Lecture Notes on Linear Algebra

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Contents

1 Vector Space

1.1 Properties of Linearly Dependent and Independent sets

Theorem 1.1

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $S = \{x_1, \cdots, x_n\} \subseteq \mathbb{V}$. Then S is linearly dependent iff there exits k s.t. x_k is linear combination of $x_1, \cdots, x_{k-1}, x_{k+1}, \ldots, x_n$ (that is, there exists x_k such that $x_k \in \text{LS}(S - \{x_k\})$).

Proof. Since $x_1, \cdots, x_n$ be linearly dependent, there exist scalars $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$ not all zero such that $\sum\limits_{i=1}^{n} \alpha_i x_1 = 0$. Without loss of generality we assume that $\alpha_k \neq 0$. Then $x_k = -\frac{1}{\alpha_k}(\sum\limits_{i=1, i\neq k}^{n} \alpha_i x_1)$. So x_k is linear combination of $x_1, \cdots, x_{k-1}, x_{k+1}, \ldots, x_n$.

Conversely, there exits k s.t. x_k is a linear combination of $x_1, \cdots, x_{k-1}, x_{k+1}, \ldots, x_n$. That means $x_k = c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} + c_{k+1}x_{k+1} + \cdots + c_nx_n$ for some scalars $c_i \in \mathbb{F}$ for $i = 1, \ldots, k-1, k+1, \ldots, n$. Then $c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} - x_k + c_{k+1}x_{k+1} + \cdots + c_nx_n = 0$. This implies S is linearly dependent. □

Remark 1.1. The above theorem gives you a technique to check whether a set is linearly dependent or not. That is, you just have to check whether a vector from that set is a linear combination of remaining vectors of S or not.

The following theorem says something more.

Theorem 1.2

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $S = \{x_1, \ldots, x_n\} \subseteq \mathbb{V}$ and $x_1 \neq 0$. Then S is linearly dependent iff then $\exists k > 1$ s.t. x_k is a linearly combination of $x_1, \ldots, x_{k-1}$.

Proof. Consider $\{x_1\}, \{x_1, x_2\}, \cdots, \{x_1, x_2, \cdots, x_n\}$ one by one. Take the smallest $k > 1$ s.t. $S_k = \{x_1, \cdots, x_k\}$ is linearly dependent. So S_{k-1} is linearly independent(since k is the smallest). In that case if $\sum_{i=1}^{k} \alpha_i x_i = 0$ $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$ not all zero, then $\alpha_k \neq 0$. Otherwise S_k will be linearly independent. Therefore x_k is linear combination of $x_1, \ldots, x_{k-1}$.

There exists $k > 1$ such that x_k is linear combination of $x_1, \ldots, x_{k-1}$. Therefore $x_k = c_1 x_1 + \cdots + c_{k-1} x_{k-1}$ where $c_1, c_2, \ldots, c_{k-1} \in \mathbb{F}$ not all zero. Then

$$c_1 x_1 + \cdots + c_{k-1} x_{k-1} - x_k + 0 x_{k+1} + 0 x_{k+2} + \cdots + 0 x_n = 0$$

where $c_1, c_2, \ldots, c_{k-1}$ are not all zero. Hence S is linearly dependent. □

Theorem 1.3

Every subset of a finite linearly independent set is linearly independent.

Proof. Let S be a linearly independent set. Let $S = \{x_1, \ldots, x_k\}$. Let $S_1 \subseteq S$. We have to show that S_1 is linearly independent. Suppose S_1 is linearly dependent. Then there exists a vector say x_m in S_1 such that $x_m \in \text{LS}(S_1 - \{x_m\})$. Since $S_1 - \{x_m\} \subseteq S - \{x_m\}$. Then $x_m \in \text{LS}(S - \{x_m\})$. Hence S_1 is linearly independent, a contradiction. Therefore S_1 is linearly independent. □

Remark 1.2. The above theorem says that if you will be able to find out a subset of a set S which is linearly dependent, then S is linearly dependent. But if you will find out a subset of S which is linearly independent, then you can not conclude anything about S.

Remark 1.3. Every subset of a finite linearly dependent set need not be linearly dependent. In $\mathbb{R}^2$, the vectors $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, e_1, e_2$, are linearly dependent, as $x - e_1 - 2e_2 = 0$. The subset $\{e_1, e_2\}$ is linearly independent.

