Lecture-6
20/1/2017

Linear Mapping (Linear Transformation).

- Linear Algebra (LA) → Schaum series book. (~~Lipschutz~~) S. LIPSCHUTZ
- Elementary LA → Howard Anton.

$\vec{F} = m\vec{a}$
(force) (acceleration)

$(F_x, F_y, F_z) = m(a_x, a_y, a_z)$.

rotation, ~~translation~~ → examples of LT.

Let V, W → vector spaces over a field $F(\mathbb{R})$.
Then $T: V \to W$ is a LT. if

1) $T(\underset{\sim}{v}_1 + \underset{\sim}{v}_2) = T(\underset{\sim}{v}_1) + T(\underset{\sim}{v}_2)$, $\forall \underset{\sim}{v}_1, \underset{\sim}{v}_2 \in V$
2) $T(c\underset{\sim}{v}) = cT(\underset{\sim}{v})$; $\underset{\sim}{v} \in V$, $c \in R$

Note. Put $c = 0$ in 2)

$T(0 . \underset{\sim}{v}) = 0 . T(\underset{\sim}{v})$.

or, $T(\underset{\sim}{0}_V) = \underset{\sim}{0}_W$.

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$0(1,2,3) = (0,0,0) \in \mathbb{R}^3$

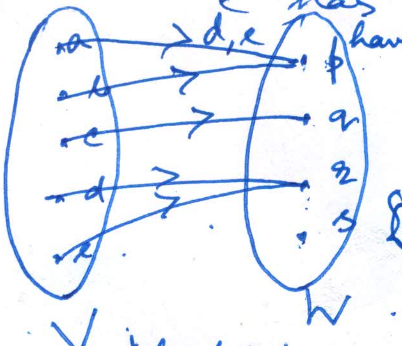
$0(4,5) = (0,0) \in \mathbb{R}^2$

Thm: ~~T is a LT~~ $T: V \to W$ is a LT.

if 1) $T(\underset{\sim}{0}_V) = \underset{\sim}{0}_W$.

2) $T(c_1\underset{\sim}{v}_1 + c_2\underset{\sim}{v}_2) = c_1 T(\underset{\sim}{v}_1) + c_2 T(\underset{\sim}{v}_2)$

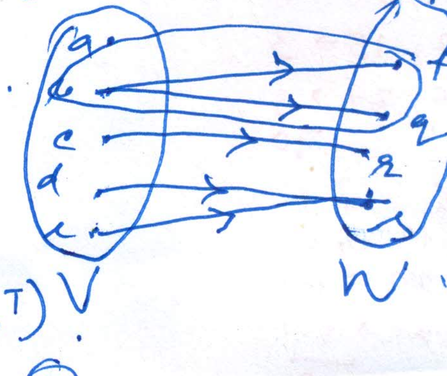
a, b have image p.
e has '' q
d, e have '' r.



$\{p, q, r\} = Im(T)$

V Mapping W

not a mapping



V W

$W = \{p, q, r, s\} = Im\{T\}$

Ex. Let $T : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$T(x, y) = xy$.

1) $T(0,0) = 0.0 = 0$.

$\therefore$ $\underset{\sim}{0}_V$ (here $(0,0)$) is mapped into $\underset{\sim}{0}_W$ (here 0).

~~Bent~~ 2) Let $\underset{\sim}{v}_1, \underset{\sim}{v}_2 \in \mathbb{R}^2$.

$\underset{\sim}{v}_1 = (x_1, y_1)$, $\underset{\sim}{v}_2 = (x_2, y_2)$.

$\underset{\sim}{v}_1 + \underset{\sim}{v}_2 = (x_1 + x_2, y_1 + y_2)$

$$T(\underset{\sim}{v}_1 + \underset{\sim}{v}_2) = T\left[(x_1 + x_2, y_1 + y_2)\right]$$
$$= (x_1 + x_2)(y_1 + y_2)$$

$T(\underset{\sim}{v}_1) = T(x_1, y_1) = x_1 y_1$; $T(\underset{\sim}{v}_2) = x_2 y_2$.

$$\therefore T(\underset{\sim}{v}_1) + T(\underset{\sim}{v}_2) = x_1 y_1 + x_2 y_2$$
$$\neq (x_1 + x_2)(y_1 + y_2) = T(\underset{\sim}{v}_1 + \underset{\sim}{v}_2)$$

2). $V \to$ a vector space of all polynomials.

Let $D : V \to V$ be defined by, $D p(x) = \frac{d p(x)}{dx}$.

