# Assignment 3



1\. (a) $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

The characteristic polynomial is $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

ie. $\lambda^3 - 7\lambda^2 + 36 = 0$

The eigen values are, $\lambda = -2, 3, 6$.

Now, $(A - \lambda I)X = 0$, where $X = (x, y, z)^T$

$\underline{\lambda = -2}$, $(A + 2I)X = 0$ gives

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - \frac{1}{3}R_1} \begin{pmatrix} 3 & 1 & 3 \\ 0 & 20/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

System of eqns. reduces to $\begin{pmatrix} 3 & 1 & 3 \\ 0 & 20/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

which gives $(x, y, z) = c(-1, 0, 1)$, $c \neq 0$, $c \in \mathbb{R}$

$\therefore$ The eigenvectors corresponding to $\lambda = -2$ are

$c(-1, 0, 1)$

Similarly, the eigen vectors corresponding to the eigen values 3 & 6 are $c(1, -1, 1)$ and $c(1, 2, 1)$ respectively.

(b) $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$. The characteristic polynomial is

$\lambda^3 - 12\lambda - 16 = 0$.

The eigen values are, $\lambda = -2, -2, 4$.

For, $\underline{\lambda = -2}$, $(A - \lambda I)X = 0$ gives

$$\begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{pmatrix} 3 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The reduced system is $x - y + z = 0$.

take $y = c$, $z = d$, $x = c - d$

$\therefore$ $(x, y, z) = (c-d, c, d) = c(1,1,0) + d(-1,0,1)$
where $c, d \in \mathbb{R}$ & $(c,d) \neq (0,0)$

$\therefore$ Two linearly independent eigen vectors corresponding to $\lambda = -2$ are $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda = 4}$ $(A - \lambda I)X = 0$ gives

$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \xrightarrow[R_3 + 2R_1]{R_2 + R_1} \begin{pmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

Reduced system is $\begin{pmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

or, $-x - y + z = 0$
$-2y + z = 0$

take $y = c$; then $z = 2c$, $x = c$
i.e. $(x, y, z) = c(1, 1, 2)$, $c \in \mathbb{R}$, $c \neq 0$

$\therefore$ The eigen vectors corresponding to $\lambda = 4$ are $c(1, 1, 2)$.

(c) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. The characteristic eqn is

$$\lambda^3 - 1 = 0.$$

The eigen values are

$$\lambda = 1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2} i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

For $\lambda = 1$, $(A - \lambda I)X = 0$ gives

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \xrightarrow{R_3 + R_1} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

System of eqns reduced to

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or, $-x + y = 0$

$-y + z = 0$, $z = c$,

ie $x = y = z = c$; say, where $c \in \mathbb{R}, c \neq 0$

$\therefore (x, y, z) = c(1, 1, 1)$.

$\therefore$ The eigen vectors corresponding to $\lambda = 1$ are $c(1, 1, 1)$.

For $\lambda = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$(A - \lambda I)X = 0$ gives

$$\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2} i & 1 & 0 \\ 0 & \frac{1}{2} - \frac{\sqrt{3}}{2} i & 1 \\ 1 & 0 & \frac{1}{2} - \frac{\sqrt{3}}{2} i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2}-\frac{\sqrt{3}}{2}i & 1 & 0 \\ 0 & \frac{1}{2}-\frac{\sqrt{3}}{2}i & 1 \\ 1 & 0 & \frac{1}{2}-\frac{\sqrt{3}}{2}i \end{pmatrix} \xrightarrow{R_3 - \frac{1}{\frac{1}{2}-\frac{\sqrt{3}}{2}i}R_1} \begin{pmatrix} \frac{1}{2}-\frac{\sqrt{3}}{2}i & 1 & 0 \\ 0 & \frac{1}{2}-\frac{\sqrt{3}}{2}i & 1 \\ 0 & -\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) & \frac{1}{2}-\frac{\sqrt{3}}{2} \end{pmatrix}$$

i.e. $\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)x + y = 0$

$\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)y + z = 0$

$x + \left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)z = 0.$

Solving, we get,

$$(x, y, z) = c\left(1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i\right),$$

$c \in \mathbb{R}, c \neq 0.$

$\therefore$ The eigen vectors corresponding to $\lambda = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$ are

$$c\left(1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i\right).$$

Similarly, for $\lambda = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$; the corresponding eigen vectors are $c\left(1, -\frac{1}{2}-\frac{\sqrt{3}}{2}i, -\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$.

