C.F.

$f(D)y = 0$

Auxiliary equation $\quad f(m) = 0$

roots $\alpha_1, \alpha_2, \ldots \alpha_n$.

Case I: Roots are real and non-repeated

$$C.F. = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + \ldots + C_n e^{\alpha_n x}$$

Case II: Roots are real but repeated, say.

$\alpha_1 = \alpha_2 = \alpha$; $\alpha_3, \alpha_4 \ldots \alpha_n$

$$C.F. = (C_1 + C_2 x) e^{\alpha x} + C_3 e^{\alpha_3 x} + \ldots + C_n e^{\alpha_n x}$$

Case III: Roots are complex and non-repeated

$\alpha \pm i\beta, \alpha_3, \alpha_4 \ldots \alpha_n$

$$C.F. = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + C_3 e^{\alpha_3 x} + \ldots + C_n e^{\alpha_n x}$$

Case IV: Roots are complex and repeated

$\alpha \pm i\beta, \alpha \pm i\beta, \alpha_5, \alpha_6, \ldots \alpha_n$

$$C.F. = e^{\alpha x}\left((C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x\right) + C_5 e^{\alpha_5 x} + \ldots$$
$$\ldots + C_n e^{\alpha_n x}.$$

Evaluation of C.F. :

Ex. Solve the differential equation

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$$

In operator form:

$$(D^4 - 2D^3 + 5D^2 - 8D + 4)\,y = 0$$

Auxiliary equation: $m^4 - 2m^3 + 5m^2 - 8m + 4 = 0$

Its roots: $m = 1, 1, 2i, -2i$

The general solution:

$$y = (C_1 + C_2 x)e^{x} + C_3 \cos 2x + C_4 \sin 2x.$$

Ex. Suppose roots of the auxiliary eq. are

$$1, 2, 2, 1 \pm 2i, 1 \pm 2i$$

GENERAL SOLUTION :

$$y = c_1 e^{x} + (c_2 + c_3 x)e^{2x} + e^{x}\left[(c_4 + c_5 x)\cos 2x + (c_6 + c_7 x)\sin 2x\right]$$

Determination of particular integral:

Diff. Eq. $\quad f(D)\,y = X$

$$\boxed{P.I. = \frac{1}{f(D)}.X}$$

1. General method of getting P.I.

$$\boxed{\frac{1}{(D-\alpha)}X = e^{\alpha x}\int X e^{-\alpha x}\,dx}$$

Proof: $\quad$ Let $\quad y = \dfrac{1}{D-\alpha}X$

On operating $D-\alpha$ both sides, we get

$$(D-\alpha)\,y = X$$

$$\Rightarrow \frac{dy}{dx} - \alpha y = X \qquad \text{(linear equation in } y\text{)}$$

$$I.F. = e^{\int -\alpha\,dx} = e^{-\alpha x}$$

$$\Rightarrow \quad y \cdot e^{-\alpha x} = \int X e^{-\alpha x}\,dx + c$$

$$\Rightarrow \quad \boxed{y = e^{\alpha x}\int X e^{-\alpha x}\,dx} + c\,e^{\alpha x}$$

Ex. Solve $(D^2+a^2)y = \sec ax$

Auxiliary equation: $m^2+a^2=0 \Rightarrow m = \pm ai$

$$\text{C.F.} = C_1 \cos ax + C_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2+a^2} \sec ax = \frac{1}{(D-ia)(D+ia)} \cdot \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax$$

Consider $\frac{1}{D-ia} \sec ax = e^{iax} \int \sec ax \, e^{-iax} dx$

$$= e^{iax} \int \sec ax \left[\cos ax - i \sin ax\right] dx$$

$$= e^{iax} \int \left(1 - i \frac{\sin ax}{\cos ax}\right) dx$$

$$= e^{iax}\left[x + \frac{i}{a} \ln|(\cos ax)|\right]$$

Similarly $\frac{1}{D+ia} \sec ax = e^{-iax}\left[x - \frac{i}{a} \ln|(\cos ax)|\right]$

Hence,

$$\text{P.I.} = \frac{1}{2ia}\left[e^{iax}\left\{x + \frac{i}{a} \ln(\cos ax)\right\} - e^{-iax}\left\{x - \frac{i}{a} \ln(\cos ax)\right\}\right]$$

$$= \frac{1}{2ia}\left[x\left(e^{iax} - e^{-iax}\right) + \frac{i}{a} \ln|(\cos ax)|\left\{e^{iax} + e^{-iax}\right\}\right]$$

$$= \frac{x}{a} \cdot \sin ax + \frac{1}{a^2} \ln|(\cos ax)| \cdot \cos ax.$$

GENERAL SOLUTION:

$$y = C_1 \cos ax + C_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a^2} \ln|(\cos ax)| \cos ax$$

2. Short Methods for finding P.I. (Proofs: Shanti Narayan)

- X is of the form $\boxed{e^{ax}}$

i) $$\boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}} \qquad \text{where } f(a) \neq 0$$

ii) If $f(a) = 0$, then $f(D)$ must have a factor of the type $(D-a)^r$.

