Solution Model for Quiz 1

Transform Calculus

section 4

Sol^n 1 :-) $\mathcal{L}^{-1}\left\{\dfrac{s/2 + 5/3}{s^2+4s+6}\right\}$.

We observe that the quadratic polynomial s^2+4s+6 does not have real zeros & so has no real linear factors. In this situation, we complete the square:

$$\frac{s/2 + 5/3}{s^2+4s+6} = \frac{s/2 + 5/3}{(s+2)^2+2}$$

$$\text{Numerator} = s/2 + 5/3 = \tfrac{1}{2}(s+2) + \tfrac{5}{3} - \tfrac{2}{2} = \tfrac{1}{2}(s+2) + 2/3.$$

$$\therefore \frac{s/2 + 5/3}{(s+2)^2+2} = \frac{\left(\frac{1}{2}\right)(s+2) + 2/3}{(s+2)^2+2} = \frac{1}{2}\cdot\frac{s+2}{(s+2)^2+2} + \frac{2}{3}\cdot\frac{1}{(s+2)^2+2}.$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}.$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\bigg|_{s\to s+2}\right\} + \frac{2}{3\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\bigg|_{s\to s+2}\right\}$$

$$= \frac{1}{2}e^{-2t}\cos(\sqrt{2}\,t) + \frac{\sqrt{2}}{3}e^{-2t}\sin(\sqrt{2}\,t).$$

Sol^n 2 :-)

$$\mathcal{L}\left[\int_0^t e^{\tau}\sin(t-\tau)\,d\tau\right]$$

$$= \mathcal{L}\{e^t * \sin t\} = \mathcal{L}\{e^t\}\cdot\mathcal{L}\{\sin t\}$$

$$= \frac{1}{(s-1)}\cdot\frac{1}{(s^2+1)} = \frac{1}{(s-1)(s^2+1)}.$$

Sol^n 3 :-) Given $f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}\,d\tau$,

We have, $h(t-\tau) = e^{t-\tau}$, so that $h(t) = e^t$.

We take the Laplace Transform of each term.

$$\mathcal{L}\{f(t)\} = 3\mathcal{L}\{t^2\} - \mathcal{L}\{e^{-t}\} - \mathcal{L}\left\{\int_0^t f(\tau)e^{t-\tau}\,d\tau\right\}$$

$$\Rightarrow F(s) = 3\cdot\frac{2}{s^3} - \frac{1}{(s+1)} - \mathcal{L}\{f(t) * e^t\}$$

[as $\mathcal{L}\{f(t)\} = F(s)$ & $\mathcal{L}\{e^t\} = \frac{1}{(s-1)}$]

$$\Rightarrow F(s) = 3\cdot\frac{2}{s^3} - \frac{1}{(s+1)} - F(s)\cdot\frac{1}{(s-1)}$$

$$\Rightarrow \left[1 + \frac{1}{s-1}\right]F(s) = \frac{6}{s^3} - \frac{1}{(s+1)} \Rightarrow \frac{sF(s)}{(s-1)} = \frac{6}{s^3} - \frac{1}{(s+1)}$$

$$\Rightarrow F(s) = \frac{6(s-1)}{s^4} - \frac{(s-1)}{s(s+1)}$$

$$\Rightarrow F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}$$

$$\therefore f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= 3\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} - \mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\Rightarrow \boxed{f(t) = 3t^2 - t^3 + 1 - 2e^{-t}}$$ is the given sol^n

[$\frac{s-1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$

$\Rightarrow s-1 = A(s+1) + Bs$

Putting $s=0, -1$ and get

$-1 = A \Rightarrow A = -1$

$-2 = -B \Rightarrow B = 2$

$\therefore \frac{(s-1)}{s(s+1)} = \frac{(-1)}{s} + \frac{2}{(s+1)}$]

—x—