SUFFICIENT CONDITIONS FOR A FUNCTION TO HAVE MINIMA/MAXIMA

For simplicity, we set

$$r = f_{xx}(a,b), \quad s = f_{xy}(a,b), \quad t = f_{yy}(a,b)$$

Let a function $f(x,y)$ be continuous and have first and second order partial derivatives at a point $P(a,b)$. If (a,b) is a critical point, then the point P is a point of

i) local maximum if $rt - s^2 > 0$ and $r > 0$

ii) local minimum if $rt - s^2 > 0$ and $r < 0$

iii) saddle point if $rt - s^2 < 0$

iv) may be a local minimum, local maximum or a saddle point if $rt - s^2 = 0$.

Proof: consider $\Delta f = f(a+h, b+k) - f(a,b)$

Note that $(a+h, b+k)$ is a point in the neighbourhood of (a,b)

By Taylor's series expansion

$$\Delta f = \left(h f_x + k f_y\right)_{(a,b)} + \frac{1}{2}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{(a,b)} + \cdots$$

As (a,b) is a critical point, meaning $f_x|_{(a,b)} = f_y|_{(a,b)} = 0$

$$\Rightarrow \Delta f = \frac{1}{2}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]_{(a,b)} + R$$

$$= \frac{1}{2}\left[h^2 r + 2hks + k^2 t\right] + R$$

$$= \frac{1}{2r}\left[h^2 r^2 + 2hkrs + k^2 rt\right] + R \quad (\text{Assuming } r \neq 0)$$

$$= \frac{1}{2r}\left[(hr+ks)^2 - k^2s^2 + k^2rt\right] + R$$

$$= \frac{1}{2r}\left[(hr+ks)^2 + k^2(rt-s^2)\right] + R$$

$\left(\text{OR } \frac{1}{2t}\left[(hs+kt)^2 + h^2(rt-s^2)\right] + \right.$

same conclusions follow if $r=0$ & $t \neq 0$

Since $(hr+ks)^2$, the sufficient condition for the expression

$[(hr+ks)^2 + k^2(rt-s^2)]$ to be positive is that

$$rt - s^2 > 0$$

$\Rightarrow$ If $rt-s^2>0$, then

i) $\Delta f > 0$ if $r>0$

ii) $\Delta f < 0$ if $r<0$

$\Rightarrow$ The point (a,b) is a point of $\begin{cases} \text{minimum if } (rt-s^2)>0 \text{ \& } r>0 \\ \text{maximum if } (rt-s^2)>0 \text{ \& } r<0 \end{cases}$

iii) If $rt-s^2<0$, then the sign of Δf depends on h & k.

For example,

Let $k \to 0$ & $h \neq 0$ $\Rightarrow$ $\Delta f > 0$ if $r>0$

and if $k \neq 0$ & we choose h such that $hr+ks=0$

$\Rightarrow \Delta f < 0$ for $r>0$

Hence no maximum/minimum of f can occur at $P(a,b)$.

$\Rightarrow P(a,b)$ is a saddle point

(iv) If $rt - s^2 = 0$, then

$$\Delta f = \frac{1}{2r}\left[(hr+ks)^2\right] + R$$

If we take h & k such that $hr = -ks$ i.e, $\frac{h}{k} = -\left(\frac{s}{r}\right)$, then the whole second order terms of right hand side will vanish.

Therefore for these points in the neighbourhood we have to consider third order terms in the remainder. Other than these points we have

$\Delta f > 0$ for $r > 0$ and

$\Delta f < 0$ for $r < 0$.

Thus the conclusion will depend on the higher order terms.

$\Rightarrow$ A FURTHER INVESTIGATION IS REQUIRED.

WORKING RULES:

1) FIND CRITICAL POINTS OR STATIONARY POINTS $f_x = 0$ & $f_y = 0$.

