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Hence $\{b, b_1, b_2, \dots, b_k\} \subseteq B$ is LD. A contradiction that B is LI.

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3. Let $S \subset \mathbb{R}^n$ be linearly independent and $|S| = n$. Then S is maximal linearly independent.

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