Date
08/08/2017

-1-

Lecture 8

Short Impulses or Dirac's Delta function

Impulses :- In Mechanics, the impulse of a force $f(t)$ over a time interval say, $a \le t \le a+k$, is defined to be the integral of $f(t)$ from a to $a+k$.

$$\int_a^{a+k} \underline{\qquad} \qquad \boxed{k \to 0}$$

[Phenomena of an impulsive nature, such as the action of very large forces over very short intervals of time, are of great practical interest since they arise in various applications. eg, when a tennis ball is hit by a racket etc.]

Let

$$f_k(t-a) = \begin{cases} 1/k, & \text{if } a \le t \le a+k \\ 0, & \text{otherwise} \end{cases} \quad \longrightarrow (1)$$

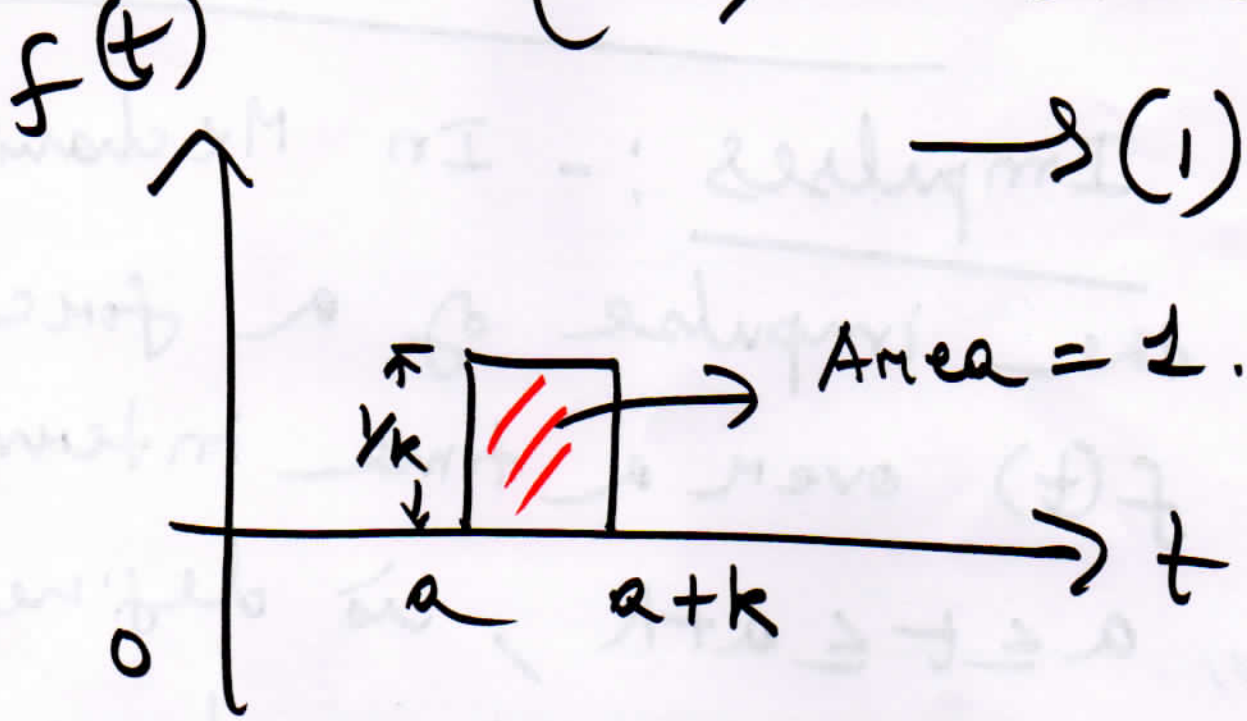

Fig 1 :- The function $f_k(t-a)$

Its impulse I_k is 1 (how?)

$$I_k = \int_0^\infty f_k(t-a)\,dt$$

$$= \int_a^{a+k} (1/k)\,dt = 1/k\,[t]_a^{a+k}$$

$= 1/k \cdot k = 1.$

$$\therefore \quad I_k = \int_0^\infty f_k(t-a)\,dt$$

$$= 1. \longrightarrow (2)$$

$$f_k(t-a) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right]$$

(how?)

