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Lax Method :

Replace $U_j^n = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right)$

in the FTCS to get :

$$\frac{U_j^{n+1} - \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right)}{\delta t} + c \cdot \frac{U_{j+1}^n - U_{j-1}^n}{2\delta x} = 0$$

which is an explicit scheme and stable

$$|\nu| = |c\,\delta t / \delta x| \leq 1$$

modified eq^n for the Lax method is :

$$U_t + c \cdot U_x = \frac{c}{2} \cdot \delta x \left(\frac{1}{\nu} - \nu\right) U_{xx} + \frac{c \cdot (\delta x)^3}{3!} \cdot (1-\nu^2) U_{xxx}$$

dissipation error appear as the coefficient of U_{xx} is :

$$\frac{c}{2} \cdot \delta x \left(\frac{1}{\nu} - \nu\right) > 0 \quad \text{when} \quad \nu < 1.$$

The dissipation error is zero if $\nu = 1$.

T.E is $O\left(\frac{\delta x^2}{\delta t}, \delta t\right)$

Lax scheme provides higher dissipation error as the coefficient of U_{xx} is large.

Lax-Wendroff method :

Expand U_j^{n+1} which $U(x,t)$ at x_j, t_{n+1} by the Taylor Series about (x_j, t_n)

$$U_j^{n+1} = U_j^n + \delta t \cdot U_t\Big|_j^n + \frac{\delta t^2}{2} \cdot U_{tt}\Big|_j^n + O(\delta t^3)$$

replace U_t, U_{tt} by using the PDE

$$U_t + c \cdot U_x = 0$$

$$\left.\begin{aligned} U_{tx} + c \cdot U_{xx} &= 0 \\ U_{tt} + c \cdot U_{xt} &= 0 \end{aligned}\right\}$$

$$\Rightarrow -c^2 \cdot U_{xx} + U_{tt} = 0$$

$$\Rightarrow \boxed{U_{tt} = c^2 \cdot U_{xx}}$$

$$U_j^{n+1} = U_j^n - \delta t \cdot c \, U_x\Big|_j^n + \frac{\delta t^2}{2} c^2 \cdot U_{xx}\Big|_j^n + O(\delta t^3)$$

Discretize the x-derivatives by central diff. scheme,

$$U_j^{n+1} = U_j^n - \frac{c \cdot \delta t}{2 \cdot \delta x}\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{c^2 \cdot \delta t^2}{2 \delta x^2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right) + O(\delta t^3)$$

The scheme,

$$U_j^{n+1} = U_j^n - \frac{\nu}{2} \cdot \left(U_{j+1}^n - U_{j-1}^n\right) + \frac{\nu^2}{2}\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$$

which is single-step Lax-Wendroff explicit method.

the modified eq^n as:

$$U_t + C \cdot U_x = -C \cdot \frac{\delta x^2}{6}(1-\gamma^2) U_{xxx} - C \cdot \frac{\delta x^3}{8} \cdot \gamma(1-\gamma^2) U_{xxxx} + \cdots$$

T.E is $O(\delta t^2, \delta x^2)$.
and the method is not dissipative, a small amount of dispersion error involves as the modified eq^n has U_{xxx}.

Two-step Lax-Wendroff scheme:

$n \to n+1/2$

Step-1: Apply Lax method at the mid-pt $(n, j+\frac{1}{2})$ to get the solution at $(n+\frac{1}{2})$ time step.

$$\frac{U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \frac{1}{2}\left(U_{j+1}^{n} + U_{j}^{n}\right)}{(\delta t/2)} + C \cdot \frac{U_{j+1}^{n} - U_{j}^{n}}{2 \cdot \frac{\delta x}{2}} = 0$$

$$\Rightarrow U_{j+1/2}^{n+1/2} = \frac{1}{2}\left(U_{j+1}^{n} + U_{j}^{n}\right) - \frac{\gamma}{2} \cdot \left(U_{j+1}^{n} - U_{j}^{n}\right) = 0 \quad \text{—(I)}$$

Step-2: $n+\frac{1}{2} \to n+1$

Central diff. scheme for t at $n+1/2$
Central diff. for j with $\frac{\delta x}{2}$.

