PARABOLIC PDEs $\quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Possible approximations:

i) $$\frac{u_m^{n+1} - u_m^n}{k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Schmidt method (explicit)

ii) $$\frac{u_m^n - u_m^{n-1}}{k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Lawson method (Implicit)

iii) $$\frac{u_m^{n+1} - u_m^{n-1}}{2k} = \frac{u_{m-1}^n - 2u_m^n + u_{m+1}^n}{h^2}$$ Richardson (leap frog) method

iv) $$\frac{u_m^{n+1} - u_m^{n-1}}{2k} = \frac{u_{m-1}^n - (u_m^{n+1} + u_m^{n-1}) + u_{m+1}^n}{h^2}$$ Dufort-Frankel method

v) $$\frac{u_m^{n+1} - u_m^n}{k} = \frac{1}{2}\left[\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} + \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2}\right]$$

Simplified forms by setting $\frac{k}{h^2} = \lambda$ (mesh ratio parameter)

i) $$u_m^{n+1} = (1-2\lambda)u_m^n + \lambda(u_{m-1}^n + u_{m+1}^n)$$

ii) $$-\lambda u_{m-1}^{n+1} + (1+2\lambda)u_m^{n+1} - \lambda u_{m+1}^{n+1} = u_m^n$$

iii) $$u_m^{n+1} = u_m^{n-1} + 2\lambda(u_{m-1}^n - 2u_m^n + u_{m+1}^n)$$

iv) $$u_m^{n+1} = \frac{(1-2\lambda)}{(1+2\lambda)} u_m^{n-1} + \frac{2\lambda}{1+2\lambda}(u_{m-1}^n + u_{m+1}^n)$$

v) $$-\lambda u_{m-1}^{n+1} + (2+2\lambda)u_m^{n+1} - \lambda u_{m+1}^{n+1} = \lambda u_{m-1}^n + (2-2\lambda)u_m^n + \lambda u_{m+1}^n$$

Ex Solve the heat equation by explicit method (FTCS):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1, \qquad 0 \le t \le 0.05$$

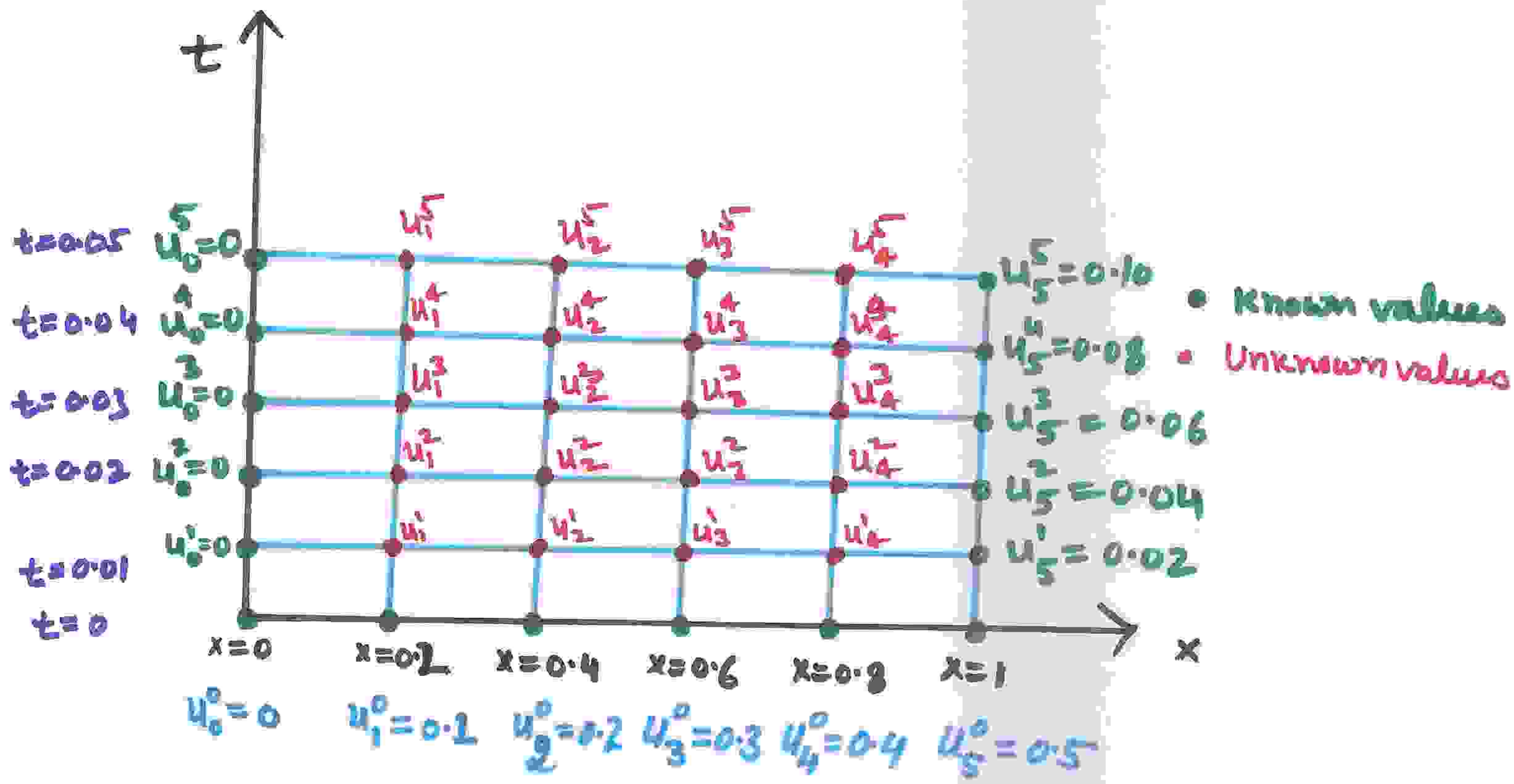
Subject to the BCs: $\underbrace{u(0,t) = 0 \ \& \ u(1,t) = 2t}_{t>0}$ and

