Vector Calculus

Lecture - 19

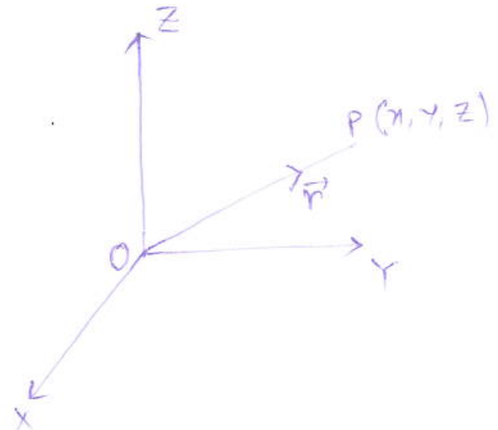
Position vector $\vec{r}$.

$\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$(\hat{i}, \hat{j}, \hat{k}) \rightarrow$ unit vectors along

x, y, z - axes.

$\therefore |\hat{i}| = |\hat{j}| = |\hat{k}| = 1.$



$|\vec{r}| = +\sqrt{x^2 + y^2 + z^2}$

- Vector product

1. Dot product.

let $\vec{A} = (A_1, A_2, A_3)$, $\vec{B} = (B_1, B_2, B_3)$.

$= A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ $\quad = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$\vec{A}\cdot\vec{B} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot(B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$

$= A_1B_1 + A_2B_2 + A_3B_3.$

$= |\vec{A}|\cdot|\vec{B}|\cos\theta$; θ is the angle between $\vec{A}$ & $\vec{B}$.

$0 \leq \theta \leq \pi$

$\Rightarrow \cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}|\cdot|\vec{B}|}$

Note.
$\hat{i}\cdot\hat{i} = 1 = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k}$
$\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$, $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.

Geometric Significance

Dot product of two vectors means the projection of one ~~the~~ vector on another vector multiplied by the magnitude of the latter vetor.

2. Cross product

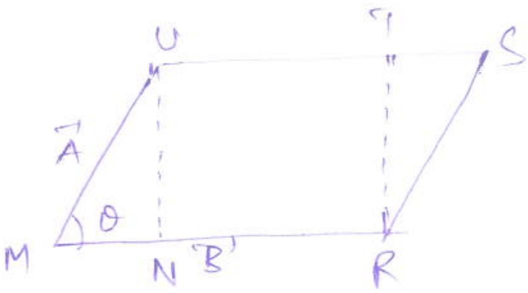
$\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}| \sin\theta \cdot \hat{n}$ unit vector : $\vec{A}, \vec{B}, \vec{n}$

$\hookrightarrow$ make a rt. handed system.

$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Note $|\vec{A} \times \vec{B}| = |\vec{A}|.|\vec{B}|.|\sin\theta|$

Area of the parallelogram.



$= $ Area of ΔUMN

$+$ Area of ΔTSR

$+$ Area of UNRT

$= 2 \times \frac{1}{2} MN \times |\vec{A}| \sin\theta + UN \times NR$

$= MN\, |\vec{A}| \sin\theta + |\vec{A}| \sin\theta \times NR$

$= |\vec{A}| \sin\theta \times MR = |\vec{A}| \sin\theta \cdot |\vec{B}|$

Way of calculating $\vec{A} \times \vec{B}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Gradient, divergence, curl

$\vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \rightarrow$ vector differential operator

Let $\phi(x,y,z)$ be a scalar function

$\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{i} + F_2(x,y,z)\,\hat{j} + F_3(x,y,z)\,\hat{k}$.

be a vector function.

$\vec{\nabla}\phi$ = gradiant of $\phi \rightarrow$ a vector quantity

$\vec{\nabla} \cdot \vec{F}$ = divergence of $\vec{F} \rightarrow$ a scalar quantity

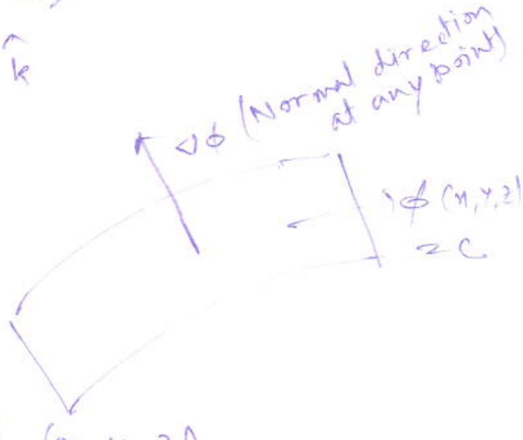
$\vec{\nabla} \times \vec{F}$ = Curl of $\vec{F} \rightarrow$ a vector quantity

Gradiant of a scalar function $\phi(x,y,z)$

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x,y,z)$$

$$= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$|\vec{\nabla}\phi| = \sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}.$$



$\vec{\nabla}\phi$ is along the normal direction to the surface $\phi(x,y,z) = c$ at (x,y,z).