2. (a) $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$. The characteristic equ$^n$ is

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

and eigen values are $\lambda = 1, 1, -1$

1 is an eigen value of algebraic multiplicity 2 and $-1$ is of algebraic multiplicity 1.

$\underline{\lambda = 1}$ $\quad (A - \lambda I)X = 0$ gives,

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & -1 \\ 3 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

③

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & -1 \\ 3 & 2 & -3 \end{pmatrix} \xrightarrow{R_{12}} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \\ 3 & 2 & -3 \end{pmatrix} \xrightarrow{R_3 - 3R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow{R_3 - R_2}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Reduced system of equns. is

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

ie. $x + y - z = 0$

$-y = 0.$

take, $z = c$, $x = c$, $y = 0$ , $c \neq 0$, $c \in \mathbb{R}$

$\therefore \quad (x, y, z) = c(1, 0, 1)$

$\therefore$ The eigen vectors corresponding to $\lambda = 1$ are $c(1, 0, 1)$.

The rank of the characteristic subspace is 1.

$\therefore$ The geometric multiplicity of $\lambda = 1$ is 1.

$\underline{\lambda = -1}$ in a similar way, the eigen vectors corresponding to $\lambda = -1$ are $c(1, 2, 7)$; $c \neq 0$, $c \in \mathbb{R}$

Therefore the geometric multiplicity of $\lambda = -1$ is 1.

b) $A = \begin{pmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{pmatrix}$. The characteristic equation is

$(\lambda - 1)(\lambda^2 - \lambda) = 0$

The eigen ~~vector~~ values are $\lambda = 1, 1, 0$.

$\therefore$ The eigen value 1 is of algebraic multiplicity 2 and 0 is of algebraic multiplicity 1.

$\underline{\lambda = 1}$, the corresponding eigen vectors are

$$c(1,0,0) + d(2,-3,0) \quad ; \quad \begin{array}{l} c, d \in \mathbb{R} \\ (c,d) \neq (0,0) \end{array}$$

The rank of the characteristic subspace is 2.

Therefore, geometric multiplicity of $\lambda = 1$ is 2.

$\underline{\lambda = 0}$, the corresponding eigen vectors are $c(-2,1,-2)$

The geometric multiplicity of $\lambda = 0$ is 1.

(3) (i) If $\lambda$ is an eigen value of $A$ and $X$ is the corresponding eigen vector, then $AX = \lambda X$ — ①

Then, $A^m X = A^{m-1}(AX) = A^{m-1}(\lambda X)$ (using ①)

$$= \lambda A^{m-1} X$$
$$= \lambda A^{m-2}(AX)$$
$$= \lambda A^{m-2}(\lambda X) \quad \text{(using ①)}$$
$$= \lambda^2 A^{m-2} X$$
$$\vdots$$
$$= \lambda^m X$$

Therefore, $\lambda^m$ is an eigen value of $A^m$.

(ii) $A$ is no-singular, $A^{-1}$ exists. $\lambda \neq 0$, therefore $\lambda^{-1}$ exists

since $\lambda$ is an eigen value of $A$ then

$$AX = \lambda X \quad \text{For } X \neq 0$$

or, $A^{-1}(AX) = A^{-1}(\lambda X)$

or, $I \cdot X = \lambda A^{-1} X$

or, $\lambda^{-1} X = \lambda^{-1} \lambda A^{-1} X$

or, $\lambda^{-1} X = A^{-1} X$

or, $A^{-1} X = \lambda^{-1} X$.

$\therefore \lambda^{-1}$ is an eigen value of $A^{-1}$.

Now, by similar ~~manner~~ manner as in (i),

$\lambda^{-m}$ is also an eigen value of $A^{-m}$.

(4) Since $\lambda$ is an eigen value of $A$, if $X$ is the corresponding eigen vector, then $AX = \lambda X$.