$$\frac{U_j^{n+1} - U_j^{n}}{2 \cdot \frac{\delta t}{2}} + c \cdot \frac{U_{j+1/2}^{n+1/2} - U_{j-1/2}^{n+1/2}}{2 \cdot (\delta x / 2)} = 0$$

$$\boxed{U_j^{n+1} = U_j^{n} - \nu \left(U_{j+1/2}^{n+1/2} - U_{j-1/2}^{n+1/2} \right)} \quad \text{— (II)}$$

H.T: Replace $U_{j+1/2}^{n+1/2}$ & $U_{j-1/2}^{n+1/2}$ in (II) by using (I) to get the single-step Lax-Wendroff scheme.

Q- $U_t + U_x = \nu \cdot U_{xx}$, $\nu > 0$

$$U(x,0) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}$$

1.) $U(-L/2, 0) = 1$, $U(L/2, 0) = 0$

$L = 10$

$U_t + U \cdot U_x = \nu \cdot U_{xx}$

2) $U(x,0) = \sin(\pi x)$, $0 < x < 1$

$U(0, t) = U(1, t) = 0$, $t > 0$

Let $\nu = 1$

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$$\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$

$$F = \frac{1}{2}u^2$$

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0$$

(I.) Upwind scheme :

(II.) Lax Scheme :

$$u_j^n = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{\delta t}{2\delta x}\left(F_{j+1}^n - F_{j-1}^n\right)$$

(III.) Lax Wendroff Scheme :

$$u(x, t+\delta t) = u(x,t) + \delta t \cdot \frac{\partial u}{\partial t} + \frac{\delta t^2}{2} \cdot \frac{\partial^2 u}{\partial t^2} + \cdots$$

$$\frac{\partial u}{\partial t} = -\frac{\partial F}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial t}\right)$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial t} = A \cdot \frac{\partial u}{\partial t} = -A \cdot \frac{\partial F}{\partial x}$$

$$u_{tt} = \frac{\partial}{\partial x}\left(A \cdot \frac{\partial F}{\partial x}\right)$$

$A = \frac{\partial F}{\partial U}$, Jacobian

Let,

* $F = (F_1, F_2, F_3)$

$U = U_1, U_2, U_3$

$A = \frac{\partial(F_1, F_2, F_3)}{\partial(U_1, U_2, U_3)} \longrightarrow$ Jacobian

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$$U_j^{n+1} = U_j^n - \delta t \cdot \frac{\partial F}{\partial x}\Big|_j^n + \frac{\delta t^2}{2} \cdot \underbrace{\frac{\partial}{\partial x}\left(A \cdot \frac{\partial F}{\partial x}\right)}\Big|_j^n$$

$$\frac{\partial}{\partial x}\left(A \cdot \frac{\partial F}{\partial x}\right)\Big|_j^n = \frac{A \cdot \frac{\partial F}{\partial x}\Big|_{j+1/2}^n - A \cdot \frac{\partial F}{\partial x}\Big|_{j-1/2}^n}{2 \cdot (\delta x/2)}$$

$$\boxed{U_j^{n+1} = U_j^n - \frac{\delta t}{\delta x} \cdot \left(\frac{F_{j+1}^n - F_{j-1}^n}{2}\right) + \frac{\delta t^2}{2\delta x^2}\left[A_{j+1/2}^n \left(F_{j+1}^n - F_j^n\right) - A_{j-1/2}^n \left(F_j^n - F_{j-1}^n\right)\right] + O(\delta t^2, \delta x^2)}$$

$$A_{j+1/2}^n = A\left(\frac{U_j^n + U_{j+1/2}^n}{2}\right)$$

$$A_{j-1/2}^n = A\left(\frac{U_j^n + U_{j-1}^n}{2}\right)$$

$\uparrow$ Lax-Wendroff single-step method.

