Tutorial Problems set-III

Note: All these problems can be solved using the results of Chapter-3.

[0.0.1] *Exercise* Can you construct a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ such that $R(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$?

Sol. We can check that the dimension of $R(T)$ is 3. That is, $rank(T)$ is 3. Since T is a linear transformation from a finite dimensional vector space to another space. Using rank nullity theorem we have $2 = nullity(T) + 3$. Hence $nullity(T) = -1$ which is not possible, Hence there is no such linear transformation.

[0.0.2] *Exercise* Can you construct a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $R(T) = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$?

Sol. The $rank(T)$ is 2 (check!). Using rank nullity theorem , nullity of such transformation if it exists should be 0. This says that it is possible to have such type of linear transformation.

$T(x_1, x_2) = (x_1 + x_2, -x_1, -x_2)$, it is clear that this T is a linear transformation.

$$R(T) = \{T(x_1, x_2) : (x_1, x_2) \in \mathbb{R}^2\}$$

$$= \{(x_1 + x_2, -x_1, -x_2) : x_1, x_2 \in \mathbb{R}\}$$

$$= \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$$

[0.0.3] *Exercise* Let $\mathbb{V} = \mathbb{R}^n$ and A be a $n \times n$ matrix. If $Ax = 0$ has a unique solution, then show that $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.

Sol. Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(x) = Ax$. Notice that $Ker(T) = \{x \in \mathbb{R}^n : T(x) = 0\} = \{x \in \mathbb{R}^n : Ax = 0\}$. It is given that $Ax = 0$ system of equation has a unique solution which is trivial. Hence $Ker(T) = \{0\}$. Therefore nullity of T is 0.

The map T is one-one and $T : \mathbb{R}^n \to \mathbb{R}^n$ then T is onto. That is, $R(T) = \{T(x) : x \in \mathbb{R}^n\} = \{Ax : x \in \mathbb{R}^n\} = \mathbb{R}^n$. This implies for each $b \in \mathbb{R}^n$, there exists unique $x \in \mathbb{R}^n$ such that $Ax = b$.

[0.0.4] *Exercise* Let $T : \mathbb{V} \to \mathbb{V}$ be a linear map such that $R(T) = Ker(T)$. What can you say about T^2?

Sol. We prove that the map T^2 is a zero linear transformation. Here T^2 means $T \circ T$. Let $x \in \mathbb{V}$. Then $T^2(x) = T(T(x)) = 0$ as $Ker(T) = R(T)$. Hence T^2 is zero linear transformation.

Converse is not true. That is, there is a LT $T : \mathbb{V} \to \mathbb{V}$ such that $T^2 \equiv 0$ but $R(T) \neq Ker(T)$. $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (z, 0, 0)$. Then $T^2 \equiv 0$. Notice that $Ker(T) = \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $R(T) = \{(x, 0, 0) : x \in \mathbb{R}$. It is clear that $Ker(T) \neq R(T)$.

[0.0.5] *Exercise* Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the field $\mathbb{Q}$. f is a map from $\mathbb{V}$ to $\mathbb{W}$ such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{V}$. Sow that f is a linear transformation.

Sol. Given that $\mathbb{V}$ and $\mathbb{W}$ are two vector spaces over the field $\mathbb{Q}$. To show that T is a linear transformation. That is to show $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in \mathbb{V}$ and for all $\alpha, \beta \in \mathbb{Q}$. It is enough to show $T(\alpha x) = \alpha x$ for all $x \in \mathbb{V}$ and for all $\alpha \in \mathbb{Q}$. Take α is a positive integer, that is $\alpha = m$. Then

$$f(mx) = f(x + \cdots + x)$$

$$= f(x) + \cdots + f(x)$$

$$= mf(x).$$

We now show that $f(-x) = -f(x)$.
$$f(x + (-x)) = f(x) + f(-x).$$

$$\implies f(0) = f(x) + f(-x) \text{ (We prove that } f(0_{\mathbb{V}}) = 0_{\mathbb{W}}).$$

$$\implies f(-x) = -f(x).$$

Take α is a negative integer, that is $\alpha = -m$.

