2 Partial Differential Equations on Unbounded Domains

In our study of PDEs we noted the differences among the types of equations: parabolic, hyperbolic, and elliptic. Those classifications dictate the types of initial and boundary conditions that should be imposed to obtain a well-posed problem. There is yet another division that makes one *method* of solution preferable over another, namely, the nature and extent of the spatial domain. Spatial domains may be bounded, like a bounded interval, or unbounded, like the entire set of real numbers. It is a matter of preference which type of domain is studied first. It seems that boundaries in a problem, which require boundary conditions, make a problem more difficult. Therefore, we first investigate problems defined on unbounded domains.

2.1 Cauchy Problem for the Heat Equation

We begin with the heat, or diffusion, equation on the real line. That is, we consider the initial value problem

$$u_t = ku_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{2.1}$$

$$u(x,0) = \phi(x), \quad x \in \mathbb{R}. \tag{2.2}$$

Physically, this problem is a model of heat flow in an infinitely long bar where the initial temperature $\phi(x)$ is prescribed. In a chemical or biological con-

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text, the equation governs density variations under a diffusion process. Notice that there are no boundaries in the problem, so we do not prescribe boundary conditions explicitly. However, for problems on infinite domains, conditions at infinity are sometimes either stated explicitly or understood. Such a condition might require boundedness of the solution or some type of decay condition on the solution to zero as $x \to \pm\infty$. In mathematics, a pure initial value problem like (2.1)–(2.2) is often called a **Cauchy problem**.

Deriving the solution of (2.1)–(2.2) can be accomplished in two steps. First we solve the problem for a step function $\phi(x)$, and then we construct the solution to (2.1)–(2.2) using that solution. Therefore, let us consider the problem

$$w_t = kw_{xx}, \quad x \in \mathbb{R}, \ t > 0, \tag{2.3}$$

$$w(x,0) = 0 \ \text{ for } x < 0; \ \ w(x,0) = u_0 \ \text{ for } x > 0, \tag{2.4}$$

where we have taken the initial condition to be a step function with jump u_0.

We motivate our approach with a simple idea from the subject of dimensional analysis. Dimensional analysis deals with the study of units (seconds, meters, kilograms, and so forth) and dimensions (time, length, mass, and so forth) of the quantities in a problem and how they relate to each other. Equations must be dimensionally consistent (one cannot add apples to oranges), and important conclusions can be drawn from this fact. The cornerstone result in dimensional analysis is called the *pi theorem*. The pi theorem guarantees that whenever there is a physical law relating dimensioned quantities $q_1, \ldots, q_m$, then there is an equivalent physical law relating the independent dimensionless quantities that can be formed from $q_1, \ldots, q_m$. By a dimensionless quantity we mean one in which all the dimensions (length, time, mass, etc.) cancel out. For a simple example take the law

$$h = -\frac{1}{2}gt^2 + vt,$$

which gives the height h of an object at time t when it is thrown upward with initial velocity v; the constant g is the acceleration due to gravity. Here the dimensioned quantities are h, t, v, and g, having dimensions length, time, length per time, and length per time-squared. This law can be rearranged and written equivalently as

$$\frac{h}{vt} = -\frac{1}{2}\left(\frac{gt}{v}\right) + 1$$

in terms of the two dimensionless quantities

$$\pi_1 \equiv \frac{h}{vt} \quad \text{and} \quad \pi_2 \equiv \frac{gt}{v}.$$

For example, h is a length and vt, a velocity times a time, is also a length; so π_1, or h divided by vt, has no dimensions. Similarly, $\pi_2 = gt/v$ is dimensionless.

A law in dimensioned variables can always be reformulated in dimensionless quantities. So the physical law can be written as $\pi_1 = -\frac{1}{2}\pi_2 + 1$.

We use similar reasoning to guess the form of the solution of the initial value problem (2.3)–(2.4). First we list all the variables and constants in the problem: x, t, w, u_0, k. These have dimensions length, time, degrees, degrees, and length-squared per time, respectively. We notice that w/u_0 is a dimensionless quantity (degrees divided by degrees); the only other independent dimensionless quantity in the problem is $x/\sqrt{4kt}$ (the "4" is included for convenience). By the pi theorem we expect that the solution can be written as some combination of these dimensionless variables, or

$$\frac{w}{u_0} = f\left(\frac{x}{\sqrt{4kt}}\right)$$

for some function f yet to be determined. In fact, this is the case. So let us substitute

$$w = f(z), \quad z = \frac{x}{\sqrt{4kt}}$$

into the PDE (2.3). We have taken $u_0 = 1$ for simplicity. The chain rule allows us to compute the partial derivatives as

$$\begin{aligned} w_t &= f'(z)z_t = -\frac{1}{2}\frac{x}{\sqrt{4kt^3}}f'(z), \\ w_x &= f'(z)z_x = \frac{1}{\sqrt{4kt}}f'(z), \\ w_{xx} &= \frac{\partial}{\partial x}w_x = \frac{1}{4kt}f''(z). \end{aligned}$$