Note: To find u_j^{n+1} we need u_{j-1}^n, u_j^n, u_{j+1}^n.
(Three previous step solution)

Two-Step Lax Wendroff Scheme:

Step-I: $$\bar{U}_j = \frac{1}{2}\left(U_j^n + U_{j+1}^n\right) - \frac{\delta t}{2\cdot\delta x}\left(F_{j+1}^n - F_j^n\right)$$
(Predictor step)

Step-II: $$U_j^{n+1} = U_j^n - \frac{\delta t}{\delta x}\left[F(\bar{U}_j) - F(\bar{U}_{j-1})\right]$$
(Corrector step)

Ex: $U_t + U\cdot U_x = 0$, $x>0, t>0$

$U(x,0) = x$, $U(0,t) = 0$

$\delta x = 0.2$, $\nu = \frac{\delta t}{\delta x} = 0.5$

Use single-step Lax-Wendroff scheme.

Sol^n: $\delta t = 0.5\times 0.2 = \boxed{0.1}$

$$U_j^{n+1} = U_j^n - \nu\left[\frac{F_{j+1}^n - F_{j-1}^n}{2}\right] + \frac{\nu^2}{2}\left[A_{j+1/2}^n\left(F_{j+1}^n - F_j^n\right) - A_{j-1/2}^n\left(F_j^n - F_{j-1}^n\right)\right]$$

$F = \frac{1}{2}U^2$; $A = \frac{\partial F}{\partial U} = \boxed{U}$

$$\Rightarrow U_j^{n+1} = U_j^n - \overset{0.25}{\cancel{\quad}}\left[F_{j+1}^n - F_{j-1}^n\right] + 0.125\left[A_{j+1/2}^n\left(F_{j+1}^n - F_j^n\right) - A_{j-1/2}^n\left(F_j^n - F_{j-1}^n\right)\right]$$

$$A_{j+1/2}^n = A\left(\frac{U_j^n + U_{j+1/2}^n}{2}\right) = U\left(\frac{U_j^n + U_{j+1/2}^n}{2}\right)$$

Sol^n: 0.182, 0.364, ...

Mac-Cormack Scheme :

The two step Mac-Cormack scheme is based on one sided difference at each step.

Predictor step -

$$\bar{U}_j^{n+1} = U_j^n - \frac{\delta t}{\delta x}\cdot\left(F_{j+1}^n - F_j^n\right) \quad \text{— (I)}$$

Corrector step :

$$U_j^{n+1} = \frac{1}{2}\left(\bar{U}_j^{n+1} + U_j^n\right) - \frac{\delta t}{2\cdot\delta x}\left(\bar{F}_j^{n+1} - \bar{F}_{j-1}^{n+1}\right) \quad \text{— (II)}$$

Q- $U_t + \left(\frac{1}{2}U^2\right)_x = 0$

$$U(x,0) = \sqrt{x}, \quad 0 \le x \le 1$$
$$= 0, \quad \text{otherwise.}$$

$U(0,t)=0$

$\delta x = 0.2 \quad ; \quad \nu = 0.5$

Sol^n:

Predictor step:

$$\bar{U}_j^{n+1} = U_j^n - \frac{1}{4}\left[\left(U_{j+1}^n\right)^2 - \left(U_j^n\right)^2 \right]$$

Corrector step:

$$U_j^{n+1} = \frac{1}{2}\left(\bar{U}_j^{n+1} + U_j^n \right) - \frac{1}{8}\left[\left(\bar{U}_j^{n+1}\right)^2 - \left(\bar{U}_{j-1}^{n+1}\right)^2 \right]$$

Given, $U(x,0) = \begin{cases} \sqrt{x}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

$U(0,t)=0$ | $U_j^0 = \sqrt{x_j}, \quad j=0,1,\dots 5$
$\qquad = 0, \quad j=6,\dots$

From Predictor step, $U_0^n = 0$

For n=0:

$\cancel{\bar{U}_j^{n+1} = U_j^n - \frac{1}{4}}$ $\quad \bar{U}_j^1 = U_j^0 - \frac{1}{4}\left[\left(U_{j+1}^0\right)^2 - \left(U_j^0\right)^2 \right]$

j=0: $\bar{U}_0^1 = U_0^0 - \frac{1}{4}\left[\left(U_1^0\right)^2 - \left(U_0^0\right)^2 \right]$

$$= 0 - \frac{1}{4}\left[\left(U_1^0\right)^2 \right] = -\frac{1}{4} \cdot \left(U_1^0 \right)^2$$

$$= -\frac{1}{4} \times x_1 = \boxed{-0.05}$$