Assignment-2 (Solutions) (1)

ASSIGNMENT-2

1.

(i) 0 is the zero element in F

$0+0 = 0$ in F

$\Rightarrow (0+0)v = 0v$ in V

$\Rightarrow 0v + 0v = 0v$

Now, $-0v \in V$, since $0v \in V$

Therefore, $-0v + (0v + 0v) = -0v + 0v$

or, $(-0v + 0v) + 0v = \mathbf{0}$

or, $\mathbf{0} + 0v = \mathbf{0}$

or, $0v = \mathbf{0}$

(ii) If $c=0$, its trivially holds.
Let $cv = \mathbf{0}$ and let $c \neq 0$. Then c^{-1} exists in F.

Now $cv = \mathbf{0} \Rightarrow c^{-1}(cv) = c^{-1}\mathbf{0}$

$\Rightarrow (c^{-1}c)v = c^{-1}\mathbf{0}$

$\Rightarrow 1v = \mathbf{0}$

$\Rightarrow v = \mathbf{0}$

Therefore $cv = \mathbf{0}$ and $c \neq 0 \Rightarrow v = \mathbf{0}$

~~Now If $c=0$, then~~

2. (i) Let $a, b \in S$ where $a = (0, y_1, z_1)$
$b = (0, y_2, z_2)$

For $c_1, c_2 \in \mathbb{R}$,

$c_1 a + c_2 b = c_1(0, y_1, z_1) + c_2(0, y_2, z_2)$
$= (0, c_1 y_1 + c_2 y_2, c_1 z_1 + c_2 z_2) \in S$

since $c_1y_1+c_2y_2,\ c_1z_1+c_2z_2 \in \mathbb{R}$

Therefore, S is a subspace of $\mathbb{R}^3$.

(ii) S is a not a subspace of $\mathbb{R}^3$, since, for

$\alpha=(1,0,0), \beta=(1,1,0) \in S$,

$$\alpha+\beta = (1,0,0)+(1,1,0)$$
$$= (2,1,0) \notin S$$

(iii) S is not a subspace of $\mathbb{R}^3$.

because, for $\alpha=(1,0,0), \beta=(0,1,0) \in S$,

$$\alpha+\beta = (1,0,0)+(0,1,0)$$
$$= (1,1,0) \notin S$$

(iv) $S=\{(x,y,z)\in\mathbb{R}^3 : x+y+z=0\}$

Let $\alpha=(x_1,y_1,z_1), \beta=(x_2,y_2,z_2) \in S$.

Then $x_1+y_1+z_1=0$

$x_2+y_2+z_2=0$

For $c_1, c_2 \in \mathbb{R}$,

$$c_1\alpha+c_2\beta = c_1(x_1,y_1,z_1)+c_2(x_2,y_2,z_2)$$
$$=(c_1x_1, c_1y_1, c_1z_1)+(c_2x_2, c_2y_2, c_2z_2)$$
$$=(c_1x_1+c_2x_2,\ c_1y_1+c_2y_2,\ c_1z_1+c_2z_2)$$
$$= (x,y,z), \text{ say}$$

where $x=c_1x_1+c_2x_2$, $y=c_1y_1+c_2y_2$, $z=c_1z_1+$

Now, $x+y+z = c_1x_1+c_2x_2+c_1y_1+c_2y_2+c_1z_1+c_2z_2$

$$= c_1(x_1+y_1+z_1)+c_2(x_2+y_2+z_2)$$

$$= c_1\cdot 0 + c_2\cdot 0 = 0$$

Therefore, $c_1\alpha + c_2\beta \in S$, hence S is a subspace of $\mathbb{R}^3$.

(v) $S = \{(x,y,z) \in \mathbb{R}^3 : x+y+z=1\}$

S is not a subspace of $\mathbb{R}^3$, since $(0,0,0) \notin S$.

(vi) $S = \{(x,y,z) \in \mathbb{R}^3 : x+2y-z=0,\ 2x-y+z=0\}$

Let $\alpha = (\alpha_1, \beta_1, \gamma_1)$, $\beta = (\alpha_2, \beta_2, \gamma_2) \in S$

then $\alpha_1 + 2\beta_1 - \gamma_1 = 0$, $2\alpha_1 - \beta_1 + \gamma_1 = 0$

$\alpha_2 + 2\beta_2 - \gamma_2 = 0$, $2\alpha_2 - \beta_2 + \gamma_2 = 0$

For $c_1, c_2 \in \mathbb{R}$

$$c_1\alpha + c_2\beta = c_1(\alpha_1,\beta_1,\gamma_1) + c_2(\alpha_2,\beta_2,\gamma_2)$$

$$= (c_1\alpha_1 + c_2\alpha_2,\ c_1\beta_1 + c_2\beta_2,\ c_1\gamma_1 + c_2\gamma_2)$$

$$= (x, y, z), \text{ say}$$

where $x = c_1\alpha_1 + c_2\alpha_2$, $y = c_1\beta_1 + c_2\beta_2$, $z = c_1\gamma_1 + c_2\gamma_2$