Identity element of V is 0. $\left[x^2 \in V. \; Dx^2 = 2x \in V \right]$.

1) $D.0 = 0$.

2). $p(x), q(x) \in V$.

$$D(c_1 p(x) + c_2 q(x)) = \frac{d}{dx}\left[c_1 p(x) + c_2 q(x)\right]$$
$$= c_1 \frac{d}{dx} p(x) + c_2 \frac{d}{dx} q(x)$$
$$= c_1 D p(x) + c_2 D q(x).$$

$\therefore D$ is a LT from V to V.

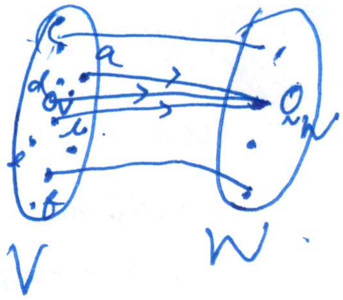
Kernel and Image (range) of T.

Schwarm

Anton

Kernel ~~of~~ a LT $= \ker\{T\}$

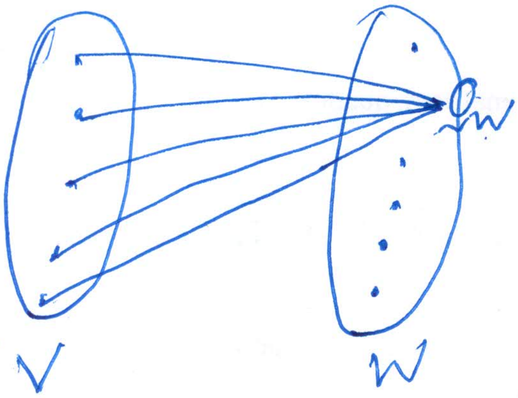
$$\ker\{T\} = \{\underset{\sim}{v} \in V : T(\underset{\sim}{v}) = 0_W\}$$



Ex1. Zero transformation from V to W.

$\ker T = \{a, b, \underset{\sim}{0}_v\}$.

$T: V \to W$ is such that $T(\underset{\sim}{v}) = \underset{\sim}{0}_W \quad \forall\, \underset{\sim}{v} \in V$



$\ker\{T\} = V$

$\mathrm{Im}\{T\} = \underset{\sim}{0}_W$.

Image (range) of LT. $= \mathrm{Im}\{T\}$

$$= \{\underset{\sim}{w} \in W : T(\underset{\sim}{v}) = \underset{\sim}{w} \text{ for some } \underset{\sim}{v} \in V\}$$

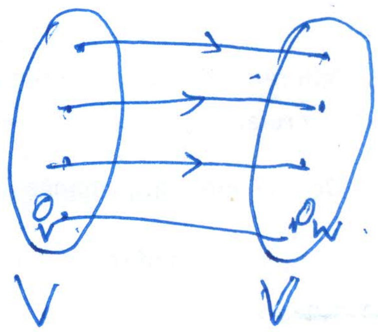
Ex2. Identity transformation

$T: V \to V$.

$T(\underset{\sim}{v}) = \underset{\sim}{v}$

$\ker\{T\} = \{\underset{\sim}{0}_v\}$.

$\mathrm{Im}\{T\} = V$



Ex-3 Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the projection map into the xy-plane defined by $T(x, y, z) = (x, y, 0)$.

$\ker T = \{(0, 0, k) : k \in \mathbb{R}\} = z$-axis.

$\operatorname{Im} T = \{(a, b, 0) ; a, b \in \mathbb{R}\}$ i.e the entire xy-plane.

Im of T = entire (x, y) plane. $(0, 0, k)$

because every (x, y) has ~~some~~ image of some pt. $\in \mathbb{R}^3$.

Th. $\ker\{T\}$ is a subspace of V.

$\operatorname{Im}\{T\}$ " " " of W.

Definition Dimension of $\ker\{T\}$ = nullity of T.

" " $\operatorname{Im}\{T\}$ = rank of T.

Thm. rank T + nullity T = dimension of V

Thm. Let $T : V \to W$. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of V. Then $\{Te_1, Te_2, \ldots, Te_n\}$ spans the $\operatorname{Im}\{T\}$.

Ex 1. Determine the LT $T: \mathbb{R}^4 \to \mathbb{R}^3$ which maps the basis vectors $\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3, \underset{\sim}{e}_4$ of $\mathbb{R}^4$ to $(1,1,1), (-1,0,1), (1,2,3), (1,-1,-3)$. Verify that Rank T + nullity T = dim. $\mathbb{R}^4$.