2) FOR EACH CRITICAL POINT, EVALUATE

$$r = f_{xx}, \quad s = f_{xy}, \quad t = f_{yy}$$

3) IDENTIFICATION:

i) If $rt - s^2 > 0$ & $r < 0$ $\rightarrow$ Maximum

ii) If $rt - s^2 > 0$ & $r > 0$ $\rightarrow$ Minimum

iii) If $rt - s^2 < 0$ $\rightarrow$ Saddle point

iv) If $rt - s^2 = 0$ $\rightarrow$ Doubtful, needs further investigation

Ex. Discuss the local extrema of the function

$$f(x,y) = (4x^2+y^2)\, e^{-x^2-4y^2}$$

Sol:

$$f_x(x,y) = e^{-x^2-4y^2}\left[8x - 2x(4x^2+y^2)\right]$$

$$= e^{-x^2-4y^2}\left[8x - 8x^3 - 2xy^2\right]$$

$$= e^{-x^2-4y^2}(2x)\left[4 - 4x^2 - y^2\right]$$

$$f_y(x,y) = e^{-x^2-4y^2}\left[2y - 8y(4x^2+y^2)\right]$$

$$= e^{-x^2-4y^2}(2y)\left[1 - 16x^2 - 4y^2\right]$$

CRITICAL POINTS: $f_x = 0$ & $f_y = 0$

i) $x=0,\ y=0$

ii) $x=0,\quad 1-4y^2=0 \Rightarrow y = \pm\frac{1}{2}$

$\Rightarrow (0, \frac{1}{2})$ & $(0, -\frac{1}{2})$

iii) let $x \neq 0,\ y=0$

$\Rightarrow 4-4x^2=0 \Rightarrow x = \pm 1.$

$(1,0) \quad (-1,0)$

iv) $x\neq 0,\ y\neq 0 \Rightarrow \left.\begin{array}{l} 4x^2+y^2=4 \\ \& \ 4x^2+y^2=\frac{1}{4} \end{array}\right\}$ NO SOLUTION

Hence the critical points are:

$P_1 = (0,0)$, $P_2 = (0,\frac{1}{2})$ $P_3 = (0,-\frac{1}{2})$ $P_4 = (1,0)$ $P_5 = (-1,0)$

Second order derivatives:

$$r = f_{xx} = e^{-x^2-4y^2}\left[8-24x^2-2y^2+(8x-8x^3-2xy^2)(-2x)\right]$$

$$= 2e^{-x^2-4y^2}\left[4-20x^2+8x^4-y^2+2x^2y^2\right]$$

$$t = f_{yy} = e^{-x^2-4y^2}\left[2-32x^2-24y^2+(2y-32x^2y-8y^3)(-8y)\right]$$

$$= 2e^{-x^2-4y^2}\left[1-20y^2-16x^2-128x^2y^2+32y^4\right]$$

$$s = f_{xy} = e^{-x^2-4y^2}\left[-4xy+(8x-8x^3-2xy^2)(-8y)\right]$$

$$= 4xy\,e^{-x^2-4y^2}\left[-17+16x^2+4y^2\right]$$

<u>Identification:</u>

<u>$P_1(0,0)$:</u> $r = 8$ $s = 0$ $t = 2$

$$rt - s^2 = 16 > 0 \ \& \ r > 0$$

$\Rightarrow$ The point P_1 is a local minima.

<u>$P_2(0,\frac{1}{2})$ & $P_3(0,-\frac{1}{2})$:</u>

$$r = 2e^{-1}\left[4-\tfrac{1}{4}\right] = \frac{15}{2e}$$

$$s = 0$$

$$t = 2e^{-1}\left[1-5+2\right] = -\frac{4}{e}$$

$$rt - s^2 = -\frac{30}{e^2} < 0$$

$\Rightarrow$ P_2 & P_3 are saddle points.

$P_4(1,0)$ & $P_5(-1,0)$

$$r = 2e^{-1}[4-20+8] = -16e^{-1}$$

$$s = 0$$

$$t = 2e^{-1}[1-16] = -30e^{-1}$$

$$rt - s^2 = \frac{480}{e^2} > 0, \quad r < 0$$

Hence P_4 & P_5 are the point of local Maximum.

EXAMPLE: $f(x,y) = y^2 + x^2y + x^4$.

