## Tutorial Problems set-IV

**Note:** All these problems can be solved using the results of Chapter-4.

[**0.0.1**] ***Exercise*** A matrix $A \in \mathbb{M}_n(\mathbb{F})$ is said to be **nilpotent** if $A^k = 0_n$ for some positive integer $k$. Show that $A$ is nilpotent if and only if the eigenvalues of $A$ are 0.

[**0.0.2**] ***Exercise*** Let $A$ be nilpotent.

1. If $A \neq 0_n$, show that $A$ is not diagonalizable.
2. What can you say about the minimal polynomial of $A$?

[**0.0.3**] ***Exercise*** Let $A, B \in \mathbb{M}_n(\mathbb{C})$ and let $C = AB - BA$. Show that $I - C$ is not nilpotent.

[**0.0.4**] ***Exercise*** What is the minimal polynomial of $A = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}$?

[**0.0.5**] ***Exercise*** Let $C = \begin{bmatrix} A & 0_n \\ 0_n & B \end{bmatrix}$ be a block diagonal matrix where $A, B \in \mathbb{M}_n(\mathbb{C})$. Prove that the minimal polynomial of $C$ is the L.C.M (least common multiple) of the minimal polynomial of $A$ and $B$.

[**0.0.6**] ***Exercise*** Let $A \in \mathbb{M}_n(\mathbb{C})$. Show that $\{I, A, A^2, \ldots, A^n\}$ is a linearly dependent set in the vector space $\mathbb{M}_n(\mathbb{C})$.

[**0.0.7**] ***Exercise*** Let $A = uu^*$ where $u$ is a non-zero column vector.

1. Show that the distinct eigenvalues of $A$ are 0 and $u^*u$.
2. Show that $u^*u$ is a simple eigenvalue of $A$.
3. Write down the $E_{(\lambda=0)}$ and $E_{(\lambda=u^*u)}$.
4. Compute the minimal polynomial of $A$.
5. Show that $A$ is diagonalizable.

[**0.0.8**] ***Exercise*** The characteristic polynomial of a matrix $A \in \mathbb{M}_5(\mathbb{R})$ is given by $x^5 + \alpha x^4 + \beta x^3$ , where $\alpha$ and $\beta$ are non-zero real numbers. What are the possible values of the rank of $A$?

[**0.0.9**] ***Exercise*** Write down all the eigenvalues (along with their multiplicities) of the matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ where $a_{ij} = 1$ for all $1 \leq i, j \leq n$.

[**0.0.10**] ***Exercise*** Let $A \in \mathbb{M}_3(\mathbb{C})$ be a matrix such that $A^3 = I$. Then prove that $A$ is diagonalizable.

[**0.0.11**] ***Exercise*** Let $A \in \mathbb{M}_3(\mathbb{C})$ be a matrix such that $A^2 = A$ (idempotent matrix). Then prove that $A$ is diagonalizable.

[**0.0.12**] ***Exercise*** Let $A \in \mathbb{M}_3(\mathbb{C})$ be a matrix such that $A^2 = I$ (idempotent matrix). Then prove that $A$ is diagonalizable.

[**0.0.13**] ***Exercise*** Find the minimal polynomial of the following matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

[**0.0.14**] ***Exercise*** Let $A \in \mathbb{M}_n(\mathbb{C})$. Then show that the following are equivalent.

1. $A$ is diagonalizable.
2. $P(A)$ is nilpotent $\implies P(A) = 0_n$ for any polynomial $P$ with complex co-efficient.

[**0.0.15**] ***Exercise*** Let $A, B \in \mathbb{M}_n(\mathbb{C})$.

1. If $AX - XB = 0_n$, then show that $P(A)X - XP(B) = 0_n$ for any polynomial $P$.
2. If $A$ and $B$ do not have common eigenvalues, then show that $AX - XB = 0_n \implies X = 0_n$.

**Sol.** Let $P(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_0$.

[**0.0.16**] ***Exercise*** Let $A, B \in \mathbb{M}_n(\mathbb{C})$ such that $A = AB - BA$. Let $v$ be an eigenvector of $B$ with eigenvalue $\lambda$

1. Prove that either $Av$ is zero or an eigenvector of $B$.
2. Prove that there exists a natural $k$ such that $A^k v = 0_n$.