Example: The following IVP is given as

$$y' = t + y \qquad y(0) = 1.$$

a) If the error in $y(t)$ obtained from the first four terms of the Taylor series is to be less than 1×10^{-6}, find t.

b) Determine the number of terms, in the Taylor series required to obtain results with error less than 5×10^{-6} for $t \le 0.1$

c) Use Taylor's series method (second order) to get $y(0.3)$ with step size $h = 0.1$.

<u>Solution</u>: a) $y' = t + y \Rightarrow y'(0) = 0 + 1 = 1$

$$y'' = 1 + y' \Rightarrow y''(0) = 1 + 1 = 2$$

$$y''' = y'' \Rightarrow y'''(0) = 2$$

$$\vdots$$

$$y^{(r)} = y^{(r-1)} \Rightarrow y^{(r)}(0) = 2, \quad r = 2, 3, \dots$$

Writting the full Taylor series

$$y(t) = y(0) + t\, y'(0) + \frac{t^2}{\lfloor 2} y''(0) + \frac{t^3}{\lfloor 3} y'''(0) + \dots + \frac{t^p}{\lfloor p} y^{(p)}(0) + \dots$$

$$= 1 + t + \frac{t^2}{\lfloor 2} 2 + \frac{t^3}{\lfloor 3} \cdot 2 + \dots + \frac{t^p}{\lfloor p} 2 + \dots$$

We further note that

$$y^{(p+1)}(t) = 2 + 2 \cdot \frac{t}{\lfloor 1} + 2 \frac{t^2}{\lfloor 2} + \dots$$

$$y^{(p+1)}(t) = 2e^t$$

The error relationship gives :

$$\left|\frac{t^4}{\lfloor 4} y^{(4)}(\xi)\right| < 10^{-6} \quad ; \quad \xi \in [0,t)$$

$$\Rightarrow \frac{t^4}{\lfloor 4} \cdot 2 \cdot e^{t} < 10^{-6} \qquad \text{as } y^4(\xi) < 2e^{t}$$

$$\Rightarrow t^4 e^{t} < 12 \times 10^{-6}$$

$$\Rightarrow t^4 e^{t} < 1.2 \times 10^{-5}$$

$$\Rightarrow \boxed{t \leq 0.058}$$

b) Again with the error formula, we have

$$\frac{t^{p+1}}{\lfloor p+1} \max_{t \in [0,0.1]} \left|y^{(p+1)}(t)\right| < 5 \times 10^{-6}$$

$$\Rightarrow \frac{(0.1)^{p+1}}{\lfloor p+1} \cdot 2e^{0.1} < 5 \times 10^{-6}$$

$$\Rightarrow \frac{\lfloor p+1}{(0.1)^{p+1}} > \frac{2e^{0.1}}{5} \times 10^{6} = 4.42 \times 10^{5}$$

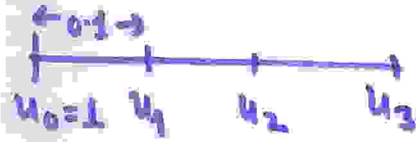
$$\Rightarrow p \geq 4$$

The number of terms required $= 5$.

c) The second order Taylor series method is given as

$$u_{n+1} = u_n + h\,u_n' + \frac{h^2}{2!}u_n'' \quad ; \quad n = 0, 1, 2$$

$$u_n' = t_n + u_n$$



$$u_n'' = 1 + u_n' = 1 + t_n + u_n$$

$\underline{n=0}$:

$$u_1 = u_0 + h\,u_0' + \frac{h^2}{2!}u_0''$$

$$= 1 + 0.1\times(0+1) + \frac{(0.1)^2}{2!}\cdot(1+0+1)$$

$$= 1 + 0.1 + \frac{0.01}{2}\times 2 = 1.11$$

$\underline{n=1}$:

$$u_2 = u_1 + h\,u_1' + \frac{h^2}{2!}\cdot u_1''$$

$$= 1.11 + (0.1)\times(0.1+1.11) + \frac{(0.1)^2}{2!}(1+0.1+1.11)$$

$$= 1.24205$$

$n=2$:

$$u_3 = u_2 + h\,u_2' + \frac{h^2}{2!}u_2''$$

$$= 1.24205 + 0.1\times(0.2+1.24205) + \frac{0.1^2}{2!}(1+0.2+1.24205)$$

$$= 1.39846525$$

t	exact y	Numerical y
0.1	1.110341836	1.11
0.2	1.242805516	1.24205
0.3	1.399717615	1.39846525

exact solution

$$y = -1 - t + 2e^t$$