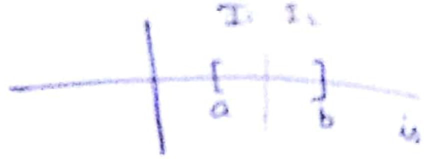
$\Rightarrow$ Every k-cell is compact :

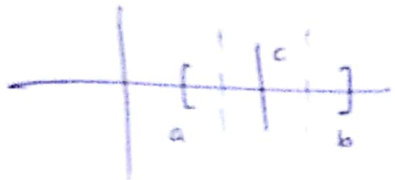
$\{G_\alpha\}$



is compact

$[a,b] = I_1 \cup I_2$

2^k - sub interval



$[a,b] = [a,c] \cup [c,b]$

further dividing.

for k, 2^k sub interval.

Proof :

let $I = \{(x_1, x_2, \dots, x_k) \mid x_i \in \mathbb{R}\}$

$a_i \le x_i \le b_i$

To show that I is compact.

Suppose I is not compact.

so

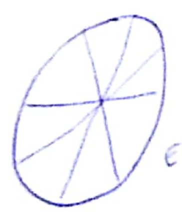
$\exists$ an open covering $\{G_\alpha\}$ of I s.t. I is not covered with finite no. G_i.

i.e $\exists$ no $n \in \mathbb{N}$ s.t. $\boxed{I \subset \bigcup_{i=1}^{n} G_i}$

$E \subset X$

Diameter E

$= \sup\{ d(p,q) \mid p,q \in E \}$



put $\delta = \text{diam } I = \sqrt{\sum_{j=1}^{k} (b_j - a_j)^2}$

any $x, y \in I \Rightarrow |x-y| \leq \delta$.

δ is the diameter of I.

put $c_j = \dfrac{a_j + b_j}{2}$. The intervals $[a_j, c_j]$

and $[c_j, b_j]$ determine 2^k k-cells

Q_i whose union is I.

Atleast one of the sets Q_i, say I_1 cannot be covered by any finite subcollection of $\{G_\alpha\}$.

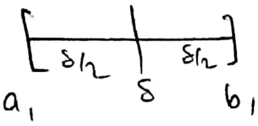
(otherwise I could be so covered).

Next, we subdivide I_1 and continue the process, we obtain a sequence $\{I_n\}$ with following properties.

(a)- $I \supset I_1 \supset I_2 \supset I_3 \ldots \supset I_n \ldots$

(b)- I_n is not covered by any finite subcollection $\{G_\alpha\}$.

(c)- If $x \in I_n$ and $y \in I_n$ then $|x-y| \leq \dfrac{\delta}{2^k}$.



By (a) $\exists$ x^* which lies in every I_n.

$\cap I_n \neq \phi$.

For some α, $x^* \in G_\alpha$. (as $I \subset \bigcup_\alpha G_\alpha$).

As G_α is open., $\exists$ $r > 0$ s.t.

$|y - x^*| < r \Rightarrow y \in G_\alpha$.

Scanned by CamScanner

If n is so large that $2^{-n}\delta < r$ (to show)

$$\frac{\delta}{2^n} < r.$$

d as $\frac{1}{n}$.

Suppose for all n, $2^{-n}\delta \geq r$.

i.e $\delta \geq r \cdot 2^n$.

$\frac{\delta}{r} \geq 2^n$ i.e $\left[2^n \leq \frac{\delta}{r}.\right]$

= which is a contradiction.

(as 2^n is unbounded).

so, $2^{-n}\delta \geq r$ — ✗

so, $\frac{\delta}{2^n} < r$ ✓

Since $\mathbb{R}$ is archimedean.

(c)- If $x \in I_n$ and $y \in I_n$.

$$|x-y| \leq \frac{\delta}{2^k}.$$

$$|x-y| \leq \frac{\delta}{2^n} < r.$$

$\Rightarrow$ $I_n \subset G_\alpha$. which contradicts to (b).