Substituting into (2.3) gives, after some cancelation, an ordinary differential equation,

$$f''(z) + 2zf'(z) = 0,$$

for $f(z)$. This equation is easily solved by multiplying through by the integrating factor e^{z^2} and integrating to get

$$f'(z) = c_1 e^{-z^2},$$

where c_1 is a constant of integration. Integrating from 0 to z gives

$$f(z) = c_1 \int_0^z e^{-r^2} dr + c_2,$$

where c_2 is another constant of integration. Therefore we have determined solutions of (2.3) of the form

$$w(x,t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-r^2} dr + c_2.$$

Next we apply the initial condition (2.4) (taking $u_0 = 1$) to determine the constants c_1 and c_2. For a fixed $x < 0$ we take the limit as $t \to 0$ to get

$$0 = w(x, 0) = c_1 \int_0^{-\infty} e^{-r^2} dr + c_2.$$

For a fixed $x > 0$ we take the limit as $t \to 0$ to get

$$1 = w(x, 0) = c_1 \int_0^{\infty} e^{-r^2} dr + c_2.$$

Recalling that

$$\int_0^{\infty} e^{-r^2} dr = \frac{\sqrt{\pi}}{2},$$

we can solve the last two equations to get $c_1 = 1/\sqrt{\pi}$, $c_2 = 1/2$. Therefore, the solution to (2.3)–(2.4) with $u_0 = 1$ is

$$w(x, t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-r^2} dr. \tag{2.5}$$

This solution can be written nicely as

$$w(x, t) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right) \tag{2.6}$$

in terms of a special function called the "erf" function, which is defined by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^{z} e^{-r^2} dr.$$

Figure 2.1 shows a graph of several time snapshots of the solution (2.6).

Now we will use (2.5) and a physical argument to deduce a solution to the Cauchy problem (2.1)–(2.2). Later, in Section 2.7, we present an analytical argument based on Fourier transforms. We make some observations. First, if a function w satisfies the heat equation, then so does w_x, the partial derivative of that function with respect to x. This is easy to see because

