3.7 Group Action Continued ...

Theorem 3.7.1. *Let G be a finite group acting on a set X. Let N denote the number of distinct orbits of X under the action of G. Then*

$$N = \frac{1}{|G|}\sum_{x\in X}|G_x|.$$

Proof. By Lemma 3.6.8, note that $\sum\limits_{x\in\mathcal{O}(y)}|G_x| = |G|$, for all $y \in X$. Let $x_1, x_2, \ldots, x_N$ be the representative of the distinct orbits of X under the action of G. Then

$$\frac{1}{|G|}\sum_{x\in X}|G_x| = \frac{1}{|G|}\sum_{i=1}^{N}\sum_{y\in\mathcal{O}(x_i)}|G_{x_i}| = \frac{1}{|G|}\sum_{i=1}^{N}|G| = \frac{1}{|G|}N\cdot|G| = N. \qquad \blacksquare$$

Example 3.7.2. *Let us come back to Example 3.6.3.2. Check that the number of distinct colorings are*

$$\frac{1}{|G|}\sum_{i=1}^{16}|G_{x_i}| = \frac{1}{8}(8+2+2+2+2+2+4+2+2+4+2+2+2+2+2+8) = 6.$$

Observation: As the above example illustrates, we are able to find the number of distinct configurations using this method. But it is important to observe that this method requires us to list all elements of X. That is, if we need to list all the elements of X then we can already pick the ones that are distinct. So, the question arises what is the need of Theorem 3.7.1. Also, if we color the vertices of the square with 3 colors, then $|X| = 3^4 = 81$, whereas the number of elements of D_4 (the group that acts as the group of symmetries of a square) remains 8. So, one feels that the calculation may become easy if one has to look at the elements of the group D_4 as one just needs to look at 8 elements of D_4. So, the question arises, can we get a formula that relates the number of distinct orbits with the elements of the group, in place of the elements of the set X? This query has an affirmative answer and is given as our next result.

Lemma 3.7.3 (Cauchy-Frobenius-Burnside's Lemma)**.** *Let G be a finite group acting on a set X. Let N denote the number of distinct orbits of X under the action of G. Then*

$$N = \frac{1}{|G|}\sum_{g\in G}|F_g|.$$

Proof. Consider the set $S = \{(g,x) \in G\times X \;:\; g\star x = x\}$. We calculate $|S|$ by two methods. As the first method, let us fix $x \in X$. Then, for each fixed $x \in X$, G_x gives the collection of elements of G that satisfy $g\star x = x$. So, $|S| = \sum\limits_{x\in X}|G_x|$.

As the second method, let us fix $g \in G$. Then, for each fixed $g \in G$, F_g gives the collection of elements of X that satisfy $g\star x = x$. So, $|S| = \sum\limits_{g\in G}|F_g|$. Thus, using two separate methods, one has $\sum_{x\in X}|G_x| = |S| = \sum_{g\in G}|F_g|$. Hence, using Theorem 3.7.1, we have

$$N = \frac{1}{|G|}\sum_{g\in G}|F_g|. \qquad \blacksquare$$

Example 3.7.4. *Let us come back to Example 3.6.3.2. Check that* $|F_e| = 16,\ |F_r| = 2,\ |F_{r^2}| = 4, |F_{r^3}| = 2,\ |F_f| = 4,\ |F_{rf}| = 8, |F_{r^2f}| = 4$ *and* $|F_{r^3f}| = 8$. *Hence, the number of distinct configurations are*

$$\frac{1}{|G|}\sum_{g\in G}|F_g| = \frac{1}{8}(16+2+4+2+4+8+4+8) = 6.$$

It seems that we may still need to know all the elements of X to compute the above terms. But it will be shown in the next section that to compute $|F_g|$, for any $g \in G$, we just need to know the decomposition of g as product of disjoint cycles.