Indian Institute of Technology, Kharagpur

Mid Autumn Semester Examination 2009

Department of Mathematics

Sub No: MA 40001/41007 Sub Name: Functional Analysis

Time: Two hours. Full Marks: 30

Answer all questions, the questions are of equal values

1(a) Let $\|.\|_\infty$ denote the sup norm on $C[0,1]$, i.e. $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|, x \in C[0,1]$. Let $x_0 \in C[0,1]$ be fixed function. Prove that the map $L : C[0,1] \mapsto \mathbb{R}$ defined by $L(h) = \int_0^1 2x_0(t)h(t)\,dt$, $h \in C[0,1]$ is bounded linear transformation from $(C[0,1], \|.\|_\infty)$ to $(\mathbb{R}, |.|)$.

1(b) Let X be a normed linear space, and let (T_n) be a convergent sequence with limit $T \in B(X)$. If $S \in B(X)$, then show that (ST_n) is convergent in $B(X)$ with limit ST.

2(a) Show that l^∞ is not separeble.

2(b) Lets X be the set of all continuous real-valued functions on $J = [0,1]$, and let $d(x,y) = \int_0^1 |x(t) - y(t)|\,dt$. Show that the metric space (X,d) is not complete.

3(a) If X is a finite dimensional normed linear space and Y be any normed linear space, show that every linear map from X to Y is continuous.

3(b) If X is an infinite dimensional normed linear space, show that there exist (i) a linear one-to-one map $F : X \mapsto X$ which is not continuous. (ii) a linear functional $f : X \mapsto K$ which is not continuous.

4(a) For $x = (\xi_j) \in l^p$, $y = (\eta_j) \in l^p$ and $p \geq 1$, show that

$$(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{\infty} |\xi_l^p|)^{\frac{1}{p}} + (\sum_{m=1}^{\infty} |\eta_m|^p)^{\frac{1}{p}}.$$

4(b) Let $T : D(T) \subset X \mapsto Y$ be a bounded linear operator, where X is a normed linear space and Y is a Banach space. Show that T has an extension on the closure of $D(T)$ which preserves the norm.

5(a) Let X be a normed linear space and E be a proper closed subspace of X. Show that there is a x in X with $\|x\| = 1$ and $\|y - x\| > \frac{1}{3}$, for all $y \in E$.

5(b) Let $T : X \mapsto Y$ be a linear operator, where X and $Y$are norm spaces. Show that the following conditions ar equivalent (i) T is continuous on X (ii) T sends Cauchy sequence in X in to Cauchy sequence in Y

6 Let Y be a finite dimensional proper subspace of a normed space X. Show that there is x_1 in X such that $\|x_1\| =$ and $dist(x_1, Y) = 1$, that is Riesz lemma holds with $r = 1$.

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