MA10002 Mathematics-II : Assignment - 7

1. Find the value of integrals (i) $\int_0^\infty e^{-x^2}dx$ (ii) $\int_0^\infty e^{-x}x^{\frac{3}{2}}dx$ (iii) $\int_0^\infty x^m e^{-ax^n}dx$, where m, n, and a are positive integers. (iv) $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^4 x dx$ (v) $\int_r^s (x-r)^{k-1}(s-x)^{l-1}dx$.

2. Given $\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}$, prove that $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$ where $0 < n < 1$.

3. Show that (i) $\int_0^1 \sqrt{1-x^4}dx = \frac{\{\Gamma(\frac{1}{4})\}^2}{6\sqrt{2\pi}}$ (ii) $\int_0^{\frac{\pi}{2}} \sqrt{\tan x}dx = \frac{\pi}{\sqrt{2}}$ (iii) $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}dx = \frac{(2\pi)^{\frac{3}{2}}}{[\Gamma(\frac{1}{4})]^2}$.

4. (i) Show that $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$, where $m > -1, n > -1$. (ii) If m is a nonnegative integer and n is a positive constant, then show that $\int_0^\infty x^m n^{-x}dx = \frac{m!}{(\log n)^{m+1}}$.

5. Show that if m is a positive integer then $\Gamma(m+\frac{1}{2}) = \frac{(2m-1)(2m-3)(2m-5)\cdots\cdots(3)(1)\sqrt{\pi}}{2^m}$.

6. If m is positive integer and $x - m \neq 0, -1, -2, -3, \cdots\cdots$, then find the value of $\frac{\Gamma(x+m)}{\Gamma(x-m)}$.

7. Show that if m is a positive integer then

   (i) $2 . 4 . 6 . 8 . 10. 12 ., \cdots\cdots, .2m = 2^m\Gamma(m+1)$.

   (ii) $1 . 3 . 5 . 7 . 9 . 11 ., \cdots\cdots, .(2m-1) = \frac{2^{1-m}\Gamma(2m)}{\Gamma(m)}$.

8. Show that $\sqrt{\pi}\Gamma(2m+1) = 2^{2m}\Gamma(m+\frac{1}{2})\Gamma(m+1)$ for any positive integer m. Hence, deduce the Legendre's duplication formula $\sqrt{\pi}\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m+\frac{1}{2})$.

9. Show that $\int_0^\infty \frac{x^m}{x^n+a}dx = \frac{1}{na^{\left(\frac{n-m-1}{n}\right)}}\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(1-\frac{m+1}{n}\right)$, where the constants m, n, and a are such that $a > 0$ and $n > m+1 > 0$.

10. Show that if m is a positive integer then $\Gamma(\frac{1}{m})\Gamma(\frac{2}{m})\cdots\cdots\Gamma(\frac{m-1}{m}) = \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}}$.

Q : 1 Find the value of the integrals (i) $\int_0^\infty e^{-x^2} dx$

(ii) $\int_0^\infty e^{-x} x^{3/2} dx$ (iii) $\int_0^\infty x^m e^{-ax^n} dx$, $m, n, a \notin \mathbb{N}$

(iv) $\int_0^{\pi/2} \sin^4 x \cos^4 x \, dx$ (v) $\int_r^s (x-r)^{k-1} (s-x)^{l-1} dx$

Soln:

(i) $\int_0^\infty e^{-x^2} dx$, Substituting $x^2 = z$, we get,

$$\frac{1}{2}\int_0^\infty e^{-z} z^{-1/2} dz = \frac{1}{2}\int_0^\infty e^{-z} z^{1/2-1} dz = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

Hence $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$.

