Hyperbolic Partial Differential Equation

Explicit method: Let us consider the following initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad t>0, \quad 0<x<1 \qquad \text{—(1)}$$

where the initial conditions are

$$\left.\begin{aligned} u(x,0) &= f(x) \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= g(x) \qquad 0<x<1 \end{aligned}\right\} \text{—(2)}$$

and the boundary conditions

$$\left.\begin{aligned} u(0,t) &= \varphi(t) \\ u(1,t) &= \psi(t) \qquad t\geq 0 \end{aligned}\right\} \text{— (3)}$$

The central-difference approximations for u_{xx} and u_{tt} at the grid point (x_m, t_n) are

$$u_{xx} = \frac{1}{h^2}\left(u_{m-1}^{n} - 2u_m^n + u_{m+1}^{n}\right) + O(h^2)$$

$$u_{tt} = \frac{1}{k^2}\left(u_m^{n-1} - 2u_m^n + u_m^{n+1}\right) + O(k^2)$$

$m, n = 0, 1, 2 \ldots$

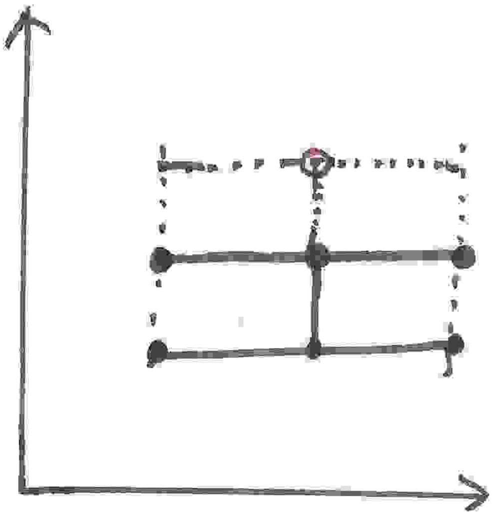
The equation (1) becomes

$$\frac{1}{k^2}\left(u_m^{n-1} - 2u_m^n + u_m^{n+1}\right) = \frac{c^2}{h^2}\left(u_{m-1}^{n} - 2u_m^n + u_{m+1}^{n}\right)$$

$$\Rightarrow u_m^{n+1} = r^2 u_{m-1}^{n} + 2(1-r^2)u_m^n + r^2 u_{m+1}^{n} - u_m^{n-1} \qquad \text{—(4)}$$

where $r = \dfrac{ck}{h}$

The schematic diagram:



In order to start the computations, we need the data on two previous time line.

The information required on the line $t=K$ is obtained by using a suitable approximation to the inital condition

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Using second order central approximation:

$$\frac{1}{2K}\left(u_m^1 - u_m^{-1}\right) = g_m \quad \text{———} \quad (4a)$$

Putting $n=0$ in (4):

$$u_m^1 = r^2 u_{m-1}^0 + 2(1-r^2)\, u_m^0 + r^2 u_{m+1}^0 - u_m^{-1}$$

$$= r^2 f_{m-1} + 2(1-r^2)\, f_m + r^2 f_{m+1} - u_m^{-1} \quad \text{—} (4b)$$

where $f_m = u_m^0$

Eliminating u_m^{-1} between (4a) & (4b) we obtain the expression for u along $t=K$, i.e.: for $n=1$ as

$$u_m^1 = r^2 f_{m-1} + 2(1-r^2)\, f_m + r^2 f_{m+1} + 2K g_m - u_m^1$$

$$\Rightarrow \quad u_m' = \frac{1}{2}\left[r^2 f_{m-1} + 2(1-r^2) f_m + r^2 f_{m+1} + 2k g_m\right]$$

This gives the values of u for $n=1$. For $n = 2, 3, \ldots$ the values are obtained from (4).

The truncation error of this method is $O(h^2+k^2)$ and the formula (4) is convergent for $0 < r \le 1$.

