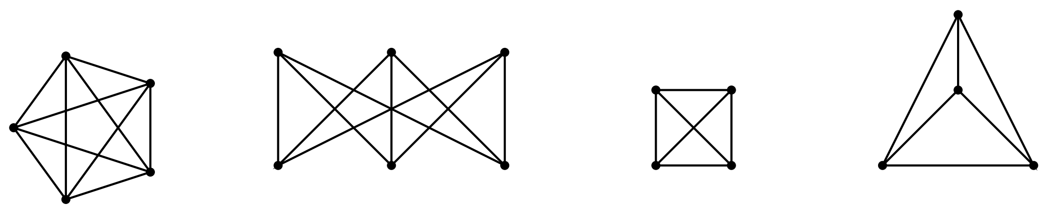
4.6 Planar Graphs

This section deals with planar graphs, one of the first topics in topological graph theory. All our graphs in this chapter will be connected graphs. To start with, a graph is said to be *planar* if the graph can be drawn on a plane so that its edges intersect only at their end vertices. A graph that is not planar is called *non-planar*. A drawing of a planar graph so that the edges do not cross each other, except at their end vertices, is called a *planar embedding* of the graph. We explain this with the examples given in Figure 4.13.



K_5-Non-planar $K_{3,3}$-Non-planar K_4 K_4 - Planar embedding

Figure 4.13: Planar and non-planar graphs

The next two results, give a direct proof of the fact that K_5 and $K_{3,3}$ are non-planar. In Section 4.7, another proof of these results will be given by using the result of Euler for convex polyhedrons.

Theorem 4.6.1. *The graph K_5 is non-planar.*

Proof. On the contrary, let us assume that K_5 is planar. Let the vertices of K_5 be denoted by $1, 2, 3, 4$ and 5. Then, in any planar drawing of K_5, the cycle [123451], of K_5, must appear as a cycle. Let us draw it in the form of a pentagon as shown in Figure 4.14. Then the edge $\{1, 4\}$ will either lie completely inside the pentagon or completely outside it. Without loss of generality, let us assume that the edge $\{1, 4\}$ lies completely inside the pentagon. Therefore, both the edges $\{2, 5\}$ and $\{3, 5\}$ will lie completely outside the pentagon (or else they will intersect the edge $\{1, 4\}$). Now, look at the drawing of the edges $\{1, 3\}$ and $\{2, 4\}$. If $\{1, 3\}$ is drawn inside the pentagon than the edge $\{2, 4\}$ cannot be drawn without intersecting either the edge $\{1, 3\}$ or the edge $\{3, 5\}$. A similar argument is valid if the edge $\{2, 4\}$ is drawn inside the pentagon. ■

Theorem 4.6.2. *The graph $K_{3,3}$ is non-planar.*

Proof. On the contrary, let us assume that $K_{3,3}$ is planar. Let the vertices of $K_{3,3}$ be denoted by $a, b, c, 1, 2$ and 3. Then, in any planar drawing of $K_{3,3}$, the cycle $[1a2b3c1]$, of $K_{3,3}$, must appear as a cycle. Let us draw it in the form of a hexagon as shown in Figure 4.14. Then the edge $\{1, b\}$ will either lie completely inside the hexagon or completely outside it.

Without loss of generality, let us assume that the edge $\{1, b\}$ lies completely inside the hexagon. Now, consider the edges $\{2, c\}$ and $\{3, a\}$. They need to be drawn completely outside

