1.5 Relations, Partitions and Equivalence Relation

We start with the definition of cartesian product of two sets and to define relations.

Definition 1.5.1 (Cartesian Product)**.** *Let A and B be two sets. Then their cartesian product, denoted $A \times B$, is defined as $A \times B = \{(a, b) : a \in A, b \in B\}$.*

Example 1.5.2. *1. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Then*

$$\begin{aligned} A \times A &= \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}. \\ A \times B &= \{(a,1),(a,2),(a,3),(a,4),(b,1),(b,2),(b,3),(b,4),(c,1),(c,2),(c,3),(c,4)\}. \end{aligned}$$

2. The Euclidean plane, denoted $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}\}$.

Definition 1.5.3 (Relation)**.** *A relation R on a non-empty set A, is a subset of $A \times A$.*

Example 1.5.4. *1. Let $A = \{a, b, c, d\}$. Then, some of the relations R on A are:*

(a) $R = A \times A$.

(b) $R = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(b,c)\}$.

(c) $R = \{(a,a),(b,b),(c,c)\}$.

(d) $R = \{(a,a),(a,b),(b,a),(b,b),(c,d)\}$.

(e) $R = \{(a,a),(a,b),(b,a),(a,c),(c,a),(c,c),(b,b)\}$.

(f) $R = \{(a,b),(b,c),(a,c),(d,d)\}$.

2. Consider the set $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. Some of the relations on $\mathbb{Z}^*$ are as follows:

(a) $R = \{(a, b) \in \mathbb{Z}^* \times \mathbb{Z}^* : a|b\}$.

(b) Fix a positive integer n and define $R = \{(a, b) \in \mathbb{Z}^2 : n$ divides $a - b\}$.

(c) $R = \{(a, b) \in \mathbb{Z}^2 : a \le b\}$.

(d) $R = \{(a, b) \in \mathbb{Z}^2 : a > b\}$.

3. Consider the set $\mathbb{R}^2$. Also, let us write $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Then some of the relations on $\mathbb{R}^2$ are as follows:

(a) $R = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\mathbf{x}|^2 = x_1^2 + x_2^2 = y_1^2 + y_2^2 = |\mathbf{y}|^2\}$.

(b) $R = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 : \mathbf{x} = \alpha\mathbf{y}$ for some $\alpha \in \mathbb{R}^*\}$.

(c) $R = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 : 4x_1^2 + 9x_2^2 = 4y_1^2 + 9y_2^2\}$.

(d) $R = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 : \mathbf{x} - \mathbf{y} = \alpha(1, 1)$ for some $\alpha \in \mathbb{R}^*\}$.

(e) Fix a $c \in \mathbb{R}$. Now, define $R = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 : y_2 - x_2 = c(y_1 - x_1)\}$.

(f) $R = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\mathbf{x}| = \alpha|\mathbf{y}|\}$, for some positive real number α.

4. *Let A be the set of triangles in the plane. Then $R = \{(a,b) \in A^2 : a \sim b\}$, where $\sim$ stands for similarity of triangles.*

5. *In $\mathbb{R}$, define a relation $R = \{(a,b) \in \mathbb{R}^2 : |a-b| \text{ is an integer}\}$.*

6. *Let A be any non-empty set and consider the set $\mathcal{P}(A)$. Then one can define a relation R on $\mathcal{P}(A)$ by $R = \{(S,T) \in \mathcal{P}(A) \times \mathcal{P}(A) : S \subset T\}$.*

Now that we have seen quite a few examples of relations, let us look at some of the properties that are of interest in mathematics.

Definition 1.5.5. *Let R be a relation on a non-empty set A. Then R is said to be*

1. *reflexive if $(a,a) \in R$, for all $a \in A$.*

2. *symmetric if $(b,a) \in R$ whenever $(a,b) \in R$.*

3. *anti-symmetric if, for all $a, b \in A$, the conditions $(a,b), (b,a) \in R$ implies that $a = b$ in A.*

4. *transitive if, for all $a, b, c \in A$, the conditions $(a,b), (b,c) \in R$ implies that $(a,c) \in R$.*

We are now ready to define a relation that appears quite frequently in mathematics. Before doing so, let us either use the symbol $\sim$ or $\overset{R}{\sim}$ for relation. That is, if $a, b \in A$ then $a \sim b$ or $a \overset{R}{\sim} b$ will stand for $(a,b) \in R$.

Definition 1.5.6. *Let $\sim$ be a relation on a non-empty set A. Then $\sim$ is said to form an equivalence relation if $\sim$ is reflexive, symmetric and transitive.*

The equivalence class containing $a \in A$, denoted $[a]$, is defined as $[a] := \{b \in A : b \sim a\}$.

Example 1.5.7.

