ANALYTIC FUNCTION: If the derivative $f'(z)$ exists at all points z of a domain D (open), then $f(z)$ is said to be analytic in D.

The terms regular, and holomorphic are also used for analytic.
A function $f(z)$ is said to be analytic at a point z_0 if there exists a neighbourhood $|z-z_0|<\delta$ at all points of which $f'(z)$ exists.

CAUCHY-RIEMANN EQUATIONS (C-R Equations)

A necessary condition that $f(z) = u(x,y) + i\,v(x,y)$ be analytic in a domain D is that u & v satisfy C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad —(1)$$

in D.

Moreover, if the partial derivatives in (1) are continuous in D then the C-R equations are sufficient conditions for analyticity of $f(z)$ in D.

PROOF: Necessary conditions (C-R. Equations)

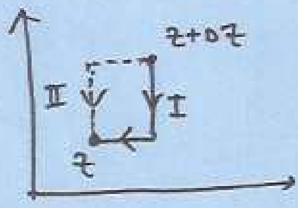
Assume $f'(z)$ exists at z.

We need to prove $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Begin with

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y)] - [u(x,y) + i v(x,y)]}{\Delta x + i \Delta y}.$$

Since $f'(z)$ exist, the right hand side limit should be the same along all path $\Delta z \to 0$



Along path I: $\Delta y \to 0$ and then $\Delta x \to 0$

$$\Rightarrow f'(z) = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right]$$

$$\Rightarrow f'(z) = u_x + i v_x \quad — ①$$

Along path II:

$$f'(z) = \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y)}{i \Delta y} + \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} \right]$$

$$\Rightarrow f'(z) = v_y - i u_y \quad — ②$$

Note that existence of $f'(z)$ implies existence of u_x, u_y, v_x, v_y.

Comparing ① & ②:

$$u_x = v_y \quad \& \quad v_x = -u_y.$$

NOTE 1: If we know the existence of the derivative, we can use the following formula

$$f'(z) = u_x + i v_x = v_y - i u_y.$$

Not 2: The CR equations are necessary condition for f to be differentiable at a point. If they are not satisfied, then $f'(z)$ does not exist at this point.

If the CR equations hold at a point z, then f may or may not be differentiale at z.

Example: Let $f(z) = \bar{z} = x - iy$ (Nowhere differentiable)

$$\Rightarrow u(x,y) = x \quad \& \quad v(x,y) = -y$$

$$u_x = 1 \qquad v_y = -1$$

$\Rightarrow$ CR equations do not hold at any point and therefore f is not differentiable at any point. (CR eq. is a necessary condition)

Example: Let $f(z) = z\,\mathrm{Re}(z)$. (Differentiable at origin but not analytic)

$$= (x+iy)x$$
$$= x^2 + ixy$$

$$u(x,y) = x^2 \qquad v(x,y) = xy$$

$$u_x = 2x \qquad v_y = x$$

$$u_y = 0 \qquad v_x = y$$

CR equations donot hold at any point except $z = 0$. (CR Eq. is a necessary condition)

$\Rightarrow$ f is not differentiable at z if $z \neq 0$, but may have a derivative at 0. Differentiability at $z=0$ needs to be checked.

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z\,\mathrm{Re}(\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \mathrm{Re}(\Delta z) = 0.$$

$\Rightarrow$ The function is differentiable at $z = 0$.

Example: $f(z) = |z|^2$ (Diff at origin but not analytic

$= x^2+y^2$

$\Rightarrow u(x,y) = x^2+y^2 \quad v(x,y) = 0$

$\Rightarrow u_x = 2x \qquad u_y = 2y$

$v_x = 0 \qquad v_y = 0$

CR equations are satisfied only at $z=0$, nowhere else. (CR equations is a necessary condition)

$\Rightarrow$ f is not differentiable at z if $z \neq 0$, but may have a derivative at $z=0$. If fact this function is differentiable at $z=0$, since

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|\Delta z|^2}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = 0.$$

Example: Show that $f(z) = \sqrt{|xy|}$ is not differentiable at the origin, although CR equations are satisfied at the point. (CR equations are not suff. for differentiability)

Sol: $u(x,y) = \sqrt{|xy|} \qquad v(x,y) = 0$

At the origin: $$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{0-0}{x} = 0.$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = 0$$

$$\frac{\partial v}{\partial x} = 0 \quad , \quad \frac{\partial v}{\partial y} = 0$$

$\Rightarrow$ CR equations are satisfied.

Again: consider

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{\sqrt{|xy|}-0}{x+iy}$$

Take $z\to 0$ along the path $y=mx$, then

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{(1+im)x} = \lim_{x\to 0} \frac{\sqrt{|m|}}{1+im}$$

which depends on $m \Rightarrow f'(0)$ does not exist.

