Linear Algebra

SYSTEMS OF LINEAR EQUATIONS:

A completely general system of m linear equations with n unknowns $x_1, x_2, \dots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \;\;\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Consider these four arrays:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

All of these arrays are examples of matrices.

$$\boxed{\text{A matrix is a rectangular array of numbers}}$$

The above system of equations can be expressed in matrix form as

$$Ax = b$$

A is called coefficient matrix & b is called right hand side vector.

To track of the numerical data we define augmented matrix:

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right].$$

Def: A system of equations is consistent if it has at least one solution, and inconsistent if it has no solution.

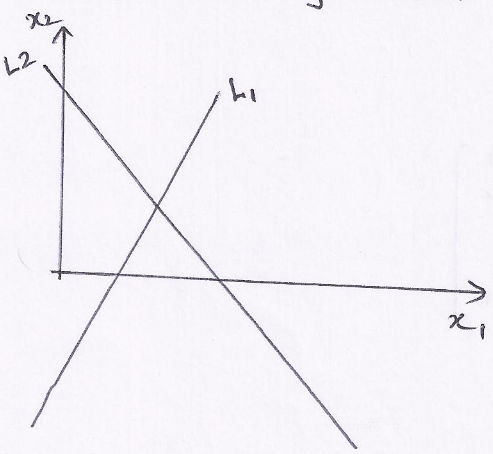
SOLUTION OF SYSTEM OF LINEAR EQUATIONS:

consider the system of two unknowns:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}\right\} \text{ represents straight lines}$$

Possible cases of solution:

a)



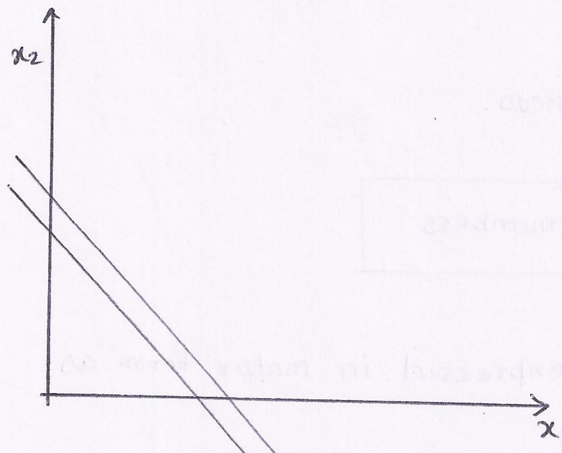
Unique solution

lines intersect

Ex: $-x+y=1$

$2x+y=4$

b)



Lines are parallel.

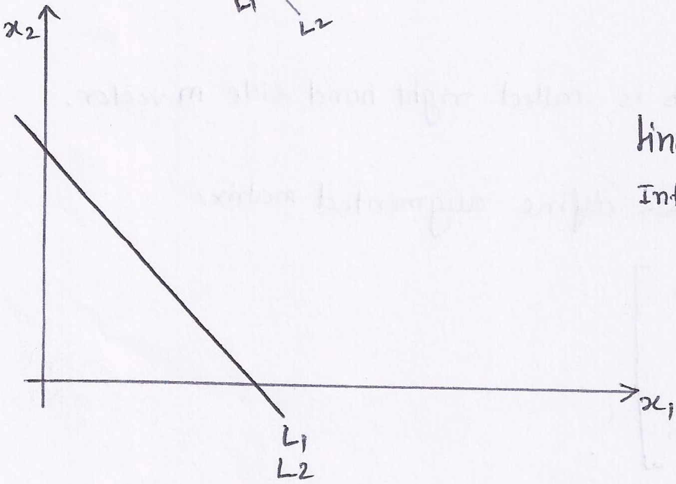
NO SOLUTION

Ex:

$-x+y=1$

$2x-2y=4$

c)



lines coinside

Infinitely many solutions

Ex $-x+y=1$

$2x-2y=-2$

similarly case of three unknowns can be interpreted with the help of plane (Hyperplanes).

SOLUTION METHODS:

a) Method of determinants — Cramer's rule
b) Matrix inversion method $Ax = b \Rightarrow x = A^{-1}b$.
→ c) Gauss Elimination method

} Direct method

→ d) Numerical method (iterative method) Jacobi and Gauss seidel method

GAUSS ELIMINATION METHOD:

Ex:

$$6x + 4y = 2 \quad -(1)$$
$$3x - 5y = -34 \quad -(2)$$

STEP I: multiply eq. (1) by $\frac{1}{2}$ and substract it from (2)

$$6x + 4y = 2 \quad -(3)$$
$$-7y = -34-1 \quad -(4)$$

STEP II: Solution:

$$y = 5$$
$$x = (2-4*5)/6$$
$$= -3$$

In augmented matrix form:

$$\left[\begin{array}{cc|c} 6 & 4 & 2 \\ 3 & -5 & -34 \end{array}\right]$$ → corresponding to system (1) & (2)

$R_2 \to R_2 - \frac{1}{2}R_1$ ↓

$$\left[\begin{array}{cc|c} 6 & 4 & 2 \\ 0 & -7 & -35 \end{array}\right]$$ → corresponding to the system (3) & (4)

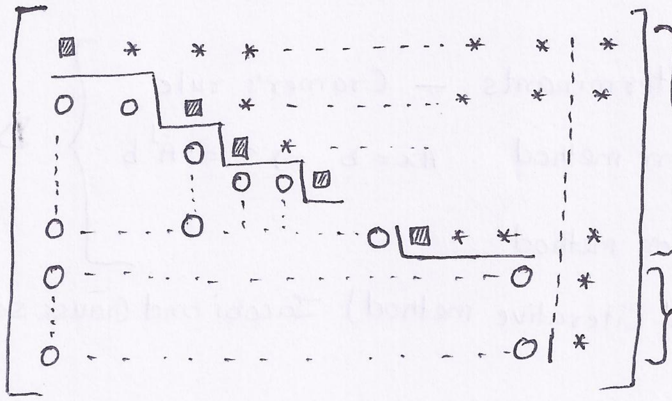
↳ this is called Echelon form.

