INDIAN INSTITUTEOF TECHNOLOGY, KHARAGPUR

Date :- FN/AN Time: -2 Hrs. Full Marks :- 30 Dept. :- Mathematics
No. of Students :- 88 Mid Term (Autumn) Semester Examination 2016-17

Sub. No. :- MA41007 Sub. Name :- Functional Analysis

Course :-1st Yr. M.Sc. (Dual) Mathematics & 4th Yr. M.Sc. (Int.), Mathematics & Computing

Instruction :- Attempt ALL questions.

1(a). Let (X,d) be a metric space and Y be a separable subspace of (X,d). If closure of Y is X i.e. $\overline{Y}=X$, then prove that (X,d) is separable. [2 M]

1(b). Let $d_1=\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a mapping defined by

$$d_1(x,y)=\begin{cases}1+|y-x| & \textit{if one and only one of the real numbers } x \,\&\, y \textit{ is positive}\\ |y-x| & \textit{otherwise}\end{cases}$$

Prove that d_1 is a metric on $\mathbb{R}$. [2 M]

1(c). Show that , by given an example , that a complete and incomplete metric space may be Homeomorphic. [2 M]

2(a). Let X be the set of all continuous real valued functions on $J=[0,1]$, and let $d(x,y)=\int_0^1 |x(t)-y(t)|\, dt$. Show that (x_n) where $x_n(t)=\begin{cases} n & ,\text{if } 0\le t\le \frac{1}{n^2}\\ t^{-\frac{1}{2}} & ,n^{-2}\le t\le 1\end{cases}$

is a Cauchy sequence in (X,d) but is not convergent. [3 M]

2(b). Show that the space C of all convergent sequences $x=(\xi_i)$ of complex numbers, with the metric induced from the space l_∞ is complete. [3 M]

3(a). A normed space $(X,\|.\|)$ is complete if every absolutely convergent series in $(X,\|.\|)$ is convergent .Prove it. [2M]

3(b). Let X denote the linear space of all polynomials $p(t)$ in one variable t with coefficients in $\mathbb{R}$ *or* $\mathbb{C}$. For $p\in X$ with $p(t)=a_0+a_1t+.....+a_nt^n$, let

$\|p\|=\sup\{|p(t)|:0\le t\le 1\}$ & $\|p\|_1=|a_0|+|a_1|+....+|a_n|$

Are these equivalent norms .Justify. [2 M]

3(c). If a Cauchy sequence has a convergent subsequence , then prove that the whole sequence is convergent. [2 M]

4(a). Prove that every finite dimensional subspace Y of a normed space $(X, \|.\|)$ is complete. [2 M]

4(b). Let (C_0 , d) be the space of null sequences (x_n) of complex numbers converging to zero with $d(x,y) = \sup|x_n - y_n|$. Show that the space C_0 is unbounded. [2 M]

4 (c). Give an example of a metric space which is not a normed space. [2 M]

5(a). In a finite dimensional normed space , prove that any subset M $\subset$ X is compact if it is closed and bounded. [2 M]

5(b). Prove that set of integers $\mathbb{Z}$ are nowhere dense set in $\mathbb{R}$. [2 M]

5(c). Show that $\mathbb{R}^n$ is locally compact but not compact. [2 M]

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