(ii) $$\int_0^\infty e^{-x} x^{3/2} dx = \int_0^\infty e^{-x} x^{5/2-1} = \Gamma\left(\frac{5}{2}\right)$$

$$= \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2) = \frac{3}{4}\sqrt{\pi}.$$

(iii) $\int_0^\infty x^m e^{-ax^n} dx$, put $ax^n = z \Rightarrow anx^{n-1}dx = dz$

Then, $$\int_0^\infty e^{-ax^n} x^m dx = \lim_{\substack{\epsilon \to 0+ \\ b \to \infty}} \int_\epsilon^b x^m e^{-ax^n} dx$$

$$= \frac{1}{na}\int_{a\epsilon^n}^{ab^n} \left(\frac{a}{z}\right)^{\frac{m+1}{n}-1} e^{-z} dz = \frac{1}{na^{\frac{m+1}{n}}}\int_0^\infty e^{-z} z^{\frac{m+1}{n}-1} dz$$

$$\therefore \int_0^\infty x^m e^{-ax^n} dx = \frac{1}{na^{\frac{m+1}{n}}}\Gamma\left(\frac{m+1}{n}\right).$$

iv) $\int_0^{\pi/2} \sin^4 x \cos^4 x \, dx$

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta \quad x, y > 0$$

Put $2x-1 = m$ and $2y-1 = n$, then

$$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)} \quad m, n > -1$$

Now, put $m = n = 4$, $\theta = x$

$$\int_0^{\pi/2} \sin^4 x \cos^4 x \, dx = \frac{1}{2} \frac{\Gamma(5/2)\Gamma(5/2)}{\Gamma(5)}$$

$$= \frac{1}{2} \cdot \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2) \cdot \frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)}{\lfloor 4} = \frac{3\pi}{256}.$$

v) $\int_r^s (x-r)^{K-1} (s-x)^{l-1} \, dx$

Let $I = \int_r^s (x-r)^{K-1} (s-x)^{l-1} \, dx$

Put $x = r\cos^2\theta + s\sin^2\theta$, then

$$I = \int_0^{\pi/2} 2(s-r)^{K+l-1} \sin^{2K-1}\theta \cos^{2l-1}\theta \, d\theta$$

$$= (s-r)^{K+l-1} B(K, l).$$

Q: 2 Given $\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}$, Prove that

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \quad \text{where } 0<n<1.$$

Solⁿ: Let $I = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx \qquad 0<n<1$

Put $x = \frac{z}{1-z}$, then

$$I = \int_0^1 z^{n-1}(1-z)^{-n}\,dz = \int_0^1 z^{n-1}(1-z)^{-n}\,dz$$

$$= B(n, 1-n) = \Gamma(n)\Gamma(1-n) \quad [0<n<1]$$

Hence, $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \quad 0<n<1.$

Q: 3 Show that (i) $\int_0^1 \sqrt{1-x^4}\,dx = \frac{[\Gamma(\frac{1}{4})]^2}{6\sqrt{2\pi}}$

(ii) $\int_0^{\pi/2} \sqrt{\tan x}\,dx = \frac{\pi}{\sqrt{2}}$ (iii) $\int_0^{\pi/2} \sqrt{\cos x}\,dx = \frac{(2\pi)^{3/2}}{[\Gamma(\frac{1}{4})]^2}$

(i) $\int_0^1 \sqrt{1-x^4}\,dx$, Put $x^4 = z$, Then

$$\int_0^1 \sqrt{1-x^4}\,dx = \frac{1}{4}\int_0^1 z^{-3/4}(1-z)^{\frac{1}{2}}\,dz = \frac{1}{4}\left[B\left(\frac{1}{4},\frac{3}{2}\right)\right]$$

$$= \frac{1}{4}\,\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{2})}{\Gamma(7/4)} = \frac{1}{4}\,\frac{\Gamma(\frac{1}{4})\frac{1}{2}\Gamma(\frac{1}{2})}{\frac{3}{4}\Gamma(\frac{3}{4})}$$

$$= \frac{\sqrt{\pi}}{6}\,\frac{[\Gamma(\frac{1}{4})]^2}{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})} = \frac{\sqrt{\pi}}{6}\,\frac{[\Gamma(\frac{1}{4})]^2}{\pi}\sin\frac{\pi}{4} = \frac{[\Gamma(\frac{1}{4})]^2}{6\sqrt{2\pi}}$$

