

### Solution of Tutorial Problems set-I

**Note:** All these problems can be solved using the results of Chapter-I.

[0.0.1] **Exercise** Check that each of the following sets are vector space with respect to usual addition and scalar multiplication.

- (i) The set of all real sequences over the field  $\mathbb{F} = \mathbb{R}$ .
- (ii) The set of all bounded real sequences over the field  $\mathbb{R}$ .
- (iii) The set of all convergent real sequences over the field  $\mathbb{R}$ .
- (iv)  $\{(a_n) \mid a_n \in \mathbb{R}, a_n \rightarrow 0\}$  over the field  $\mathbb{R}$ .
- (v) The set of all **eventually** 0 sequences over the field  $\mathbb{R}$ . We call  $(x_n)$  eventually 0 if  $\exists k$  s.t.  $x_n = 0$  for all  $n \geq k$ .
- (vi)  $\mathbb{P}(x) = \{p(x) \mid p(x) \text{ is a real polynomial in } x\}$  over the field  $\mathbb{R}$ .
- (vii)  $\mathbb{P}_5(x) = \{p(x) \in \mathbb{R}[x] \mid \text{degree of } p(x) \leq 5\}$  over the field  $\mathbb{R}$ .
- (viii)  $\{A_{n \times n} \mid a_{ij} \in \mathbb{R}, A \text{ upper triangular}\}$  over the field  $\mathbb{R}$ .

[0.0.2] **Exercise** Consider  $\mathbb{P}_n(x)$  and  $\mathbb{P}(x)$  over  $\mathbb{R}$ . Check that each of the following sets is subspace or not.

- (i)  $\{P(x) \in \mathbb{P}_3(x) \mid P(x) = ax + b, a, b \in \mathbb{R}\}$ .
- (ii)  $\{P(x) \in \mathbb{P} \mid P(0) = 0\}$ .
- (iii)  $\{P(x) \in \mathbb{P} \mid P(0) = 1\}$ .
- (iv)  $\{P(x) \in \mathbb{P} \mid P(-x) = P(x)\}$ .
- (v)  $\{P(x) \in \mathbb{P} \mid P(-x) = -P(x)\}$ .

[0.0.3] **Exercise** Fix  $A \in \mathcal{M}_n(\mathbb{R})$ . Let  $\mathbb{U} = \{B \in \mathcal{M}_n(\mathbb{R}) : AB = BA\}$ .

- a) Show that  $\mathbb{U}$  is a subspace of  $\mathcal{M}_n(\mathbb{R})$ .
- b) Let  $\mathbb{W} = \{a_0 I + a_1 A + \cdots + a_n A^m \mid m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, a_i \in \mathbb{R}\}$ . Show that  $\mathbb{W}$  is a subspace of  $\mathbb{U}$ .

**Sol.** a) Let  $B, C \in \mathbb{U}$  and  $\alpha, \beta \in \mathbb{R}$ . To show that  $\alpha B + \beta C \in \mathbb{U}$ .  $A[\alpha B + \beta C] = \alpha AB + \beta AC = \alpha BA + \beta CA = [\alpha B + \beta C]A$ . Therefore  $\alpha B + \beta C \in \mathbb{U}$ . So  $\mathbb{U}$  is a subspace.

b) Note that if  $r(A)$  is any polynomial, then  $Ar(A) = r(A)A$ . Hence  $r(A) \in \mathbb{U}$ . Thus  $\mathbb{W} \subseteq \mathbb{U}$ .  
Let  $p(A)$  and  $q(A)$  be two polynomials in  $A$ . Then  $\alpha p(A) + \beta q(A)$  is also polynomial in  $A$ . So  $\alpha p(A) + \beta q(A) \in \mathbb{W}$ . So  $\mathbb{W}$  is a subspace.

[0.0.4] **Exercise** Find basis and dimension for each of the following vector spaces.