→ Every infinite bounded subset of $\mathbb{R}^k$.

$E \subset \mathbb{R}^k$

infinite bounded

→ Weirstrass Theorem :

Every infinite bounded subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

→ Heine Borel theor. :

Every closed and bounded subset of $\mathbb{R}^k$ is compact.

i.e $E \subset \mathbb{R}^n$ is compact

6/9/18 iff E is closed and bounded.

→ $X = \mathbb{R}^n$

$\dim X = \infty$, this result does not hold in general.

→ Let $E \subset \mathbb{R}^k$. If E has one of the following properties, then it has the other two properties:

(a) - E is closed and bounded.

(b) - E is compact.

(c) - Every infinite subset of E has a limit pt. in E.

Proof :

$(a) \Rightarrow (b)$, E is closed and bounded.

$E \subset I$, I is a k-cell. I is compact.

Closed subset of a compact set is compact.

So, E is compact.

i.e $(a) \Rightarrow (b)$.

Suppose (b) holds, to show (c) also holds,

i.e to show $(b) \Rightarrow (c)$

given E is compact.

Let S be an infinite subset of E.

So, S has a limit point in E.

$(b) \Rightarrow (c)$.

To show (c) $\Rightarrow$ (a)

Proof : Suppose (a) is not true.

i.e. E is not bounded.

then E contains some pts x_n

$$|x_n| > n \qquad (n = 1, 2, \ldots)$$

Set $S = \{x_n\}$ is infinite and has no limit point in $\mathbb{R}^k$, and hence not in E. (as $E \subset \mathbb{R}^k$).

i.e E must be bounded.

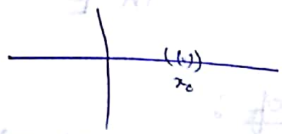
$\rightarrow$ suppose E is not closed.

i.e. there is a point $x_0 \in \mathbb{R}^k$ which is a limit point of E and $x_0 \notin E$

For $n = 1, 2, \ldots$, there are points $x_n \in E$

s.t. $|x_n - x_0| < \frac{1}{n}$

$$r = \frac{1}{n}$$



$n = 1, 2, 3, \ldots$

Let $p = \{x_n\}$, then p is infinite.

(Otherwise $|x_n - x_0| =$ constant for infinitely many n which is not true).

p is an infinite subset of E.

So, p has x_0 as a limit point.

$\rightarrow$ To show that p has no other limit point in $\mathbb{R}^k$.

For $y \in \mathbb{R}^k$, $y \neq x_0$, $d(y, x_0) > 0$

$$|x_n - y| = |x_n - x_0 + x_0 - y|$$

Scanned by CamScanner

$\geq |x_0 - y| - |x_n - x_0|$

$\geq |x_0 - y| - \frac{1}{n}$

$\geq \frac{1}{2}|x_0 - y|$

$|x_n - x_0| < \frac{1}{n}$

$-|x_n - x_0| > -\frac{1}{n}$

$\frac{1}{n} < \epsilon$

$n \in N$

$\lim_{n \to \infty} \frac{1}{n} = 0$

$\epsilon > 0, \exists N$

$\epsilon_0 < \frac{1}{2}|x_0 - y|$

$N_{\epsilon_0}(y)$ will not intersect none of the points of p.

i.e. y is not a limit of p.

So, p has no limit point in E, so

E must be closed.

7/9/18

Theorem (Weirstrass): Every bounded infinite subset of $\mathbb{R}^k$ has a limit pt. in $\mathbb{R}^k$.

Proof:

Let E be an infinite bounded subset of $\mathbb{R}^k$.

So, $E \subset I$, I is a k-cell in $\mathbb{R}^k$.

As E is an infinite subset of the compact set I.

E has a limit pt. in I and hence E has a limit pt. in $\mathbb{R}^k$.

$$E = \left\{\frac{1}{n}\right\}, \quad n \in \mathbb{N}$$

E is infinite.

