TEST

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MATHEMATICAL METHODS

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SOLUTION

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(1.) (a) To Prove: $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

Proof

We know that

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2 \cdot 2 \cdot (n+1)} + \frac{x^4}{2 \cdot 4 \cdot 2^2 (n+1)(n+2)} - \cdots \right]$$

Now, LHS = $J_{-1/2}(x)$

$$= \frac{x^{-1/2}}{2^{-1/2} \Gamma(1/2)} \left[1 - \frac{x^2}{2 \cdot 2 \cdot (1/2)} + \frac{x^4}{2 \cdot 4 \cdot 2^2 (1/2)(3/2)} - \frac{x^6}{2 \cdot 4 \cdot 6 \cdot 2^3 (1/2)(3/2)(5/2)} \right]$$

$$= \sqrt{\frac{2}{\pi x}} \left(1 - \frac{x^2}{2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \right)$$

$$= \sqrt{\frac{2}{\pi x}} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \cdots \right)$$

$$= \sqrt{\frac{2}{\pi x}} \cos x = \text{RHS}$$

thus proved

(b) To Prove: $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

Proof: We have $J_{3/2}(x) = \frac{x^{3/2}}{2^{3/2} \cdot \Gamma(3/2)} \left(1 - \frac{x^2}{2 \cdot 2 \left(\frac{5}{2}\right)} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 7} \right)$

$$= \frac{x^{3/2} \cdot 2}{2\sqrt{2} \cdot \sqrt{\pi}} \left(1 - \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 4 \cdot 5} \right)$$

$\frac{x^2}{3}$

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$$= \sqrt{\frac{2}{\pi x}} \left(\frac{x^2}{3} - \frac{x^4}{2\cdot5\cdot3} + \frac{x^6}{2\cdot4\cdot5\cdot7\cdot3} \dots \right)$$

$$= \sqrt{\frac{2}{\pi x}} \left((1-1) + \left(\frac{x^2}{3!} - \frac{x^2}{2!} \right) + (\right)$$

$$= \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$

(c) Now, we know that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

& $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

we have to Prove, $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{-\cos x}{x} - \sin x \right)$

$$J_{-3/2}(x) = \frac{x^{-3/2}}{2^{-3/2}\,\Gamma(-3/2)} \left(1 - \frac{x^2}{2\cdot2\cdot(-1/2)} + \frac{x^4}{2\cdot4\cdot2^2(-1/2)(1/2)} \right)$$

$$= \frac{2\sqrt{2}\times3}{x\sqrt{x}\,4\sqrt{\pi}} \left(1 + \frac{x^2}{2} - \frac{x^4}{2\cdot4} + \dots \right)$$

$$= \sqrt{\frac{2}{\pi x}}\,\frac{3}{x} \left(1 + \frac{x^2}{2} - \frac{x^4}{2\cdot4} \dots \right)$$

$$= \sqrt{\frac{2}{\pi x}} \left(-\left(\frac{1}{x} - \frac{x^2}{2!} + \frac{x^3}{4!} + \dots \right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) \right)$$

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$$= \sqrt{\frac{2}{\pi x}} \left(\frac{-\cos x}{x} - \sin x \right)$$

2 (a) We have to show that $J_n(x) = 0$ has no repeated roots except at $x=0$.

We can prove this by contradiction.

Let $J_n(x)=0$ has repeated root.

let the repeated root be k.

$\therefore$ $J_n(k)=0$ & $J_n'(k)=0$ — (P)

Now using recurrence relation, we have

$$J_{n-1}(x) = \left(\frac{n}{x}\right) J_n(x) + J_n'(x) \quad \text{— (A)}$$

$$J_{n+1}(x) = \left(\frac{n}{x}\right) J_n(x) - J_n'(x) \quad \text{— (B)}$$

Now, using (P),

$J_{n+1}(k) = 0$ & $J_{n-1}(k) = 0$.

except when at $x=0$

this is not possible

as $J_{n+1}(k) = 0 = J_{n-1}(k)$

thus our assumption is wrong

$J_n(x) = 0$ has no repeated roots except at $x=0$.