- (i)  $\mathbb{M}_n(\mathbb{C})$  over  $\mathbb{R}$ .
- (ii)  $\mathbb{H}_n(\mathbb{C})$ ,  $n \times n$  Hermitian matrices, over  $\mathbb{R}$ .
- (iii)  $\mathbb{S}_n(\mathbb{C})$ ,  $n \times n$  Skew-Hermitian matrices, over  $\mathbb{R}$ .

**Sol.** (i)  $2n^2$ .

(ii)  $n^2$ .

(iii)  $n^2$ .

[0.0.5] **Exercise** Check whether the following vector space is finite dimensional or infinite dimensional.

(i) The set of all real sequences over the field  $\mathbb{F} = \mathbb{R}$ .

(ii) The set of all bounded real sequences over the field  $\mathbb{R}$ .

(iii) The set of all convergent real sequences over the field  $\mathbb{R}$ .

(iv)  $\{(a_n) \mid a_n \in \mathbb{R}, a_n \rightarrow 0\}$  over the field  $\mathbb{R}$ .

(v) The set of all **eventually** 0 sequences over the field  $\mathbb{R}$ . We call  $(x_n)$  eventually 0 if  $\exists k$  s.t.  $x_n = 0$  for all  $n \geq k$ .

(vii)  $\mathbb{P}(x) = \{p(x) \mid p(x) \text{ is a real polynomial in } x\}$  over the field  $\mathbb{R}$ .

(viii)  $\mathbb{P}_5(x) = \{p(x) \in \mathbb{R}[x] \mid \text{degree of } p(x) \leq 5\}$  over the field  $\mathbb{R}$ .

[0.0.6] **Exercise** Write 4 nontrivial subspaces of  $\mathbb{R}^4$ .

**Sol.** a) Put  $v_1 = [1 \ 1 \ 1 \ 1]^t$ . Then  $\text{LS}(v_1)$  is a nontrivial subspace.

b) Put  $v_2 = [0 \ 1 \ 1 \ 1]^t$ . Then  $\text{LS}(v_1, v_2)$  is a nontrivial subspace.

c) Put  $v_3 = [0 \ 0 \ 1 \ 1]^t$ . Then  $\text{LS}(v_1, v_2, v_3)$  is a nontrivial subspace.

d) Any nontrivial subspace of  $\mathbb{R}^4$  must be a span of either a) one nonzero vector or b) two linearly independent vectors or c) three linearly independent vectors. Why?

Let  $\mathbb{W}$  be a nontrivial subspace. So  $\exists$  a nonzero  $v_1 \in \mathbb{W}$ . If  $\mathbb{W} = \text{LS}(v_1)$ , it is of type a) and we are done. So let  $v_2 \in \mathbb{W} - \text{LS}(v_1)$ . By our theorem,  $v_1, v_2$  are linearly independent. If  $\mathbb{W} = \text{LS}(v_1, v_2)$ , it is of type b) and we are done. So let  $v_3 \in \mathbb{W} - \text{LS}(v_1, v_2)$ . By our theorem,  $v_1, v_2, v_3$  are linearly independent. If  $\mathbb{W} = \text{LS}(v_1, v_2, v_3)$ , it is of type c) and we are done. So let  $v_4 \in \mathbb{W} - \text{LS}(v_1, v_2, v_3)$ . By our theorem,  $v_1, v_2, v_3, v_4$  are linearly independent. So  $\text{LS}(v_1, v_2, v_3, v_4) \subseteq \mathbb{W}$ . But  $\text{LS}(v_1, v_2, v_3, v_4) = \mathbb{R}^4$ , by use of GJE. In this case  $\mathbb{W}$  is trivial.

[0.0.7] **Exercise** Show that  $u_1, \dots, u_k \in \mathbb{R}^n$  are linearly independent iff  $Au_1, \dots, Au_k$  are linearly independent for any invertible  $A_n$ .

**Sol.** First we assume that  $Au_1, \dots, Au_k$  are linearly independent. To show  $u_1, \dots, u_k \in \mathbb{R}^n$  are linearly independent.

