Mathematics - II (MA 10002)

Lecture 1
5/1/17

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1. Linear algebra.
2. Numerical Analysis.
3. Integral calculus.
4. Vector Calculus.

Part-1, 2, 4.
Text Book:
Kreyszig - vol. 8/9/10 Adv. Eng. Maths.

Part-3.
Text Book.
Piskunov / Shantinarayan & Mittal Integral calculus.

Linear Algebra.

In linear algebra we will study about
- Vector space & linear mappings between these spaces.

Some sets whose elements satisfy certain properties under certain defined 'operations'.

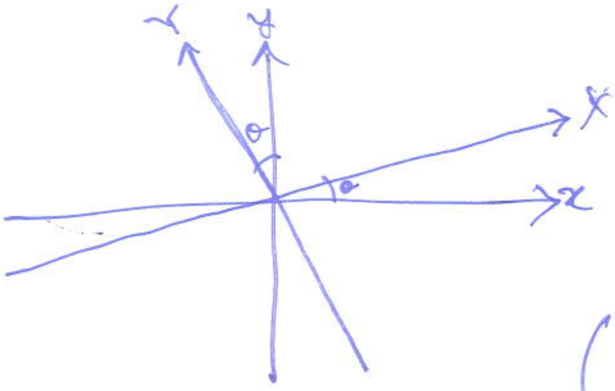
$2+3=5$ $\quad (2,3)+(-1,5)=(1,8)$

$y = 3x$, linear mapping. $T: R \to R$.

$\therefore T(x) = 3x$ ($T \to$ linear).

$y = x^2$ $\quad T: R \to R$, $T(x) = x^2$ ($T \to$ non-linear).

$T(x,y) = (x+1, y+1)$ $\quad T: \mathbb{R}^2 \to \mathbb{R}^2$ ($T \to$ non-linear)



$X = x\cos\theta - y\sin\theta$

$Y = x\sin\theta + y\cos\theta$.

(linear transformation)

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

System of equations & matrices play an important role in linear algebra

Gaussian elimination method for solving system of equations.

non-homogeneous system

Ex-
$$\boxed{\begin{aligned} E_1&: x + y + 2z = 9 \\ E_2&: 2x + 4y - 3z = 1 \\ E_3&: 3x + 6y - 5z = 0 \end{aligned}}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}}_{\text{coefficient matrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

Keep E_1 unchanged.

step 1 Eliminate x from E_2 and E_3.

$$\left.\begin{aligned} E_2&: -2E_1 + 1\times E_2 \\ E_3&: -3E_1 + 1\times E_3 \end{aligned}\right\} E_i \to -a_{i1}E_1 + a_{11}E_i$$

$E_1: x + y + 2z = 9$.

$E_2: 2y - 7z = -17$.

$E_3: 3y - 11z = -27$.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -7 \\ 0 & 3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -17 \\ 27 \end{bmatrix}$$

step 2. Keep E_1 & E_2 unchanged.

$E_i \to -a_{i2}E_2 + a_{22}E_i \quad (i=3)$

$(E_3 \to -\overset{(=3)}{a_{32}}E_2 + \overset{(=2)}{a_{22}}E_3)$.

$$\boxed{\begin{gathered} x=1,\ y=2 \\ z=3. \end{gathered}}$$

$E_1: x + y + 2z = 9$

$E_2: 2y - 7z = -17$.

$E_3: -z = -3$.

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -17 \\ -3 \end{bmatrix}$$

step-3 Back substitution

From E_3, $z = 3$, From E_2, $y = \frac{7z-17}{2} = 2$

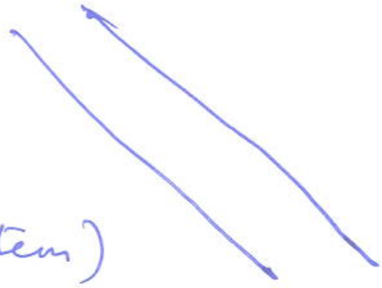
From E_1, $x = 9 - y - 2z = 1$.