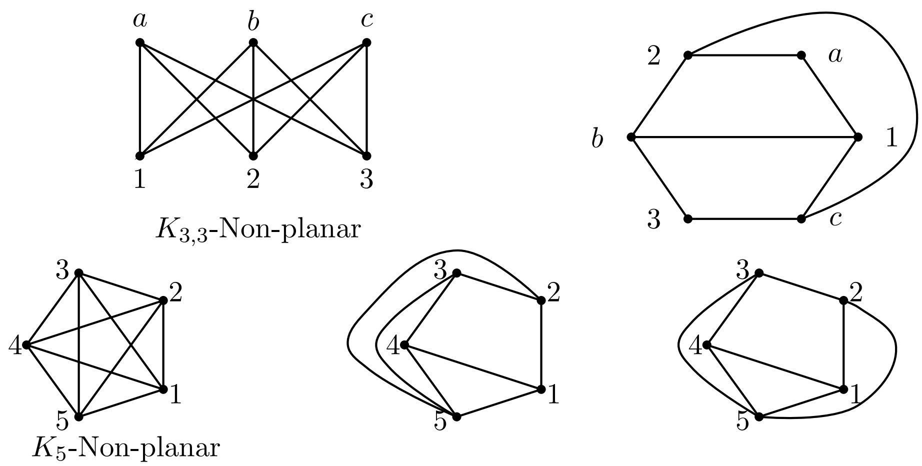


Figure 4.14: Non-planarity of the graphs K_5 and $K_{3,3}$

the hexagon (or else they will intersect the edge $\{1, b\}$). Once $\{2, c\}$ is drawn outside the hexagon then the edge $\{3, a\}$ cannot be drawn without intersecting either the edge $\{1, b\}$ or the edge $\{2, c\}$. A similar argument is valid if the edge $\{3, a\}$ is drawn outside the hexagon. ■

Thus, we have shown that the graphs K_5 and $K_{3,3}$ are non-planar. It can be easily observed that if X is a graph that has K_5 or $K_{3,3}$ as its subgraph then X will also be non-planar. A necessary and sufficient condition for a graph to be non-planar was given by Kuratowski. To understand the statement of the theorem, we need to know the condition under which two graphs are homeomorphic.

Let $X = (V, E)$ be a graph. Let $e = \{u, v\}$ be an edge of X and let $z \notin V$. Then, the graph $Y = (V', E')$ is said to be obtained from the graph $X = (V, E)$ by *insertion* of the vertex z if $V' = V \cup \{z\}$ and $E' = (E \setminus e) \cup \{\{u, z\}, \{z, v\}\}$. That is, Y has been obtained from X by dividing the edge $e = \{u, v\}$ into two edges $\{u, z\}$ and $\{z, v\}$, or equivalently, a vertex z of degree 2 has been introduced on the edge e. This process of insertion of a new vertex of degree 2 is sometimes called *subdivision* of an edge. The inverse of this process is called *edge merging.*

A repeated application of edge subdivision leads to the following definition.

Definition 4.6.3. *Let X and Y be two graphs. Then they are said to be* homeomorphic *if both X and Y can be obtained from a graph, say Z, by inserting new vertices of degree* 2 *on the edges of Z.*

For example, all the paths are homeomorphic to each other and in particular to the graph K_2. Similarly, all the cyclic graphs are homeomorphic to the cycle C_3, when one is interested in the study of simple graphs. In general, one can say that all cyclic graphs are homeomorphic to a graph $X = (V, E)$, where $V = \{v\}$ and $E = \{e, e\}$ (*i.e.,* a graph having exactly one vertex and a loop). Also, note that if two graphs are isomorphic then they are also homeomorphic. Figure 4.15 gives examples of homeomorphic graphs that are different from a path or a cycle.

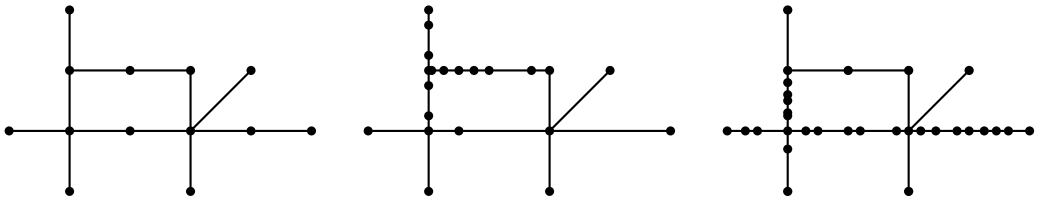


Figure 4.15: Homeomorphic graphs

With this understanding, the theorem of Kuratowski states that "a simple connected graph is planar if and only if it does not contain any subgraph that is homeomorphic to either K_5 or $K_{3,3}$". The proof of the theorem is out of the scope of these notes. The interested readers can see the book "graph theory" by Harary [6] for a proof. We end this section with the following observations which directly follow from Kuratowski theorem.

Observations:

1. Among all simple connected non-planar graphs

   (a) the complete graph K_5 has minimum number of vertices.

   (b) the complete bipartite graph $K_{3,3}$ has minimum number of edges.

2. If Y is a non-planar subgraph of a graph X then X is also non-planar.