Routh-Hurwitz Criterion

The roots of the characteristic equation

$$b_0 z^k + b_1 z^{k-1} + \dots + b_k = 0$$

have negative real part iff all the principal diagonal minors of the Hurwitz matrix are positive provided $b_0 > 0$.

If one or more of b_i's are equal to zero and other b_j's are positive, then it indicates that a root lies on the circle $|\xi| = 1$.

If one or more of b_j's are negative, then there is atleast one root for which $|\xi_i| > 1$.

Ex: Check if ~~all~~ real part the roots of the characteristic equation

$$z^4 + 2z^3 + 4z^2 + 7z + 3 = 0$$

are negative.

Sol:

$b_0 = 1 \quad b_1 = 2 \quad b_2 = 4 \quad b_3 = 7 \quad b_4 = 3.$

$$D = \begin{bmatrix} b_1 & b_3 & 0 & 0 \\ b_0 & b_2 & b_4 & 0 \\ 0 & b_1 & b_3 & 0 \\ 0 & b_0 & b_2 & b_4 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 0 & 0 \\ 1 & 4 & 3 & 0 \\ 0 & 2 & 7 & 0 \\ 0 & 1 & 4 & 3 \end{bmatrix}$$

$\Delta_1 = 2 > 0$; $\quad \Delta_2 = \begin{vmatrix} 2 & 7 \\ 1 & 4 \end{vmatrix} = 1 > 0$

$$\Delta_3 = \begin{vmatrix} 2 & 7 & 0 \\ 1 & 4 & 3 \\ 0 & 2 & 7 \end{vmatrix} = 2(28-6) - 7(7)$$
$$= 44 - 49 = -5 < 0.$$

Hence some root(s) has/have ~~positive~~ non-negative real part.

Ex: Find the general solution of the difference equation

$$u_{n+2} - 5u_{n+1} + 6u_n = 0$$

Sol: Substituting $u_n = A\xi^n$, we get the characteristic equation

$$\xi^2 - 5\xi + 6 = 0 \Rightarrow \xi = 2, 3.$$

general solution: $u_n = c_1(2)^n + c_2(3)^n$.

Ex: Find the range of α, so that the roots of the characteristic equation of the difference equations

$$(1-5\alpha)\, y_{n+2} - (1+8\alpha)\, y_{n+1} + \alpha\, y_n = 0$$

are less than 1 in magnitude.

Sol: The characteristic equation

$$(1-5\alpha)\xi^2 - (1+8\alpha)\xi + \alpha = 0$$

setting $\xi = \frac{1+z}{1-z}$, we get transformed characteristic equation

$$(1-5\alpha)\frac{(1+z)^2}{(1-z)^2} - (1+8\alpha)\frac{1+z}{1-z} + \alpha = 0$$

$$\Rightarrow (1-5\alpha)(1+z)^2 - (1+8\alpha)(1+z)(1-z) + \alpha(1-z)^2 = 0$$

$$\Rightarrow (2+4\alpha)z^2 + (2-12\alpha)z - 12\alpha = 0$$

The Routh-Hurwitz Criterion is satisfied if

$$2+4\alpha > 0, \qquad 2-12\alpha > 0, \qquad -12\alpha > 0$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\underbrace{\alpha > -\tfrac{1}{2} \qquad \alpha < \tfrac{1}{6} \qquad \alpha < 0}$$

$$\alpha \in \left(-\tfrac{1}{2}, 0\right)$$

Therefore $|\xi| < 1$ for all $\alpha \in \left(-\frac{1}{2}, 0\right)$

Stability Analysis:

Consider the linear multi-step method:

$$u_{j+1} = \sum_{i=1}^{K} a_i u_{j-i+1} + h \sum_{i=0}^{K} b_i u'_{j-i+1} \qquad \text{——} \ (1)$$

OR

$$\rho(E)\, u_{j-K+1} - h\, \sigma(E)\, u'_{j-K+1} = 0$$

where

$$\rho(\xi) = \xi^{K} - a_1 \xi^{K-1} - \cdots\cdots - a_{K-1}\xi - a_K$$

$$\sigma(\xi) = b_0 \xi^{K} + b_1 \xi^{K-1} + \cdots\cdots + b_{K-1}\xi + b_K$$

Applying (1) to the test equation $y' = \lambda y$, we get

$$u_{j+1} = \sum_{i=1}^{K} a_i u_{j-i+1} + h\lambda \sum_{i=0}^{K} b_i u_{j-i+1} \qquad \text{——} \ (2)$$

