,Q + \frac{1}{s^2}\,Q.$$

$$= \frac{s+1}{s^2-1} + \frac{1}{s^2(s^2-1)}$$

$$= \frac{1}{s-1} + \left[\frac{1}{s^2-1} - \frac{1}{s^2}\right]$$

3rd step :- From this expression for Y we obtain the sol^n

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) - \mathcal{L}^{-1}\left(1/s^2\right).$$

$$= \boxed{e^t + \sinh t - t}$$

$$y(t) = \boxed{e^{t} + \sinh t - t}$$

t - space

Given problem

$y'' - y = t$

$y(0) = 1$

$y'(0) = 1$

$\longrightarrow$

s - space

Subsidiary eq^n

$(s^2-1)Y = s + \frac{1}{s^2} + 1$

sol^n of the given problem

$y(t) = e^{t} + \sinh t - t$

$\longleftarrow$

sol^n of subsidiary eq^n

$Y = \frac{1}{(s-1)} + \frac{1}{(s^2-1)} - \frac{1}{s^2}$

Comparison

C.W.

EX/ Solve the IVP

$$y'' + 2y' + y = e^{-t}$$

$$y(0) = -1, \quad y'(0) = 1.$$

$(s^2+2s+1)\,Y = (s+1)^2 Y = -s-1+\frac{1}{(s+1)}$

$$Y = \frac{-s-1}{(s+1)^2} + \frac{1}{(s+1)^3}.$$

$$\therefore y(t) = \mathcal{L}^{-1}(Y(s))$$

$$= -\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$$

$$= -e^{-t} + \frac{1}{2}t^2 e^{-t}. \quad [\text{how?}]$$

$$\therefore y(t) = \left(\frac{1}{2}t^2 - 1\right)e^{-t}.$$

By First shifting theorem.

$$\mathcal{L}(t^2) = 2/s^3$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}[e^{-t}.\, t^2] = 2/(s+1)^3$$

check the other way. H.W

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} = \left(\frac{e^{-t}.\, t^2}{2}\right)$$

Shifted data Problems:

(I.V.P)

$t = t_0$ instead of $t = 0$.

First sol^n method:-

Find the General sol^n by L.T & from it find the sol^n of the problem as in the classical method.

Second Method:-

Let $t = \tilde{t} + t_0$,

so that $t = t_0$, gives $\boxed{\tilde{t} = 0}$

& the L.T becomes applicable throughout.

Solve :- $y''(t) + y(t) = 2t$,

$y(\pi/4) = \pi/2$, $y'(\pi/4) = 2-\sqrt{2}$.

Sol^n: we have $t_0 = \pi/4$.

& we set $t = \tilde{t} + \pi/4$

Then the given problem is

$$\tilde{y}'' + \tilde{y} = 2\left(\tilde{t} + \pi/4\right)$$

where $\tilde{y}(\tilde{t}) = y(t)$.

as $\left[y(t) = y(\tilde{t} + \pi/4) = \tilde{y}(\tilde{t}) \rightarrow \text{new } f^n \right]$

$$\tilde{y}(0) = \pi/2, \quad \tilde{y}'(0) = 2-\sqrt{2}$$

$\boxed{\text{when } t = \pi/4, \ \pi/4 = \tilde{t} + \pi/4}$ $\Rightarrow \tilde{t} = 0$

1st step :-

$$\mathcal{L}\{\tilde{y}''\} + \mathcal{L}\{\tilde{y}\} = 2\,\mathcal{L}(\tilde{t}) + 2\,\mathcal{L}(\pi/4)$$

