4.8 Stereographic Projection

Consider the two graphs in Figure 4.16. The faces have been denoted with f_i, for some positive integer i. Note that in the embedding of both the graphs, there is a face, denoted f_1, which seems to be an infinite face, whereas the other faces seem to be enclosed by the edges of the graphs. It turns out that the face f_1, which seems to form an infinite face, can be made into a closed face by embedding the graph into a sphere and then projecting it back on to a plane after rotating the sphere in such a way that a point of the face f_1 touches the plane. This idea will be clear at the end of this section. This projection is called the *stereographic projection.* An example of the same has been shown in Figure 4.17.

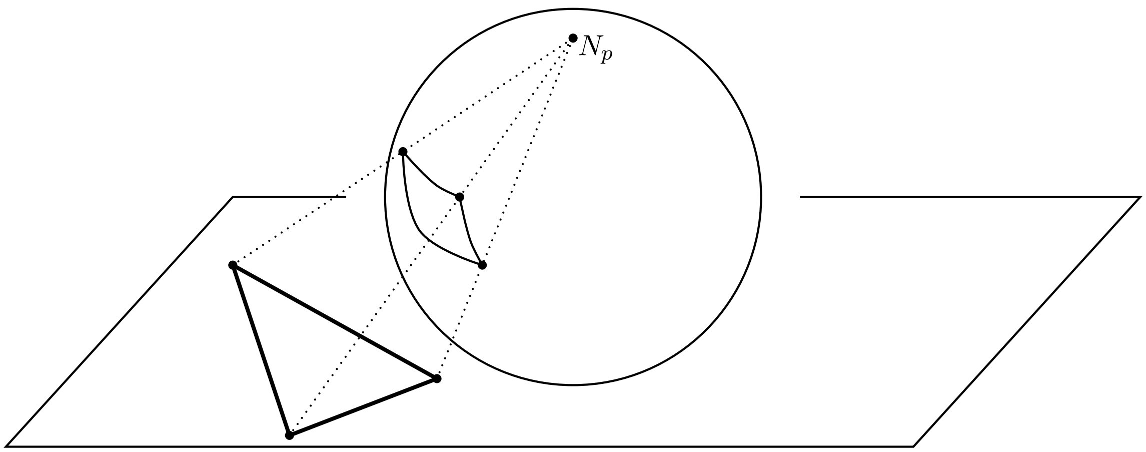

Figure 4.17: Stereographic projection

The stereographic projection consists of the following:

- place a sphere on the plane and let S_p be the point on the sphere that touches the plane. The point S_p is called the *south pole.*

- Now, draw the line passing through the point S_p and the center of the sphere. This line intersects the sphere at a point, say N_p. The point N_p is called the *north pole* of the sphere. See Figure 4.17 for the north pole.

- Now, given any point x on the sphere, draw a straight line that passes through N_p and x. This line will intersect the plane at a unique point, say y. Then the stereographic projection of the point x is defined to be the point y on the plane.

It can be easily observed that the stereographic projection, defined above, has an inverse map, which maps any point on the plane to a point on the sphere. To do this, let y_1 be a point on the plane. Join the point y_1 with N_p, the north pole. Then this line will intersect the sphere at a unique point. This point will be the inverse image of the point y_1. Also, the image of the point N_p, the north pole, under the stereographic projection is the point at infinity in the extended plane (i.e., $\mathbb{R}^2 \cup \{\infty\}$).

Before proceeding with the result that states that every face of planar embedding of a planar graph X can be made into an infinite face and/or bounded face, we observe that using the stereographic projection, a planar embedding and embedding on a sphere are one and the same.

Theorem 4.8.1. *Let v be a vertex of a connected planar graph X. Then X can be embedded in the plane in such a way that v is on the exterior face of the embedding.*

Proof. Let $\tilde{X}$ be a planar embedding of X. Use the stereographic projection to project this embedding on the sphere. Let this embedding on the sphere be called $\tilde{Z}$. It is clear that this embedding exists, as explained in previous paragraphs. Now let f be the face that has v as one of its vertices. Take a point z in f and fix it. Now, place the sphere, on a plane, in such a way that the point z (on the sphere) acts as the north pole. Then the projection of the embedding $\tilde{Z}$ with the point z as the north pole, gives a planar embedding of X with the required property.
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