1.8 Binomial Theorem

Lemma 1.8.1. *Let N be a finite set consisting of n elements. Then the number of distinct subsets of N, of size k, $1 \le k \le n$, equals* $\dfrac{n!}{k!\,(n-k)!}$.

Proof: It can be easily verified that the result holds for $k = 1$. Hence, we fix a positive integer k, with $2 \le k \le n$. Then observe that any one-to-one function $f : \{1, 2, \ldots, k\} \longrightarrow N$ gives rise to the following:

1. a set $K = \mathrm{Im}(f) = \{f(i) : 1 \le i \le k\}$. The set K is a subset of N and $|K| = k$ (as f is one-to-one). Also,

2. given the set $K = \mathrm{Im}(f) = \{f(i) : 1 \le i \le k\}$, one gets a one-to-one function $g : \{1, 2, \ldots, k\} \longrightarrow K$, defined by $g(i) = f(i)$, for $1 \le i \le k$.

Therefore, we define two sets A and B by

$$A = \{f : \{1, 2, \ldots, k\} \longrightarrow N \mid f \text{ is one-to-one}\}, \text{ and}$$

$$B = \{K \subset N \mid |K| = k\} \times \{f : \{1, 2, \ldots, k\} \longrightarrow K \mid f \text{ is one-to-one}\}.$$

Thus, the above argument implies that there is a bijection between the sets A and B and therefore, using Item 3 on Page 25, it follows that $|A| = |B|$. Also, using Lemma 1.7.3, we know that $|A| = n_{(k)}$ and $|B| = |\{K \subset N \mid |K| = k\}| \times k!$. Hence

$$\text{Number of subsets of } N \text{ of size } k = |\{K \subset N \mid |K| = k\}| = \frac{n_{(k)}}{k!} = \frac{n!}{(n-k)! \cdot k!}.$$

■

Remark 1.8.2. *Let N be a set consisting of n elements.*

1. *Then, for $n \ge k$, the number $\dfrac{n!}{k!\,(n-k)!}$ is generally denoted by $\binom{n}{k}$, and is called "n choose k". Thus, $\binom{n}{k}$ is a positive integer and equals "Number of subsets, of a set consisting of n elements, of size k".*

2. *Let K be a subset of N of size k. Then $N \setminus K$ is again a subset of N of size $n - k$. Thus, there is one-to-one correspondence between subsets of size k and subsets of size $n - k$. Thus, $\binom{n}{k} = \binom{n}{n-k}$.*

3. *The following conventions will be used:*

$$\binom{n}{k} = \begin{cases} 0, & \text{if } n < k, \\ 1, & \text{if } k = 0. \end{cases}$$

Lemma 1.8.3. *Fix a positive integer n. Then, for any two commuting symbols x and y*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof: The expression $(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\cdots\cdot(x+y)}_{n\text{times}}$. Note that the above multiplication is same as adding all the 2^n products (appearing due to the choice of either choosing x or choosing y, from each of the above n-terms). Since either x or y is chosen from each of the n-terms, the product looks like $x^k y^{n-k}$, for some choice of $k, 0 \le k \le n$. Therefore, for a fixed $k, 0 \le k \le n$, the term $x^k y^{n-k}$ appears $\binom{n}{k}$ times as we need to choose k places from n places, for x (and thus leaving $n-k$ places for y), giving the expression $\binom{n}{k}$ as a coefficient of $x^k y^{n-k}$.

Hence, the required result follows. ■

Remark 1.8.4. *Fix a positive integer n.*

1. *Then the numbers $\binom{n}{k}$ are called* BINOMIAL COEFFICIENTS *as they appear in the expansion of $(x+y)^n$ (see Lemma 1.8.3).*

2. *Substituting $x = y = 1$, one gets $2^n = \sum\limits_{k=0}^{n} \binom{n}{k}$.*

3. *Observe that $(x+y+z)^n = \underbrace{(x+y+z)\cdot(x+y+z)\cdot\cdots\cdot(x+y+z)}_{n times}$. Note that in this expression, we need to choose, say*

   (a) i places from the n possible places for x $(i \ge 0)$,

   (b) j places from the remaining $n-i$ places for y $(j \ge 0)$ and

   thus leaving the $n-i-j$ places for z (with $n-i-j \ge 0$). Hence, one has

$$(x+y+z)^n = \sum_{i,j \ge 0, i+j \le n} \binom{n}{i} \cdot \binom{n-i}{j} x^i y^j z^{n-i-j}.$$

4. *The expression $\binom{n}{i} \cdot \binom{n-i}{j} = \dfrac{n!}{i!\; j!;\,(n-i-j)!}$ is also denoted by $\binom{n}{i,j,n-i-j}$.*

5. *Similarly, if $i_1, i_2, \ldots, i_k$ are non-negative integers, such that $i_1 + i_2 + \cdots + i_k = n$, then the coefficient of $x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$ in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$ equals*

$$\binom{n}{i_1, i_2, \ldots, i_k} = \frac{n!}{i_1! \cdot i_2! \cdots i_k!}.$$

   That is,

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{i_1,\ldots,i_k \ge 0 \\ i_1+i_2+\cdots+i_k=n}} \binom{n}{i_1, i_2, \ldots, i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}.$$

   These coefficient and called MULTINOMIAL COEFFICIENTS.