CAUCHY'S INTEGRAL THEOREM:

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,dz = 0$$

Proof: Take an additional assumption that the derivative $f'(z)$ is continuous.

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy) \quad -(1)$$

We know from the C-R equations

$$f'(z) = u_x + i v_x = v_y - i u_y \quad -(2)$$

Since $f'(z)$ is assumed to be continuous then it implies continuity of u_x, v_x, v_y, u_y.

Hence by Green's theorem* (next page)

$$\oint_C u\,dx - v\,dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$: R is the region bnd by C.

Using C-R equations $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, we get

$$\oint_C u\,dx - v\,dy = 0$$

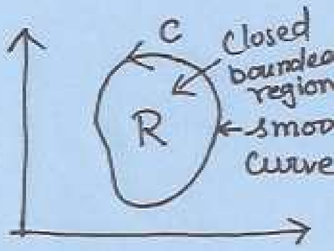
Similarly we can show that $\oint_C v\,dx + u\,dy = 0$

Hence $\oint_C f(z)\,dz = 0$

GREEN'S THEOREM: (Transformation between double integrals and line integrals)

Let $F_1(x,y)$ & $F_2(x,y)$ be continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ & $\frac{\partial F_2}{\partial x}$ everywhere in some domain containing R, then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C \left(F_1\,dx + F_2\,dy\right)$$



<u>Remark</u>: Cauchy's integral theorem has been proved using Green's theorem with the added restriction that $f'(z)$ be continuous in D. However, Goursat gave a proof which removed these restrictions. Sometimes Cauchy integral theorem is called Cauchy-Goursat theorem.

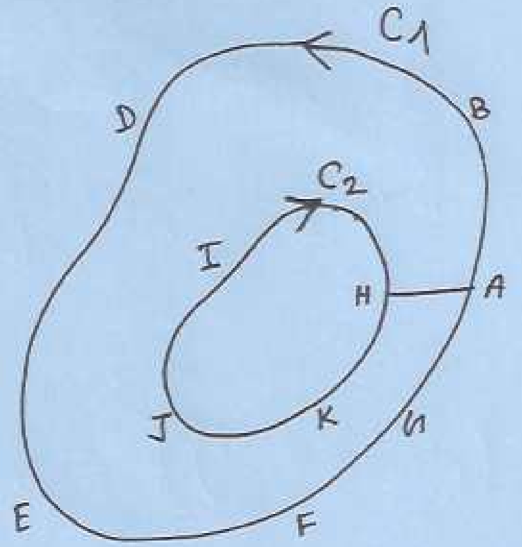
REMARK 1: Cauchy's theorem can also be applied to muliply-connected domain

Construct cross-cut AH.

Then the region bounded by

ABDEFGAHKJIHA

is simply connected.



Then Cauchy's theorem implies:

$$\oint_{ABD \cdots IHA} f(z)\, dz = 0$$

Hence

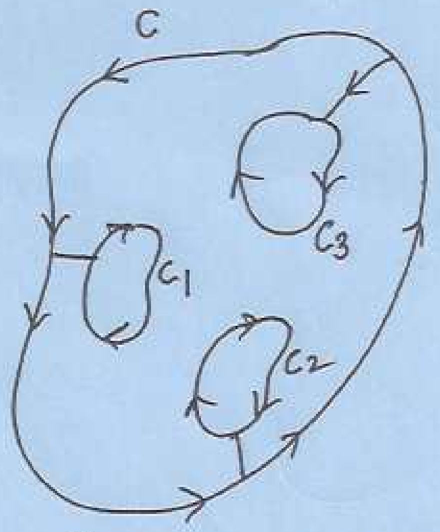
$$\int_{ABDEFGA} f(z)\, dz + \int_{AH} f(z)\, dz + \int_{HKJIH} f(z)\, dz + \int_{HA} f(z)\, dz = 0$$

Using $\int_{AH} f(z)\, dz = -\int_{HA} f(z)\, dz$ it becomes:

$$\underset{\text{anticlockwise}}{\int_{ABDEFGA}} f(z)\, dz + \underset{\text{clockwise}}{\int_{HKJIH}} f(z)\, dz = 0$$

$$\int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz = 0$$

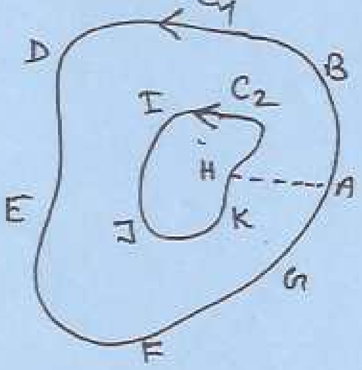
More general result:



$$\int_C f(z)\,dz + \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz = 0$$

Remark 2: As a consequence of above remark we have following result:

Let $f(z)$ be analytic in a domain D bounded by two simple closed curve C_1 & C_2 and also on C_1 & C_2. Then $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$, where C_1 & C_2 are both traversed counter clockwise.