Till now we have seen linear independency and dependency for finite set. Next we are going to define linear independency and dependency for infinite set and we hire the concepts of linear independency and dependency of finite to define for infinite set.

Definition 1.1. Let $\mathbb{V}$ be a vector space and let $S \subseteq \mathbb{V}$ be infinite. We say S is **linearly dependent** if it contains a finite linearly dependent set, otherwise it is **linearly independent**.

Theorem 1.4

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $S = \{v_1, \ldots, v_n\} \subseteq \mathbb{V}$ and $T \subseteq \text{LS}(S)$ such that $m = |T| > |S|$. Then T is linearly dependent.

Proof. Let $T = \{u_1, u_2, \ldots, u_m\}$. Since $u_j \in \text{LS}(S)$, there exists A_{ij} in $\mathbb{F}$ such that $u_j = \sum_{i=1}^{n} A_{ij} v_i$ for $j = 1, \ldots, m$. For any m scalars $x_1, x_2, \ldots, x_m$ we have

$$\begin{aligned} x_1u_1 + x_2u_2 + \cdots + x_mu_m &= \sum_{j=1}^{m} x_j u_j \\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{n} A_{ij} v_i \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} (A_{ij} x_j) v_i \\ &= \sum_{i=1}^{n} (\sum_{j=1}^{n} A_{ij} x_j) v_i \end{aligned}$$

There exist scalars $x_1, x_2, \ldots, x_m$ not all zero such that

$$\sum_{j=1}^{n} A_{ij} x_j = 0 \text{ for } i = 1, \ldots, n$$

(using system of homogeneous equations, here the co-efficient matrix is A of size $n \times m$ and rank of A is less than m as $n < m$).

Therefore we have $x_1, x_2, \ldots, x_m$ in $\mathbb{F}$ not all zero such that $x_1u_1 + x_2u_2 + \cdots + x_mu_m = \sum_{j=1}^{m} x_j u_j = 0$. Hence $T = \{u_1, u_2, \ldots, u_m\}$ is linearly dependent. $\square$

The above theorem is quite important. This theorem says that if you have a finite subset S of a vector space containing n elements. Then any finite subset of $\text{LS}(S)$ containing more than n elements is linearly dependent. For example, consider the vector space $\mathbb{R}^3$. Take $S = \{(1,1,0),(1,0,0)\}$. Take $T = \{(1,1,0),(1,0,0),(3,1,0)\}$. You can easily check that $T \subseteq \text{LS}(S)$ and T contains three elements. T is linearly dependent.

Corollary 1.1. *Any $n+1$ vectors in $\mathbb{R}^n$ is linearly dependent.*

Proof. Follows as $\mathbb{R}^n = \text{LS}(e_1, \ldots, e_n)$. □

The following corollary is quite important. This theorem gives you a technique to extend a linearly independent set to a larger linearly independent set. This technique will be used frequently throughout this course. So keep it in your mind.

Corollary 1.2. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $S \subseteq \mathbb{V}$ be linearly independent and $x \in \mathbb{V} \setminus S$. Then $S \cup \{x\}$ is linearly independent iff $x \in \mathbb{V} - \text{LS}(S)$.*

Corollary 1.3. *Let $S \subseteq \mathbb{V}$ be linearly independent. Then $\text{LS}(S) = \mathbb{V}$ iff each proper superset of S is linearly dependent.*

1.2 Properties of Basis and Dimension

Theorem 1.5

Every vector space has a basis.

Proof. Let $\mathbb{V}$ be any vector space over some field $\mathbb{F}$. Let X be the set of all linearly independent subsets of $\mathbb{V}$.

The set X is nonempty since the empty set is an independent subset of $\mathbb{V}$, and it is partially ordered by inclusion, which is denoted, as usual, by $\subseteq$.

Let Y be a subset of X that is totally ordered by $\subseteq$, and let L_Y be the union of all the elements of Y (which are themselves certain subsets of $\mathbb{V}$).

Since $(Y, \subseteq)$ is totally ordered, every finite subset of L_Y is a subset of an element of Y, which is a linearly independent subset of $\mathbb{V}$, and hence L_Y is linearly independent. Thus L_Y is an element of X. Therefore, L_Y is an upper bound for Y in $(X, \subseteq)$ it is an element of X, that contains every element of Y.