$$0 = (w_t - kw_{xx})_x = (w_x)_t - k(w_x)_{xx}.$$

Therefore, since $w(x, t)$ solves the heat equation, the function

$$G(x, t) \equiv w_x(x, t)$$

solves the heat equation. By direct differentiation of $w(x, t)$ we find that

$$G(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}. \tag{2.7}$$

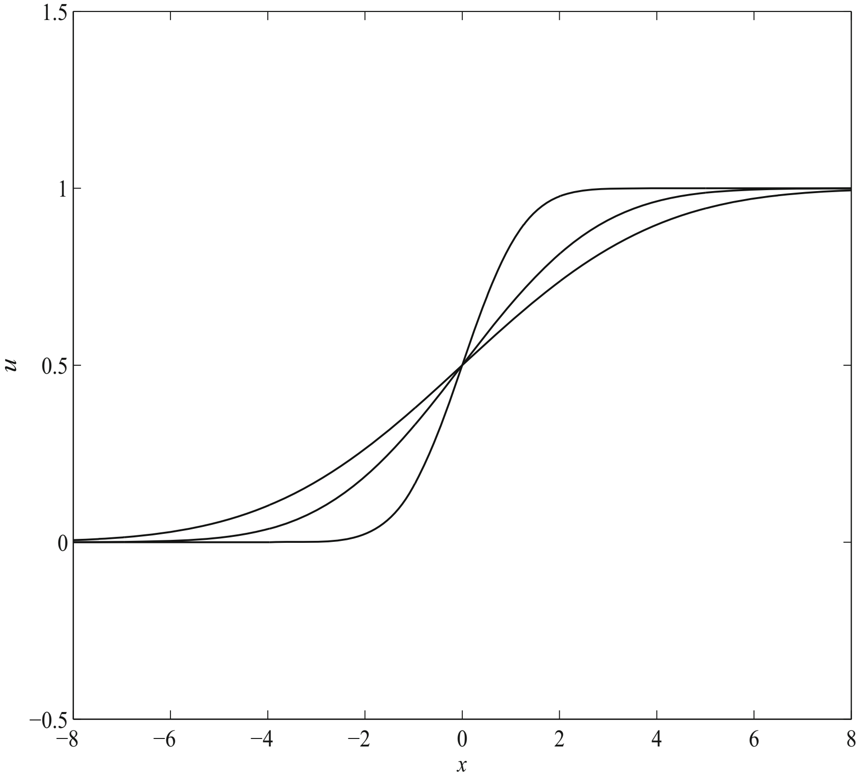

Figure 2.1 Temperature profiles $u = \frac{1}{2}\left(1 + \operatorname{erf}\left(x/\sqrt{4kt}\right)\right)$ at three different times t when the initial temperature is a step function and $k = 1$. As time increases, the profiles are smearing out

The function G is called the **heat kernel** or **fundamental solution** to the heat equation; the reader will note that for each $t > 0$ it graphs as a bell-shaped curve (see Exercise 1, Section 1.1), and the area under the curve for each $t > 0$ is one; that is,

$$\int_{-\infty}^{\infty} G(x,t)dx = 1, \quad t > 0.$$

$G(x,t)$ is the temperature surface that results from an initial unit heat source, i.e., injecting a unit amount of heat at $x = 0$ at time $t = 0$. We further observe that shifting the temperature profile again leads to a solution to the heat equation. Thus, $G(x-y,t)$, which is the temperature surface caused by an initial unit heat source at y, solves the heat equation for any fixed, but arbitrary, y. If $\phi(y)$, rather than unity, is the magnitude of the source at y, then $\phi(y)G(x-y,t)$ gives the resulting temperature surface; the area under a

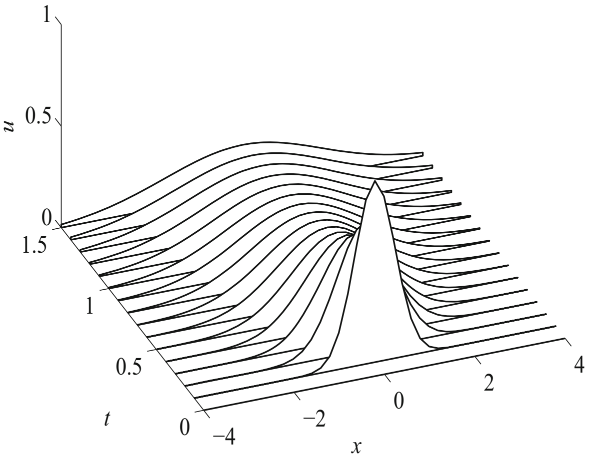

Figure 2.2 Plot of the fundamental solution $u = G(x, t)$ to the diffusion equation (2.7). As $t \to 0^+$ the solution approaches a unit 'point source' at $t = 0$

temperature profile is now $\phi(y)$, where y is the location of the source. Now, let us regard the initial temperature function ϕ in (2.2) as a continuous distribution of sources $\phi(y)$ for each $y \in \mathbb{R}$. Then, superimposing all the effects $\phi(y)G(x-y, t)$ for all y gives the total effect of all these isolated sources; that is,

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \phi(y) G(x-y,t)\,dy \\ &= \int_{-\infty}^{\infty} \phi(y) \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/(4kt)}\,dy \end{aligned}$$

is a solution to the Cauchy problem (2.1)–(2.2) for reasonable assumptions on the initial condition ϕ. More precisely:

Theorem 2.1

Consider the initial value problem for the heat equation,

$$\begin{aligned} u_t &= k u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= \phi(x), \quad x \in \mathbb{R}, \end{aligned}$$

where ϕ is a bounded continuous function on $\mathbb{R}$. Then

$$u(x,t) = \int_{-\infty}^{\infty} \phi(y) \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/(4kt)}\,dy \tag{2.8}$$

is a solution to the heat equation for $x \in \mathbb{R}$, $t > 0$, and it has the property that $u(x,t) \to \phi(x)$ as $t \to 0^+$. If ϕ is piecewise continuous, i.e., it has only finitely many jump discontinuities in any bounded interval, then $u(x,t)$ is a solution to the heat equation on $x \in \mathbb{R}$, $t > 0$; and, as $t \to 0^+$, the solution approaches the average value of the left and right limits at a point of discontinuity of ϕ; in symbols,

$$u(x,t) \to \frac{1}{2}\left(\phi(x^-) + \phi(x^+)\right) \text{ as } t \to 0^+. \quad \square$$

This discussion of the Cauchy problem for the heat equation has been intuitive, and it provides a good basis for understanding why the solution has the form it does.

