Lecture - 3 : ~~Fr~~ Thursday - 3-5 p.m.

14.1.16

Subspace.

V be a vector space with respect to '+', (.).

Let W be a nonempty subset of V.

\# If W is itself a vector space with respect to the same operations '+' & (.) imposed on V

\# If W is closed under the operations '+' & (.) imposed on V, then W is a subspace.

Theorem. A nonempty subset W of V (vector space) is a subspace if and only if

$\langle i \rangle$ $\underset{\sim}{0} \in W$, $\langle ii \rangle$ $u, v \in W \Rightarrow u+v \in W$

$\langle iii \rangle$ $u \in W$, $c \in F \; (=R) \Rightarrow cu \in W$.

$\langle ii \rangle$ & $\langle iii \rangle$ can be merged as $c_1 u + c_2 v \in W \;\; \forall u, v \in W$ and $c_1, c_2 \in R$.

Ex.1 $\{\underset{\sim}{0}\} \rightarrow$ subspace of any vector space.

$\quad \hookrightarrow$ trivial vector space.

Ex.2 $W = \{(a, b, 0)\} \subset \mathbb{R}^3$

1) $(0, 0, 0) \in \mathbb{R}^3$

2) Let $\omega_1 = (a_1, b_1, 0) \in W$, $\omega_2 = (a_2, b_2, 0) \in W$

$$c_1 w_1 + c_2 w_2 = c_1 (a_1, b_1, 0) + c_2 (a_2, b_2, 0)$$
$$= (c_1 a_1, c_1 b_1, 0) + (c_2 a_2, c_2 b_2, 0)$$
$$= (c_1 a_1 + c_2 a_2, c_1 b_1 + c_2 b_2, 0) \in W.$$

$\therefore$ W is a subspace of $\mathbb{R}^3$.

Ex 2.a. $W = \{(a_1, b_1, 1)\} \subset \mathbb{R}^3$

$(0,0,0) \notin W \Rightarrow W$ is not a subspace of $\mathbb{R}^3$.

Ex.3 $M_{2\times 2}$ = a vector space of all 2×2 matrices.

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \right\} \subset M_{2\times 2}.$$

Here 1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$

2) Take $w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ det $w_1 = 0 \quad \therefore w_1 \in W$

$w_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ det $w_2 = 0 \quad \therefore w_2 \in W$

$w_1 + w_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ det $(w_1 + w_2) = 1 \neq 0$

$\Rightarrow w_1 + w_2 \notin W$

$\therefore$ W is not a subspace.

W = set of all 2×2 symmetric matrices

$\Rightarrow$ W is a subspace of $M_{2\times 2}$.

Ex.4. $\mathcal{P}(t)$ = a vector space of set of all polynomials $\{a_0 + a_1 t + \cdots + a_n t^n\}$

Let $P_n(t) \to$ set of all polynomials of degree $\leq n$.

Then $P_n(t)$ is a subspace of $\mathcal{P}(t)$.

$Q_n(t) \to$ set of all polynomials of degree $= n$.

then $Q_n(t)$ is not a subspace of $\mathcal{P}(t)$.

Let us consider $Q_3(t)$.

let $v_{3_1}(t) = 2 + 6t - 3t^2 + 4t^3 \in Q_3(t)$

$v_{3_2}(t) = 1 - 5t + 4t^2 - 4t^3 \in Q_3(t)$

$v_{3_1}(t) + v_{3_2}(t) = 3 + t + t^2 \notin Q_3(t)$.

Ex.5. $V(f)$ be the vector space of all continuous function $f: R \to R$, $\underset{\sim}{O}(x) = 0$

$W = \{f : f(5) = f(2)\}$

$O(5) = 0$, $O(2) = 0 \Rightarrow O(5) = O(2)$

$\therefore O \in W$.

$f_1, f_2 \in W$ $\quad \left.\begin{array}{l} f_1(5) - f_1(2) = 0 \\ f_2(5) - f_2(2) = 0 \end{array}\right\}$

show that $c_1 f_1 + c_2 f_2 \in W$.

Exercises 1. Show that $W = \{f : f(5) = f(2) + 3\}$ is not a subspace.

