Indian Institute of Technology Kharagpur

Department of Mathematics

Supplementary Test 2013

MA 20101 — Transform Calculus

Full Marks: 50 — Time: 3 hrs

Notations. Laplace Transform: $L[f(t)] \equiv F(s) = \int_0^\infty e^{-st} f(t)dt\,, \mathrm{Re}(s) > a > 0, f(t) = O(e^{-at})$

Fourier Transform: $\mathcal{F}[f(x)] \equiv \bar{F}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)dx$

Fourier Cosine Transform: $\mathcal{F}_c[f(x)] \equiv \bar{F}_c(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(\xi x)dx$

Fourier Sine Transform: $\mathcal{F}_s[f(x)] \equiv \bar{F}_s(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(\xi x)dx$

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1. Show that $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds = -\int_0^s F(s)ds + \int_0^\infty \frac{f(t)}{t}dt\,,$
Hence find the value of $I = \int_0^\infty \frac{\sin t}{t}dt\,.$ 5

2. Find $L^{-1}\left[\frac{1}{1+\sqrt{1+s}}\right]$. 5

3. From the definition of Laplace Transform, show that for a periodic function $f(t)$ of period $2K$,
$L[f(t)] = \frac{1}{1-e^{-2Ks}} \int_0^{2K} e^{-st} f(t)dt\,,$
Hence find $L[f(t)]$ for the following periodic function
$$f(t) = \begin{cases} 1, & 2n < t \le 2n+1 \\ 2, & 2n+1 < t \le 2n+2 \end{cases}$$
where $n = 0, 1, 2, \ldots$ 5

4. Solve the wave equation $u_{tt} = c^2 u_{xx}\,, 0 < x < 1\,, t > 0\,(c > 0)$ subject to the initial conditions $u(x,0) = \sin \pi x = -u_t(x,0)$ and boundary conditions $u(0,t) = u(1,t) = 0\,.$ 5

5. If $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t g(\tau)f(t-\tau)d\tau$, show that $L[f(t) * g(t)] = F(s)G(s)\,.$
Using this result, show that $L^{-1}\left[\frac{e^{s^2/2}}{s^2}\right] = t\,\mathrm{erf}(t/\sqrt{2}) + \sqrt{\frac{2}{\pi}}(e^{-t^2/2} - 1)\,.$ 5

6. Find $\mathcal{F}[e^{-a^2x^2}]$. Determine the value of a for which the shape of the function remains identical. 5

7. Show that for $f(x) = e^{-a|x|}\,(a > 0)\,, \mathcal{F}[f(x)] = \mathcal{F}_c[e^{-ax}]$, Hence evaluate $I = \int_0^\infty \frac{\cos \lambda x}{\lambda^2+4}d\lambda\,.$ 5

8. (a) Find Inverse Fourier Transform of $\bar{F}(\xi) = \frac{\xi}{(\xi^2+a^2)^2}$.
   (b) Evaluate $J = \int_0^\infty \frac{\omega \sin \omega x}{(\omega^2+1)^2}d\omega\,.$ 5

9. Show that $\mathcal{F}^{-1}[\bar{F}(\xi)\bar{G}(\xi)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)g(x-\tau)d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\tau)f(x-\tau)d\tau\,.$ 5

10. (a) Find $\mathcal{F}[\frac{e^{-x^2}}{x}]$.
    (b) Show that $2\mathcal{F}[f(x)\cos \omega x] = \bar{F}(\xi+\omega) + \bar{F}(\xi-\omega)\,.$ 5

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