E is bounded.

0 is a limit pt. in E.

but $0 \notin E$. So, open set.

$\Rightarrow$ **Perfect Set:**

→ Every perfect set in $\mathbb{R}^k$ is uncountable

$$E = [a,b], \quad a<b$$

is perfect

→ $[a,b]$ is uncountable.

Proof:

let P be a perfect set. So, it must be infinite.

Suppose P is countable.

i.e. $P = \{x_1, x_2, x_3, \dots\}$

As x_1 is a limit point of P.

$\exists$ a nbhd V_1 of x_1

s.t. $\boxed{V_1 \cap P \neq \phi}$

Scanned by CamScanner

$V_1 = \{ y \in \mathbb{R}^k \mid |y - x| < r \}$

$\overline{V_1} = \{ y \in \mathbb{R}^k \mid |y - x_1| \leq r \}$

Construct a sequence of $\{V_n\}$ of nhbd, as follows.

$V_1 = \{ y \in \mathbb{R}^k \mid |y - x_1| < r \}$

$\overline{V_1} = \{ y \in \mathbb{R}^k \mid |y - x_1| \leq r \}$

V_n has been constructed, so that $V_n \cap P \neq \phi$

Since, every pt. of P is a limit point, there is a nhbd V_{n+1} s.t.

a). $V_{n+1} \subset V_n$

b). $x_n \notin \overline{V_{n+1}}$

c). $V_{n+1} \cap P \neq \phi$

$\{K_i\}$ 

$\bigcap_{i=1}^{n} K_i$

$\rightarrow$ $K_n = \overline{V_n} \cap P$

$\boxed{\overline{V_n}, V_n \text{ closure}}$

$\overline{V_n}$ is compact (as it is closed and bounded).

$\boxed{\begin{array}{l} K_n \subset \overline{V_n} \\ K_n \text{ is closed and closed subset of the compact set is compact.} \end{array}}$

Since, $x_n \notin K_{n+1}$, no pt. of P lies in $\bigcap_{n=1}^{\infty} K_n$.

Since, $K_n \subset P \Rightarrow \bigcap_{n=1}^{\infty} K_n = \phi$,

$\rightarrow$ P is uncountable.

i.e. every perfect set of $\mathbb{R}^k$ is uncountable.

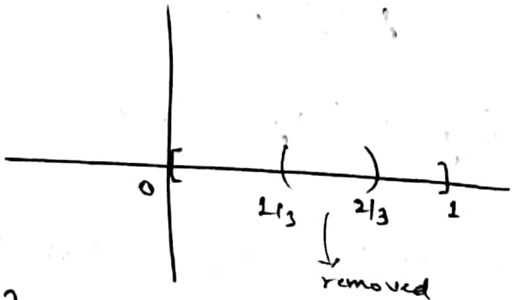
Corollary: Set $k = 1$

$[a,b]$ is a perfect set in $\mathbb{R}$.

$\Rightarrow$ $[a,b]$ is uncountable.

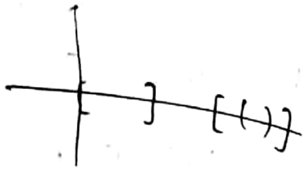
$[0,1]$ is uncountable.

$\longrightarrow$



$F_0 = [0,1]$

$F_1 = [0, 1/3] \cup [2/3, 1]$



$F_2 =$

$F_n =$

$P = \bigcap_{n=1}^{\infty} F_n$.

compact,
(closed & bounded)

$\Rightarrow$ Cantor set:

The cantor set F can be described by removing a sequence of open intervals from the closed unit interval $F_0 = [0,1]$.