(ii) $$\int_0^{\pi/2} \sqrt{\tan x}\, dx = \frac{1}{2}\int_0^{\pi/2} 2 \sin^{1/2} x \cos^{-1/2} x\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} 2 \sin^{2\cdot\frac{3}{4}-1} x \cos^{2\cdot\frac{1}{4}-1} x\, dx$$

$$= \frac{1}{2} B\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$= \frac{\pi}{2\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}.$$

(iii) $$\int_0^{\pi/2} \sqrt{\cos x}\, dx = \frac{1}{2}\int_0^{\pi/2} 2 \sin^0 x \cos^{1/2} x\, dx$$

$2m-1 = 0$ and $2n-1 = 1/2$

$\Rightarrow m = \frac{1}{2} \quad n = \frac{3}{4}$

$$\int_0^{\pi/2} \sqrt{\cos x}\, dx = \frac{1}{2} B\left(\frac{1}{2}, \frac{3}{4}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{2\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{\sqrt{\pi}\,\Gamma(3/4)}{2\cdot\Gamma(5/4)} = \frac{\sqrt{\pi}\,\Gamma(3/4)}{\frac{1}{4}\cdot 2\Gamma\left(\frac{1}{4}\right)} = \frac{2\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)\Gamma(3/4)}{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}$$

$$= \frac{2\sqrt{\pi} \times \pi\sqrt{2}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^2} = \frac{(2\pi)^{3/2}}{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}.$$

Q:4 A function $\Gamma : (0,\infty) \longrightarrow \mathbb{R}$ is defined by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$. Show that $\log \Gamma$ is a Convex function.

$$\Gamma(\lambda x + (1-\lambda) y) = \int_0^\infty e^{-t} t^{\lambda x + (1-\lambda) y - 1} dt \quad \lambda \in (0,1)$$

$$= \int_0^\infty \left(e^{-t} t^{x-1}\right)^\lambda \left(e^{-t} t^{y-1}\right)^{1-\lambda}$$

$$\leq \left\{ \int_0^\infty \left(e^{-t} t^{x-1}\right) dt \right\}^\lambda \left\{ \int_0^\infty \left(e^{-t} t^{y-1}\right) dt \right\}^{1-\lambda}$$

[By Holder's inequality)

$$= \left[\Gamma(x)\right]^\lambda \left[\Gamma(y)\right]^{1-\lambda}.$$

Hence $\log \Gamma(\lambda x + (1-\lambda) y) \leq \lambda \log \Gamma(x) + (1-\lambda) \log \Gamma(y)$

Q:5 (i) Show that $\int_0^1 x^m \left[\log\left(\frac{1}{x}\right)\right]^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$, $m > -1$, $n > -1$ (ii) If m is positive integer and n is positive constant, then show that

$$\int_0^\infty x^m n^{-x} dx = \frac{m!}{(\log n)^{m+1}}.$$

(i) $\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx$, Let $\log\frac{1}{x} = t$

Then $\frac{1}{x} = e^t \Rightarrow x = e^{-t} \Rightarrow x^m = e^{-mt}$

$$\Rightarrow \int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx$$

$$= \int_\infty^0 e^{-mt}\, t^n \left(-e^{-t}\right) dt = \int_\infty^0 t^n \left(-e^{-mt}\right) e^{-t} dt$$

$$= \int_0^\infty t^n e^{-(m+1)t} dt$$

Put $(m+1)t = z \Rightarrow dt = \frac{dz}{m+1}$

$$= \int_0^\infty e^{-z} z^{n+1-1} dz \Big/ (m+1)^{n+1}$$

$$= \frac{\Gamma(n+1)}{(m+1)^{n+1}}$$

(ii) $\int_0^\infty x^m n^{-x} dx$

We Know that $n^{-x} = \left(e^{\log n}\right)^{-x} = e^{-(\log n)x}$

Thus, $\int_0^\infty x^m n^{-x} dx = \int_0^\infty x^m e^{-(\log n)x} dx$

Now put $(\log n)x = z \Rightarrow dx = \frac{dz}{\log n}$

Thus, $\int_0^\infty x^m e^{-(\log n)x} dx = \int_0^\infty e^{-z} z^m dz \Big/ (\log n)^{m+1}$

