Beta - Gamma function
(continuation)

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\, dx \quad (\alpha > 0) \qquad \Bigg| \qquad B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\, dt$$

$m > 0,\ n > 0.$

$$B(m,n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta \ \sin^{2n-1}\theta \ d\theta$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

1) $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

proof -

L.H.S. $\displaystyle \int_0^\infty e^{-x}\, x^{(\alpha+1)-1}\, dx$

$$= \int_0^\infty e^{-x} x^{\alpha}\, dx$$

$$= -e^{-x}x^{\alpha}\Big|_0^\infty + \alpha\int_0^\infty e^{-x}\, x^{\alpha-1}\, dx$$

$(\alpha > 0)$

$$= \alpha\,\Gamma(\alpha).$$

2) $n \longrightarrow$ integer ; $\Gamma(n+1) = n!$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2)$$

$$= \cdots\cdots = n(n-1)(n-2)\ldots 3.2.1.\,\Gamma(1)$$

$$= n!$$

$$\Gamma(1) = \int_0^\infty e^{-x} x^{\alpha-1}\, dx = e^{-x}\Big|_\infty^0 = 1$$

3) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\quad B\left(\frac{1}{2},\frac{1}{2}\right) = \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \dfrac{\sqrt{\pi}.\sqrt{\pi}}{1} = \pi.$

So, $\Gamma\left(\frac{1}{2}\right) = \sqrt{B\left(\frac{1}{2},\frac{1}{2}\right)} = \sqrt{\pi}$

Duplication Formula

Prove

$$\sqrt{\pi}\;\Gamma(2m) = 2^{2m-1}\,\Gamma(m)\Gamma\left(m+\tfrac{1}{2}\right)$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Now consider once for $n=m$, then for $n=\frac{1}{2}$.

$$B(m,m) = \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}$$

&

$$B\left(m,\tfrac{1}{2}\right) = \frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = \frac{\sqrt{\pi}\;\Gamma(m)}{\Gamma(m+\frac{1}{2})}$$

$$B(m,m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2m-1}\theta\,d\theta$$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1}2\theta\,d\theta$$

put $2\theta = \phi$.

$$B(m,m) = \frac{1}{2^{2m-2}}\int_0^{\pi} \sin^{2m-1}\phi\,\frac{d\phi}{2}$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1}\phi\,d\phi + \underbrace{\int_{\pi/2}^{\pi} \sin^{2m-1}\phi\,d\phi}_{I_2}$$

$$I_2 = -\int_{\pi/2}^{0} \sin^{2m-1}(\pi-\psi)\,d\psi$$

$$= \int_0^{\pi/2} \sin^{2m-1}\psi\,d\psi.$$

$$= \int_0^{\pi/2} \sin^{2m-1}\phi\,d\phi.$$

So, $$B(m,m) = \frac{1}{2^{2m-1}} \times 2\int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi.$$

$$= \frac{1}{2^{2m-1}} B\left(m, \frac{1}{2}\right)$$

$$B(m,m) = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)}, \quad B\left(m, \frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\therefore \frac{\frac{1}{2^{2m-1}} \cdot B\left(m, \frac{1}{2}\right)}{B\left(m, \frac{1}{2}\right)} = \frac{\{\Gamma(m)\}^2}{\Gamma(2m)} \times \frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(m)}$$

$$\Rightarrow \sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\,\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)$$

$$\boxed{\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}}$$

Prob Prove that $\int_0^1 \log \Gamma(x)\, dx$ converges. Hence find its value.

proof
$$\Gamma(x+1) = x\,\Gamma(x)$$

$$\int_0^1 \log \Gamma(x+1)\, dx = \int_0^1 \log x \, dx + \int_0^1 \log \Gamma(x)\, dx.$$

$$\int_0^1 \log \Gamma(x)\, dx = \underbrace{\int_0^1 \log\Gamma(x+1)\, dx}_{\text{proper}} - \int_0^1 \log x \, dx.$$

$$\underset{t\to 0}{Lt} \int_{\epsilon}^{1} \log x \, dx \longrightarrow \text{get value.}$$

or, $g(x) = \dfrac{1}{\sqrt{x}}$.

$$\underset{x\to 0}{Lt} \frac{f}{g} = \underset{x\to 0}{Lt} \sqrt{x}\log x \longrightarrow \frac{\log x}{\frac{1}{\sqrt{x}}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= 0. \text{ [By L'Hospital]}$$

$\int_0^1 \frac{dx}{\sqrt{x}}$ converges

$\Rightarrow \int_0^1 \log x \, dx$ converges.