If we remove the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ of E_0 to obtain two closed sets

$[0, 1/3]$ and $[2/3, 1]$

let $F_1 = [0, 1/3] \cup [2/3, 1]$

We next remove the open middle third of each of the two closed intervals to obtain,

$F_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$

We see that F_2 is union of $2^2 = 4$ closed intervals, each of which is of the form $\left[\frac{k}{3^2}, \frac{k+1}{3^2}\right]$

$\begin{cases} k = 0, \\ k = 2 \\ k = 6, \\ k = 8 \end{cases}$

We next remove the open middle thirds of the four sets to get F_3, which is union of $2^3 = 8$ sets, we continue in this way to get F_n, which is union of 2^n closed set (intervals) of the form $\left[\frac{k}{3^n}, \frac{k+1}{3^n}\right]$, then we construct F_{n+1} by removing $\frac{1}{3}$ of each of the closed intervals.

$$F_0 \supset F_1 \supset F_2 \supset F_3 \supset \ldots \ldots F_n \supset F_{n+1} \ldots$$

$\left[F = \bigcap_{n=1}^{\infty} F_n \neq \phi, \right]$ is called the Cantor set.

$\downarrow$

F is compact (Heine - Borel Theorem)

(length of $E_0 = 1$)

$\Rightarrow$ To show that the total length of the removed interval is 1.

$\rightarrow$ 1^{st} middle third is of length $1/3$, the next two middle thirds have lengths that add up to $\frac{2}{3^2}$, the next four middle thirds have length add up to $\frac{2^2}{3^3}$

i.e G.P $\rightarrow$ $\frac{1}{3} + \frac{2}{3^2} + \frac{2^2}{3^3} + \ldots + \frac{2^n}{3^{n+1}}$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1$$

→ Total length of the closed intervals that make up F_n is $\left(\frac{2}{3}\right)^n$?

→ $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

Since $F \subseteq F_n$ $\forall n$

=

$$\left[\begin{array}{c} r < 1/n \\ \lim_{n\to\infty} r^n = 0 \end{array} \right]$$

We see that F can be said to have "length" it must have length 0.

$\boxed{\begin{array}{l} I = (-\epsilon, \epsilon) \quad \epsilon > 0 \\ l = 2\epsilon \end{array}}$

(2). The set F contains no nonempty open intervals as subset.

Proof : If F contains a non-empty open intervals

$J : = (a, b)$, then

$J \subset F_n \quad \forall n$

$0 < b - a \leq \left(\frac{2}{3}\right)^n \quad \forall n.$

$\boxed{\begin{array}{l} I_1 \subset I_2 \\ l(I_1) \leq l(I_2) \end{array}}$

→ so, $b - a = 0$

i.e $a = b$

$\boxed{\begin{array}{l} 0 \leq a < \epsilon \\ \text{so, } a = 0 \end{array}}$

← in left

$\Rightarrow\Leftarrow$

$\epsilon = \left(\frac{2}{3}\right)^n$

$\boxed{\begin{array}{l} \text{if } |x| < \epsilon \\ \quad \forall \epsilon > 0 \\ \quad \text{so, } x = 0 \\ \text{using above} \end{array}}$

$\boxed{\begin{array}{l} |a - b| < \epsilon \\ \text{so, } \boxed{a = b} \end{array}}$

Scanned by CamScanner

⇒ F is uncountable, because it is perfect set

As F is closed and it also contains all its limit points

$\boxed{MF = 0}$

(any major 0 set can't be countable).

⇒ **Separated Sets :**

Two sub sets A and B of a metric space X is said to be separated if

$$A \cap \bar{B} = \phi$$

$$\bar{A} \cap B = \phi$$

(Separate set is disjoint. Not, vice versa)

→ A set E is said to be connected if there is no separation.

→ Separated sets are disjoint but disjoint sets need not be separated.

ex:- $A = [0,1]$, $B = (1,2)$

$$\bar{A} \cap B = \phi$$

$$A \cap \bar{B} = \{1\} \neq \phi$$

ex:- $(0,1)$ $(1,2)$

12/9/18

⇒ Connected and Separated :

$A, B \subset X$ (metric)

$A \neq \phi$, $B \neq \phi$ subsets of X.