initial condition

$$u(x,0) = \frac{1}{2}x \qquad 0 \le x \le 1$$

Take $h = 0.2$, $K = 0.01$.

Solution:



$$\lambda = \frac{K}{h^2} = \frac{0.01}{0.04} = \frac{1}{4}$$

Explicit method: $u_m^{n+1} = (1-2\lambda)u_m^n + \lambda(u_{m-1}^n + u_{m+1}^n)$

$$= \frac{1}{2}u_m^n + \frac{1}{4}(u_{m-1}^n + u_{m+1}^n)$$

$$= \frac{1}{4}\left[u_{m-1}^n + 2u_m^n + u_{m+1}^n\right].$$

Then $u_1^1 = \frac{1}{4}(u_0^0 + 2u_1^0 + u_2^0) = \frac{1}{4}(0 + 0.2 + 0.2) = 0.1$

$u_2^1 = \frac{1}{4}(u_1^0 + 2u_2^0 + u_3^0) = \frac{1}{4}(0.1 + 0.4 + 0.3) = 0.2$

and so on....

$u_3^1 = 0.30 \qquad u_4^1 = 0.40$

$u_1^2 = 0.25 \qquad u_2^2 = 0.4250 \qquad u_3^2 = 0.60 \qquad u_4^2 = 0.28$

$u_1^3 = 0.9250 \qquad u_2^3 = 1.4750 \qquad u_3^3 = 1.5113 \qquad u_4^3 = 0.300$

$u_1^4 = 3.6625 \qquad u_2^4 = 5.2737 \qquad u_3^4 = 4.544 \qquad u_4^4 = 0.5428$

$u_1^5 = 14.1363 \qquad u_2^5 = 18.7825 \qquad u_3^5 = 14.7506 \qquad u_4^5 = 1.4275$

Ex: Using Crank-Nicolson method and the central differences for the boundary conditions, solve the initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = 1; \quad 0 \le x \le 1$$

$$\frac{\partial u}{\partial x}(0,t) = u(0,t) \qquad \frac{\partial u}{\partial x}(1,t) = -u(1,t), \quad t > 0$$