From remark 1 we have

$$\oint_{ABDEFGA} f(z)\,dz + \oint_{\substack{HEJKH \\ HKJIH}} f(z)\,dz = 0$$

$$\Rightarrow \oint_{ABDEFGA} f(z)\,dz = \oint_{HIJKH} f(z)\,dz$$

$$\Rightarrow \boxed{\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz}$$

Deformation of path.

Recall:
$$\oint_C (z-z_0)^m \, dz = \begin{cases} 2\pi i & m=-1 \\ 0 & m \neq -1 \text{ \& } m \text{ is an integer} \end{cases}$$

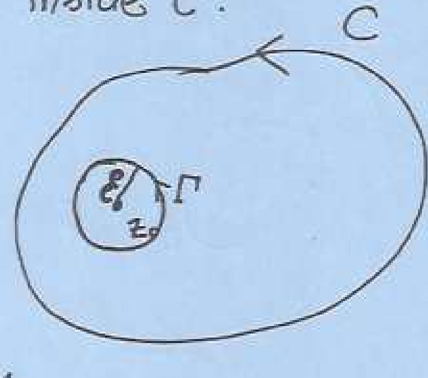
C: Circle of radius ρ and center z_0.

Above result can be generalized for any simple closed curve C due to Remark ②.

If z_0 is outside C then

$$\oint_C (z-z_0)^m \, dz = 0$$ since the function $f(z)$ is analytic everywhere inside and on C. Hence by Cauchy theorem we get the result.

If z_0 is inside C then let Γ be a circle of radius ε with center at $z = z_0$ so that Γ is inside C.



By remark ②:

$$\oint_C f(z)\, dz = \oint_\Gamma f(z)\, dz$$

$$\Rightarrow \oint_C (z-z_0)^m dz = \int_\Gamma (z-z_0)^m dz = \begin{cases} 2\pi i & m=-1 \\ 0 & m \neq -1 \text{ \& } m \text{ integer.} \end{cases}$$

We have following result:

Let C be any simple closed curve C then the counter clockwise integrati

$$\oint_C (z-z_0)^m dz = \begin{cases} 0 & \text{if } z_0 \text{ is outside C} \\ 2\pi i & m=-1 \text{ \& } z_0 \text{ is inside C} \\ 0 & m\neq -1 \text{ \& } m \text{ is integer \& } z_0 \text{ is inside C.} \end{cases}$$

Note: Some important results from above general result:

1. $$\oint_C \frac{1}{z-z_0}\,dz = 2\pi i \qquad \text{if } z_0 \text{ is inside } C.$$

2. $$\oint_C \frac{1}{(z-z_0)^n}\,dz = 0 \qquad n = 2, 3, \ldots \qquad z_0 \text{ is inside } C.$$

Remark: The result $\oint_C \frac{1}{(z-z_0)^n}\,dz = 0$ does not follow from Cauchy's theorem as $\frac{1}{(z-z_0)^n}$ is not analytic in D.

Hence the condition that $f(z)$ is analytic in D is sufficient for $\oint f(z)\,dz = 0$ rather than necessary.

Example: Evaluate

$$\int_C \frac{z+4}{z^2+2z+5}\,dz \quad \text{where } C \text{ is the circle } |z+1| = 1.$$

Sol:

Let $$f(z) = \frac{z+4}{z^2+2z+5}.$$

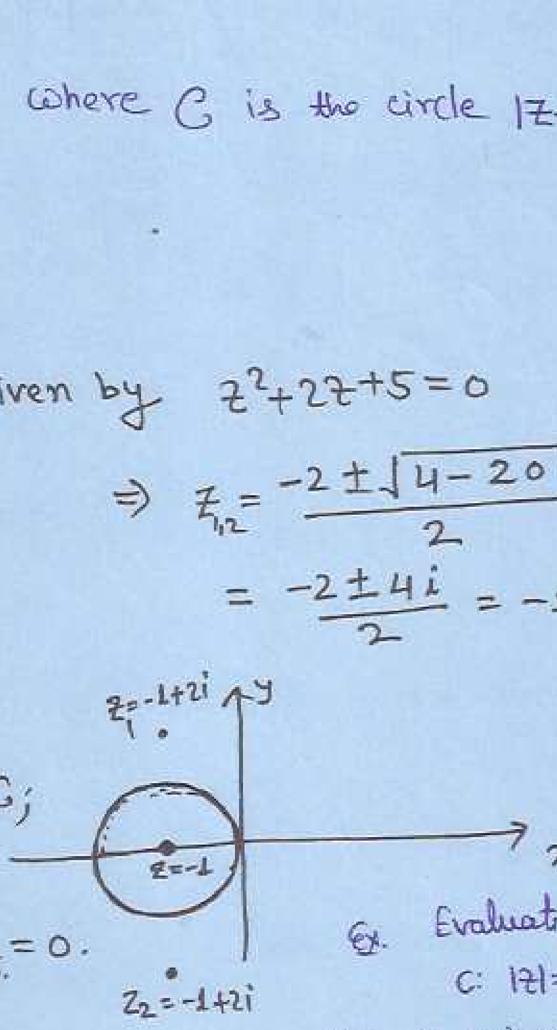
Singularities of $f(z)$ are given by $z^2+2z+5=0$

