FINITE DIFFERENCE METHOD

(I) DISCRETIZATION OF THE DOMAIN:

We devide the interval $[a, b]$ into $(N+1)$ subintervals such that

$$x_j = a + jh \qquad j = 0, 1, 2, \cdots, N+1$$

where $\quad x_0 = a, \quad x_{N+1} = b, \quad h = \dfrac{b-a}{N+1}$

(II) FINITE DIFFERENCE APPROXIMATION OF DERIVATIVES

a) Expanding $u(x_j+h)$ in Taylor's series we get

$$u(x_j+h) = u(x_j) + h\,u'(x_j) + \frac{h^2}{2}u''(x_i) + O(h^3) \quad \text{—①}$$

$$\Rightarrow \frac{u(x_j+h) - u(x_j)}{h} = u'(x_j) + \frac{h}{2}u''(x_i) + O(h^2)$$

OR

$$\boxed{u'(x_j) \simeq \frac{u(x_j+h) - u(x_j)}{h}}$$

This is called finit forward difference formula.

This difference formula provides a first order approximation to $u'(x_j)$ with respect to h.

b) Expanding $u(x_j-h)$ in Taylor's series we get

$$u(x_j-h) = u(x_j) - h\,u'(x_j) + \frac{h^2}{2}u''(x_j) + O(h^3) \quad \text{— (2)}$$

$$\Rightarrow \quad \frac{u(x_j-h)-u(x_j)}{-h} = u'(x_j) - \frac{h}{2}u''(x_j) + O(h^3)$$

OR

$$\boxed{u'(x_j) \simeq \frac{u(x_j)-u(x_{j-1})}{h}}$$

This is called backward difference formula. This diff. formula provides a first order approximation to $u'(x_j)$ with respect to h.

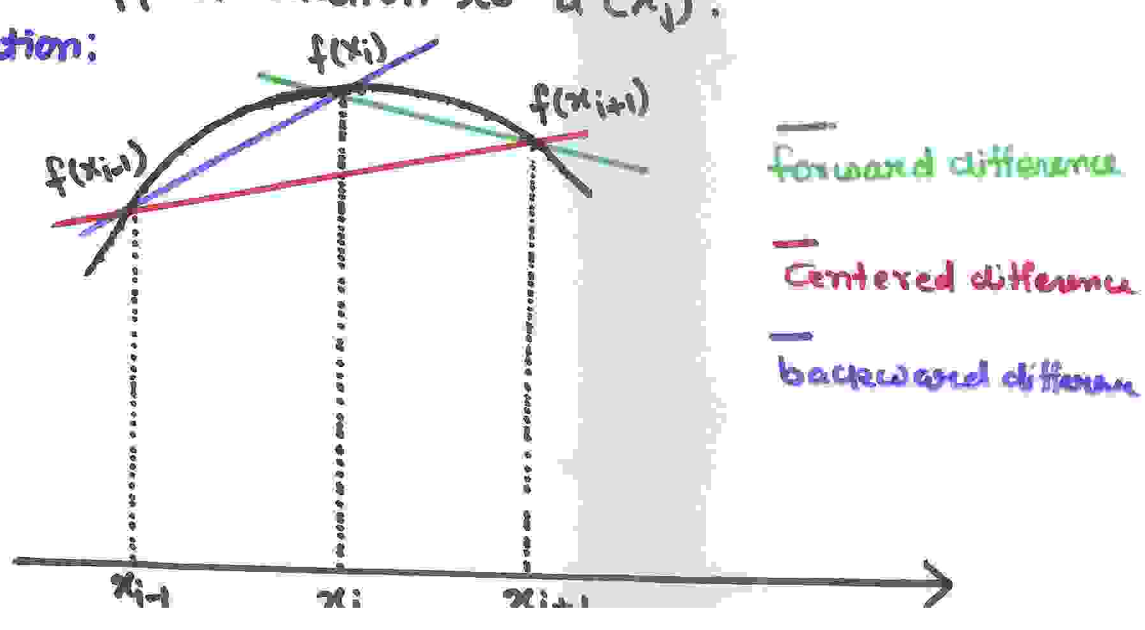
c) Subtracting (2) from (1):

$$u(x_j+h) - u(x_j-h) = 2h\,u'(x_j) + O(h^3)$$

$$\Rightarrow \quad \boxed{u'(x_j) \simeq \frac{u(x_j+h)-u(x_j-h)}{2h}}$$

This is called centered difference formula. This gives a second order approximation to $u'(x_j)$.

Physical interpretation:



d)

adding ① and ②:

$$u(x_j+h) + u(x_j-h) = 2u(x_j) + h^2 u''(x_j) + O(h^4)$$

$$\Rightarrow \quad \boxed{u''(x_j) \simeq \frac{u(x_j+h) - 2u(x_j) + u(x_j-h)}{h^2}}$$

This is called centered finite difference approximation for second order derivative.