→ Two sets A and B are said to be separated if

$\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$

Every separated sets are disjoint sets

$[0,1)$, $(1,2)$

$A = [0,1)$

$B = (1,2)$

$\bar{A} \cap B = \phi$

but $A \cap \bar{B} = \{1\} \neq \phi$

→ A set E is said to be connected if E is not a union of two non-empty separated sets.

In general, ϕ, $\mathbb{R}$ are connected sets.

→ $\mathbb{R}$ is connected in general any interval is connected

$E \subset \mathbb{R}$, E is connected iff E is an interval.

$A = (0,1) \cup (2,3)$

not connected (as it is not a interval)

Interval

I , $x, y \in I$

$x < z < y$, $z \in I$.

⇒ Theorem :

A subset E of $\mathbb{R}$ is connected iff E is an interval

i.e, $x \in E$, $y \in E$

and $x < z < y \Rightarrow z \in E$

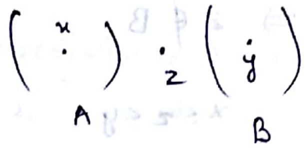
Proof :

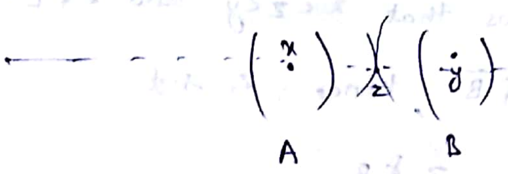
let E be a connected subset of $\mathbb{R}$. To show that E is an interval.

Suppose E is not an interval

i.e $\exists$ $x \in E$, $y \in E$ and some $z \in (x,y)$

s.t. $z \notin E$.





Now $E = A_z \cup B_z$ where $A_z = E \cap (-\infty, z)$, $B_z = E \cap (z, \infty)$

$x \in A_z$ and $y \in B_z$.

A_z and B_z are non-empty

$$A_z \cup B_z = (E \cap (-\infty, z)) \cup (E \cap (z, \infty))$$

$$= E \cap ((-\infty, z) \cup (z, \infty)) \quad \forall\, z \in \mathbb{R}$$

$$= E \cap \underbrace{\{\mathbb{R} - \{z\}\}} \longrightarrow \{\mathbb{R}\}$$

$$= E$$

E is not connected.

⇒ our assumption is wrong.

i.e E is an interval.

Converse

$\Leftarrow$ Given E is an interval, to show that E is connected

→ Suppose E is not connected.

~~As E is an interval $x, y \in E$~~

$E = A \cup B$, $A \neq \phi$, $B \neq \phi$ A and B are separated.

As E is an interval $x, y \in E$, $x < z < y \Rightarrow z \in E$.

pick $x \in A$, $y \in B$ and assume (w.l.g.) that $x < y$. Define $z = \sup(A \cap [x, y])$. $\{z \in \bar{A}\}$

$z \in \bar{A}$ and $z \in [x, y]$.

As $\bar{A} \cap B = \phi \Rightarrow z \notin B$

In particular $x \leq z < y$:

If $z \notin A$, it follows that $x < z < y$ and $z \notin E$.

If $z \in A$, then $z \notin \bar{B}$, hence z_1 s.t.

$z < z_1 < y$ and $z_1 \notin B$.

Then,

$x < z_1 < y$ $z_1 \notin E$.

$\Rightarrow\Leftarrow$

So, assumption is wrong.

So, E is connected.

Note: A is said to be nowhere dense.

if $(\bar{A})^\circ = \phi$

→ Suppose, $A \subset \mathbb{R}$

A is nowhere dense in $\mathbb{R}$ if A does not contain an interval.

* First category metric space.
2nd " " "

13/9/18

⇒ Nowhere dense set:

$A \subset X$ is nowhere dense if

$$(\bar{A})^{\circ} = \phi$$

If $X = \mathbb{R}$, $\bar{A}$ does not contain an interval.