(f(z) is not defined or f(z) is not analytic)

$$\Rightarrow z_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Both singularities lies outside the circle $|z+1| = 1$. Hence $f(z)$ is analytic everywhere within and on C; Hence by Cauchy's theorem, we get

$$\oint_C f(z)\,dz = 0 \quad \text{ie} \quad \oint_C \frac{z+4}{z^2+2z+5} = 0.$$



Ex. Evaluate $\oint_C \frac{z^2+z+1}{z-1}\,dz$

C: $|z| = \frac{1}{2}$

Ans: 0, using C-I theorem.

CAUCHY INTEGRAL FORMULA:

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0, we have

$$\oint_C \frac{f(z)}{(z-z_0)}\,dz = 2\pi i\, f(z_0) \quad \text{OR} \quad f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)}\,dz$$

Proof:

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_C \frac{f(z_0)+f(z)-f(z_0)}{z-z_0}\,dz$$

$$= f(z_0)\underbrace{\oint_C \frac{1}{z-z_0}\,dz}_{=2\pi i} + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

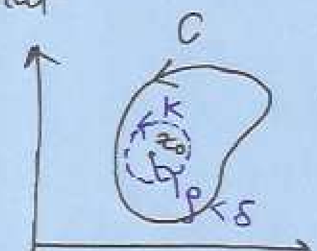
$$= f(z_0)\cdot 2\pi i + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

Now we consider $\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz$

Since $f(z)$ is analytic and therefore continuous. Hence for given $\varepsilon>0$ we can find a $\delta>0$ such that

$$|f(z)-f(z_0)| < \varepsilon \quad \text{—} \circledast$$

for all $|z-z_0|<\delta$



Using principle of deformation

$$\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = \oint_K \frac{f(z)-f(z_0)}{z-z_0}\,dz \quad : K \text{ is a circle of radius } \rho,\ \rho<\delta$$

Using $\circledast$ we have $\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\varepsilon}{\rho}$

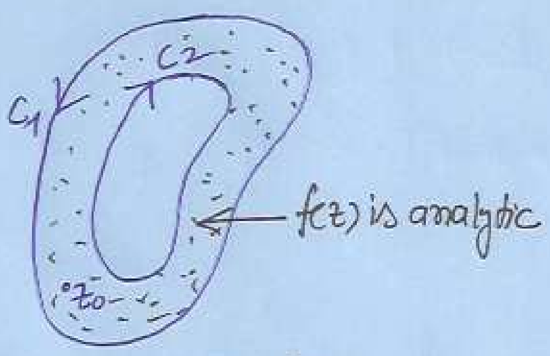
Using M-L inequality $\left|\int_C f(z)\,dz\right| \le ML$, we get

$$\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz < \frac{\varepsilon}{\rho}\cdot 2\pi\rho = 2\pi\varepsilon$$

Since ε can be chosen arbitrarily small, we have $\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$

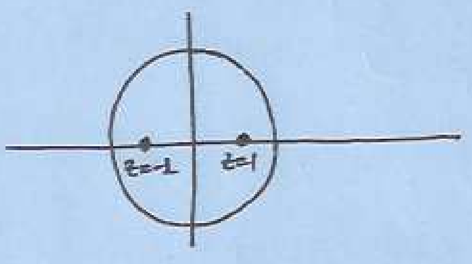
$\square$.

CACHY INTEGRAL FORMULA FOR MULTIPLY CONNECTED DOMAIN



$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}\,dz$$

<u>Example:</u> $\displaystyle\oint_C \frac{\tan z}{(z^2-1)}\,dz \qquad C:\ |z| = 3/2$



Singularities of $f(z)$:
$z = 1, -1, \pm\pi/2, \pm\frac{3\pi}{2}, \ldots$
Points $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}\ldots$, does not lie inside $|z| = 3/2$.

$$\Rightarrow \oint_C \frac{\tan z}{(z-1)(z+1)}\,dz = \oint_C \frac{\tan z}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]dz$$

$$= \frac{1}{2}\oint_C \frac{\tan z}{z-1}\,dz - \frac{1}{2}\oint_C \frac{\tan z}{z+1}\,dz$$

$$= \frac{1}{2}\cdot 2\pi i \tan 1 - \frac{1}{2}\,2\pi i \tan(-1)$$

Not that $\tan z$ is analytic inside $|z| = 3/2$

$$= 2\pi i \tan 1 .$$