$$f(-mx) = -f(mx)$$

$$= -mf(x)$$

Take $\alpha = \frac{m}{n}$ where n is positive integer.
$$f(mx) = f(n \times \frac{m}{n}x)$$

$$\implies mf(x) = nf(\frac{m}{n}x)$$

$$\implies f(\frac{m}{n}x) = \frac{m}{n}f(x).$$

We have proved that $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{Q}$ and for all $x \in \mathbb{V}$.

$$f(\alpha x + \beta y) = f(\alpha x) + f(\beta y)$$

$$= \alpha f(x) + \beta f(y).$$

Hence f is a linear transformation.

[0.0.6] *Exercise* Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces over the field $\mathbb{R}$. f is a map from $\mathbb{V}$ to $\mathbb{W}$ such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{V}$. Is f a linear transformation.

Sol. There is a non-linear map from $\mathbb{R}(\mathbb{R})$ to $\mathbb{R}(\mathbb{R})$ such that $f(x + y) = f(x) + f(y)$. We know that $\mathbb{R}(\mathbb{Q})$ is an infinite dimensional space. Let B be a basis of $\mathbb{R}(\mathbb{Q})$.

We can construct a basis B' from B which contains 1 and $\sqrt{2}$. This basis contains exactly one rational number which is 1. If B' contains two rationals 1 and q, then they are linearly dependent which is not possible.

We can construct a linear transformation f $\mathbb{R}(\mathbb{Q})$ to $\mathbb{R}(\mathbb{Q})$ such that

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in B' \text{ and irrational} \end{cases}$$

Since f is a map from $\mathbb{R}$ to $\mathbb{R}$ and it satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ as f is a linear transformation from $\mathbb{R}(\mathbb{Q})$ to $\mathbb{R}(\mathbb{Q})$.

The map f is not a linear transformation from $\mathbb{R}(\mathbb{R})$ to $\mathbb{R}(\mathbb{R})$. Assume that f is a linear transformation from $\mathbb{R}(\mathbb{R})$ to $\mathbb{R}(\mathbb{R})$. Then $f(\sqrt{2}) = \sqrt{2}f(1)$ which implies $0 = \sqrt{2}$ a contradiction. Hence f is a non-linear from $\mathbb{R}(\mathbb{R})$ to $\mathbb{R}(\mathbb{R})$ satisfying $f(x + y) = f(x) + f(y)$.

[0.0.7] *Exercise* Let f be a linear transformation from $\mathbb{V}$ to $\mathbb{W}$. If S is a subspace of $\mathbb{V}$ then $f(S)$ is a subspace of $\mathbb{W}$. Moreover, if $x_1, \ldots, x_k$ generates S then $f(x_1), \ldots, f(x_k)$ generates $f(S)$.

Sol. Let $y_1, y_2 \in f(S)$ and $\alpha.\beta \in \mathbb{F}$. Then we have $x_1 x_2 \in S$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Take

$$\alpha y_1 + \beta y_2$$

$$= \alpha f(x_1) + \beta f(x_2)$$

$$= f(\alpha x_1 + \beta x_2).$$

Since S is a subsapce, we have $\alpha x_1 + \beta x_2 \in S$. Then $f(\alpha x_1 + \beta x_2) \in f(S)$. That is $\alpha y_1 + \beta y_2 \in f(S)$. Hence $f(S)$ is a subspace of $\mathbb{W}$.

Let $y \in f(S)$. Then $f(x) = y$ for some $x \in S$. Since $\{x_1, \ldots, x_k\}$ generates S, we have $x = c_1 x_1 + \cdots + c_k x_k$. $y = f(x) = f(c_1 x_1 + \cdots + c_k x_k) = c_1 f(x_1) + \cdots + c_k f(x_k)$. Then y is a linear combination of $f(x_1), \cdots, f(x_k)$ and $f(x_1), \cdots, f(x_k)$ are elements of $f(S)$. Hence $\{f(x_1), \ldots, f(x_k)\}$ generates $f(S)$.

