**Probability and Statistics**
**Hints/Solutions to Assignment 5**

1. $P(1 < X + Y < 2) = 2e^{-1} - 3e^{-2}$. $P(X < Y \mid X < 2Y) = ¾$, $P(0 < X < 1 \mid Y = 2) = 1 - e^{-1}$. $P(X + Y < m) = ½$, gives m as a solution of $2(m+1) = e^m$, or approximately 1.6784.

2. The marginal distributions are
$f_X(x) = x + 0.5, \quad 0 < x < 1,$ and $f_Y(y) = y + 0.5, \quad 0 < y < 1,$
$= 0, \quad$ elsewhere, $= 0, \quad$ elsewhere.
So E(X) = E(Y) = 7/12, $E(X^2) = E(Y^2) = 5/12$, V(X) = V(Y) = 11/144, E(XY) = 1/3, Cov(X, Y) = -1/144. Hence V(X + Y) = 5/36, Corr(X, Y) = -1/11.
$$f_{X|Y=y}(x \mid y) = \frac{2(x+y)}{1+2y}, \quad 0 < x < 1, 0 < y < 1,$$
$$= 0, \quad \text{elsewhere.}$$

$$E(X \mid Y = y) = \frac{2(2+3y)}{(1+2y)}, E(X^2 \mid Y = y) = \frac{2(3+4y)}{(1+2y)}, V(X \mid Y = y) = \frac{2(8y^2+4y-1)}{(1+2y)^2}.$$

3. $f_X(x) = \frac{3-x}{4}, \quad 0 < x < 2,$ and $f_Y(y) = \frac{5-y}{4}, \quad 2 < y < 4.$
$$f_{X|Y=y}(x \mid y) = \frac{6-x-y}{2(5-y)}, \quad 0 < x < 2, 2 < y < 4,$$
$$f_{Y|X=x}(y \mid x) = \frac{6-x-y}{2(3-x)}, \quad 2 < y < 4, 0 < x < 2$$
Expectations etc. can be easily calculated now.

4. Similar to Q. 5.

5. The conditional distribution of $X_1 \mid X_2 = 0.098$ is N(1999.13, 607.25).
So $P(X_1 > 1950 \mid X_2 = 0.098) = P(Z > -1.99) = 0.9767$.

6.

7. $f_Y(y) = e^{-y}, y > 0, \quad f_{X|Y=y}(x \mid y) = \frac{1}{y}e^{-x/y}, x > 0, y > 0.$ So E(Y) = 1, V(Y) = 1,
E(X) = EE(X|Y) = E(Y) =1,
V(X) = EV(X|Y) + VE(X|Y) = $E(Y^2)$ + V(Y) = 2 + 1 = 3.,
E(XY) = E{YE(X|Y)} = $E(Y^2)$ = 2, Cov(X, Y) =1.
So , Corr $(X, Y) = \frac{1}{\sqrt{3}}$.

8. Let X be the marks in the first exam and Y in the second exam. Then the conditional distribution of Y | X = 80 is N(86.2, 5.76).
So P(Y > 80|X = 80) = P(Z > -2.58) = 0.9951.
When ρ = -0.8, then the conditional distribution of Y | X = 80 is N(79.8, 5.76).
So P(Y > 80|X = 80) = P(Z > 0.08) = 0.4681.

9. Similar to Q. 8.

10. $f_X(x) = 1,\ 0 < x < 1,$ and $f_Y(y) = 1,\ 0 < y < 1.$ So E(X) = E(Y) = ½,
V(X) = V(Y) = 1/12, Cov (X, Y) = -α/36, $\rho_{X,Y}$ = -α/3. -1 ≤ α ≤1.
Clearly $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for all x, y iff $\alpha = 0,$ that is, $\rho_{X,Y} = 0.$

11. (i) 0.356
(ii) $p_X(0) = 0.210, p_X(1) = 0.298, p_X(2) = 0.277, p_X(3) = 0.215.$
$p_Y(1) = 0.267, p_Y(2) = 0.397, p_Y(3) = 0.302, p_Y(1) = 0.034.$
E(X) = 1.497, V(X) = 1.1, E(Y) = 2.103, V(Y) = 0.6944,
Cov(X, Y) = 0.130809, $\rho_{X,Y}$ = 0.1497.
(iii) 0.6879.

12. (a) $P(Y < \frac{1}{2}, Y < X < \frac{1}{4}) = \int_0^{1/2} \int_y^{y+1/4} 8xy\ dx\ dy = \frac{11}{96}.$

(b) $f_{X|Y=y}(x \mid y) = \frac{2x}{1-y^2},\ y < x < 1,\ 0 < y < 1,$
$= 0,$ elsewhere.

$P(X < \frac{3}{4} \mid Y = \frac{1}{6}) = \int_{1/6}^{3/4} \frac{72}{35} x\ dx = \frac{77}{140}.$

(c) $E(X \mid Y = \frac{1}{6}) = = \int_{1/6}^{3/4} x \frac{72}{35} x\ dx = \frac{43}{63}.$

(d) E(X) = 4/5, $E(X^2)$ = 2/3, V(X) = 2/75, E(Y) = 8/15, $E(Y^2)$ = 1/3, V(Y) = 11/225,
E(XY) = 4/9, Cov (X, Y) = 4/225, Corr(X, Y) = 0.4926.

13. P(2X < Y) = ½, P(|X + Y| < 1) = 13/16. Also X, Y ~ U(-1, 1).
So E(X) = E(Y) = 0, V(X) = V(Y) = 1/3, Cov (X, Y) = 1/9, ρ = 1/3.