(Setting up of the subsidiary eqn) 15-

$$\Rightarrow s^2\tilde{Y} - s\underbrace{\tilde{y}(0)}_{(=\pi/2)} - \underbrace{\tilde{y}'(0)}_{(=2-\sqrt{2})} + \tilde{Y} = 2/s^2 + \frac{\pi/2}{s}.$$

where, $\tilde{Y} = \mathcal{L}\{\tilde{y}\}$

2nd step. - Soln of the subsidiary eqn

$$\Rightarrow \tilde{Y} = \frac{2}{(s^2+1)s^2} + \frac{\pi/2}{s(s^2+1)} + \underbrace{\tilde{y}(0)}_{(=\pi/2)}\cdot\frac{s}{s^2+1} + \underbrace{\tilde{y}'(0)}_{(=2-\sqrt{2})}\cdot\frac{1}{s^2+1}$$

$\left[\frac{(s^2+1)-s^2}{s(s^2+1)}\right.$ (by partial fraction)

$$\Rightarrow \tilde{y} = \mathcal{L}^{-1}(\tilde{Y})$$

$$= 2(\tilde{t} - \sin\tilde{t}) + \pi/2(1-\cos\tilde{t}) + \pi/2\cos\tilde{t} + (2-\sqrt{2})\sin\tilde{t}$$

substituting $\boxed{\tilde{t} = t - \pi/4}$

cancelling terms

& using $\cos \pi/4 = \sin \pi/4 = \frac{1}{\sqrt{2}}$

we obtain

EX

$$\boxed{y(t) = 2t - \sin t + \cos t}$$

— x —

Unit Step Function

$$u(t-a) \quad \mathcal{U}(t-a)$$

Def^n:- $u(t-a)$ is 0 for $t<a$ & has a jump of size 1 at $t=a$ (where we can leave it undefined) & is 1 for $t>a$.

$$u(t-a) = \begin{cases} 0, & \text{if } t<a \\ 1, & \text{if } t>a \end{cases}$$

$a \geq 0$

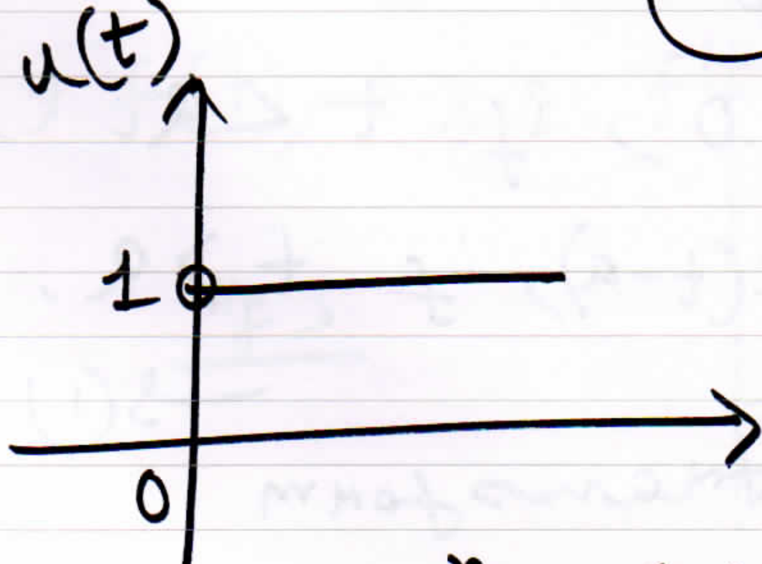


Fig1 Unit step f^n $u(t)$

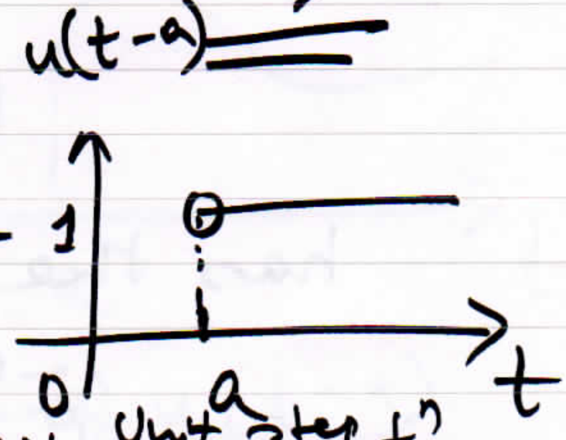


Fig2:- Unit step f^n $u(t-a)$.