In short:

$$[A|b] \xrightarrow[\text{Elimination}]{\text{Gauss}} [A'|b'] \quad \text{(Echelon form)}$$

ECHELON FORM:

$[A'|b'] =$



Pivot rows (r)

(m-r) null rows (zero rows)

▨ Pivot element $\neq 0$

$*$ other elements (can be 0)

Example: Solve the equations:

$$x_1 + x_2 + x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + x_2 + 3x_3 = 13.$$

Augmented matrix

$$[A|b] = \tilde{A} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array}\right]$$

$R_2 \to R_2 - 3R_1.$
$R_3 \to R_3 - 2R_1.$

$$\tilde{A} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$R_2 \Leftrightarrow R_1$

$$\tilde{A} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Back substitution:

$$x_3 = 2$$
$$x_2 = 1$$
$$x_1 = 3.$$

Ex:

$$\begin{aligned} 4y + 3z &= 8 \\ 2x \quad - z &= 2 \\ 3x + 2y \quad &= 5 \end{aligned} \iff \begin{bmatrix} 0 & 4 & 3 \\ 2 & 0 & -1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}.$$

$$\tilde{A} = [A|b] = \left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

$$R_3 \to R_3 - \tfrac{3}{2}R_1 \quad \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & \boxed{4} & 3 & 8 \\ 0 & 2 & \frac{3}{2} & 2 \end{array}\right]$$

$$R_3 \to R_3 - \frac{R_2}{2} \quad \sim \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

This shows that the system has no solution. (last equation $0 = -2$)

Question: What will happen if

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} \boxed{2} & 0 & -1 & 2 \\ 0 & \boxed{4} & 3 & 8 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

$\Rightarrow$ x_3 can be taken arbitrarily.

let us take $x_3 = \alpha$. then $x_2 = \frac{1}{4}(8 - 3\alpha)$ & $x_1 = \dfrac{2+\alpha}{2}$

One can also write in vector form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} \frac{1}{2} \\ -3/4 \\ 1 \end{bmatrix}.$$

ELEMENTARY ROW OPERATIONS OR TRANSFORMATIONS for MATRICES:

1. interchange of the ith and jth rows $R_i \leftrightarrow R_j$

2. multiplication of the ith row by a non-zero number λ

$$R_i \rightarrow \lambda R_i$$

3. addition of λ times the jth row to the ith row

$$R_i \rightarrow R_i + \lambda R_j$$

EQUIVALANCE OF MATRICES:

If B be $m \times n$ matrix obtained from $m \times n$ matrix A by finite number of elementary transformation of A, then A is called equivalent to B, denoted by $A \sim B$. (A is equivalent to B).

PROPERTIES OF AN EQUIVALANCE RELATION: $\sim$:

(i) Reflexivity: $A \sim A$

(ii) Symmetry: if $A \sim B$ then $B \sim A$

(iii) Transitivity: if $A \sim B$, $B \sim C$ then $A \sim C$.

EXAMPLE: Solve the system of equations $Ax = b$ with

$$[A|b] = \begin{bmatrix} \boxed{1} & 2 & -2 & -1 & 1 & 1 \\ 2 & 4 & -4 & 0 & 3 & 2 \\ -1 & -2 & 3 & 3 & 4 & 3 \\ 3 & 6 & -7 & 1 & 1 & \beta \end{bmatrix} \qquad \beta \in \mathbb{R}.$$

$$\begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \\ R_4 \rightarrow R_4 - 3R_1 \end{matrix} \quad \sim \begin{bmatrix} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & -1 & 4 & -2 & \beta-3 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \sim \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 4 & -2 & \beta-3 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccccc:c} \boxed{1} & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 5 & 4 \\ 0 & 0 & 0 & \boxed{2} & 1 & 0 \\ 0 & 0 & 0 & 6 & 3 & \beta+1 \end{array}\right]$$

$R_4 \to R_4 - 3R_3$

$$\sim \left[\begin{array}{ccccc:c} 1 & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 5 & 4 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta+1 \end{array}\right]$$

Case I: $\beta \neq -1$: No solution

Case II: $\beta = -1$:

$$\begin{array}{cccccc} x_1 & & x_3 & x_4 & & \\ \boxed{1} & 2 & -2 & -1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 2 & 5 & 4 \\ 0 & 0 & 0 & \boxed{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & \uparrow & & & \uparrow & \\ & x_2 & & & x_5 & \end{array}$$

Take $x_2 = \alpha_1$ & $x_5 = \alpha_2$

$x_4 = -\frac{1}{2}\alpha_1 \qquad x_3 = 4 - 4\alpha_2$

$x_1 = 9 - 2\alpha_1 - 9.5\alpha_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -9.5 \\ 0 \\ 4 \\ -0.5 \\ 1 \end{bmatrix}. \qquad \alpha_1, \alpha_2 \in \mathbb{R}$$

Assignment: Investigate the values of λ and μ so that the equations

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

have (i) no solution (ii) a unique solution and (iii) an infinite number of solutions.