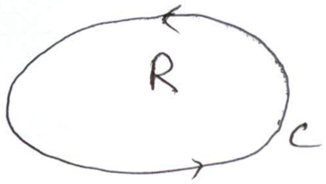
Green's theorem in plane

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Verification (Important)



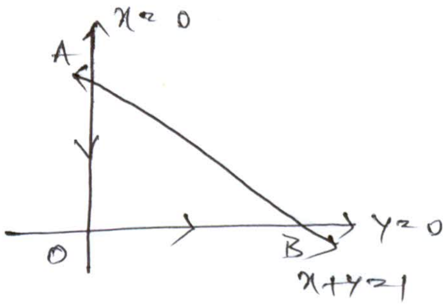
Prob 1

Verify Green's theorem in the plane for

$$\oint_C \Big[\underbrace{(3x^2 - 8y^2)}_{P}\, dx + \underbrace{(4y - 6xy)}_{Q}\, dy\Big]$$

$C \to$ boundary of the triangle enclosed by $x=0$, $y=0$, $x+y=1$.

Line integral



$$\oint_C P\,dx + Q\,dy$$

$$= \left(\int_B^A + \int_A^O + \int_O^B\right)(P\,dx + Q\,dy)$$

For $\int_{B\,(x+y=1)}^{A}$ take $y = t \Rightarrow dy = dt$

$x = 1-t \Rightarrow dx = -dt$

$$= \int_{t=0}^{1} \left\{3(1-t)^2 - 8t^2\right\}(-dt) + \left(4t - 6t(1-t)\right)dt$$

$$= \frac{8}{3}$$

$$\int_A^O (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_{y=1}^{0} 4y\,dy = 2y^2\Big|_1^0 = -2.$$

$$\int_O^B (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_{x=0}^{1} 3x^2\,dx = x^3\Big|_0^1 = 1.$$

$$\left(\int_B^A + \int_A^0 + \int_0^B\right)(P\,dx + Q\,dy) = \frac{8}{3} - 1 = \frac{5}{3}$$

$$\iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} 10\,y\,dx\,dy = \frac{5}{3}$$

Since, $\frac{\partial Q}{\partial x} = -6y$, $\frac{\partial P}{\partial y} = -16y$.

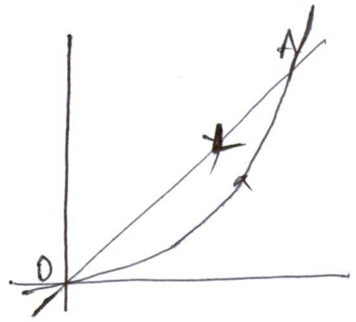
Prob. 2

$P = xy + y^2$, $Q = x^2$

C = closed boundary of the region enclosed by $y = x$ and $y = x^2$.

$$\oint_C P\,dx + Q\,dy$$

$$= \int_{A\,(\text{along } y=x)}^{O} + \int_{O\,(\text{along } y=x^2)}^{A} \quad \text{— (1)}$$



$$\iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy \quad \text{— (2)}$$

~~Verify~~

Now, (1) gives $\int_{t=1}^{0}(3t^2)\,dt + \int_{t=0}^{1}(t^3 + t^4)\,dt + t^2 \cdot 2t\,dt$

$$= \left[t^3\right]_1^0 + \left[\frac{3}{4}t^4 + \frac{t^5}{5}\right]_0^1$$

$$= -1 + \frac{3}{4} + \frac{1}{5} = -\frac{1}{20}$$

(2) gives $\int_{y=0}^{1}\int_{x=y}^{\sqrt{y}}(2x - x - 2y)\,dx\,dy = \int_{y=0}^{1}\left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}}dy$

$$= \int_{y=0}^{1}\left[\frac{y}{2} - 2y\sqrt{y} - \frac{y^2}{2} + 2y^2\right]dy$$

$$= \left[\frac{y^2}{4} - \frac{4}{5}\left(y^{5/2}\right) - \frac{y^3}{6} + \frac{2}{3}y^3\right]_0^1 = -\frac{1}{20}$$

Verify Green's thm :

3. H.W $\oint_C \left[(y^2-x^2)\,dx + (x^2+y^2)\,dy\right]$

$C \to$ triangle bounded by $y=0$, $x=3$, $y=x$.