Suppose  $u_1, \dots, u_n$  are linearly dependent. Then there exists  $\alpha_1, \dots, \alpha_k$  in  $\mathbb{F}$  not all zero s.t.  $\sum \alpha_i u_i = 0$ . Multiplying both side by  $A$ . So  $A \sum \alpha_i u_i = \sum \alpha_i (Au_i) = 0$ . So  $Au_1, \dots, Au_n$  are linearly dependent. A contradiction because  $Au_1, \dots, Au_k$  are linearly independent. Hence  $u_1, \dots, u_k \in \mathbb{R}^n$  are LI.

We now assume that  $u_1, \dots, u_k \in \mathbb{R}^n$  are linearly independent. To show  $Au_1, \dots, Au_k$  are linearly independent.

Suppose  $Au_1, \dots, Au_k$  are linearly dependent. Then  $\exists \alpha \neq 0$  s.t.  $\sum \alpha_i (Au_i) = 0$ . So  $0 = \sum \alpha_i (Au_i) = A \sum \alpha_i u_i$ . So  $A^{-1}0 = A^{-1}A \sum \alpha_i u_i = \sum \alpha_i u_i$ . So  $u_1, \dots, u_k$  are linearly dependent. A contradiction because  $u_1, \dots, u_k \in \mathbb{R}^n$  are linearly independent. Hence  $Au_1, \dots, Au_k$  are LI.

**[0.0.8] Exercise** Show that  $u_1, \dots, u_k \in \mathbb{V}$  is linearly independent iff  $\sum_{i=1}^k a_{i1}u_i, \dots, \sum_{i=1}^k a_{ik}u_i$  are linearly independent for any invertible  $A_{k \times k}$ . Show that  $\{u, v\}$  is linearly independent iff  $\{u + v, u - v\}$  is linearly independent.

**Sol.** We first assume that  $u_1, \dots, u_k$  are LI.

Put  $w_r = \sum_{i=1}^k a_{ir}u_i$ .

Then  $\begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & & \vdots \\ a_{1k} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} = A^t \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$ . To show that  $w_1, \dots, w_k$  are LI.

Suppose  $w_1, \dots, w_k$  are linearly dependent. Then exist  $\alpha_1, \dots, \alpha_k$  not all zero s.t.  $\begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_k \end{bmatrix} = 0$ . So

$$0 = \begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & & \vdots \\ a_{1k} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} = \begin{bmatrix} \beta_1 & \cdots & \beta_k \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix},$$

where  $\begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} A^t = \begin{bmatrix} \beta_1 & \cdots & \beta_k \end{bmatrix} \neq 0$ . Thus  $u_1, \dots, u_k$  are linearly dependent.

We now assume that  $w_1, \dots, w_k$  are LI. To show that  $u_1, \dots, u_k$  are LI.

Suppose  $u_1, \dots, u_k$  are LD. Then there exists  $c_1, \dots, c_k \in \mathbb{F}$  not all zero such that  $c_1u_1 + c_2u_2 + \cdots + c_ku_k = 0$ .

Multiplying both side by  $A^t$ .  $A^t(c_1u_1 + c_2u_2 + \cdots + c_ku_k) = 0$

$$\begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{k1} \\ \vdots & & \vdots \\ a_{1k} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} = 0.$$

$$\begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} \begin{bmatrix} \sum_{i=1}^k a_{i1}u_i \\ \vdots \\ \sum_{i=1}^k a_{ik}u_i \end{bmatrix} = 0.$$

This implies that  $\sum_{i=1}^k a_{i1}u_i, \sum_{i=1}^k a_{i2}u_i, \dots, \sum_{i=1}^k a_{ik}u_i$  are LI. A contradiction. Hence  $u_1, \dots, u_k$  are LI.

[0.0.9] **Exercise** Let  $\mathbb{V}$  be a vector space over  $\mathbb{F}$ . Let  $A$  and  $B$  be two non-empty subsets of  $\mathbb{V}$ . Prove or disprove:  $\text{LS}(A) \cap \text{LS}(B) \neq \{0\} \implies A \cap B \neq \phi$ .