(b) We have to Express $J_4(x)$ interms of J_0 & J_1

Using Recurrence $J_{n+1}(x) = \left(\frac{2n}{x}\right) J_n(x) - J_{n-1}(x)$

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$$J_4(x) = \frac{2\times 3}{x} J_3(x) - J_2(x) \quad \text{-(i)}$$

$$J_3(x) = \frac{2\times 2}{x} J_2(x) - J_1(x) \quad \text{-(ii)}$$

$$J_2(x) = \frac{2 J_1(x)}{x} - J_0(x) \quad \text{-(iii)}$$

Using (i), (ii) & (iii), we have,

$$J_4(x) = \frac{6}{x}\left[\frac{4}{x}\left\{\frac{2J_1(x)}{x} - J_0(x)\right\} - J_1(x)\right] - \left[\frac{2J_1}{x} - J_0\right]$$

$$= \frac{24}{x^2}\times\frac{2J_1}{x} - \frac{24}{x^2}J_0 - \frac{6}{x}J_1 - \frac{2J_1}{x} + J_0$$

$$= \left[\frac{48}{x^3} - \frac{8}{x}\right]J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$$

(3) To Prove: $J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \dots) = 1$

~~We know the generating function of $J_n(x)$:~~

$$\cancel{g(x,t) = e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n}$$

From the recurrence relation ~~(iii) & (iv)~~, we have

$$J_n' = \left(\frac{-n}{x}\right)J_n + J_{n-1}$$

$$J_n' = \left(\frac{n}{x}\right)J_n - J_{n+1}$$

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$$\Rightarrow J_{n+1}' = \frac{-(n+1)}{x}J_{n+1} + J_n$$

$$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = 2J_nJ_n' + 2J_{n+1}J_{n+1}'$$

$$= 2J_n\left(\frac{n}{x}J_n - \cancel{J_{n+1}}\right)$$

$$+ 2J_{n+1}\left(\frac{-(n+1)}{x}J_{n+1} + \cancel{J_n}\right)$$

$$= 2\left(\frac{n}{x}J_n^2 - \frac{(n+1)}{x}J_{n+1}^2\right)$$

Now, if $n=0,1,2,\dots$

$$\frac{d}{dx}\left[J_0^2 + J_1^2\right] = 2\left(0 - \frac{J_1^2}{x}\right)$$

$$\frac{d}{dx}\left[J_1^2 + J_2^2\right] = 2\left(\frac{J_1^2}{x} - \frac{2J_2^2}{x}\right)$$

Adding up we get

$$\frac{d}{dx}\left[J_0^2 + 2(J_1^2 + J_2^2 + \dots)\right] = 0$$

$$\Rightarrow J_0^2 + 2(J_1^2 + J_2^2 + \dots) = K.$$

at $x=0$, $J_1 = J_2 = \dots = 0$

& $J_0 = 1$

$\therefore K = 1$

$$\therefore J_0^2 + 2(J_1^2 + J_2^2 + \dots) = 1 \quad \text{Proved}$$

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(b) From prev. part, we have

$J_0^2 + 2(J_1^2 + J_2^2 + \cdots) = 1.$

as $J_n^2(x) \geq 0 \quad \forall\, n \in \mathbb{N}.$

$\therefore \quad 2(J_1^2 + J_2^2 + \cdots) \geq 0$

$\Rightarrow \quad J_0^2 + 2(J_1^2 + J_2^2 + \cdots) \geq J_0^2$

$\Rightarrow \quad J_0^2 \leq 1$

$\Rightarrow \quad |J_0^2| \leq 1$

$\Rightarrow \quad |J_0| \leq 1$

thus proved $\boxed{|J_0|} \leq 1$

(c) To Prove: $|J_n(x)| \leq 2^{-1/2} \quad \forall\, n \geq 1$

We know $J_0^2 + 2(J_1^2 + J_2^2 + \cdots) = 1$

$\Rightarrow \quad 2(J_n^2) \leq 1$

$(\because \ \cancel{J_0^2} \ J_i^2 \geq 0)$

$\Rightarrow \quad (J_n^2) \leq 2^{-1}$

$\Rightarrow \quad |J_n^2| \leq 2^{-1}$

$\Rightarrow \quad |J_n| \leq 2^{-1/2}$

thus $|J_n| \leq 2^{-1/2}$ prove

(4) (a) $4x(1-x)y'' + y' + 8y = 0$ about $x = 0$

The given DE can be written as

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$\Rightarrow x(1-x)\dfrac{d^2y}{dx^2} + \dfrac{y'}{4} + 2y = 0.$

Now, comparing it to the hypergeometric fn

$x(1-x)y'' + (\gamma - (\alpha+\beta+1)x)y' - \alpha\beta y = 0.$

we have.

