Differentiation under integral sign:

Proof of Leibnitz Rule:

Let $$\Phi(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\,dx .$$

$$\Delta\Phi = \Phi(\alpha+\Delta\alpha) - \Phi(\alpha)$$

$$= \int_{u_1(\alpha+\Delta\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx - \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\,dx$$

$$= \int_{u_1(\alpha+\Delta\alpha)}^{u_1(\alpha)} f(x,\alpha+\Delta\alpha)\,dx + \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha+\Delta\alpha)\,dx + \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx$$
$$- \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\,dx$$

$$= \int_{u_1(\alpha)}^{u_2(\alpha)} [f(x,\alpha+\Delta\alpha) - f(x,\alpha)]\,dx + \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx$$
$$- \int_{u_1(\alpha)}^{u_1(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx$$

Using mean value theorem:

$$\int_{u_1(\alpha)}^{u_2(\alpha)} [f(x,\alpha+\Delta\alpha) - f(x,\alpha)]\,dx = \Delta\alpha \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\xi)\,dx$$

$$\int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx = f(\xi_2,\alpha+\Delta\alpha)[u_2(\alpha+\Delta\alpha) - u_2(\alpha)]$$

$$\int_{u_1(\alpha)}^{u_1(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx = f(\xi_1,\alpha+\Delta\alpha)[u_1(\alpha+\Delta\alpha) - u_1(\alpha)]$$

$$\xi \in \,]\alpha, \alpha+\Delta\alpha[\,, \quad \xi_1 \in \,]u_1(\alpha), u_2(\alpha+\Delta\alpha)[\,, \quad \xi_2 \in \,]u_2(\alpha), u_2(\alpha+\Delta\alpha)[$$

Then, $$\frac{\Delta \phi}{\Delta \alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x, \xi)\, dx + f(\xi_2, \alpha+\Delta\alpha)\frac{\Delta u_2}{\Delta \alpha}$$

$$- f(\xi_1, \alpha+\Delta\alpha)\frac{\Delta u_1}{\Delta\alpha}$$

Taking the limit as $\Delta\alpha \to 0$,

$$\frac{d\phi}{d\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\alpha)\, dx + f[u_2(\alpha), \alpha]\frac{du_2}{d\alpha}$$

$$- f[u_1(\alpha), \alpha]\frac{du_1}{d\alpha}.$$

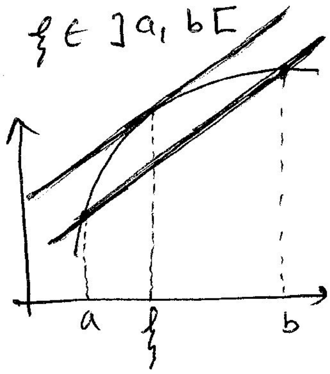
$\square$

<u>Note</u>:

We have used the following mean value theorems in the above proof:

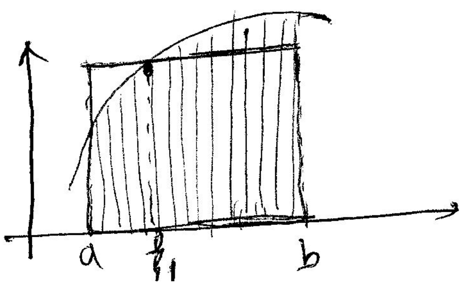
I. Lagrange mean value theorem:

$$\frac{f(b) - f(a)}{b-a} = f'(\xi) \quad ; \quad \xi \in]a, b[$$



II. Mean value theorem of the integral Calculus:

$$\int_a^b f(x)\, dx = (b-a) f(\xi_1) \quad ; \quad \xi_1 \in]a, b[$$



Differentiation under integral sign:

Leibnitz rule: Let

$$\phi(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\, dx \qquad \text{where} \quad u_1(\alpha) \text{ and } u_2(\alpha)$$

posses continuous first order derivatives with respect to α.

Then:

$$\frac{d\phi(\alpha)}{d\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx + f(u_2(\alpha),\alpha)\frac{du_2}{d\alpha} - f(u_1(\alpha),\alpha)\frac{du_1}{d\alpha}$$

Proof:

$$\Delta\phi = \phi(\alpha+\Delta\alpha) - \phi(\alpha) = \cdots$$

Book by R.C. WREDE, SPIEGEL ADVANCED CALCULUS, Schaum's outlines.

