INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR
END SEMESTER EXAMINATION

Date: 27-11-2017 AN | Time : 3 hours | Full Marks : 100
Autumn Semester: 2017-2018 | Department : Mathematics
Subject No.: MA31005 | Subject Name : Real Analysis
No. of Students: 75 | Course: M.Sc. 3rd Year (Maths. & Computing)/Breadth

Instructions: Answer all questions. Give complete arguments in support of your answers. No queries will be entertained during the exam. No marks will be awarded if explanations for your answers are not given. If some theorem is being used, that may be quoted.

1. (a) Let $y_n = \sqrt{(n+1)} - \sqrt{n}$, $n = 1,2,3,\ldots$ Show that sequences $\{y_n\}$ and $\{\sqrt{n}\, y_n\}$ are convergent. Find their limits.

   (b) Let $a_n = \sin\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{2}\right)$, $n = 1,2,3,\ldots$ Find $\liminf\limits_{n\to\infty} a_n$ and $\limsup\limits_{n\to\infty} a_n$.

   (c) Let $a_n = \frac{n}{4} - \left[\frac{n}{4}\right]$, $n = 1,2,3,\ldots$, where $[y]$ denotes the greatest integer function for real y. Find the set of all subsequential limits of $\{a_n\}$. 12M

2. For any sequence $\{c_n\}$ of positive real numbers, it is known that

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} (c_n)^{1/n} \le \limsup_{n\to\infty} (c_n)^{1/n} \le \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

(a) Use the above result to obtain $\lim\limits_{n\to\infty} \frac{n}{(n!)^{1/n}}$.

(b) Prove that $\sum\limits_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent. 4M

3. (a) Let $P(x) = ax^2 + bx + c$; $a,b,c \in \mathbb{R}$. Find $\lim\limits_{n\to\infty} \frac{P(n+1)}{P(n)}$. 2M

   (b) Prove that $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$. 4M

4. Let $e = \sum\limits_{n=0}^{\infty} \frac{1}{n!}$. Prove the following statements:

   (a) e is well defined (convergent) (b) $e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{1/n}$ (c) e is irrational

   (ratio)

   8M

5. (a) Let X and Y be metric spaces. Prove that a mapping $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y. 6M

(b) Let X, Y, Z be metric spaces, $E \subset X$, $f : E \to Y$, $g : f(E) \to Z$ and $h : E \to Z$ be defined by $h(x) = g(f(x))$, $x \in E$. If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, prove that h is continuous at the point p. 4M

6. (a) Find limits of the following functions as $x \to 0$:

(i) $\dfrac{\sqrt{(4+x)} - \sqrt{(1+x)}}{x(x+2)}$, $x \neq 0$ X (ii) $\left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$, $x \neq 0$. X LHL $-\infty$ RHL 0 4M

(b) Let $f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that f can be differentiated infinite number of times at $x = 0$. Find $f^{(n)}(0)$, $n = 1, 2, 3, \ldots$ 6M

(c) Let $f(x) = \begin{cases} (\sin x)^n, & x \geq 0 \\ 0 & x < 0 \end{cases}$. Check differentiability of f at $x = 0$ for different values of $n = 1, 2, 3, \ldots$ No ∀ $n > 1$ 4M

7. (a) Let f be a real valued function on $[a,b]$, differentiable on (a,b) and have local maximum at a point $c \in (a,b)$. Show that $f'(c) = 0$. 6M

(b) Let $f : [0, \infty)$ and $g : [0, \infty)$ be real valued functions differentiable for all $x > 0$, $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x > 0$. Show that $f(x) \leq g(x)$ for all $x \geq 0$. 4M

8. (a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $\alpha : [a,b] \to \mathbb{R}$ be monotonic increasing. For a partition P, let P^* be its refinement. Show that

(i) $L(P, f, \alpha) \leq L(P^*, f, \alpha)$ (ii) $U(P^*, f, \alpha) \leq U(P, f, \alpha)$ 6M

(b) Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $\alpha : [a,b] \to \mathbb{R}$ be monotonic increasing. Further, let α be continuous at a point $x_0 \in [a,b]$ and $f(x) = \begin{cases} 1, & x = x_0 \\ 0, & x \neq x_0 \end{cases}$. Prove that $f \in \mathfrak{R}(\alpha)$ and $\int_a^b f \, d\alpha = 0$. 6M

(c) Let $f \in \mathfrak{R}[a,b]$ and let F be a differentiable function on $[a,b]$ such that $F'(x) = f(x)$ on $[a,b]$. Show that

$$\int_a^b f(x)\, dx = F(b) - F(a).$$ 6M

(d) Let $f : [a,b] \to \mathbb{R}$ be a bounded and continuous function, $\int_a^b f(x)\, dx = 0$. Prove that $f(x) = 0$ on $[a,b]$. $\boxed{f \geq 0}$? 6M

9. (a) Let $\{K_n\}$ be a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_n\}$ is nonempty. Prove that $\bigcap_{n=1}^{\infty} K_n$ is nonempty. Cantor's Thm 6M

(b) If E is an infinite subset of a compact set K, show that E has a limit point in K. 6M