Directional derivative of ϕ along a direction

$\vec{A} = (A_1, A_2, A_3) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$.

is defined as, $D_{\vec{A}}\phi = \vec{\nabla}\phi \cdot \frac{\vec{A}}{|\vec{A}|}$.

= Rate of Change of ϕ along the vetor $\vec{A}$.

Thm

D.D of ϕ is maximum along the normal to the surface $\phi(x,y,z) = C$.

Thus, $\underset{\vec{A}}{\mathrm{Max}}\, D_{\vec{A}}\phi = D_{\vec{\phi}}\phi = \vec{\nabla}\phi \cdot \frac{\vec{\nabla}\phi}{|\vec{\phi}|} = \frac{|\vec{\nabla}\phi|^2}{|\vec{\nabla}\phi|}$

$= |\vec{\nabla}\phi|.$

Ex 1

Find the gradiant of $f = xy - z^2$ at the pt. $(4,1,2)$. And hence find the unit normal vector to the surface $xy - z^2 = c$ at the pt. $(4,1,2)$

Ans

$$\vec{\nabla} f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy - z^2)$$

$$= y\hat{i} - 2z\hat{k} + x\hat{j}$$

$$\vec{\nabla} f\Big|_{(4,1,2)} = \hat{i} + 4\hat{j} - 4\hat{k}$$

Unit normal at $(4,1,2) = \dfrac{\hat{i} + 4\hat{j} - 4\hat{k}}{\sqrt{1+16+16}} = \dfrac{1}{\sqrt{33}}(\hat{i} + 4\hat{j} - 4\hat{k})$

Ex 2

Find the gradiand of $f = 4x^2y + z^3$ at the pt $(1,-1,2)$. Find the unit normal to the surface $4x^2y + z^3 = c$ at the pt. $(1,-1,2)$

Ans

$$\vec{\nabla} f\Big|_{(1,-1,2)} = -8\hat{i} + 4\hat{j} + 12\hat{k}$$

Unit normal $= -\dfrac{2\hat{i}}{\sqrt{14}} + \dfrac{1}{\sqrt{14}}\hat{j} + \dfrac{3}{\sqrt{14}}\hat{k}$.

Ex. 3 Find the D.D. of $f = xy^2 + yz^3$ at the pt. $(2,-1,1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$. Find the max value of the D. D. of f at $(2,-1,1)$

$$\vec{\nabla} f\Big|_{(2,-1,1)} = \hat{i} - 3\hat{j} - 3\hat{k}$$

Now, $D_{(1,2,2)} f = (\vec{\nabla} \cdot f) \cdot \dfrac{(\hat{i} + 2\hat{j} + 2\hat{k})}{3} = \dfrac{y^2 + 2(2xy + z^3) + 6yz^2}{3}$

Now, Max $D_{(1,2,2)} f$ at $(2,-1,1) = |\vec{\nabla} f|_{(2,-1,1)} = \sqrt{19}$

Ex-4 In what direction from $(3, 1, -2)$ in the D.D of $\phi = x^2 y^2 z^4$ max & what is the magnitude.

Ans. $\vec{\nabla}\phi = 2xy^2z^4\hat{i} + 2x^2yz^4\hat{j} + 4x^2y^2z^3\hat{k}$

$(\vec{\nabla}\phi)_{(3,1,-2)} = 96\hat{i} + 288\hat{j} - 288\hat{k}$ (Required direction)

$|\vec{\nabla}\phi| = 96\sqrt{19}$

Ex-5 find the angle between the surfaces $x^2+y^2+z^2=9$. & $z = x^2+y^2-3$ at $(2,-1,2)$

$\vec{\nabla}f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$. $(\vec{\nabla}f)_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$.