2. Show that $W = \{(a, b, 0) : a \leq 0\} \subseteq \mathbb{R}^3$ not a subspace of $\mathbb{R}^3$.

Linear Span.

$S = \{v_1, v_2, \ldots, v_n\} \subset V$ (vector space)

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \to$ linear combination of the vectors $v_1, v_2, \ldots, v_n$; c_i's are scalars.

Any set of the form $\{c_1 v_1 + c_2 v_2 + \ldots + c_n v_n\}$ is called a linear span of S & is denoted by $L(S)$ i.e. elements of $L(S)$ are some linear combinations of the vectors in S.

$S = \{(2,3), (3,4)\}$

$L(S) = \{c_1(2,3) + c_2(3,4)\}$

$= \{2c_1 + 3c_2, 3c_1 + 4c_2\}$

$S = \{(2,3)\}$

$L(S) = \{c(2,3)\} = \{x, y\}$

$x = 2c$, $y = 3c$. Geometrically linear span of $\{(2,3)\}$ is a straight line passing through the origin & the point $(2,3)$.

$3x - 2y = 0$.

Linear dependence & independence.

$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \underset{\sim}{0}$ — (1).

If (1) holds when all c_i's are 0 (zero), then $\{v_1, v_2, \ldots, v_n\}$ are linearly independent (l.i).

If (1) holds for atleast one non zero c_i, then $v_1, v_2, \ldots, v_n$ are linearly dependent.

Ex. $v_1 = (1,1)$, $v_2 = (0,1)$ $\in \mathbb{R}^2$

$c_1 v_1 + c_2 v_2 = (0,0)$

$\Rightarrow c_1(1,1) + c_2(0,1) = (0,0)$

$\Rightarrow (c_1, c_2) + (0, c_2) = (0,0)$

$\Rightarrow (c_1, c_1 + c_2) = (0,0)$

$\Rightarrow c_1 = 0$

$c_1 + c_2 = 0 \Rightarrow c_2 = 0$.

$\therefore$ $(1,1), (0,1)$ are linearly independent.

$S_1 = \{ v_1 = (1,1) , v_2 = (3,3) \}$, $v_1, v_2 \in \mathbb{R}^2$

$S_2 = \{ v_1 = (1,1), v_2 = (2,3) , v_3 = (3,4) \}$, $v_1, v_2, v_3 \in \mathbb{R}^2$

Both S_1, S_2 are linearly dependent.

Theorem 1. A set of vectors $\{v_1, v_2, \dots, v_n\}$ is linearly dependent, if any of them is a linear combination of other vectors.

$S = \{ v_1, v_2, v_3, v_2 + v_3 \}$.

Here $v_2 + v_3 = 0.v_1 + 1.v_2 + 1.v_3$. is a linear combination of v_1, v_2, v_3. So S is linearly dependent.

Theorem 2. Any singelton set containing a non zero element is linearly independent.

i.e. a set $\{v\}$ is l.i, where $v \neq \underset{\sim}{0}$

$c.v = \underset{\sim}{0}$

holds only when $c = 0$, $\because v \neq \underset{\sim}{0}$.

Theorem 3. Any set containing the zero / identity vector is linearly dependent.

$\{ v_1, v_2, \dots, v_{r-1}, \underset{\sim}{0}, v_{r+1}, \dots, v_n \}$ is linearly dependence because

$c_1 v_1 + c_2 v_2 + \dots + c_{r-1} v_{r-1} + c_r \underset{\sim}{0} + c_{r+1} v_{r+1} + \dots + c_n v_n = \underset{\sim}{0}$

will hold if all c_i's $= 0$ except c_r.

Theorem 4. If a set of vectors is linearly independent, then any subset of these vectors is linearly independent.

Theorem 5. If a set of vectors S is L.d., then any superset of these vectors is L.d. set containing S.

Ex [redacted] 1. $S = \{v_1, v_2, v_3, v_4\} \subset \mathbb{R}^3$.

$v_1 = (3, 0, -3) \quad v_2 = (-1, 1, 2) \quad v_3 = (4, 2, -2) \quad v_4 = (2, 1, 1)$

Check for linear dependence or independence.

