DIFFERENCE EQUATIONS

A difference equation of order k is given as

$$F(u_n, u_{n+1}, \cdots u_{n+k}) = 0 \quad \text{—(1)}$$

(a relation among $u_n, u_{n+1}, \cdots u_{n+k}$)

Order = largest index − smallest index

$= n+k-n = k$

Ex: $u_{n+2} + u_{n+1} + 3u_n = 0$

order $= n+2-n = 2$

If F in (1) is linear then the difference equation is called linear, otherwise non-linear.

A general linear difference equation of order k can be written as

$$a_0 u_{n+k} + a_1 u_{n+k-1} + \cdots\cdots + a_k u_n = g_n \quad a_0 \neq 0 \quad \text{—(2)}$$

if $a_0, a_1, \cdots a_k$ are constants then the difference equation is called a linear difference equation with constant coefficients.

If $g_n = 0$ homogeneous

$g_n \neq 0$ non-homogeneous (inhomogeneous)

The general solution of (2) is of the form

$$u_n = u_n^{(H)} + u_n^{(P)}$$

where $u_n^{(H)}$ is the solution of the associated homogeneous difference equation

$$a_0 u_{n+k} + a_1 u_{n+k-1} + \cdots + a_k u_n = 0$$

and $u_n^{(P)}$ is any particular solution of (2).

For solving homogeneous equation we assume

$$u_n = A\xi^n \quad \text{where } A \neq 0 \text{ is constant.}$$

Substituting into the difference equation

$$A\left[a_0 \xi^k + a_1 \xi^{k-1} + \cdots + a_k\right]\xi^n = 0$$

$$\Rightarrow a_0 \xi^k + a_1 \xi^{k-1} + \cdots + a_k = 0 \qquad \text{--- (3)}$$

The equation (3) is called a characteristic equation.

Let $\xi_1, \xi_2, \cdots \xi_k$ be the roots of (3), then we have the following cases:

(I) Real and Distinct roots:

$$u_n^{(H)} = c_1\xi_1^n + c_2\xi_2^n + \cdots + c_k\xi_k^n.$$

(II) Real and Repeated roots:

Let $\xi_1 (= \xi_2)$ be a double root and $\xi_3, \xi_4, \cdots, \xi_k$ are distinct. Then

$$u_n^{(H)} = (c_1 + nc_2)\xi_1^n + c_3\xi_3^n + \cdots + c_k\xi_k^n.$$

(III) Complex roots:

The complex roots occur as conjugate pair.

Let $\xi_1 = \alpha + i\beta = re^{i\theta}$ and $\xi_2 = \alpha - i\beta = re^{-i\theta}$

where $r = \sqrt{\alpha^2+\beta^2}$ $\theta = \tan^{-1}(\beta/\alpha)$.

Then
$$u_n^{(H)} = [c_1 \cos(n\theta) + c_2 \sin(n\theta)]|\xi_1|^n + c_3 \xi_3^n + \cdots$$
$$\cdots + c_k \xi_k^n .$$

The particular solution depends on the form of g_n.

If $g_n = g$ (a constant)

then
$$u_n^{(p)} = \left(\frac{g}{a_0 + a_1 + \cdots + a_k}\right)$$

Some usefull observation:

- Suppose we require $u_n^h \to 0$ as $n \to \infty$, then the necessary and sufficient condition is:
$$|\xi_i| < 1$$

- Suppose we require u_n^h to be BOUNDED as $n \to \infty$. the the necessary and sufficient condition is:

ξ_i lie inside the unit circle in the complex plane and are simple if they lie on the unit circle.

[Root condition].

Routh-Hurwitz Criterion :

It is not always possible to find roots of characteristic equation to check $|\xi_i| < 1$, specially when the degree of the characteristic equation is high.

This can be done without calculating roots of the characteristic equation explicitely using Routh-Hurwitz Criterion.

a) consider the following mapping :

$$\xi = \frac{1+z}{1-z} \qquad \text{OR} \qquad z = \frac{\xi - 1}{\xi + 1}$$

which maps the interior of the unit circle $|\xi| = 1$ on to the left half plane $Re(z) < 0$, and the unit circle $|\xi| = 1$ onto the imaginary axis.