$2x + 3y = 6.$

$6x + 9y = 19.$

(no solutions to this system)

geometrically, these represent two $\parallel$ lines,



$2x + 3y = 6.$

$6x + 9y = 18.$

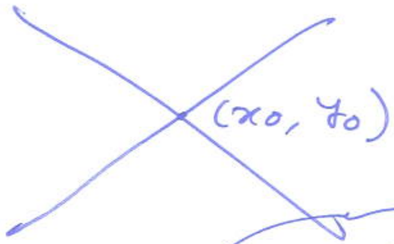
(infinitely many solutions)

(geometrically, these represent a single line).

$2x + 3y = 6.$

$x - 2y = 4.$

$x = 3\frac{3}{7}, \; y = -\frac{2}{7}.$



(geometrically this system represents, that two lines meet at a unique point)

$2x + 3y = 6$

$2x - 4y = 8$

$7y = -2$

$x = \frac{6 - 3y}{2} \quad y = -\frac{2}{7}.$

$= \frac{6 + \frac{6}{7}}{2} = 3 + \frac{3}{7}.$

2. Solve

E_1: $\underset{}{\overset{a_{11}}{2}}x + y - 2z + 3w = 1$

E_2: $\overset{a_{21}}{3}x + 2y - z + 2w = 4$

E_3: $\underset{a_{31}}{3}x + 3y + 3z - 3w = 5$

Solution. Step-1 Keep E_1 unchanged.

eliminate x from E_2, E_3.

$E_i \to -a_{i1} E_1 + a_{11} E_i$; $i = 2, 3$.

E_1: $2x + y - 2z + 3w = 1$

E_2: $y + 4z - 5w = 5$

E_3: $3y + 12z - 15w = 7$.

step. 2. Keep E_1 and E_2 unchanged

eliminate y from E_3. $E_3 \to -3E_2 + 1 \times E_3$.

E_1: $2x + y - 2z + 3w = 1$.

E_2: $y + 4z - 5w = 5$

E_3. $0 = 8$

(absurd!!!)

$\therefore$ system has no solution.

3. $E_1: x + 2y - 2z + 3w = 2$.

$E_2: 2x + 4y - 3z + 4w = 5$

$E_3: 5x + 10y - 8z + 11w = 12$

Eliminate x from E_2, E_3.

~~Sol~~ step 1: $E_1: x + 2y - 2z + 3w = 2$

$(-2E_1 + 1.E_2)$ $E_2: \quad z - 2w = 1$.

$(-5E_1 + 1.E_3)$ $E_3: \quad 2z - 4w = 2$.

Eliminate $\cancel{x}$ z from E_3.

$E_1: x + 2y - 2z + 3w = 2$

$E_2: \quad z - 2w = 1$

$E_3: \quad 0 = 0$.

$(-2E_2 + 1 \times E_3)$.

Put $w = 0$, $y = 0$, $z = 1$ (from E_2), $x = 2 + 2 = 4$. (from E_1)

$(4, 0, 1, 0)$

Put $w = 1$, $z = 3$, $y = -1$, $x = 2 - 2y + 2z - 3w$

$x = 2 + 2 + 6 - 3 = 7$.

$(7, -1, 3, 1)$

$w = d$ (arbitrary), $y = b$ (arbitrary)

($y, w \to$ free variables)

$z = 1 + 2w = 1 + 2d$

$x = 2 - 2y + 2\cancel{y}z - 3w$

$= 2 - 2b + 2(1 + 2d) - 3d$

$x = 4 - 2b + d$. $(x, y, z, w) = (4 - 2b + d, b, 1 + 2d, d)$

Thm: A non-homogeneous system of linear equations where no. of unknowns is more than the no. of equations, the system either has no solution or has infinitely many solutions.