As X is nonempty, and every totally ordered subset of $(X, \subseteq)$ has an upper bound in X, Zorn's lemma asserts that X has a maximal element. In other words, there exists some element L_{max} of X satisfying the condition that whenever $L_{max} \subseteq L$ for some element L of X, then $L = L_{max}$.

It remains to prove that L_{max} is a basis of $\mathbb{V}$. Since L_{max} belongs to X, we already know that L_{max} is a linearly independent subset of $\mathbb{V}$.

If there were some vector w of $\mathbb{V}$ that is not in the span of L_{max}, then w would not be an element of L_{max} either. Let $L_w = L_{max} \cup \{w\}$. This set is an element of X, that is, it is a linearly independent subset of $\mathbb{V}$ (because w is not in the span of L_{max}, and L_{max} is independent). As $L_{max} \subseteq L_w$, and $L_{max} \neq L_w$ (because L_w contains the vector w that is not contained in L_{max}), this contradicts the maximality of L_{max}. Thus this shows that L_{max} spans $\mathbb{V}$.

Hence L_{max} is linearly independent and spans $\mathbb{V}$. It is thus a basis of $\mathbb{V}$, and this proves that every vector space has a basis. □

Remark 1.4. Every non-trivial vector space has infinitely many basis. That means basis is not unique.

Theorem 1.6: Dimension theorem for vector spaces

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Any two bases of $\mathbb{V}$ have the same cardinality.

Proof. Let A and B be two bases of $\mathbb{V}$. Suppose A is finite. We now show that $|A| = |B|$ ($|A|$ stands for cardinality of A). We consider A is basis and B is linearly independent, then using Theorem 1.1 $|B| \leq |A|$. Then we consider B is basis and A is linearly independent set, then using Theorem 1.1 $|A| \leq |B|$. Therefore A and B have same cardinality.

Consider A is infinite set. Then $|A| \geq \aleph_0$. We want to show that $|A| = |B|$. We just now show that B is also infinite set as A is infinite. Then $|B| \geq \aleph_0$. For each $a \in A$ can be expressed as a finite linear combination of elements of B, so let B_a be the collection of these elements. Now, B_a is uniquely determined by a, as B is a basis. Also, B_a is finite. Let

$$B' = \cup_{a \in A} B_a$$

Since A spans $\mathbb{V}$, so does B'. If $B' \neq B$, pick $b \in B - B'$, so that b is a linear combination of elements of B'. Moving b to the other side of the expression and we have expressed 0 as a non-trivial linear combination of elements of B, contradicting the linear independence of B. Therefore $B' = B$. This means

$$|B| = |\cup_{a \in A} B_a| \leq \aleph_0 |A| = |A|$$

Similarly, every element in B is expressible as a finite linear combination of elements in A, and using the same argument as above,

$$|A| \leq \aleph_0 ||B|$$

.

Therefore, $|A| = |B|$. □

Theorem 1.7

Let $\mathbb{V}$ be a finite-dimensional vector space with $\dim(V) = n$. If $\mathbb{U}$ is a subspace of V , then $\dim(\mathbb{U}) \leq \dim(\mathbb{V})$, the equality holds if and only if $\dim(\mathbb{U}) = n$.

Definition 1.2. A subset $S \subseteq \mathbb{V}$ is called **maximal linearly independent** if
i) S is linearly independent
ii) no proper super set of S is linearly independent.

Example 1.1. The following examples must be verified by the reader.

1. In $\mathbb{R}^3$, the set $\{e_1, e_2\}$ is linearly independent but not maximal linearly independent.

2. In $\mathbb{R}^3$, the set $\{e_1, e_2, e_3\}$ is maximal linearly independent.

3. Let $S \subseteq \mathbb{R}^n$ be linearly independent and $|S| = n$. Then S is maximal linearly independent.

Theorem 1.8

A set $S \subseteq \mathbb{V}$ is maximal linearly independent, then $\textsc{ls}(S) = \mathbb{V}$.

