MA10002 Mathematics-II : Assignment - 7

1. Find the value of integrals (i) $\int\limits_0^\infty e^{-x^2}dx$ (ii) $\int\limits_0^\infty e^{-x}x^{\frac{3}{2}}dx$ (iii) $\int\limits_0^\infty x^m e^{-ax^n}dx$, where $m, n,$ and a are positive integers. (iv) $\int\limits_0^{\frac{\pi}{2}} \sin^4 x\cos^4 x dx$ (v) $\int\limits_r^s (x-r)^{k-1}(s-x)^{l-1}dx$.

2. Given $\int\limits_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}$, prove that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$ where $0 < n < 1$.

3. Show that (i) $\int\limits_0^1 \sqrt{1-x^4}dx = \frac{\{\Gamma(\frac{1}{4})\}^2}{6\sqrt{2\pi}}$ (ii) $\int\limits_0^{\frac{\pi}{2}} \sqrt{\tan x}dx = \frac{\pi}{\sqrt{2}}$ (iii) $\int\limits_0^{\frac{\pi}{2}} \sqrt{\cos x}dx = \frac{(2\pi)^{\frac{3}{2}}}{[\Gamma(\frac{1}{4})]^2}$.

4. (i) Show that $\int\limits_0^1 x^m\left(\log\frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$, where $m > -1, n > -1$. (ii) If m is a nonnegative integer and n is a positive constant, then show that $\int\limits_0^\infty x^m n^{-x}dx = \frac{m!}{(\log n)^{m+1}}$.

5. Show that if m is a positive integer then $\Gamma(m+\frac{1}{2}) = \frac{(2m-1)(2m-3)(2m-5)\cdots\cdots(3)(1)\sqrt{\pi}}{2^m}$.

6. If m is positive integer and $x-m \neq 0, -1, -2, -3, \cdots\cdots$, then find the value of $\frac{\Gamma(x+m)}{\Gamma(x-m)}$.

7. Show that if m is a positive integer then
   (i) $2 \,.\, 4 \,.\, 6 \,.\, 8 \,.\, 10 \,.\, 12 \,.,\cdots\cdots,.2m = 2^m\Gamma(m+1)$.
   (ii) $1 \,.\, 3 \,.\, 5 \,.\, 7 \,.\, 9 \,.\, 11 \,.,\cdots\cdots,.(2m-1) = \frac{2^{1-m}\Gamma(2m)}{\Gamma(m)}$.

8. Show that $\sqrt{\pi}\Gamma(2m+1) = 2^{2m}\Gamma(m+\frac{1}{2})\Gamma(m+1)$ for any positive integer m. Hence, deduce the Legendre's duplication formula $\sqrt{\pi}\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m+\frac{1}{2})$.

9. Show that $\int\limits_0^\infty \frac{x^m}{x^n+a}dx = \frac{1}{na^{\left(\frac{n-m-1}{n}\right)}}\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(1-\frac{m+1}{n}\right)$, where the constants $m, n,$ and a are such that $a > 0$ and $n > m+1 > 0$.

10. Show that if m is a positive integer then $\Gamma(\frac{1}{m})\Gamma(\frac{2}{m})\cdots\cdots\Gamma(\frac{m-1}{m}) = \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}}$.