$\vec{\nabla}g = 2x\hat{i} + 2y\hat{j} - \hat{k}$. $(\vec{\nabla}g)_{(2,-1,2)} = 4\hat{i} - 2\hat{j} - \hat{k}$.

$$\phi = \cos^{-1}\left(\frac{\vec{\nabla}f \cdot \vec{\nabla}g}{|\vec{\nabla}f|\,|\vec{\nabla}g|}\right) = \cos^{-1}\left(\frac{(4\hat{i}-2\hat{j}+4\hat{k})\cdot(4\hat{i}-2\hat{j}-\hat{k})}{\sqrt{4^2+2^2+4^2}\cdot\sqrt{4^2+2^2+1}}\right)$$

$$= \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right) = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Ex-6 A paraboloid has the equation $2z = x^2+y^2$. Find the equation of normal line & tangent plane to the surface at the pt $(1, 3, 5)$

$f = x^2+y^2-2z$

$(\vec{\nabla}f)_{(1,3,5)} = 2\hat{i} + 6\hat{j} - 2\hat{k}$

Let $P(x,y,z)$ be a point on the normal line

then $\vec{AP} = (x-1)\hat{i} + (y-3)\hat{j} + (z-5)\hat{k}$

Now, $\vec{AP} \times (\vec{\nabla}f) = \vec{0} \Rightarrow \frac{x-1}{-1} = \frac{y-3}{-3} = \frac{z-5}{1}$

P.T.O

Now let, $P(x,y,z)$ be a pt on the tangent ~~line~~ plane. Then $\vec{AP}\cdot\vec{\nabla}f = 0 \Rightarrow 2(x-1)+6(y-3)-2(z-5)=0$

or, $x+3y-z=5$

Divergence of $\vec{F}$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl of $\vec{F}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

Notes

$\vec{F}$ is solenoidal if $\text{div}.\vec{F} = 0$

$\vec{F}$ is irrotational if ~~curl~~ $\text{curl}\vec{F} = 0$.

again -

Curl $\vec{F} = 2\vec{\omega}$ where $\vec{\omega}$ represents the angular velocity of a particle.

- $\text{Curl}(\text{grad}\,\phi) = 0$
- $\text{div}(\text{curl}\,\vec{F}) = 0$.

Ex

Find $\text{div}(3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k})$ at the pt. $(1,2,3)$.

Ex Find the value of λ, if

$$\vec{f} = (2x^2y^2 + z^2)\hat{i} + (3xy^3 - x^2z)\hat{j} + (\lambda xy^2z + xy)\hat{k}$$

is solenoidal.

Ans

$$\operatorname{div}\vec{f} = 4xy^2 + 9xy^2 + \lambda xy^2 = 0$$

$$\Rightarrow (13+\lambda)xy^2 = 0 \Rightarrow \lambda = -13$$

Exercise 1. Show that (i) $\operatorname{curl}\operatorname{grad}\phi = 0$

(ii) $\operatorname{div}\operatorname{curl}\vec{F} = 0$

2. Show that $r^n\vec{r}$ is irrotational

3. Show that $\operatorname{div}(\operatorname{grad} r^n) = n(n+1)r^{n-2}$

Hint. $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

Ex

$$\vec{F}(x,y,z) = (y^2z^3\cos x - 4x^3z)\hat{i} + 2z^3y\sin x\,\hat{j} + (3y^2z^2\sin x - x^4)\hat{k}$$

$\vec{F}$ is conservative?

$\Rightarrow$ Hint If possible let $\vec{F} = \vec{\nabla}\phi$

Now check wheather we can find suitable ϕ.

Thm $\operatorname{Curl}(\operatorname{grad}\phi) = 0$. Conversely, if $\operatorname{curl}\vec{F} = 0$, then $\vec{F}$ must be gradiant of some scalar function ϕ.

Problem 1

Show that curl $\vec{F} = 0$. Where $\vec{F} = (x^2 + yz)\hat{i} + (y^2 + zx)\hat{j} + (z^2 + xy)\hat{k}$

and find ϕ such that $\vec{F} = \vec{\nabla}\phi$

$$\Rightarrow \quad \text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+yz & y^2+zx & z^2+xy \end{vmatrix} = 0$$

Now

$$\vec{F} = \vec{\nabla}\phi \Rightarrow \frac{\partial \phi}{\partial x} = x^2 + yz, \quad \frac{\partial \phi}{\partial y} = y^2 + zx$$

$$\frac{\partial \phi}{\partial z} = z^2 + xy$$

$$\frac{\partial \phi}{\partial x} = x^2 + yz \Rightarrow \phi = \left(\frac{x^3}{3} + yzx\right) + f_1(y, z)$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = zx + \frac{\partial f_1}{\partial y} = y^2 + zx \Rightarrow \frac{\partial f_1}{\partial y} = y_2$$

$$\therefore f_1 = \frac{y^3}{3} + f_2(z)$$

$$\therefore \phi = \frac{x^3 + y^3}{3} + xyz + f_2(z)$$

$$\frac{\partial \phi}{\partial z} = xy + f_2'(z) = xy + z^2 \Rightarrow f_2'(z) = z^2$$

$$\Rightarrow f_2(z) = \frac{z^3}{3} + c$$

$$\therefore \phi = \frac{x^3 + y^3 + z^3}{3} + xyz + c$$