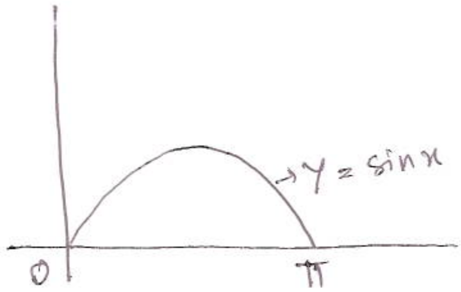
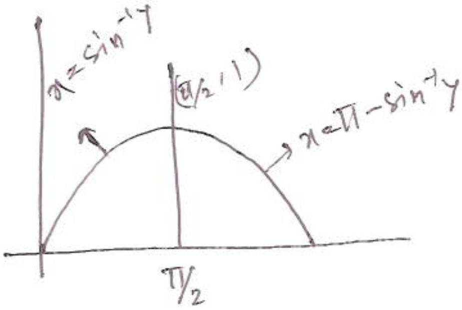
Change in order of integration

$$\int_0^{\pi}\int_{y=0}^{\sin x} f(x,y)\,dy\,dx.$$



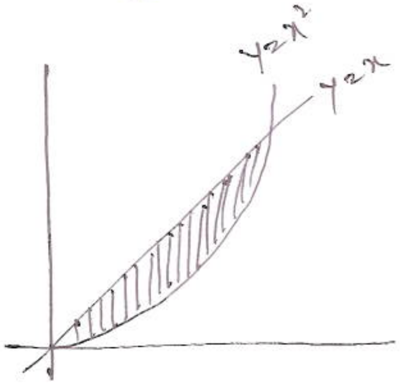


$0 \leq y \leq 1$
$0 \leq x \leq \frac{\pi}{2}$

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	0	$1/2$	$\sqrt{3}/2$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6} = \frac{5\pi}{6} = \pi - \frac{\pi}{6}$$

Q. Change the order of integration in $\int_{x=0}^{1}\int_{y=x^2}^{x} f(x,y)\,dy\,dx$



$$\int_0^1\int_{y=x^2}^{x} f(x,y)\,dy\,dx.$$

$$= \int_{y=0}^{1}\int_{x=y}^{\sqrt{y}} f(x,y)\,dx\,dy.$$

Change of variables in double integrals

$$I = \int_0^\infty \frac{dx}{1+x^2} \qquad x = \tan\theta$$

$$I = \int_0^{\pi/2} d\theta = \frac{\pi}{2}.$$

$$I = \iint_{D_{xy}} f(x,y)\, dx\, dy. \qquad (x,y) \to (u,v)$$

$$\downarrow$$

$$= \iint_{D'_{uv}} f\big(x(u,v), y(u,v)\big)\, \underbrace{|J|}_{\text{amplification factor.}}\, du\, dv$$

$J =$ Jacobian of the transformation from the region D_{xy} in xy-plane to D'_{uv} in u, v, plane.

where, $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

$(u,v) \to (x,y)$

$$J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{1}{J} \qquad JJ' = 1$$

Ex 1

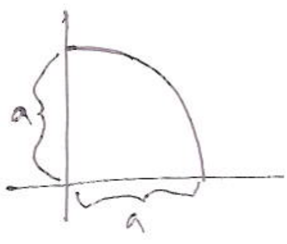
$$I = \iint \sqrt{a^2 - x^2 - y^2}\, dx\, dy.$$

$D =$ 1st quadrant of the circle

$$x^2 + y^2 = a^2$$

$$\iint f(x^2+y^2)\, dx\, dy.$$

$x = r\cos\theta$, $y = r\sin\theta$.



$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

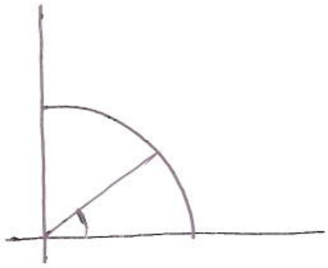
Here $u = r$, $v = \theta$.

P.T.O

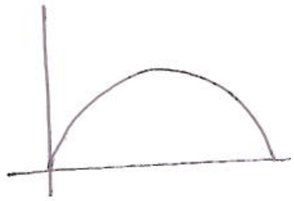
$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} \sqrt{a^2-r^2}\; r\, dr\, d\theta$$

$$= \frac{\pi}{2}\,(a^2-r^2)^{3/2}\cdot\frac{2}{3}\cdot\frac{1}{2}\Big|_a^{a}$$

$$= \frac{\pi a^3}{6}$$



2. Evaluate $\displaystyle\int_0^{2a}\int_0^{y=\sqrt{2ax-x^2}} (x^2+y^2)\, dy\, dx.$



$x = r\cos\theta,\ y = r\sin\theta\ ;\ J = r$

$$y^2 = 2ax - x^2$$

$$\Rightarrow (x-a)^2 + y^2 = a^2.$$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r^2\, r\, dr\, d\theta$$

$$= \frac{1}{4}\, a^4 \times 2^3 \times 2 \int_0^{\pi/2} \cos^4\theta\, d\theta$$