Hence, $\int_0^\infty x^m n^{-x} = \frac{\Gamma(m+1)}{(\log n)^{m+1}} = \frac{\lfloor m}{(\log n)^{m+1}}$.

Q:6 Show that if m is a positive integer, then

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{(2m-1)(2m-3)\cdots 3\cdot 1\sqrt{\pi}}{2^m}.$$

Soln: $\Gamma\left(m+\frac{1}{2}\right) = \Gamma\left(\frac{2m+1}{2}\right)$

$$= \left(\frac{2m+1}{2}-1\right)\Gamma\left(\frac{2m+1}{2}-1\right) \quad \left[\Gamma(x) = (x-1)\Gamma(x-1)\right]$$

$$= \left(\frac{2m+1}{2}-1\right)\Gamma\left(\frac{2m+1}{2}-1\right)$$

$$= \left(\frac{2m+1}{2}-1\right)\left(\frac{2m+1}{2}-2\right)\Gamma\left(\frac{2m-3}{2}\right)$$

$$= \left(\frac{2m-1}{2}\right)\left(\frac{2m-3}{2}\right)\Gamma\left(\frac{2m-3}{2}\right)$$

We can say that if we start with $\Gamma\left(m+\frac{1}{2}\right)$ we have to continue decreasing the argument by unity m times in order to reach $\frac{1}{2}$.

Thus, $\Gamma\left(m+\frac{1}{2}\right) = \left(\frac{2m-1}{2}\right)\left(\frac{2m-3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$

$$\Rightarrow \Gamma\left(m+\frac{1}{2}\right) = \frac{(2m-1)(2m-3)\cdots 3\cdot 1\sqrt{\pi}}{2^m}.$$

Q:7 If m is positive integer and $x-m \neq 0, 1, 2 \cdots$, then find the value of $\frac{\Gamma(x+m)}{\quad}$.

Soln :- $\Gamma(x+m) = (x+m-1)\Gamma(x+m-1)$

By repeating the above process, we get

$$\frac{\Gamma(x+m)}{\Gamma(x-m)} = \frac{(x+m-1)(x+m-2)\cdots(x+m-2m)\Gamma(x+m-2m)}{\Gamma(x-m)}$$

$$= (x+m-1)(x+m-2)\cdots(x-m).$$

Q:8 Show that if m is positive integer, then

(i) $2\cdot 4\cdot 6 \cdots 2m = 2^m\Gamma(m+1)$

(ii) $1\cdot 3\cdot 5 \cdots (2m-1) = \dfrac{2^{1-m}\Gamma(2m)}{\Gamma(m)}$.

Soln: (i) $2\cdot 4\cdot 6 \cdots 2m$

$$= (2.1)(2.2)\cdot(2.3)\cdots(2.m)$$

$$= 2^m \underline{|m} = 2^m\Gamma(m+1)$$

(ii) $1\cdot 3\cdot 5 \cdots (2m-1)$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2m-2)(2m-1)}{2\cdot 4\cdot 6\cdots(2m-2)}$$

$$= \frac{\underline{|2m-1}}{2^{m-1}\Gamma(m)} = \frac{2^{1-m}\Gamma(2m)}{\Gamma(m)}.$$

Q:9 Show that $\sqrt{\pi}\,\Gamma(2m+1) = 2^{2m}\,\Gamma\left(m+\frac{1}{2}\right)\Gamma(m+1)$ $\forall m \in \mathbb{N}$. Hence deduce the Legendre's — duplication formula $\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\,\Gamma(m)\Gamma(m+$