**Sol.** Disprove. Consider  $\mathbb{R}^2$  and  $A = \{(1,0), (0,1)\}$  and  $B = \{(1,2), (2,1)\}$ . Notice that  $\text{LS}(A) = \mathbb{R}^2$  and  $\text{LS}(B) = \mathbb{R}^2$ . Hence  $\text{LS}(A) \cap \text{LS}(B) \neq \{0\}$ . But  $A \cap B = \phi$

[0.0.10] **Exercise** Show that a vector space  $\mathbb{V}$  over  $\mathbb{F}$  has a unique basis if and only if either  $\text{DIM}(\mathbb{V}) = 0$  or  $\text{DIM}(\mathbb{V}) = 1$  and  $|\mathbb{F}| = 2$ .

**Sol.** First assume that  $\mathbb{V}(\mathbb{F})$  has a unique basis.

There are two cases. Either  $\mathbb{V}$  is trivial or non-trivial.

**Case I.** If  $\mathbb{V}$  is trivial, then we are done.

**Case II.**  $\mathbb{V}$  is non-trivial. First we show that  $\text{DIM} \mathbb{V} = 1$ . Suppose that  $\text{DIM} \mathbb{V} \geq 2$ . There are two cases.

**Case II(a).**  $\mathbb{V}$  is finite dimensional. Let  $\{x_1, \dots, x_k\}$  be a basis of  $\mathbb{V}$  where  $k \geq 2$ .

Consider  $B = \{x_1, x_2, \dots, x_i, \dots, x_{j-1}, x_j + x_i, x_{j+1}, \dots, x_k\}$ . To show that  $B$  is also a basis of  $\mathbb{V}$ . Using Problem 0.0.8 in tutorial sheet, you can easily show that  $B$  is LI. Since  $B$  is LI and  $|B| = \text{DIM}(\mathbb{V})$ . Hence  $B$  is basis. A contradiction that  $\mathbb{V}$  has unique basis. Hence  $\text{DIM}(\mathbb{V}) = 1$ .

**Case II(b).**  $\mathbb{V}$  is infinite dimensional. Let  $S = \{x_\alpha : \alpha \in I\}$  be a basis of  $\mathbb{V}$  where  $I$  is an index set. Take  $x_\gamma, x_\beta \in S$ . Then  $x_\gamma + x_\beta \in \mathbb{V}$  as  $\mathbb{V}$  is a vector space. But  $x_\gamma + x_\beta \notin S$  as  $S$  is LI, otherwise there is a vector  $x_\gamma + x_\beta$  which is a linear combination of  $x_\gamma$  and  $x_\beta$ .

Consider  $B = \left( \{x_\alpha : \alpha \in I\} - \{x_\gamma\} \right) \cup \{x_\gamma + x_\beta\}$ .

To show that  $B$  is a basis of  $\mathbb{V}$ . Let  $A$  be a finite subset of  $B$ . There are two cases.

**Case I.**  $A = \{x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_k}\}$  that means  $x_\gamma + x_\beta$  is not in  $A$ . Therefore  $A \subseteq S$ . Since  $S$  is LI, then  $A$  is LI.

**Case II.**  $A = \{x_{\alpha_1}, x_{\alpha_2}, \dots, x_\gamma + x_\beta, \dots, x_{\alpha_k}\}$ . Then applying the same techniques as of finite case, we have  $A$  is LI. Therefore each finite subset of  $B$  is LI. Hence  $B$  is LI.

To show  $\text{LS}(B) = \mathbb{V}$ . It is enough to show  $\text{LS}(B) = \text{LS}(S)$ .  $B - \{x_\gamma + x_\beta\} = S - \{x_\gamma\}$  and you know that  $x_\gamma + x_\beta$  is in  $B$ . Using this you can easily show that  $\text{LS}(B) = \text{LS}(S)$ .