Now, $x+2y-z$

$$= c_1\alpha_1 + c_2\alpha_2 + 2(c_1\beta_1 + c_2\beta_2) - (c_1\gamma_1 + c_2\gamma_2)$$

$$= c_1(\alpha_1 + 2\beta_1 - \gamma_1) + c_2(\alpha_2 + 2\beta_2 - \gamma_2) = 0$$

and $2x - y + z$

$$= 2(c_1\alpha_1 + c_2\alpha_2) - (c_1\beta_1 + c_2\beta_2) + (c_1\gamma_1 + c_2\gamma_2)$$

$$= c_1(2\alpha_1 - \beta_1 + \gamma_1) + c_2(2\alpha_2 - \beta_2 + \gamma_2)$$

$$= 0.$$

Therefore, $c_1\alpha + c_2\beta \in S$ and hence, S is a subspace of $\mathbb{R}^3$

(vii) $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$

S is not a subspace of $\mathbb{R}^3$

because, for $\alpha = (1, 0, 0)$, $\beta = (0, 1, 0) \in S$,

$$\alpha + \beta = (1, 0, 0) + (0, 1, 0) = (1, 1, 0) \notin S, \text{ as } 1^2 + 1^2 + 0^2 \not\leq 1$$

3. (i) Let $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2} : a + b = 0 \right\}$

Let $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in S$

Therefore, $a_1 + b_1 = 0$
$a_2 + b_2 = 0$

For $t_1, t_2 \in \mathbb{R}$

$$t_1 A + t_2 B = t_1 \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + t_2 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$= \begin{pmatrix} t_1 a_1 + t_2 a_2 & t_1 b_1 + t_2 b_2 \\ t_1 c_1 + t_2 c_2 & t_1 d_1 + t_2 d_2 \end{pmatrix} = ($$

, sa

Now, $x+y = (t_1 a_1 + t_2 a_2) + (t_1 b_1 + t_2 b_2)$

$$= t_1(a_1+b_1) + t_2(a_2+b_2)$$

$$= 0$$

Therefore, $t_1 A + t_2 B \in S$ and hence S is a subspace of $M_{2\times 2}(\mathbb{R})$.

(ii) <u>same as (i)</u>.

(iii) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2} : \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \right\}$

S is not a subspace of $M_{2\times 2}$, since for $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in S$, but

$A + B \notin S$.

(iv) $S = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in M_{2\times 2} : x, y \in \mathbb{R} \right\}$

S is a subspace of $M_{2\times 2}(\mathbb{R})$.

(v) $S = \left\{ \begin{pmatrix} a & c \\ c & d \end{pmatrix} \in M_{2\times 2} : a, c, d \in \mathbb{R} \right\}$.

S is a subspace of $M_{2\times 2}(\mathbb{R})$

(vi) $S = \left\{ \begin{pmatrix} 0 & c \\ -c & c \end{pmatrix} \in M_{2\times 2} : c \in \mathbb{R} \right\}$

S is a subspace of $M_{2\times 2}(\mathbb{R})$.

4. (i) Let $\alpha = (1,2,3,-1)$, $\beta = (3,7,1,-2)$, $\gamma = (1,3,7,-4)$

Let us consider $c_1\alpha + c_2\beta + c_3\gamma = 0$, where $c_1, c_2, c_3 \in \mathbb{R}$

Then $c_1(1,2,3,-1) + c_2(3,7,1,-2) + c_3(1,3,7,-4)$
$= (0,0,0,0)$

Therefore,
$$c_1 + 3c_2 + c_3 = 0$$
$$2c_1 + 7c_2 + 3c_3 = 0$$
$$3c_1 + c_2 + 7c_3 = 0$$
$$-c_1 - 2c_2 - 4c_3 = 0$$
giving $c_1 = c_2 = c_3 = 0$

Therefore, the given vectors are linearly independent

(ii) Let $\alpha = (1,3,1,-2)$, $\beta = (2,5,-1,3)$, $\gamma = (1,3,7,-2)$

Let consider $c_1\alpha + c_2\beta + c_3\gamma = 0$, where $c_1, c_2, c_3 \in \mathbb{R}$

Then $c_1(1,3,1,-2) + c_2(2,5,-1,3) + c_3(1,3,7,-2)$
$= (0,0,0,0)$

Therefore,
$$c_1 + 2c_2 + c_3 = 0$$
$$3c_1 + 5c_2 + 3c_3 = 0$$
$$c_1 - c_2 + 7c_3 = 0$$
$$-2c_1 + 3c_2 - 2c_3 = 0$$
giving $c_1 = c_2 = c_3 = 0$

Therefore, the given vectors are linearly independent

5. (i) given set of vectors will be linearly dependent if $\begin{vmatrix} x & y & y \\ y & x & y \\ y & y & x \end{vmatrix} = 0$

or, $x(x^2-y^2) - y(xy-y^2) + y(y^2-xy) = 0$

or, $x^3 - 2xy^2 + 2y^3 - xy^2 = 0$

or, $x^3 - xy^2 - 2xy^2 + 2y^3 = 0$

or, $x(x^2-y^2) - 2y^2(x-y) = 0$

or, $x(x-y)(x+y) - 2y^2(x-y) = 0$

or, $(x-y)(x^2+xy-2y^2) = 0$

or, $(x-y)(x^2-y^2+xy-y^2) = 0$

or, $(x-y)\{(x+y)(x-y) + y(x-y)\} = 0$

or, $(x-y)(x-y)(x+2y) = 0$

$\therefore\ x = y,\ x = -2y.$

Therefore, the given set of vectors are linearly dependent if $x = y$ or $x = -2y$.

