17/07/2017

Lecture 1

Subject Name: Transform Calculus

Subject Code: - MA20101.
(3-0-0)

Mid sem: 30 %

End sem: 50 %

Class Test: 10 %

TA — 10 %

based on

- Assignment Submission.
- Quizzes
- Attendence in the class
- Response in the class

Class: Monday 8 – 10 am

Room No. NR422

Tuesday: ~~12 – 1 pm~~

~~1 – 2 pm~~

5 – 6 pm

Syllabus :-

Laplace Transform :-

Definition of Laplace Transform

Linearity property,

conditions for existence

of Laplace Transform.

First & second shifting

properties, Laplace Transform of derivatives & integrals, unit step functions, Dirac delta-function, error function; Differentiation & integration of transforms, convolution theorem, Inversion, periodic functions, Evaluation of integrals by Laplace Transform. Solution of initial & boundary value problems.

Fourier Series:

Periodic functions, Fourier series representation of a function, half-range series, sine & cosine series, Fourier Integral formula, Parseval's identity.

Fourier Transform:

Fourier sine & cosine transforms, Linearity, scaling, frequency shifting

- time shifting properties.
Self reciprocity of
Fourier Transform, Convolution
theorem. Applications
to boundary value problems.

[Brief Introduction of
Z - Transform, Mellin
transform &
Wavelet Transform.

Books

1) Advanced Engineering
Maths by Erwin
Kreyszig (10th Ed.)

2) Integral transforms & their Applications by Lokenath Debnath & Dambaru Bhatta (CRC press)

3) Integral Transforms & their Applications by Brian Davies (Springer)

4) Introduction to Partial Differential Equations by K. Sankara Rao (Prentice Hall)

5) Laplace Transforms & Fourier Analysis — by Schaum's Outlines (Tata Mc. Graw Hill)

6) An Introduction to Laplace Transforms & Fourier Series by P. P. G. Dyke (Springer)

*** $\boxed{7)}$ Integral transform / Fourier transform by I. N. Snedon.

Laplace Transform :-

Heaviside $\frac{d}{dx} x^2 = 2x$ $\int x^2 \, dx = x$

to the O.D. eqn.

$\Rightarrow$ E : Engineering.

Note :-

Laplace Transform is essentially a mathematical tool which can be used to solve several problems in science & Engineering which involves differential eqns & corresponding initial & boundary value probems.

2) Why one should learn Laplace transform technique when other techniques are available?

$\frac{d}{dx}(x^2) = 2x.$

$\frac{d(2x)}{dx} = 2$

If, $x^{1.85} = 3$

Taking logarithms on both sides

$1.85 \ln x = \ln 3.$

$\Rightarrow \ln x = \frac{\ln 3}{1.85}$

$\Rightarrow x = \ln^{-1}\left(\frac{\ln 3}{1.85}\right)$

$= \underline{\qquad}$

NB:-

The Laplace Transform method has two main advantages over the other methods.

① Problems are solved more directly:

Initial Value Problems are solved without first determining a General solution.

Non-homogeneous O.D. eqn are solved without first solving the corresponding homogeneous O.D. eqn.

