PARTIAL DERIVATIVE

DEF. The usual derivative of a function of several variables with respect to one of the independent variables keeping all other independent variables as constan[t] is called the partial derivative of the function with respect to that variable.

Let $z = f(x, y)$; $(x, y) \in \mathbb{R}^2$, $z \in \mathbb{R}$

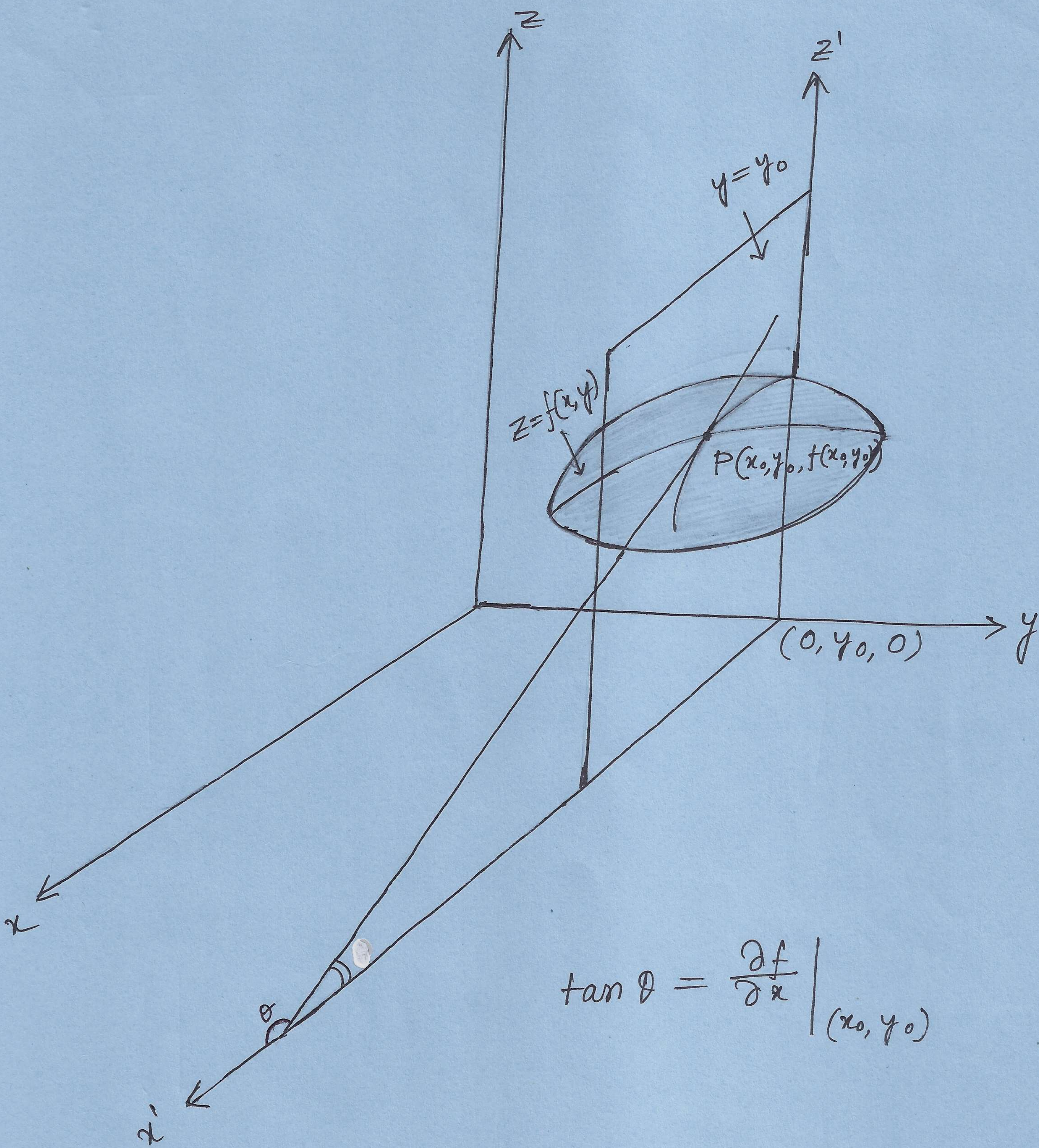
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$= \left.\frac{d}{dx} f(x, y_0)\right|_{x = x_0}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

$$= \left.\frac{d}{dy} f(x_0, y)\right|_{y = y_0}$$

GEOMETRIC INTERPRETATION:



$$\tan\theta = \left.\frac{\partial f}{\partial x}\right|_{(x_0,\, y_0)}$$

Ex. Find the value of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (x, y) of the following function

(i) $f(x,y) = ye^{-x}$ (ii) $f(x,y) = \sin(2x+3$

from the first principles.

SOL: (i)

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{ye^{-(x+\Delta x)} - ye^{-x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{ye^{-x}\{e^{-\Delta x} - 1\}}{\Delta x}$$

$$= ye^{-x} \lim_{\Delta x \to 0} \frac{\{1 - \Delta x - \frac{\Delta x^2}{2!} - \cdots - 1\}}{\Delta x}$$

$$= -ye^{-x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(y+\Delta y)e^{-x} - ye^{-x}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} e^{-x}$$

$$= e^{-x}$$

ii) $$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\sin\{2(x+\Delta x)+3y\} - \sin(2x+3y)}{\Delta x}$$

$$\left[\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right.$$

$$= \lim_{\Delta x \to 0} \frac{2.\cos(2x+3y+\Delta x) \cdot \sin \Delta x}{\Delta x}$$

$$= 2\cos(2x+3y)$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{\sin(2x + 3(y + \Delta y)) - \sin(2x+3y}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2 \cdot \cos\left(2x + 3y + \frac{3\Delta y}{2}\right) \cdot \sin\left(\frac{3}{2} \cdot \right.}{\frac{2}{3} \cdot \left(\frac{3}{2} \Delta y\right)}$$

$$= 3 \cos(2x + 3y) \cdot \lim_{\Delta y \to 0} \frac{\sin\left(\frac{3}{2} \Delta y\right)}{\frac{3}{2} \Delta y}$$

$$= 3 \cos(2x + 3y)$$