$$\mathcal{L}\{f_k(t-a)\} = \frac{1}{k}\,\mathcal{L}\{u(t-a)\}$$

$$- \frac{1}{k}\,\mathcal{L}\{u(t-(a+k))\}$$

$$= \frac{1}{ks}\left[e^{-as} - e^{-(a+k)s}\right]$$

$$= e^{-as}\left[\frac{(1-e^{-ks})}{ks}\right] \longrightarrow (3)$$

The limit of $f_k(t-a)$ as $k \to 0$ $(k>0)$ is denoted by $\delta(t-a)$,

i.e., $$\boxed{\delta(t-a) = \lim_{k \to 0} f_k(t-a)}$$

$\delta(t-a)$ is called the <u>Dirac delta function</u>

or, <u>Unit impulse function</u>

$$\mathcal{L}\{\delta(t-a)\} = \mathcal{L}\left\{\lim_{k \to 0} f_k(t-a)\right\}$$

$$= e^{-as} \cdot \left\{\lim_{k \to 0} \frac{1-e^{-ks}}{ks}\right\}$$

$$\boxed{\mathcal{L}\{\delta(t-a)\} = e^{-as}.}$$

$\underline{\delta(t-a)}$ is $\underline{\text{not}}$ a function in the ordinary sense as used in Calculus

$$\lim_{k \to 0} \left(\frac{1 - e^{-ks}}{ks} \right)$$

(L' Hospital's) $\left(\frac{0}{0} \text{ form} \right)$

Diff. N & D w.r.t k

$$= \lim_{k \to 0} \frac{s e^{-ks}}{s}$$

$$= 1.$$

$\boxed{\text{" generalized function "}}$

because from eq^n (1), we get

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a)$$

$$= \begin{cases} \infty, & \text{if } t = a \\ 0, & \text{otherwise.} \end{cases}$$

$$= \int_0^\infty \underline{\underline{\delta}}(t-a)\,dt$$

$$= \int_0^\infty \lim_{k \to 0} f_k(t-a)\,dt.$$

$$= 1. \quad (\text{how ??}).$$

But an ordinary f^n that is everywhere 0 except at a single point must have the integral $\underline{0}$.

(Investigate ??)

Ex 1/ Determine the response of the damped mass-spring system governed by

$$y'' + 3y' + 2y = r(t), \quad y(0) = y'(0) = 0,$$

where $r(t)$ is

a) the square wave $r(t) = u(t-1) - u(t-2)$

b) the unit impulse at time $t = 1$, ie, $\delta(t-1)$.

Sol^n:- a) $\mathcal{L}\{y''\} + 3\,\mathcal{L}\{y'\} + 2\,\mathcal{L}\{y\} = \mathcal{L}\{r(t)\}$

$$\Rightarrow \left[s^2 Y - s\,y(0) - y'(0) \right] + 3\left[sY - y(0) \right] + 2Y = \mathcal{L}\{r(t)\}$$

$$\Rightarrow \underbrace{(s^2+3s+2)}_{=(s+1)(s+2)} Y(s) = \frac{1}{s}\left[e^{-s} - e^{-2s} \right]$$

(how??)

$$\Rightarrow Y(s) = \frac{1}{s(s+1)(s+2)} \left(e^{-s} - e^{-2s} \right)$$

is the subsidiary eqn.

$$\Rightarrow Y(s) = F(s) \left(e^{-s} - e^{-2s} \right)$$

where

$$F(s) = \frac{1}{s(s+1)(s+2)}$$

In terms of partial fractions

$$F(s) = \frac{1/2}{s} - \frac{1}{(s+1)} + \frac{1/2}{(s+2)}$$

$$f(t) = \mathcal{L}^{-1}\left(F(s)\right)$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$+ \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\Rightarrow f(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \rightarrow (1)$$

By <u>Second shifting theorem</u> (why ??) we have

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{F(s)\cdot\left(e^{-s} - e^{-2s}\right)\right\}$$

$$= \mathcal{L}^{-1}\left\{F(s)\cdot e^{-s} - F(s)e^{-2s}\right\}$$

$$\boxed{y(t) = f(t-1)\,u(t-1) - f(t-2)\,u(t-2).}$$

$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$

$\Rightarrow f(t-a)\,u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$

Here, $a=1$ & $a=2$

$$\Rightarrow y(t) = \begin{cases} 0, & 0<t<1 \\ f(t-1), & 1<t<2 \\ f(t-1) - f(t-2), & t>2 \end{cases}$$

$$= \begin{cases} 0, & 0<t<1 \\ \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}, & 1<t<2 \\ -e^{-(t-1)} + e^{-(t-2)} + \frac{1}{2}e^{-2(t-1)} - \frac{1}{2}e^{-2(t-2)}, & t>2. \end{cases}$$

