Numerical Analysis

Books

1. Atkinson
2. Jain Iyengar
3. Hildebrand
4. Scarborough
5. Conte and de Boor.

Iterative solution of system of linear equations :-

$$A_{n\times n} X_{n\times 1} = B_{n\times 1} \longrightarrow (i)$$

(1) (i) has a unique solution

(2) None of the diagonal elements $= 0$.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \text{—— (ii)}$$

We can write above system of equations in the form —

$$\text{(iii)} \left\{\begin{array}{l} x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - \cdots - a_{1n}x_n\right] \\ x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n\right] \\ \vdots \\ x_n = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n(n-1)}x_{n-1}\right] \end{array}\right.$$

Iterative method

Given some initial approximation for the solution $(x_1, x_2, \dots, x_n)$ as $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$.

Improve the solution at every step through some iterative method (algorithm)

1) (Gauss-) Jacobi Method 2) Gauss-Seidel Method

Jacobi method

Given $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$

To find successive iterates $x_1^{(k+1)}, x_2^{(k+1)}, \dots, x_n^{(k+1)}$

$k = 0, 1, 2, 3, \dots$

stop where —

$$\left| x_i^{(k+1)} - x_i^{(k)} \right| < \epsilon \longrightarrow \text{where } \epsilon \text{ is a small quantity say } 10^{-5}$$

Now,

$$x_1^{(k+1)} = \frac{1}{a_{11}} \left[b_1 - a_{12} x_2^{(k)} - \dots - a_{1n} x_n^{(k)} \right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}} \left[b_2 - a_{21} x_1^{(k)} - a_{23} x_3^{(k)} - \dots - a_{2n} x_n^{(k)} \right]$$

$$\vdots$$

$$x_n^{(k+1)} = \frac{1}{a_{nn}} \left[b_n - a_{n1} x_1^{(k)} - a_{n2} x_2^{(k)} - \dots - a_{nn-1} x_{n-1}^{(k)} \right]$$

In summation form which can be written as —

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j \right], \quad i = 1(1)n.$$

Example 1

Solve the system of equations —

$$5x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

By Jacobi Method correct upto 2-decimal places.
Given $x_1^{(0)} = 0 = x_2^{(0)} = x_3^{(0)}$.

$$\Rightarrow \left.\begin{aligned} x_1^{(k+1)} &= \frac{1}{5}\left[-1 + 2x_2^{(k)} - 3x_3^{(k)}\right] \\ x_2^{(k+1)} &= \frac{1}{9}\left[2 + 3x_1^{(k)} - x_3^{(k)}\right] \\ x_3^{(k+1)} &= \frac{1}{7}\left[-3 + 2x_1^{(k)} - x_2^{(k)}\right]. \end{aligned}\right\} \quad (*)$$

taking $k = 0$ in $(*)$ we get —

$$x_1^{(1)} = \frac{1}{5}\left[-1 + 2\times 0 - 3\times 0\right] = -0.2$$
$$x_2^{(1)} = \frac{1}{9}\left[2 + 3\times 0 - 0\right] = 0.222$$
$$x_3^{(1)} = \frac{1}{7}\left[-3 + 2\times 0 - 0\right] = -0.4285$$

put $k = 1$ in $(*)$ and so on —
tabular values are given bellow —

n	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0	0	0
1	-0.2	0.222	-0.4285
2	0.1459	0.20316	-0.5174
⋮			
4	0.181	0.332	-0.421
5	0.185	0.329	-0.424

Solutions are —

$x_1 = 0.18$
$x_2 = 0.33$
$x_3 = -0.42$

Correct to 2 decimal places.

Rule of rounding-off

Take $0 \cdot d_1 d_2 \cdots d_n d_{n+1} \cdots d_r$ where d_i's are digits from 0 to 9.

Suppose you've to round off the no. correct to n-decimal places.