(ii) Given set of vectors will be linearly dependent if $\begin{vmatrix} x & y & 1 \\ y & 1 & x \\ 1 & x & y \end{vmatrix} = 0$

or, $x(y-x^2) - y(y^2-x) + 1(xy-1) = 0$

or, $xy - x^3 - y^3 + xy + xy - 1 = 0$

or, $x^3 + y^3 - 3xy + 1 = 0$

or, $(x+y+1)(x^2+y^2+1-xy-x-y)=0$

or, $\frac{1}{2}(x+y+1)\{x^2-2xy+y^2+x^2-2x+1$
$+y^2-2y+1\}=0$

or, $\frac{1}{2}(x+y+1)\{(x-y)^2+(x-1)^2+(y-1)^2\}=0$

$\therefore$ $x+y+1=0$ or, $x-y=0$
$x-1=0$
$y-1=0$

i.e. $x+y+1=0$ or, $x=y=1$

$\therefore$ The given set of vectors will be linearly depende
if $x+y+1=0$ or, $x=y=1$.

6. ~~α, β, γ are linearly independent vectors of $V(F)$. Then there exists $c_1, c_2, c_3 \in F$ which are not all zero such that~~

$$\cancel{c_1\alpha + c_2\beta + c_3\gamma = \theta}$$

$c_1\alpha+c_2\beta+c_3\gamma=0$
$\Rightarrow c_1=c_2=c_3=0$

(i) $\{(\alpha+\beta)(\beta+\gamma),(\gamma+\alpha)\}$

Let $x=\alpha+\beta$, $y=\beta+\gamma$, $z=\gamma+\alpha$.

Let $c_1, c_2, c_3 \in F$, then

$$c_1x+c_2y+c_3z = c_1(\alpha+\beta)+c_2(\beta+\gamma)+c_3(\gamma+\alpha)$$
$$=c_1\alpha+c_2\beta+c_3\gamma+c_1\beta+c_2\gamma+c_3\alpha$$
$$=(c_1+c_3)\alpha+(c_2+c_1)\beta+(c_3+c_1)\beta$$

Now, $c_1 x + c_2 y + c_3 z = 0$

$\Rightarrow (c_1 + c_3)\alpha + (c_2 + c_1)\beta + (c_3 + c_1)\gamma = 0$

since, α, β, γ are linearly independent vectors in $V(F)$,

$$c_1 + c_3 = 0, \quad c_2 + c_1 = 0, \quad c_3 + c_1 = 0$$

gives, $c_1 = c_2 = c_3 = 0$

Therefore, $\{(\alpha+\beta), (\beta+\gamma), (\gamma+\alpha)\}$ is linearly independent

(ii) let $x = \alpha + \beta,\ y = \alpha - \beta,\ z = \alpha - 2\beta + 2\gamma$

Let $c_1, c_2, c_3 \in F$ such that

$c_1 x + c_2 y + c_3 z = 0$

$\Rightarrow c_1(\alpha+\beta) + c_2(\alpha - \beta) + c_3(\alpha - 2\beta + 2\gamma) = 0$

$\Rightarrow (c_1 + c_2 + c_3)\alpha + (c_1 - c_2 - 2c_3)\beta + 2c_3\gamma = 0$

Since α, β, γ are linearly independent vectors in $V(F)$,

$$c_1 + c_2 + c_3 = 0$$
$$c_1 - c_2 - 2c_3 = 0$$
$$2c_3 = 0$$
$$\Rightarrow c_1 = c_2 = c_3 = 0$$

Therefore $\{(\alpha+\beta), (\gamma-\beta), (\alpha - 2\beta + 2\gamma)\}$ is a set of linearly independent vectors

7. $S = \{\alpha, \beta, \gamma\}$

$T = \{\alpha, \alpha+\beta, \alpha+\beta+\gamma\}$

$U = \{\alpha+\beta, \beta+\gamma, \gamma+\alpha\}$

Let ~~u ∈ S~~, ~~then where~~ ~~u = (u₁, u₂, u₃)~~.

$u \in L(S)$. Then for $c_1, c_2, c_3 \in \mathbb{R}$,

$$u = c_1\alpha + c_2\beta + c_2\gamma = a\alpha + b(\alpha+\beta) + c(\alpha+\beta+\gamma)$$

$a+b+c = c_1$

b

$$= (c_1 - c_2)\alpha + (c_2 - c_3)(\alpha+\beta) + c_3(\alpha+\beta+\gamma)$$

$$\in L(T)$$

$$\therefore \quad L(S) \subset L(T) \quad \text{— (1)}$$

Now, let $v \in L(T)$, then for $c_1, c_2, c_3 \in \mathbb{R}$,

$$v = c_1(\alpha) + c_2(\alpha+\beta) + c_3(\alpha+\beta+\gamma)$$

$$= (c_1 + c_2 + c_3)\alpha + (c_2 + c_3)\beta + c_3\gamma$$

$$\in L(S)$$

$$\therefore \quad L(T) \subset L(S). \quad \text{— (2)}$$

From (1) & (2), $L(S) = L(T)$ — (*)