4. H.W $\oint_C \left[(2xy-x^2)\,dx + (x^2+y^2)\,dy\right]$

$C \to$ boundary of the region enclosed by $y=x^2$, $y^2=x$.

$\boxed{\text{Note}}$ – In case of Green's theorem if we take

$P = \frac{x}{2}$ and $Q = -\frac{y}{2}$

then R.H.S become $\iint \frac{1}{2} - \left(-\frac{1}{2}\right) dx\,dy = \iint_R dx\,dy$

$=$ area of the region

L.H.S

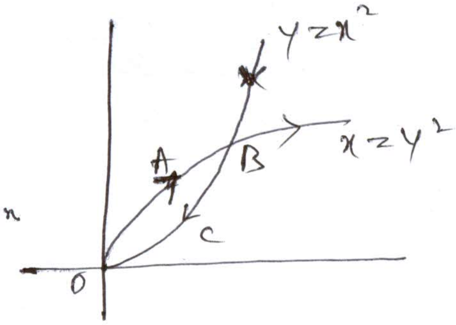
$= \frac{1}{2}\oint_C (x\,dy - y\,dx) =$ area of the region R whose boundary is C.

EX Using line integrals, compute the area of the region bounded by $y=x^2$ & $x=y^2$.

To find area of OABC using line integral

$\frac{1}{2}\oint_C (x\,dy - y\,dx)$ $\left[\text{Since } \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\iint (1+1)\,dx\,dy\right]$

where $C \to$ the boundar



$= \frac{1}{2}\int_B^O x\,dy - y\,dx + \frac{1}{2}\int_O^B (x\,dy - y\,dx) = \frac{1}{3}$ sq. unit

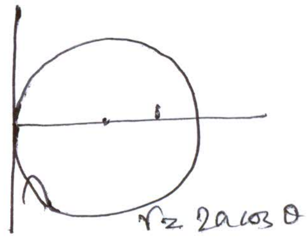
P.T.O

$$\frac{1}{2}\oint_C (x\,dy - y\,dx) \rightarrow \text{some area}$$

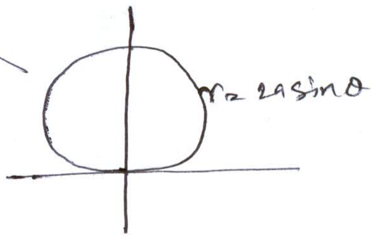
$x = r\cos\theta$, $dx = -r\sin\theta\,d\theta$.
$y = r\sin\theta$, $dy = r\cos\theta\,d\theta$

$$= \frac{1}{2}\oint_C (r\cos\theta . r\cos\theta\,d\theta + r\sin\theta . r\sin\theta\,d\theta)$$

$$= \frac{1}{2}\oint_C r^2\,d\theta$$



$$= \frac{1}{2}\int_{\theta=0}^{\pi} (2a\sin\theta)^2\,d\theta$$



$$= 2a\int_{\theta=0}^{\pi} 2\sin^2\theta\,d\theta$$

$$= 2a\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= 2a\pi$$

Surface integrals

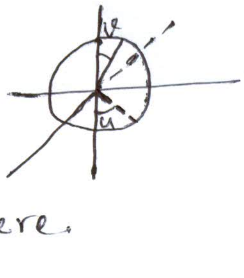
$$\iint_S f\,ds \qquad \iint_{D_{xy}} f(x,y)\,dx\,dy.$$

$$\iint_S \vec{F}(x,y,z)\,d\vec{s} = \iint_S \vec{F}(x,y,z)\,\vec{n}.\,ds.$$

Ex $\iint_S f(x,y,z)\,ds.$ $x = t$, $y = t^2$, $z = t^3$
curve $x^2+y^2+z^2 = a^2$.

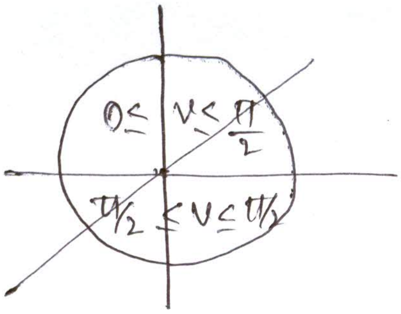
$x = a\cos\phi\sin\theta$, $y = a\sin\phi\sin\theta$, $z = a\cos\theta$

$(x,y,z) \rightarrow$ is any pt. lying on the surface of the sphere



$x = a\cos u\sin v$, $y = a\sin u\sin v$, $z = a\cos v$
$0 \le v \le \frac{\pi}{2}$, $0 \le u \le 2\pi \rightarrow$ upper hemisphere
$\pi/2 \le v \le \pi$, $0 \le u \le 2\pi \rightarrow$ lower hemisphere.

