Ex. Discuss the local extrema of the function

$$f(x,y) = (4x^2+y^2)\, e^{-x^2-4y^2}$$

Sol:

$$f_x(x,y) = e^{-x^2-4y^2}\left[8x - 2x(4x^2+y^2)\right]$$

$$= e^{-x^2-4y^2}\left[8x - 8x^3 - 2xy^2\right]$$

$$= e^{-x^2-4y^2}(2x)\left[4 - 4x^2 - y^2\right]$$

$$f_y(x,y) = e^{-x^2-4y^2}\left[2y - 8y(4x^2+y^2)\right]$$

$$= e^{-x^2-4y^2}(2y)\left[1 - 16x^2 - 4y^2\right]$$

CRITICAL POINTS: $f_x = 0$ & $f_y = 0$

i) $x=0, \quad y=0$

ii) $x=0, \quad 1-4y^2=0 \Rightarrow y = \pm\frac{1}{2}$

$\Rightarrow (0,\frac{1}{2})$ & $(0,-\frac{1}{2})$

iii) Let $x \neq 0, \quad y=0$

$\Rightarrow 4-4x^2 = 0 \Rightarrow x = \pm 1.$

$(1,0) \quad (-1,0)$

iv) $x\neq 0, \ y\neq 0 \quad \Rightarrow \quad \left.\begin{array}{l} 4x^2+y^2 = 4 \\ \& \ 4x^2+y^2 = \frac{1}{4} \end{array}\right\}$ NO SOLUTION

Hence the critical points are:

$P_1 = (0,0)$, $P_2 = (0,\frac{1}{2})$ $P_3 = (0,-\frac{1}{2})$ $P_4 = (1,0)$ $P_5 = (-1,0)$

Second order derivatives:

$$r = f_{xx} = e^{-x^2-4y^2}\left[8 - 24x^2 - 2y^2 + (8x - 8x^3 - 2xy^2)(-2x)\right]$$

$$= 2e^{-x^2-4y^2}\left[4 - 20x^2 + 8x^4 - y^2 + 2x^2y^2\right]$$

$$t = f_{yy} = e^{-x^2-4y^2}\left[2 - 32x^2 - 24y^2 + (2y - 32x^2y - 8y^3)(-8y)\right]$$

$$= 2e^{-x^2-4y^2}\left[1 - 20y^2 - 16x^2 - 128x^2y^2 + 32y^4\right]$$

$$s = f_{xy} = e^{-x^2-4y^2}\left[-4xy + (8x - 8x^3 - 2xy^2)(-8y)\right]$$

$$= 4xy\,e^{-x^2-4y^2}\left[-17 + 16x^2 + 4y^2\right]$$

<u>Identification:</u>

<u>$P_1(0,0)$:</u> $r = 8$ $s = 0$ $t = 2$

$rt - s^2 = 16 > 0$ & $r > 0$

$\Rightarrow$ The point P_1 is a local minima.

<u>$P_2(0,\frac{1}{2})$ & $P_3(0,-\frac{1}{2})$:</u>

$$r = 2e^{-1}\left[4 - \tfrac{1}{4}\right] = \frac{15}{2e}$$

$$s = 0$$

$$t = 2e^{-1}\left[1 - 5 + 2\right] = -\frac{4}{e}$$

$$rt - s^2 = -\frac{30}{e^2} < 0$$

$\Rightarrow$ P_2 & P_3 are saddle points.

$P_4(1,0)$ & $P_5(-1,0)$

$$r = 2e^{-1}[4-20+8] = -16e^{-1}$$

$$s = 0$$

$$t = 2e^{-1}[1-16] = -30e^{-1}$$

$$rt - s^2 = \frac{480}{e^2} > 0, \quad r < 0$$

Hence P_4 & P_5 are the point of local maximum.

EXAMPLE: $f(x,y) = y^2 + x^2y + x^4$.