Sln: $2^{2m}\,\Gamma\left(m+\frac{1}{2}\right)\Gamma(m+1)$

$$= \frac{2^{2m}(2m-1)(2m-3)\cdots 3\cdot 1\sqrt{\pi}\,\Gamma(m+1)}{2^m}$$

$$= \frac{2^{2m}(2m)(2m-1)(2m-2)\cdots 4\cdot 2\cdot 1\sqrt{\pi}\,\Gamma(m+1)}{2^m(2m)(2m-2)(2m-4)\cdots 4\cdot 2}$$

$$= \frac{2^{2m}\,\underline{|2m}\,\sqrt{\pi}\,\Gamma(m+1)}{2^m(2m)(2m-2)\cdots 4\cdot 2}$$

$$= \frac{2^{2m}\,\underline{|2m}\,\sqrt{\pi}\,\Gamma(m+1)}{2^m\,2^m\,\underline{|m}} = \frac{\underline{|2m}\,\sqrt{\pi}\,\Gamma(m+1)}{\underline{|m}}$$

$$= \sqrt{\pi}\,\Gamma(2m+1)$$

Hence, $\sqrt{\pi}\,\Gamma(2m+1) = 2^{2m}\,\Gamma\left(m+\frac{1}{2}\right)\Gamma(m+1)$

Deduction: Put $\Gamma(2m+1) = 2m\,\Gamma(2m)$ and $\Gamma(m+1) = m\,\Gamma(m)$, then

$$\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\,\Gamma(m)\,\Gamma\left(m+\tfrac{1}{2}\right)$$

Q :10 Given $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad x>0, y>0$

Show that (i) $B(x,y) = \int_0^{\pi/2} 2\sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta$,

(iv) hence deduce $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

(ii) $B(x,y) = \int_0^{\infty} \frac{u^{x-1}}{(u+1)^{x+y}} du$, $x,y>0$

(iii) $B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$

Soln: (i) $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$

Let $t = \sin^2\theta \Rightarrow dt = 2\sin\theta\cos\theta \, d\theta$

$$\Rightarrow B(x,y) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{x-1}(\cos^2\theta)^{y-1} 2\sin\theta\cos\theta \, d\theta$$

$$\Rightarrow B(x,y) = \int_0^{\frac{\pi}{2}} 2\sin^{2x-1}\theta \cos^{2y-1}\theta \, d\theta$$

$$B(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^{\pi/2} 2\,d\theta = \pi$$

$$\Rightarrow \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \pi$$

$$\Rightarrow \left[\Gamma(\tfrac{1}{2})\right]^2 = \pi \Rightarrow \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

(ii)

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x, y > 0$$

Let $t = u/_{u+1}$, Then $\frac{dt}{du} = \frac{1}{(u+1)^2}$ and

$1-t = \frac{1}{u+1}$

Thus, $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$

$$= \int_0^{\infty} \frac{u^{x-1}\,du}{(u+1)^{x+y}} \quad x, y > 0$$

(iii) $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Sol^n: $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt \quad x > 0$

Let $t = m^2 \Rightarrow dt = 2m\,dm$

Thus, $\Gamma(x) = \int_0^{\infty} m^{2(x-1)} e^{-m^2} 2m\,dm$

$$= 2\int_0^{\infty} m^{2x-1} e^{-m^2} dm$$

Thus, $\Gamma(p) = 2\int_0^{\infty} x^{2p-1} e^{-x^2} dx$

$$\Gamma(q) = 2\int_0^{\infty} y^{2q-1} e^{-y^2} dy$$

$$\Gamma(p)\Gamma(q) = 4\left[\int_0^\infty x^{2p-1} e^{-x^2}\,dx\right]\left[\int_0^\infty y^{2q-1} e^{-y^2}\,dy\right]$$

