Th-5/ (Differentiation of Transform)

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}\{F(s)\}$$

& in general,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}.$$

Proof:- $F(s) = \mathcal{L}\{f(t)\}$

$$= \int_0^\infty e^{-st} f(t)\,dt$$

Differentiating under the integral sign w.r.t s, we have

$$F'(s) = \frac{dF}{ds} = \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right)$$

$$= \int_0^\infty (-t)\, e^{-st} f(t)\,dt.$$

(how??)

$$= -\int_0^\infty e^{-st} \underbrace{\{t \cdot f(t)\}}\,dt$$

$$= -\mathcal{L}\{t f(t)\},$$

assuming absolute convergence of the f^n $f(t)$.

Hence, $\mathcal{L}\{t f(t)\} = -F'(s)$

$$= -\frac{d}{ds}\{F(s)\}$$

or, $\mathcal{L}^{-1}\{F'(s)\} = -t f(t).$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$$

$\Rightarrow \int_0^\infty e^{-st} t^n f(t)\, dt$

Diff. w.r.t s.

$$\int_0^\infty -t^{n+1} \cdot e^{-st} f(t)\, dt$$

$$= (-1)^n \frac{d^{n+1}}{ds^{n+1}}\{F(s)\}$$

$$\Rightarrow \mathcal{L}\{t^{n+1} f(t)\} = (-1)^{n+1} \frac{d^{n+1}}{ds^{n+1}}\{F(s)\}$$

proved by induction

Q) Find the L.T of the f^n $t \sin(t)$.

$$\mathcal{L}\{t \sin t\} = -\frac{d}{ds}\left(\frac{1}{1+s^2}\right)$$

$$= \boxed{\frac{2s}{(1+s^2)^3}}$$

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1}$$

Heaviside's Unit step Function

or, Unit step function

is defined as

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Since $H(t)$ is precisely the same as 1 for $t>0$,

$\therefore$ the $\mathcal{L}\{H(t)\}$ must be the same as $\mathcal{L}\{1\} = \frac{1}{s}$ $(s>0)$

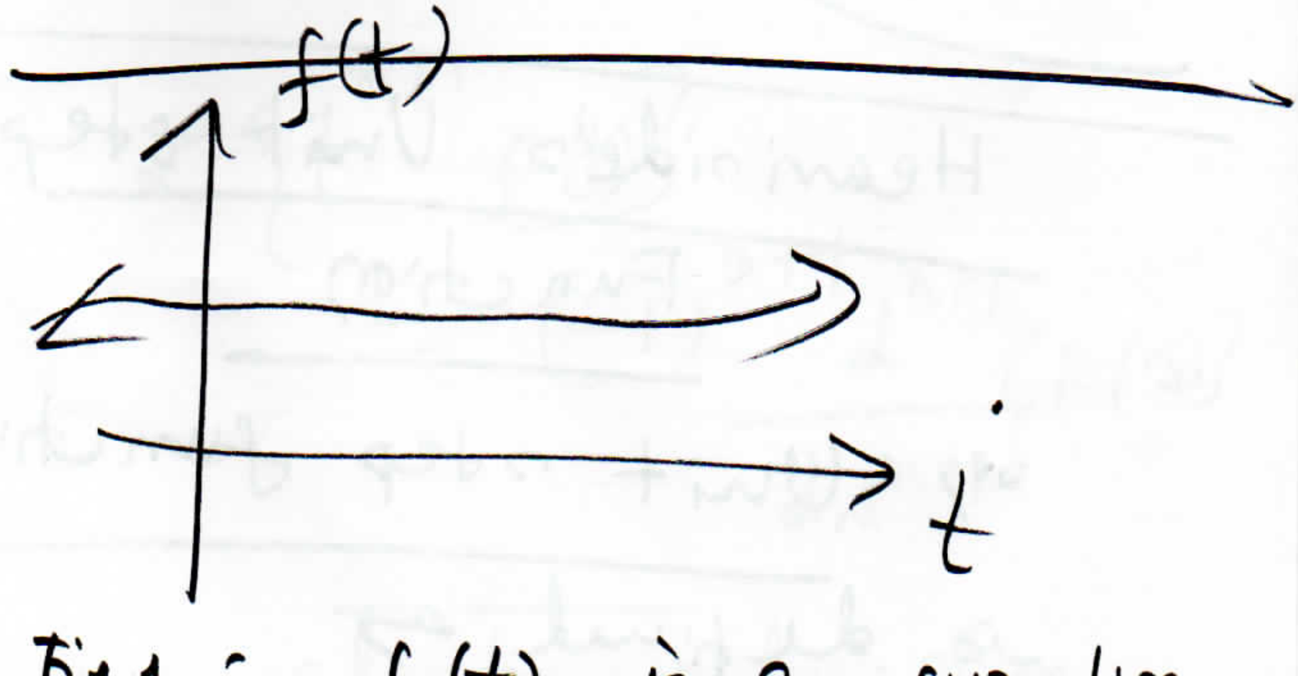

Fig 1 :- $f(t)$ is a function with no well-defined starting value.

