PARTIAL DIFFERENTIAL EQUATIONS

We consider the general PDE of the form

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad -(1)$$

If A, B, C are function of $x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, then (1) is called quasilinear PDE. If A, B & C are functions of x, y and F is a linear function of $u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ then (1) is called linear.

A linear PDE is written as

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu + G = 0 \quad -(2)$$

Here A, B, C, D, E, F, G are functions of x & y or they are constants.

The equation (2) is called homogeneous if $G = 0$ otherwise non homogeneous.

The PDE (2) is said to be

i) hyperbolic at a point (x, y) if $B^2 - 4AC > 0$ at (x, y)

ii) parabolic at a point (x, y) if $B^2 - 4AC = 0$ at (x, y)

iii) elliptic at a point (x, y) if $B^2 - 4AC < 0$ at (x, y).

Example: Classify the partial differential equation

$$y u_{xx} - 2u_{xy} - x u_{yy} - u_x + \cos(y) u_y - 4 = 0$$

Sol: $A = y \quad B = -2 \quad C = -x$

$$B^2 - 4AC = 4 + 4xy = 4(1+xy)$$

The equation is hyperbolic for all (x,y) such that $xy > -1$

The equation is parabolic for all (x,y) such that $xy = -1$

The equation is elliptic for all (x,y) such that $xy < -1$.

Example: Classify the region where the following PDE is hyperbolic, parabolic, elliptic.

$$(1+y) z_{xx} + 2x z_{xy} + (1-y) z_{yy} = z_x \quad — (1)$$

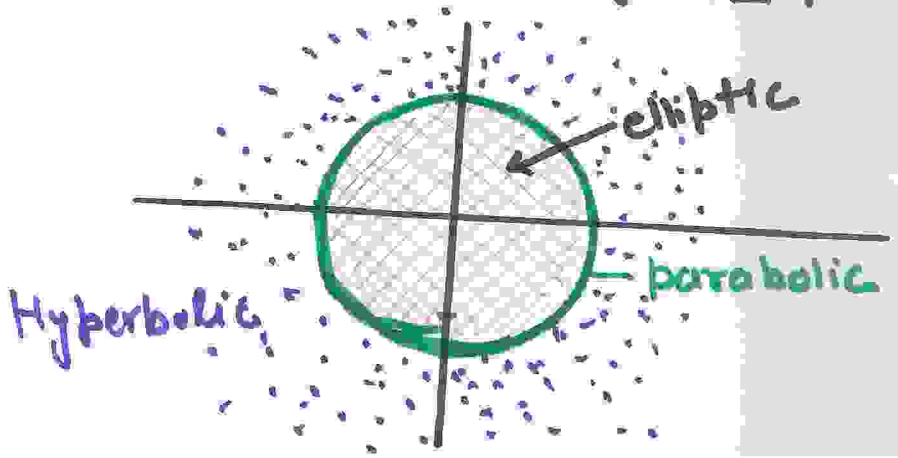
Sol: Here $A = (1+y) \quad B = 2x \quad C = (1-y)$

$$B^2 - 4AC = 4x^2 - 4(1+y)(1-y)$$

$$= 4x^2 - 4(1-y^2)$$

$$= 4(x^2+y^2-1)$$

The equation is hyperbolic in the region $x^2+y^2 > 1$, parabolic in the region $x^2+y^2 = 1$ and elliptic in the region $x^2+y^2 < 1$.



Example: Classify the following PDEs

a) $u_{xx} - u_{tt} = 0$ b) $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$ c) $u_{xx} + u_{yy} = 0$

Sol:

a) $A = 1$ $B = 0$ $C = -1$

$B^2 - 4AC = 4 > 0$ hyperbolic

- Wave equation
- transverse vibration of a string

b) $A = K,$ $B = 0,$ $C = 0$

$B^2 - 4AC = 0$ parabolic

- Heat conduction in a solid
- HEAT EQUATION

c) $A = 1$ $B = 0$ $C = 1$

$B^2 - 4AC = 0 - 4 < 0$ (elliptic)

- Laplace equation
- Steady state heat equation

CANONICAL FORMS

We assume that the given PDE is of single type in a given domain. It does not change its nature at different point.