There is another, standard way to write the solution (2.8) to the Cauchy problem (2.1)–(2.2). If we change variables in the integral using the substitution $r = (x-y)/\sqrt{4kt}$, then $dr = -dy/\sqrt{4kt}$, and (2.8) becomes

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} \phi(x - r\sqrt{4kt})dr. \tag{2.9}$$

This formula is called the **Poisson integral representation**.

We make several observations. First, the solution of the Cauchy problem is an integral representation. Although the formula is not complicated, for most initial conditions $\phi(x)$ the integration cannot be performed analytically. Therefore, numerical or computer evaluation of the integral is ultimately required if temperature profiles are desired. Also, notice that the temperature $u(x,t)$ is nonzero for every real x, even if ϕ is zero outside a small interval about the origin. Thus, a signal propagated by the heat, or diffusion, equation travels infinitely fast; according to this model, if odors diffuse, a bear would instantly smell a newly opened can of tuna ten miles away. Next, although we do not give a proof, the solution given by (2.8) is very smooth; that is, u is infinitely differentiable in both x and t in the domain $t > 0$; this is true even if ϕ is piecewise continuous. Initial signals propagated by the heat equation are immediately smoothed out.

Finally, we note that the heat kernel $G(x,t)$ defined in (2.7) is also called the **Green's function** for the Cauchy problem. In general, the Green's function for a problem is the response of a system, or the effect, caused by a point source. In heat flow on the real line, $G(x,t)$ is the response, i.e., the temperature surface caused by a unit, point heat source given to the system at $x = 0,\ t = 0$. Some of the references discuss the construction of a Green's functions for a variety of problems. Because of the basic role this function plays in diffusion problems, $G(x,t)$ is also called the **fundamental solution** to the heat equation. The reader should review Section 1.4 for a discussion of the origin of the fundamental solution from a probability discussion.

EXERCISES

1. Solve the Cauchy problem (2.1)–(2.2) for the following initial conditions.

   a) $\phi(x) = 1$ if $|x| < 1$ and $\phi(x) = 0$ if $|x| > 1$.

   b) $\phi(x) = e^{-x}$, $x > 0$; $\phi(x) = 0$, $x < 0$.

   In both cases write the solutions in terms of the erf function. Hint: In (b) complete the square with respect to y in the exponent of e.

2. If $|\phi(x)| \leq M$ for all x, where M is a positive constant, show that the solution u to the Cauchy problem (2.1)–(2.2) satisfies $|u(x,t)| \leq M$ for all x and $t > 0$. Hint: Use the calculus fact that the absolute value of an integral is less than or equal to the integral of the absolute value: $|\int f| \leq \int |f|$.

3. Consider the problem (2.3)–(2.4) with $u_0 = 1$. For a fixed $x = x_0$, what is the approximate temperature $w(x_0, t)$ for very large t? Hint: Expand the integrand in the formula for the solution in a power series and integrate term by term.

4. Show that if $u(x,t)$ and $v(x,t)$ are any two solutions to the heat equation (2.1), then $w(x,y,t) = u(x,t)v(y,t)$ solves the two-dimensional heat equation $w_t = k(w_{xx} + w_{yy})$. Can you guess the solution to the two-dimensional Cauchy problem

$$\begin{aligned} w_t = k(w_{xx} + w_{yy}), \quad (x,y) \in \mathbb{R}^2,\ t > 0, \\ w(x,y,0) = \psi(x,y), \quad (x,y) \in \mathbb{R}^2? \end{aligned}$$

5. Let the initial temperature in the Cauchy problem (2.1)–(2.2) be given by $\phi(x) = e^{-|x+2|} + e^{-|x-2|}$, with $k = 1$. Use the numerical integration operation in a computer algebra package to draw temperature profiles at several times to illustrate how heat flows in this system. Exhibit the temperature profiles on a single set of coordinate axes.

6. Verify that

$$\int_{-\infty}^{\infty} G(x,t)dx = 1, \quad t > 0.$$

   Hint: Change variables as in the derivation of Poisson's integral representation.

