Interpolation

x	0	0.1	0.3	0.5	0.6
y	0	2	2.9	5.2	9.8

x	0	0.1	0.2	0.3	0.4
$y=f(x)$	0	2	2.9	3.2	4.5

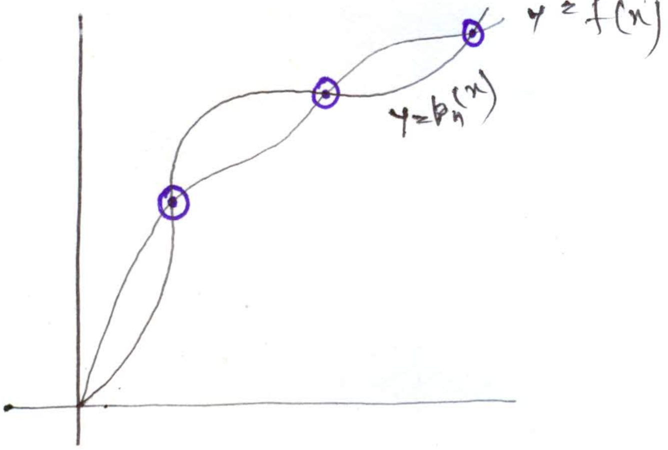
$\longrightarrow$ x Values equidistant

differences $\quad \Delta f(x) = f(x+h) - f(x)$

$\nabla f(x) = f(x) - f(x-h).$

$\Delta y_i = y_{i+1} - y_i, \quad \nabla y_i = y_i - y_{i-1}$

$\left.\begin{array}{l}\text{Newton's forward} \\ \text{Newton's backward}\end{array}\right\}$ interpolating polynomials



x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

We need $f(x_j) = p_n(x_j) \quad j = 0, 1, 2, \ldots, n$

$f(x) \approx p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

$\cancel{f(x) \approx p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}$

$f(x) \approx F_n(x) = \cancel{a_0 + a_1 x + a_2 x^2} \; a_0 \sin x + a_1 \sin 2x + \cdots$

$= a_0 e^x + a_1 e^{2x} + \cdots$

Lagrange's polynomial

Can be used when x- points / nodes are either equidistance or arbitrary

x	x_0	x_1	$\cdots$	x_n
y	y_0	y_1	$\cdots$	y_n

Lagrange polynomial $L_n(x)$

1st step is to form the function $l_i(x)$

$l_i(x)$ are such that —

$$l_i(x_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

$$l_i(x) = c_0 (x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)$$

Now, $l_i(x_i) = 1$

$$\Rightarrow 1 = c_0 (x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)$$

$\therefore$

$$l_i(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}$$

$$L_n(x) = \sum_{i=0}^{n} l_i(x) f(x_i)$$

$$L_n(x_j) = \sum_{\substack{i=0 \\ i \neq j}}^{n} \underbrace{l_i(x_j)}_{=0} f(x_i) + \underbrace{l_j(x_j)}_{=1} f(x_j) = f(x_j)$$

Ex 1 Find the Lagrange polynomial from the table

	x_0	x_1	x_2
x	1	3	4
y	1	27	64
	f_0	f_1	f_2

. Hence find $f(1.6)$

$$L_2(x) = \sum_{i=0}^{2} l_i(x) f(x_i)$$

$$= l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

$$= \frac{(x-3)(x-4)}{(1-3)(1-4)} \times 1 + \frac{(x-1)(x-4)}{(3-1)(3-4)} 27 + \frac{(x-1)(x-3)}{(4-1)(4-3)} 64$$

$$= 8x^2 - 19x + 12.$$

$$f(1.6) \simeq L_2(1.6) = 2.08$$

Newton's divided difference formula

Divided difference of f w.r.to the arguments x_0, x_1

$$f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f(x_1, x_0)$$

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}.$$

$\vdots$

$$f(x_0, x_1, \ldots, x_n) = \frac{f(x_0, x_1, \ldots, x_{n-1}) - f(x_1, x_2, \ldots, x_n)}{x_0 - x_n}.$$

