1.7 Distinguishable Balls

In the previous chapter, we had seen the following:
Let A and B be two non-empty finite disjoint subsets of a set S. Then

1. $|A \cup B| = |A| + |B|$.

2. $|A \times B| = |A| \cdot |B|$.

3. A and B have the same cardinality if there exists a one-one and onto function $f : A \longrightarrow B$.

Lemma 1.7.1. *Let M and N be two sets such that $|M| = m$ and $|N| = n$. Then the total number of functions $f : M \longrightarrow N$ equals n^m.*

Proof: Let $M = \{a_1, a_2, \ldots, a_m\}$ and $N = \{b_1, b_2, \ldots, b_n\}$. Since a function is determined as soon as we know the value of $f(a_i)$, for $1 \le i \le m$, a function $f : M \longrightarrow N$ has the form

$$f \leftrightarrow \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ f(a_1) & f(a_2) & \cdots & f(a_m) \end{pmatrix},$$

where $f(a_i) \in \{b_1, b_2, \ldots, b_n\}$, for $1 \le i \le m$. As there is no restriction on the function f, $f(a_1)$ has n choices, $b_1, b_2, \ldots, b_n$. Similarly, $f(a_2)$ has n choices, $b_1, b_2, \ldots, b_n$ and so on. Thus, the total number of functions $f : M \longrightarrow N$ is

$$\underbrace{n \cdot n \cdot \cdots \cdot n}_{m \text{ times}} = n^m.$$

■

Remark 1.7.2. *Observe that Lemma 1.7.1 is equivalent to the following question:* In how many ways can m **distinguishable/distinct** balls be put into n **distinguishable/distinct** boxes? *Hint: Number the balls as $a_1, a_2, \ldots, a_m$ and the boxes as $b_1, b_2, \ldots, b_n$.*

Lemma 1.7.3. *Let M and N be two sets such that $|M| = m$ and $|N| = n$. Then the total number of distinct one-to-one functions $f : M \longrightarrow N$ is $n(n-1)\cdots(n-m+1)$.*

Proof: Observe that "f is one-to-one" means "whenever $x \neq y$ we must have $f(x) \neq f(y)$". Therefore, if $m > n$, then the number of such functions is 0.

So, let us assume that $m \le n$ with $M = \{a_1, a_2, \ldots, a_m\}$ and $N = \{b_1, b_2, \ldots, b_n\}$. Then by definition, $f(a_1)$ has n choices, $b_1, b_2, \ldots, b_n$. Once $f(a_1)$ is chosen, there are only $n-1$ choices for $f(a_2)$ ($f(a_2)$ has to be chosen from the set $\{b_1, b_2, \ldots, b_n\} \setminus \{f(a_1)\}$). Similarly, there are only $n-2$ choices for $f(a_3)$ ($f(a_3)$ has to be chosen from the set $\{b_1, b_2, \ldots, b_n\} \setminus \{f(a_1), f(a_2)\}$), and so on. Thus, the required number is $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-m+1)$. ■

Remark 1.7.4. 1. *The product $n(n-1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$ is denoted by $n!$, and is commonly called "n factorial".*

2. *By convention, we assume that* $0! = 1$.

3. *Using the factorial notation* $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-m+1) = \dfrac{n!}{(n-m)!}$. *This expression is generally denoted by* $n_{(m)}$, *and is called the falling factorial of* n. *Thus, if* $m > n$ *then* $n_{(m)} = 0$ *and if* $n = m$ *then* $n_{(m)} = n!$.

4. *The following conventions will be used in these notes:*

$$0! = 0_{(0)} = 1, \ 0^0 = 1, \ n_{(0)} = 1 \text{ for all } n \geq 1, \ 0_{(m)} = 0 \text{ for } m \neq 0.$$

The proof of the next corollary is immediate from Lemma 1.7.3 and hence the proof is omitted.

Corollary 1.7.5. *Let* M *and* N *be two sets such that* $|M| = |N| = n$ *(say). Then the number of one-to-one functions* $f : M \longrightarrow N$ *equals* $n!$, *called "n-factorial".*