Proof. First we assume that S is maximal linearly independent set. To show $\textsc{ls}(S) = \mathbb{V}$. Suppose $\textsc{ls}(S) \neq \mathbb{V}$. Then there exists $\alpha \in \mathbb{V}$ but not in $\textsc{ls}(S)$. Take $S_1 = S \cup \{\alpha\}$.

Claim: The set S_1 is linearly independent. Suppose it is dependent. Then S_1 has a finite subset which is linearly dependent, say, R. There are two cases.

CASE I: $\alpha \notin R$. Then R is a finite subset of S. A contradiction that R is linearly dependent (every subset of a linearly independent set is linearly independent ans S is linearly independent).

CASE II: $\alpha \in R$. Let $R = \{\alpha_1, \ldots, \alpha_k, \alpha\}$. Since R is linearly dependent then there exist $c_1, c_2, \ldots, c_k, c$ not all zero in $\mathbb{F}$ such that

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k + c\alpha = 0$$

. If $c = 0$, then $c_i = 0$ for $i = 1, \ldots, k$ (since $\{\alpha_1, \ldots, \alpha_k\}$ is a subset of S which is linearly independent). So $c \neq 0$. Then $\alpha = \frac{1}{c}(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k)$. This implies $\alpha \in \textsc{ls}(S)$, a contradiction.

Hence we have proved our claim that S_1 is linearly independent.

We notice that $S \subsetneq S_1$. A contradiction that S is maximal. Hence $\textsc{ls}(S) = \mathbb{V}$. □

Theorem 1.9

A subset $S \subseteq \mathbb{V}$ is a basis of $\mathbb{V}$. Then S is maximal linearly independent set.

Proof. Suppose that S is not maximal. Then there exits a linearly independent set S_1 such that $S \subsetneq S_1$ and $S_1 \subseteq \textsc{ls}(S)$. By using Theorem 1.1, S_1 is dependent, a contradiction. Hence S is maximal. □

Remark 1.5. It is clear that every basis is maximal linearly independent set.

Remark 1.6. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and let B be a basis of $\mathbb{V}$. There exist unique $\alpha_1, \ldots, \alpha_k \in B$ and unique $c_1, \ldots, c_k \in \mathbb{F}$ such that $x = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k$.

Definition 1.3. Let $\mathbb{V}$ be a vector space. Then $\mathbb{V}$ is called **finite dimensional** if it has a basis B which is finite. The dimension of $\mathbb{V}$ is the cardinality of B and it is denote by $\dim(V)$.

Remark 1.7. $\dim(\{0\}) = 0$. The zero vector space is spanned by the vector 0, but $\{0\}$ is a linearly dependent set and not a basis. For this reason, we shall agree that the zero vector space has dimension 0. Alternatively, we could reach the same conclusion by arguing that the empty set is a basis for the zero vector space. The empty set spans $\{0\}$, because the intersection of all subspaces containing the empty set is $\{0\}$, and the empty set is linearly independent because it contains no vectors.

Theorem 1.10: Basis Deletion Theorem

If $\mathbb{V} = \textsc{ls}(\{\alpha_1, \ldots, \alpha_k\})$. Then some v_i can be removed to obtain a basis for $\mathbb{V}$.

Proof. Let $S = \{\alpha_1, \ldots, \alpha_k\}$. If S is linearly independent, then S is basis of $\mathbb{V}$. If S is linearly dependent, then there exists $\alpha_i \in S$ such that α_i is linear combination of rest of the vectors of S. That is $\alpha_i = b_1\alpha_1 + \cdots + b_{i-1}\alpha_{i-1} + b_{i+1}\alpha_{i+1} + \cdots + b_k\alpha_k$ where $b_p \in \mathbb{F}$ for $p = 1, \ldots, i-1, i+1, \ldots, k$ and not all zero. We assume that $b_t \neq 0$. Take $S_1 = S - \{\alpha_t\}$. Using Corollary 1.3, $\textsc{ls}(S_1) = \mathbb{V}$. If S_1 is linearly independent, then S_1 is basis of $\mathbb{V}$. Continuing this process we get a linearly independent subset consisting of $k - p = n$ vectors(since n is finite existence of such p is possible. Therefore S_p is a basis of $\mathbb{V}$. □

The following is an application of Basis deletion theorem.

Theorem 1.11

Let $\mathbb{V}$ be a finite dimensional vector space and let $\dim \mathbb{V} = n$. Let $S = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{V}$ such that $\textsc{ls}(S) = \mathbb{V}$. Then S is a basis for $\mathbb{V}$.

