INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR

Date__________FN/AN, Time : 3 Hrs., Full Marks _50_, Deptt.______Mathematics_____

No. of Students __81__, End Autumn Semester Examination ________2012-13___________

Sub. No. ______MA41007__________ Sub. Name ___Functional Analysis___4th_ Yr. Integ. M.Sc./ Ist yr. M.Sc.(2Yr.) & 3rd Yr. CS (Elective).

Instruction : *Attempt any* **FIVE** *Questions*

1(a) Discuss the cannonical mapping (embedding) of a vector space X into its 2^{nd} algebraic dual.

(b) Let X be an n-dimensional vector space with $\{e_1, e_2, ..., e_n\}$ as basis. Find its algebraic dual basis $\{f_1, f_2, ..., f_n\}$ and show that $\dim X^* = \dim X$, where X^* denote algebraic dual of X.

(c) Prove that a finite dimensional vector space is algebraically reflexive.

2(a) Prove that the dual space of l^1 is l^∞.

(b) If X is a normed space and $\dim X = \infty$, show that the dual space X' is not identical with its algebraic dual X^*.

(c) Determine the null space of the operator $T : R^3 \to R^2$ represented by $\begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 0 \end{bmatrix}$.

3(a) Is $C[a,b]$ an Hilbert space? Justify your answer.

(b) Show that every subset of a separable inner product space is separable.

(c) Let X be an IPS and $Y \neq \phi$ be a complete subspace of X and $x \in X$ fixed. Then prove that the vector $x - y$ is orthogonal to Y, where $y \in Y$.

4(a) State and prove Bessel's inequality (in case of IPS)

(b) Let (e_k) be an orthogonal sequence in an IPS X. Show that for any $x, y \in X$,

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle \langle y, e_k\rangle| \leq \|x\| \|y\|$$

(P.T.O.)

(c) Define the Hilbert Adjoint operator T^* of a bounded linear operator $T: H_1 \to H_2$ where H_1, H_2 are Hilbert spaces . Prove that T^* exists, unique & $\|T^*\| = \|T\|$.

5(a) Prove that : Let H_1, H_2 be Hilbert spaces and $h: H_1 \times H_2 \to K$ be a bounded sequilinear form. Then h has a representation $h(x,y) = \langle Sx, y \rangle$ where is $S: H_1 \to H_2$ is a bounded linear operator, S is uniquely determined by h and has norm $\|S\| = \|h\|$.

(b) Let $T: C^n \to C^n$ be a linear operator. A basis of C^n is given. Let T is represented by means of a square matrix A of order n. Find the matrix representation of Hilbert adjoint operator T^* in terms of matrix A.

(c) Show that if $T: H \to H$ is a bounded self adjoint linear operator, so is T^n , where n is a positive integer.

6(a) State Uniform Boundedness Theorem. Using this theorem , prove that the normed space X of all polynomials with norm defined by , $\|x\| = \max_j |\alpha_j|$, $(\alpha_0, \alpha_1, \ldots$ the coefficients of x) is not a complete normed space.

(b) Show that $T = R^2 \to R$ defined by $(\xi_1, \xi_2) \to \xi_1$ is open. Is the mapping $R^2 \to R^2$ given by $(\xi_1, \xi_2) \to (\xi_1, 0)$ an open mapping ?

(c) Let $X = C[0,1]$ and $T = D(T) \to X$ such that $Tx(t) = \dfrac{dx}{dt}$. Is T closed ? bounded ? where $D(T)$ denote the subspace of functions $x \in X$ which has continuous derivatives.