Thm

Divided difference is symmetric w.r.to the arguments.

$$f(x_0, x_1, x_2) = f(x_1, x_0, x_2) = f(x_2, x_1, x_0)$$

Divided difference interpolating polynomial

$$p_n(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + \cdots + (x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}) f(x_0, x_1, \cdots x_n)$$

$$p_n(x) = f(x_n) + (x-x_n) f(x_n, x_{n-1}) + (x-x_n)(x-x_{n-1}) f(x_n, x_{n-1}, x_{n-2}) + \cdots + (x-x_n)(x-x_{n-1})\cdots(x-x_1) f(x_n, x_{n-1}, \ldots, x_0)$$

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f(x_0)}{h}.$$

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2} = \frac{\frac{\Delta f(x_0)}{h} - \frac{\Delta f(x_1)}{h}}{-2h}$$

$$= \frac{\Delta f(x_1) - \Delta f(x_0)}{2h^2} = \frac{\Delta^2 f(x_0)}{2!\, h^2}.$$

$$f(x_0, x_1, x_2, x_3) = \frac{\Delta^3 f(x_0)}{3!\, h^3}.$$

Absolute error

$$|f(x) - p_n(x)| = |E_n(x)| = \frac{w(x) f^{(n+1)}(\xi)}{(n+1)!};$$

$w(x) = (x-x_0)(x-x_1)\cdots(x-x_n)$ ↳ polynomial of degree $n+1$.

$x_0 < \xi < x_n$, if interpolation

$x < \xi < x_0$ or, $x_n < \xi < x$. if extrapolation.

If f is known to you, You can compute only

$$\max_{a<\xi<b} \frac{w(x) f^{(n+1)}(\xi)}{(n+1)!}$$

x	x_0	$\cdots$	x_n	x_{n+1}
y	y_0		y_n	

If f is not known

$$|E_n(x)| = \frac{w(x)\Delta^{n+1} f(x_0)}{h^{n+1}} \text{ (NFI fromula)} = \frac{w(x)\nabla^{n+1} f(x_n)}{h^{n+1}},$$

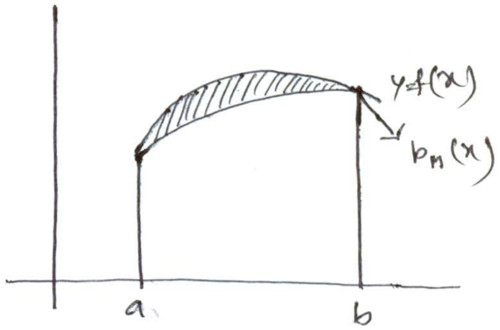
(NBI formula)

Numerical Integration

$\int_a^b f(x)\,dx$, where $f(x)$ is integrable.

$$\int_a^b f(x)\,dx \approx \int_a^b p_n(x)\,dx$$

$p_n(x)$: interpolating polynomial of degree n.



→ Newton-Cotes formula.

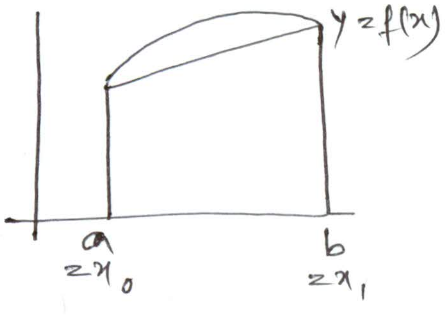
$n=1$, Trapezoidal Rule

$n=2$, Simpson's $\frac{1}{3}$rd Rule

$n=3$, Simpson's $\frac{3}{8}$th Rule.

