ASSIGNMENT - 2

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LINEAR ALGEBRA

1. We have to find all linear transformations f from $M_n(\mathbb{R})$ to $\mathbb{R}$ s.t.

$$f(AB) = f(BA). \quad \forall A, B \in M_n(\mathbb{R})$$

Since f is a linear transformation

$$\therefore f(\alpha A + \beta B) = \alpha f(A) + \beta f(B).$$

Now, consider the Basis of M_n

$$= \{ e_{ij},\ 1 \le i, j \le n \}$$

e_{ij} is defined s.t the $[i,j]^{th}$ element in the matrix is 1 & rest are 0.

So, if A is a matrix, it can be written as

$$A = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} e_{ij}$$

where a_{ij} are the elements at (i,j) of A.

Now, we have

$$f(A) = f\left(\sum_j \sum_i a_{ij} e_{ij}\right)$$

$$= \sum_j \sum_i a_{ij} f(e_{ij})$$

Now, we have to realise that e_{ij} can be written as

$$e_{ij} = e_{ii} \cdot e_{ij}$$

So, our equation becomes

$$f(A) = \sum_j \sum_i a_{ij} f(e_{ii} e_{ij}).$$

Now, $\because f(AB) = f(BA)$

$$\therefore f(e_{ii} e_{ij}) = f(e_{ij} \cdot e_{ii}).$$

thus,

$$f(A) = \sum_j \sum_i a_{ij} f(e_{ij} e_{ii}).$$

Now we have to observe the values of $e_{ij} e_{ii}$

Here, if $i = j$, then $e_{ii} e_{ii} = e_{ii}$

else if $i \ne j$ then $e_{ij} e_{ii} = 0$ (matrix)

So, in the summation, all the terms where $i \ne j$

$$f(e_{ij} e_{ii}) = f(0) = 0$$

thus the summation becomes

$$f(A) = \sum_i a_{ii} f(e_{ii})$$

Now, we need to find $f(e_{ii})$

Let $E_i = e_{i1} + e_{1i}$ (if $i \ne 1$) — (I)

So, $E_i e_{ii} E_i = (e_{i1} + e_{1i})\, e_{ii}\, (e_{i1} + e_{1i})$

$$= (0 + e_{1i})(e_{i1} + e_{1i})$$

$$= e_{11} + 0$$

$$= e_{11}$$

& Also observe that $E_i^2 e_{ii} = e_{ii}$ — (II)

$$(e_{i1} + e_{1i})(e_{i1} + e_{1i}) e_{ii}$$

$$= (e_{i1} e_{i1} + e_{i1} e_{1i} + e_{1i} e_{i1} + e_{1i} e_{1i}) e_{ii}$$

$$= (0 + e_{ii} + e_{11}) e_{ii}$$

$$= e_{ii} + 0$$

Now, $f(e_{11}) = f(E_i e_{ii} E_i) = f((E_i e_{ii}) E_i)$

$$= f((E_i)^2 e_{ii})$$

$$= f(e_{ii}) \quad \text{from (II)}$$

thus, our eq^n becomes

$$f(A) = \sum_i a_{ii} f(e_{11})$$

$$= f(e_{11}) \sum a_{ii} = f(e_{11})\, tr(A)$$

$$= k\, tr(A)$$

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We can assume $f(e_{11})$ to be some const t.

$\therefore$

$$f(A) = t \cdot tr(A) \quad \text{for some } t \in \mathbb{R}$$

(2) $\mathbb{V} \to$ finite dimentional vector space.
& $\mathbb{W}$ is a subspace of $\mathbb{V}$.
To Prove :- $\mathbb{W}$ has a unique complement iff
$\mathbb{W} = \{0\}$ or $\mathbb{W} = \mathbb{V}$.

(a) if $\mathbb{W} = \{0\}$ or $\mathbb{W} = \mathbb{V}$.

So, case I . $\mathbb{W} = \{0\}$

$\dim(\mathbb{W}) = 0$ $\quad (\mathbb{W} \oplus \overline{\mathbb{W}} = \mathbb{V})$

& $\dim(\mathbb{V}) = \dim(\mathbb{W}) + \dim(\overline{\mathbb{W}})$.

here $\overline{\mathbb{W}}$ is a complement of $\mathbb{W}$.

$\Rightarrow \dim(\overline{\mathbb{W}}) = \dim(\mathbb{V})$.

So $\overline{\mathbb{W}}$ is the complete vector space $\mathbb{V}$.
Because its basis has the same dimension
& so it will span the entire vector space.
this is unique.

Case II . $\mathbb{W} = \mathbb{V}$.

then $\dim(\mathbb{V}) = \dim(\mathbb{W})$

$\therefore \dim(\overline{\mathbb{W}}) = 0$

so $\overline{\mathbb{W}} = \{0\}$ which is unique again.

So, if $\mathbb{W} = \{0\}$ or $\mathbb{W} = \mathbb{V}$, then it has unique complement.

(b) Now, the converse part.

$\mathbb{W}$ has a unique complement $\overline{\mathbb{W}}$

$\Rightarrow$ $\mathbb{W} = \{0\}$ or $\mathbb{W} = \mathbb{V}$.

let U be a subspace of $\mathbb{V}$ such that

$U \neq \{0\}$ & $U \neq \mathbb{V}$

and U has a unique complement $\overline{U}$.

Now, let's pick an element $K \in \overline{U}$, $K \neq 0$.

& let $u \in U$ s.t $u \neq 0$.

Let B be the basis of $\overline{U}$ containing K.

now, we can modify the basis by replacing K with $K+u$.

Now, the new Linear Span of this new basis be called $(\overline{U})'$.

this new $(\overline{U})'$ is also a complement of U.

this contradicts the fact that U had a unique complement.

thus,

$\mathbb{W}$ has a unique complement iff $\mathbb{W} = \{0\}$ or $\mathbb{W} = \mathbb{V}$.

Note for Q 1

Now, we have to show that

$f(A) = t.\,tr(A)$ satisfies the relation

$f(AB) = f(BA)$

$$f(AB) = t \cdot tr(AB) = t \cdot \sum_i \sum_j a_{ij} b_{ji} = t \sum_j \sum_i b_{ji} a_{ij}$$

$$= t \sum_{j=1}^{n} (BA)_{jj}$$

$$= t \cdot tr(BA)$$

thus, we have shown that the set of all such L.T is

$f(A) = t \cdot tr(A)$