Hint: for $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = (0, 0, 0)$

Ans : L. D. solve for c_1, c_2, c_3, c_4.

Ex 2. Determine whether, $v = (-2, 5, 3)$ belongs to $L(S)$ where $S = \{v_1 (1, -3, 2), v_2 = (2, -4, -1), v_3 (1, -5, 7)\}$.

Hint: for $(-2, 5, 3) = c_1 v_1 + c_2 v_2 + c_3 v_3$.

Check whether solutions for c_1, c_2, c_3 exist.

Ans → NO.

Example. $v_1 = (3, 0, -3)$, $v_2 = (-1, 1, 2)$, $v_3 = (4, 2, -2)$, $v_4 = (2, 1, 1)$.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -3 \\ -1 & 1 & 2 \\ 4 & 2 & -2 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -3 \\ 0 & 3 & 3 \\ 0 & 0 & -24 \\ 0 & 0 & 0 \end{pmatrix}$$

If in echelon form no. of non zero rows ≠ no. of given vectors, vectors are linearly dependent. and if equal then vectors are linearly independent.

Theorem. If a vector space V be spanned by a l.d. set $\{v_1, v_2, \dots, v_n\}$, then V can also be spanned by a proper subset of $\{v_1, v_2, \dots, v_n\}$ i.e. some vectors can be deleted from a l.d. spanning set of V.

Ex. $S = \{v_1 = (1,2,0),\ v_2 = (3,-1,1),\ v_3 = (4,1,1)\}$

Show that $S = \{v_1, v_2, v_3\}$ linearly dependent.

Note. $v_3 = v_1 + v_2 \Rightarrow v_1 = v_3 - v_2$.

$\therefore$ S is l.d.

$$L(S) = \{c_1v_1 + c_2v_2 + c_3v_3\}.$$
$$= \{c_1v_1 + c_2v_2 + c_3(v_1+v_2)\}.$$
$$= \{(c_1+c_3)v_1 + (c_2+c_3)v_2\} = \{d_1v_1 + d_2v_2\} = L(v_1, v_2)$$

$$L(S) = L(v_1, v_2, v_3) = \{c_1(v_3 - v_2) + c_2v_2 + c_3v_3\}$$
$$= \{(-c_1 + c_2)v_2 + (c_1 + c_3)v_3\} = L(v_2, v_3).$$

Basis.

$S = \{v_1, v_2, \dots, v_n\} \subset V$ (vector space)

Then S is said to be a basis for the vector space V if

1) S is a linear independent set
2) S spans (generates) V [every element of V is a l.c. of the elements of S]

No. of vectors in a basis is called the dimension of V.

if the no. n is finite, V is said to be finite dimensional.

1. space of all polynomials $\{a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n\}$
'n' not specified.
→ an infinite dimensional vector space.

2. space of all polynomials of degree $\leq m$
$\{a_0 + a_1 t + \dots + a_m t^m\}$ is generated by $m+1$ l.i
polynomials $1, t, t^2, \dots, t^m$.
So dimension of this space $= m+1$.