Again, let $w \in L(U)$, then for $c_1, c_2, c_3 \in \mathbb{R}$

$$w = c_1(\alpha+\beta) + c_2(\beta+\gamma) + c_3(\gamma+\alpha)$$

$$= (c_1 + c_3)\alpha + (c_1 + c_2)\beta + (c_2 + c_3)\gamma$$

$$\in L(S).$$

$$\therefore \quad L(U) \subset L(S) \quad \text{— (3)}$$

Let, $z \in L(S)$, then for $c_1, c_2, c_3 \in \mathbb{R}$,

$$z = c_1\alpha + c_2\beta + c_3\gamma$$

$$= \frac{c_1+c_2}{2}(\alpha+\beta) + \frac{c_2+c_3}{2}(\beta+\gamma) + \frac{c_3+c_1}{2}(\gamma+\alpha)$$

$$\in L(U)$$

$$\therefore \quad L(S) \subset L(U) \quad — (4)$$

From, (3) & (4), $L(S) = L(U)$. — (**)

From, (*) & (**), $L(S) = L(T) = L(U)$

8.

(i) Let $\alpha = (1,2,3,0)$, $\beta = (2,3,0,1)$, $\gamma = (3,0,1,2)$

Let $c_1, c_2, c_3 \in \mathbb{R}$,

then $c_1\alpha + c_2\beta + c_3\gamma = 0$

$$\Rightarrow c_1(1,2,3,0) + c_2(2,3,0,1) + c_3(3,0,1,2) = (0,0,0,0)$$

Then,

$$c_1 + 2c_2 + 3c_3 = 0 \quad — (i)$$

$$2c_1 + 3c_2 + 0\cdot c_3 = 0 \quad — (ii)$$

$$3c_1 + 0c_2 + c_3 = 0 \quad — (iii)$$

$$0c_1 + c_2 + 2c_3 = 0 \quad — (iv)$$

$(ii) - 2\times(i) \Rightarrow \quad -c_2 - 6c_3 = 0 \quad — (v)$

Solving, (v) & (iv), $c_2 = c_3 = 0$ and hence $c_1 = 0$

$$\therefore \quad c_1 = c_2 = c_3 = 0$$

Therefore, α, β, γ are linearly independent vectors

(ii) Let $\alpha = (1,1,1,0)$, $\beta = (1,1,0,1)$, $\gamma = (1,0,1,1)$.
$\delta = (0,1,1,1)$.

Let $c_1, c_2, c_3, c_4 \in \mathbb{R}$

$$c_1\alpha + c_2\beta + c_3\gamma + c_4\delta = 0$$

$$\Rightarrow c_1(1,1,1,0) + c_2(1,1,0,1) + c_3(1,0,1,1) + c_4(0,1,1,1) = (0,0,0,0)$$

Therefore,

$$0 \cdot c_1 + c_2 + c_3 + c_4 = 0 \quad \text{—(i)}$$

$$c_1 + c_2 + 0 \cdot c_3 + c_4 = 0 \quad \text{—(ii)}$$

$$c_1 + 0 \cdot c_2 + c_3 + c_4 = 0 \quad \text{—(iii)}$$

$$c_1 + c_2 + c_3 + 0 \cdot c_4 = 0 \quad \text{—(iv)}$$

(iv) − (ii) $\Rightarrow$ $c_3 - c_4 = 0 \Rightarrow c_3 = c_4$

(iii) − (i) $\Rightarrow$ $c_1 - c_2 = 0 \Rightarrow c_1 = c_2$

putting, $c_4 = c_3$ and $c_2 = c_1$ in ~~(iv)~~ (ii) we get

$$2c_1 + c_3 = 0 \Rightarrow 2c_1 = -c_3$$

$$\cancel{c_1 + 2c_3 = 0}$$

Putting the values of c_3, c_4 & c_2 in (i), we get

$$c_1 - 2c_1 - 2c_1 = 0$$

$$\Rightarrow c_1 = 0 \text{ and hence } c_2 = c_3 = c_4 = 0$$

$$\therefore c_1 = c_2 = c_3 = c_4 = 0.$$

Therefore, the given sets of vector is linearly independen

9 (a) (i)

The set $\{(K,1,K),(0,K,1),(1,1,1)\}$ is a set of 3 vectors of $\mathbb{R}^3$.

The set will be a basis if they are L.I

i.e $\begin{vmatrix} K & 1 & K \\ 0 & K & 1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$

or, $K(K-1)-1(0-1)+K(0-K) \neq 0$

or, $\cancel{K^2}-K+1-\cancel{K^2} \neq 0$

or, $K \neq 1$.

Hence when $K \neq 1$, the following set form a basis

(ii) $\begin{vmatrix} K & 0 & 1 \\ 1 & K+1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$

or, $K(K+1-1)+1(1-1) \neq 0$

or, $K^2 \neq 0$ or, $K \neq 0$.

Hence when $K \neq 0$

$\{(K,0,1), (1,K+1,1), (1,1,1)\}$

are L.I

These are 3 L.I vectors in $\mathbb{R}^3$

Hence they form a basis

for $K \neq 0$.

(10). (10) Consider

(i) $$c_1(c\alpha) + c_2(c\beta) + c_3(c\gamma) = 0$$

$$\Rightarrow (c_1c)\alpha + (c_2c)\beta + (c_3c)\gamma = 0$$

Now since $\{\alpha, \beta, \gamma\}$ is L.I,

Hence $$\left.\begin{array}{l} c_1c = 0 \\ c_2c = 0 \\ c_3c = 0 \end{array}\right\} \begin{array}{l} \Rightarrow c_1 = 0 \\ \Rightarrow c_2 = 0 \\ \Rightarrow c_3 = 0 \end{array} \quad \because c \neq 0$$

Hence $\{c\alpha, c\beta, c\gamma\}$ is L.I.

