Implicit Runge - Kutta Methods:

The implicit Runge-Kutta method using n-slopes is given as

$$K_i = f\left(t_j + c_i h,\ u_j + h \sum_{m=1}^{n} a_{im} K_m\right) \qquad i = 1, 2, \dots, n.$$

$$u_{j+1} = u_j + h \sum_{m=1}^{n} \omega_m K_m$$

Case $n=1$:

$$u_{j+1} = u_j + h\,\omega_1 K_1 \qquad \text{--- (1)}$$

$$K_1 = f(t_j + c_1 h,\ u_j + h\,a_{11} K_1)$$

Taylor's series expansion of K_1:

$$K_1 = f(t_j, u_j) + (c_1 h f_t + h\,a_{11} K_1 f_y)_{t_j} + O(h^2)$$

Substituting in (1):

$$u_{j+1} = u_j + h\omega_1\left(f(t_j, u_j) + h(c_1 f_t + a_{11} K_1 f_y)_{t_j}\right) + O(h^3) \qquad \text{--- (2)}$$

Taylor's series of the solution:

$$y(t_{j+1}) = y(t_j) + h\,y'(t_j) + \frac{h^2}{\underline{|2}} y''(t_j) + \cdots$$

$$= y(t_j) + h f(t_j, y(t_j)) + \frac{h^2}{2}(f_t + f f_y)_{t_j} + \cdots \qquad (3)$$

Comparing (2) & (3):

$\omega_1 = 1.$ $\qquad \omega_1 c_1 = \frac{1}{2}$ $\qquad a_{11}\omega_1 = \frac{1}{2}$

$\Rightarrow \omega_1 = 1$ $\qquad c_1 = \frac{1}{2}$ $\qquad a_{11} = \frac{1}{2}.$

Hence the second order Runge-Kutta method becomes:

$$u_{j+1} = u_j + K_1 h \quad ; \quad K_1 = f\left(t_j + \frac{h}{2},\ u_j + \frac{h}{2} K_1\right)$$

To obtain K_1, we need to solve the nonlinear equation for K_1.

Case $n=2$:

$$u_{j+1} = u_j + \frac{h}{2}(K_1+K_2)$$

$$K_1 = f\left(t_j + \frac{3-\sqrt{3}}{6}h,\ u_j + \frac{h}{4}K_1 + \frac{3-2\sqrt{3}}{12}hK_2\right)$$

$$K_2 = f\left(t_j + \frac{3+\sqrt{3}}{6}h,\ u_j + \frac{3+2\sqrt{3}}{12}hK_1 + \frac{h}{4}K_2\right)$$

The order of the method is 4.

Ex: Using the implicit Runge-Kutta method

$$u_{n+1} = u_n + K_1 h$$

$$K_1 = f\left(t_n + \frac{h}{2},\ u_n + \frac{h}{2}K_1\right)$$

to find the solution of the initial value problem

$$y' = t^2 + y^2, \quad y(1) = 2, \quad 1 \le t \le 1.2 \text{ with } h = 0.1.$$

Solution: Since $f(t,y) = t^2+y^2$, we have

$$K_1 = \left(t_n + \frac{h}{2}\right)^2 + \left(u_n + \frac{h}{2}K_1\right)^2$$

$n=0$: $\quad h = 0.1,\ t_0 = 1,\ u_0 = 2$

$$K_1 = (1+0.05)^2 + (2+0.05\times K_1)^2$$

$$\Rightarrow K_1 = (1.05)^2 + (2+0.05K_1)^2$$

$$\Rightarrow K_1 = 1.1025 + 4 + .0025K_1^2 + 0.2K_1$$

$$\Rightarrow 0.0025K_1^2 - 0.8K_1 + 5.1025 = 0$$

This can be solved by Newton's Raphson method:

Let $F(K_1) = 0.0025\,K_1^2 - 0.8\,K_1 + 5.1025$

$$F'(K_1) = 0.0050\,K_1 - 0.8$$

NR iterations:

$$K_1^{(s+1)} = K_1^{(s)} - \frac{F(K_1^{(s)})}{F'(K_1^{(s)})} \qquad s = 0, 1, 2 \ldots$$

$$K_1^{(0)} = f(t_0, u_0) = 1 + 4 = 5$$

$K_1^{(1)} = 6.5032258$

$K_1^{(2)} = 6.51058650$

$K_1^{(3)} = \underline{6.510586}68$

$$u_1 = u_0 + h\,K_1 = 2 + 0.1 \times 6.510586 = 2.6510586.$$

<u>n=1</u>: $K_1 = f(1.1 + 0.05,\ 2.6510586 + 0.05\,K_1)$

$\Rightarrow K_1 = (1.15)^2 + (2.6510586 + 0.05\,K_1)^2$

$\Rightarrow 0.0025\,K_1^2 + 0.26510586\,K_1 - K_1 + 8.350611701 = 0$

$\Rightarrow 0.0025\,K_1^2 - 0.73489414\,K_1 + 8.350611701 = 0$

<u>NR Method</u>: $K_1^{(1)} = 11.793142933$

$K_1^{(2)} = 11.839886962$

$K_1^{(3)} = 11.8398950463$

$K_1^{(0)} = f(1.1, 2.6510586) = 8.238111701$

$$u_2 = u_1 + h\,K_1 = 2.6510586 + 0.1 \times 11.8398950$$

$$= \underline{3.8350481}.$$