$$\underbrace{\int_0^1 \log\Gamma(x)\,dx}_{\text{converges}} = \underbrace{\int_0^1 \log\Gamma(x+1)\,dx}_{\text{proper}} - \underbrace{\int_0^1 \log x\,dx}_{\text{converges}}.$$

$$\int_0^1 \log\Gamma(x)\,dx = \int_0^1 \log\Gamma(1-z)\,dz \quad \text{putting } x = 1-z$$

$$= \int_0^1 \log\Gamma(1-x)\,dx.$$

$$\int_0^1 \log\Gamma(x)\,dx = \frac{1}{2}\left[\int_0^1 \log\Gamma(x)\,dx + \int_0^1 \log\Gamma(1-x)\,dx.\right]$$

$$= \frac{1}{2}\int_0^1 \log\Gamma(x)\,\Gamma(1-x)\,dx.$$

$$= \frac{1}{2}\int_0^1 \log\frac{\pi}{\sin\pi x}\,dx.$$

$$= \frac{1}{2}\int_0^1 \log\pi\,dx - \frac{1}{2}\int_0^1 \log\sin\pi x\,dx = \frac{1}{2}\log 2\pi$$

Prove that -

$$\frac{\Gamma(n+1)\Gamma(1-m)}{\Gamma(n-m+2)} = \frac{1}{n+1} + \frac{m}{n+2} + \frac{m(m+1)}{2!} \times \frac{1}{n+3} + \frac{m(m+1)(m+2)}{3!} \times \frac{1}{n+4}.$$

Hint

$$\frac{\Gamma(n+1)\Gamma(1-m)}{\Gamma(n-m+2)} = B(n+1, 1-m).$$

$$= \int_0^1 x^n \left(1 + mx + \frac{m(m+1)}{2!}x^2 + \frac{m(m+1)(m+2)}{3!}x^3 + \cdots \right) dx.$$

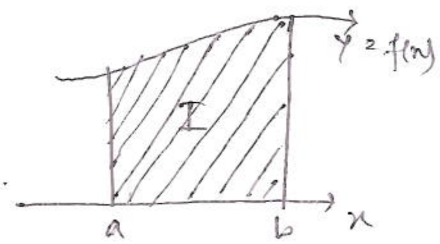
$$= \frac{1}{n+1} + \frac{m}{n+2} + \frac{m(m+1)}{2!} \frac{1}{n+3} \cdots$$

Multible Integrals

$$\int_a^b f(x)\,dx.$$

$[f(x)\,dx]$

$= [y\,dx] = [y][dx] = L \cdot L = L^2$

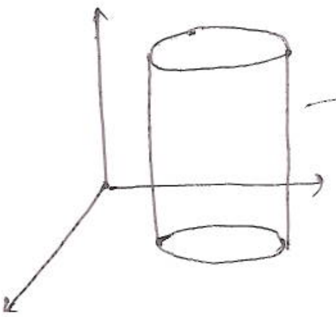


$$\int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dx\,dy. \quad , \quad \iiint_{l\,c\,a}^{m\,d\,b} f(x,y,z)\,dx\,dy\,dz.$$

$Z = f(x,y) \to$ Surface

Ex $Z = \sqrt{a^2 - x^2 - y^2} = f_1(x,y)$ or $Z = -\sqrt{a^2-x^2-y^2} = f_2(x,y)$

are the parts of ~~the~~ surface of the sphere $x^2+y^2+z^2 = a^2$



$\longrightarrow Z = f(x,y) = \sqrt{4 - x^2 - y^2}$

$Z = 1, \quad \sqrt{4 - x^2 - y^2} = 1$

$x^2 + y^2 = 3.$

$Z = 0, \quad x^2 + y^2 = 4.$

- **Books**
  - Jain - Iyengar
  - Howard Auton [vector calculus]
  - Apostol - Vol II.

A. Constant limits

$$\int_{y=c}^{d}\int_{a}^{b} f(x,y)\,dx\,dy \qquad D = R = \{(x,y) : a \le x \le b, \; c \le y \le d\}$$

Ex

$$\iint_D \frac{x^2}{1+y^2}\, dx\, dy \qquad D = \{0 \le x \le 1, 0 \le y \le 1\}$$

Note: $f(x,y) = A(x) B(y)$.

$$= \left(\int_0^1 \frac{dy}{1+y^2}\right)\left(\int_0^1 x^2\, dx\right) = \frac{\pi}{12}.$$