If F is closed and F does not contain any open interval, F is nowhere dense.

$$A = \{1, 1/2, \tfrac{1}{3}, \dots, \tfrac{1}{n}, \dots\}$$
$$\bar{A} = X$$

⇒ A is everywhere dense.

$\mathbb{Q}$ is dense in $\mathbb{R}$.

$$\bar{\mathbb{Q}} = \mathbb{R}$$

A metric space X is said to be separable if it contains a countable dense subsets.

→ $\mathbb{R}^k$ is separable:

$$\mathbb{Q}^k = \{(a_1, a_2, \dots, a_k) \mid a_i \in \mathbb{Q}\}$$

$\mathbb{Q}^k$ is countable.

and $\bar{\mathbb{Q}^k} = \mathbb{R}^k$.

→ Ex: $I^+ = \{1, 2, \dots, n\}$
Cantor sets
are examples of set in $\mathbb{R}$ which are nowhere dense.

→ Give an example to show that the notion being nowhere dense does not imply that the set is everywhere dense.

$A = \{1, 2, \ldots\}$, $\bar{A} = \{1, 2, \ldots\}$

$\bar{A} = A \neq \mathbb{R}$

$\bar{A}$ does not contain any interval, hence (it) set is nowhere dense.

:

→

$\cup G_\alpha$ is open.

$\cap F_j$ is closed.

→

F_σ G_δ

A subset D of $\mathbb{R}$ is said to be of type F_σ if it is the union of countable number of closed sets.

$$D = \bigcup_{j=1}^{\infty} F_j$$

F_σ

A subset D of $\mathbb{R}$ is said to be of type G_δ if it is the intersection of countable collection of open sets. i.e. $D = \bigcap_{j=1}^{\infty} G_j$.

<u>Definition</u>: The subset D of $\mathbb{R}$ is said to be of <u>first category</u> if $D = \bigcup_{n=1}^{\infty} E_n$ where E_n is nowhere dense in $\mathbb{R}$.

If D is not first category, then it is said to be of <u>2nd category</u>.

<u>$\mathbb{Q}$ is 1st category</u>

$$\mathbb{Q} = \bigcup_{j=1}^{\infty} \{a_j\}$$

$\boxed{\text{Set of rational number is of 1st category.}}$

⇒ Theorem: (The Baire category theorem):

The set $\mathbb{R}$ is of 2nd category i.e. we can't write

$$\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$$

with $\overline{F_n}^{\circ} = \phi$

44/16

Q.→ Let $\mathbb{Q}$ be the set of rational numbers and a metric, with $d(p,q) = |p-q|$

$$E = \{ p \in \mathbb{Q} : 2 < p^2 < 3 \}.$$

Show that E is closed and bounded in $\mathbb{Q}$ but not compact. Is E open in $\mathbb{Q}$??

F^c open
⇒ F is closed.
F^c is closed
⇒ F is open.

($\mathbb{Q} \setminus E$ → g.e)

Soln: suppose $x \in \mathbb{Q} \setminus E$

$\Rightarrow 2 \geq x^2$, and $x^2 \geq 3$

As x is rational,

$2 > x^2$ and $x^2 > 3$

If $x = 0$, let $\delta = 1$, otherwise

let $\delta = \min\left(\sqrt{\frac{2-x^2}{3}}, \frac{2-x^2}{3|x|}\right)$

if $y \in (x-\delta, x+\delta)$

we have $y^2 < 2$, is to be if $x = 0$, $\delta = 1$

In other cases, $y = x+h$, $|h| < \delta$.

$$y^2 = (x+h)^2 = x^2 + 2hx + y^2 < x^2 + 2h|x| + y^2$$
$$< x^2 + \frac{2}{3}(2-x^2) + \frac{2-x^2}{3} = 2$$

$E \subset [-2,2]$

E is not compact,

$G_n = \{ p : \mathbb{Q}\ 2 < p^2 < 3 - \frac{1}{n} \}$

is open.