AKS

Indian Institute of Technology, Kharagpur

Date..................... FN/AN Time: 3 Hrs Full Marks: 50 No. of Students: 61
End (Spring) Semester 2016-17 Subject Name: Discrete Mathematics Deptt: MA/CS
(MA20013)

Instruction: Answer **all** questions. Notations used are as explained in the class.

Question 1 [$10 \times 2 = 20$ marks]

a) Prove or disprove: The sets A and B are equipotent where

$$A = \{x \in R \mid 0 \leq x \leq 1\}, \qquad B = \{x \in R \mid a \leq x \leq b\}.$$

b) Show that strong induction is a valid method of proof by showing that it follows from the well-ordering property.

c) Generalize the following patterns and show correctness by using induction.

$$\begin{aligned} 1 \cdot 2 \cdot 3 \cdot 4 &= 5^2 - 1, \\ 2 \cdot 3 \cdot 4 \cdot 5 &= 11^2 - 1, \\ 3 \cdot 4 \cdot 5 \cdot 6 &= 19^2 - 1, \\ 4 \cdot 5 \cdot 6 \cdot 7 &= 29^2 - 1, \\ &\vdots \end{aligned}$$

d) Use structural induction to show that $l(T)$, the number of leaves of a full binary tree T, is 1 more than $i(T)$, the number of internal vertices of T.

e) Solve the recurrence relation

$$a_n = 10a_{n-1} - 25a_{n-2} + 5^{n+1}, n \geq 2$$

subject to the initial values $a_0 = 5, a_1 = 15$.

f) From the Binet formula for Fibinacci numbers, derive the relation

$$f_{2n+2}f_{2n-1} - f_{2n}f_{2n+1} = 1, n \geq 1,$$

where f_n denotes the n-th Fibonacci number.

g) The "second order" Fibonacci sequence is defined by the rule

$$\mathcal{F}_0 = 0, \mathcal{F}_1 = 1, \mathcal{F}_{n+2} = \mathcal{F}_{n+1} + \mathcal{F}_n + f_n,$$

where f_n denotes the n-th Fibonacci number. Express $\mathcal{F}_n$ in terms of f_n and f_{n+1}.

h) Determine the number of ways to color the squares of a 1-by-n chessboard using the colors, red, white and blue, if an even number of squares is colored red; using generating function.

i) Let $(S, \circ)$ be a semi group. If for $x, y \in S$, $x^2 \circ y = y = y \circ x^2$, prove that $(S, \circ)$ is an abelian group.

j) Find all elements of order 8 in the group $(Z_{24}, +)$.

——P.T.O.——

Question 2 [$1+1+3=5$ marks]

Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ be the n-th Harmonic number.

a) Prove that $H_{2^m} \geq 1 + \frac{m}{2}$

b) Prove that $\sum_{k=1}^{n} H_k = (n+1)H_n - n$.

c) Find the generating function for $\{\sum_{0<k<n} \frac{1}{k(n-k)}\}$; differentiate it and express the coefficient in terms of harmonic numbers.

Question 3 [$1+2+2=5$ marks]

a) Define Stirling number of the second kind $S(n,k)$.

b) Show by a combinatorial argument that $S(n,k) = kS(n-1,k) + S(n-1,k-1)$.

c) Prove that $S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n$.

Question 4 [$3+4+3=10$ marks]

a) Prove that the number of different ways to compute the product of $n+1$ matrices is given by: $\binom{1/2}{n+1}(-1)^n\, 2^{2n+1}$.

b) State and prove the principle of inclusion and exclusion.

c) Let f be an increasing function that satisfies the recurrence relation $f(n) = af\left(\frac{n}{b}\right) + c$ whenever n is divisible by b, where $a > 1$, b is an integer greater than 1 and c is a positive real number. Prove that $f(n) = O(n^{\log_b a})$.

Question 5 [$2+3+3+2=10$ marks]

a) The following table defines a binary composition $*$ on the set $S = \{a, b, c\}$.

$*$	a	b	c
a	a	b	c
b	b	a	c
c	c	a	b

Examine if $(S, *)$ is a group.

b) Let $(G, \circ)$ be an even order group. Prove that G contains an odd number of elements of order 2.

c) Let E be the modular elliptic curve defined by $y^2 = x^3 + 2x + 1 \bmod 11$.

(i) Find all points on E (including the point at infinity).

(ii) Compute $\mathsf{ord}_E((3,1))$.

d) Construct the field $\mathsf{GF}(3^2)$ using the primitive polynomial $x^2 + x + 2$.

——-The End———