Look at the digit –

d_{n+1}

- $d_{n+1} > 5$
  - Round-off no. $0 \cdot d_1 d_2 \cdots (d_n + 1)$
- $d_{n+1} < 5$
  - Rounded-off no. $0 \cdot d_1 d_2 \cdots d_n$
- $d_{n+1} = 5$
  - d_n even
    - R. of no. $0 \cdot d_1 d_2 \cdots d_n$
  - d_n odd
    - R. off no. $0 \cdot d_1 d_2 \cdots (d_n + 1)$

0·357
0·3567
0·3575
0·4982
0·3685

Round-off these numbers correct to 3 decimal places.

Jacobi Method in matrix form

$$a_{11} x_1^{(k+1)} + a_{12} x_2^{(k)} + \cdots + a_{1n} x_n^{(k)} = b_1$$
$$a_{21} x_1^{(k)} + a_{22} x_2^{(k+1)} + \cdots + a_{2n} x_n^{(k)} = b_2$$
$$\vdots$$
$$a_{n1} x_1^{(k)} + a_{n2} x_2^{(k)} + \cdots + a_{nn} x_n^{(k+1)} = b_n$$

$$\Rightarrow \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \vdots \\ \vdots & & a_{n-1\,n-1} & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & & \vdots \\ a_{31} & a_{32} & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & a_{12} & a_{13} \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots \\ \vdots & & \ddots & a_{n-1\,n} \\ 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ x_n^{(k)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or, $DX^{(k+1)} + (L+U)X^{(k)} = B$

where,
$D \rightarrow$ Diagonal matrix
$L \rightarrow$ Lower triangular matrix with 0 as diagonal entries
$U \rightarrow$ Upper triangular matrix with 0 as diagonal entries.

or, $X^{(k+1)} = D^{-1}B - D^{-1}(L+U)X^{(k)}$

or, $X^{(k+1)} = M + NX^{(k)}$

where. $M = D^{-1}B$, $N = -D^{-1}(L+U)$

In example ① - $M = \left[-\frac{1}{5} \quad \frac{2}{9} \quad -\frac{3}{7}\right]^T$

$$N = \begin{bmatrix} 0 & 2/5 & -3/5 \\ 3/9 & 0 & -1/9 \\ -3/7 & 2/7 & 0 \end{bmatrix}.$$

Gauss- Seidel Method

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - \cdots - a_{1n}x_n^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \cdots - a_{2n}x_n^{(k)}\right]$$

$$\vdots$$

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} - \cdots - a_{nn-1}x_{n-1}^{(k+1)}\right]$$

Or we can write it as—

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1 \\ j<i}}^{n} a_{ij}x_j^{(k+1)} - \sum_{\substack{j=1 \\ j>i}}^{n} a_{ij}x_j^{(k)}\right]$$

$i = 1, 2, \ldots, n.$

Matrix from of Gauss-Seidal method

$$a_{11} x_1^{(k+1)} + a_{12} x_2^{(k)} + \cdots + a_{1n} x_n^{(k)} = b_1$$

$$a_{21} x_1^{(k+1)} + a_{22} x_2^{(k+1)} + a_{23} x_3^{(k)} + \cdots + a_{2n} x_n^{(k)} = b_2$$

$$\vdots$$

$$a_{n1} x_1^{(k+1)} + a_{n2} x_2^{(k+1)} + \cdots \quad + a_{nn} x_n^{(k+1)} = b_n.$$

$$\Rightarrow (L+D) X^{(k+1)} + U X^{(k)} = B$$

$$\Rightarrow X^{(k+1)} = (L+D)^{-1} B + (L+D)^{-1} U X^{(k)}$$

$$\Rightarrow X^{(k+1)} = M_1 + N_1 X^{(k)}$$

$$M_1 \rightarrow (L+D)^{-1} B, \quad N_1 = (L+D)^{-1} U.$$

▨ Solve ~~Gauss-Se~~ Example 1 by Gauss-Seidal Method.