Hence  $B$  is a basis of  $\mathbb{V}$ . Therefore  $\mathbb{V}$  has two basis a contradiction. Hence  $\text{DIM}(\mathbb{V}) = 1$ .

We now show that  $|\mathbb{F}| = 2$ . Suppose that  $|\mathbb{F}| \geq 2$ . Then  $|\mathbb{F}|$  has at least one element which is other than additive identity and multiplicative identity. Let  $\{x\}$  be a basis of  $\mathbb{V}$  and let  $\alpha \in \mathbb{F}$  such that  $\alpha$  is neither 0 nor 1. Then  $\{x\}$  and  $\{\alpha x\}$  both are basis of  $\mathbb{V}$ . A contradiction that  $\mathbb{V}$  has unique basis. Hence  $|\mathbb{F}| = 2$ .

Assume that either  $\text{DIM}(\mathbb{V}) = 0$  or  $\text{DIM}(\mathbb{V}) = 1$  and  $|\mathbb{F}| = 2$ . If  $\text{DIM}(\mathbb{V}) = 0$ , then  $\mathbb{V} = \{0\}$ . Hence its has unique basis.

If  $\text{DIM}(\mathbb{V}) = 1$  and  $|\mathbb{F}| = 2$ . Suppose that  $B$  has two bases they are  $\{x\}$  and  $\{y\}$ . Since  $y \in \mathbb{V}$  and  $\{x\}$  is basis, then  $y = \alpha x$  where  $\alpha \in \mathbb{F}$ . Since  $\mathbb{F}$  has two elements, then  $\alpha = 1$  (multiplicative identity). Therefore  $y = 1(x) = x$ . Hence  $\mathbb{V}$  has unique basis.

**[0.0.11] Exercise** Let  $\mathbb{V}$  be an  $n$  dimensional vector space over  $\mathbb{F}$  and let  $\mathbb{F}$  has exactly  $p$  elements. Then show that  $|\mathbb{V}| = p^n$ .

**Sol.** Let  $B = \{u_1, u_2, \dots, u_n\}$  be a basis of  $\mathbb{V}$ . Any element in  $\mathbb{V}$  can be written as a **unique** linear combination of  $u_1, u_2, \dots, u_n$ . That is  $c_1 u_1 + c_2 u_2 + c_3 u_3 + \dots + c_n u_n$  where  $c_i \in \mathbb{F}$ . For each  $c_i$  we have  $p$  choices and the choice of  $c_i$  does not depend on the choice of  $c_j$  for  $i \neq j$ . Hence  $|\mathbb{V}| = p^n$ .

**[0.0.12] Exercise** Check whether vector space  $\mathbb{R}$  (set of real numbers) over the field  $\mathbb{Q}$  (set rational number) is infinite dimensional or finite dimensional.

**Sol.** If we are able to show that for each  $n \in \mathbb{N}$  there exists a LI subset of  $\mathbb{R}(\mathbb{Q})$  containing  $n+1$  elements. Then we are done. Let  $\alpha$  be a transcendental number. To show that  $\{1, \alpha, \alpha^2, \alpha^3, \dots, \alpha^n\}$  is LI. Take  $a_0 + a_1 \alpha + a_2 \alpha^2 + \dots + a_n \alpha^n = 0$ . We never have non-trivial solution of this equation as  $\alpha$  is transcendental. Hence  $\{1, \alpha, \alpha^2, \alpha^3, \dots, \alpha^n\}$  is LI. For each  $n$  we have a LI set of  $n+1$  vectors. Therefore  $\mathbb{R}(\mathbb{Q})$  is infinite dimensional.

**[0.0.13] Exercise** Let  $S = \left\{ \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} a \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \right\}$ . Find the values of  $a$  for which  $\text{LS}(S) \neq \mathbb{R}^3$ .

**Sol.** We find the values of  $a$  for which  $\text{LS}(S) = \mathbb{R}^3$ . Since  $S$  contains exactly three and  $\text{DIM}(S) = \mathbb{R}^3$ , then  $S$  is a basis of  $\mathbb{R}^3$ . This implies  $S$  is LI.