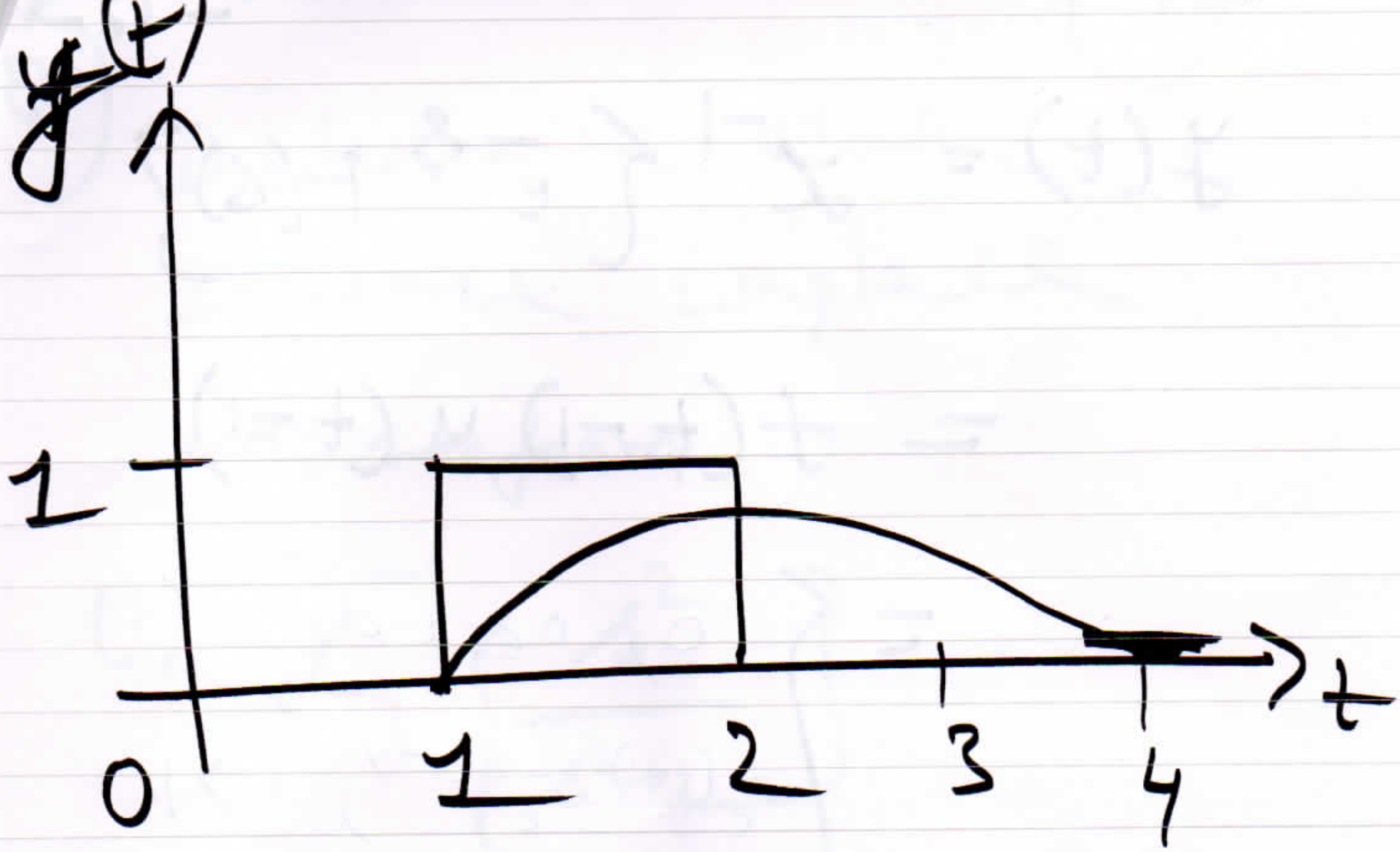

Fig 2 Square wave & response

(B) $y'' + 3y' + 2y = \delta(t-1)$

$\Rightarrow Y(s) = F(s) \cdot e^{-s}$

where

$$F(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

$$y(t) = \mathcal{L}^{-1}\left\{ e^{-s} F(s) \right\}$$

$$= f(t-1)\, u(t-1)$$

$$= \begin{cases} 0, & 0 \le t < 1 \\ e^{-(t-1)} - e^{-2(t-1)}, & t > 1 \end{cases}$$