Th-10

Second shifting theorem / t-shifting / Second Translation theorem

If $f(t)$ has the transform $F(s)$, then the "shifted function"

$$\tilde{f}(t) = f(t-a)\,u(t-a)$$

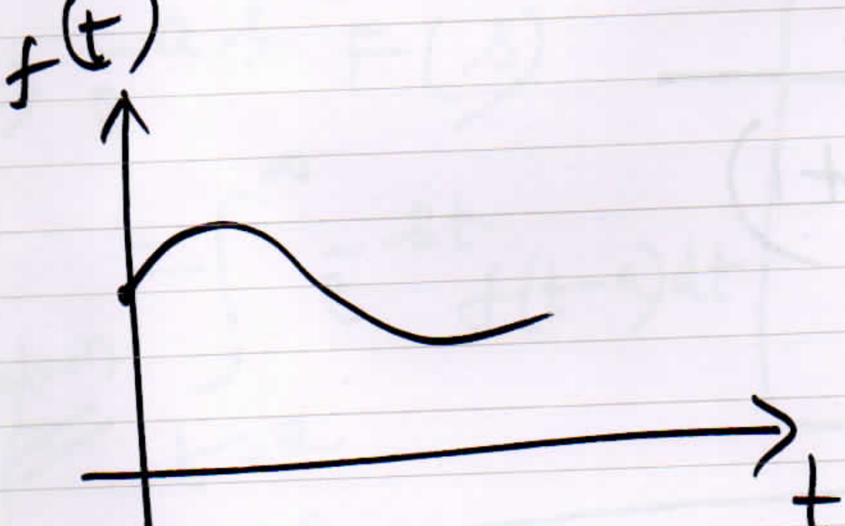
$$= \begin{cases} 0, & \text{if } t < a \\ f(t-a), & \text{if } t > a. \end{cases} \longrightarrow (1)$$

has the transform $e^{-as} F(s)$.

ie, $$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s) \longrightarrow (2)$$

or, if we take the inverse on both sides, we can write

$$f(t-a)\,u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}.$$



(a) $f(t),\ t \geq 0$

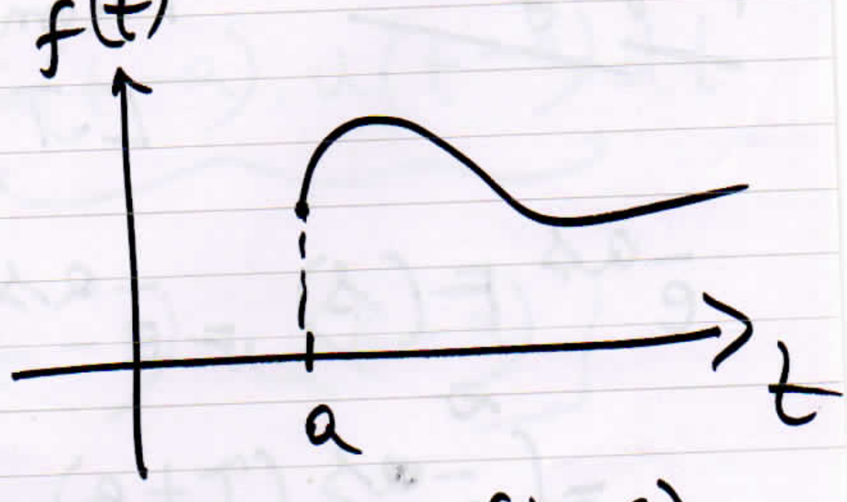


fig 3

(b) $f(t-a)\,u(t-a)$

Shift on the t-axis.