Stationary points: $f_x = 0$ & $f_y = 0$

$$\Rightarrow 2xy + 4x^3 = 0 \quad \& \quad 2y + x^2 = 0$$

$$\Rightarrow x = 0 \ \& \ y = 0.$$

$$r = f_{xx}|_{(0,0)} = (2y + 12x^2)|_{(0,0)} = 0$$

$$s = f_{xy}|_{(0,0)} = 2x|_{(0,0)} = 0$$

$$t = f_{yy}|_{(0,0)} = 2|_{(0,0)} = 2.$$

$rt - s^2 = 0$ further investigation is required.

$$\Delta f = f(0+h, 0+k) - f(0,0) = k^2 + h^2k + h^4$$

$$= \left(\frac{k}{2}\right)^2 + h^2k + h^4 + \frac{3}{4}k^2$$

$$= \left(\frac{k}{2} + h^2\right)^2 + \frac{3}{4}k^2 > 0 \quad \forall \ h \neq 0, \ k \neq 0$$

$\Rightarrow (0,0)$ is a point of LOCAL MINIMUM.

Ex. Find local minima/maxima of the function

$$f(x,y) = 2x^4 - 3x^2y + y^2$$

Sol.

$$f_x = 8x^3 - 6xy$$

$$f_y = -3x^2 + 2y$$

Stationary points: $8x^3 - 6xy = 0$ & $-3x^2 + 2y = 0$

$$\Rightarrow 8x^3 - 3x(3x^2) = 0 \Rightarrow x = 0.$$

$$\Rightarrow y = 0$$

Stationary point $(0,0)$.

$$r = f_{xx}\big|_{(0,0)} = (24x^2 - 6y)\big|_{(0,0)} = 0$$

$$s = f_{xy}\big|_{(0,0)} = -6x\big|_{(0,0)} = 0$$

$$t = f_{yy}\big|_{(0,0)} = 2$$

$rt - s^2 = 0$ test fails,

$$\Delta f = f(h,k) - f(0,0)$$

$$= 2h^4 - 3h^2k + k^2$$

$$= 2h^4 - 2h^2k - h^2k + k^2$$

$$= 2h^2(h^2 - k) - k(h^2 - k)$$

$$= (2h^2 - k)(h^2 - k)$$

$\left.\begin{array}{ll} \underline{\text{For } k<0}: & \Delta f > 0 \\ \underline{\text{For } h^2<k<2h^2}: & \Delta f < 0 \end{array}\right\}$ sign changes

$\Rightarrow (0,0)$ is a saddle point.

Ex. The function $f(x,y) = (y-x^2)^2 + x^5$ has a stationary point at the origin. Characterize the function at the point $(0,0)$.

Sol: $f_x = 2(y-x^2)(-2x) + 5x^4 \Rightarrow f_{xx} = -4[(y-x^2) + x(-2x)] + 20x^3$

$r = f_{xx}|_{(0,0)} = 0$ $\qquad f_{xy} = -4x$.

$f_y = 2(y-x^2) \Rightarrow f_{yy} = 2$

$s = f_{xy}|_{(0,0)} = 0$

$t = 2$.

$rt - s^2 = 0$ test fails!

However, we can readily see that the function has no extreme value there, as the function assumes both positive and negative values in the neighbourhood of the origin.

Ex. Find and characterize the extreme values of the function

$$f(x,y) = (x-y)^4 + (y-1)^4.$$

Sol. $f_x = 4(x-y)^3 \qquad f_{xx} = 12(x-y)^2 \qquad f_{xy} = -24(x-y)$

$f_y = -4(x-y)^3 + 4(y-1)^3 \qquad f_{yy} = +12(x-y)^2 + 12(y-1)$

Critical points: $(x-y)^3 = 0$ & $-(x-y)^3 + (y-1)^3 = 0$

$\Rightarrow x = 1,\ y = 1.$

$r = f_{xx}|_{(1,1)} = 0 \qquad s = f_{xy}|_{(1,1)} = 0 \qquad t = f_{yy}|_{(1,1)} = 0$

Criterion fails!

