Assignment - 2 (for practice, no need to submit)

1. What is the cardinality of the following sets;

(a) $S = \{ f \mid f : \{0,1\}^n \to \{0,1\} \}$

(b) $S = \{ x \in \mathbb{Z} \mid 3 \text{ does not divide } x \}$

(c) $S = \{ a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{Q}, i = 0,1,2,\ldots,k \}$

(d) $S = \{ M \mid M = (m_{ij})_{500 \times 500}, m_{ij} \in \mathbb{Q} \}$

(e) $\mathbb{Q} \times \mathbb{Z} \times \mathbb{N}$

(f) $\mathbb{R} - \mathbb{Q}$

2. Show that for fix countable sets A, B, the set X of all fun^{ns} from finite subsets of B into A is countable.

3. Prove or disprove: The sets A and B are equipotent when

$$A = \{ x \in \mathbb{R} \mid 0 \le x \le 1 \}$$
$$B = \{ x \in \mathbb{R} \mid a \le x \le b \}$$

4. Show that strong induction is a valid method of proof by showing that it follows from the well-ordering principle.

5. Find a formula for the following pattern and justify your answer with the help of induction.

$9 \times 0 + 8 = 8$
$9 \times 9 + 7 = 88$
$9 \times 98 + 6 = 888$
$9 \times 987 + 5 = 8888$
$9 \times 9876 + 4 = 88888$
$9 \times 98765 + 3 = 888888$
$9 \times 987654 + 2 = 8888888$
$9 \times 9876543 + 1 = 88888888$
$9 \times 98765432 + 0 = 888888888$
$9 \times 987654321 - 1 = 8888888888$

6. Find the flaw with the following "proof" that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Basis Step: $a^0 = 1$ is true by the defn. of a^0

Inductive Step: Assume that $a^j = 1$ for all non-negative integers j with $j \leq k$. Then note that

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

7. Establish the Bernoulli inequality: If $1+a>0$, then $(1+a)^n \geq 1+na$.

8. From the Binet formula for Fibonacci numbers, derive the relation

$$f_{2n+2} f_{2n-1} - f_{2n} f_{2n+1} = 1, \quad n \geq 1.$$

9. Prove each of the following statements, explicitly mentioning the method of proof:

(a) If k is odd, then 2^{n+2} divides $k^{2^n} - 1$ for all natural number n.

(b) $6 - \sqrt{35} < \frac{1}{10}$ (assume that a calculator is not available).

10. Show that the sum of the squares of the first n Fibonacci numbers is given by the formula

$$f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = f_n f_{n+1}.$$

11. The Lucas numbers are defined by the same recurrence formula as the Fibonacci number,

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 3$$

but with $L_1 = 1$ and $L_2 = 3$; this gives the sequence

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \ldots$$

For the Lucas ~~sequence~~ numbers, derive each of the following identities; where f_n denotes the n-th Fibonacci number.

~~(a) $L_n = u_{n+1} + u_{n-1}, n \geq 2$~~

(a) $L_n = f_{n+1} + f_{n-1} = f_n + 2f_{n-1}, \quad n \geq 2$

(b) $L_{n+1} + L_{n-1} = 5f_n, \quad n \geq 2$

(c) $L_n^2 = f_n^2 + 4f_{n+1}f_{n-1}, \quad n \leq 2$

12. If $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, obtain the Binet formula for the Lucas numbers

$$L_n = \alpha^n + \beta^n, \quad n \geq 1.$$

13. Use the Binet formulas to obtain the relations below

(a) $L_n^2 - 5f_n^2 = 4(-1)^n, \quad n \geq 1$

(b) $L_{2n+1} = 5f_nf_{n+1} + (-1)^n, \quad n \geq 1.$

14. Prove that no integer in the following sequence is a perfect square:

$$11, 111, 1111, 11111, \ldots$$

15. Show that Cantor's Ternary set T where T is the set of real nos. of the form $\frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots + \frac{a_n}{3^n} + \cdots$ where each a_i is either 0 or 2, is not countable.

16. Justify that the following sets are countable:

(a) $\{3, 3^2, 3^3, 3^4, \ldots, 3^n, \ldots\}$

(b) the set of prime numbers.

17. Each of the numbers

$1 = 1$

$3 = 1+2$

$6 = 1+2+3$

$10 = 1+2+3+4$

$\vdots$

represents the number of dots that can be arranged evenly in an equilateral triangle:

This led the ancient Greeks to call a number triangular if it is the sum of consecutive integers, beginning with 1. Show that the difference between the squares of two consecutive triangular numbers is always a cube.

18. Each of the numbers

1

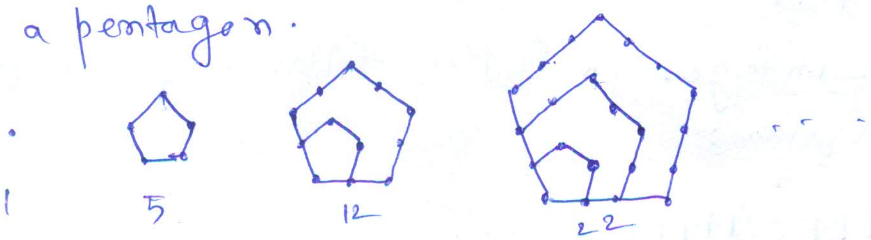
$5 = 1+4$

$12 = 1+4+7$

$22 = 1+4+7+10$

$\ldots$

represents the number of dots that can be arranged evenly in a pentagon.



The ancient Greeks called these pentagonal numbers. If p_n denotes the n-th pentagonal number, where $p_1 = 1$ and $p_n = p_{n-1} + (3n-2)$ for $n \geq 2$, prove that $p_n = \frac{n(3n-1)}{2}$, $n \geq 1$.