7. Consider the Cauchy problem for the heat equation

$$u_t = ku_{xx}, \ \ x \in \mathbb{R},\ t > 0; \ \ u(x,0) = e^{-x},\ x \in \mathbb{R}.$$

   Verify that $u(x,t) = e^{-x+kt}$ is an unbounded solution. Is this a contradiction to the theorem?

2.2 Cauchy Problem for the Wave Equation

The one-dimensional wave equation is

$$u_{tt} - c^2 u_{xx} = 0. \tag{2.10}$$

We observed in Section 1.5 that it models the amplitude of transverse displacement waves on a taut string as well as small amplitude disturbances in acoustics. It also arises in electromagnetic wave propagation, in the mechanical vibrations of elastic media, as well as in other problems. It is a hyperbolic equation and is one of the three fundamental equations in PDEs (along with the diffusion equation and Laplace's equation). Under the transformation of variables (to characteristic coordinates)

$$\xi = x - ct, \quad \tau = x + ct,$$

the wave equation is transformed into the canonical form

$$U_{\tau\xi} = 0, \quad U = U(\xi, \tau),$$

which can be integrated twice to obtain the general solution

$$U(\xi, \tau) = F(\xi) + G(\tau),$$

where F and G are arbitrary functions. Thus, the general solution to (2.10) is

$$u(x,t) = F(x - ct) + G(x + ct). \tag{2.11}$$

Hence, solutions of the wave equation are the superposition (sum) of right- and left-traveling waves moving at speed c.

The Cauchy problem for the wave equation is

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in , t > 0, \tag{2.12}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R}. \tag{2.13}$$

Here, f defines the initial displacement of an infinite string, and g defines its initial velocity. The equation is second-order in t, so both the position and velocity must be specified initially.

There is a simple analytical formula for the solution to the Cauchy problem (2.12)–(2.13). It is called **d'Alembert's formula**, and it is given by

$$u(x,t) = \frac{1}{2}(f(x - ct) + f(x + ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{2.14}$$

If f'' and g' are continuous, then it is a straightforward exercise in differential calculus, using Leibniz's formula, to verify that this formula solves (2.12)–(2.13). The formula can be derived (see Exercise 1) by determining the two functions F and G in (2.11) using the initial data (2.13).

Example 2.2

Insight into the behavior of solutions comes from examining the special case where the initial velocity is zero and the initial displacement is a bell-shaped curve. Specifically, we consider the problem (with $c = 2$)

$$u_{tt} - 4u_{xx} = 0, \quad x \in , t > 0,$$
$$u(x, 0) = e^{-x^2}, \quad u_t(x, 0) = 0, \quad x \in \mathbb{R}.$$

The exact solution is, by d'Alembert's formula,

$$u(x, t) = \frac{1}{2}(e^{-(x-2t)^2} + e^{-(x+2t)^2}).$$

Either the solution surface or wave profiles can be graphed easily using a computer algebra package. Figure 2.3 shows the solution surface; observe how the initial signal at $t = 0$ splits into two smaller signals, and those travel off in opposite directions at speed $c = 2$. In the exercises the reader is asked to examine the case where $f = 0$ and $g \neq 0$; this is the case where the initial displacement is zero and the string is given an initial velocity, or impulse, by, say, striking the string with an object. $\square$

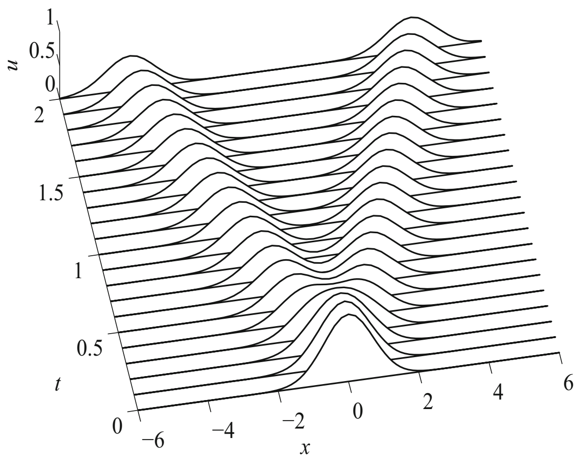

Figure 2.3 Time profiles of the solution surface. The initial signal splits into two signals which move at speeds c and $-c$ along the positive and negative characteristics, $x - ct = \text{const.}$, $x + ct = \text{const.}$, respectively

Close examination of d'Alembert's formula reveals a fundamental property of the wave equation. If the initial disturbance is *supported* in some interval

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