Proof. If we show that S is linearly independent then we are done. If S is linearly dependent, then by Deletion theorem, we can reduce S to a basis for $\mathbb{V}$ which contains less than n elements. A contradiction that $\dim(\mathbb{V}) = n$. Hence S is basis of $\mathbb{V}$. □

Theorem 1.12: Basis Extension Theorem

Every linearly independent set of vectors in a finite-dimensional vector space $\mathbb{V}$ can be extended to a basis of V .

Proof. Let $\dim(\mathbb{V}) = n$. Let $S = \{\alpha_1, \ldots, \alpha_k\}$ be a linearly independent set. If $\mathrm{LS}(S) = \mathbb{V}$, then S is basis of $\mathbb{V}$. If $\mathrm{LS}(S) \neq \mathbb{V}$, then there exists a vector $\beta_1 \in \mathbb{V}$ but not in $\mathrm{LS}(S)$. Take $S_1 = S \cup \{\beta_1\}$. Using Corollary 1.2, the set S_1 is linearly independent. If $\mathrm{LS}(S_1) = \mathbb{V}$, then S_1 is basis for $\mathbb{V}$. If $\mathrm{LS}(S_1) \neq \mathbb{V}$, then there exists a vector $\beta_2 \in \mathbb{V}$ but not in $\mathrm{LS}\, S_1$. Take $S_2 = S_1 \cup \{\beta_2\}$. Using Corollary 1.2, the set S_2 is linearly independent. If $\mathrm{LS}(S_2) = \mathbb{V}$,then S_2 is basis for $\mathbb{V}$. If $\mathrm{LS}(S_2) \neq \mathbb{V}$, then there exists a vector $\beta_3 \in \mathbb{V}$ but not in $\mathrm{LS}\, S_2$. Take $S_3 = S_1 \cup \{\beta_3\}$. Continuing this process we get a linearly independent subset consisting of $k + p = n$ vectors (since n is finite existence of such p is possible. Therefore $S_p = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_p\}$. We notice that S_p is linearly independent and it contains n vectors. So S_p is basis of $\mathbb{V}$. □

The following is an application of Basis extension theorem.

Theorem 1.13

Let $\mathbb{V}$ be a finite dimensional vector space and $\dim \mathbb{V} = n$. Let $S = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{V}$ such that S is linearly independent. Then S is a basis of $\mathbb{V}$.

Proof. If we show that $\mathrm{LS}(S) = \mathbb{V}$, then we are done. If $\mathrm{LS}(S) \neq \mathbb{V}$, then by using Extension theorem, we can extend S to be a basis for $\mathbb{V}$ and which contains at least $n + 1$ elements. A contradiction that $\dim(\mathbb{V}) = n$. Hence S is a basis of $\mathbb{V}$. □

Definition 1.4. Let $\mathbb{V}$ be a finite dimensional vector spaces and let $B = \{x_1, \ldots, x_n\}$ be a basis. Let $x \in \mathbb{V}$. Then there exists unique $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$. Then $(\alpha_1, \ldots, \alpha_n)$ is called **co-ordinate** of x.

Remark 1.8. A vector space is infinite dimensional if it is not finite dimensional.

1.3 Properties of Subspaces

In this subsection we are going to discuss some results related to subspace. It is known to us that arbitrary intersection of subspaces of a vector space again a subspace of that vector space. But union of two subspaces of a vector space need not be a subspace of that vector space, see the following example.

Example 1.2. Let $\mathbb{V} = \mathbb{R}^2$be a vector space over the filed $\mathbb{R}$. Let $\mathbb{U} = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$ and let $\mathbb{V} = \{x \in \mathbb{R}^2 \mid x_1 = 0\}$. We can easily check that both are subspaces $\mathbb{R}^2$. Take $\mathbb{U} \cup \mathbb{V} = \{x \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$ which is not a subspace.

Exercise 1.1. *Give two subspaces $\mathbb{U}, \mathbb{W}$ of $\mathbb{R}^3$ for each: a) $\mathbb{U} \cup \mathbb{W}$ is a subspace of $\mathbb{V}$. b) $\mathbb{U} \cup \mathbb{W}$ is not a subspace of $\mathbb{V}$.*

The following theorem supplies a necessary and sufficient condition to make union of two subspaces again a subspace.