However, if we consider: $f(1+h, 1+k) - f(1,1)$

$= (1+h-1-k)^4 + (1+k-1)^4$

$= (h-k)^4 + k^4 > 0 \quad \forall\ h,k \neq 0$

$\Rightarrow f$ has a minimum at the point $x=1,\ y=1$.

LAGRANGE'S METHOD OF UNDETERMINED COEFFICIENTS

Method of Lagrange Multipliers

Find the maxima/minima of the function

$$u = f(x, y) \quad \text{— (1)}$$

with the following constraint

$$\varphi(x, y) = 0 \quad \text{— (2)}$$

From equation (1), we have using chain rule of composite function

$$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

(we can write because x & y are related from relation (2))

At the point of extremum

$$\frac{du}{dx} = 0$$

(one variable problem)

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \text{— (3)}$$

Also, equation (2) satisfies at any point; so at the point of extremum

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \cdot \frac{dy}{dx} = 0 \quad \text{— (4)}$$

(Differentiation of implicit function)

In order to avoid calculation of $\frac{dy}{dx}$, aim is to eliminate $\frac{dy}{dx}$ from (3) and (4). We assume that an extremum point the two partial derivatives φ_x & φ_y do not both vanish. Assuming $\varphi_y \neq 0$, and multiplying (4) by $\lambda = -f_y/\varphi_y$ and add it to equation (3), we get

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0$$

By the definition of λ, the equation

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \quad \text{holds}$$

Hence, at the extremum point, three equations are satisfied;

$$\left.\begin{aligned} \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} &= 0 \\ \varphi(x,y) &= 0 \end{aligned}\right\} \quad (5)$$

Out of these three equations, we determine x, y & λ.

LAGRANGE'S RULE:

We can write the system (5) using an auxiliary function of the form

$$F(x,y,\lambda) = f(x,y) + \lambda\varphi(x,y)$$

and now writting the necessary condition of an extreme value as

$$F_x = 0 \Rightarrow f_x + \lambda\varphi_x = 0$$
$$F_y = 0 \Rightarrow f_y + \lambda\varphi_y = 0$$
$$F_\lambda = 0 \Rightarrow \varphi = 0.$$

GENERAL CASE:

Find extremum of $f(x_1, x_2, \dots, x_n)$ und the conditions
$\varphi_i(x_1, x_2, \dots x_n) = 0 \quad i = 1, 2, \dots k.$

Construct the auxiliary function

$$F(x_1, x_2, \dots x_n, \lambda_1, \lambda_2, \dots \lambda_k) = f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{k} \lambda_i \varphi_i(x_1, x_2, \dots, x_n)$$

Find stationary points of F:

$$\frac{\partial F}{\partial x_1} = 0 = \frac{\partial F}{\partial x_2} = \cdots = \frac{\partial F}{\partial x_n} = \frac{\partial F}{\partial \lambda_1} = \cdots = \frac{\partial F}{\partial \lambda_k}$$

↳ $(n+k)$ equations and $(n+k)$ unknowns.

Note that, using method of Lagrange multiplier, we obtain stationary points. We donot determine the nature of the stationary point. The second derivative test for constrained problem is more theoretical importance than practical. In practice we usually are interested in finding max/min value of a function under some given constraints

Example: Find maximum/minimum of the function

$$x^2-y^2-2x$$

in the region $x^2+y^2 \leq 1$

Sol: I] local extrema in the interior domain $x^2+y^2<1$

Let $f(x,y) = x^2-y^2-2x$

$f_x = 0 \Rightarrow$

$2x-2=0 \Rightarrow x=1$

$f_y = 0 \Rightarrow -2y=0 \Rightarrow y=0$

Critical point $(1,0)$, however this point lies on the boundary

so no extrema in the interior.