$\gamma = 1/4; \quad \alpha + \beta + 1 = 0, \quad \alpha\beta = -2.$

$\beta = -1-\alpha \Rightarrow -\alpha(1+\alpha) = -2$

$\alpha^2 + \alpha - 2 = 0$

$(\alpha+2)(\alpha-1) = 0$

$\alpha = 1, -2$

$\beta = -2, 1.$

if $\alpha = 1, \ \beta = -2$

$\therefore y = c_1\, {}_2F_1(\alpha, \beta; \gamma; x) + c_2 x^{1-\gamma}\, {}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma, x)$

$\Rightarrow y = (c_1)\, {}_2F_1(1, -2; 1/4; x) + (c_2)\, x^{3/4}\, {}_2F_1\left(\frac{7}{4}, \frac{-5}{4}; \frac{7}{4}, x\right)$

$y = (c_1)\left(1 - 8x + \frac{32x^2}{5}\right) + c_2\, x^{3/4}\, {}_2F_1\left(\frac{7}{4}, \frac{-5}{4}, \frac{7}{4}, x\right)$

Here, c_1 & c_2 are arbitrary constants

(b) $\lim\limits_{a,b\to\infty} {}_2F_1\left(a, b; \frac{1}{2}; \frac{x^2}{4ab}\right).$

we know that

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r} \frac{x^r}{r!}$$

$$\Rightarrow {}_2F_1\left(a, b, \tfrac{1}{2}, \frac{x^2}{4ab}\right) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(\frac{1}{2})_r} \left(\frac{x^2}{4ab}\right)^r \frac{1}{r!}$$

$$= \lim_{a,b\to\infty} 1 + \frac{\cancel{ab}}{1\cdot\frac{1}{2}\cdot\cancel{4ab}}\,x^2 + \frac{a(a+1)\,b(b+1)}{1\cdot 2\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots}\,x^4$$

$$= \lim_{a,b\to\infty} 1 + \frac{x^2}{2} + \frac{(a+1)(b+1)\,x^4}{24ab} + \cdots$$

$$= \lim_{a,b\to\infty}\left(1 + \frac{x^2}{2} + \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\frac{x^4}{24} + \cdots\right)$$

$$= \left(1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)$$

$$= \cosh(x)$$

Ans

5 (a) A tensor is called symmetric with respect to 2 contravariant or 2 covariant indices if its components remain unchanged upon the interchange of the indices.

∴ $A^{ij} = A^{ji}$ for contravariant

& $A_{ji} = A_{ij}$ for contravariant

& if $A^{ijk}_{pq} = A^{jik}_{qp}$

If a tensor is symmetric with respect to any 2 contravariant & any two covariant indices, it is called 'symmetric'.

A tensor is called skew symmetric with respect to 2 contravariant or 2 covariant if there is a negative sign when we change the indices

i.e, $A^{ij} = -A^{ji}$ or $A_{ji} = -A_{ij}$

& if $A^{ijk}_{pq} = -A^{jik}_{pq}$,

it is skew-symmetric in i & j.

If A^{ij} is a 2nd order contravariant tensor that has N^2 components in the space V_N, then the components are

$$\begin{matrix} A^{11} & A^{12} & A^{13} & \cdots & A^{1N} \\ A^{21} & A^{22} & A^{23} & \cdots & A^{2N} \\ \vdots & & & & \\ A^{N1} & A^{N2} & \cdots & & A^{NN} \end{matrix}$$

Now, if it is skew symmetric, the diagonal elements are 0. For the other elements, we can define each pair as independent as $A^{ij} = -A^{ji}$

∴ thus, the total number of independent components is $\frac{N(N-1)}{2}$

(b) Christoffel symbol or bracket of 1st & 2nd kind is

$$[i,j,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

$$\& \left\{ {l \atop i\,j} \right\} = g^{lk}[i,j,k]$$

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It is another symbol for $\{ij\ l\}$.
These are not tensors
we have
$$\begin{bmatrix} l \\ ij \end{bmatrix} = g^{lk}[ij,k].$$

Now, for each element, we can define the inner multiplicat- given by
$$g_{lm}\begin{bmatrix} l \\ ij \end{bmatrix} = g_{lm}\, g^{lk}[ij,k]$$
$$= \delta^k_m [ij\ k] = [i\ j, k]$$
$$(\underline{\delta^k_m A^m = \underline{A^k}})$$

Now, we have to determine the number of independent components of the Christoffel Symbols.

$\therefore$ for a metric tenson we know that there are $\frac{n(n+1)}{2}$ independent components g_{ij}.
thus for each independent compound there are n symbols.
$\therefore$ by symmetr
there are $\frac{n^2(n+1)}{2}$ number of Christoffel Symbols which have idep. compount