LINEAR ALGEBRA

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ASSIGNMENT-1

(1) (c) Uncountably infinite. If we look at the neighbourhood, Then it has a linear combination of continuous functions.
For eg. consider $f_{\alpha_1} = \sin\alpha_1(x-1)$, $f_{\alpha_2} = \sin\alpha_2(x-1)$ $(\alpha_1 \neq \alpha_2)$
$f_{\alpha_1}(x)$ & $f_{\alpha_2}(x)$ are L.I so, let's assume they are in the basis.
So in the basis f_{α_i} must be present $\forall\, \alpha_i \in \mathbb{R}$.
but $\mathbb{R}$ is uncountably infinite. thus, card. of basis is C.I.

(2) (c) A norm $\|\cdot\|$ is induced by an inner product iff it satisfies the $\|$gm law.

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

if we take $x = (5,5)$ & $y = (1,2)$ in $\mathbb{R}^2$

LHS = $49 + 16 = 65$

RHS = $2\times(25+4) = 2\times(29) = 58$

Thus, there is no inner product on $\mathbb{R}^n$ which induces the norm.

(3). (c) We know that every Vector Space has a basis.
So. let the Basis be $B = \{v_1, v_2\}$.
Then $LS(B) = V$
$LS(B) = \{v_1, v_2, v_1+v_2, 0.\}$
Now, in order to create other bases, select any two L.I elements from V.
So, no of Bases = ${}^3C_2 = 3$
V has exactly 3 bases

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(4) (c) $V = M_{3\times 2}(\mathbb{R})$. , $x = \begin{bmatrix}1\\1\end{bmatrix}$.
$W = \{A \in V ; Ax = 0\}$.

let $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ $\quad Ax = \begin{bmatrix} a+b \\ c+d \\ e+f \end{bmatrix} = 0$

$\Rightarrow a = -b \quad c = -d, \; e = -f$

So, $W = LS\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}\right)$.

thus $\dim(W) = 3$

(5) (b) Given: u_1, u_2, u_3 are L.I.
Now, to show that v_1, v_2 & v_3 are LI,
we must show that $c_1v_1 + c_2v_2 + c_3v_3 = 0$
only if $c_1 = c_2 = c_3 = 0$ ie, trivial sol^n exists

$\Rightarrow c_1(u_1 + 2u_2 + 3u_3) + c_2(au_2 + 5u_3) + c_3\, 2u_3 = 0$

$\Rightarrow (c_1)u_1 + (2c_1 + ac_2)u_2 + (3c_1 + 5c_2 + 2c_3)u_3 = 0$

$\because u_1, u_2$ & u_3 are L.I

$\therefore c_1 = 0$

$2c_1 + ac_2 = 0$

$3c_1 + 5c_2 + 2c_3 = 0.$

thus $\begin{vmatrix} 1 & 0 & 0 \\ 2 & a & 0 \\ 3 & 5 & 2 \end{vmatrix} \neq 0 \quad \Rightarrow 2a \neq 0 \quad \Rightarrow a \neq 0$

Also if $a = 0$ then. $\frac{v_2}{5} = \frac{v_3}{2}$ ie, L.D.
thus $a \neq 0$

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(6) (c) $\mathbb{W}$ ~~is~~ = $\{A \in \mathbb{V} : A \to$ upper t. $tr(A) = 0\}$.

Now, $\dim(\mathbb{W}) = \frac{n(n-1)}{2} + (n-1)$

$$= \frac{n^2+n-2}{2}$$

$$\dim(\mathbb{W}^\perp) = \dim(\mathbb{V}) - \dim(\mathbb{W})$$

$$= n^2 - \frac{(n^2+n-2)}{2} = \frac{n^2-n+2}{2}$$

(7) (c) $(\mathbb{V}, \langle,\rangle)$ - IPS. over $\mathbb{K}$. $\|x\| = \sqrt{\langle x, x\rangle}$.

Now $\|x+y\|^2 = \langle x+y, x+y\rangle$

$$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

$$= \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2$$

it is given that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

$\therefore \langle x,y\rangle + \langle y,x\rangle = 0$.

if $\mathbb{K} = \mathbb{C}$. then $\langle x,y\rangle = 0$

if $\mathbb{K} = \mathbb{R}$, $\langle x,y\rangle + \overline{\langle x,y\rangle} = 0$.

thus. x is orthogonal to y + $\langle x,y\rangle \in \mathbb{R}$.

(8) (a) $\mathbb{W} = \{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = 0\}$. & $\mathbb{W}$ is a subspace of $\mathbb{R}^n$.

$\dim(\mathbb{W}) = n-1 \geq \left(\frac{n}{2}\right)$ (for $n \geq 2$)

$$\dim(\mathbb{W}) \geq \frac{\dim(\mathbb{R}^n)}{2}$$

$\therefore$ $\mathbb{W}$ has 2 virtually disjoint components

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(9) (a) $\{\phi\}^\perp = \mathbb{V}$, $\{\phi\}$ is an empty set

$\{\phi\}^\perp = \{u \in \mathbb{V} \mid \langle u,v\rangle = 0 \ \forall \ v \in \{\phi\}\}$.

& $\because$ $\{\phi\}$ is empty

$\therefore \forall u \in \mathbb{V}, \langle u,v\rangle = 0$.

$\therefore \{\phi\}^\perp = \mathbb{V}$.

(10) (b) let $f(x) = x^2 g(x)$. (more even powers of x can also be multiplied to make $f(x)=0$ at $x=0$)

Now we are finding the orthogonal complement of U

so, $U^\perp = \{u \in \mathbb{V} \mid \langle u,v\rangle = 0 \ \forall \ v \in \mathbb{Q}(U)\}$

Now, let $g(x)$ be an element of $U^\perp$.

$\therefore \langle f,g\rangle = 0$

$$\Rightarrow \int_{-1}^{1} f(x)g(x) = 0 \Rightarrow \int_{-1}^{1} x^2 (g(x))^2 = 0.$$

Now, $\because x^2 \geq 0 \ \forall \ x \in [-1,1]$ & $(g(x))^2 \geq 0$

$\therefore$ this integral can only be 0 iff $g(x) = 0$

thus there cannot be any other unique $g(x)$ other than 0, S.T., $\langle f,g\rangle = 0$

thus $U^\perp = \{0\}$

so (a) & (b) ~~are~~ false too.