II.] Auxiliary function for the problem Max/Min x^2-y^2-2x

subject to $x^2+y^2=1$.

$$F(x,y,\lambda) = (x^2-y^2-2x)+\lambda(x^2+y^2-1) = 0$$

$F_x = 0 \Rightarrow 2x-2+2\lambda x = 0$ — ①

$F_y = 0 \Rightarrow -2y+2\lambda y = 0 \Rightarrow 2y(\lambda-1)=0 \Rightarrow y=0,\ \lambda=1$

If $y=0$, then $x^2+y^2=1$ gives $x=\pm 1$, Points: $(1,0)$ & $(-1,0)$

If $\lambda=1$. then ① $\Rightarrow 4x-2=0 \Rightarrow x=\frac{1}{2}$

if $x=\frac{1}{2}$ then $x^2+y^2=1 \Rightarrow y^2 = 1-\frac{1}{4} = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$

Points: $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ & $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$

Function values at critical points:

1. $(1,0)$: $f(x,y) = -1$
2. $(-1,0)$: $f(x,y) = \boxed{3}$ ← MAX
3. $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$: $f(x,y) = \frac{1}{4} - \frac{3}{4} - 1 = \boxed{-\frac{3}{2}}$ ← MIN

Ex. Find the maximum and minimum of

$$f(x,y) = x^2+2y^2$$

on the disk $x^2+y^2 \leq 1$.

Sol: I] Find local maxima/minima in $x^2+y^2<1$?

$$f_x = 2x \quad \& \quad f_y = 4y$$

Critical point $(0,0)$.

Clearly $(0,0)$ is absolute (Global) minimum of the function $f(x,y)$.

II] Find max/min on the circle $x^2+y^2=1$.

auxiliary function: $F(x,y,\lambda) = (x^2+2y^2) + \lambda(x^2+y^2-1)$

Critical point: $F_x = 0 \Rightarrow 2x + 2x\lambda = 0 \Rightarrow 2x(1+\lambda) = 0$ — ①

$F_y = 0 \Rightarrow 4y + 2y\lambda = 0 \Rightarrow 2y(\lambda+2) = 0$ — ②

$F_\lambda = 0 \Rightarrow x^2+y^2-1 = 0$ — ③

① $\Rightarrow \lambda = -1$, ② $\Rightarrow y = 0$ ③ $\Rightarrow x = \pm 1$

① $\Rightarrow x = 0$, ② $\Rightarrow \lambda = -2$ ③ $\Rightarrow y = \pm 1$

Critical point are $(\pm 1, 0)$ & $(0, \pm 1)$.

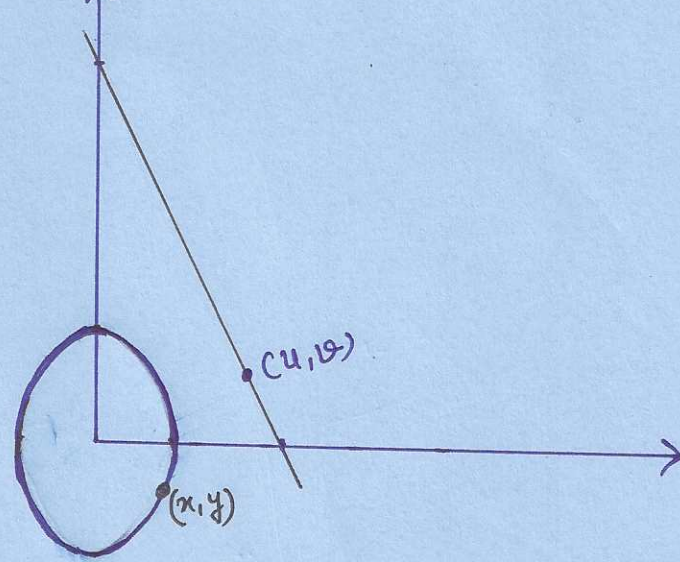
Functional value: $f(\pm 1, 0) = 1$

$f(0, \pm 1) = 2$

Global maximum: $\boxed{2}$ at $(0, \pm 1)$

Global minimum: $\boxed{0}$ at $(0,0)$.