Solut. $T(x, y, z, w) = (\_\_b\_\_, \_\_c\_\_, \_\_d\_\_)$

$\because \{e_1, e_2, e_3, e_4\}$ is a basis of (x, y, z, w),

then, $(x, y, z, w) = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4$

$$= x(1,0,0,0) + y(0,1,0,0) + z(0,0,1,0) + w(0,0,0,1)$$

$$(x, y, z, w) = x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + z\underset{\sim}{e}_3 + w\underset{\sim}{e}_4$$

$$\therefore T(x, y, z, w) = T(x\underset{\sim}{e}_1 + y\underset{\sim}{e}_2 + z\underset{\sim}{e}_3 + w\underset{\sim}{e}_4)$$

$$= xT\underset{\sim}{e}_1 + yT\underset{\sim}{e}_2 + zT\underset{\sim}{e}_3 + wT\underset{\sim}{e}_4$$

$$= x(1,1,1) + y(-1,0,1) + z(1,2,3) + w(1,-1,-3)$$

$$\therefore T(x, y, z, w) = (x - y + z + w,\ x + 2z - w,\ x + y + 3z - 3w)$$

$$\ker\{T\} = \{(x, y, z, w) : T(x, y, z, w) = (0,0,0)\}$$

$$\therefore T(x, y, z, w) = \qquad = (0, 0, 0)$$

$$\therefore \begin{aligned} x - y + z + w &= 0 \\ x + 2z - w &= 0 \\ x + y + 3z - 3w &= 0 \end{aligned} \qquad \begin{aligned} x - y + z + w &= 0. \\ y + z - 2w &= 0. \\ 2y + 2z - 4w &= 0. \end{aligned}$$

$x - y + z + w = 0.$

$y + z - 2w = 0$

Let $w = d,\ z = c.\ y = 2w - z$
$= 2d - c.$

$x = y - z - w = 2d - c - c - d = d - 2c.$

$\therefore (x, y, z, w) = (d - 2c,\ 2d - c,\ c,\ d)$

$= c(-2, -1, 1, 0) + d(1, 2, 0, 1)$

Also. $\begin{pmatrix} -2 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} -2 & -1 & 1 & 0 \\ 0 & 3 & 1 & 2 \end{pmatrix}$

$\therefore \underbrace{(-2, -1, 1, 0)}_{= \underset{\sim}{v}_1},\ \underbrace{(1, 2, 0, 1)}_{= \underset{\sim}{v}_2}$ are l.i & span $\ker\{T\}$

$\therefore \{\underset{\sim}{v}_1, \underset{\sim}{v}_2\}$ forms a basis for $\ker\{T\}$

$\therefore$ nullity $T = 2.$

$T\underset{\sim}{e}_1,\ T\underset{\sim}{e}_2,\ T\underset{\sim}{e}_3,\ T\underset{\sim}{e}_4$ span Im T.

i.e $(1, 1, 1),\ (-1, 0, 1),\ (1, 2, 3),\ (1, -1, -3)$ span Im T.

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore (1, 1, 1),\ (0, 1, 2)$ form a basis for Im T.

$\therefore$ rank $T = 2.$

$\therefore$ ~~dim~~ nullity T + rank $T = \dim \mathbb{R}^4 = 4.$

(verified)

Ex. Determine the LT

$T : \mathbb{R}^3 \to \mathbb{R}$ defined by

$T(1,1,1) = 3$, $T(0,1,-2) = 1$, $T(0,0,1) = -2$

Find dimensions of $\ker T$ & $\operatorname{Im} T$ to verify

rank T + nullity T = dim V

Hint. Note $(1,1,1)$, $(0,1,-2)$, $(0,0,1)$

form a basis for $\mathbb{R}^3$. (why?)

$\therefore$ each (x,y,z) can be expressed as

$$(x,y,z) = c(1,1,1) + d(0,1,-2) + e(0,0,1)$$

$$= (c, c+d, c-2d+e)$$

$c = x$, $c+d = y$, $c-2d+e = z$

$d = y-x$, $e = z+2d-c$

$$= z+2y-2x-x$$

$$= -3x+2y+z$$

$$(x,y,z) = x(1,1,1) + (y-x)(0,1,-2) + (-3x+2y+z)(0,0,1)$$

$$T(x,y,z) = T(\qquad)$$

$$= xT(1,1,1) + (y-x)T(0,1,-2) + (-3x+2y+z)T(0,0,1)$$

$$T(x,y,z) = 8x - 3y - 2z$$

$$\ker\{T\} = \{(x,y,z) : 8x-3y-2z=0\}$$