DERIVATIVE BOUNDARY CONDITIONS:

Consider the third kind BCs:

$$a_0 u(a) - a_1 u'(a) = \gamma_1$$

$$b_0 u(b) + b_1 u'(b) = \gamma_2$$

The finite difference approximation of the linear differential equation $-u'' + p(x) u' + q(x) u = r(x)$ gives:

$$A_j u_{j-1} + B_j u_j + C_j u_{j+1} = \frac{h^2}{2} r(x_j) \quad j = 1, 2, \dots, N \qquad (1)$$

System (1) contains $(N+2)$ unknowns. We have only N-equations. We need to have two more equations in order to solve the system uniquely.

Discretizing the BCs using a second order approximation we get at $x = x_0$:

$$a_0 u_0 - a_1 \frac{u_1 - u_{-1}}{2h} = \gamma_1$$

or $$u_{-1} = u_1 + \frac{2h}{a_1}(\gamma_1 - a_0 u_0) = -\frac{2a_0 h}{a_1} u_0 + u_1 + \frac{2h}{a_1}\gamma_1$$

At $x = x_{N+1}$:

$$b_0 u_{N+1} + b_1 \frac{u_{N+2} - u_N}{2h} = \gamma_2$$

$$\Rightarrow \quad u_{N+2} = u_N - \frac{2hb_0}{b_1} u_{N+1} + \frac{2h}{b_1}\gamma_2$$

Here u_{-1} and u_{N+2} are the approximations at x_{-1} and x_{N+2}.

The nodes x_{-1} and x_{N+2} lie outside the interval $[a,b]$ and are called fictitious nodes.

Now if we assume that the approximation ① holds at $x = x_0$ & $x = x_{N+1}$ and use the value of fictitious nodes from above, we get

<u>$j=0$:</u>

$$A_0 u_{-1} + B_0 u_0 + C_0 u_1 = \frac{h^2}{2} r(x_0)$$

$$\Rightarrow A_0\left[-\frac{2a_0h}{a_1}u_0 + u_1 + \frac{2h}{a_1}\gamma_1\right] + B_0 u_0 + C_0 u_1 = \frac{h^2}{2} r(x_0)$$

$$\Rightarrow \left(B_0 - \frac{2ha_0}{a_1}A_0\right)u_0 + (A_0 + C_0)u_1 = \frac{h^2}{2} r(x_0) - \frac{2h}{a_1}\gamma_1 A_0$$

<u>$j = N+1$:</u>

$$A_{N+1}u_N + B_{N+1}u_{N+1} + C_{N+1}u_{N+2} = \frac{h^2}{2}r(x_{N+1})$$

$$\Rightarrow A_{N+1}u_N + B_{N+1}u_{N+1} + C_{N+1}\left[u_N - \frac{2hb_0}{b_1}u_{N+1} + \frac{2h}{b_1}\gamma_2\right] = \frac{h^2}{2}r(x_{N+1})$$

$$\Rightarrow [A_{N+1} + C_{N+1}]u_N + \left[B_{N+1} - \frac{2hb_0}{b_1}C_{N+1}\right]u_{N+1} = \frac{h^2}{2}r(x_{N+1}) - \frac{2h}{b_1}\gamma_2 C_{N+1}$$

ALTERNATIVE APPROACH: (WITHOUT USING FICTITIOUS POINTS)

We can discretize BCs using forward and backward difference formula:

For $j=0$:

$$a_0 u_0 - a_1\left[\frac{u_1-u_0}{h}\right] = \gamma_1$$

FORWARD DIFFERENCE

or

$$[a_0 h + a_1]u_0 - a_1 u_1 = h\gamma_1$$

For $j=N+1$:

$$b_0 u_{N+1} + b_1\left[\frac{u_{N+1}-u_N}{h}\right] = \gamma_2$$

or

$$[b_0 h + b_1]u_{N+1} - b_1 u_N = h\gamma_2$$

Since the difference approximation used there are of first order, the method may not retain the second order.

SOLUTION OF TRIDIAGONAL SYSTEM:

a) Gauss elimination may be used.

However for a tridiagonal system much cheaper algorithm like Thomas Algorithm may be used.

THOMAS ALGORITHM:

Consider

$$\underbrace{\begin{bmatrix} b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & a_3 & b_3 & c_3 \\ 0 & 0 & a_4 & b_4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}}_{f}$$

The matrix A can be factorize as

$$A = LU$$

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & a_3 & b_3 & c_3 \\ 0 & 0 & a_4 & b_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta_2 & 1 & 0 & 0 \\ 0 & \beta_3 & 1 & 0 \\ 0 & 0 & \beta_4 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 & c_1 & 0 & 0 \\ 0 & \alpha_2 & c_2 & 0 \\ 0 & 0 & \alpha_3 & c_3 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & c_1 & 0 & 0 \\ \beta_2\alpha_1 & \beta_2 c_1 + \alpha_2 & c_2 & 0 \\ 0 & \beta_3\alpha_2 & \beta_3 c_2 + \alpha_3 & c_3 \\ 0 & 0 & \beta_4\alpha_3 & \beta_4 c_3 + \alpha_4 \end{bmatrix}$$

Comparison gives

$$\alpha_1 = b_1 \qquad \beta_2 = \frac{a_2}{\alpha_1} \qquad \alpha_2 = b_2 - \beta_2 c_1$$

$$\beta_3 = \frac{a_3}{\alpha_2} \qquad \alpha_3 = b_3 - \beta_3 c_2$$

$$\beta_4 = \frac{a_4}{\alpha_3} \qquad \alpha_4 = b_4 - \beta_4 c_3$$

In general for $n \times n$ matrix:

$$\alpha_1 = b_1 \qquad \beta_i = \frac{a_i}{\alpha_{i-1}} \qquad \alpha_i = b_i - \beta_i c_{i-1}$$

$i = 2, 3, \cdots, n.$

We have

$$L \underbrace{U x}_{=: y} = f$$

Solve:

$$Ux = y \quad \& \quad Ly = f$$

$$Ly = f \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta_2 & 1 & 0 & 0 \\ 0 & \beta_3 & 1 & 0 \\ 0 & 0 & \beta_4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

$$\Rightarrow \quad y_1 = r_1$$

$$y_2 = r_2 - \beta_2 y_1$$

$$y_3 = r_3 - \beta_3 y_2$$

$$y_4 = r_4 - \beta_4 y_3$$

In general: $y_i = r_i - \beta_i y_{i-1}, \quad i = 2, \cdots, n.$

$$Ux = y \Rightarrow \begin{bmatrix} \alpha_1 & c_1 & 0 & 0 \\ 0 & \alpha_2 & c_2 & 0 \\ 0 & 0 & \alpha_3 & c_3 \\ 0 & 0 & 0 & \alpha_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$x_4 = \frac{y_4}{\alpha_4} \qquad x_i = \frac{y_i - c_i x_{i+1}}{\alpha_i} \qquad i = 3, 2, 1.$$

general form:

$$x_n = \frac{y_n}{\alpha_n}; \qquad x_i = \frac{y_i - c_i x_{i+1}}{\alpha_i}; \quad i = n-1, \cdots, 1.$$