$$x_1^{(k+1)} = \frac{1}{5}\left[-1 + 2x_2^{(k)} - 3x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{9}\left[2 + 3x_1^{(k+1)} - x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{7}\left[-3 + 2x_1^{(k+1)} - x_2^{(k+1)}\right]$$

considering $x_1^{(0)} = 0 = x_2^{(0)} = x_3^{(0)}$

n	x_1	x_2	x_3
0	0	0	0
1	-0.2	0.1556	0.5079
2	0.167	0.334	-0.429
3	0.191	0.333	-0.422
4	0.186	0.331	-0.423

$\Rightarrow$ $x_1 = 0.19$, $x_2 = 0.33$, $x_3 = -0.42$

$\boxed{\text{Note}}$ → Correct to 3 decimal places one can check.

G-J needs → 10 steps
G-S needs → 5 steps

Application →

Sparse system where most of the elements of the elements of the coefficient matrix are zero → We apply iterative methods

Example 2

$$\left.\begin{aligned} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6 \end{aligned}\right\} (*)$$ It has solution (1, 1)

Apply Gauss-Jacobi Method.

⇒

n	x_1	x_2
0	-4	-6
1	-34	-34
2	-174	-244
3	-1244	-1244

Writing equations in the form —
$x_1 = -4 + 5x_2$
$x_2 = -6 + 7x_2$
and considering $x_1^{(0)} = 0 = x_2^{(0)}$ we got these tabular values.

So, Here we can see the values of the roots are diverging in consecutive iterations.

One may check that this happens because system (*) is not in diagonally dominant form.

$AX = B$ → is said to be strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

We can write the given system as —

$$\left.\begin{aligned} 7x_1 - x_2 &= 6 \\ x_1 - 5x_2 &= -4 \end{aligned}\right\} (**)$$

Note that coefficient matrix $\begin{bmatrix} 7 & -1 \\ 1 & -5 \end{bmatrix}$ is stricly diag. dominant.

In this set up the method will converge.

Sufficient condition for the convergence of the Gauss-Jacobi or Gauss-Seidel method is that, the coefficient matrix A is strictly diagonally dominant.

Note- Strictly diagonally dominant $\Rightarrow$ Convergence
Convergence may not $\Rightarrow$ Strictly Diagonally Dominant.

Iterative solution of a Single non-linear Equation

$x = \cosh x$

You find initial guess x_0

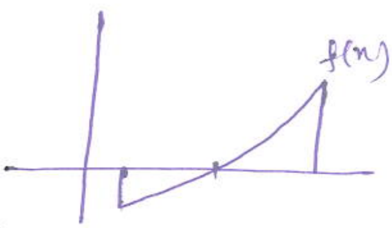
$\alpha = \cosh \alpha$. considering α as a root of $f(x) = 0$.

where $f(x) = 0 \Rightarrow x - \cosh x = 0$

that means $f(\alpha) = 0$

Now, initial guess x_0 is given -

How to generate $x_1, x_2, \ldots$



The method is said to converge, if

$$x_n \to \alpha \text{ as } n \to \infty.$$

$$|x_n - \alpha| < \epsilon \quad \forall\, n > N.$$

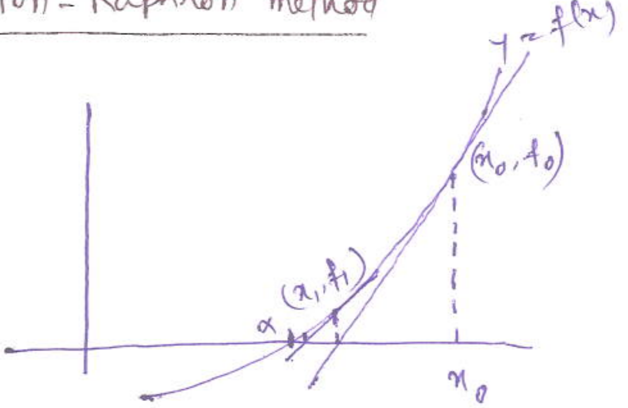
For practical purpose, one checks wheather.

$$|x_{n+1} - x_n| < \epsilon \quad \forall\, n > N.$$

We say that a method has order of convergence p, if $|\alpha - x_{n+1}| < c\,|\alpha - x_n|^p$

or, $\lim_{n\to\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^p} = c \to$ Asymptotic error constant.