Hence $f(z)$ is not differentiable at the origin although CR equations are satisfied.

Example: Prove that the function $f(z) = u+iv$ where

$$f(z) = \begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & z\neq 0 \\ 0 & z=0 \end{cases}$$

is continuous and CR equations are satisfied at the origin, yet $f'(z)$ does not exist there.

Sol:
$$f(z) = \frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}, \quad z\neq 0.$$

$$\Rightarrow u(x,y) = \frac{x^3-y^3}{x^2+y^2} \qquad v(x,y) = \frac{x^3+y^3}{x^2+y^2}$$

Continuity of $f(z)$ at $z=0$:

$$\lim_{z\to 0} u(x,y) = \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

$$\lim_{z\to 0} v(x,y) = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0.$$

Hence $f(z)$ is continuous at $z=0$.

Also, at the origin:

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} -\frac{y}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{x}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{y}{y} = 1.$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$\Rightarrow$ CR equations are satisfied at the origin.

<u>Now</u>: $$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

Along $y=mx$:

$$= \lim_{x\to 0} \frac{(x^3-m^3x^3)+i(x^3+m^3x^3)}{(x^2+m^2x^2)(x+imx)}$$

$$= \frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)} \quad \text{depends on } m.$$

$\Rightarrow f'(0)$ does not exist.

HARMONIC FUNCTIONS:

A function $u(x,y)$ which satisfies the Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in a domain D is said to be harmonic in D.

Th.: If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u & v satisfy Laplace's equation

$$u_{xx} + u_{yy} = 0 \quad \& \quad v_{xx} + v_{yy} = 0.$$

Proof:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \qquad \text{C-R equations.}$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

Given $f(z)$ analytic, Existence of $u_{xx}, u_{yy}, \ldots$ & continuity of higher order derivatives are obvious.

Th.: If u be harmonic on a domain D, then for some v, $u + iv$ defines an analytic function for $z = x + iy$ in D.

Note: u & v are called harmonic conjugate of each others.

CONSTRUCTION OF ANALYTICAL FUNCTION

Example: Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic and find v such that $f(z) = u + iv$ is analytic.

Sol:
$$\frac{\partial u}{\partial x} = -e^{-x}(x\sin y - y\cos y) + e^{-x}(\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(x\sin y - y\cos y) - e^{-x}(\sin y) - e^{-x}\sin y \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y)$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}[-x\sin y + \sin y + y\cos y + \sin y] \quad \text{— (2)}$$

(1) & (2) $\Rightarrow \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u$ is harmonic.

From C-R equations:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y \quad \text{— (3)}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -(xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y) \quad \text{— (4)}$$

Integrating (3):
$$v = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}\left[y\sin y - \int \sin y\, dy\right] + F(x)$$

$$= -e^{-x}\cancel{\cos y} + xe^{-x}\cos y + e^{-x}y\sin y + e^{-x}\cancel{\cos y} + F(x)$$

$$= xe^{-x}\cos y + ye^{-x}\sin y + F(x)$$

Now calculate
$$\frac{\partial v}{\partial x} = -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y + F'(x) \quad \text{— (5)}$$

(4) & (5) $\Rightarrow F'(x) = 0 \Rightarrow F(x) = C$ (constant)

$\Rightarrow v = xe^{-x}\cos y + ye^{-x}\sin y + C$

Example: Find an analytic function $f(z) = u(x,y) + i v(x,y)$, given that

$$v = \frac{x}{x^2+y^2} + \cosh x \cos y$$

$$\frac{\partial v}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \sinh x \cos y$$

$$\frac{\partial v}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} - \cosh x \sin y$$

C-R equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} - \cosh x \sin y$$

$$u = \int \frac{-2xy}{(x^2+y^2)^2}\, dx - \int \cosh x \sin y \, dx + F(y)$$

$$= -y \cdot \left(-\frac{1}{(x^2+y^2)}\right) - \sin y \cdot \sinh x + F(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} - \sinh x \cos y + F'(y)$$

Again C-R equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ gives.

$$\cancel{\frac{1}{x^2+y^2}} - \frac{\cancel{2}y^2}{(x^2+y^2)^2} - \cancel{\sinh x \cos y} + F'(y) = -\cancel{\frac{1}{x^2+y^2}} + \frac{\cancel{2}x^2}{(x^2+y^2)^2} \cancel{+ \sinh x \cos y}$$

$$\Rightarrow F'(y) = 0 \Rightarrow F(y) = C.$$

So. $$u = \frac{y}{x^2+y^2} - \sin y \sinh x + C$$

$$f(z) = u + iv = \frac{y}{x^2+y^2} - \sin y \sinh x + i\left(\frac{x}{x^2+y^2} + \cosh x \cos y\right)$$ ($C = 0$ for simplic