Theorem 1.14

Let $\mathbb{U}, \mathbb{W}$ be two subspaces of $\mathbb{V}$. Then $\mathbb{U} \cup \mathbb{W}$ is a subspace of $\mathbb{V}$ if and only if either $\mathbb{U} \subseteq \mathbb{W}$ or $\mathbb{W} \subseteq \mathbb{U}$.

Proof. First we assume that $\mathbb{U} \cup \mathbb{W}$ is subspace of $\mathbb{V}$. To show either $\mathbb{U} \subseteq \mathbb{W}$ or $\mathbb{W} \subseteq \mathbb{U}$. Suppose that $\mathbb{U} \not\subseteq \mathbb{W}$ and $\mathbb{W} \not\subseteq \mathbb{U}$. Then there exist two vectors x and y in $\mathbb{V}$ such that $x \in \mathbb{U}$ but not in $\mathbb{W}$ and $y \in \mathbb{W}$ but not in $\mathbb{U}$. Therefore $x, y \in \mathbb{U} \cup \mathbb{W}$. Then $x + y \in \mathbb{U} \cup \mathbb{W}$. This implies that either $x + y \in \mathbb{U}$ or $\mathbb{W}$. If $x + y \in \mathbb{U}$, then $(x + y) - x = y \in \mathbb{U}$ which is not possible. If $x + y \in \mathbb{W}$, then $(x+y)-y = x \in \mathbb{W}$ which is not possible. Therefore either $\mathbb{U} \subseteq \mathbb{W}$ or $\mathbb{W} \subseteq \mathbb{U}$.

Conversely, $\mathbb{U} \subseteq \mathbb{W}$ or $\mathbb{W} \subseteq \mathbb{U}$. Then either $\mathbb{U} \cup \mathbb{W} = \mathbb{W}$ or $\mathbb{U} \cup \mathbb{W} = \mathbb{V}$. Hence $\mathbb{U} \cup \mathbb{W}$ is subspace of $\mathbb{V}$. □

Remark 1.9. If $\mathbb{U}$ is a subspace of $\mathbb{W}$ and $\mathbb{W}$ is a subspace of $\mathbb{V}$, then $\mathbb{U}$ is a subspace of $\mathbb{V}$.

Definition 1.5. If $\mathbb{U}, \mathbb{W}$ are subspaces of $\mathbb{V}$, then $\mathbb{U} + \mathbb{W} := \{u + w \mid u \in \mathbb{U}, w \in \mathbb{W}\}$ and this is called the **sum** of $\mathbb{U}$ and $\mathbb{W}$.

Theorem 1.15

If $\mathbb{U}, \mathbb{W}$ are subspaces of $\mathbb{V}$. Then $\mathbb{U} + \mathbb{W}$ is the smallest subspace of $\mathbb{V}$ which contains both $\mathbb{U}$ and $\mathbb{W}$.

Proof. First we show that $\mathbb{U}+\mathbb{W}$ is subspace of V. Let $x, y \in \mathbb{U}+\mathbb{W}$. Then $x = x_1 + x_2$ for some $x_1 \in \mathbb{U}$ and $x_2 \in \mathbb{W}$ and $y = y_1 + y_2$ for some $y_1 \in \mathbb{U}$ and $y_2 \in \mathbb{W}$. Take $\alpha x + \beta y = \alpha(x_1 + x_2) + \beta(y_1 + y_2) = (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2)$. Since $\mathbb{U}$ and $\mathbb{W}$ are subspaces of $\mathbb{V}$. Then $\alpha x_1 + \beta y_1 \in \mathbb{U}$ and $\alpha x_2 + \beta y_2 \in \mathbb{W}$. Therefore $\alpha x + \beta y \in \mathbb{U}+\mathbb{W}$. Hence $\mathbb{U}+\mathbb{W}$ is subspace of $\mathbb{V}$.

Now we show that $\mathbb{W}, \mathbb{U} \subseteq \mathbb{V}$. Let $x \in \mathbb{U}$. Then $x = x + 0$. It is clear that $x \in \mathbb{U}+\mathbb{W}$. Hence $\mathbb{U} \subseteq \mathbb{V}$. Similarly we can show that $\mathbb{W} \subseteq \mathbb{V}$.

