Method of variation of parameters

Consider the following second order non-homogeneous linear equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = f(x) \tag{1}$$

Let $y = c_1y_1 + c_2y_2$, with c_1 and c_2 as arbitrary constants, be the general solution of the homogeneous equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$$

We assume that

$$y = C_1y_1 + C_2y_2 \tag{2}$$

is the general solution of the non-homogeneous equation (1), where C_1 and C_2 are functions of x to be so chosen that (1) is satisfied.

Differentiating (2) we get

$$y' = C_1y_1' + C_2y_2' + \underbrace{C_1'y_1 + C_2'y_2}_{=0} \tag{3}$$

For simplicity, in order to find C_1 and C_2 we assume that

$$C_1'y_1 + C_2'y_2 = 0 \tag{4}$$

Differentiating (3) again,

$$y'' = C_1y_1'' + C_2y_2'' + C_1'y_1' + C_2'y_2' \tag{5}$$

Substituting y, y' and y'' in (1) we get

$$C_1\left(y_1'' + a_1y_1' + a_2y_1\right) + C_2\left(y_2'' + a_1y_2' + a_2y_2\right) + C_1'y_1' + C_2'y_2' = f(x)$$

$$\implies \quad C_1'y_1' + C_2'y_2' = f(x) \tag{6}$$

Solving the equations (4) and (6):

$$C_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2 f(x)}{W}$$

Here W is called Wronskian. It is non-zero because y_1 and y_2 are linearly independent. Similarly

$$C_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f(x)}{W}$$

After integrating:

$$C_1 = \int -\frac{y_2 f(x)}{W} dx + d_1 \quad \text{and} \quad C_2 = \int \frac{y_1 f(x)}{W} dx + d_2$$

Hence the general solution of the non-homogeneous equation

$$\boxed{y = d_1 y_1 + d_2 y_2 + y_1 \int -\frac{y_2 f(x)}{W} dx + y_2 \int \frac{y_1 f(x)}{W} dx}$$

Example: Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x} \tag{7}$$

Solution:

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

Let $y = C_1 e^x + C_2 e^{-x}$ be the general solution of the given equation.

$$y' = C_1 e^x - C_2 e^{-x} + \underbrace{C_1' e^x + C_2' e^{-x}}_{=0}$$

$$y'' = C_1 e^x + C_2 e^{-x} + C_1' e^x - C_2' e^{-x}$$

Substituting in (7)

$$C_1' e^x - C_2' e^{-x} = \frac{2}{1+e^x}$$

The Wronskian

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Hence

$$C_1 = -\frac{1}{2} \int -e^{-x} \frac{2}{1+e^x} dx + d_1 = \int \frac{e^{-x}}{1+e^x} dx + d_1$$

Substitute $e^x = z \Rightarrow e^x dx = dz$

$$C_1 = \int \frac{1}{z^2(1+z)} dz + d_1 = \int \frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z} dz + d_1$$

$$C_1 = -\frac{1}{z} - \ln z + \ln(1+z) + d_1 = -e^{-x} - x + \ln(1+e^x) + d_1$$

Similarly

$$C_2 = -\frac{1}{2}\int e^x \frac{2}{1+e^x}dx + d_1 = -\ln(1+e^x) + d_2$$

The general solution of the differential equation

$$\boxed{y = d_1 e^x + d_2 e^{-x} - 1 - xe^x + (e^x - e^{-x})\ln(1+e^x)}$$