Mrs

Indian Institute of Technology Kharagpur

Department of Mathematics
Mid Semester Examination 2015-2016 No. of Students: 100

Date:——— FN/AN, Time: 2 Hours Full Marks:30
Sub No: MA30003 & MA41003 Sub Name: Linear Algebra
—— Yr. B.Tech(H)/B.Arch(H)/M.Sc.

Declaration:
a) Answer without proper justification carries no marks.
b) No queries will be entertained during examination.

---

1. True or False. (As always, to say 'true' you must prove it, while to say 'false' you must produce a counterexample).

   (a) Let $0 \neq A \in \mathbb{R}^{n\times n}$. If $\{v_1, v_2, \ldots, v_k\}, k \leq n$ is a linearly independent sub set of $\mathbb{R}^n$ then $\{Av_1, Av_2, \ldots, Av_k\}$ is linearly independent. [2]

   (b) A matrix $A \in \mathbb{R}^{n\times n}$ has rank one if and only if $A = uv^T$ for some column vectors u and v in $\mathbb{R}^n$. [2]

   (c) If the null space (or kernel) of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

   $$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 2y \\ 0 \end{bmatrix}$$

   is spanned by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then the range space of T is spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$.

   (Note: do not use matrix representation of T in your argument.) [2]

   (d) Any normed linear space is an inner product space. [1]

   (e) $\mathbb{R}^3$ is a vector space over $\mathbb{R}$ under the following operations [2]

   $$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \alpha \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, x_i, y_i, z_i, x, y, z, \alpha \in \mathbb{R}, i = 1:2.$$

(f) There exists a set of four vectors in $\mathbb{R}^3$, any three of which form a linearly independent set in the vector space $\mathbb{R}^3$ over $\mathbb{R}$. [2]

(g) Subset of a linearly dependent set is linearly dependent. [1]

(h) Let A be an $m \times n$ matrix whose entries are real numbers. Then, the dimension of the span of columns of A (in $\mathbb{R}^m$) equals dimension of the span of rows of A (in $\mathbb{R}^n$). [3]

2. Determine a basis for the vector space of skew-symmetric 3×3 matrices with real coefficients, over $\mathbb{R}$ (A square matrix A is skew-symmetric if $A^T = -A$). [2]

3. Find the dimension of the vector space $M^\perp = \{v \in \mathbb{R}^3 | v^T u = 0 \text{ for all } u \in \mathbb{R}^3\}$ where

$$M = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,|\, x = 0,\, y + z = 0 \right\}$$

is a vector space over $\mathbb{R}$. [3]

4. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{R}$. Suppose $u, v \in V$ such that $\|u\| = 3, \|u + v\| = 4$ and $\|u - v\| = 6$. What number must $\|v\|$ be equal to? [2]

5. Determine a vector $v \in \mathbb{R}^3$ using Gram-Schmidt process such that v is orthogonal to both $\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$. [3]

6. Determine the matrix representation of a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ (without using the expression of rotation matrix directly) which rotates any non-zero vector in $\mathbb{R}^2$ by $\pi/6$ radians, and then translates the resultant vector by $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. [2]

7. Let $v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $W = \left\{ u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \,|\, y + z = 0 \right\}$. Determine the vector $\widehat{u} \in W$ such that $\|\widehat{u} - v\|_2$ provides the minimum value of $\|u - v\|_2$ for all $u \in W$. [3]

All The Best!