Exact Differential Equations

If M and N are functions of x and y, the equation $Mdx + Ndy = 0$ is called exact when there exists a function $f(x,y)$ such that

$$d(f(x,y)) = Mdx + Ndy$$

or

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

Theorem: The necessary and sufficient condition for the differential equation

$$Mdx + Ndy = 0$$

to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad — ①$$

Proof: The condition is necessary '⇒'

Let the equation be exact, then

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

Equating coefficients of dx & dy, we get:

$$M = \frac{\partial f}{\partial x} \qquad N = \frac{\partial f}{\partial y}$$

Assuming f to be continuous upto 2nd order partial derivatives, we obtain

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the equation is exact then M & N satisfy ①.

Now we show that the condition ① is sufficient.

We assume the $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and show that the equation $M\,dx + N\,dy$ is exact.

That means we find a function $f(x,y)$ such that

$$df = M\,dx + N\,dy.$$

Let $g(x,y) = \int M\,dx$ be the partial integral of M such that

$$\frac{\partial g}{\partial x} = M.$$

We first prove that $\left(N - \frac{\partial g}{\partial y}\right)$ is a function of y only.

Consider $$\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x \partial y}$$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y \partial x} \qquad \left(\text{assuming } \frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Now consider:

$$f = g(x,y) + \int\left(N - \frac{\partial g}{\partial y}\right)dy \text{ and then}$$

$$df = dg + d\left(\int\left(N - \frac{\partial g}{\partial y}\right)dy\right) = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy + \overbrace{\frac{\partial}{\partial x}\Big(\underbrace{\int\left(N - \frac{\partial g}{\partial y}\right)dy}_{\text{function of } y}\Big)}^{=0}dx$$

$$+ \frac{\partial}{\partial y}\left(\int\left(N - \frac{\partial g}{\partial y}\right)dy\right)dy$$

$$= \frac{\partial g}{\partial x}\,dx + \cancel{\frac{\partial g}{\partial y}\,dy} + N\,dy - \cancel{\frac{\partial g}{\partial y}\,dy}$$

$$= M\,dx + N\,dy.$$

$\Rightarrow$ The given differential equation is exact.

<u>Remark</u>: The solution of an exact differential equation $Mdx + Ndy = 0$ can be written as

$$f = c$$

i.e.,

$$\underset{(y\text{ const.})}{\int M\,dx} + \int \underbrace{\left(N - \frac{\partial g}{\partial y}\right)}_{\text{function of } y \text{ alone}} dy = c$$

OR

$$\underset{(y\text{ const})}{\int M\,dx} + \int (\text{terms of } N \text{ not containing } x)\, dy = c.$$

<u>Example</u>: Solve $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

<u>Sol</u>: $M = x^2 - 4xy - 2y^2 \qquad N = y^2 - 4xy - 2x^2$

$$\frac{\partial M}{\partial y} = -4x - 4y = \frac{\partial N}{\partial x} \Rightarrow \text{the equation is exact.}$$

Hence, there exists a function $f(x,y)$ such that

$$d(f(x,y)) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M\,dx + N\,dy$$

$$\Rightarrow \frac{\partial f}{\partial x} = x^2 - 4xy - 2y^2 \quad \& \quad \frac{\partial f}{\partial y} = y^2 - 4xy - 2x^2$$

Int. of 1st w.r.t. x
$$\Rightarrow f = \frac{x^3}{3} - 2x^2y - 2xy^2 + c_1(y)$$

On differentiation w.r.t y:

from above.

$$\frac{\partial f}{\partial y} = -2x^2 - 4xy + c_1'(y) = y^2 - 4xy - 2x^2$$

$$\Rightarrow c_1'(y) = y^2 \Rightarrow c_1(y) = \frac{y^3}{3} + c_2$$

Hence: $f = c_3 \Rightarrow \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} + c_2 = c_3$

$$\Rightarrow \boxed{x^3 - 6xy(x+y) + y^3 = c}$$

Example: Show that the differential equation

$$(3xy+y^2)\,dx + (x^2+xy)\,dy = 0$$

is not exact and hence it cannot be solved by the method discussed above.

Sol: Check: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$3x+2y \neq 2x+y$$

So the given equation is not exact.