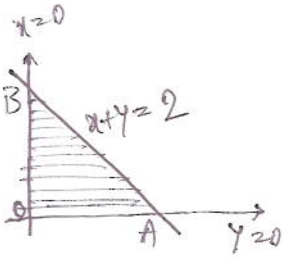
$$= \cancel{\frac{1}{4}}\,\frac{8}{4}\,a^4\, B\left(\frac{5}{2},\frac{1}{2}\right)$$, since $2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\, d\theta = B(m,n)$ and $B(m,n) = B(n,m)$.

$$= 2\cancel{8}\,a^4\, \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)}$$

$$= 2a^4\sqrt{\pi}\;\frac{\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)}{2} = \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2}\cdot 2a^4\sqrt{\pi}$$

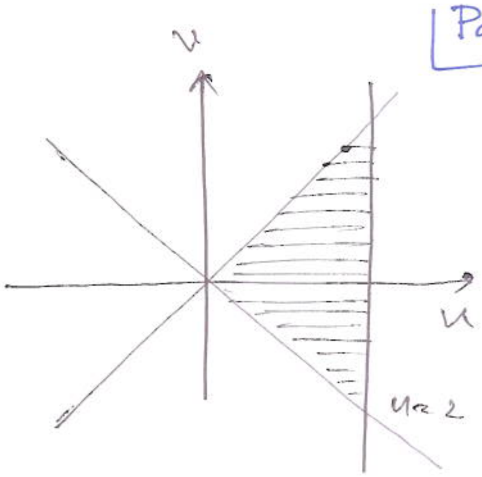
$$= \frac{3}{4}\,a^4\,\pi$$

<u>Ex</u> $\displaystyle\iint_D e^{\frac{y-x}{y+x}}\, dx\, dy.$, $D = \Delta$ bounded by the line $x+y=2,\ x=0,\ y=0.$



$$\left.\begin{array}{l} y+x = u \\ y-x = v \end{array}\right\}\quad x = \frac{u-v}{2},\quad y = \frac{u+v}{2}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

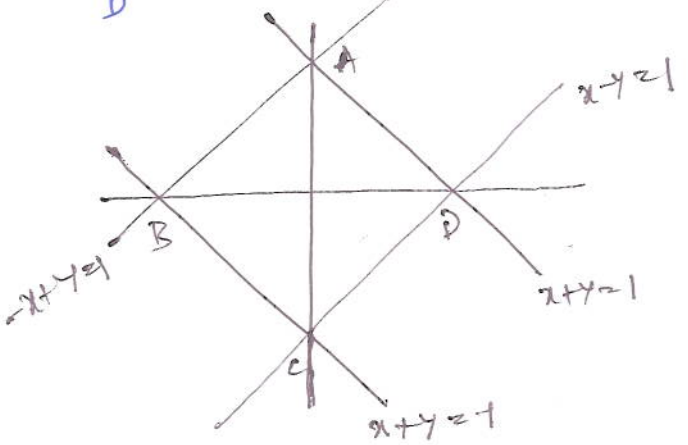


$$= \frac{1}{2} \iint_{D'} e^{\frac{v}{u}}\, du\, dv$$

$$= \frac{1}{2} \int_{u=0}^{2} \int_{v=-u}^{u} e^{v/u}\, dv\, du$$

$$= \frac{1}{2} \int_{u=0}^{2} \left[e^{v/u} \times u \right]_{-u}^{u} du = \frac{1}{2}\left(e - e^{-1}\right)\cdot \frac{2^2}{2} = e - \frac{1}{e}.$$

Ex $\iint_D e^{x+y}\, dx\, dy, \quad D = \left\{(x,y) : |x| + |y| \le 1\right\}$

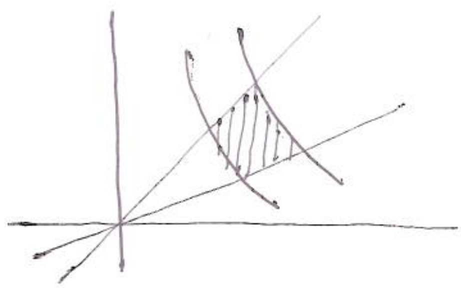


$y + x = u$
$y - x = v$

$$\int_{v=-1}^{1} \int_{u=-1}^{1} e^{u} \frac{1}{2}\, du\, dv = e - \frac{1}{e}$$

Ex $\iint_D x^2 y^2\, dx\, dy.$ D = portion in the 1st quadrant bounded by the curves $xy = 1$, $xy = 2$, $y = x$, $y = 4x$.

