SIMILARITY OF MATRICES:

An $n\times n$ matrix B is called similar to an $n\times n$ matrix A if

$$B = P^{-1}AP$$

for some non-singular matrix P.

Th: If B is similar to A then B has the same eigen values as A.
If x is an eigenvector of A then $y = P^{-1}x$ is an eigenvector of B corresponding to the same eigenvalue.

Proof:

$$Ax = \lambda x$$

$$\Rightarrow \lambda P^{-1}x = P^{-1}Ax$$

$$\Rightarrow \lambda P^{-1}x = P^{-1}A(PP^{-1})x = B(P^{-1}x)$$

$\Rightarrow$ λ is an eigenvalue of B and $P^{-1}x$ a corresponding eigenvector.

Note: Similar matrices have the same determinant.

DIAGONALIZATION OF A MATRIX

A matrix A is diagonalizable if there exists a non-singular matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

THEOREM: An $n\times n$ matrix A is diagonalizable iff it has n linearly independent evectors.

THEOREM: An $n\times n$ matrix A is diagonalizable if its all the eigenvalues are real and distinct.

Remark: The matrix P which diagonalizes A is called model matrix of A whose columns are the eigenvectors corresponding to different eigenvalues.

Ex: 1:

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

eigenvalues $1 \quad 6$

Eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \qquad P^{-1} = -\frac{1}{5}\begin{bmatrix} 1 & -4 \\ -1 & -1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}.$$

Ex: 2:

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Eigen values $2 \quad 2 \quad 8$

Eigenvectors: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

Ex: 3:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Eigenvalues $2 \quad 2 \quad 3$

Eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The given matrix is not diagonalizable.

Application of Diagonalization:

a) Power of matrices:

$$P^{-1}AP = D$$

$$\Rightarrow \quad A = PDP^{-1}$$

$$A^2 = (PDP^{-1})(PDP^{-1})$$

$$= PD(P^{-1}P)DP^{-1}$$

$$A^2 = PD^2P^{-1}$$

$$A^3 = (PD^2P^{-1})(PDP^{-1})$$

$$A^3 = PD^3P^{-1} \quad \cdots \quad \boxed{A^n = PD^nP^{-1}}$$

Ex: Find A^5 for $A = \begin{bmatrix} 1 & 4 \\ 1/2 & 0 \end{bmatrix}$

Eigenvalues $\quad -1 \quad 2$

Eigenvectors $\quad \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$$

$$A^5 = \frac{1}{6}\begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^5 & 0 \\ 0 & 2^5 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 32 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 44 \\ 5.5 & 10 \end{bmatrix}.$$

b)

system of linear differential equation

$$\dot{x}(t) = A x(t)$$

- $D = P^{-1} A P \Rightarrow A = P D P^{-1}$

$$\Rightarrow \dot{x}(t) = P D P^{-1} x(t)$$

$$\Rightarrow [P^{-1} x(t)]' = D [P^{-1} x(t)]$$

Substitute: $P^{-1} x(t) =: y(t)$

then: $\dot{y}(t) = D y(t)$

$$\Rightarrow \begin{bmatrix} \dot{y}_1(t) \\ \vdots \\ \dot{y}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

$$\Rightarrow \dot{y}_i(t) = \lambda_i y_i(t) \Rightarrow y_i(t) = c_i e^{\lambda_i t} \quad \forall i$$

$$\Rightarrow x(t) = P y(t)$$

Quadratic forms:

A function $q(x_1, x_2, \ldots, x_n)$ from $\mathbb{R}^n$ to $\mathbb{R}$ of the form

$$q(x_1, x_2, \ldots, x_n) = \sum_{i \le j} a_{ij} x_i x_j$$

is called a quadratic form.

A quadratic form can be written as

$$q(x_1, x_2, \ldots x_n) = q(x) = x^T A x$$

for a symmetric matrix A.

Ex: Consider the quadratic form:

$$q(x_1, x_2) = 3x_1^2 + 10 x_1 x_2 + 2x_2^2.$$

Find a symmetric matrix A such that $q(x_1, x_2) = x^T A x$.

Sol: $A = \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix}$. Check: $x^T A x = [x_1\ x_2] \begin{bmatrix} 3 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$= [x_1\ x_2] \begin{bmatrix} 3x_1 + 5x_2 \\ 5x_1 + 2x_2 \end{bmatrix}$$

$$= 3x_1^2 + 5x_1x_2 + 5x_2x_1 + 2x_2^2$$

$$= 3x_1^2 + 10x_1x_2 + 2x_2^2$$

- Note that making A symmetric is important for a unique representation of A.

Ex: Consider the quadratic form $q(x_1, x_2, x_3) = 9x_1^2 + 7x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 - 6x_2x_3$.

Find the corresponding matrix A.

Sol:

$$A = \begin{bmatrix} 9 & -1 & 2 \\ -1 & 7 & -3 \\ 2 & -3 & 3 \end{bmatrix}$$

The general 3 dimensional quadratic form:

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3$$

Can be written as

$$[x_1 \ \ x_2 \ \ x_3] \begin{bmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Def: Let A be an $n\times n$ symmetric matrix, then A is: (valid for non-symmetric matrices)

a) Positive definit if $x^TAx > 0$ for all $x \neq 0$ in $\mathbb{R}^n$

b) Positive semidefinite if $x^TAx \geq 0$ for all $x \neq 0$ in $\mathbb{R}^n$

c) negative definit if $x^TAx < 0$ for all $x \neq 0$ in $\mathbb{R}^n$

d) negative semidefinite if $x^TAx \leq 0$ for all $x \neq 0$ in $\mathbb{R}^n$ and

e) indefinit if $x^TAx < 0$ for some x in $\mathbb{R}^n$ and $x^TAx > 0$ for some other x in $\mathbb{R}^n$.

The Principal Axes Theorem: Let A be an $n\times n$ symmetric matrix. Then there is an (orthogonal) change of variable, $x = Py$ that transforms the quadratic form x^TAx into a quadratic form y^TDy with no cross product term.

Ex: Find out what type of cure the equation $8x_1^2 - 4x_1x_2 + 5x_2^2 = 1$ represents:

By the transformation $x = Py$ we can transform the given equatio to $9y_1^2 + 4y_2^2 = 1$ where 9 & 4 are the eigenvalue of $A = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$.

This represents the ellipse.