Now we show that $\mathbb{U}+\mathbb{W}$ is the smallest subspace of $\mathbb{V}$ which contains both $\mathbb{U}$ and $\mathbb{W}$. Suppose that there is a subspace $\mathbb{Z}$ containing $\mathbb{U}$ and $\mathbb{W}$ such that $\mathbb{Z} \subsetneq \mathbb{U}+\mathbb{W}$. Then there exist a vector $x \in \mathbb{U} \subseteq \mathbb{V}$ but not in $\mathbb{Z}$. Since $x \in \mathbb{U} \subseteq \mathbb{V}$, $x = x_1 + x_2$ where $x_1 \in \mathbb{U}$ and $x_2 \in \mathbb{W}$. Notice that $x_1, x_2 \in \mathbb{Z}$. Therefore $x = x_1 + x_2 \in \mathbb{Z}$. This is a contradiction. Hence $\mathbb{U}+\mathbb{W}$ is the smallest subspace of $\mathbb{V}$ which contains both $\mathbb{U}$ and $\mathbb{W}$. □

Theorem 1.16

If $\mathbb{U}, \mathbb{W}$ are subspaces of a finite-dimensional vector space $\mathbb{V}$, then $\dim(\mathbb{U}+\mathbb{W}) = \dim(\mathbb{U}) + \dim(\mathbb{W}) - \dim(\mathbb{U} \cap \mathbb{W})$.

Proof. Since $\mathbb{V}$ is finite dimensional, $\mathbb{U}$ and $\mathbb{W}$ both are finite dimensional. Let $B = \{v_1, \ldots, v_k\}$ be a basis of $\mathbb{U} \cap \mathbb{W}$. By using Extension theorem we extend B_1 to a basis for $\mathbb{U}$ which is $\{v_1, \ldots, v_k, u_1, \ldots, u_m\}$ and basis for $\mathbb{W}$ which is $\{v_1, \ldots, v_k, w_1, \ldots, w_p, \}$.

Let $x \in \mathbb{U}+\mathbb{W}$. Then $x = x_1 + x_2$ for some $x_1 \in \mathbb{U}$ and $x_2 \in \mathbb{W}$. Therefore

$$x_1 = \sum_{i=1}^{k} a_i v_i + \sum_{j=1}^{m} b_j u_j$$

and

$$x_2 = \sum_{i=1}^{k} c_i v_i + \sum_{j=1}^{p} d_j w_j$$

.

Then $x = \sum\limits_{i=1}^{k} a_i v_i + \sum\limits_{j=1}^{m} b_j u_j + \sum\limits_{i=1}^{k} c_i v_i + \sum\limits_{j=1}^{p} d_j w_j$.

This implies $x \in \text{LS}(\{v_1, \ldots, v_k, u_1, \ldots, u_m, v_1, \ldots, v_k, w_1, \ldots, w_p, \})$

We now show that $\{v_1, \ldots, v_k, u_1, \ldots, u_m, w_1, \ldots, w_p, \})$ is linearly independent.

To show that we take,

$$\sum_{i=1}^{k} f_i v_i + \sum_{j=1}^{m} g_j u_j + \sum_{l=1}^{p} h_j w_j = 0 \tag{1}$$

Therefore,

$$\sum_{i=1}^{k} f_i v_i + \sum_{j=1}^{m} g_j u_j = -\sum_{l=1}^{p} h_j w_j$$

This implies $\sum\limits_{i=1}^{k} f_i v_i + \sum\limits_{j=1}^{m} g_j u_j \in \mathbb{U} \cap \mathbb{W}$. Then $\sum\limits_{i=1}^{k} f_i v_i + \sum\limits_{j=1}^{m} g_j u_j \in \mathbb{U} \cap \mathbb{W} = \sum\limits_{i=1}^{k} \alpha_i v_i$.

Therefore, $\sum\limits_{i=1}^{k} (f_i - \alpha_i) v_i + \sum\limits_{j=1}^{m} g_j u_j = 0$. Then $f_i - \alpha_i = 0$ for $i = 1, \ldots k$ and $g_j = 0$ for $j = 1, \ldots, m$ as $\{v_1, \ldots, v_k, u_1, \ldots, u_m\}$ is linearly independent. Put the values of g_i in Equation (1), then

$$\sum_{i=1}^{k} f_i v_i + \sum_{l=1}^{p} h_l w_l = 0$$

.