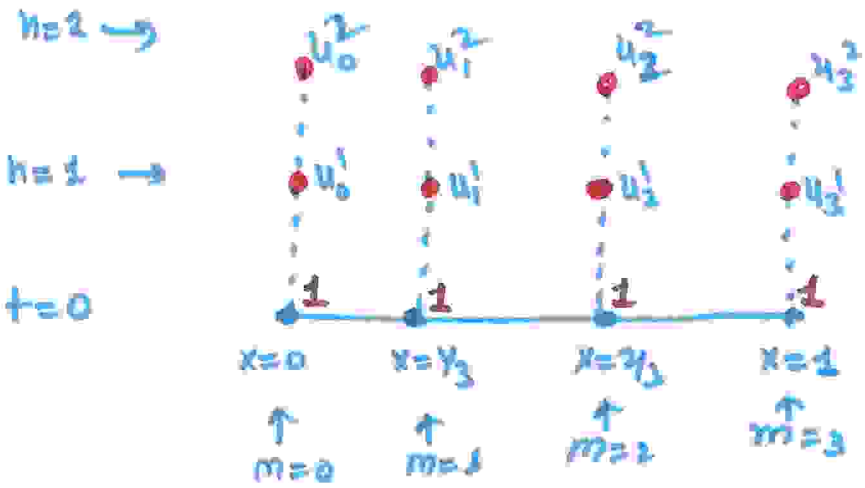
with steplength $h = 1/3$ and $\lambda = 1/3$

Integrate upto two time levels.

Sol: Crank-Nicolson Method:

$$-\lambda u_{m-1}^{n+1} + (2+2\lambda) u_m^{n+1} - \lambda u_{m+1}^{n+1} = \lambda u_{m-1}^{n} + (2-2\lambda) u_m^{n} + \lambda u_{m+1}^{n}$$

$$\Rightarrow -\frac{1}{3} u_{m-1}^{n+1} + \frac{8}{3} u_m^{n+1} - \frac{1}{3} u_{m+1}^{n+1} = \frac{1}{3} u_{m-1}^{n} + \frac{4}{3} u_m^{n} + \frac{1}{3} u_{m+1}^{n} \quad \text{—(1)}$$



equation (1) for

$m=0$: $$-\frac{1}{3} u_{-1}^{n+1} + \frac{8}{3} u_0^{n+1} - \frac{1}{3} u_1^{n+1} = \frac{1}{3} u_{-1}^{n} + \frac{4}{3} u_0^{n} + \frac{1}{3} u_1^{n} \quad \text{—(2)}$$

$m=1$: $$-\frac{1}{3} u_0^{n+1} + \frac{8}{3} u_1^{n+1} - \frac{1}{3} u_2^{n+1} = \frac{1}{3} u_0^{n} + \frac{4}{3} u_1^{n} + \frac{1}{3} u_2^{n} \quad \text{—(3)}$$

$m=2$: $$-\frac{1}{3} u_1^{n+1} + \frac{8}{3} u_2^{n+1} - \frac{1}{3} u_3^{n+1} = \frac{1}{3} u_1^{n} + \frac{4}{3} u_2^{n} + \frac{1}{3} u_3^{n} \quad \text{—(4)}$$

$m=3$: $$-\frac{1}{3} u_2^{n+1} + \frac{8}{3} u_3^{n+1} - \frac{1}{3} u_4^{n+1} = \frac{1}{3} u_2^{n} + \frac{4}{3} u_3^{n} + \frac{1}{3} u_4^{n} \quad \text{—(5)}$$

Using the central diff., the B.C. at $x=0$:

$$\frac{u_1^s - u_{-1}^s}{2h} = u_0^s \Rightarrow u_{-1}^s = u_1^s - 2h\, u_0^s$$

$$\Rightarrow u_{-1}^s = u_1^s - \frac{2}{3} u_0^s \quad \text{—(6)}$$

Now we can replace u_{-1}^{n} & u_{-1}^{n+1} in (2) using (6)

$$-\frac{1}{3}\left(u_1^{n+1}-\frac{2}{3}u_0^{n+1}\right)+\frac{8}{3}u_0^{n+1}-\frac{1}{3}u_1^{n+1}=\frac{1}{3}\left(u_1^{n}-\frac{2}{3}u_0^{n}\right)+\frac{4}{3}u_0^{n}+\frac{1}{3}u_1^{n}$$

$$\Rightarrow \frac{26}{9}u_0^{n+1}-\frac{2}{3}u_1^{n+1}=\frac{10}{9}u_0^{n}+\frac{2}{3}u_1^{n}$$

$$\Rightarrow \frac{13}{9}u_0^{n+1}-\frac{1}{3}u_1^{n+1}=\frac{5}{9}u_0^{n}+\frac{1}{3}u_1^{n} \quad \text{———} \quad (7)$$

