Method of variation of parameters

Consider the following second order non-homogeneous linear equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x) \tag{1}$$

Let $y = c_1y_1 + c_2y_2$, with c_1 and c_2 as arbitrary constants, be the general solution of the homogeneous equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$$

We assume that

$$y = C_1y_1 + C_2y_2 \tag{2}$$

is the general solution of the non-homogeneous equation (1), where C_1 and C_2 are functions of x to be so chosen that (1) is satisfied.

Differentiating (2) we get

$$y^{'} = C_1y_1^{'} + C_2y_2^{'} + \underbrace{C_1^{'}y_1 + C_2^{'}y_2}_{=0} \tag{3}$$

For simplicity, in order to find C_1 and C_2 we assume that

$$C_1^{'}y_1 + C_2^{'}y_2 = 0 \tag{4}$$

Differentiating (3) again,

$$y^{''} = C_1y_1^{''} + C_2y_2^{''} + C_1^{'}y_1^{'} + C_2^{'}y_2^{'} \tag{5}$$

Substituting y, y' and y'' in (1) we get

$$C_1\left(y_1^{''} + a_1y_1^{'} + a_2y_1\right) + C_2\left(y_2^{''} + a_1y_2^{'} + a_2y_2\right) + C_1^{'}y_1^{'} + C_2^{'}y_2^{'} = f(x)$$

$$\implies \quad C_1^{'}y_1^{'} + C_2^{'}y_2^{'} = f(x) \tag{6}$$

Solving the equations (4) and (6):

$$C_1^{'} = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2^{'} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix}} = -\frac{y_2f(x)}{W}$$

Here W is called Wronskian. It is non-zero because y_1 and y_2 are linearly independent. Similarly

$$C_2^{'} = \frac{\begin{vmatrix} y_1 & 0 \\ y_1^{'} & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix}} = \frac{y_1 f(x)}{W}$$

After integrating:

$$C_1 = \int -\frac{y_2 f(x)}{W} dx + d_1 \quad \text{and} \quad C_2 = \int \frac{y_1 f(x)}{W} dx + d_2$$

Hence the general solution of the non-homogeneous equation

$$\boxed{y = d_1 y_1 + d_2 y_2 + y_1 \int -\frac{y_2 f(x)}{W} dx + y_2 \int \frac{y_1 f(x)}{W} dx}$$

Example: Apply the method of variation of parameters to solve

$$\frac{d^2 y}{dx^2} - y = \frac{2}{1+e^x} \tag{7}$$

Solution:

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

Let $y = C_1 e^x + C_2 e^{-x}$ be the general solution of the given equation.

$$y' = C_1 e^x - C_2 e^{-x} + \underbrace{C_1^{'} e^x + C_2^{'} e^{-x}}_{=0}$$

$$y'' = C_1 e^x + C_2 e^{-x} + C_1^{'} e^x - C_2^{'} e^{-x}$$

Substituting in (7)

$$C_1^{'} e^x - C_2^{'} e^{-x} = \frac{2}{1+e^x}$$

The Wronskian

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

Hence

$$C_1 = -\frac{1}{2} \int -e^{-x} \frac{2}{1+e^x} dx + d_1 = \int \frac{e^{-x}}{1+e^x} dx + d_1$$

Substitute $e^x = z \Rightarrow e^x dx = dz$

$$C_1 = \int \frac{1}{z^2(1+z)} dz + d_1 = \int \frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z} dz + d_1$$

$$C_1 = -\frac{1}{z} - \ln z + \ln(1+z) + d_1 = -e^{-x} - x + \ln(1+e^x) + d_1$$

Similarly

$$C_2 = -\frac{1}{2}\int e^x \frac{2}{1+e^x} dx + d_1 = -\ln(1+e^x) + d_2$$

The general solution of the differential equation

$$\boxed{y = d_1 e^x + d_2 e^{-x} - 1 - x e^x + (e^x - e^{-x})\ln(1+e^x)}$$

Cauchy-Euler Equations

A linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X \tag{8}$$

or

$$(x^n D^n + a_1 D^{n-1} + \cdots + a_n) y = X \tag{9}$$

is called Euler-Cauchy equation.

Working Rule: To solve equation (8) we change the variable from x to z by putting $x = e^z$ i.e. $z = \ln(x)$.

$$z = \ln(x) \Rightarrow \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dz}}$$

We define a new operator

$$x\frac{d}{dx} \equiv \frac{d}{dz} \equiv D_1$$

Again

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D_1(D_1 - 1)y$$

Thus we have the following formulas for $D \equiv \frac{d}{dx}$ and $D_1 \equiv \frac{d}{dz}$

$$\begin{aligned}
&xD = D_1 \\
&x^2D^2 = D_1(D_1 - 1) \\
&x^3D^3 = D_1(D_1 - 1)(D_1 - 2) \\
&\vdots \\
&x^nD^n = D_1(D_1 - 1)(D_1 - 2)\ldots(D_1 - n + 1)
\end{aligned}$$