Therefore $f_i = 0$ for $i = 1, \ldots, k$ and $h_j = 0$ for $j = 1, \ldots, p$ as $\{v_1, \ldots, v_k, w_1, \ldots, w_p, \}$. Hence $\{v_1, \ldots, v_k, u_1, \ldots, u_m, v_1, \ldots, v_k, w_1, \ldots, w_p, \})$ is linearly independent.

Then $\{v_1, \ldots, v_k, u_1, \ldots, u_m, v_1, \ldots, v_k, w_1, \ldots, w_p, \})$ is basis of $\mathbb{U} + \mathbb{W}$. Therefore $\dim(\mathbb{U} + \mathbb{W}) = k + m + P + k - k = \dim(\mathbb{U}) + \dim(\mathbb{W}) - \dim(\mathbb{U} \cap \mathbb{W})$. $\square$

Definition 1.6. When $\mathbb{U} \cap \mathbb{W} = \{0\}$, it is called the **internal direct sum** of $\mathbb{U}$ and $\mathbb{W}$. Notation: $\mathbb{U} \oplus \mathbb{W}$.

Remark 1.10. If $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$, then $\mathbb{W}$ is called **complement** of $\mathbb{U}$.

Theorem 1.17

Let $\mathbb{U}, \mathbb{W}$ be two subpaces of $\mathbb{V}$. Then $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$ iff for each $v \in \mathbb{V}$, there exists unique $u \in \mathbb{U}$ and there exists unique $w \in \mathbb{W}$ such that $v = u + w$.

Proof. First we assume that $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$, that means, $\mathbb{U} \cap \mathbb{W} = \{0\}$. Let $x \in \mathbb{V}$. Then there exist $x_1 \in \mathbb{U}$ and $x_2 \in \mathbb{W}$ such that $x = x_1 + x_2$. Now we show that x_1 and x_2 are unique. Suppose there two vectors $y_1(\neq x_1) \in \mathbb{U}$ and $y_2(\neq y_1) \in \mathbb{W}$ such that $x = y_1 + y_1$. Then $x_1 + x_2 = y_1 = y_2$ this implies $x_1 - y_1 = y_2 - x_2$. Therefore $x_1 - y_1, y_2 - x_2 \in \mathbb{U} \cap \mathbb{W}$. This is only possible when $x_1 - y_1 = y_2 - x_2 = 0$. Then $x_1 = y_1$ and $x_2 = y_2$.

Conversely, $v \in \mathbb{V}$, there exists unique $u \in \mathbb{U}$ and there exists unique $w \in \mathbb{W}$ such that $v = u + w$. If we show that $\mathbb{U} \cap \mathbb{W} = \{0\}$, then we are done. Let $x \in \mathbb{U} \cap \mathbb{W}$. Then $x = x + 0$ where $x \in \mathbb{U}$ and $0 \in \mathbb{W}$, and $x = 0 + x$ where $0 \in \mathbb{U}$ and $x \in \mathbb{W}$. This is possible only when $x = 0$ otherwise the hypothesis is wrong. Then $\mathbb{U} \cap \mathbb{W} = \{0\}$. Hence $\mathbb{V} = \mathbb{U} \oplus \mathbb{W}$. □

The following definition says that how to create a new vector space from a two vector spaces over the same field.

Definition 1.7. Let $\mathbb{V}, \mathbb{W}$ be vector spaces over $\mathbb{F}$. $\mathbb{V} \times \mathbb{W}$ is Cartesian product of $\mathbb{V}$ and $\mathbb{W}$. Addition and and scalar multiplication on $\mathbb{V} \times \mathbb{W}$ defined by

$$(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2)$$

and

$$\alpha(v, w) = (\alpha v, \alpha w), \alpha \in \mathbb{F}$$

.

Exercise 1.2. *Then* $\mathbb{V} \times \mathbb{W}$ *is a vector space over* $\mathbb{F}$.

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2 Inner Product Space

3 Linear Transformation

4 Eigenvalues and Eigenvectors

5 Types of Operators