However, if we proceed with the method given above, we get

$$\frac{\partial f}{\partial x} = 3xy+y^2 \qquad \frac{\partial f}{\partial y} = x^2+xy$$

$$\Rightarrow f = \frac{3}{2}x^2y + y^2x + f_1(y)$$

$$\frac{\partial f}{\partial y} = \frac{3}{2}x^2 + 2y\,x + f_1'(y) = x^2+xy$$

$$\Rightarrow f_1'(y) = \underbrace{-\frac{x^2}{2} - xy}_{}$$

depends on x & y. (Not possible to solve)

Thus, there is no $f(x,y)$ exists and hence it can not be solved in this way.

Exact Differential Equations: Integrating Factors

If an equation of the form $M\,dx + N\,dy = 0$ is not exact, it is sometimes possible to choose a function of x & y such that after multiplying all terms of the equation, it becomes exact. Such a multiplier is called an integrating factor. That is, if $I(x,y)$ is an integrating factor then the differential equation

$$I(x,y)\,M(x,y)\,dx + I(x,y)\,N(x,y)\,dy = 0$$

becomes exact.

Note: Although an equation of the form $M\,dx + N\,dy = 0$ always has integrating factor(s), there is not general rule of finding them. We now discuss some methods of finding integrating factors.

Rule I: By inspection

This method is based on recognition of some standard exact differentials that occur frequently in practice.

i) $d(xy) = y\,dx + x\,dy$

ii) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$ or $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

iii) $d\left(\ln \frac{y}{x}\right) = \frac{x\,dy - y\,dx}{xy}$ or $d\left(\ln \frac{x}{y}\right) = \frac{y\,dx - x\,dy}{xy}$

iv) $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{x^2+y^2}$ or $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2+x^2}$

v) $d(\ln xy) = \frac{y\,dx + x\,dy}{xy}$

Ex. Solve the differential equation

$$y(y^2+1)\,dx + x(y^2-1)\,dy = 0$$

(Check! it is not exact D.E.)

Sol: Rewritting:

$$y^2(y\,dx + x\,dy) + y\,dx - x\,dy = 0$$

Divinding it by y^2: (I.F.)

$$y\,dx + x\,dy + \frac{y\,dx - x\,dy}{y^2} = 0$$

$$\Rightarrow d(xy) + d\left(\frac{x}{y}\right) = 0$$

$$\Rightarrow xy + \frac{x}{y} = c \quad \Rightarrow \boxed{xy^2 + x = cy}$$

Ex. Solve $(y^2e^x + 2xy)\,dx - x^2\,dy = 0$

(Check! it is not exact D.E.)

We know that

$$d\left(\frac{x^2}{y}\right) = \frac{2x}{y}\,dx - \frac{x^2}{y^2}\,dy$$

Diving the given equation by y^2, we get:

$$\left(e^x + \frac{2x}{y}\right)dx - \frac{x^2}{y^2}\,dy = 0$$

$$\Rightarrow d(e^x) + d\left(\frac{x^2}{y}\right) = 0$$

$$\Rightarrow \boxed{e^x + \frac{x^2}{y} = c}$$

Rule II: $Mdx + Ndy = 0$ is homogeneous and $Mx + Ny \neq 0$.

In this case $I(x,y) = \dfrac{1}{Mx+Ny}$ is an integrating factor.

Example: $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0 \quad —①$

Sol.

$$M = x^2y - 2xy^2 \qquad N = -(x^3 - 3x^2y)$$

$$Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2$$

$$= x^2y^2 \neq 0.$$

$$\text{I.F.} = \frac{1}{x^2y^2}$$

Multiplying ① by I.F.

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad —②$$

Now equation ② is an exact differential equation (check!)

