Lagrange's mean value theorem:

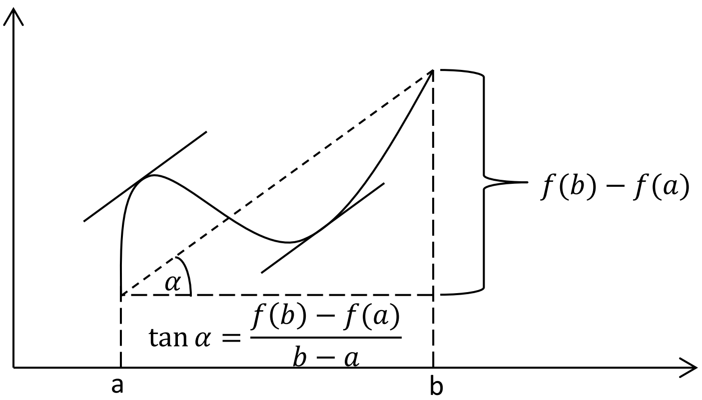
If a function f is

a) continuous in [a, b]
b) differentiable in (a, b)

then there exists at least one value $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

In other words, there is at least one tangent line in the interval that is parallel to the secant line that goes through the endpoints of the interval.



Proof: Define a function

$$\phi(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right]x$$

Note that the function $\phi(x)$ satisfies all the conditions of Rolle 's Theorem as $\phi(a) = \phi(b)$, and continuity and differentiability follows from the continuity and differentiability of $f(x)$. Rolle 's Theorem gives

$\phi'(c) = 0$ for some $c \in (a, b) \Rightarrow f'(c) - \frac{f(b)-f(a)}{b-a} = 0.$

Generalized mean value theorem (Cauchy mean value theorem):

If $f(x)$ and $g(x)$ are two functions continuous in [a, b] and differentiable in (a, b), and $g'(x)$ does not vanish anywhere inside the interval then ∃ a point c in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Define

$$\phi(x) = \big(f(x) - f(a)\big) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right]\big(g(x) - g(a)\big)$$

Note that $g(b) \neq g(a)$ because g' does not vanish in (a, b). If $g(b) = g(a)$ then Rolle's Theorem implies $g'(c) = 0$, which contradicts the assumption that $g'(x) \neq 0$.

$\phi(x)$ satisfies all hypotheses of the Rolle's theorem on the interval [a, b]. Then there exists a point $c \in (a, b)$ such that $c \in (a, b)$ and $\phi'(c) = 0$.

$$\Rightarrow f'(c) - \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] g'(c) = 0$$

$$\Rightarrow \left[\frac{f(b) - f(a)}{g(b) - g(a)}\right] = \frac{f'(c)}{g'(c)} \ .$$

Notice that:

$$\text{Generalized MVT} \xRightarrow{g(x)=x} \text{Lagrange MVT} \xRightarrow{f(a)=f(b)} \text{Rolle's Theorem}$$

Ex: Using mean value theorem show that

$|\cos e^x - \cos e^y| \leq |x - y|$ for $x, y \leq 0$ (equality holds for x = y)

Sol: Consider $f(t) = \cos e^t$ in the interval$[x, y]$. Using Lagrange mean value theorem

$$\frac{\cos e^x - \cos e^y}{x - y} = f'(c), \qquad c \in (x, y)$$

$$\Rightarrow |\cos e^x - \cos e^y| \leq |x - y| \max_{c \in (x,y)} f'(c) < |x - y|$$

as $f'(t) = -\mathrm{e}^{\mathrm{t}} \sin e^t \Rightarrow |f'(t)| = |e^t||\sin e^t| < 1 \text{ for } t < 0$

Ex: Using mean value theorem show that

$$\ln(1 + x) \leq \frac{x}{\sqrt{(1+x)}} \text{ for } x \geq 0.$$

Hint: Consider $f(t) = \ln(1 + t) - \frac{t}{\sqrt{(1+t)}}$ in the interval $[0, x]$.