This formula provides a second-order approximation to $u''(x_j)$ with respect to h.

$$\boxed{u'(x_j) \simeq \frac{u(x_{j+1}) - u(x_j)}{h}}$$

$$\boxed{u'(x_j) \simeq \frac{u(x_j) - u(x_{j-1})}{h}}$$

$$\boxed{u'(x_j) \simeq \frac{u(x_{j+1}) - u(x_{j-1})}{2h}}$$

$$\boxed{u''(x_j) = \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2}}$$

III) SOLVING THE EQUATIONS

Consider the second order linear differential equation:

$$-u'' + p(x)u' + q(x)u = r(x) \qquad a<x<b \quad — \text{①}$$

$$u(a) = \gamma_1 \qquad u(b) = \gamma_2 \qquad (1a)$$

Using the second order difference approximations at $x = x_j$ we obtain the difference equation

$$-\frac{1}{h^2}\left[u_{j+1} - 2u_j + u_{j-1}\right] + p(x_j)\frac{u_{j+1}-u_{j-1}}{2h} + q(x_j)u_j = r(x_j)$$

$$j = 1, 2, \dots N \qquad — \text{②}$$

Note that u_j is an approximation of $u(x_j)$.

The BCs (1a) become:

$$u_0 = \gamma_1 \qquad u_{N+1} = \gamma_2$$

N equations and N unknowns.

Multiplying ② by $\frac{h^2}{2}$ we obtain

$$\frac{-u_{j+1} + 2u_j - u_{j-1}}{2} + \frac{h}{4}p(x_j)\left(u_{j+1} - u_{j-1}\right) + \frac{h^2}{2}q(x_j)u_j$$

$$= \frac{h^2}{2}r(x_j)$$

OR

$$-\frac{1}{2}\left(1+\frac{h}{2}p(x_j)\right)u_{j-1}+\left(1+\frac{h^2}{2}q(x_j)\right)u_j-\frac{1}{2}\left(1-\frac{h}{2}p(x_j)\right)u_{j+1}$$

$$=\frac{h^2}{2}r(x_j)$$

Definding

$$A_j=-\frac{1}{2}\left(1+\frac{h}{2}p(x_j)\right)$$

$$B_j=1+\frac{h^2}{2}q(x_j)$$

$$C_j=-\frac{1}{2}\left(1-\frac{h}{2}p(x_j)\right)$$

we get

$$A_j u_{j-1}+B_j u_j+C_j u_{j+1}=\frac{h^2}{2}r(x_j)$$

$$j=1,2,\dots,N \quad \text{—— ③}$$

The system of equations ③ can be written in matrix notation:

$$\begin{bmatrix} B_1 & C_1 & \cdots & & 0 \\ A_2 & B_2 & C_2 & \cdots & 0 \\ \vdots & & & & \\ & & A_{N-1} & B_{N-1} & C_{N-1} \\ \cdots & & \cdots & A_N & B_N \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_N \end{bmatrix}=\frac{h^2}{2}\begin{bmatrix} r(x_1)-\frac{2}{h^2}A_1\gamma_1 \\ r(x_2) \\ \vdots \\ r(x_{N-1}) \\ r(x_N)-\frac{2}{h^2}C_N\gamma_2 \end{bmatrix}$$

$$A\bar{u}=\bar{b}$$

The solution of this system gives the finite difference solution of the BVP satisfying BCs.

LOCAL TRUNCATION ERROR:

The local truncation error of the finite difference scheme discussed above is defined as

$$T_j = A_j u(x_{j-1}) + B_j u(x_j) + C_j u(x_{j+1}) - \frac{h^2}{2} r(x_j)$$

$$= -\frac{1}{2}\left[1 + \frac{h}{2} p(x_j)\right]\left[\cancel{u(x_j)} - h u'(x_j) + \frac{h^2}{2} u''(x_j) - \frac{h^3}{\lfloor 3} u'''(x_j) + \frac{h^4}{\lfloor 4} u^{(iv)}(x_j) + \cdots\right]$$

$$+ \left[\cancel{1} + \frac{h^2}{2} q(x_j)\right] u(x_j)$$

$$- \frac{1}{2}\left[1 - \frac{h}{2} p(x_j)\right]\left[\cancel{u(x_j)} + h u'(x_j) + \frac{h^2}{2} u''(x_j) + \frac{h^3}{\lfloor 3} u'''(x_j) + \frac{h^4}{\lfloor 4} u^{(iv)}(x_j) + \cdots\right] - \frac{h^2}{2} r(x_j)$$

$$= \cancel{\frac{h}{2} u'(x_j)} + \underline{\frac{h^2}{4} p(x_j) u'(x_j)} - \underline{\frac{h^2}{4} u''(x_j)} - \cancel{\frac{h^3}{8} p(x_j) u''(x_j)}$$

$$+ \cancel{\frac{h^3}{12} u'''(x_j)} + \frac{h^4}{24} u'''(x_j) p(x_j) - \frac{h^4}{48} u^{(iv)}(x_j) + \cdots$$

$$+ \underline{\frac{h^2}{2} q(x_j) u(x_j)} - \cancel{\frac{h}{2} u'(x_j)} + \underline{\frac{h^2}{4} p(x_j) u'(x_j)} - \underline{\frac{h^2}{4} u''(x_j)}$$

$$+ \cancel{\frac{h^3}{8} u''(x_j) p(x_j)} - \cancel{\frac{h^3}{12} u'''(x_j)} + \frac{h^4}{24} p(x_j) u'''(x_j)$$

$$- \frac{h^4}{48} u^{(iv)}(x_j) - \underline{\frac{h^2}{2} r(x_j)}$$

$$= \frac{h^4}{12} p(x_j) u'''(x_j) - \frac{h^4}{24} u^{(iv)}(x_j) + O(h^5)$$

$$= O(h^4).$$

The order of a method is the largest integer p for which $\left|\frac{1}{h^2} T_j\right| = O(h^p)$.

The above method is a 2nd order method