Fatou's Lemma Suppose $\{f_n\}$ is a sequence of non-neg. measurable then s.t. $f_n \to f$ ptwise. Then $\int f \le \liminf \int f_n$

Proof Let g be a bounded measurable function having support of finite measure s.t. $0 \le g \le f$.

Let $g_n = \min\{g, f_n\}$

g_n's are measurable function having finite measure.

$g_n \to g$ as $n \to \infty$ (ptwise)

$\Rightarrow \int g_n \to \int g$.

$\Rightarrow g_n \le f_n$

$\Rightarrow \int g_n \le \int f_n$

$\Rightarrow \int g \le \liminf \int f_n$

Take supremum over all $g \le f$ in (*).

$\Rightarrow \int f \le \liminf \int f_n$

Scanned by CamScanner

$$\int \overset{\overset{\eta_1}{\uparrow}}{g_1} + \int \overset{\overset{\eta_2}{\uparrow}}{g_2} = \int g_1 + g_2 \leq \int (f+g)$$

Take supremum in the LHS over all $g_1 \leq f$ & $g_2 \leq g$

Then, we have $\int f + \int g \leq \int (f+g)$ — ①

Conversely, let $\eta \leq f+g$, η is non-ve bdd meas. having finite supp.

Set $\eta_1 = \min\{f, \eta\}$

$\eta_2 = \eta - \eta_1$

Then $0 \leq \eta_1 \leq f$, $0 \leq \eta_2 \leq g$

for non-neg functions.

$$f_n(x) = \begin{cases} n, & 0 < x \leq 1/n \\ 0, & 1/n \leq x < 1 \end{cases}$$

$\int f_n = 1$ for each n

$f_n \to 0$ pt wise.

$1 = \int f_n \not\to \int 0 = 0$

$\int f \leq \liminf \int f_n$

Date

$$\int_{[a,b]}^{L} \tilde{\psi} - \int_{[a,b]}^{L} \tilde{\phi} = \lim_{k\to\infty} \left[\int_{[a,b]}^{L} \psi_k - \int_{[a,b]}^{L} \phi_k \right]$$

$$= \lim_{k\to\infty} \left[\int_{[a,b]}^{R} \psi_k - \int_{[a,b]}^{R} \phi_k \right] \le 0$$

$$\tilde{\psi} - \tilde{\phi} \ge 0$$

$\Rightarrow$ $\tilde{\psi} = \tilde{\phi}$ a.e.

$\Rightarrow$ $\tilde{\psi} = \tilde{\phi} = f \Rightarrow f$ is measurable fun.

$$\int_{[a,b]}^{L} f = \lim_{k\to\infty} \int_{[a,b]}^{L} \varphi_k$$

$$= \lim_{k\to\infty} \int_{[a,b]}^{R} \varphi_k = \int_{[a,b]}^{R} f$$

Lebesgue Integral of non-neg. measurable funs.

Suppose $f \ge 0$, f measurable.

$$\int f = \sup_{0 \le g \le f} \int g$$ g is a bounded measurable function supported on a set of finite measure.

We say f is integrable if $\int f < \infty$.

Proving Linearity:

$$\int f + \int g \le \int (f+g)$$

$g_1 \le f$

$g_2 \le g$

Page No.

Scanned by CamScanner

$$E = \bigcup_{i=1}^{\infty} E_k$$

$$m(E) = 0$$

$$E = \{x \in R \mid f(x) > 0\}$$

x x x

Prove that every Riemann integrable function in $[a,b]$ is measurable & lebesgue integ-

$$\overset{R}{\int_{[a,b]}} f(x)\,dx = \overset{L}{\int_{[a,b]}} f(x)\,dx$$

roof: Since $f \in R[a,b]$, we get the two sequences of simple functions namely $\{\phi_k\}$ and $\{\psi_k\}$ s.t.

$$\phi_1 \leq \phi_2 \leq \dots \leq f \leq \dots \leq \psi_2 \leq \psi_1$$

with $\lim_{k\to\infty} \overset{R}{\int_{[a,b]}} \phi_k = \overset{R}{\int_{[a,b]}} f = \lim_{k\to\infty} \overset{R}{\int_{[a,b]}} \psi_k$

Observe $\overset{R}{\int_{[a,b]}} \phi_k = \overset{L}{\int_{[a,b]}} \phi_k$ and $\overset{R}{\int_{[a,b]}} \psi_k = \overset{L}{\int_{[a,b]}} \psi_k$

Suppose for $k \geq 1$, $\phi_k \to \tilde{\phi}_k$ & $\psi_k \to \tilde{\psi}_k$

Then $\tilde{\phi}_k \leq f \leq \tilde{\psi}_k$

& $\lim_{k\to\infty} \overset{L}{\int_{[a,b]}} \phi_k = \overset{L}{\int_{[a,b]}} \tilde{\phi}$, $\lim_{k\to\infty} \overset{L}{\int_{[a,b]}} \psi_k = \overset{L}{\int_{[a,b]}} \tilde{\psi}$ (BCT)

Scanned by CamScanner

$\Rightarrow I_n$ must converge, i.e. $\lim_{n\to\infty} \int \varphi_n$ exists.