Ex. Find the shortest distance between the line $y = 10 - 2x$ and the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$



Sol. Shortest distance between the line and the ellipse :

Min

$$f(x,y,u,v) = \sqrt{(x-u)^2 + (y-v)^2}$$

subject to

$$\phi_1(x,y) = \frac{x^2}{4} + \frac{y^2}{9} - 1 = 0 \quad \text{—} (1)$$

$$\phi_2(u,v) = 2u + v - 10 = 0 \quad \text{—} (2)$$

Auxiliary function

$$F(x, y, u, v, \lambda_1, \lambda_2) = (x-u)^2 + (y-v)^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{9} - 1\right) + \lambda_2(2u + v - 10)$$

(for simplicity, we have taken $f(x,y,u,v) = (x-u)^2 + (y-v)^2$)

For critical points:

$$F_x = 0 \Rightarrow 2(x-u) + \frac{x}{2}\lambda_1 = 0 \Rightarrow -\lambda_1 x = 4(x-u)$$

$$F_y = 0 \Rightarrow 2(y-v) + \frac{2y}{9}\lambda_1 = 0 \Rightarrow -\lambda_1 y = 9(y-v)$$

$$\Rightarrow 4(x-u)y = 9(y-v)x \quad \text{—} (3)$$

$$F_u = 0 \Rightarrow -2(x-u) + 2\lambda_2 = 0 \Rightarrow \lambda_2 = (x-u)$$

$$F_v = 0 \Rightarrow -2(y-v) + \lambda_2 = 0 \Rightarrow \lambda_2 = 2(y-v)$$

$$\Rightarrow x - u = 2(y-v) \quad \text{—} (4)$$

$F_{\lambda_1} = 0 \Rightarrow \phi_1(x,y) = 0$ & $F_{\lambda_2} = 0 \Rightarrow \phi_2(u,v) = 0$.

From ③ & ④ $\quad 4y = \frac{9}{2}x \Rightarrow 8y = 9x$

① $\Rightarrow \quad \frac{x^2}{4} + \frac{1}{9}\cdot\frac{9^2x^2}{8^2} - 1 = 0 \Rightarrow x = \pm\frac{8}{5}$

$$y = \pm\frac{9}{5}$$

For : $x = \frac{8}{5}, \ y = \frac{9}{5}$

④ $\Rightarrow \frac{8}{5} - u = 2\left(\frac{9}{5} - v\right) \Rightarrow 2v - 2 = u$.

② $\Rightarrow 2(2v-2) + v - 10 = 0 \Rightarrow v = \frac{14}{5}$

$$u = \frac{18}{5}$$

One critical point: $(x,y) = \left(\frac{8}{5}, \frac{9}{5}\right) \ (u,v) = \left(\frac{18}{5}, \frac{14}{5}\right)$

The distance in this case: $\sqrt{\left(\frac{8}{5} - \frac{18}{5}\right)^2 + \left(\frac{9}{5} - \frac{14}{5}\right)^2} = \sqrt{5}$

For $x = -\frac{8}{5}, \ y = -\frac{9}{5}$

$$\left.\begin{array}{ll} ④ \Rightarrow & u = 2v + 2 \\ ② \Rightarrow & v = \frac{6}{5} \end{array}\right\} \Rightarrow (u,v) = \left(\frac{22}{5}, \frac{6}{5}\right)$$

The distance in this case: $\sqrt{\left[\left(-\frac{8}{5}\right) - \frac{22}{5}\right]^2 + \left[\left(-\frac{9}{5}\right) - \left(\frac{6}{5}\right)\right]^2} = 3\sqrt{5}$

Hence the shortest distance between the line and the ellipse is $\boxed{\sqrt{5}}$.