Simple Trapezoidal rule

$$\int_a^b f(x)\,dx \simeq \int_a^b p_1(x)\,dx$$

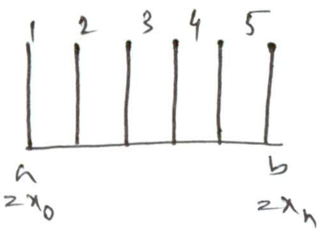


$$L_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$

$$= \frac{x-b}{-(b-a)} f(a) + \frac{x-a}{b-a} f(b)$$

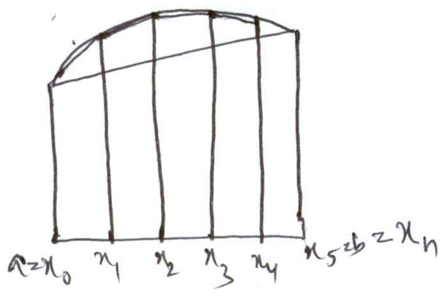
$$\int_a^b L_1(x)\,dx = \int_a^b \frac{(x-a)f(b)-(x-b)f(a)}{(b-a)}\,dx.$$

$$= \frac{f(b)}{b-a}\left.\frac{(x-a)^2}{2}\right|_a^b - \frac{f(a)}{b-a}\left.\frac{(x-b)^2}{2}\right|_a^b$$



$$= \frac{f(b)}{b-a} \times \frac{(b-a)^2}{2} + \frac{f(a)}{b-a}\frac{(b-a)^2}{2}$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \longrightarrow \text{Simple Trapezoidal rule.}$$



Composite Trapezoidal Rule

$$\int_{a=x_0}^{b=x_n} f(x)\,dx \simeq \int_{x_0}^{x_1} L_1^{01}(x)\,dx + \int_{x_1}^{x_2} L_1^{12}(x)\,dx + \int_{x_2}^{x_3} L_1^{23}(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} L_1^{n-1,n}(x)\,dx.$$

$$= \frac{x_1-x_0}{2h}\left\{\frac{f(x_0)+f(x_1)}{2}\right\} + (x_2-x_1)\left\{\frac{f(x_1)+f(x_2)}{2}\right\}$$

$$+ (x_3-x_2)\left\{\frac{f(x_2)+f(x_3)}{2}\right\} + \cdots + (x_{n-1}-x_{n-2})\left\{\frac{f(x_{n-2})+f(x_{n-1})}{2}\right\}$$

$$+ (x_n-x_{n-1})\left\{\frac{f(x_n)+f(x_{n-1})}{2}\right\}$$

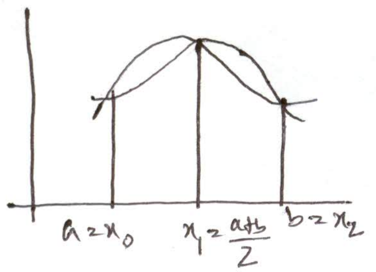
$$= \frac{h}{2}\left[f(x_0) + 2\left\{f(x_1)+f(x_2)+\cdots+f(x_{n-1})\right\} + f(x_n)\right]$$

$\rightarrow$ Composite Trapezoidal Rule.

Simpson's $\frac{1}{3}$rd rule.

$$\int_a^b f(x)\,dx \approx \int_a^b L_2(x)\,dx$$

a x_0	$\frac{a+b}{2}$ x_1	b x_2
y_0	y_1	y_2



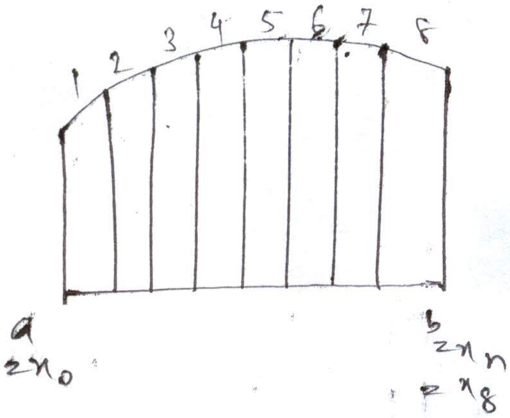
$$I = \int_a^b \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0)\,dx + \int_a^b \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)\,dx$$

$$+ \int_a^b \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)\,dx$$

$$x_1-x_0 = x_2-x_1 = \frac{b-a}{2} = h$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$\boxed{h = \frac{b-a}{2}}$ $\longrightarrow$ Simple simpson's $\frac{1}{3}$rd rule