We shall consider

$$A u_{xx} + B u_{xy} + C u_{yy} = H(x, y, u, u_x, u_y) \quad \text{--- (1)}$$

Under a suitable transformation (non-singular)

$$\xi = \xi(x,y) \ \ \& \ \ \eta = \eta(x,y)$$

the above PDE (1) can be transformed to one of the following forms (called canonical forms):

(i) $w_{\xi\xi} - w_{\eta\eta} = \Phi(\xi, \eta, w, w_\xi, w_\eta)$

or $w_{\xi\eta} = \Phi(\xi, \eta, w, w_\xi, w_\eta)$ for hyperbolic case

ii) $w_{\xi\xi} + w_{\eta\eta} = \Phi(\xi, \eta, w, w_\xi, w_\eta)$ for elliptic case

iii) $w_{\xi\xi} = \Phi(\xi, \eta, w, w_\xi, w_\eta)$

or

$w_{\eta\eta} = \Phi(\xi, \eta, w, w_\xi, w_\eta)$ for parabolic case.

SOLUTION OF $P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z)$

OR

$$Pp + Qq = R$$

LAGRANGE METHOD

I. Write Lagrange auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

II. Solve Lagrange equations to get

$$u(x,y,z) = c_1 \quad \& \quad v(x,y,z) = c_2$$

two independent solutions

III. Write the general solution of PDE as

$$v = f(u)$$

OR

$$u = f(v)$$

or

$$f(u,v) = 0$$

where f is an aritrary function.

* For simplicity we can write a solution of the PDE as

$$v = u$$

CANONICAL FORMS (Simplify equations by coordinate transformation)

Let us consider the general second order PDE

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial y \partial x} + C\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad -(1)$$

Let $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ be a nonsingular transformation. Note that for a nonsingular transformation we have

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} \neq 0$$, and then we can

find $x = x(\xi,\eta)$ & $y = y(\xi,\eta)$.

So we change the independent variables (x,y) to (ξ,η). Let us write

$$w(\xi,\eta) = u\big(x(\xi,\eta), y(\xi,\eta)\big)$$

Using the chain rule we find

$$u_x = w_\xi \xi_x + w_\eta \eta_x$$

$$u_y = w_\xi \xi_y + w_\eta \eta_y$$

$$u_{xx} = (w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x)\xi_x + w_\xi \xi_{xx} + (w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x)\eta_x + w_\eta \eta_{xx}$$

$\Rightarrow$

$$u_{xx} = \omega_{\xi\xi}\xi_x^2 + 2\omega_{\xi\eta}\xi_x\eta_x + \omega_{\eta\eta}\eta_x^2 + \omega_\xi \xi_{xx} + \omega_\eta \eta_{xx}$$

Similarly:

$$u_{xy} = \omega_{\xi\xi}\xi_x\xi_y + \omega_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + \omega_{\eta\eta}\eta_x\eta_y + \omega_\xi \xi_{xy} + \omega_\eta \eta_{xy}$$

$$u_{yy} = \omega_{\xi\xi}\xi_y^2 + 2\omega_{\xi\eta}\xi_y\eta_y + \omega_{\eta\eta}\eta_y^2 + \omega_\xi \xi_{yy} + \omega_\eta \eta_{yy}$$

Substituting into (1):

$$A\left(\omega_{\xi\xi}\xi_x^2 + 2\omega_{\xi\eta}\xi_x\eta_x + \omega_{\eta\eta}\eta_x^2\right) + B\left(\omega_{\xi\xi}\xi_x\xi_y + \omega_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + \omega_{\eta\eta}\eta_x\eta_y\right) + C\left(\omega_{\xi\xi}\xi_y^2 + 2\omega_{\xi\eta}\xi_y\eta_y + \omega_{\eta\eta}\eta_y^2\right) + G(\xi, \eta, \omega, \omega_\xi, \omega_\eta) = 0$$

$$\Rightarrow \omega_{\xi\xi}\left(\underline{A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2}\right) + \underline{2}\,\omega_{\xi\eta}\left(\underline{A\xi_x\eta_x + \tfrac{1}{2}B\xi_x\eta_y + \tfrac{1}{2}B\xi_y\eta_x + C\xi_y\eta_y}\right) + \omega_{\eta\eta}\left(\underline{A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2}\right) + G(\xi, \eta, \omega, \omega_\xi, \omega_\eta) = 0$$

$$\Rightarrow \boxed{\bar{A}\omega_{\xi\xi} + \bar{B}\omega_{\xi\eta} + \bar{C}\omega_{\eta\eta} + G(\xi, \eta, \omega, \omega_\xi, \omega_\eta) = 0} \quad -(2)$$