Symbolic Logic
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Lecture - 30
More on Derivations
Proof of Tautologies

Hello! And welcome back again to Module 30 of this course Symbolic Logic and we are still on the topic of the derivations.

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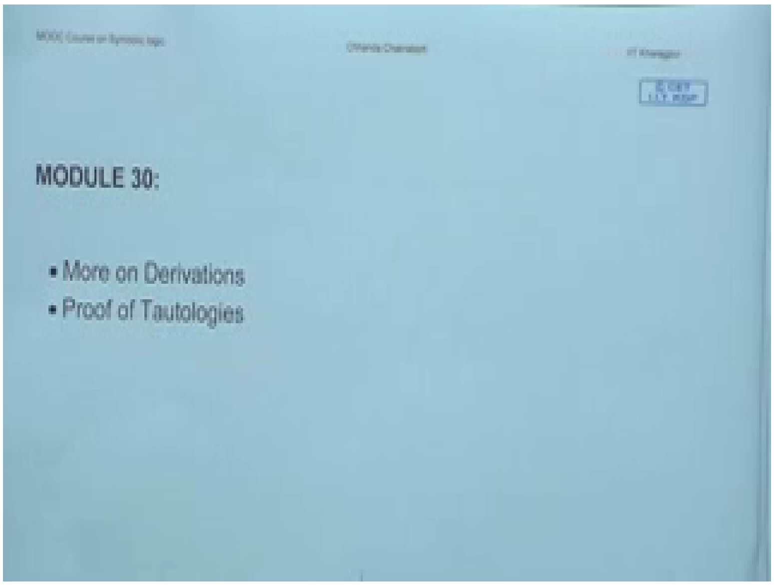

And we are going to finish this topic today by the way. We have been looking into various proof procedures and discussing these, but today we sort of close this discussion. So, on our agenda today in module 30, we have whatever we want to discuss about the derivation, we will close that and then will learn this : How to prove, how to construct a proof of tautology. By proof, I mean the formal derivation.

So, this is what we are going to learn and the connection of all of this is again through the completeness claim.

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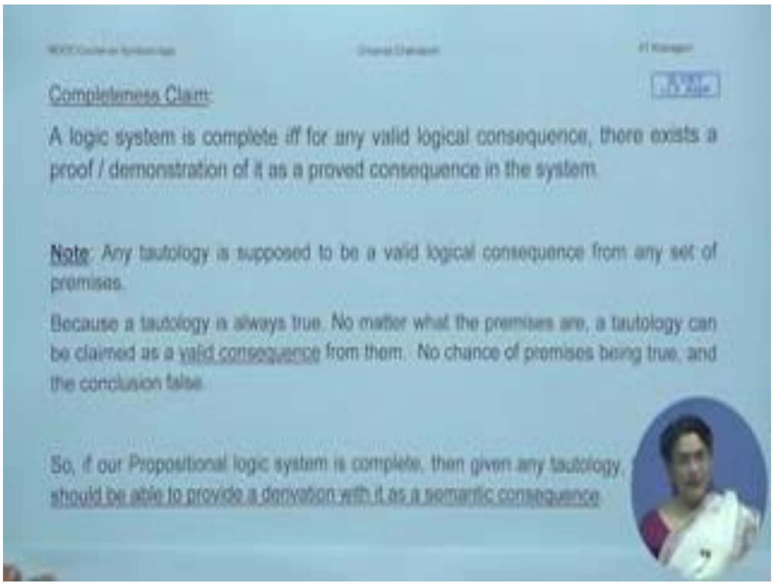

So, let us revisit this extremely important property of completeness. We remind ourselves that a logic system will be called complete if and only if for any valid logical consequence there is also a proof or a demonstration of it in, as a formal derivation, available in this system. So true consequence, valid consequence and proven consequence. Ok? And you show the derivation how it can be a proven consequence. So, that is completeness claim.