Then, $$\boxed{\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{\lfloor r} e^{ax}}$$

Ex. $$P.I. = \frac{1}{D^3 - D^2 - D + 1} \cdot e^x$$

$$= \frac{1}{(D-1)^2 (D+1)} e^x$$

$$= \frac{1}{(D-1)^2} \frac{1}{2} e^x$$

$$= \frac{1}{2} \cdot \frac{x^2}{\lfloor 2} \cdot e^x = \frac{1}{4} x^2 e^x .$$

Ex. $$P.I. = \frac{1}{D^2 + D + 5} \cdot 7$$

$$= \frac{1}{D^2 + D + 5} \cdot 7 e^{0x}$$

$$= 7 \cdot \frac{1}{D^2 + D + 5} \cdot e^{0x} = \frac{7}{5} .$$

Proof of:

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad \text{where } f(a) \neq 0$$

Let $f(D) = D^n + C_1 D^{n-1} + \cdots + C_{n+1} D + C_n$

Consider

$$f(D)\, e^{ax} = \left[D^n + C_1 D^{n-1} + \cdots + C_{n-1} D + C_n\right] e^{ax}$$

$$= \left[a^n + C_1 a^{n-1} + \cdots + C_{n-1} a + C_n\right] e^{ax}$$

$$f(D)\, e^{ax} = f(a)\, e^{ax}$$

Operating both side by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)} f(D)\, e^{ax} = \frac{1}{f(D)} \left(f(a)\, e^{ax}\right)$$

$$\Rightarrow e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$$

$$\Rightarrow \boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}$$

- X is $\cos ax$ or $\sin ax$

$$\boxed{P.I. = \frac{1}{f(D)}\cos ax = \frac{1}{\varphi(D^2)}\cos ax = \frac{1}{\varphi(-a^2)}\cos ax}$$

provided $\varphi(-a^2) \neq 0$

$\boxed{\text{Replace } D^2 \text{ by } -a^2}$

Ex. $P.I. = \dfrac{1}{D^4+D^2+1}\cos 2x = \dfrac{1}{(D^2)^2+D^2+1}\cos 2x$

$$= \frac{1}{16-4+1}\cos 2x = \frac{1}{13}\cos 2x$$

Ex. $P.I. = \dfrac{1}{D^2-2D+1}\cos 3x$

$$= \frac{1}{-9-2D+1}\cos 3x = \frac{1}{-2D-8}\cos 3x$$

$$= -\frac{1}{2}\cdot\frac{1}{D+4}\cdot\cos 3x$$

$$= -\frac{1}{2}\,\frac{D-4}{D^2-16}\cos 3x$$

$$= \frac{1}{50}(D-4)\cos 3x$$

$$= \frac{1}{50}\cdot(-3\cdot\sin 3x - 4\cos 3x)$$

$$= -\frac{1}{50}\left(3\sin 3x + 4\cos 3x\right).$$

If $\varphi(-a^2) = 0$:

Ex. $\frac{1}{D^2+a^2} \sin ax$

$$= \text{imag}\left\{ \frac{1}{D^2+a^2}\cos ax + i \frac{1}{D^2+a^2}\sin ax \right\}$$

$$= \text{imag}\left\{ \frac{1}{D^2+a^2} \cdot e^{iax} \right\}$$

Consider. $\frac{1}{D^2+a^2} e^{iax} = \frac{1}{(D-ai)(D+ia)} \cdot e^{iax}$

$$= \frac{1}{D-ai} \cdot \frac{1}{2ia} e^{iax}$$

$$= \frac{1}{2ia} \cdot \frac{x}{1} \cdot e^{iax}$$

$$= \frac{x}{2ia}\{\cos ax + i \sin ax\}$$

$$= \frac{x}{2a}\sin ax - i \frac{x}{2a}\cos ax$$

P. I. $= -\frac{x}{2a}\cos ax$.

Rules:

$$\boxed{\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax}$$

$$\boxed{\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax}$$

Ex. Solve $(D^2+4)y = \sin^2 x$

$$C.F. = C_1 \cos 2x + C_2 \sin 2x$$

$$P.I. = \frac{1}{D^2+4} \sin^2 x$$

$$= \frac{1}{D^2+4} \cdot \frac{1}{2}(1-\cos 2x)$$

$$= \frac{1}{2} \cdot \left[\frac{1}{4} - \frac{1}{D^2+4} \cdot \cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} - \frac{x}{2 \cdot 2} \sin 2x\right]$$

$$= \frac{1}{8}\left[1 - x \sin 2x\right]$$

General solution:

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8}\left[1 - x \sin 2x\right].$$

- X is x^m or a polynomial of degree m.

Take out the lowest degree term from $f(D)$, so as to reduce it in the form

$$[1 \pm F(D)]^{\alpha}.$$

Take it to numerator and expand it.

Ex. Find $\frac{1}{D^3-D^2-6D}\cdot(x^2+1)$

$$= \frac{1}{-6D\left(1+\frac{D}{6}-\frac{D^2}{6}\right)}(x^2+1)$$

$$= -\frac{1}{6D}\left[1+\left(\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}(x^2+1)$$

$$= -\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2-\cdots\right](x^2+1)$$

$$= -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{D^2}{6}+\frac{D^2}{36}\cdots\right](x^2+1)$$

$$= -\frac{1}{6D}\left[(x^2+1)-\frac{1}{6}(2x)+\frac{7}{36}\cdot 2\right]$$

$$= -\frac{1}{6D}\left[x^2-\frac{x}{3}+\frac{25}{18}\right]$$

$$= -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right]$$

- X is $e^{ax}V$, where V is any function of x.

Rule: $$\boxed{\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V.}$$

Ex.

$$P.I. = \frac{1}{D^2+3D+2} e^{2x} \sin x$$

$$= e^{2x} \cdot \frac{1}{(D+2)^2+3(D+2)+2} \cdot \sin x$$

$$= e^{2x} \cdot \frac{1}{D^2+7D+12} \sin x$$

$$= e^{2x} \cdot \frac{1}{7D+11} \sin x$$

$$= e^{2x} \frac{7D-11}{49D^2-121} \sin x$$

$$= e^{2x} \cdot \frac{7D-11}{-170} \sin x$$

$$= -\frac{e^{2x}}{170} \left(7 \cdot \cos x - 11 \sin x \right)$$

$$= \frac{e^{2x}}{170} \left(11 \sin x - 7 \cos x \right).$$

- X is xV

Rule $$\boxed{\frac{1}{f(D)}(x\cdot V) = x\cdot\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V}$$

Ex.

$$P.I. = \frac{1}{D^2-2D+1}\cdot x\sin x$$

$$= x\frac{1}{D^2-2D+1}\sin x - \frac{(2D-2)}{(D^2-2D+1)^2}\sin x$$

$$= x\frac{1}{-2D}\sin x - \frac{(2D-2)}{(-2D)^2}\sin x$$

$$= +\frac{x}{2}\cos x - \frac{(2D-2)}{4(-1)}\sin x$$

$$= \frac{x}{2}\cos x + \frac{1}{2}(\cos x - \sin x)$$

$$= \frac{1}{2}(x\cos x + \cos x - \sin x).$$

$$f(D)y = X$$

$$P.I. = \frac{1}{f(D)} X$$

1. General rule $\frac{1}{D-\alpha} X = e^{\alpha x} \int X e^{-\alpha x} \, dx$

2. Short Methods:

a) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$; $f(a) \neq 0$

a') If $f(a) = 0$; $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{\lfloor r} e^{ax}$;

$$f(D) = (D-a)^r \, \Phi(D)$$

b) $\frac{1}{\Phi(D^2)} \cos ax = \frac{1}{\Phi(-a^2)} \cos ax$; $\Phi(a^2) \neq 0$

b') $\frac{1}{\Phi(D^2)} \sin ax = \frac{1}{\Phi(-a^2)} \sin ax$; $\Phi(-a^2) \neq 0$

b'') $\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$

b''') $\frac{1}{D^2+a^2} \sin ax = \frac{x}{2a} \sin ax$

c) $\frac{1}{f(D)}\left(e^{ax} V\right) = e^{ax} \frac{1}{f(D+a)} V$

d) $\frac{1}{f(D)}(xV) = x \cdot \frac{1}{f(D)} V - \frac{f'(D)}{\{f(D)\}^2} V$