(ii) $|I_n| \leq \varepsilon\, m(E) + M\varepsilon$ → bcoz $f=0$ a.e. given

$\Rightarrow \int f = 0$

Defn: If f is a non-neg. & bdd measurable function supported on a set of finite measure, then

$$\int f = \lim_{n\to\infty} \int \varphi_n$$

where φ_n's are simple, $\mathrm{Supp}(\varphi_n) \subseteq \mathrm{Supp}(f)$ & $\varphi_n \to f$ ptwise.

Proposition

(i) $\int (af+bg) = a\int f + b\int g$ (Linearity)

(ii) $f \leq g \Rightarrow \int f \leq \int g$.

(iii) $\int_{E\cup F} f = \int_E f + \int_F f$, $E \cap F = \varphi$.

(iv) $\left| \int_E f \right| \leq \int_E |f|$ (T.E.)

Page No

$f \geq 0$ measurable, $|f(x)| < M$ $\forall x \in E$, $\text{Supp}(f) \subseteq E$. We can find a sequence of simple functions ϕ_n such that $\text{supp}(\phi_n) \subseteq E$ $\forall n$ & $\phi_n \to f$ ptwise $(|\phi_n| < M)$

Lemma

Let f be a nonneg bounded measurable function supported on a set E of finite measure.
Let $\phi_n \to f$ (ϕ_n's are as above).
Then (i) $\lim_{n\to\infty} \int \phi_n$ exists.

(ii) If $f = 0$ a.e., then $\lim_{n\to\infty} \int \phi_n = 0$.

Proof (i) By Egorov's theorem. We can find a closed set $A_\varepsilon \subset E$ s.t. $\phi_n \to f$ uniformly on A_ε & $m(E \setminus A_\varepsilon) < \varepsilon$.
Define $I_n = \int \phi_n$

$$|I_n - I_m| = \left|\int_E (\phi_n - \phi_m)\right|$$
$$\leq \int_{A_\varepsilon} |\phi_n - \phi_m| + \int_{E\setminus A_\varepsilon} |\phi_n - \phi_m|$$
$$\leq \int_{A_\varepsilon} |\phi_n - \phi_m| + 2M\, m(E \setminus A_\varepsilon)$$
$$\leq \int_{A_\varepsilon} |\phi_n - \phi_m| + 2M\varepsilon$$

Choose m, n large enough s.t. $|\phi_n - \phi_m| < \varepsilon$ on A_ε (bcoz of uniform convergence of $\{\phi_n\}$)

Scanned by CamScanner

$$\varphi = \sum_{k=1}^{M} c_k \chi_{E_k} \quad , c_k \neq 0 \, , \, E_k \cap E_j = \phi$$

Lebesgue integral.

$$\int \varphi \, dx = \sum_{k=1}^{M} c_k \, m(E_k)$$

→ Integral of bounded measurable functions supported on a set of finite measure.

Defⁿ: (Support of f) $= \{ x \in E \mid f(x) \neq 0 \}$

* If f is measurable, then the $\mathrm{supp}(f)$ is also so.

Page No.

Scanned by CamScanner

Bounded Convergence Theorem

Suppose $\{f_n\}$ is a sequence of measurable function supported on a set of finite measure E and bounded by M s.t. $f_n \to f$ ptwise. Then f is measurable, bdd and supported on E s.t. $\int f_n \to \int f$.

$$\int_E |f_n - f| \leq \int_{A_\epsilon} |f_n - f| + \int_{E \setminus A_\epsilon} |f_n - f|$$

$$\leq \epsilon\, m(E) + 2M \underbrace{m(E \setminus A_\epsilon)}_{\epsilon}$$

$$\therefore \left|\int f_n - \int f\right| \leq \int |f_n - f| \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \int f_n \to \int f.$$

Exercise Suppose $f \geq 0$, $supp(f) \subseteq E$, $m(E) < \infty$. If $\int f = 0$, then prove that $f = 0$ a.e.

$E_k = \{x \in E \mid f(x) \geq 1/k\}$, $k = 1, 2, \ldots$

$$0 = \int f \geq \int_{E_k} f \geq \frac{1}{k} m(E_k)$$

$$\Rightarrow m(E_k) = 0$$