If u is the exact differential of ② then:

$$\frac{\partial u}{\partial x} = \frac{1}{y} - \frac{2}{x} \Rightarrow u = \frac{x}{y} - 2\ln x + \varphi(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{x}{y^2} + \varphi'(y) = -\frac{x}{y^2} + \frac{3}{y} \Rightarrow \varphi'(y) = \frac{3}{y}$$

$$\Rightarrow \varphi(y) = 3\ln y + C_1$$

$$\Rightarrow \boxed{\frac{x}{y} - 2\ln x + 3\ln y + C_1 = C_2}$$

or $\boxed{\dfrac{x}{y} - 2\ln x + 3\ln y = C}$

Rule III: $Mdx+Ndy=0$ is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$ then $\frac{1}{Mx-Ny}$ is an integrating factor provided $Mx-Ny \neq 0$

Ex: Solve

$$(xy\sin xy + \cos xy)\,y\,dx + (xy\sin xy - \cos xy)\,x\,dy = 0$$

$$M = (xy\sin xy + \cos xy)\,y \qquad N = (xy\sin xy - \cos xy)\,x$$

$$Mx - Ny = (xy\sin xy + \cos xy)\,xy - (xy\sin xy - \cos xy)\,xy$$

$$= 2xy\cos xy \neq 0$$

$$IF = \frac{1}{2xy\cos xy}$$

Multiplying the given equation by I.F.

$$\frac{1}{2}\left(y\tan xy + \frac{1}{x}\right)dx + \frac{1}{2}\left(x\tan xy - \frac{1}{y}\right)dy = 0$$

it must be exact (check!)

Solution: $\boxed{\frac{x}{y}\sec xy = c}$

Rule IV: An integrating factor for an equation of the form

$$x^a y^b (my\,dx + nx\,dy) + x^r y^s (py\,dx + qx\,dy) = 0$$

is $x^h y^k$ where h & k can be obtained by applying the condition that after multiplication by $x^h y^k$ the equation must become exact. Here a, b, m, n, r, s, p, q are constants.

Example: Solve $(3x + 2y^2)y\,dx + 2x(2x + 3y^2)\,dy = 0$

This equation can be rewritten as

$$x(3y\,dx + 4x\,dy) + y^2(2y\,dx + 6x\,dy) = 0$$

multiplying the integrating factor $x^h y^k$, we get

$$(3x^{h+1}y^{k+1} + 2x^h y^{k+3})\,dx + (4x^{h+2}y^k + 6x^{h+1}y^{k+2})\,dy = 0$$

If it is exact we must have

$$3(k+1)x^{h+1}y^k + 2(k+3)x^h y^{k+2} = 4(h+2)x^{h+1}y^k + 6(h+1)x^h y^{k+2}$$

This is satisfied if

$$3(k+1) = 4(h+2) \quad \&$$
$$2(k+3) = 6(h+1)$$

Solving these we get $h = 1$, $k = 3$.

Integrating factor is xy^3.

Solution: $\boxed{x^3y^4 + x^2y^6 = c}$

Rule V : Most general approach :

The idea is to multiply the given differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

by a function $I(x,y)$ and then try to choose $I(x,y)$ so that the resulting equation

$$I(x,y)\,M(x,y)\,dx + I(x,y)\,N(x,y)\,dy = 0 \quad \text{—— ①}$$

becomes exact.

The above equation is exact if and only if

$$\frac{\partial(IM)}{\partial y} = \frac{\partial(IN)}{\partial x} \quad \text{—— } (*)$$

If a function I satisfying $(*)$ can be found then the given equation ① will be exact. However solving $(*)$ is very difficult so we consider some special cases.

i) An integrating factor I that is either as function of x alone or
ii) a function of y alone.

In the case i), the equation $(*)$ reduces to

$$IM_y = IN_x + NI_x \Rightarrow I_x = \frac{IM_y - IN_x}{N}$$

If $\dfrac{M_y - N_x}{N}$ is a function of x only, say $f(x)$ then

$I(x) = e^{\int f(x)\,dx}$ is an integrating factor. (by solving $\frac{dI}{I} = f(x)\,dx$)

In the case ii) If $\frac{1}{M}(N_x - M_y)$ is a function of y alone, say $f(y)$

then $I(y) = e^{\int f(y)\,dy}$ is an I.F.

Example: Solve $(x^2+y^2+x)\,dx + xy\,dy = 0$ —①

$$M = x^2+y^2+x \qquad N = xy$$

$$\frac{\partial M}{\partial y} = 2y \quad \& \quad \frac{\partial N}{\partial x} = y$$

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y-y) = \frac{1}{x}$$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = x.$$

Multiplying ① by x:

$(x^3+xy^2+x^2)\,dx + x^2y\,dy = 0$ This must be an exact D.E.