Now, where does a tautology feature in all of this? Because, we have been hearing about proof of tautologies. So, formal derivation is one thing, and then we are talking about tautologies. And the first question that should ring in your mind is that tautologies are not arguments even. So how can we prove them? What do you mean by proving a tautology? I am going to explain that. But wait; first let's take it in that a tautology is a valid logical consequence of *any* set of premises. In a sort of trivial manner, tautology always follows as a valid consequence, no matter what the premises are. Why is that? Because a tautology is always true. Always true . So it doesn't matter what premises you have and what their values are, truth values are. What you do not have is, there is never going to be a situation when if the premises are true, the conclusion is going to be false. Because the tautology is always true.

So, if you have a tautology as a conclusion, you are never going to have a situation when the premises are true, the conclusion is false. Correct? So you are avoiding that invalidity condition very, very clearly. Therefore, in a trivial and an empty sort of a way, you can say that a tautology is a valid consequence of *any* set of premises. So if that is the case, a tautology, any tautology ABCD that you have, is a true consequence and a valid consequence, then if our system is complete, there should be a formal derivation that can show how it is to be derived. Right? This is what we claim.

So, we need to have a derivation standing to back up our claim that we are now complete and the crucial test here will be: How can you prove any given tautology? So that's what we going to slowly look into that.

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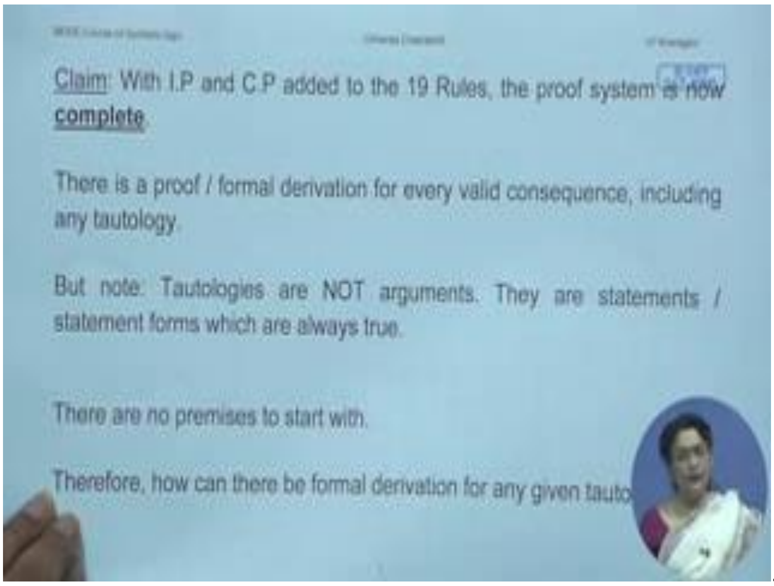

We have now added the Indirect Proof and the Conditional Proof, both versions of it, of course, with the nineteen rules in our proof system. And this is our claim that there exists a proof or formal derivation for every valid consequence, including any given tautology.

Now, let us pause and just take that earlier question : But how can we prove a tautology as a formal derivation? Because the formal derivation depends on two components; that you are going to have a premise, you are going to have a conclusion. But a tautology is not an argument; it's a statement or a statement form. That has …the only truth value that it allows is true, right? So what do you mean by formal derivation of tautology? So no premises to start with, and how do we do formal derivation in that case? And this is where I will remind you that what we have now are the Limited Scope Assumption procedures.

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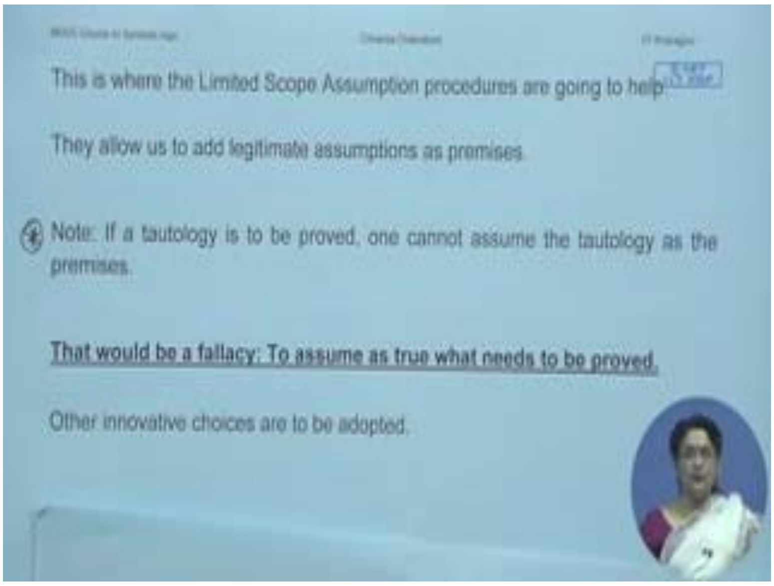