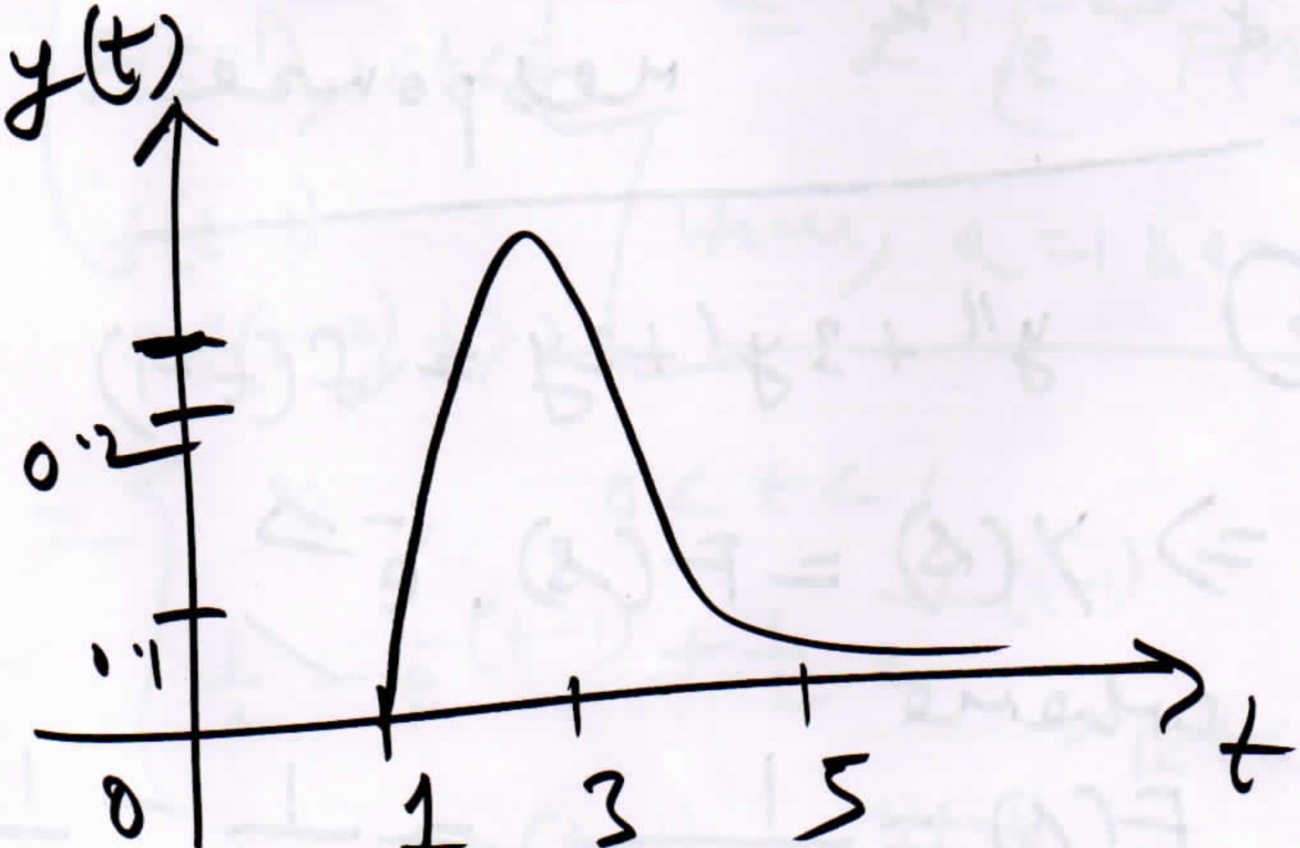

Fig3:- Response to a hammer blow at t=1

Q) Determine the Inverse Laplace transform

(i) $\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\}$

(ii) $\mathcal{L}^{-1}\left\{\frac{s^3}{s^2+1}\right\}$.

— x —