Consider  $A = \begin{bmatrix} 4 & a & 4 \\ 5 & 2 & 3 \\ 6 & 4 & 2 \end{bmatrix}$ . Since  $S$  is LI, then  $A$  is invertible. Hence  $\text{DET}(A) \neq 0$  and  $\text{DET}(A) = 8a$ . Therefore we have  $8a \neq 0$ . This implies  $a \neq 0$ .

We have seen that if  $a \in \mathbb{R} - \{0\}$ , then  $\text{LS}(S) = \mathbb{R}^3$ .

Then So  $\text{LS}(S) \neq \mathbb{R}^3$  iff  $a = 0$ .

**[0.0.14] Exercise** Give 2 bases for the trace 0 real symmetric matrices of size  $3 \times 3$ . Extend these bases to bases of the real symmetric matrices of size  $3 \times 3$ . Extend these bases to bases of the real matrices of size  $3 \times 3$ .

**[0.0.15] Exercise** Consider  $\mathbb{W} = \{v \in \mathbb{R}^6 | v_1 + v_2 + v_3 = 0, v_2 + v_3 + v_4 = 0, v_4 + v_5 + v_6 = 0\}$ . Supply a basis for  $\mathbb{W}$  and extend it to a basis of  $\mathbb{R}^6$ .

**Sol.** For the system:  $\left[ \begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right]$ . Get the RREF:  $\left[ \begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right]$ .

Free variables:  $v_3, v_5, v_6$ . Obtain linearly independent solutions by putting a free variable 1 and other free variables 0:

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

This is a basis for  $\mathbb{W}$ . Note that  $e_1$  cannot be a linear combination of these three vectors.!!

To extend add  $e_1, e_2, e_4$ . These three with the first basis vector will give  $e_3$ . Similarly..... So these 6 will span  $\mathbb{R}^6$ . They are linearly independent.

**[0.0.16] Exercise** For what values  $\alpha$  are the vectors  $(0, 1, \alpha), (\alpha, 1, 0)$  and  $(1, \alpha, 1)$  in  $\mathbb{R}^3$  linearly independent?

**[0.0.17] Exercise** If  $S$  and  $T$  are two subspaces of a vector spaces having a common complement set  $W$ , does it follow that  $S = T$ ?

**Sol.** Not necessarily. Consider  $\mathbb{R}^2(\mathbb{R})$ . Take  $S = \{(x, 0) : x \in \mathbb{R}\}$  and  $T = \{(0, y) : y \in \mathbb{R}\}$ . Consider  $W = \text{LS}(\{(1, 1)\})$ . It is easy to check  $W$  is complement of  $S$  and  $T$ . But  $S \neq T$ .

**[0.0.18] Exercise** In the vector space  $\mathbb{R}^4$ , find two different complements of the subspace  $S = \{(x_1, x_2, x_3, x_4) : x_3 - x_4 = 0\}$

**Sol.** We first find a basis of  $\mathbb{W}$ . Let  $(x_1, x_2, x_3, x_4) \in \mathbb{W}$ . Then  $x_3 - x_4 = 0$ . Therefore  $(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_3) = x_1(1, 0, 0, 0) + x_2(0, 1, 0, 0) + x_3(0, 0, 1, 1)$ . This implies that  $\text{LS}(\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1)\})$ . One can easily prove that  $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1)\}$  is LI. Hence  $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1)\}$  is basis of  $\mathbb{W}$ .

We now extend  $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1)\}$  to a basis of  $\mathbb{R}^4$ . Take  $(0, 0, 0, 1) \in \mathbb{R}^3 - \mathbb{W}$ . Using basis extension theorem  $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 0, 0, 1)\}$  is LI. It has exactly four elements. Then  $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1), (0, 0, 0, 1)\}$  is a basis of  $\mathbb{R}^4$ . Therefore  $S_1 = \text{LS}(\{(0, 0, 0, 1)\})$  is a complement of  $\mathbb{W}$ .