The exact solution satisfies:

$$y(t_{j+1}) = \sum_{i=1}^{K} a_i\, y(t_{j-i+1}) + \underbrace{\lambda h}_{=:\bar{h}} \sum_{i=0}^{K} b_i\, y(t_{j-i+1}) + T_{j+1} \qquad \text{——} \ (3)$$

Substracting (3) from (2) and setting $\varepsilon_j = u_j - y(t_j)$, we get

$$\varepsilon_{j+1} = \sum_{i=1}^{K} a_i \varepsilon_{j-i+1} + \bar{h} \sum_{i=0}^{K} b_i \varepsilon_{j-i+1} - T_{j+1}$$

OR $$\left[\rho(E) - \bar{h}\sigma(E)\right] \varepsilon_{j-K+1} + T_{j+1} = 0 \qquad \text{——} \ (4)$$

This is a Kth order, linear, non-homogeneous difference equation with constant coefficients. For simplicity, we assume $T_{j+1} = T$ (some constant).

Solution of difference equation (4):

we first find the solution of the homogeneous equation

$$[\rho(E) - \bar{h}\,\sigma(E)]\,\varepsilon_{j-k+1} = 0 \quad \text{——} \quad (4')$$

The characteristic equation is given as

$$\rho(\xi) - \bar{h}\,\sigma(\xi) = 0 \quad \text{———} \quad (5)$$

Let the roots are $\xi_{1h}, \xi_{2h}, \ldots, \xi_{kh}$ and they are distinct. Then, the solution of (4') is given by

$$\varepsilon_j^H = c_1 \xi_{1h}^j + c_2 \xi_{2h}^j + \cdots + c_k \xi_{kh}^j .$$

The particular solution is given as

$$\varepsilon_j^P = -\frac{T}{[\rho(1) - \bar{h}\,\sigma(1)]}$$

For a consistent method, we have $\rho(1) = 0$ & $\sigma(1) = \rho'(1)$, then

$$\varepsilon_j^P = \frac{T}{\bar{h}\,\rho'(1)}$$

Hence, the general solution of (4) is given as

$$\varepsilon_j = c_1 \xi_{1h}^j + c_2 \xi_{2h}^j + \cdots + c_k \xi_{kh}^j + \frac{T}{\bar{h}\,\rho'(1)} \quad \text{——} \quad (6)$$

For $h \to 0$, the roots of the ch. equation (5) approaches to the roots of $\boxed{\rho(\xi) = 0}$ ———— (7)

The equation (7) is called reduced characteristic equation.

If $\xi_1, \xi_2, \dots, \xi_k$ are the roots of $\rho(\xi) = 0$, then for sufficiently small $\bar{h}$, we may write

$$\xi_{ih} = \xi_i \left(1 + \bar{h}\kappa_i + O(|\bar{h}|^2)\right), \quad i = 1, 2, \dots k. \qquad -(8)$$

The coefficient κ_i's are called the growth parameters.

Subst. (8) into the characteristic equation (5)

$$\rho\left(\xi_i + \bar{h}\kappa_i\xi_i + O(|\bar{h}|^2)\right) - \bar{h}\sigma\left(\xi_i + \bar{h}\kappa_i\xi_i + O(|\bar{h}|^2)\right) = 0$$

Expanding into the Taylor's series, we get

$$\rho(\xi_i) + \bar{h}\kappa_i\xi_i\rho'(\xi_i) - \bar{h}\sigma(\xi_i) + O(|\bar{h}|^2) = 0$$

Since $\rho(\xi_i) = 0$ we get

$$\boxed{\kappa_i \coloneqq \frac{\sigma(\xi_i)}{\xi_i\,\rho'(\xi_i)}}$$

Remark: Since the method is consistent, $\rho(1) = 0$, $\rho'(1) = \sigma(1)$

$\Rightarrow \xi_1 = 1$ & $\rho'(1) = \sigma(1)$.

Then $\kappa_1 \coloneqq \dfrac{\sigma(1)}{\rho'(1)} = 1$

Now consider the error equation

$$\varepsilon_j = c_1\xi_{1h}^j + c_2\xi_{2h}^j + \dots + c_k\xi_{kh}^j + \frac{T}{\bar{h}\,\rho'(1)}.$$

- If any of the roots $\xi_{ih}, i = 1, 2, \dots k$ satisfy $|\xi_{ih}| > 1$, then the error $|\varepsilon_j|$ grows unboundedly.
- If there is a multiple root of magnitude unity, then again $|\varepsilon_j|$ grows unboundedly.
- If the roots ξ_{ih} are simple and some of them have magnitude unity, then a fixed amount of error is retained in the numerical solution.