$$= 4 \iint_Q x^{2p-1} y^{2q-1} e^{-(x^2+y^2)}\,dA$$

Q denotes the entire first quadrant of xy plane

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty \int_0^\infty x^{2p-1} y^{2q-1} e^{-(x^2+y^2)}\,dx\,dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$

$$\Gamma(p)\Gamma(q) = 4\iint_Q \sin^{2p-1}\theta \cos^{2q-1}\theta \, e^{-r^2} r^{2p+2q-2}\,dA$$

$$= \int_0^{\pi/2} 2\sin^{2p-1}\theta \cos^{2q-1}\theta\,d\theta \int_0^\infty 2e^{-r^2} r^{2p+2q-2}\,dr$$

$$= B(p,q)\,\Gamma(p+q)$$

$$\Rightarrow B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Q : 11 Show that $\displaystyle\int_0^\infty \frac{x^m}{x^n+a}\,dx = \frac{1}{n a^{\left(\frac{n-m-1}{n}\right)}}$ ⊗

$\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(1-\frac{m+1}{n}\right)$, $n > m+1 > 0$, $a > 0$

Let $x^n = a\tan^2\theta \Rightarrow x = a^{1/n}\tan^{2/n}\theta$

$$\Rightarrow x^m = a^{m/n}\tan^{2m/n}\theta$$

$$dx = a^{1/n}\tan^{2/n-1}\theta\,(2/n)\sec^2\theta\,d\theta$$

Thus, $\displaystyle\int_0^\infty \frac{x^m}{x^n+a}\,dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\left[a^{m/n}\tan^{2m/n}\theta\right] a^{1/n}\left(\frac{2}{n}\right)\tan^{2/n-1}\theta\,\sec^2\theta\,d\theta}{a\tan^2\theta+a}$$

$$= \frac{1}{n\,a^{(n-m-1)/n}}\int_0^{\pi/2} 2\tan^{\frac{2(m+1)}{n}-1}\theta\,d\theta$$

$$= \frac{1}{n\,a^{(n-m-1)/n}}\int_0^{\frac{\pi}{2}} 2\sin^{\frac{2(m+1)}{n}-1}\theta\,\cos^{-\frac{2(m+1)}{n}+1}\theta\,d\theta$$

Let $2x-1 = \frac{2(m+1)}{n}-1$ and $2y-1 = -\frac{2(m+1)}{n}+1$

$$= \frac{1}{n\,a^{(n-m-1)/n}}\,B(x,y),\quad x = \frac{m+1}{n},\ y = 1-\frac{m+1}{n}$$

$$= \frac{1}{n\,a^{(n-m-1)/n}}\,\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$= \frac{1}{n\,a^{(n-m-1)/n}}\,\Gamma\left(\frac{m+1}{n}\right)\Gamma\left(1-\frac{m+1}{n}\right).$$

(7)

Q:12 Show that if m is a positive integer, then

$$\int_0^{\pi/2} \sin^{2m}\theta \, d\theta = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9 \cdots (2m-1)}{2\cdot 4\cdot 6 \cdots 2m} \frac{\pi}{2}$$

Solⁿ: $$\int_0^{\pi/2} \sin^{2m}\theta \, d\theta = \frac{1}{2}\int_0^{\pi/2} 2\sin^{2m}\theta \cos^0\theta \, d\theta$$

Let $2x-1 = 2m$ and $2y-1 = 0$

Thus, $x = m + \frac{1}{2}$ $y = \frac{1}{2}$

Hence, $$\int_0^{\pi/2} \sin^{2m}\theta \, d\theta = \frac{1}{2} B\left(m+\frac{1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m+1)}$$

$$= \frac{\left[(2m-1)(2m-3)\cdots 3\cdot 1\sqrt{\pi}\right]/2^m \cdot \Gamma\left(\frac{1}{2}\right)}{\left[2\cdot 4\cdot 6 \cdots 2m\right]/2^m} \times \frac{1}{2}$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2m-3)(2m-1)}{2\cdot 4\cdot 6 \cdots 2m} \left(\frac{\pi}{2}\right)$$