Take  $(1, 1, 1, 0) \in \mathbb{R}^3 - \mathbb{W}$ .

Using basis extension theorem  $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1), (1, 1, 1, 0)\}$  is LI. It has exactly four elements. Then  $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 1), (1, 1, 1, 0)\}$  is a basis of  $\mathbb{R}^4$ . Therefore  $S_2 = \text{LS}(\{(1, 1, 1, 0)\})$  is a complement of  $\mathbb{W}$ .

**[0.0.19] Exercise** Show that a non-trivial subspace  $S$  of a finite dimensional vector space  $\mathbb{V}$  has two virtually disjoint complements iff  $\text{DIM}(S) \geq \frac{\text{DIM}(\mathbb{V})}{2}$ .

**Sol.** We first assume that  $\text{DIM}(S) \geq \frac{\text{DIM}(\mathbb{V})}{2}$ . To show  $S$  has two virtually disjoint complements. Let  $\text{DIM}(\mathbb{V}) = n$  and  $\text{DIM}(S) = k$ . Then  $k \geq \frac{n}{2}$ . Let  $\{x_1, \dots, x_k\}$  be a basis of  $S$ . We extend it to basis of  $\mathbb{V}$  that is  $\{x_1, \dots, x_k, x_{k+1}, \dots, x_n\}$ . Take  $S_1 = \text{LS}(x_{k+1}, \dots, x_n)$ . Then  $S_1$  is a complement of  $S$ .

Consider  $\{x_1, \dots, x_k, x_{k+1} + x_1, x_{k+2} + x_2, \dots, x_n + x_{n-k}\}$ . This is possible as  $k \geq \frac{n}{2}$ . To show this set is basis of  $\mathbb{V}$ . Take  $S_2 = \text{LS}(\{x_{k+1} + x_1, x_{k+2} + x_2, \dots, x_n + x_{n-k}\})$ . This is also a complement of  $S$ . To show  $S_1 \cap S_2 = \{0\}$ .

Let  $x \in S_1 \cap S_2 = \{0\}$ . Then  $x \in S_1$  and  $x \in S_2$ .

$$x = c_1 x_{k+1} + c_2 x_{k+2} + \dots + c_{n-k} x_n \text{ and}$$

$$x = b_1(x_{k+1} + x_1) + b_2(x_{k+2} + x_2) + \dots + b_{n-k}(x_n + x_{n-k})$$

$$b_1 x_1 + \dots + b_{n-k} x_{n-k} + (b_1 - c_1)x_{k+1} + \dots + (b_{n-k} - c_{n-k})x_n = 0$$

$\{x_1, \dots, x_{n-k}, x_{k+1}, \dots, x_n\}$  is LI

Then  $b_i = c_i = 0$  for  $i = 1, \dots, n - k$ . Hence  $x = 0$ .

We now assume that  $S$  has two virtually disjoint complements.

To show  $\text{DIM}(S) \geq \frac{\text{DIM}(\mathbb{V})}{2}$ . Suppose that  $\text{DIM}(S) < \frac{\text{DIM}(\mathbb{V})}{2}$ . Let  $\text{DIM}(\mathbb{V}) = n$  and  $\text{DIM}(S) = k$ . Let  $S_1$  and  $S_2$  be two virtually disjoint complements. Then  $\text{DIM}(S_1) = n - k = \text{DIM}(S_2)$ .

$\text{DIM}(S_1 + S_2) = \text{DIM}(S_1) + \text{DIM}(S_2) = 2(n - k) > 2(n - \frac{n}{2}) = n$ , a contradiction. Hence  $\text{DIM}(S) \geq \frac{\text{DIM}(\mathbb{V})}{2}$ .

[0.0.20] **Exercise** Find a complement of the subspace  $\{(x_1, x_2, \dots, x_n) : \sum_{i=1}^n x_i = 0\}$  in  $\mathbb{R}^n$ .

**Sol.** Same as Exercise 0.0.18