The second B.C. gives:

$$\frac{u_4^s-u_2^s}{2h}=-u_3^s \Rightarrow u_4^s=u_2^s-\frac{2}{3}u_3^s$$

(5) ⇒

$$-\frac{1}{3}u_2^{n+1}+\frac{8}{3}u_3^{n+1}-\frac{1}{3}\left(u_2^{n+1}-\frac{2}{3}u_3^{n+1}\right)=\frac{1}{3}u_2^{n}+\frac{4}{3}u_3^{n}+\frac{1}{3}\left(u_2^{n}-\frac{2}{3}u_3^{n}\right)$$

$$\Rightarrow -\frac{2}{3}u_2^{n+1}+\frac{26}{9}u_3^{n+1}=\frac{2}{3}u_2^{n}+\frac{10}{9}u_3^{n}$$

$$\Rightarrow -\frac{1}{3}u_2^{n+1}+\frac{13}{9}u_3^{n+1}=\frac{1}{3}u_2^{n}+\frac{5}{9}u_3^{n} \quad \text{———} \quad (8)$$

The equation (3) (4) (7) & (8) in matrix form

$$\begin{bmatrix} \frac{13}{9} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 8/3 & -1/3 & 0 \\ 0 & -1/3 & 8/3 & -1/3 \\ 0 & 0 & -1/3 & 13/9 \end{bmatrix}\begin{bmatrix} u_0^{n+1} \\ u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \end{bmatrix}=\begin{bmatrix} 5/9 & 1/3 & 0 & 0 \\ 1/3 & 4/3 & 1/3 & 0 \\ 0 & 1/3 & 4/3 & 1/3 \\ 0 & 0 & 1/3 & 5/9 \end{bmatrix}\begin{bmatrix} u_0^{n} \\ u_1^{n} \\ u_2^{n} \\ u_3^{n} \end{bmatrix}$$

For $n=0$:

$$\begin{bmatrix} \frac{13}{9} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 8/3 & -1/3 & 0 \\ 0 & -1/3 & 8/3 & -1/3 \\ 0 & 0 & -1/3 & 13/9 \end{bmatrix} \begin{bmatrix} u_0^1 \\ u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = \begin{bmatrix} 5/9 & 1/3 & 0 & 0 \\ 1/3 & 4/3 & 1/3 & 0 \\ 0 & 1/3 & 4/3 & 1/3 \\ 0 & 0 & 1/3 & 5/9 \end{bmatrix} \begin{bmatrix} u_0^0 = 1 \\ u_1^0 = 1 \\ u_2^0 = 1 \\ u_3^0 = 1 \end{bmatrix}$$

$$\Rightarrow \quad u_0^1 = 0.8409$$

$$u_1^1 = 0.9773$$

$$u_2^1 = 0.9773$$

$$u_3^1 = 0.8409$$

For $n=1$:

$$\begin{bmatrix} \frac{13}{9} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 8/3 & -1/3 & 0 \\ 0 & -1/3 & 8/3 & -1/3 \\ 0 & 0 & -1/3 & 13/9 \end{bmatrix} \begin{bmatrix} u_0^2 \\ u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix} = \begin{bmatrix} 5/9 & 1/3 & 0 & 0 \\ 1/3 & 4/3 & 1/3 & 0 \\ 0 & 1/3 & 4/3 & 1/3 \\ 0 & 0 & 1/3 & 5/9 \end{bmatrix} \begin{bmatrix} 0.8409 \\ 0.9773 \\ 0.9773 \\ 0.8409 \end{bmatrix}$$

$$\Rightarrow \quad u_0^2 = 0.7629$$

$$u_1^2 = 0.9272$$

$$u_2^2 = 0.9272$$

$$u_3^2 = 0.7629$$