<u>Example</u>: Solve the wave equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

with B.C.

$$u(0,t) = u(1,t) = 0 \qquad t > 0,$$

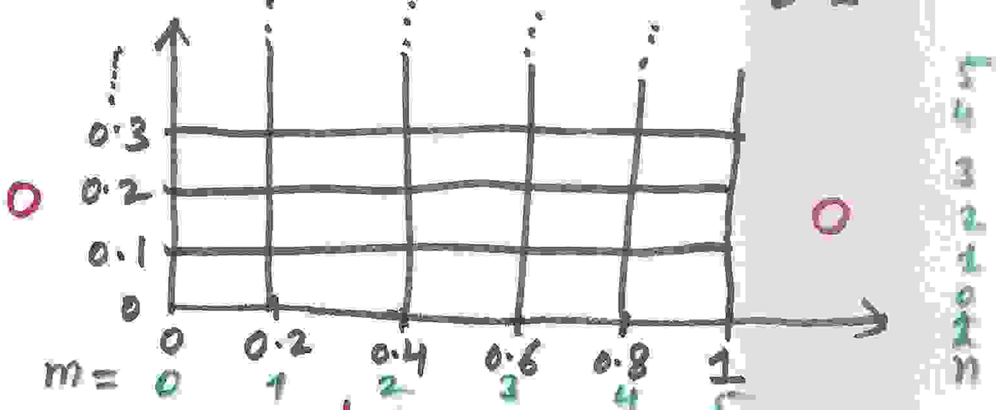
and I.C.
$$u(x,0) = \frac{1}{2}\sin \pi x$$
$$u_t(x,0) = 0 \qquad 0 \le x \le 1$$

for $x = 0, 0.2, \ldots, 1$ and $t = 0, 0.1, 0.2, \ldots, 0.5$.

<u>Sol.</u>: The explicit formula is

$$u_m^{n+1} = r^2 u_{m-1}^n + 2(1-r^2) u_m^n + r^2 u_{m+1}^n - u_m^{n-1}$$

$$h = 0.2 \quad K = 0.1 \qquad r = \frac{k}{h} = \frac{0.1}{0.2} = 0.5$$



$\frac{1}{2}\sin \pi x$

B.Cs: $u_0^n = u_5^n = 0, \quad n = 1, 2, \cdots 5.$

ICs: $u_m^0 = \frac{1}{2} \sin \pi x_m$

& $\frac{\partial u}{\partial t}(x_m, 0) = 0 \Rightarrow \dfrac{u_m^1 - u_m^{-1}}{2K} = 0$

$$\Rightarrow \quad u_m^1 = u_m^{-1}$$

For $r = 0.5$, the finit diff. scheme becomes:

$$u_m^{n+1} = 0.25\, u_{m-1}^n + 1.5\, u_m^n + 0.25\, u_{m+1}^n - u_m^{n-1} \quad -(1)$$

For $n = 0$:

$$u_m^1 = 0.25\, u_{m-1}^0 + 1.5\, u_m^0 + 0.25\, u_{m+1}^0 - u_m^{-1}$$

Since $u_m^1 = u_m^{-1}$, then

$$u_m^1 = 0.125\, u_{m-1}^0 + 0.75\, u_m^0 + 0.125\, u_{m+1}^0$$

The above formula gives the values of u for $n = 1$

For $n = 2, 3 \cdots$ the value are obtained form (1).

See the computed values in Table 1.

$t=0.5$ $n=5$	0	0.0057	0.0093	0.0093	0.0057	0
$t=0.4$ $n=4$	0	0.0952	0.1539	0.1539	0.0952	0
$t=0.3$ $n=3$	0	0.1755	0.2840	0.2840	0.1755	0
$t=0.2$ $n=2$	0	0.2391	0.3869	0.3869	0.2391	0
$t=0.1$ $n=1$	0	0.2799	0.4528	0.4528	0.2799	0
$t=0$ $n=0$	0	0.2939	0.4755	0.4755	0.2939	0
	$x=0$ $m=0$	$x=0.2$ $m=1$	$x=0.4$ $m=2$	$x=0.6$ $m=3$	$x=0.8$ $m=4$	$x=1$ $m=5$