SINGULARITIES:

A point $z = z_0$ at which the function is not defined or the function is not analytic is called a singular point of $f(z)$.

The singular point $z = z_0$ is called isolated singular point of $f(z)$ if $z = z_0$ has a neighbourhood without further singular points.

If no such a neighbourhood exists, then the singular point $z = z_0$ is called a non-isolated singular point.

Example: (i) $f(z) = \tan z$

isolated singularities at $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

(ii) $f(z) = \tan\left(\frac{1}{z}\right) = \dfrac{\sin\left(\frac{1}{z}\right)}{\cos\left(\frac{1}{z}\right)}$

Singular point $\cos\left(\frac{1}{z}\right) = 0 \Rightarrow \frac{1}{z} = (2n+1)\frac{\pi}{2}$

or $z = \dfrac{2}{(2n+1)\pi}, n = 0, \pm 1, \ldots$

These points are isolated singular points.

Note that the function $f(z)$ is not defined at $z = 0$, therefore $z = 0$ is also a singular point of $f(z)$. Further, $\lim_{n \to \infty} \dfrac{2}{(2n+1)\pi} = 0$. Therefore every nbd. of $z = 0$ contains many singular points. Thus $z = 0$ is non-isolated s.p.

In the following discussion all singularities are assumed to be isolated.

Isolated singularities of $f(z)$ at $z = z_0$ can be further classified:

REMOVABLE SINGULARITY:

If a single valued function $f(z)$ is not defined at $z = z_0$ but $\lim_{z \to z_0} f(z)$ exists, then $z = z_0$ is called a removable singularity.

In this case we defined $f(z)$ at $z = z_0$ as equal to $\lim_{z \to z_0} f(z)$ and $f(z)$ will then be analytic at a.

- In case of removable singularity, principal part will not appear in the Laurent series.

POLE: If the principal part of the Laurent series has only finitely many terms, i.e.

$$P.P. = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

Then the singularity of $f(z)$ at $z = z_0$ is called a Pole and m is called its order. Poles of order 1 are called simple poles.

- If z_0 is a singular point and we can find a positive integer m such that

$$\lim_{z \to z_0} (z-z_0)^m f(z) = A \neq 0$$

then $z = z_0$ is called a pole of order m.

ISOLATED ESSENTIAL SINGULARITY

If the principle part of the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

has infinitely many terms, we say that $f(z)$ has at $z = z_0$ an isolated essential singularity.

- A isolated singularity that is not a pole, or removable singularity is called an essential singularity.

CLASSIFICATION OF ISOLATED SINGULARITIES of $f(z)$ at $z = z_0$

Using Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \qquad -①$$

valid in $0 < |z-z_0| < R$

If principal part of ① has

i) no term (removable singularity)

ii) a finite number of terms (Pole)

iii) an infinite number of terms (essential sing.)

Using limits:

i) Removable: if $\lim_{z \to z_0} f(z)$ exists finitely.

ii) Pole of order m if

$$\lim_{z \to z_0} (z-z_0)^m f(z) = A \neq 0$$

iii) An isolated essential singularity that is not a pole, or removable singularity.

Example: $f(z) = \frac{\sin z}{z}$, then $z=0$ is a removable singularity since $f(0)$ is not defined but $\lim_{z \to 0} \frac{\sin z}{z} = 1$.

Example: The function

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z = 0$ and a bole of order 5 at $z = 2$.

Example: The function $e^{1/z}$ has an essential singularity at $z=0$ as

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{\underline{|2}} \cdot \frac{1}{z^2} + \frac{1}{\underline{|3}} \cdot \frac{1}{z^3} + \cdots$$

Q. Describe the singularities of the function

$$f(z) = \tan z = \frac{\sin z}{\cos z}$$

Sol: Singularities : $z = \frac{\pi}{2} + m\pi$ for $m = 0, \pm 1, \pm 2, \ldots$

i) $\lim_{z \to \frac{\pi}{2} + m\pi} f(z)$ does not exist finitely.

Hence there are no removable singularity.

ii) $$\lim_{z \to \frac{\pi}{2} + m\pi} \left[z - \left(\frac{\pi}{2} + m\pi\right)\right] \frac{\sin z}{\cos z}$$

$$= \lim_{z \to \frac{\pi}{2} + m\pi} \frac{z - \left(\frac{\pi}{2} + m\pi\right)}{\frac{\cos z}{\sin z}} \qquad \frac{0}{0} \text{ form}$$

Using L'Hospital rule

$$= \lim_{z \to \frac{\pi}{2} + m\pi} \frac{1}{\frac{-\sin z \sin z - \cos z \cos z}{\sin^2 z}}$$

$$= -1$$

Hence $f(z) = \tan z$ has a simple pole at $z = \frac{\pi}{2} + m\pi$.

Q: Classify the singular points of the function

$$f(z) = \frac{z}{(z^2+4)^2}$$

METHOD - I

Sol: Write

$$f(z) = \frac{z}{[z^2-(2i)^2]^2} = \frac{z}{(z+2i)^2(z-2i)^2}$$

We write the Laurent series for $z = 2i$ [powers of $(z-2i)$]

$$f(z) = \frac{1}{8i}\left[\frac{1}{(z-2i)^2} - \frac{1}{(z+2i)^2}\right]$$

$$= \frac{1}{8i}\left[\frac{1}{(z-2i)^2} - \frac{1}{(z-2i+4i)^2}\right]$$

$$= \frac{1}{8i}\left[\frac{1}{(z-2i)^2} + \frac{1}{16}\left\{1 + \frac{z-2i}{4i}\right\}^{-2}\right]$$

$$= \frac{1}{8i}\left[\frac{1}{(z-2i)^2} + \frac{1}{16}\left\{1 - \frac{z-2i}{2i} + \text{higher powers of } (z-2i)\right\}\right]$$

Hence $z = 2i$ is a pole of order 2.

Similarly $z = -2i$ is a pole of order 2.

METHOD 2:

$$\lim_{z\to 2i} (z-2i)^2 f(z) = \lim_{z\to 2i} \frac{z}{(z+2i)^2}$$

$$= \frac{2i}{16i^2} = \frac{1}{8i} \neq 0.$$

$\Rightarrow$ $z = 2i$ is a pole of order 2.

Similarly for $z = -2i$.