Hint
take $u = xy$; $v = \frac{y}{x}$ $\Big\}$ $y = \sqrt{uv}$, $x = \sqrt{\frac{u}{v}}$



Triple integrals

$$\iiint_{R} f(x,y,z)\,dx\,dy\,dz$$

R = solid region

1. parallelopiped
2. Cylinder
3. sphere
4. paraboloid
5. Cone
6. Tetrahedron.

$$f = A(x)\,B(y)\,C(z)$$

1. $$\iiint xyz^2\,dx\,dy\,dz$$

$$R = \{(x,y,z) \mid 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}.$$

$$= \left(\int_{x=0}^{1} x\,dx\right)\left(\int_{y=-1}^{2} y\,dy\right)\left(\int_{z=0}^{3} z^2\,dz\right)$$

$$= \frac{1}{2}\cdot\frac{3}{2}\cdot 9 = \frac{27}{4}$$

2. $$\iiint (x^2+yz)\,dx\,dy\,dz.$$

$$R = \{(x,y,z) \mid 0 \le x \le 2,\ -3 \le y \le 0,\ -1 \le z \le 1\}$$

$$\left(\int_0^2 x^2\,dx\right)\left(\int_{-3}^{0} dy\right)\left(\int_{-1}^{1} dz\right) + 0 = 8/3 * 3 \times 2 = 16$$

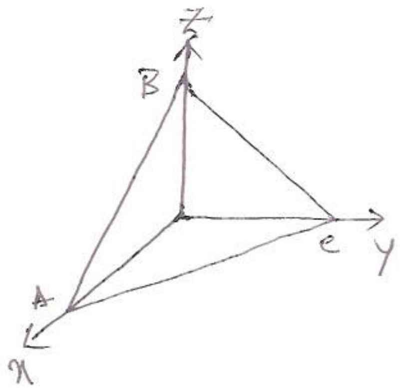
$$\left[\text{Since } \int_0^2 dx \int_{-3}^{0}\int_{-1}^{1} yz\,dz\,dy \right.$$

$$= \int_0^2 2x \int_{-3}^{0} \left[\frac{yz^2}{2}\right]_{-1}^{1} dy$$

$$\left. = 0 \right]$$

P.T.O

EX $\iiint z \, dx\, dy\, dz$.



$R \to$ region in the 1st octant cut by the plane $x+y+z=1$.

R can be written as

$R: \phi_1(x,y) \le z \le \phi_2(x,y)$; $x, y \in D_{xy}$.

projection of R on xy-plane.

or

$R: \psi_1(z,x) \le y \le \psi_2(z,x)$; $z, x \in D_{zx}$.

projection of R on zx-plane

$x^2+y^2+z^2=a^2$

let $a^2=16$ that means $x^2+y^2+z^2=16$.

projection of this on xy-plane $x^2+y^2=16$.

" " sphere on the plane $z=1$ — $x^2+y^2=15$.

In the above problem

$$D = \left\{ (x,y,z) \mid x+y+z \le 1,\ x \ge 0,\ y \ge 0,\ z \ge 0 \right\}$$

$$\int_{y=0}^{1}\int_{x=0}^{1-y}\int_{z=0}^{1-x-y} z \, dz\, dx\, dy$$

$$\iint_{D_{xy}} dx\, dy$$

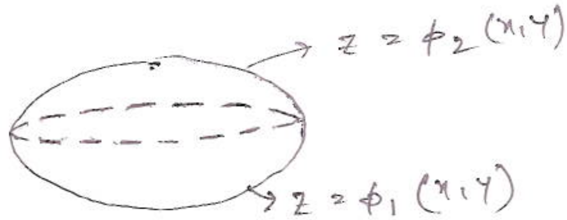
$$= \text{area of } D_{xy}$$

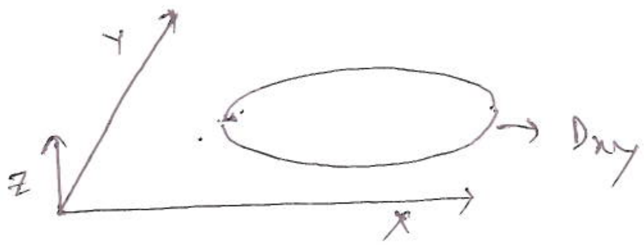
$[dx][dy]$
$= L^2$

$$\iiint_R dx\, dy\, dz = \text{Volume of } R$$

$[dx][dy][dz]$
$= L^3$

$$R = \left\{ \phi_1(x,y) \le z \le \phi_2(x,y) : x, y \in D_{xy} \right\}$$





$$\iiint_R dx\, dy\, dz = \iint_{D_{xy}} \int_{z=\phi_1}^{\phi_2} dz\, dx\, dy$$

$$= \iint_{D_{xy}} \phi_2(x,y)\, dx\, dy - \iint_{D_{xy}} \phi_1(x,y)\, dx\, dy.$$