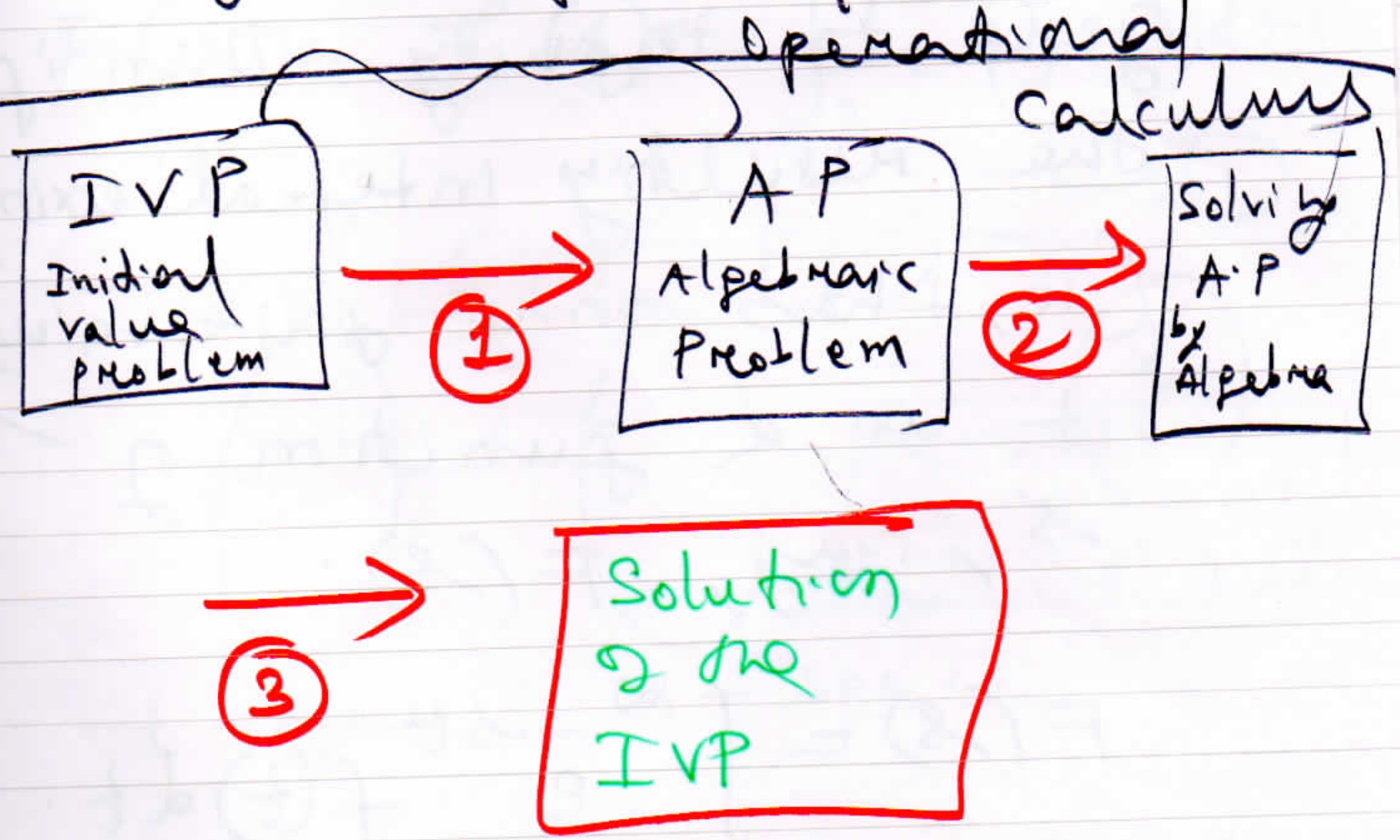
2) More importantly, the ~~use~~ of the unit step function (Heaviside function) & Dirac's delta make the method particularly powerful for problems with inputs (driving forces) that have discontinuities or represent short impulses or complicated periodic functions.

Note:- When the Laplace transform technique is applied to a Partial Differential equation, it reduces the no. of independent variables by one.

The process of solution consists of three main steps:

1st step:- The given "hard" problem is transformed into a "simple" eq^n (Subsidiary equation) nd step :- The subsidiary eqn is solved by purely algebraic manipulation

3rd step :- The soln of the subsidiary eqn is transformed back to obtain the soln of the given problem.



*** Laplace Transform

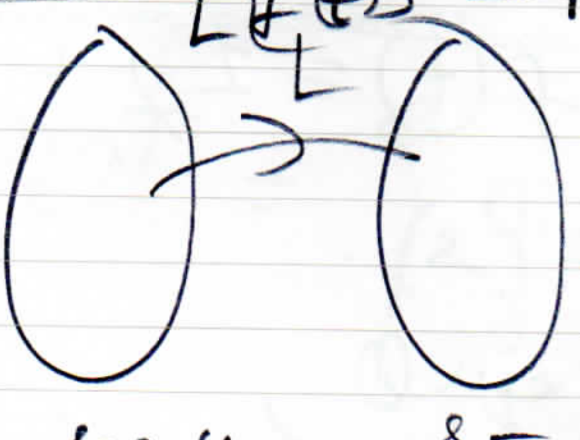
Defn:- Let $f(t)$ be a given function that is defined for all $t \geqslant 0$. We multiply $f(t)$ by e^{-st} & integrate w.r.t 't' from zero to infinity. Then if the resulting integral exists (i.e., it has some finite value) it is a function of s, say, $F(s)$:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

This function $F(s)$ of the variable s is called the Laplace Transform of the original function $f(t)$ & will be denoted by $\mathcal{L}(f)$. Thus

$$\boxed{F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt \quad \to (1)}$$

$L\{f(t)\} = F(s)$



The original f^n 'f' depends on t & the new f^n 'F' (its transform) depends on s

t - space   s - space

Fig 1:- Laplace Transform as a mapping.