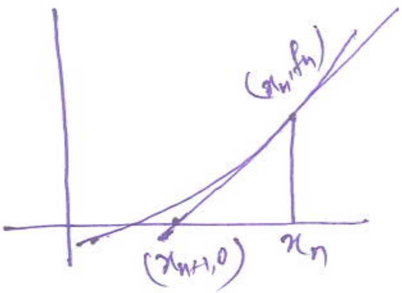
Newton-Raphson method



Equation of tangent at (x_n, f_n)

$$\frac{f_n - 0}{x_n - x_{n+1}} = f'(x_n) \Rightarrow x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad n = 0, 1, 2, 3 \ldots$$



$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and so on.

Through NR method we actually linearize some non-linear function i.e. given the curve $y = f(x)$. We are approximating this by a straight line (tangent line) in NR method. This is also done in secant and R-F method.

Example

Compute the root of $f(x) = 10^x + x - 4$ correct to 6 decimal places. Given $x_0 = 0.5$, (Use NR method)

$\Rightarrow$ $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, Now, $f(x) = 10^x + x - 4$

$\therefore f'(x) = 10^x \log 10 + 1$

$x_0 = 0.5$ and $f'(x_0) = 8.281$.

n	x_n	$f(x_n)$
0	.5	−0.3377
1	.54	0.007369
2	.5391	−0.0007096
3	.5391857	0.000059
4	.5391786	−0.00000468
5	.5391791	−0.000000198
6	.539179	−0.000001095

The result correct to six significant figure is obtained at the 6th step.

Advantage

Order of convergence of NR method is 2.

Disadvantages

1. This method is not guranteed to converge.
2. It may be difficult to compute $f'(x)$.
3. Even if $f'(x)$ exists $f'(x_n)$ may be zero
   $\rightarrow$ there the method will fail.

NR Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

let $x - \frac{f(x)}{f'(x)} = g(x)$.

Then NR method can be expressed as—

$$x_{n+1} = g(x_n).$$

→ Fixed-point iteration scheme.

Take $g(x) = x^2$. Find fixed points of $g(x)$

Def. Fixed points are those x for which—
$g(x) = x$.

$x^2 = x$ or $x(x-1) = 0$

$\Rightarrow x = 0, 1.$

$[-2, -1] \rightarrow$ no fixed pt. of $g(x) = x^2$

$[-1, \frac{1}{2}] \rightarrow 1$ fixed pt. of $g(x) = x^2$.

$[-1, 2] \rightarrow 2$ fixed pts of $g(x) = x^2$.

let $f(x) = x - g(x)$

α is a root of $f(x) = 0 \Leftrightarrow \alpha$ is a fixed pt. of $g(x)$

∴ Root finding problem is equivalent to finding fixed pt. of some function.

To find root of $f(x) = x^2 - 3 = 0$

P.T.O

$x^2 = 3, \quad x = \frac{3}{x} = g_1(x)$

$$x = \frac{1}{2}\left(x + \frac{3}{x}\right) = g_2(x)$$

$$x = x + c\left(x^2 - 3\right) = g_3(x)$$

Which $g(x)$ will you take?

Answer - That $g(x)$ for which

$|g'(x)| < 1 \; \forall x$ in some prescribed interval.

or, $|g'(x_0)| < 1$, for working purpose, where x_0 is the initial given.

Now, In above cases —

$|g_1'(x_0)| = \frac{1}{2}$, $\quad |g_2'(x_0)| = 1.5$, $\times$

$|g_3'(x_0)| = \frac{1}{6}$.

Take $g_3(x)$. Since $\frac{1}{6} < \frac{1}{2}$, so in this case method will converge faster.