Since $\dim V = 3$ any linearly independent set of 3 vectors is a basis of V.

Hence $\{c\alpha, c\beta, c\gamma\}$ is a basis of V.

(ii) Consider $$c_1(\alpha + c\beta) + c_2\beta + c_3\gamma = 0$$

$$\Rightarrow c_1\alpha + (c_1c + c_2)\beta + c_3\gamma = 0$$

$$\Rightarrow \left.\begin{array}{l} c_1 = 0 \\ c_1c + c_2 = 0 \\ c_3 = 0 \end{array}\right\} \quad \because \{\alpha, \beta, \gamma\} \text{ is L.I}$$

$$\Rightarrow c_1 = c_2 = c_3 = 0$$

Hence $\{\alpha + c\beta, \beta, \gamma\}$ is 3 independent vector. Hence this is basis of V.

(iii) Consider

$$c_1(\alpha + c\beta) + c_2(\beta + c\gamma) + c_3(\gamma + c\alpha) = 0$$

$$\Rightarrow (c_1 + c_3c)\alpha + (cc_1 + c_2)\beta + (cc_2 + c_3)\gamma = 0$$

Since $\{\alpha, \beta, \gamma\}$ is L.I

$\therefore$ $c_1 + c_3 c = 0$ — ①

$c c_1 + c_2 = 0$ — ②

$c c_2 + c_3 = 0$ — ③

① $\times c$ − ② $\times 1$

$\Rightarrow$ $c^2 c_3 - c_2 = 0$ — (4)

(4) $\times c$ + (3) $\times 1$

$\Rightarrow$ $c_3(1 + c^3) = 0$.

$\Rightarrow$ either $c_3 = 0$ or $1 + c^3 = 0$.

$\Rightarrow$ either $c_3 = 0$ or $c = -1$.

When $c_3 \neq 0$, $c = -1$

then the set $\{\alpha - \beta, \beta - \gamma, \gamma - \alpha\}$ is not L.I

Since $(\alpha - \beta) + (\beta - \gamma) = -(\gamma - \alpha)$

Hence when $c = -1$

the set dose not form a basis.

when $c \neq -1$, its form a basis.

(iv) Given $\beta_1 = \alpha_1 + \alpha_3$,

$\beta_2 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3$,

$\beta_3 = \alpha_1 + 2\alpha_2 + 3\alpha_3$

Consider

$c_1\beta_1 + c_2\beta_2 + c_3\beta_3 = 0$.

$\Rightarrow$ $c_1(\alpha_1 + \alpha_3) + c_2(2\alpha_1 + 3\alpha_2 + 4\alpha_3)$

$+ c_3(\alpha_1 + 2\alpha_2 + 3\alpha_3) = 0$.

$\Rightarrow \alpha_1 (c_1 + 2c_2 + c_3) + \alpha_2 (3c_2 + 2c_3)$ ⑨

$+ \alpha_3 (c_1 + 4c_2 + 3c_3) = 0$

Since $\{\alpha_1, \alpha_2, \alpha_3\}$ is linearly independent

$$\therefore \left.\begin{array}{r} c_1 + 2c_2 + c_3 = 0 \\ 3c_2 + 2c_3 = 0 \\ c_1 + 4c_2 + 3c_3 = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = c_3 = 0$$

Hence $\{\beta_1, \beta_2, \beta_3\}$ is linearly independent.

(11)

(i) $\mathbb{R}^4$ is a vector space of dimension 4.
The standard basis for $\mathbb{R}^4$ is $\{e_1, e_2, e_3, e_4\}$

where $e_1 = (1,0,0,0)$
$e_2 = (0,1,0,0)$
$e_3 = (0,0,1,0)$
$e_4 = (0,0,0,1)$

Let $\alpha = (1,0,1,0)$ and $\beta = (0,1,0,1)$

Then $\alpha = 1 \cdot e_1 + 0 \cdot e_2 + 1 \cdot e_3 + 0 \cdot e_4$.

Since the coefficient of e_1 in the representation of α is non-zero, hence α can be replace in the basis and $\{\alpha, e_2, e_3, e_4\}$ is a new basis of $\mathbb{R}^4$.

Let $\beta = c_1\alpha + c_2 e_2 + c_3 e_3 + c_4 e_4$.

$\therefore (0,1,0,1) = c_1(1,0,1,0) + c_2(0,1,0,0)$
$+ c_3(0,0,1,0)$
$+ c_4(0,0,0,1)$

Therefore

$c_1 = 0,$
$c_2 = 1$
$c_1 + c_3 = 0 \Rightarrow c_3 = 0.$
$c_4 = 1.$

Since the the coefficient of e_2 is non-zero hence β can be replace e_2 in the basis $\{\alpha, e_2, e_3, e_4\}$ and $\{\alpha, \beta, e_3, e_4\}$ is the new basis of $\mathbb{R}^4$.

(ii) Proceed similary as above one may get $\alpha = (1,1,0,0) = 1\cdot e_1 + 1\cdot e_2 + 0\cdot e_3 + 0\cdot e_4$.