Furthermore, the original function $f(t)$ by eq^n (1) is called the <u>inverse transform</u> or, inverse of $F(s)$ & will be denoted by $\mathcal{L}^{-1}(F)$, ie, we shall write

$$f(t) = \mathcal{L}^{-1}(F)$$

**** [<u>Note</u>:- Original f^ns are denoted by lowercase letters & their transforms by the same letters in capitals, so that $F(s)$ denotes the transform of $f(t)$, $Y(s)$ denotes the transform of $y(t)$ & so on.]

<u>EX1</u>/ Let $f(t) = 1$, when $t \geq 0$

Find $F(s)$.

<u>Sol^n</u>:- From eq^n (1)

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \mathcal{L}(1) = \int_0^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$\therefore \mathcal{L}(f) = \frac{1}{s} \; (s > 0)$

(why?)

Correct

$$\int_0^\infty e^{-st}\,dt = \lim_{T\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^T$$

$$= \lim_{T\to\infty}\left[-\frac{1}{s}e^{-sT} + \frac{1}{s}e^{0}\right]$$

$$= \lim_{T\to\infty}\left[1 - e^{-st}\right]_0^T$$

$$\lim_{T\to\infty} e^{-sT} = \begin{cases} 0, & s > 0 \\ \infty, & s < 0 \end{cases}$$

$$\therefore \mathcal{L}(1) = F(s) = \begin{cases} 1/s, & \text{if } s > 0 \\ \text{diverges, if } s \leq 0. \end{cases}$$

C. V.
Ex2/ L.T of the exponential functions

Let $f(t) = e^{at}$, $t \geq 0$ where a is a constant.

Find $\mathcal{L}(f)$.

Soln:-

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at}\, dt = \lim_{T\to\infty}\left[\frac{e^{(a-s)t}}{(a-s)}\right]_0^T$$

$$= \int_0^\infty e^{-(s-a)t}\, dt = \lim_{T\to\infty}\left[\frac{e^{(a-s)T} - 1}{(a-s)}\right]$$

The limit depends on whether $s > a$ or $s < a$

as $\lim_{T\to\infty} e^{(a-s)T} = \begin{cases} 0, & \text{if } s > a \\ \infty, & \text{if } s < a \end{cases}$

$$= \begin{cases} \dfrac{1}{s-a}, & \text{when } s > a \\ \text{diverges}, & \text{if } s \leq a. \end{cases}$$

Exitence of Laplace Transform

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt. \quad \rightarrow (1)$$

For a fixed s, the integral in (1) will exist if the whole integrand $e^{-st} f(t)$ $\rightarrow 0$ fast enough as $t \rightarrow \infty$

$\Rightarrow$ the function $f(t)$ itself should not grow faster than, say e^{kt}.

eg, not like e^{t^2}

eg, t^n $(n \in \mathbb{Z})$ is of exponential order as $t^n e^{-bt} \to 0$ as $t \to$

(EX). $\left[\underline{EX}\ \lim_{t\to\infty} \frac{t^n}{e^{bt}} = 0\right.$

Defn:- By exponential order as $t \to \infty$ we mean $\exists$ a constant α such that

$$\lim_{t\to\infty} e^{-\alpha t} f(t)$$

is finite.

Note:- The function $f(t)$ need not be continuous. it is sufficient if it is piecewise continuous.

Fig 2:- Example of a piece-wise continuous f^n.

xxxxxx

Th-1/ (Existence theorem for Laplace Transform) (Sufficient condition)

Let $f(t)$ be a function that is piecewise continuous on every finite interval in the range $t \geq 0$ & satisfies

$$|f(t)| \leq M \cdot e^{kt}, \forall t \geq$$

$\longrightarrow$ (2)

& for some constants k & M.

Then the Laplace transform of $f(t)$ exist for all $s > k$.