1. *Let $a, b \in \mathbb{Z}$. Then $a \sim b$, if 10 divides $a - b$. Then verify that $\sim$ is an equivalence relation. Moreover, the equivalence classes can be taken as $[0], [1], \ldots, [9]$. Observe that, for $0 \le i \le 9$, $[i] = \{10n + i : n \in \mathbb{Z}\}$.* This equivalence relation in modular arithmetic is written as $a \equiv b \pmod{10}$.

   In general, for any fixed positive integer n, the statement "$a \equiv b \pmod{n}$" (read "a is equivalent to b modulo n"*) is equivalent to saying that $a \sim b$ if n divides $a - b$.*

2. *Determine the equivalence relations that appear in Example 1.5.4. Also, for each equivalence relation, determine a set of equivalence classes.*

Definition 1.5.8 (Partition of a set)**.** *Let A be a non-empty set. Then a partition Π of A, into m-parts, is a collection of non-empty subsets $A_1, A_2, \ldots, A_m$, of A, such that*

1. *$A_i \cap A_j = \emptyset$ (empty set), for $1 \le i \ne j \le m$ and*

2. $\bigcup\limits_{i=1}^{m} A_i = A$.

Example 1.5.9. *1. The partitions of $A = \{a, b, c, d\}$ into*

(a) 3-parts are $a|\ b|\ cd$, $a|\ bc|\ d$, $ac|\ b|\ d$, $a|\ bd|\ c$, $ad|\ b|\ c$, $ab|\ c|\ d$, where the expression $a|bc|d$ represents the partition $A_1 = \{a\}$, $A_2 = \{b, c\}$ and $A_3 = \{d\}$.

(b) 2-parts are

$$a|\ bcd, \quad b|\ acd, \quad c|\ abd, \quad d|\ abc, \quad ab|\ cd, \quad ac|\ bd \ \text{ and } \ ad|\ bc.$$

2. Let $A = \mathbb{Z}$ and define

(a) $A_0 = \{2x : x \in \mathbb{Z}\}$ and $A_1 = \{2x + 1 : x \in \mathbb{Z}\}$. Then $\Pi = \{A_0, A_1\}$ forms a partition of Zl into odd and even integers.

(b) $A_i = \{10n + i : n \in \mathbb{Z}\}$, for $i = 1, 2, \ldots, 10$. Then $\Pi = \{A_1, A_2, \ldots, A_{10}\}$ forms a partition of $\mathbb{Z}$.

3. $A_1 = \{0, 1\}$, $A_2 = \{n \in \mathbb{N} : n \text{ is a prime}\}$ and $A_3 = \{n \in \mathbb{N} : n \geq 3, n \text{ is composite}\}$. Then $\Pi = \{A_1, A_2, A_3\}$ is a partition of $\mathbb{N}$.

4. Let $A = \{a, b, c, d\}$. Then $\Pi = \{\{a\}, \{b, d\}, \{c\}\}$ is a partition of A.

Observe that the equivalence classes produced in Example 1.5.7.1 indeed correspond to the non-empty sets A_i's, defined in Example 1.5.9.2b. In general, such a statement is always true. That is, suppose that A is a non-empty set with an equivalence relation $\sim$. Then the set of distinct equivalence classes of $\sim$ in A, gives rise to a partition of A. Conversely, given any partition Π of A, there is an equivalence relation on A whose distinct equivalence classes are the elements of Π. This is proved as the next result.

Theorem 1.5.10. *Let A be a non-empty set.*

1. Also, let $\sim$ define an equivalence relation on the set A. Then the set of distinct equivalence classes of $\sim$ in A gives a partition of A.

2. Let I be a non-empty index set such that $\{A_i : i \in I\}$ gives a partition of A. Then there exists an equivalence relation on A whose distinct equivalence classes are exactly the sets $A_i, i \in I$.

Proof. Since $\sim$ is reflexive, $a \sim a$, for all $a \in A$. Hence, the equivalence class $[a]$ contains a, for each $a \in A$. Thus, the equivalence classes are non-empty and clearly, their union is the whole set A. We need to show that if $[a]$ and $[b]$ are two equivalence classes of $\sim$ then either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Let $x \in [a] \cap [b]$. Then by definition, $x \sim a$ and $x \sim b$. Since $\sim$ is symmetric, one also has $a \sim x$. Therefore, we see that $a \sim x$ and $x \sim b$ and hence, using the transitivity of $\sim$, $a \sim b$. Thus, by definition, $a \in [b]$ and hence $[a] \subseteq [b]$. But $a \sim b$, also implies that $b \sim a$ ($\sim$ is symmetric) and hence $[b] \subseteq [a]$. Thus, we see that if $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$. This proves the first part of the theorem.

For the second part, define a relation $\sim$ on A as follows: for any two elements $a, b \in A$, $a \sim b$ if there exists an $i, i \in I$ such that $a, b \in A_i$. It can be easily verified that $\sim$ is indeed reflexive, symmetric and transitive. Also, verify that the equivalence classes of $\sim$ are indeed the sets $A_i, i \in I$. $\blacksquare$