Since coefficient of e_1 is non-zero, $\therefore$

$\{\alpha, e_2, e_3, e_4\}$ is a new basis of $\mathbb{R}^4$.

Now let $\beta = (1,1,1,0)$.

If $\beta = c_1\alpha + c_2 e_2 + c_3 e_3 + c_4 e_4$.

Then

$$(1,1,1,0) = c_1(1,1,0,0) + c_2(0,1,0,0) + c_3(0,0,1,0) + c_4(0,0,0,1)$$

Hence $c_1 = 1$

$c_1 + c_2 = 1 \Rightarrow c_2 = 0$,

$c_3 = 1$

$c_4 = 0$

Hence
$$\beta = 1 \cdot \alpha + 0 \cdot e_2 + 1 \cdot e_3 + 0 \cdot e_4$$

Since the coefficient of e_3 is non-zero, hence e_3 can be replace β and hence $\{\alpha, \beta, e_2, e_4\}$ is a new basis of $\mathbb{R}^4$.

(12) (i) $S = \{(x, y, z) \in \mathbb{R}^3 : 2x + y - z = 0\}$;

Let $\xi = (a, b, c) \in S$, then

$$2a + b - c = 0$$
$$\Rightarrow c = 2a + b$$

Therefore
$$\xi = (a, b, 2a+b)$$
$$= a(1, 0, 2) + b(0, 1, 1).$$

Let $\alpha = (1,0,2)$, $\beta = (0,1,1)$

Then $e_q = a\alpha + b\beta \in L\{\alpha, \beta\}$

Therefore $S \subset L\{\alpha, \beta\}$ —— (1)

Again $\alpha \in S$, $\beta \in S$

$\Rightarrow L\{\alpha, \beta\} \subset S$ —— (2)

from (1) & (2) we get $L\{\alpha, \beta\} = S$.

Also α, β is linearly independent, since none of them is scalar multiple of other

Hence $\{\alpha, \beta\}$ is a basis of S.

Hence $\dim S = 2$.

(ii)

$$S = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l} x+2y-z=0 \\ 2x-y+3z=0 \end{array} \right\}$$

Let $e_q = (a,b,c)$ be an arbitrary vector in S.

Then
$$a + 2b - c = 0$$
$$2a - b + 3c = 0$$

solving we have,

$$\frac{a}{5} = \frac{b}{-5} = \frac{c}{-5} = k \text{ (say)}$$

Hence u takes the form $k(5,-5,-5)$. (17)

where k is a real number.

Therefore $S = L\{(5,-5,-5)\}$

Since $\{(5,-5,-5)\}$ is L.I

Hence $\dim S = 1$.

(13). Let $S = \{(x,y,z,w) : x+y+z+w=0\}$

$T = \{(x,y,z,w) \in \mathbb{R}^4 : 2x+y-z+w = 0\}$

$\therefore S \cap T = \left\{ (x,y,z,w) \in \mathbb{R}^4 : \begin{array}{l} x+y+z+w=0 \\ 2x+y-z+w=0 \end{array} \right\}$

Let $u = (a,b,c,d) \in \mathbb{R}^4$ then,

$a+b+c+d=0$. — (1)

$2a+b-c+d=0$ — (2)

(1) + (2) gives,

$$3a+2b+2d=0$$

$$\Rightarrow d = -\left(\frac{3a+2b}{2}\right)$$

Now let $c \in \mathbb{K}$

then $c\alpha = (ca_1, ca_2)$

$$T(c\alpha) = (ca_1 + ca_2,\ ca_1 - ca_2)$$
$$= c(a_1 + a_2,\ a_1 - a_2)$$
$$= cT(\alpha)$$

Thus $T(\alpha+\beta) = T(\alpha) + T(\beta) \quad \forall \alpha, \beta \in \mathbb{R}^2$

$T(c\alpha) = cT(\alpha) \quad \forall c \in \mathbb{R}$, $\alpha \in \mathbb{R}^2$.

Hence T is a linear transformation.

(ii) Proceed similarly as above.

(iii). $T(x, y, z) = (yz, zx, xy)$

Consider $\alpha = (1, 0, 0)$

$\beta = (0, 1, 0)$

$T(\alpha) = (0, 0, 0)$

$T(\beta) = (0, 0, 0)$

$\alpha + \beta = (1, 1, 0)$

$T(\alpha+\beta) = (0, 0, 1)$

$\neq T(\alpha) + T(\beta)$

$$\boxed{\begin{array}{l}\alpha = (1,0,0)\\ \beta = (0,1,0)\end{array}}$$

$T(\alpha) = (0,0,0)$

$T(\beta) = (0,0,0)$

$\alpha+\beta = (1,1,0)$

$T(\alpha+\beta) = (0,0,1)$

$\neq \underline{T(\alpha)+T(\beta)}$

Hence T is not linear map.

(15)

$T(0,1,1) = (2,1,1)$
$T(1,0,1) = (1,2,1)$
$T(1,1,0) = (1,1,2)$

Let $u = (x,y,z)$ be an arbitrary vector of the domain space $\mathbb{R}^3$.