$0 \le u \le 2\pi, \quad 0 \le v \le \pi \longrightarrow$ entire sphere



$x = a \sin v \cos u, \quad y = a \sin v \sin u$

$z = a \cos v$

Apostol vol. II

Now if we consider v in the angle with xy-plane then

$0 \le v \le \pi/2 \longrightarrow$ upper hemisphere

$0 \ge v \ge -\pi/2 \longrightarrow$ lower "

and

$x = a \cos v \cos u$

$y = a \cos v \sin u$

$z = a \sin v$

Calculation of surface integrals

$S: \quad x = x(u,v), \ y = y(u,v), \ z = z(u,v).$

$$\vec{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

$$\vec{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

tangent vectors in the tangent plane at (x,y,z) to the surface. S.

$\vec{T}_u \times \vec{T}_v \rightarrow$ normal to S at the pt (x,y,z)

$\|\vec{T}_u \times \vec{T}_v\| =$ magnitude of $\vec{T}_u \times \vec{T}_v$

$$\iint f \, dS = \iint_{D_{uv}} f\big(x(u,v), y(u,v), z(u,v)\big) \, \|\vec{T}_u \times \vec{T}_v\| \, du \, dv$$

$z = f(x,y)$, $\quad z = +\sqrt{a^2 - x^2 - y^2} \rightarrow$ upper hemisphere.

$x^2 + y^2 + z^2 - a^2 = 0 \longrightarrow$ implicit. representation of surface; $F(x,y,z) =$ const.

$z = f(x,y) / \ y = g(z,x) / \ x = h(y,z) \rightarrow$ Explicit representation

Let $z = f(x, y)$

then $\vec{T}_u = \vec{T}_x = (1, 0, z_x) = (1, 0, f_x)$

$\vec{T}_v = \vec{T}_y = (0, 1, z_y) = (0, 1, f_y)$

$\vec{T}_x \times \vec{T}_y = -f_x \hat{i} - f_y \hat{j} + \hat{k}$

$\|\vec{T}_x \times \vec{T}_y\| = \sqrt{1 + f_x^2 + f_y^2}$

$\therefore \iint\limits_S f(x, y, z)\, dS = \iint f(x, y, z(x, y)) \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy$

$D_{xy} \rightarrow$ projected region on xy plane.

If $f = 1$, $\iint\limits_S dS =$ Surface area of S

$= \iint\limits_{D_{xy}} \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy.$

$z = f(x, y)$; $\phi(x, y, z) = z - f(x, y) = 0.$

$\vec{\nabla} \phi = -f_x \hat{i} - f_y \hat{j} + \hat{k} = \vec{T}_x \times \vec{T}_y.$

$\iint f(x, y, z)\, dS = \iint\limits_{D_{xy}} f(x, y, z)\, |\vec{\nabla} \phi|\, dx\, dy$

$$\iint \vec{F}(x, y, z) \cdot d\vec{S} = \iint \vec{F}(x, y, z) \cdot \vec{n}\, dS$$

$$= \iint \vec{F}(x, y, z) \cdot \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}\, |\vec{\nabla}\phi|\, dx\, dy$$

$$= \iint \vec{F}(x(u, v), y(u, v), z(u, v)) \cdot (\vec{T}_u \times \vec{T}_v)\, du\, dv.$$

Let $z = x^2 + y^2 - 2$

then either, $x^2 + y^2 - z - 2 = 0 \Rightarrow \vec{\nabla}\phi \rightarrow 2x\hat{i} + 2y\hat{j} - \hat{k}.$

or, $-x^2 - y^2 + z + 2 = 0 \Rightarrow \vec{\nabla}\phi \rightarrow -2x\hat{i} - 2y\hat{j} + \hat{k}.$

} outward Normal

When $z = -\sqrt{x^2 + y^2 - a^2}$

outward normals $\phi = z + \sqrt{x^2 + y^2 - a^2} = 0$

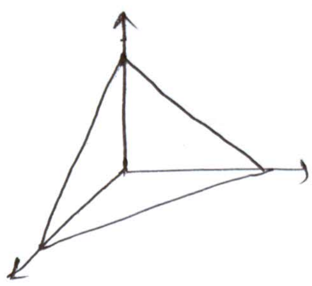
$\vec{\nabla}\phi = \frac{x}{\sqrt{x^2 + y^2 - a^2}}\hat{i} + \frac{y}{\sqrt{x^2 + y^2 - a^2}}\hat{j} + \hat{k} \rightarrow$ outward normal of the circle.

