Linear Algebra

Lecture 15

Annihilator :

Let V be a finite dimensional vector space over $\mathbb{F}$. For every <u>subset</u> S of V, the annihilator of S, denoted as S°, is defined as

$$S^\circ = \{ f \in V^* \mid f(x) = 0 \quad \forall x \in S \}$$

<u>claim</u>: S° is a subspace of V^*.

$$f_1, f_2 \in S^\circ, \quad a, b \in \mathbb{F}, \quad x \in S$$

$$(af_1 + bf_2)(x) = af_1(x) + bf_2(x)$$
$$= 0$$

$\Rightarrow$ $af_1 + bf_2 \in S^\circ$

$\Rightarrow$ S° is a subspace of V^*.

<u>claim</u>; Let W_1 and W_2 be subspaces of V.

Then $W_1 = W_2 \iff W_1^\circ = W_2^\circ$

$$W_1^\circ = \{ f \in V^* \mid f(x) = 0 \quad \forall x \in W_1 \}$$

$$W_2^\circ = \{ g \in V^* \mid g(y) = 0 \quad \forall y \in W_2 \}$$

We know that $W_1^\circ = W_2^\circ$

if $f \in W_1^\circ$

$$f(x) = 0 \quad \forall\ x \in W_1$$

$f \in W_2^\circ$

$$f(y) = 0 \quad \forall\ y \in W_2$$

<u>Exercise</u>: If W is a subspace of V and $x \notin W$. Then there exists $f \in W^\circ$ such that $f(x) \neq 0$

$$V/W \longrightarrow F$$

$$x + W \longmapsto f(x)$$

Suppose $W_1 \neq W_2$, let $x \in W_1 \setminus W_2$

$$\exists\ f \in W_2^\circ \quad \text{s.t.} \quad f(x) \neq 0$$

Example:

$$V = \mathbb{R}^2$$

$$W = \left\{ t\begin{pmatrix} 1 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$W^\circ = ??$$

Let $f \in V^* = (\mathbb{R}^2)^*$

Matrix representation of f.

$\{e_1, e_2\}$ ordered basis of $\mathbb{R}^2$

$\{1\}$ $\mathbb{R}$.

$$A_f = \begin{bmatrix} f(e_1) & f(e_2) \end{bmatrix}_{1\times 2}$$

For any vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$= A_f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} f(e_1) & f(e_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To find f such that

$$f(w) = 0 \qquad \forall\, w \in W$$

$$\begin{bmatrix} f(e_1) & f(e_2) \end{bmatrix} \begin{bmatrix} t \\ -t \end{bmatrix} = 0 \iff f(e_1) = f(e_2)$$

$\iff \forall f \in W^{\circ}$

$f(e_1) = f(e_2)$

$A_f = [t \quad t]_{1\times 2} \quad \forall t \in \mathbb{R}$

$W^{\circ} = \{(1 \quad 1)t \mid t \in \mathbb{R}\}$

Compute S° where

$S = \left\{ t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$

$\boxtimes$

Ex: $(S^{\circ})^{\circ} = \text{span}(\psi(S))$

Proposition: Let V be a finite dimensional vector space over $\mathbb{F}$ and let W be a subspace of V.

$\dim(W) + \dim(W^{\circ}) = \dim(V)$

Proof: Let $\{x_1, \ldots, x_k\}$ be an ordered basis for W.

Extend it to $\{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ an ordered basis for V.

Let $\{f_1, \ldots, f_n\}$ be the dual basis for V^* corresponding to $\{x_1, \ldots, x_n\}$.

$$f_i(x_j) = \delta_{ij}$$

<u>claim</u>: $\{f_{k+1}, f_{k+2}, \ldots, f_n\}$ is a basis for W^0.

Let $f \in W^0 \subseteq V^*$

$$f(x) = \sum_{i=1}^{n} f(x_i) f_i$$

For $x \in W$, $f(x) = 0$

$x = \alpha_1 x_1 + \cdots + \alpha_k x_k$

$+ \alpha_{k+1} x_{k+1} + \cdots + \alpha_n x_n$

$$x = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_k \\ \alpha_{k+1} \\ \vdots \\ \alpha_n \end{pmatrix}$$

Proposition: Let V and W be finite dimensional vector spaces over $\mathbb{F}$ and $T: V \to W$ is linear. Then

$$N(T^t) = (R(T))^0$$

Proof: Let $g \in N(T^t) \subseteq W^*$

$$\Rightarrow T^t(g) = 0$$

$$\Rightarrow gT = 0$$

$$\Rightarrow \forall x \in V$$

$$(gT)v = (0)v = 0$$

$$\Rightarrow g(T(v)) = 0$$

$$\Rightarrow g \in (R(T))^0$$

$$\Rightarrow N(T^t) \subseteq (R(T))^0$$ ∎

$$\boxed{(N(T) = (R(T^t))^0)}$$

Exercise: $\quad T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$A_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$R(A_T) = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$$

$$\left(R(A_T)\right)^{0} = \{ (0, 0, z) \mid z \in \mathbb{R} \}^{*}$$

Now, $\quad T^t : (\mathbb{R}^3)^* \longrightarrow (\mathbb{R}^2)^*$

Matrix representation of T^t is

$$A_T^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$N(T^t) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$$

$$\boxed{\begin{array}{c} (0,\ 0,\ 2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} \end{array}}$$

$$v^T A = 0$$

$$\Updownarrow$$

$$A^T v = 0^T$$

Proposition: Let V and W be finite dimensional vector spaces over $\mathbb{F}$. Let $T: V \to W$ be a linear transformation. Then

(1) T is onto $\iff$ T^t is one-to-one.

(2) T^t is onto $\iff$ T is one-to-one.

Proof:

Let T be onto.

$\Rightarrow \quad R(T) = W$

$\Rightarrow \quad (R(T))^{0} = W^{0} = \{0\}$ $\quad *$

$\quad\quad \parallel$

$\Rightarrow \quad N(T^{t}) = \{0\}$

$\Rightarrow \quad T^{t}$ is one-to-one.

Conversely, assume

T^{t} is one-to-one.

$\Rightarrow N(T^{t}) = \{0\}$

$\quad\quad \parallel$

$\Rightarrow \quad (R(T))^{0} = \{0\}^{0}$ $\quad *$ $\quad W_1^{0} = W_2^{0}$

$\Rightarrow W_1 = W_2$

$\Rightarrow \quad R(T) = W$

$\Rightarrow \quad T$ is onto.