Solution: $(3x^4 + 6x^2y^2 + 4x^3) = c$

Ex: Solve $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

$$M = 2xy^4e^y + 2xy^3 + y \qquad N = x^2y^4e^y - x^2y^2 - 3x.$$

$$\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -8xy^3e^y - 8xy^2 - 4$$

$$= -4(2xy^3e^y + 2xy^2 + 1)$$

$$= -\frac{4}{y}\cdot(2xy^4e^y + 2xy^3 + y) = -\frac{4}{y}\cdot M$$

$$\Rightarrow \text{I.F.} = e^{\int -\frac{4}{y}dy} = y^{-4}$$

Solution: $\boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c}$

Exact Differential Equations (Summary)

Necessary and sufficient condition of $M(x, y)dx + N(x, y)dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Integrating Factors

Rule I: By Inspection

Example:

$$d(xy) = ydx + xdy, \quad d\left(\ln xy\right) = \frac{ydx + xdy}{xy} \quad etc.$$

Rule II: $Mdx + Ndy = 0$ **is homogeneous and** $Mx + Ny \neq 0$ **then** $I(x, y) = \dfrac{1}{(Mx + Ny)}$ **is an integrating factor**

Rule III: $Mdx + Ndy = 0$ **is of the form** $f_1(xy)ydx + f_2(xy)xdy = 0$ **and** $Mx - Ny \neq 0$ **then** $\dfrac{1}{(Mx - Ny)}$ **is an integrating factor**

Rule IV: $Mdx + Ndy = 0$ **is of the form** $x^a y^b(mydx + nxdy) + x^r y^s(pydx + qxdy) = 0$ **then** $I(x, y) = x^h y^k$ **may be taken as an integrating factor, where** h, k **are obtained so that the differential equation after multiplication by** $I(x, y)$ **becomes exact**

Rule V: Most general approach

If $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ is function of x alone say $f(x)$, then $I(x) = e^{\int f(x)dx}$ is an I.F.

If $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ is function of y alone say $f(y)$, then $I(y) = e^{\int f(y)dy}$ is an I.F.

Linear Differential Equation:

A first order differential equation is called linear if it can be written in the form:

$$\frac{dy}{dx} + P(x)\,y = Q(x) \qquad (\text{linear in } y)$$

Rewritten as

$$\underbrace{dy + Py\,dx}_{\text{Compare with } M\,dx + N\,dy \text{ to get}} = Q(x)\,dx \qquad \text{— ①}$$

$M = Py \qquad N = 1$

Observe that $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{1}(P - 0) = P$ (function of x alone)

Hence $I.F. = e^{\int P\,dx}$

multiplying ① by $e^{\int P\,dx}$

$$e^{\int P\,dx}\,dy + Py\,e^{\int P\,dx}\,dx = Q(x)\,e^{\int P\,dx}\,dx$$

$$\Rightarrow \quad d\left(e^{\int P\,dx} \cdot y\right) = Q\,e^{\int P\,dx}\,dx$$

Integrating:

$$e^{\int P\,dx} \cdot y = \int Q\,e^{\int P\,dx}\,dx + C$$

OR

$$\boxed{y \cdot IF = \int Q \cdot I.F.\,dx + C}$$

Note: Sometimes a differential equation cannot be put in the form $\frac{dy}{dx} + P(x)\, y = Q(x)$ which is linear in y, but in the form

$$\frac{dx}{dy} + P_1(y)\, x = Q_1(y)$$

which is linear in x, then

$$I.F. = e^{\int P_1 dy}$$

and the solution

$$x \cdot I.F. = \int Q_1 \, I.F. \, dy + c$$

Ex. Solve $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\, y = \frac{4x^2}{1+x^2} \quad (\text{linear in } y)$$

$$I.F. = e^{\int \frac{2x}{1+x^2} dx} = e^{\ln(1+x^2)} = 1+x^2$$

Solution: $y \cdot I.F = \int Q \cdot I.F\, dx + C \Rightarrow y \cdot (1+x^2) = \int 4x^2\, dx + c$

$$\Rightarrow \boxed{y(1+x^2) = \frac{4}{3}x^3 + c}$$

Ex. Solve $(x+2y^3)\frac{dy}{dx} = y \Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y^2$

$$I.F. = e^{\int -\frac{1}{y} dy} = e^{-\ln y} = \frac{1}{y}$$

$$\Rightarrow x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y}\, dy + c$$

$$= \int 2y\, dy + c$$

$$\Rightarrow \boxed{\frac{x}{y} = y^2 + c}$$

Equation reducible to linear form:

An equation of the form

$$f'(y)\frac{dy}{dx} + Pf(y) = Q \qquad \text{— (1)}$$

Putting $f(y) = v \Rightarrow f'(y)\frac{dy}{dx} = \frac{dv}{dx}$

Equation (1) reduces to:

$$\frac{dv}{dx} + Pv = Q \qquad (\text{linear in } v)$$

A special case: Bernoulli's Equation

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n \qquad \text{— (2)}$$

where P & Q are constants or function of x and n is a constant except 0 & 1 is called Bernoulli's differential equation.

Note that equation (2) can be written as

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$$

Subst: $\frac{1}{y^{n-1}} = v \Rightarrow (1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{1}{(1-n)}\cdot\frac{dv}{dx} + Pv = Q$$

$$\Rightarrow \frac{dv}{dx} + P(1-n)v = Q(1-n) \qquad (\text{linear in } v)$$

Ex. $(x^2-2x+2y^2)\,dx + 2xy\,dy = 0$

Sol: Rewriting:

$$2xy\frac{dy}{dx} + x^2-2x+2y^2 = 0$$

or $$2y\frac{dy}{dx} + \frac{2y^2}{x} = \frac{2x-x^2}{x}$$

Subst. $y^2 = v \Rightarrow 2y\frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{dv}{dx} + \frac{2}{x}\cdot v = (2-x) \qquad \text{(linear in } v\text{)}$$

$$\text{I.F.} = e^{\int \frac{2}{x}\cdot dx} = x^2$$

$$\Rightarrow v\cdot x^2 = \int (2-x)\,x^2\,dx + c$$

$$\Rightarrow \boxed{y^2x^2 = \frac{2}{3}x^3 - \frac{x^4}{4} + c}$$

Ex: $\frac{dy}{dx} - y\tan x = -y^2\sec x$

Sol: Dividing by y^2:

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\cdot\tan x = -\sec x$$

putting $\frac{1}{y} = v \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow -\frac{dv}{dx} - v\tan x = -\sec x \Rightarrow \frac{dv}{dx} + v\tan x = \sec x. \qquad \text{(linear in } v\text{)}$$

$$\text{I.F.} = e^{\int \tan x\,dx} = e^{\ln \sec x} = \sec x$$

Solution: $v\cdot\sec x = \int \sec^2 x\,dx + c$

$$\Rightarrow v\cdot\sec x = \tan x + c$$

$$\Rightarrow \boxed{y^{-1}\sec x = \tan x + c}$$

Ex. Solve $\frac{dz}{dx} + \frac{z}{x}\ln z = \frac{z}{x}(\ln z)^2$, $x>0, z>0$

Sol: Dividing by z:

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x}\ln z = \frac{1}{x}(\ln z)^2$$

Subst. $\ln z = t \Rightarrow \frac{1}{z}\cdot\frac{dz}{dt} = \frac{dt}{dx}$

$$\Rightarrow \frac{dt}{dx} + \frac{1}{x}t = \frac{1}{x}t^2$$

$$\Rightarrow \frac{1}{t^2}\frac{dt}{dx} + \frac{1}{t}\cdot\frac{1}{x} = \frac{1}{x}$$ Bernoulli's equation

Subst. $\frac{1}{t} = v \Rightarrow -\frac{1}{t^2}\frac{dt}{dx} = \frac{dv}{dx}$

$$\Rightarrow -\frac{dv}{dx} + \frac{1}{x}\cdot v = \frac{1}{x}$$

$$\text{I.F.} = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

Solution: $$v\cdot\frac{1}{x} = -\int \frac{1}{x}\cdot\frac{1}{x}dx + C$$

$$\frac{v}{x} = \frac{1}{x} + C$$

$$\Rightarrow v = 1 + Cx$$

$$\Rightarrow \boxed{(\ln z)^{-1} = 1 + Cx}$$