[0.0.8] *Exercise* Check that following are linear transformations.

1. Let $\mathbb{V}$ be the vector space of all convergent real sequence. $T : \mathbb{V} \to \mathbb{R}$ be defined by $T(x_n) = \lim_{n \to \infty} x_n$

2. Let $\mathbb{D}[0,1]$ be the set of set of all continuously differentiable function on $[0, 1]$ and let $T : \mathbb{D}[0, 1] \to \mathbb{C}[0, 1]$ defined by $T(f) = f'$.

3. Let $\mathbb{V} = \mathbb{R}^n$ and $\mathbb{W}$ be the subspace given by $\mathbb{W} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$. Consider $P : \mathbb{V} \to \mathbb{W}$ given by $P(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, 0)$.

Sol. Do yourself.

[0.0.9] *Exercise* Let $B = \{[2, 1, 1]^t, [3, 4, 1]^t, [2, 2, 1]^t\}$ be a basis of $\mathbb{V} = \mathbb{R}^3$. Find the dual basis of $\mathbb{V}^*$ corresponding to basis B.

Sol. Do your self.

[0.0.10] *Exercise* Let $f_1(x) = x_1 + 2x_2 + 3x_3$, $f_2(x) = 2x_1 + 3x_2 + 5x_3$ and $f_3(x) = 3x_1 + 2x_2 + 4x_3$, $x = [x_1, x_2, x_3]^t$ be linear functionals on $\mathbb{R}^3$. Prove that $\{f_1, f_2, f_3\}$ form a basis of the dual space of $\mathbb{R}^3$. Find a basis B of $\mathbb{R}^3$ such that $\{f_1, f_2, f_3\}$ is a dual basis corresponding to B.

Sol. Do yourself.

[0.0.11] *Exercise* Let $\mathbb{V}$ be a finite dimensional vector space over the field $\mathbb{F}$. Let $f, g \in \mathbb{V}^*$ be non-zero. Then f and g are linearly independent if and only if $ker(f) = ker(g)$.

Sol. We first assume that f and g are linearly dependent. Then there exists $\alpha \in \mathbb{F} - \{0\}$ such that $f = \alpha g$. This implies $Ker(f) = Ker(g)$.

We now assume that $Ker(f) = Ker(g)$. Notice that $rank(f) = rank(g) = 1$. Then $Nullity(f) = Nullity(g) = \text{DIM}(\mathbb{V}) - 1$. Let $\text{DIM}(\mathbb{V}) = n$. Let $\{u_1, \ldots, u_{n-1}$ is a basis of $Ker(f) = Ker(g)$. We extend it to a basis for $\mathbb{V}$ which is $\{u_1, \ldots, u_{n-1}, u_n\}$. $f(u_i) = g(u_i) = 0$ for $i = 1, \ldots, n - 1$ and $f(u_n), g(u_n)$ are non-zero.

Since $f(u_n)$ and $g(u_n)$ are two nonzero elements of $\mathbb{F}$, they are LD and $f(u_n) = \alpha g(u_n)$ for some non-zero $\alpha \in \mathbb{F}$.

Let $x \in \mathbb{V}$. Then $x = c_1 u_1 + \cdots + c_{n-1} u_{n-1} + c_n u_n$.

$f(x) = c_n f(u_n)$ and $g(x) = c_1 g(u_n)$. Then $f(x) = \alpha g(x)$ for all $x \in V$.

Hence f and g are linearly dependent.

[0.0.12] Exercise Let $A \in \mathbb{M}_n(\mathbb{F})$. Let $T : \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_n(\mathbb{F})$ be a linear transformation defined by $T(B) = AB$ for all $B\mathbb{M}_n(\mathbb{F})$. Then there exists a basis Q such that $[T]_Q = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A \end{bmatrix}$.