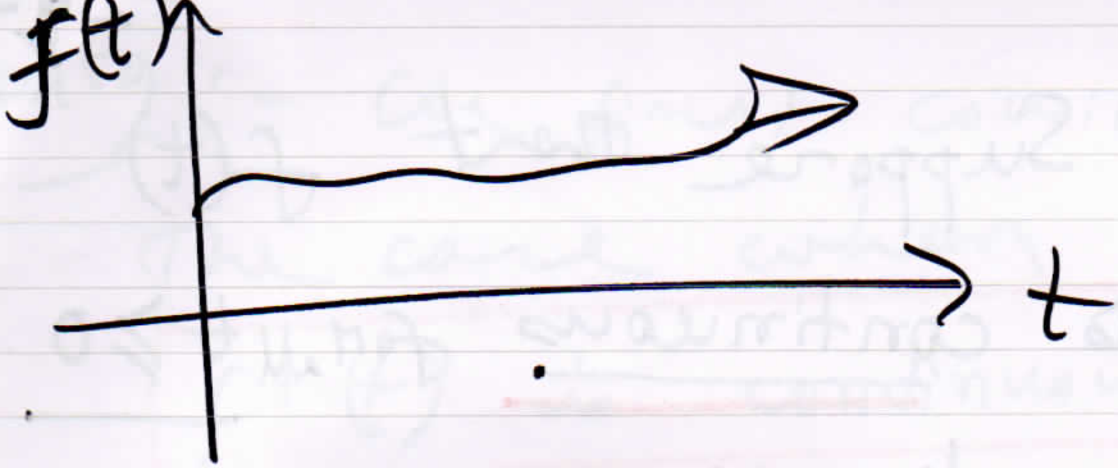

Fig 2 :- $H(t) f(t)$, the f^n is now zero before $t=0$.

$$\int_{-\infty}^{\infty} e^{-st} f(t)\, dt.$$

Th-6 / (L.T of the derivative of $f(t)$)

(or, Derivative property of the L.T)

Th-/ Suppose that $f(t)$ is continuous for all $t \geq 0$ & satisfies

$$|f(t)| \leq M e^{kt},$$

for some k & M & has a derivative $f'(t)$ that is piecewise continuous on every finite interval in the range $t \geq 0$.

Then the Laplace Transform (L.T) of the derivative $f'(t)$ exists when $s > k$

& $\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)$, $\rightarrow ①$ $(s > k)$

proof :- We first consider the case when $\boxed{f'(t) \text{ is continuous}}$ for all $t \geq 0$.

Then by the defⁿ & by integrating by parts,

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} \cdot f'(t)\, dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\underbrace{\int_0^\infty e^{-st} f(t)\, dt}_{\mathcal{L}\{f(t)\}}$$

$$= \boxed{0} - f(0) + s\,\mathcal{L}\{f(t)\}$$

$$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0)$$

Since f satisfies the cond^n $|f(t)| \le Me^{-kt}$

the integrated portion on the right is zero at the upper limit when $s > k$.

& at the lower limit, it contributes $-f(0)$. The last integral is $\mathcal{L}(f(t))$ exists when $s > k$ & eq^n ① holds

If the derivative $f'(t)$ is merely

piece-wise continuous

the proof is quite similar (EX)

In this case the range of integration must be broken up into parts such that $f'(t)$ is continuous in each such part.

Note :- This theorem may be extended to piece-wise continuous function $f(t)$ but in place of ① we then obtain the formula

(1*)

$$\mathcal{L}(f') = s\,\mathcal{L}(f) - f(0) - \left[f(a+0) - f(a-0)\right] e^{-as}$$

where $f(t)$ is continuous except for an ordinary discontinuity (finite jump) at $t = a$ (>0)

By applying (1) to the second derivative $f''(t)$ we get

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}(f') - f'(0)$$

$$\mathcal{L}(f'') = s\left[s\,\mathcal{L}(f) - f(0)\right] - f'(0)$$

$$\mathcal{L}(f'') = s^2\,\mathcal{L}(f) - s\,f(0) - f'(0)$$

s'lly, $\mathcal{L}(f''') = s^3\,\mathcal{L}(f) - s^2 f(0) - s f'(0) - f''(0)$

etc.

$$\boxed{\mathcal{L}(f^n) = s^n\,\mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}$$