Stationary points: $f_x = 0$ & $f_y = 0$

$$\Rightarrow 2xy + 4x^3 = 0 \quad \& \quad 2y + x^2 = 0$$

$$\Rightarrow x = 0 \;\&\; y = 0.$$

$$r = f_{xx}\big|_{(0,0)} = (2y + 12x^2)\big|_{(0,0)} = 0$$

$$s = f_{xy}\big|_{(0,0)} = 2x\big|_{(0,0)} = 0$$

$$t = f_{yy}\big|_{(0,0)} = 2\big|_{(0,0)} = 2.$$

$rt - s^2 = 0$ further investigation is required.

$$\Delta f = f(0+h, 0+k) - f(0,0) = k^2 + h^2k + h^4$$

$$= \left(\frac{k}{2}\right)^2 + h^2k + h^4 + \frac{3}{4}k^2$$

$$= \left(\frac{k}{2} + h^2\right)^2 + \frac{3}{4}k^2 > 0 \quad \forall \; \begin{matrix} h \neq 0 \\ k \neq 0 \end{matrix}$$

$\Rightarrow$ (0,0) is a point of LOCAL MINIMUM.

Ex. Find local minima/maxima of the function

$$f(x,y) = 2x^4 - 3x^2y + y^2$$

Sol.

$$f_x = 8x^3 - 6xy$$

$$f_y = -3x^2 + 2y$$

Stationary points: $8x^3 - 6xy = 0$ & $-3x^2 + 2y = 0$

$$\Rightarrow 8x^3 - 3x(3x^2) = 0 \Rightarrow x = 0.$$

$$\Rightarrow y = 0$$

Stationary point $(0,0)$.

$$r = f_{xx}\big|_{(0,0)} = (24x^2 - 6y)\big|_{(0,0)} = 0$$

$$s = f_{xy}\big|_{(0,0)} = -6x\big|_{(0,0)} = 0$$

$$t = f_{yy}\big|_{(0,0)} = 2$$

$rt - s^2 = 0$ test faits,

$$\begin{aligned}
\Delta f &= f(h,k) - f(0,0) \\
&= 2h^4 - 3h^2k + k^2 \\
&= 2h^4 - 2h^2k - h^2k + k^2 \\
&= 2h^2(h^2 - k) - k(h^2 - k) \\
&= (2h^2 - k)(h^2 - k)
\end{aligned}$$

$\left.\begin{array}{l}\underline{\text{For } k < 0:} \quad \Delta f > 0 \\ \underline{\text{For } h^2 < k < 2h^2:} \quad \Delta f < 0\end{array}\right\}$ sign changes

$\Rightarrow (0,0)$ is a saddle point.

Ex. The function $f(x,y) = (y-x^2)^2 + x^5$ has a stationary point at the origin. Characterize the function at the point $(0,0)$.

Sol: $f_x = 2(y-x^2)(-2x) + 5x^4 \Rightarrow f_{xx} = -4[(y-x^2)+x(-2x)] + 20x^3$

$r = f_{xx}|_{(0,0)} = 0$ $\quad f_{xy} = -4x.$

$f_y = 2(y-x^2) \Rightarrow f_{yy} = 2$

$s = f_{xy}|_{(0,0)} = 0$

$t = 2.$

$rt - s^2 = 0$ test fails!

However, we can readily see that the function has no extreme value there, as the function assumes both positive and negative values in the neighbourhood of the origin.

Ex. Find and characterize the extreme values of the function

$$f(x,y) = (x-y)^4 + (y-1)^4.$$

Sol. $f_x = 4(x-y)^3$ $\quad f_{xx} = 12(x-y)^2$ $\quad f_{xy} = -24(x-y)$

$f_y = -4(x-y)^3 + 4(y-1)^3$ $\quad f_{yy} = +12(x-y)^2 + 12(y-1)$

Critical points: $(x-y)^3 = 0$ & $-(x-y)^3 + (y-1)^3 = 0$

$\Rightarrow x = 1,\ y = 1.$

$r = f_{xx}|_{(1,1)} = 0$ $\quad s = f_{xy}|_{(1,1)} = 0$ $\quad t = f_{yy}|_{(1,1)} = 0$

Criterion fails!

However, if we consider: $f(1+h, 1+k) - f(1,1)$

$$= (1+h-1-k)^4 + (1+k-1)^4$$

$$= (h-k)^4 + k^4 > 0 \quad \forall\ h,k \neq 0$$

$\Rightarrow f$ has a minimum at the point $x=1,\ y=1.$