3) $\dim V = 2$, $|F| = 2$

$\therefore$ F has only two elements, additive and multiplicative identity i.e. 0 and 1.

$\therefore$ There are only 3 possible basis:

i) $\{(0,1), (1,0)\}$

ii) $\{(1,1), (0,1)\}$

iii) $\{(1,1), (1,0)\}$

Ans) 3c) V has exactly 3 three bases.

4) $V = M_{3\times 2}(\mathbb{R})$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$W = \{ A \in V : Ax = 0 \}$

Let $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ $\therefore \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} a+b \\ c+d \\ e+f \end{bmatrix} = 0 \Rightarrow a = -b$
$c = -d$
$e = -f$

$\therefore$ Standard basis of W:

$$\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

Ans): 4. c) Dimension of W is 3.

5) If v_1, v_2, v_3 are LI

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ has only trivial soln. $c_1 = c_2 = c_3 = 0$

$\Rightarrow c_1(u_1 + 2u_2 + 3u_3) + c_2(a u_2 + 5u_3) + c_3 . 2u_3 = 0$

$\Rightarrow (c_1) u_1 + (2c_1 + ac_2) u_2 + (3c_1 + 5c_2 + 2c_3) u_3 = 0$

u_1, u_2, u_3 are LI.

$\therefore c_1 = 0$

$\therefore 2c_1 + ac_2 = 0$

$3c_1 + 5c_2 + 2c_3 = 0$

This system of homogeneous linear equ^n's has trivial soln. iff $D \neq 0$
$(c_1 = c_2 = c_3 = 0)$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 2 & a & 0 \\ 3 & 5 & 2 \end{vmatrix} \neq 0$$

$\Rightarrow 1(2a) \neq 0 \Rightarrow a \neq 0$

Ans) 5b) v_1, v_2 and v_3 are linearily independent iff $a \neq 0$.

6) $V = M_{n\times n}(\mathbb{R})$, $\dim(V) = n^2$ (no. of places to put 1 to form standard basis)

$W = \{A \in V : A$ is an upper triangular and trace$(A) = 0\}$

$2x + n = n^2 \Rightarrow x = \frac{n^2 - n}{2}$

$\dim W = \frac{n^2-n}{2} + n - 1 = \frac{n^2}{2} + \frac{n}{2} - 1$

(no. of places to put 1 above the principal diagonal, all places below diagonal will have 0)

(no. of place to put 1 in the diagonal, $\because$ trace $= 0$ as we can consider a_{11} as $a_{11} = -(a_{22} + \dots + a_{nn})$)

$V = W \oplus W^{\perp} \Rightarrow \dim V = \dim W + \dim W^{\perp}$

$\Rightarrow \dim W^{\perp} = n^2 - \left(\frac{n^2}{2} + \frac{n}{2} - 1\right)$

$= \frac{n^2}{2} - \frac{n}{2} + 1 = \frac{n^2 - n + 2}{2}$

6 Ans) c) Dimensions of orthogonal complement of W is $\frac{n^2-n+2}{2}$.

7) $\|x\| = \sqrt{\langle x, x\rangle}$

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \langle y,x\rangle$

Given, $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

$\Rightarrow \langle x,y\rangle = \langle y,x\rangle = 0$

$\Rightarrow \overline{\langle y,x\rangle} = \langle y,x\rangle = 0$ (conjugate symmetry of IP)

$\Rightarrow x$ is orthogonal to y & $\langle x,y\rangle \in \mathbb{R}$

Ans) 7c) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 \Rightarrow x$ is orthogonal to y if $\mathbb{K} = \mathbb{R}$

8) $W = \{ (x_1, x_2, \dots, x_n) : \sum_{i=1}^{n} x_i = 0 \}$ subspace of R^n

W is non-trivial

$$\frac{\dim(\mathbb{R}^n)}{2} = \frac{n}{2}$$

$\dim(W) = n-1$ (taking x_1 ~~as de~~ $-(x_2 + \dots + x_n)$)

$\therefore \dim(W) \geq \frac{\dim(\mathbb{R}^n)}{2}$ $\left[n-1 \geq \frac{n}{2} \quad \because n \geq 2 \right]$

$\therefore$ Ans) 8 a) W has two virtually disjoint complements.

9) $\{\phi\}^{\perp} = \{ u \in V \mid \langle u, v \rangle = 0 \ \forall \ v \in \{\phi\} \}$

$\because \{\phi\}$ is an empty set.

$\therefore \forall \ u \in V, \ \langle u, v \rangle = 0, \ v \in \{\phi\}$

$\therefore \{\phi\}^{\perp} = V$

Ans) 9a) $\{\phi\}^{\perp} = V$, $\{\phi\}$ is an empty set.

10) $C[-1, 1]$ is an infinite dimensional inner product space.

$\therefore C[-1, 1] \neq U \oplus U^{\perp}$ (equality holds only for finite-dimensional IPS)

$\mathbb{U} = \{ f \in C[-1, 1] : f(0) = 0 \}$

Let $f(x) = x \in \mathbb{U}$

Let $g(x) = 1$

$$\therefore \langle f, g \rangle = \int_{-1}^{1} x \cdot 1 \, dx = \left. \frac{x^2}{2} \right|_{-1}^{1} = 0$$

$\therefore g(x) \in \mathbb{U}^{\perp}$

$\therefore 1 \in \mathbb{U}^{\perp} \Rightarrow U^{\perp} \neq \{0\}$

$\therefore$ Ans) 10c) $U^{\perp}$ is a proper and non-trivial subspace of $C[-1, 1]$.