Let $u = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$

$$= (\underbrace{c_2 + c_3}_{x},\ \underbrace{c_1 + c_3}_{y},\ \underbrace{c_1 + c_2}_{z})$$

then $c_2 + c_3 = x$
$c_3 + c_1 = y$
$c_1 + c_2 = z$

solving we have

$$c_1 = \frac{y+z-x}{2}, \quad c_2 = \frac{z+x-y}{2},$$

$$c_3 = \frac{x+y-z}{2}.$$

Since T is linear

$$\therefore T(u) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$$

$$= c_1(2,1,1) + c_2(1,2,1) + c_3(1,1,2)$$

$$= (2c_1 + c_2 + c_3,\ c_1 + 2c_2 + c_3,\ c_1 + c_2 + 2c_3)$$

$\therefore T(x,y,z) = (y+z,\ z+x,\ x+y)$

(16). Let us find the scalars c_1, c_2, c_3 such that

$$(1,3,1) = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$$

then $(1,3,1) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$

This gives

$$c_1 + c_2 + c_3 = 1$$
$$c_1 + c_2 = 3$$
$$c_1 = 1$$

solving $c_1 = 1,\ c_2 = +2,\ c_3 = -2$

$\therefore$ Co-ordinate vector of α is

$$(1, 2, -2).$$

~~(17)~~ (17)

Let $U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+b=0 \right\}$

$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c+d=0 \right\}$.

Let

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in U.$$

Then $a_1 + a_2 = 0$

$\Rightarrow a_2 = -a_1$

Hence $\dim U = 3$. (4)

Similary one may get $\dim W = 3$.

4a) $$U \cap W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} a+b=0 \\ c+d=0 \end{array} \right\}$$

Now let $E = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \in U \cap W$

Then $e_1 + e_2 = 0 \Rightarrow e_2 = -e_1$
$e_3 + e_4 = 0 \Rightarrow e_4 = -e_3$

Hence

$$E = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$$

$$= \begin{pmatrix} e_1 & -e_1 \\ e_3 & -e_3 \end{pmatrix}$$

$$= e_1 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + e_3 \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

Now let $E_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$

Hence

$$A = \begin{pmatrix} a_1 & -a_1 \\ a_3 & a_4 \end{pmatrix}$$

$$= a_1 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + a_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Now let

$$A_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then $\{A_1, A_2, A_3\}$ are linearly independent

Also, $A \in U \Rightarrow A \in L\{A_1, A_2, A_3\}$

Hence $U \subset L\{A_1, A_2, A_3\}$ ——— (1)

Since $A_1, A_2, A_3 \in U$

$\Rightarrow L\{A_1, A_2, A_3\} \subset U$ ——— (2)

From (1) & (2) we get that

$L\{A_1, A_2, A_3\} = U$

then $\{E_1, E_2\}$ are L.I. (15)

Also $E \in U \cap W$

$\Rightarrow E \in L\{E_1; E_2\}$

$\therefore U \cap W \subset L\{E_1, E_2\}$ —— ①

Now $E_1 \in U \cap W$

$E_2 \in U \cap W$

$\Rightarrow L\{E_1, E_2\} \in U \cap W$ —— ②

From ① & ② one may get that

$L\{E_1, E_2\} = U \cap W$

Hence $\{E_1, E_2\}$ is a basis of $U \cap W$.

$\therefore \boxed{\dim(U \cap W) = 2}$

$$\begin{aligned}\dim(U+W) &= \dim U + \dim W - \dim(U \cap W)\\ &= 3 + 3 - 2\\ &= 6 - 2\\ &= 4.\end{aligned}$$

(18). $T(0,1,1) = (0,1,1,1)$

$T(1,0,1) = (1,0,1,1)$

$T(1,1,0) = (1,1,0,1)$

Now let $v = (x, y, z) \in \mathbb{R}^3$

then $(x, y, z) = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)$

$$\therefore \quad c_2 + c_3 = x \quad \text{--- (1)}$$

$$c_1 + c_3 = y \quad \text{--- (2)}$$

$$c_1 + c_2 = z \quad \text{--- (3)}$$

$$\therefore \quad c_1 - c_2 = y - x$$

$$c_1 + c_2 = z$$

$$2c_1 = (y - x + z)$$

$$c_1 = \frac{y - x + z}{2}$$

Similarly

$$c_2 = \frac{x + z - y}{2}$$

$$c_3 = \frac{x + y - z}{2}$$

Since T is linear

$$\therefore T(x,y,z) = c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$$

$$= c_1(0,1,1,1) + c_2(1,0,1,1) + c_3(1,1,0,1)$$

$\therefore T(x,y,z) = (c_2+c_3,\ c_1+c_3,\ c_1+c_2,\ c_1+c_2+c_3)$

$$= \left(x, y, z, \frac{x+y+z}{2}\right)$$

Q

Let $(x,y,z) \in N(T)$

$(x,y,z) \in N(T)$
$T(x,y,z) = (0,0,0,0)$

then
$$\left.\begin{array}{l} x = 0 \\ y = 0 \\ z = 0 \\ \frac{x+y+z}{2} = 0 \end{array}\right\}$$

Hence $N(T) = \{0\}$

$\therefore \dim N(T) = 0$.

Q Range of T is a linear span of the vector $\{(0,1,1,1), (1,0,1,1), (1,1,0,1)\}$

Now consider
$$c_1(0,1,1,1) + c_2(1,0,1,1) + c_3(1,1,0,1) = 0$$

$$\Rightarrow \begin{array}{l} c_2 + c_3 = 0 \\ c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_1 + c_3 + c_2 = 0 \end{array} \Rightarrow c_1 = c_2 = c_3 = 0$$

Hence $\{(0,1,1,1), (1,0,1,1), (1,1,0,1)\}$ is a basis of $R(T)$

$\therefore \dim R(T) = 3$,

therefore $\dim N(T) + \dim R(T)$

$= 0 + 3$

$= 3$ (proved)

(19) $Dp(x) = \frac{d}{dx} p(x)$, $p(x) \in P$.