Q: Show that if m is a positive integer then,

$$\Gamma(1/m)\Gamma(2/m)\cdots\cdots\Gamma\left(\frac{m-1}{m}\right) = \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}}.$$

Soln: Let $L = \Gamma(1/m)\Gamma(2/m)\cdots\cdots\Gamma\left(\frac{m-1}{m}\right)$

Then, $L^2 = \left[\Gamma(1/m)\Gamma\left(\frac{m-1}{m}\right)\right]\left[\Gamma(2/m)\Gamma\left(\frac{m-2}{m}\right)\right]$
$\cdots\cdots\left[\Gamma\left(\frac{m-1}{m}\right)\Gamma(1/m)\right]$

$$L^2 = \left[\Gamma\left(\frac{1}{m}\right)\Gamma\left(1-1/m\right)\right]\left[\Gamma(2/m)\Gamma\left(1-2/m\right)\right]\cdots\cdots\left[\Gamma\left(\frac{m-1}{m}\right)\Gamma(1/m)\right]$$

$$= \frac{\pi}{\sin\frac{\pi}{m}}\cdot\frac{\pi}{\sin\frac{2\pi}{m}}\cdots\cdots\frac{\pi}{\sin\frac{(m-1)\pi}{m}}$$

$$= \frac{\pi^{m-1}}{\sin\frac{\pi}{m}\cdot\sin\frac{2\pi}{m}\cdots\cdots\sin\frac{(m-1)\pi}{m}} \longrightarrow (1)$$

Now $x^{2m} - 2x^m\cos 2m\theta + 1 = 0$

$$\Rightarrow x^m = \cos 2m\theta + i\sin 2m\theta$$

$$\Rightarrow x = \cos\left(2\theta + \frac{2K\pi}{m}\right) + i\sin\left(2\theta + \frac{2K\pi}{m}\right), \quad K = 0, 1, \ldots m-1$$

Therefor, $x^{2m} - 2x^m\cos 2m\theta + 1$

$$= \prod_{K=0}^{m-1}\left[x^2 - 2x\cos\left(2\theta + \frac{2K\pi}{m}\right) + 1\right]$$

Taking $x = 1$, we get

$$4\sin^2 m\theta = \prod_{K=0}^{m-1} 4\sin^2\left(\theta + \frac{K\pi}{m}\right)$$

$$\sin^2 m\theta = 4^{n-1}\sin^2\theta\,\sin^2\left(\theta+\frac{\pi}{m}\right)\cdots\cdots\sin^2\left(\theta+\frac{(m-1)\pi}{m}\right)$$

$$\sin m\theta = 2^{n-1}\sin\theta\,\sin\left(\theta+\pi/m\right)\cdots\sin\left(\theta+\frac{(m-1)\pi}{m}\right)$$

$$\frac{\sin m\theta}{\sin\theta} = 2^{n-1}\sin\left(\theta+\pi/m\right)\sin\left(\theta+\frac{2\pi}{m}\right)\cdots\cdots\sin\left(\theta+\frac{(m-1)\pi}{m}\right)$$

$$\lim_{\theta\to 0}\frac{\sin m\theta}{\sin\theta} = \lim_{\theta\to 0} 2^{n-1}\sin\left(\theta+\pi/m\right)\cdots\sin\left(\theta+\frac{(m-1)\pi}{m}\right)$$

$$\sin\left(\pi/m\right)\sin\left(2\pi/m\right)\cdots\sin\frac{(m-1)\pi}{m} = \frac{m}{2^{m-1}}$$

Hence from (1), we have

$$L^2 = \frac{\pi^{m-1}\,2^{m-1}}{m}$$

$$\Rightarrow L = \frac{(2\pi)^{\frac{m-1}{2}}}{\sqrt{m}}$$