Now, $A^2X = A(AX) = A(\lambda X) = \lambda(AX) = \lambda^2 X$

i.e. $A^2X = \lambda^2 X$

or, $\cancel{(A^2 - \lambda^2)X = 0}$

$\cancel{(A^2 - \lambda^2)}$

or, $AX = \lambda^2 X$, since $A^2 = A$

or, $\lambda X = \lambda^2 X$

or, $(\lambda^2 - \lambda)X = 0$.

or, $\lambda^2 - \lambda = 0$, since $X$ is non zero.

or, $\lambda = 0$ or $1$.

(5) consider, $\det[(P+I)P^t] = \det(P+I) \cdot \det(P^t)$

$= \det(P+I) \cdot \det(P)$

$= -1 \cdot \det(P+I)$. —— (1)

Now, $\det[(P+I)P^t] = \det[PP^t + P^t]$

$= \det[I + P^t]$

$= \det(P+I)^t$

$= \det(P+I)$ —— (2)

From (1) & (2), $\det(P+I) = 0$

or, $\det[P - (-1)I] = 0$

This implies that, $-1$ is an eigen value of $P$.

(6.) Since, the eigenvalues of a skew-Hermitian matrix are either zero or purely imaginary, then $\lambda = 0$ or $\lambda = ic$, $c \in \mathbb{R}$

If $\lambda = 0$, $\left|\frac{1-\lambda}{1+\lambda}\right| = 1$

If $\lambda = ic$, $\left|\frac{1-ic}{1+ic}\right| = \frac{\sqrt{1+c^2}}{\sqrt{1+c^2}} = 1$.

(7.) $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. The characteristic eqn is

$$\lambda^2 - 4\lambda - 5 = 0 \quad \text{—— (i)}$$

Then, $A^2 - 4A - 5I = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix} - 4\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the Cayley-Hamilton theorem satisfied

$\therefore A^2 - 4A - 5I = 0. \quad \text{—— (ii)}$

Multiplying above eqn. by $A^{-1}$, we get

$$A - 4I - 5A^{-1} = 0$$

or, $A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$.

Now, ~~$A^5 - 4A^4$~~ $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10$

$$= (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \cancel{\lambda} + 5$$

By Cayley-Hamilton theorem

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10$

$$= (A^2 - 4A - 5I)(A^3 - 2A + 3I) + A + 5I$$

$$= A + 5I \quad \text{[using (ii)]}$$

Then $\alpha = 1$, $\beta = 5$.

**(8)** $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$

The characteristic equation is $\lambda^3 - 20\lambda + 8 = 0$

By Cayley - Hamilton theorem,

$$A^3 - 20A + 8I = 0$$

i.e. $$A^{-1} = \frac{5}{2} I - \frac{1}{8} A^2$$

$$\therefore \quad A^{-1} = \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

**(9)** $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

The characteristic equation is $\lambda^3 - \lambda^2 - \lambda + 1 = 0$.

By using Cayley - Hamilton theorem,

$$A^3 - A^2 - A + I = 0$$

or, $$A^3 - A = A^2 - I$$

Now, $$A^4 - A^2 = A(A^3 - A) = A(A^2 - I) = A^3 - A$$
$$= A^2 - I$$

$\cancel{A^5 - A^3}$ $\therefore A^4 = 2A^2 - I$.

$$A^5 - A^3 = A^2(A^3 - A) = A^2(A^2 - I) = A^4 - A^2$$
$$= A^2 - I$$

$$A^6 - A^4 = A^3(A^3 - A) = A^3(A^2 - I)$$
$$= A^5 - A^3$$
$$= A^2 - I$$

$$\therefore \quad A^6 = A^4 + A^2 - I$$
$$= 2A^2 - I + A^2 - I$$
$$= \cancel{3A} \; 3A^2 - 2I.$$

Similarly, we get the recurrence relation

$$A^{2i} = iA^2 - (i-1)I \quad \forall\, i = 1, 2, 3, \ldots$$

Putting $i = 50$,

$$A^{100} = 50A^2 - 49I$$
$$= 50\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 50 & 0 & 1 \end{pmatrix}.$$