3. For $\mathbb{R}^3$, $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ forms a standard basis.

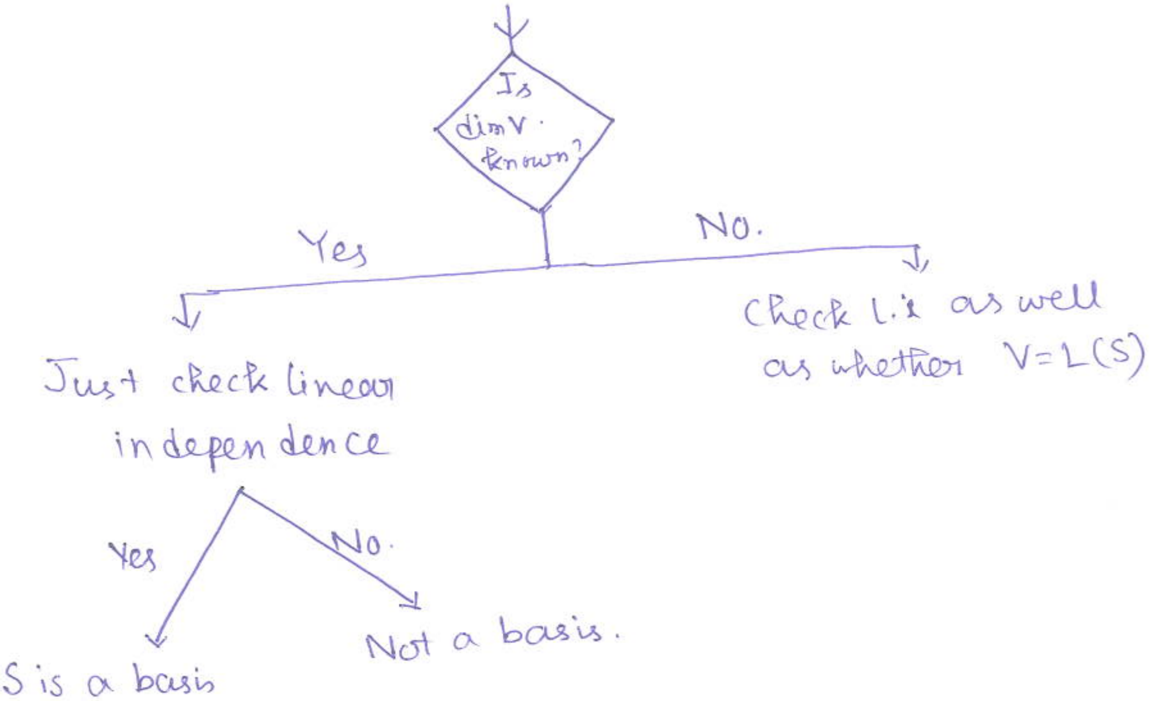
Take $(a,b,c) \in \mathbb{R}^3$.

$$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$
$$= ae_1 + be_2 + ce_3.$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\{(1,1,1), (0,2,5), (0,0,3)\}$ also a basis of $\mathbb{R}^3$.

Check $S = \{v_1, v_2, \ldots, v_m\}$ forms a basis for V.



$$V_{2\times 3} = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \right\}$$
$M_{2\times 3}$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = a\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{e_1} + b\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{e_2} + c\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{e_3}$$
$$+ d\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{e_4} + e\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{e_5} + f\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{e_6}.$$

Dimension of the vector space $= 6 = 2\times 3$.

" " " " " $V_{m\times n} = m.n$

Dimension of a subspace

Let V be a vector space of dim n.

Let W be a subspace of V.

If $\dim W = m$, then $m \leq n$.

Consider $\mathbb{R}^3$, $\dim \mathbb{R}^3 = 3$

Let $W \subset \mathbb{R}^3$

1. $\dim W = 0 \Rightarrow W = \{(0,0,0)\} \Rightarrow$ by W we mean origin.
2. $\dim W = 1 \Rightarrow W$ consists of all lines passing through origin.
3. $\dim W = 2 \Rightarrow W$ consists of all planes which contains origin.
4. $\dim W = 3 \Rightarrow W$ is the entire $\mathbb{R}^3$ space.

$W = \{ax + by + cz = 0\}$. solution space for W.

Let $W = \{(x, y, z) : ax + by + cz = 0\}$

$z = z_0$, $y = y_0$

$$x = -\frac{b}{a}y - \frac{c}{a}z = -\frac{b}{a}y_0 - \frac{c}{a}z_0 \quad (a \neq 0)$$

$$(x, y, z) = \left(-\frac{b}{a}y_0 - \frac{c}{a}z_0, y_0, z_0\right)$$

$$= y_0 \underbrace{\left(-\frac{b}{a}, 1, 0\right)}_{w_1} + z_0 \underbrace{\left(-\frac{c}{a}, 0, 1\right)}_{w_2}$$

$\{w_1, w_2\}$ is a basis of W and spans W.

$$c_1\left(-\frac{b}{a}, 1, 0\right) + c_2\left(-\frac{c}{a}, 0, 1\right) = (0, 0, 0)$$

$$\Rightarrow -\frac{b}{a}c_1 - \frac{c}{a}c_2 = 0$$

$$c_1 = 0$$
$$c_2 = 0$$

$\therefore$ w_1, w_2 are linearly independent.