Note - If nothing is mentioned assume the surface to be oriented by outward normals.

H. Anton → Book

Ex. $\iint xz \, dS$.

S → part of the plane $x+y+z=1$, that lies in the 1^{st} octant.



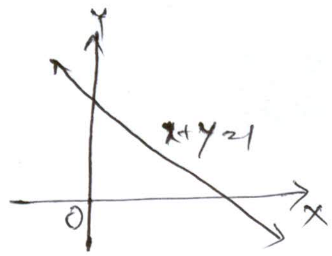
$$= \iint_{D_{uv}} xz \, \|\vec{T}_u \times \vec{T}_v\| \, du\,dv.$$

$$\iint_{D_{xy}} x(1-x-y) \, \|\vec{\nabla}\phi\| \, dx\,dy$$

$\phi = x+y+z-1=0 \Rightarrow |\vec{\nabla}\phi| = \sqrt{3}$

D_{xy} → projection of the surface $x+y+z=1$ on xy-plane at $z=0$.

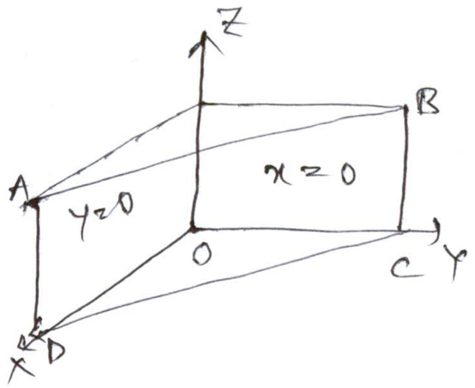
$$I = \sqrt{3}\int_{x=0}^{1}\int_{y=0}^{1-x} \{x(1-x) - xy\} \, dy\,dx = \frac{\sqrt{3}}{24}$$



Ex 2 compute $\iint_{\sigma} (x+y+z) \, dS$

σ → portion of the plane $x+y=1$ in the 1st octant bounded by $z=0$ & $z=1$.

$$I = \int_{z=0}^{1}\int_{x=0}^{1} (\cancel{x} + 1 - \cancel{x} + z) |\vec{\nabla}\phi| \, dx\,dz$$



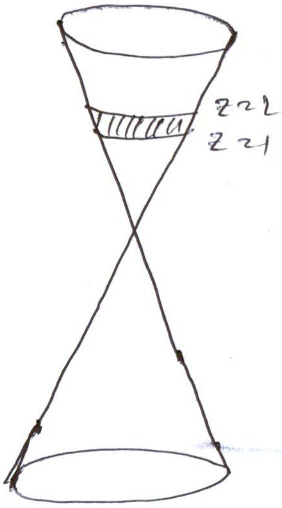
or, $$I = \int_{z=0}^{1}\int_{y=0}^{1} (1-y+y+z) \, |\vec{\nabla}\phi| \, dy\,dz$$

$$= \frac{3}{\sqrt{2}}.$$

$\iint_S y^2 z^2 \, dS$

where $S \to$ part of the cone $z^2 = x^2 + y^2$ between the planes $z = 1$, $z = 2$,

$z = +\sqrt{x^2+y^2}$



$$\iint y^2 z^2 \, |\vec{\nabla}\phi| \, dx\, dy$$

$$= \iint y^2 (x^2+y^2) \, |\vec{\nabla}| \, dx \, dy$$

$x = r\cos\theta$, $y = r\sin\theta$, $z = r$

$$\| \vec{T_r} \times \vec{T_\theta} \| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \sqrt{2}\, r$$

$$\therefore I = \int_{\theta=0}^{2\pi} \int_{r=1}^{2} r^2 \sin^2\theta . \, r^2 . \sqrt{2}\, r \, dr \, d\theta$$

$$= \frac{21\sqrt{2}}{2} \pi .$$