Ex1, Ex2, Ex3 give examples of non-homogeneous systems where atleast one of the elements in the r.h.s. of the equations is non-zero,

$$\left.\begin{aligned} 2x + y - 2z + 3w &= 0 \\ 3x + 2y - z + 2w &= 0 \\ 3x + 3y + 3z - 3w &= 0 \end{aligned}\right\} \text{homogeneous system of equation}$$

$x=0,\ y=0,\ z=0,\ w=0$ is always a solution to this homogeneous system

(Note: 0 is always a solution of homogeneous system, so it is never inconsistent)

syst. of equations (non-homogeneous)

→ consistent (has a solution)
  → unique solution (Ex-1)
  → infinitely many solutions (Ex-3)

→ inconsistent (no-solution) (Ex-2)

Homogeneous system
(always consistent)

unique solution ($\underset{\sim}{0}$ solution).

infinitely many sol.

$\left(\begin{array}{l} x = 0,\ y = 0,\ z = 0,\ w = 0. \\ (0,0,0,0) = \underset{\sim}{0} \end{array} \right)$ next class.

Thm. A homogeneous system of linear equations where no. of unknowns is more than the no. of equations, the system has infinitely many solutions.

Ex1
$$\begin{array}{c} (\text{Row}1) \\ R_1 \\ R_2 \\ R_3 \end{array}\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \xrightarrow[R_3': -3R_1 + 1 \times R_3]{R_2': -2R_1 + 1 \times R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -7 \\ 0 & 3 & -11 \end{bmatrix} \xrightarrow[+2R_3]{R_3 \to -3R_2} \begin{bmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{2} & -7 \\ 0 & 0 & \boxed{-1} \end{bmatrix}$$

distinguished elements 1, 2, −1

(echelon form)

$$\left. \begin{array}{l} R_i \leftrightarrow R_j \\ R_i \to \underset{(\text{real no.})}{k} R_i \\ R_i \to R_i + \underset{(\text{real no.})}{k'} R_j \end{array} \right\} \text{3 basic row operations}$$

$R_i \to kR_i + k'R_j$

distinguished elements 1, −2

$$\begin{bmatrix} 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{-2} \\ 0 & 0 & 0 \end{bmatrix}$$ (echelon matrix).

echelon-matrix: A matrix is said to be an echelon matrix or in echelon form if the no. of zeros (preceeding the 1st non-zero entry) in each row increases row by row untill zero row appears (if there is any)

Defn. 1st non-zero entry of each row in an echlon matrix is called distinguished element.

Row - reduced echelon matrix.

$$\begin{pmatrix} 0 & \boxed{1} & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

(echelon) (row-reduced)

$$\begin{pmatrix} 0 & \boxed{1} & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

(echelon) (not-row-reduced)

1) All the distinguished elements are 1

2) the distinguished element must be the only non-zero entry in it's respective column.

Thm. Every matrix has a unique row-reduced echelon form.

Note! Echelon forms are not unique for a matrix.

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}_{3\times 4}.$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 9 & 5 & 2 \\ 7 & 1 & 0 \end{vmatrix} = 0.$$

minor of order 3

$$\begin{vmatrix} 1 & 2 \\ 9 & 5 \end{vmatrix} \neq 0.$$

minor of order 2

rank $A = 2$.

rank of a matrix (in terms of echelon form)

No. of non-zero rows in echelon form is ~~called~~ the rank of matrix.

Ex 1

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -7 \\ 0 & 0 & -1 \end{pmatrix}_{3\times 3} \qquad (A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -1 & -3 \end{array}\right)_{3\times 4}$$

Augmented matrix.

rank $A = 3$. (there is no zero row) rank $(A|B) = 3$.

Ex 2

$$A = \begin{pmatrix} 2 & 1 & -2 & 1 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (A|B) = \left(\begin{array}{cccc|c} 2 & 1 & -2 & 1 & 1 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & 0 & 0 & 0 & -8 \end{array}\right)_{3\times 4}$$

rank $A = 2$ rank $(A|B) = 3$.

Ex-3

$$A = \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \left(\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

rank $A = 2$. rank $(A|B) = 2$

system of equations (non-homogeneous)

- Consistent – rank$[A]$ = rank$[A|B] = r$
  - unique sol. $r = n$ (no. of unknowns)
  - $r < n$.
- inconsistent – rank$[A] \neq$ rank$[A|B]$.