A Particular case: Assume $u_1(\alpha)$ & $u_2(\alpha)$ are some constants. Then.

$$\frac{d\phi(\alpha)}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx$$

or

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\, dx = \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)\, dx.$$

Note:
Leibnitz rule is not applicable, in general, in the case of improper integrals. In all example given in this section we assume that differentiation under integral sign is valid.

Example: Show that

$$\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\,dx = \tfrac{1}{2}\pi \log(1+a) \quad \text{if } a \geq 0.$$

Let $\psi(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\,dx.$

$$\Rightarrow \psi'(a) = \int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)}\cdot dx$$

$$= \int_0^\infty \frac{1}{1-a^2}\left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right] dx$$

$$= \frac{1}{(1-a^2)}\left[\tan^{-1} x - a\tan^{-1} ax\right]_0^\infty$$

$$= \frac{1}{(1-a^2)}\cdot\left(\frac{\pi}{2} - a\frac{\pi}{2}\right) = \frac{\pi}{2(1+a)}$$

Integrating.

$$\psi(a) = \frac{\pi}{2}\log(1+a) + c$$

Note that $\psi(0) = 0$

$$\Rightarrow \quad 0 = \frac{\pi}{2}\cdot \times 0 + c \Rightarrow c = 0.$$

$$\Rightarrow \psi(a) = \frac{\pi}{2}\log(1+a)$$

$\square$

Example:

Prove: $\int_0^\infty e^{-x^2}\cos\alpha x\,dx = \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}$.

$$\psi(\alpha) = \int_0^\infty e^{-x^2}\cos\alpha x\,dx$$

$$\psi'(\alpha) = -\int_0^\infty e^{-x^2}\sin\alpha x \cdot x\,dx$$

Integrating right hand side by parts.

$$\psi'(\alpha) = +\frac{e^{-x^2}}{2}\sin\alpha x\Big|_0^\infty + \int_0^\infty\left(-\frac{e^{-x^2}}{2}\right)\left(+\cos\alpha x\cdot\alpha\right)dx$$

$$= 0 - \frac{\alpha}{2}\psi(\alpha)$$

$$\Rightarrow \frac{\psi'(\alpha)}{\psi(\alpha)} = -\frac{\alpha}{2} \Rightarrow \log\psi(\alpha) = -\frac{\alpha^2}{4} + C.$$

$$\Rightarrow \psi(\alpha) = C_1 e^{-\frac{\alpha^2}{4}}.$$

Note that $\psi(0) = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

then. $\psi(0) = \frac{\sqrt{\pi}}{2} = C_1 e^{-0} \Rightarrow C_1 = \frac{\sqrt{\pi}}{2}$

$$\Rightarrow \int_0^\infty e^{-x^2}\cos\alpha x\,dx = \frac{\sqrt{\pi}}{2}\cdot e^{-\frac{\alpha^2}{4}}. \quad \square$$

Question:

Starting from a suitable integral show that

$$\int_0^x \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)}$$

Solution:

Consider

$$\psi(a,x) = \int_0^x \frac{dx}{(x^2+a^2)}$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_0^x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Diff. under integral sign.

$$\frac{\partial\psi}{\partial a} = \int_0^x -\frac{1}{(x^2+a^2)^2}\cdot 2a \cdot dx = \frac{1}{a}\cdot\frac{1}{\left(1+\frac{x^2}{a^2}\right)}\left(-\frac{x}{a^2}\right)$$

$$+\left(-\frac{1}{a^2}\right)\tan^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \boxed{\int_0^x \frac{1}{(x^2+a^2)^2}\,dx = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)}}$$

Question: $\psi(\alpha) = \int_\alpha^{\alpha^2} \frac{\sin\alpha x}{x}\,dx$ find $\psi'(\alpha)$ where $\alpha \neq 0$.

$$\psi'(\alpha) = \int_\alpha^{\alpha^2} \frac{\cos\alpha x}{x}\cdot x\,dx + 2\alpha\cdot\frac{\sin\alpha^3}{\alpha^2} - \frac{\sin\alpha^2}{\alpha}$$

$$= \frac{\sin\alpha x}{\alpha}\Big|_\alpha^{\alpha^2} + \frac{2\sin\alpha^3}{\alpha} - \frac{\sin\alpha^2}{\alpha}$$

$$= \frac{3\sin\alpha^3 - 2\sin\alpha^2}{\alpha}$$

□