Sol. We use E_{ij} to denote the matrix size n whose ijth entry is 1 and rest of the entries are zero.

Let $Q = \{E_{11}, E_{21}, \ldots, E_{n1}, E_{12}, E_{22}, \ldots, E_{n2}, \ldots, E_{1n}, E_{2n}, \ldots, E_{nn}\}$. Q is a basis of $\mathbb{M}_n(\mathbb{F})$.

Just check that $[T]_Q = \begin{bmatrix} A & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & A \end{bmatrix}$.

[0.0.13] Exercise Let $\mathbb{V}$ be a finite dimensional vector space over the field $\mathbb{F}$. Let $T : \mathbb{V} \to \mathbb{V}$ be a linear operator such that $rank(T^2) = rank(T)$. Prove that $Ker(T) \cap R(T) = \{0\}$.

Sol. We show that $R(T) = R(T^2)$. Let $y \in R(T^2)$.

Then there exists $x \in \mathbb{V}$ such that $T^2(x) = y$. $T(T(x)) = y$, this implies $y \in R(T)$.

Hence $R(T^2) \subseteq R(T)$. They have same dimension. Therefore $R(T^2) = R(T)$. Using rank-nullity theorem, we have $Nullity(T^2) = Nullity(T)$. We notice that $Ker(T) \subseteq Ker(T^2)$. Hence $Ker(T^2) = Ker(T)$. Let $x \in Ker(T) \cap R(T) = \{0\}$. To show that $x = 0$. Then $T(x) = 0$ and $T(y) = x$.

$T^2(y) = T(T(y)) = T(x) = 0$. Hence $y \in Ker(T^2)$. Therefore $y \in Ker(T)$. $T(y) = 0$. Hence $x = 0$.

[0.0.14] Exercise Let $\mathbb{V} = \mathbb{P}_2(x, \mathbb{R})$, set of all real polynomials of degree at most 2. Let $f_1, f_2, f_3 \in \mathbb{V}^*$ defined by

$$f_1(P) = \int_0^1 P(x)dx, \ f_2(P) = \int_0^2 P(x)dx \text{ and } f_3(P) = \int_0^3 P(x)dx$$

Then find the basis B of $\mathbb{V}$ such that $\{f_1, f_2, f_3\}$ is the dual basis of $\mathbb{V}^*$ corresponding to B.

[0.0.15] *Exercise* Let $\mathbb{V}$ be a finite dimensional vector space over the filed $\mathbb{F}$ and let $\mathbb{W}$ be a subspace of $\mathbb{V}$. If f is a linear functional on $\mathbb{W}$. Prove that there is a linear functional g on $\mathbb{V}$ such that $g(v) = f(v)$ for all $v \in \mathbb{V}$.

Sol. Let $\{v_1, \ldots, v_k\}$ be a basis of $\mathbb{W}$. Then $f(v_i) = \alpha_i$ for $i = 1, \ldots, k$ where $\alpha_1, \ldots, \alpha_k \in \mathbb{F}$. We extend $\{v_1, \ldots, v_k, \ldots, v_n\}$ to a basis for $\mathbb{V}$ which is $\{v_1, \ldots, v_k, \ldots, v_n\}$.

We know that there exists a unique linear transformation g such that $g(v_i) = \alpha_i$ for $i = 1, \ldots, k$ and $g(v_i) = \beta_i$ for $i = k+1, \ldots, n$, Where $\beta_i \in \mathbb{F}$ for $i = k+1, \ldots, n$.

Let $x \in \mathbb{W}$. Then $x = c_1 v_1 + \cdots + c_k v_k$. Therefore $f(x) = c_1 \alpha_1 + \cdots + c_k \alpha_k$ and

$$g(x) = g(c_1 v_1 + \cdots + c_k v_k) = c_1 g(v_1) + \cdots + c_k g(v_k)$$

$$g(x) = c_1 \alpha_1 + \cdots + c_k \alpha_k = f(x).$$