Ex 2

$$\int_{y=0}^{\pi/2}\int_{x=-1}^{1} (x \sin y - y e^{x})\, dx\, dy$$

$$= \int_{y=0}^{\pi/2}\int_{-1}^{1} x \sin y\, dx\, dy - \int_{y=0}^{\pi/2}\int_{x=-1}^{1} y e^{x}\, dx\, dy.$$

$$= \left(\frac{1}{e} - e\right)\frac{\pi^2}{8}$$

Ex 3

$$\iint_D \sin(x+y)\, dx\, dy \; ; \; D = \left\{0 \le x \le \frac{\pi}{2}, 0 \le y \le \frac{\pi}{2}\right\}.$$

$$= \int_{y=0}^{\pi/2}\int_{x=0}^{\pi/2} \sin(x+y)\, dx\, dy$$

$$= \int_{y=0}^{\pi/2} \cos(x+y)\Big|_{x=\pi/2}^{0}\, dy.$$

$$= \int_{y=0}^{\pi/2} \left[\cos(y) - \cos\left(\frac{\pi}{2}+y\right)\right] dy.$$

$$= \left[\sin y\right]_0^{\pi/2} - \left[\sin\left(y+\frac{\pi}{2}\right)\right]_0^{\pi/2}$$

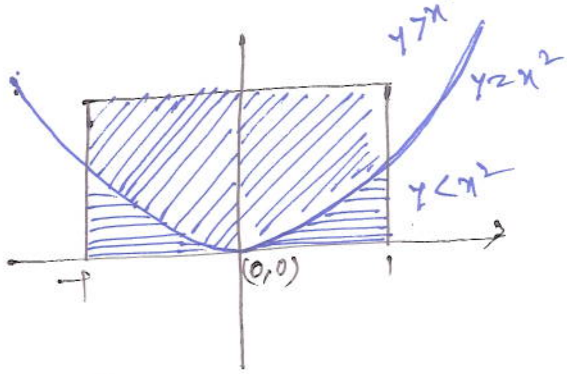
$$= 1 + 1 = 2.$$

Ex 4

$$\iint_D \sqrt{|y-x^2|}\;dx\,dy$$

where, $D=\{-1\le x\le 1,\ 0\le y\le 2\}$

$$= \iint_{D_1\,(y>x^2)} \sqrt{y-x^2}\;dx\,dy + \iint_{D_2\,(y<x^2)} \sqrt{x^2-y}\;dx\,dy$$



$$= \int_{x=-1}^{1}\int_{y=x^2}^{2}\sqrt{y-x^2}\;dy \;=\; \int_{x=-1}^{1}\int_{y=0}^{x^2}\sqrt{x^2-y}\;dx\,dy$$

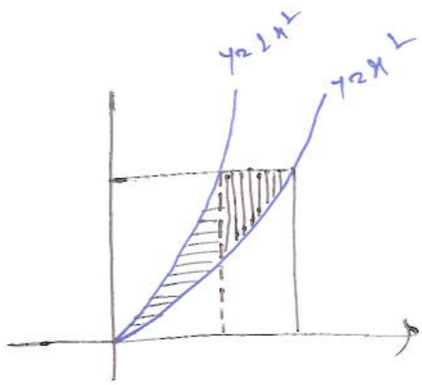
$$= \frac{\pi}{2}+\frac{4}{3}$$

Ex 5

$$I = \iint_R f(x,y)\;dx\,dy.$$

where, $R=\{0\le x\le 1,\ 0\le y\le 1\}$.

$$f(x,y)=\begin{cases} x+y, & x^2<y<2x^2 \\ 0, & \text{otherwise} \end{cases}$$



$$I = \int_{x=0}^{1/\sqrt{2}}\int_{y=x^2}^{2x^2} f(x,y)\;dx\,dy + \int_{x=1/\sqrt{2}}^{1}\int_{y=x^2}^{1}(x+y)\;dx\,dy$$

Ex $$\iint_R \frac{e^{\frac{2x}{y}}}{y^3}\,dx\,dy, \quad R = \{0 \le x \le 2;\ 1 \le y \le 2\}.$$

$$= \int_{y=1}^{2}\left(\int_{x=0}^{2} e^{2x/y}\,dx\right)\frac{dy}{y^3} = \int_{y=1}^{2} \frac{y}{2} e^{2x/y}\Big|_0^2 \frac{dy}{y^3}$$

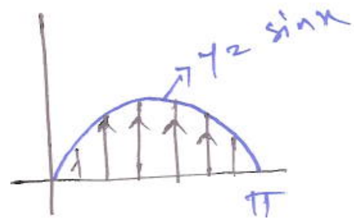
$$= \frac{1}{2}\int_1^2 \left(e^{4/y} - 1\right)\frac{dy}{y^2} = \frac{1}{2}\int_1^2 \frac{e^{4/y}}{y^2} - \frac{1}{2}\int_1^2 \frac{dy}{y^2}$$

$$= -\frac{1}{8}\left[e^2 - e^4\right] - \frac{1}{4}.$$

- **Variable limits**

$$I = \iint_D (x^2 - y^2)\,dx\,dy.$$

where, D is a region bounded by $y = \sin x$ and $[0, \pi]$ on x-axis.