Divide $(b-a)$ into n equal subintervals each of length h i.e. $b-a = nh$.

Note n must be even here.

$$\int_a^b f(x)\,dx \simeq \int_a^b L_2(x)\,dx \qquad x_i - x_{i-1} = h$$

$$= \left(\int_{a=x_0}^{x_1} + \int_{x_1}^{x_2}\right) L_2^{012}(x)\,dx + \left(\int_{x_2}^{x_3} + \int_{x_3}^{x_4}\right) L_2^{234}(x)\,dx$$

$$+ \cdots + \left(\int_{x_{n-2}}^{x_{n-1}} + \int_{x_{n-1}}^{x_n}\right) L_2^{n-2,n-1,n}(x)\,dx.$$

$$= \frac{h}{3}\left\{f(x_0) + 4f(x_1) + f(x_2)\right\} + \frac{h}{3}\left\{f(x_2) + 4f(x_3) + f(x_4)\right\}$$

$$+ \frac{h}{3}\left\{f(x_4) + 4f(x_5) + f(x_6)\right\} + \cdots$$

$$+ \frac{h}{3}\left\{f(x_{n-4}) + 4f(x_{n-3}) + f(x_{n-2})\right\} + \frac{h}{3}\left\{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right\}$$

$$I^s(f) = \frac{h}{3}\Big[f(x_0) + 4\left\{f(x_1) + f(x_3) + \cdots + f(x_{n-1})\right\} + 2\left\{f(x_2) + f(x_4) + \cdots + f(x_{n-2})\right\} + f(x_n)\Big]$$

Ex 1 Compute $\int_{2.1}^{3.6} y\,dx$ employing appropriate numerical integration formula taking $h=0.3$, using the table

x	2.1	2.4	2.7	3.0	3.3	3.6
y	3.2	2.7	2.9	3.5	4.1	5.2

$b=3.6,\ a=2.1,\ b-a=1.5 \qquad n=\frac{b-a}{h}=\frac{1.5}{.3}=5$

$$I_f^T=\frac{h}{2}\left[f(x_0)+2\{f_1+\cdots+f_4\}+f_5\right]$$

$$=\frac{0.3}{2}\left[3.2+2\{2.7+2.9+3.5+4.1\}+5.2\right]=5.22$$

Ex 2 Employ simpson's $\frac{1}{3}$rd rule to compute $\int_1^2 \frac{dx}{x}$ by taking $h=0.25$

x	1	1.25	1.5	1.75	2
y	1	0.25			0.5

$b-a=1$
$h=0.25$
$n=4$

$$I_f^s=\frac{h}{3}\left[f(1)+2f(1.5)+f(2)+4\{f(1.25)+f(1.75)\}\right]$$

$$=0.6933$$

Note:

Ex 2 can also be solved by Trapezoidal rule, but Ex 1 cannot be solved by Simpson's $\frac{1}{3}$rd rule as it requires even no. of subintervals.

Error

$$|E_n(x)|=\left|\int_a^b f(x)\,dx-\int_a^b L_n(x)\,dx\right|$$

$$|E_n^T(x)|=(b-a)\frac{h^2}{12}f^{4}(\eta) \qquad a<\eta<b$$

$$|E_n^s(x)|=(b-a)\frac{h^4}{180}f^{iv}(\eta)$$

Since $0<h<1$, $\therefore$ error in simpsons rule is less than the Trapezoidal rule.