Substituting these operator relations in the equation (9), we obtain a linear differential equation with constant coefficient

$$f(D_1)y = Z, \quad \text{where Z becomes a function of z only}$$

Example 1.

$$(x^2D^2 - xD + 2)y = x \ln x \tag{10}$$

Let $x = e^z$ so that $z = \ln x$ and $D_1 \equiv \frac{d}{dz}$ then the equation (10) becomes

$$[D_1(D_1 - 1) - D_1 + 2]\, y = ze^z$$

Auxiliary equation $m^2 - 2m + 2 = 0$ and its roots are $m = 1 \pm i$
Hence

C.F. $= e^z\, [c_1 \cos(z) + c_2 \sin(z)] = x\, [c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))]$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D_1^2 - 2D_1 + 2} ze^z = e^z \frac{1}{(D_1+1)^2 - 2(D_1+1) + 2} z \\ &= e^z \frac{1}{D_1^2 + 1} z = e^z (1 + D_1^2)^{-1} z = e^z z = x \ln(x) \end{aligned}$$

General solution

$$\boxed{y = x\, [c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))] + x \ln(x)}$$

Equations reducible to Euler-Cauchy form There can be several forms of equation which can be reduced to Euler-Cauchy form

Example 1: Solve

$$\frac{d^3y}{dx^3} - \frac{4}{x}\frac{d^2y}{dx^2} + \frac{5}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 1$$

Solution: $y = c_1x^2 + c_2x^{(5+\sqrt{21})/2} + c_3x^{(5-\sqrt{21})/2} - x^3/5$

Example 2:

$$2x^2y\frac{d^2y}{dx^2} + 4y^2 = x^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}$$

Hint: $y = z^2$

Solution:

$$y = z^2 \Rightarrow \frac{dy}{dx} = 2z\frac{dz}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\left(\frac{dz}{dx}\right)^2 + 2z\frac{d^2z}{dx^2}$$

Substituting these values in the differential equation we get

$$x^2\frac{d^2z}{dx^2} - x\frac{dz}{dx} + z = 0$$

or

$$[x^2D^2 - xD + 1]z = 0$$

Substitute $x = e^t \Leftrightarrow \ln x = t$

$$\Rightarrow x\frac{dz}{dx} = \frac{dz}{dt} \Rightarrow xD \equiv D_1$$

Similarly

$$x^2D^2 = D_1(D_1 - 1)$$

Then the equation becomes

$$[D_1^2 - 2D_1 + 1]z = 0 \Rightarrow z = [c_1 + c_2t]e^t$$

$$\Rightarrow z = [c_1 + c_2\ln(x)]x$$

$$\Rightarrow \boxed{y = (c_1 + c_2\ln(x))^2x^2}$$

Example 3: A differential equation of the form

$$(a+bx)^n\frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_ny = X$$

can be reduced to Euler-Cauchy equation by putting

$$a + bx = v \Rightarrow \frac{dv}{dx} = b$$

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = b\frac{dy}{dv}$$

Again

$$\frac{d^2y}{dx^2} = b^2\frac{d^2y}{dv^2} \text{ or in general } \frac{d^ny}{dx^n} = b^n\frac{d^ny}{dv^n}$$

Substituting these derivatives in the equation, we get

$$v^n \frac{d^n y}{dv^n} + \frac{a_1}{b} v^{n-1} \frac{d^{n-1} y}{dv^{n-1}} + \cdots + \frac{a_{n-1}}{b^{n-1}} v \frac{dy}{dv} + \frac{a_n}{b^n} y = \frac{X}{b^n}$$

which is an standard Euler-Cauchy equation.

Example 4: solve

$$(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos \ln(1+x)$$

Solution: Let $(1+x) = v \Rightarrow \frac{dv}{dx} = 1$.
Hence $\frac{dy}{dx} = \frac{dy}{dv}$ and $\frac{d^2y}{dx^2} = \frac{d^2y}{dv^2}$ and the differential equation becomes

$$v^2 \frac{d^2 y}{dv^2} + v \frac{dy}{dv} + y = 4 \cos \ln v$$

Put $v = e^z \Rightarrow \ln(v) = z$ and let $D_1 \equiv \frac{d}{dz}$

$$[D_1(D_1 - 1) + D_1 + 1]\, y = 4 \cos z$$

$$\left(D_1^2 + 1\right) y = 4 \cos z$$

$$\begin{aligned} \text{C.F.} = c_1 \cos(z) + c_2 \sin(z) &= c_1 \cos(\ln v) + c_2 \sin(\ln v) \\ &= c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+x)) \end{aligned}$$

$$\text{P.I.} = 2z \sin z = 2 \ln(v) \sin(\ln(v)) = 2 \ln(1+x) \sin(\ln(1+x)).$$

The general solution

$$\boxed{y = c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+x)) + 2 \ln(1+x) \sin(\ln(1+x)).}$$