$$= \int_{x=0}^{\pi}\int_{y=0}^{\sin x} (x^2 - y^2)\,dx\,dy$$

$$= \int_{x=0}^{\pi}\left[x^2 y - \frac{y^3}{3}\right]_0^{\sin x} dx.$$

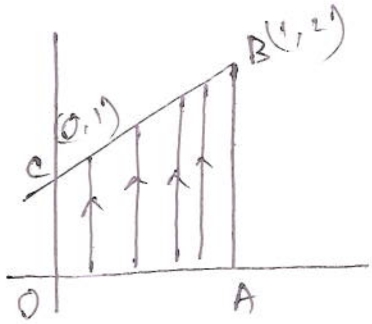
$$= \int_0^{\pi}\left[x^2 \sin x - \frac{\sin^3 x}{4}\right] dx$$

$$= \pi^2 - \frac{40}{9}$$

EX

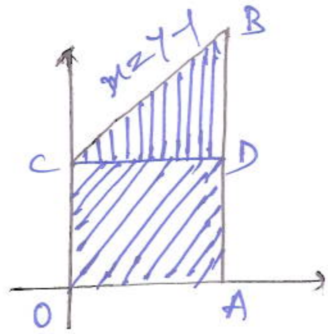
$$\iint_D (1+x)\sin y \, dx\, dy.$$

Where, D is a trapezium whose vertices are at $(0,0), (1,0), (1,2), (0,1)$



Now, Equation of the line CB is $\frac{y-2}{x-1} = \frac{2-1}{1-0} \Rightarrow y = x+1$

$$I = \int_{x=0}^{1}\int_{y=0}^{x+1} (1+x)\sin y \, dx\, dy$$



$$= \int_{y=0}^{1}\int_{x=0}^{1} f(x,y)\, dx\, dy$$

$$+ \int_{y=1}^{2}\int_{x=y-1}^{1} f(x,y)\, dx\, dy$$

EX

change the limits of the given integration

$$I = \int_{0}^{2a}\int_{0}^{y=+\sqrt{2ax-x^2}} f(x,y)\, dx\, dy$$

$y^2 = 2ax - x^2 \Rightarrow (x-a)^2 + y^2 = a^2. \Rightarrow x = a \pm \sqrt{a^2 - y^2}$

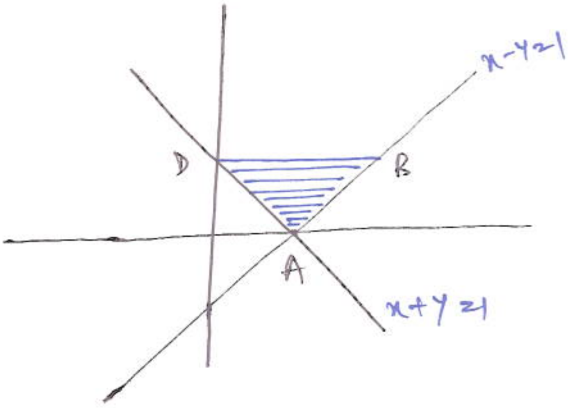
$$\therefore I = \int_{y=0}^{a}\int_{x=a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\, dx\, dy$$

Ex. Change the limits in

$$I = \int_0^1 dy \int_{1-y}^{1+y} f(x,y)\, dx\, dy$$

$x = 1+y \Rightarrow x - y = 1$

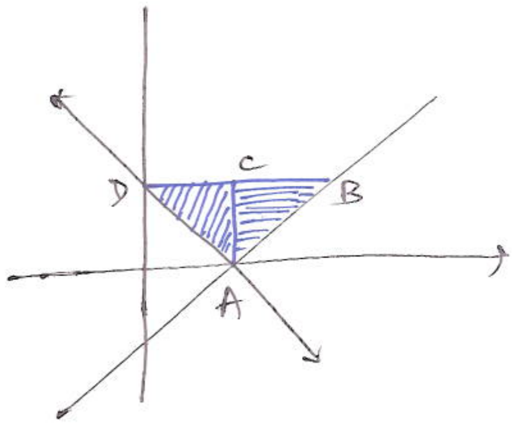
$x = 1-y \Rightarrow x + y = 1$



So, the given region is denoted by ABD

Let us divide it in two parts $ABD = ACD + ABC$

$$= \iint_{ACD} dx\,dy + \iint_{ABC} dx\,dy$$



$$= \int_{x=0}^{1}\int_{1-x}^{1} f\, dy\, dx$$

$$+ \int_{x=1}^{2}\int_{y=x-1}^{1} f\, dy\, dx.$$