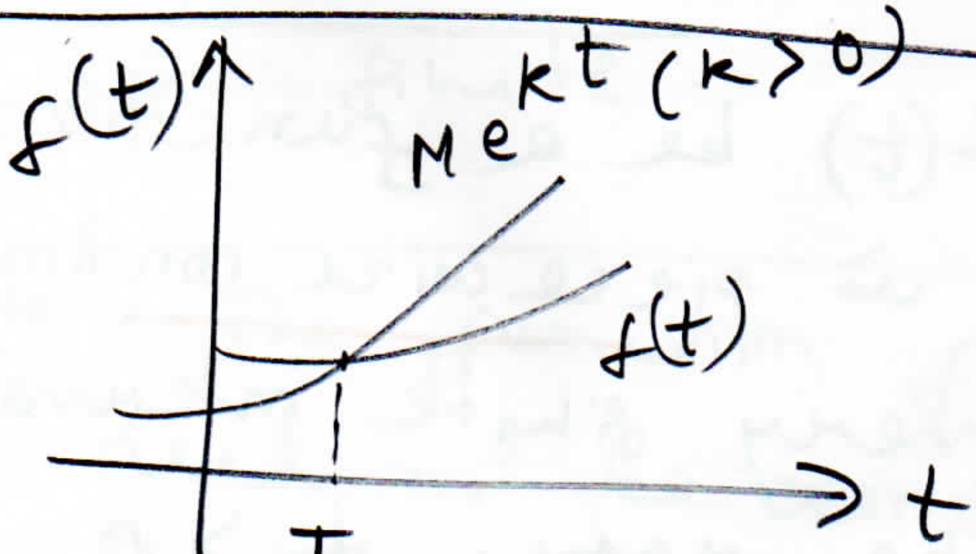


Fig 3 :- function f is of exponential order.

2 :– Since $f(t)$ is piece-wise continuous, $e^{-st} f(t)$ is integrable over any finite interval on the t-axis.

From eq ②, assuming that $s > k$, we obtain

$$|\mathcal{L}(f)| = \left| \int_0^\infty e^{-st} f(t)\,dt \right|$$

$$\leq \int_0^\infty |f(t)|\, e^{-st}\,dt$$

$$\leq \int_0^\infty M \cdot e^{kt}\, e^{-st}\,dt$$

$$= M \int_0^\infty e^{-(s-k)t}\,dt$$

$$= M/(s-k)$$

where the condition

<u>$s > k$</u> was needed

for the existence of

the last integral.

<u>EX</u>/ The conditions in

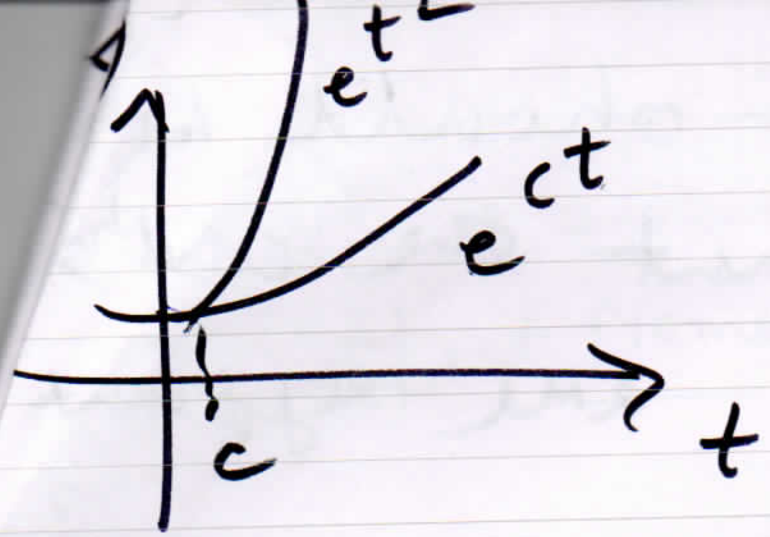
$\boxed{\text{Th} - 1}$ are sufficient

for most applications

e.g., $\cosh t < e^t$

$t^n < n!\, e^t \quad (n = 0, 1, \ldots)$

<u>EX</u> e^{t^2} ? <u>No</u> why?



as no matter how large we choose M & k in (2)

$$e^{t^2} > M e^{kt}, \forall t > t_0$$

where t_0 is a sufficiently large no. depending on M & k,

Ans

$$\left|\frac{e^{t^2}}{e^{kt}}\right| = \left|e^{t^2-kt}\right| = \left|e^{t(t-k)}\right|$$

$\longrightarrow \infty$ as $t \to \infty$

for any value of k.

<u>Note</u> :- It should be noted that the condⁿˢ in $\boxed{\text{Th-1}}$ are sufficient rather than necessary.

e.g. $f(t) = ?$

Q) Given an example of a fⁿ $f(t)$ which does not satisfy the sufficient ~~necessary~~ conditions but its Laplace transform exist?