Now, $D(\alpha p(x) + \beta q(x))$, $\alpha, \beta \in F$

$= \frac{d}{dx}(\alpha p(x) + \beta q(x))$

$= \frac{d}{dx}(\alpha p(x)) + \frac{d}{dx}(\beta q(x))$

$= \alpha \frac{d}{dx}(p(x)) + \beta \frac{d}{dx}(q(x))$

$= \alpha Dp(x) + \beta Dq(x)$

$\forall \alpha, \beta \in F$

Hence D is a linear transform.

(20) $L(\{\alpha, \beta\})$ is the set of vectors that is spanned by α and β.

$\therefore L(\{\alpha, \beta\}) = \{ c\alpha + d\beta : c \in \mathbb{R}, d \in \mathbb{R} \}$

(i) If $\gamma \in L(\{\alpha, \beta\})$ then there must be real number c, d such that

$$(2,1,3) = c(1,2,3) + d(3,1,0)$$
$$= (c+3d,\ 2c+d,\ 3c)$$

therefore
$$\left.\begin{aligned} c+3d &= 2 \\ 2c+d &= 1 \\ 3c &= 3 \end{aligned}\right\}$$

These equations are inconsistent.

$\because$ $c = 1$, $\Rightarrow$ $d = 1-2c$
$= -1$,

But it does not satisfy
$c+3d = 2$.

$\therefore$ $\boxed{\gamma \notin L(\{\alpha, \beta\})}$

(ii) If $\delta \in L(\{\alpha, \beta\})$

then
$$\left.\begin{aligned} c+3d &= -1 \\ 2c+d &= 3 \\ 3c &= 6 \end{aligned}\right\} \Rightarrow c=2,\ d=-1$$

Hence these eqn's are consistent

$\therefore$ $\boxed{\delta \in L(\{\alpha, \beta\})}$

(21)

(i) $T(1,0,0) = (3,1) = 3(1,0) + 1(0,1)$

$T(0,1,0) = (-2,-3) = -2(1,0) + (-3)(0,1)$

$T(0,0,1) = (1,-2) = 1(1,0) - 2(0,1)$

therefore the matrix of

$$T = \begin{pmatrix} 3 & 1 \\ -2 & -3 \\ 1 & -2 \end{pmatrix}^T$$

$$= \begin{pmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix}$$

(ii) $T(1,0,0) = (-2,-3) = -3(0,1) - 2(1,0)$

$T(1,0,0) = (3,1) = 1(0,1) + 3(1,0)$

$T(0,0,1) = (1,-2) = -2(0,1) + 1(1,0)$

Therefore matrix of

$$T = \begin{pmatrix} -3 & -2 \\ 1 & 3 \\ -2 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} -3 & 1 & -2 \\ -2 & 3 & 1 \end{pmatrix}$$

22 Given $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T(1,0,0) = (1,-1,0)$$
$$T(0,1,0) = (0,2,1)$$
$$T(0,0,1) = (1,1,1).$$

$$T(x,y,z) = T\big(x(1,0,0) + y(0,1,0) + z(0,0,1)\big)$$
$$= xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$$
$$= x(1,-1,0) + y(0,2,1) + z(1,1,1)$$
$$= (x+z,\ -x+2y+z,\ y+z).$$

$$\operatorname{Ker}(T) = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, T(x,y,z) = 0 \right\}$$
$$= \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \begin{array}{c} x+z=0 \\ -x+2y+z=0 \\ y+z=0 \end{array} \right\}.$$

Let $(x,y,z) \in \operatorname{Ker}(T)$

$$\Rightarrow \quad \begin{array}{c} x+z=0 \\ -x+2y+z=0 \\ y+z=0. \end{array}$$

$$\Rightarrow \quad x = -z, \quad y = -z$$

$$\therefore \quad (x,y,z) = (-z,-z,z)$$
$$= z(-1,-1,1) \quad \forall (x,y,z) \in \operatorname{Ker}(T).$$

$$\operatorname{Ker}(T) \subseteq L(\{(-1,-1,1)\}).$$

Since $(-1,-1,1) \in Ker(T)$,

$$L(\{(-1,-1,1)\}) \subseteq Ker(T).$$

$$\therefore \quad Ker(T) = L(\{(-1,-1,1)\})$$

$$\therefore \quad \dim(Ker(T)) = 1.$$

By definition of T,

$$R(T) = L(\{(1,-1,0),(0,2,1),(1,1,1)\})$$

Since $(1,1,1) = (1,-1,0) + (0,2,1)$,

$\{(1,-1,0),(0,2,1),(1,1,1)\}$ is l.d

& check that $\{(1,-1,0),(0,2,1)\}$ is l.i

$\therefore$ A basis of $R(T)$.

$$\therefore \quad R(T) = L(\{(1,-1,0),(0,2,1)\})$$

$$\therefore \quad \dim(R(T)) = 2.$$

Now $\dim(N(T)) + \dim(R(T)) = 1+2$

$$= 3.$$

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