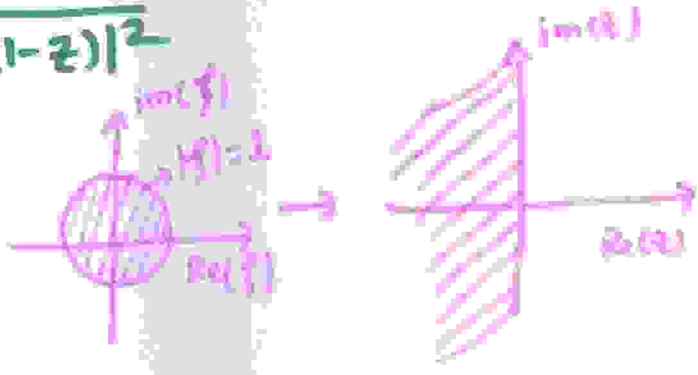
Consider

$$\xi\bar{\xi} - 1 = \frac{1+z}{1-z}\,\frac{1+\bar{z}}{1-\bar{z}} - 1$$

$$= \frac{\cancel{1} + \bar{z} + z + \cancel{z\bar{z}} - (\cancel{1} - \bar{z} - z + \cancel{z\bar{z}})}{(1-z)(1-\bar{z})}$$

$$|\xi|^2 - 1 = \frac{2(z+\bar{z})}{|(1-z)|^2}$$

$$\Rightarrow \boxed{|\xi|^2 - 1 = \frac{4\,Re(z)}{|(1-z)|^2}}$$



- $|\xi| = 1 \Rightarrow Re(z) = 0$
- $|\xi| < 1 \Rightarrow Re(z) < 0$.

b) Routh-Hurwitz Criterion: substituting $\xi = \frac{1+z}{1-z}$ into

$$a_0 \xi^k + a_1 \xi^{k-1} + \cdots + a_k = 0, \text{ we get}$$

$$b_0 z^k + b_1 z^{k-1} + \cdots + b_k = 0 \quad \text{———} \quad (*)$$

This is called transformed characteristic equation. Let $b_0 > 0$.

Denote:

$$D = \begin{bmatrix} b_1 & b_3 & b_5 & \cdots & b_{2k-1} \\ b_0 & b_2 & b_4 & \cdots & b_{2k-2} \\ 0 & b_1 & b_3 & \cdots & b_{2k-3} \\ 0 & b_0 & b_2 & \cdots & b_{2k-4} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & b_k \end{bmatrix}, \quad b_j = 0 \quad j > k$$

Routh-Hurwitz criterion states that the real part of the roots of (*) are negative if and only if the principal minors of D are positive, i.e.,

necessary condition for roots of (*) (real parts) to be negative is that all the coefficients b_i must be of same sign. Routh Hurwitz provides necessary and suff. condition.

K = 1: $b_0 > 0, \quad \underbrace{|b_1|}_{\det(b_1)} > 0 \Rightarrow b_0 > 0, \; b_1 > 0$

K = 2: $b_0 > 0; \quad b_1 > 0; \quad \begin{vmatrix} b_1 & 0 \\ b_0 & b_2 \end{vmatrix} = b_1 b_2 > 0$

$\Rightarrow b_0 > 0; \quad b_1 > 0; \quad b_2 > 0.$ (assuming necessary condition)

K = 3: $b_0 > 0; \quad b_1 > 0; \quad \underbrace{\begin{vmatrix} b_1 & b_3 \\ b_0 & b_2 \end{vmatrix} > 0}_{(b_1 b_2 - b_0 b_3) > 0}; \quad \underbrace{\begin{vmatrix} b_1 & b_3 & 0 \\ b_0 & b_2 & 0 \\ 0 & b_1 & b_3 \end{vmatrix} > 0}_{b_3 (b_1 b_2 - b_0 b_3) > 0}$

$\Rightarrow$

$$b_0 > 0; \quad b_1 > 0; \quad \underbrace{(b_1 b_2 - b_0 b_3)}_{\Rightarrow b_2 > 0} > 0; \quad b_3 > 0$$

$\Rightarrow b_0 > 0, \; b_1 > 0, \; b_2 > 0, \; b_3 > 0, \; (b_1 b_2 - b_0 b_3) > 0.$

K = 4: Similarly:

$$b_i > 0 \; (i = 0, \ldots, 4); \; (b_1 b_2 - b_0 b_3) > 0;$$

$$(b_1 b_2 b_3 - b_1^2 b_4 - b_0 b_3^2) > 0.$$