And they are going to help. How? Because, these procedures allow us to add premises. So, where there is no premise, if you follow these procedures, then you can *generate* premises, which are going to be useful for proving that tautology. Did you get that? So, tautology, you are right, does not have any premise to start. Because it is not even an argument. So if we are going to still prove the tautology as a consequence, where are we going to get the premises from? And the answer lies in the very nature of the Limited Scope Assumption procedures that we have learnt. They are going to allow us to insert premises, where there is none, and those additional premises will be what we need to prove that tautology. Alright? So, this is how we are going to move about in that.

So, what you cannot do, and I have seen it too many times, so I am going to mention it and this is very, very important is that, sometimes we forget, you know, what is it that we are trying to prove. And by mistake we *assume* what we are going *to prove*. That is not done. Isn't it? That is a big fallacy. What you are trying to prove, you cannot say let's assume that is true, and then derive the same. That is known as circularity. You have moved in a circle and which does not prove anything.

So, remember the tautology, when you are proving the tautology that is supposed to be the consequence. It cannot serve as a premise. So you should not start from what is to be proved. Got it? So that is not to be done, that cannot be added as a premise. Somewhere there has to be other creative choices available and this is where I said the (Refer Time:

07:28) acquaintance with IP and CP is going to mean a lot. Because, this is where you are going to have those creative options available to you. The more you understand how the procedure moves, more you will understand how to generate the premises for a tautology proof.

We are going to see examples, but this is an important point to sort of (Refer Time: 07:50) appreciate and also to note. Too many times because the tautology looks completely premise-free. So you get flustered and you start think maybe that's where we need to start it from. That would be a classical fallacious proof. Don't do that.

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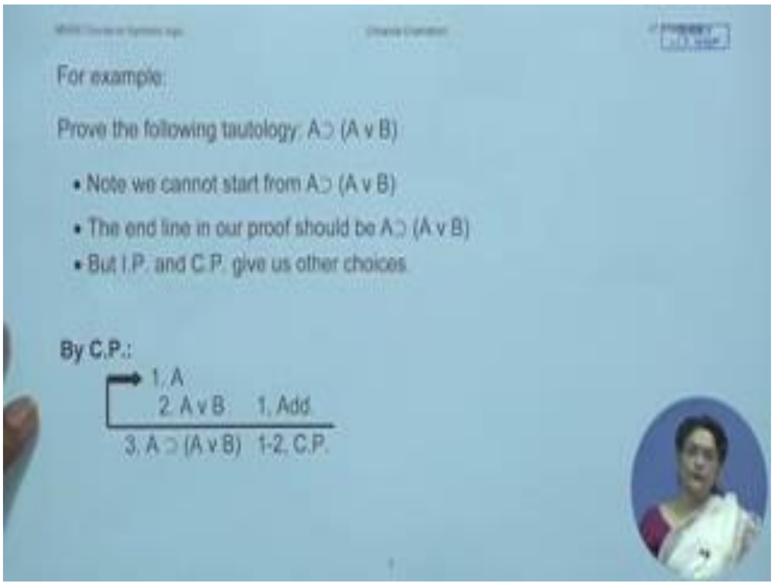

So let's start with example, but the general point stands. So we will take a look into an example. Suppose we are asked to prove this following tautology A ⊃ (A ∨ B). That is a tautology. If you are in doubt, do the truth table to see whether it is a tautology or not. The question is we are going to prove that. Right? And then comes… now you probably see what we meant is we cannot take what we are going to prove as the starting point. So, we cannot start from here. That should be the last line because that is to be the consequence. That is the conclusion and then you can say look, I have proven this tautology valid, as a valid consequence, alright?

So therefore, there has to be other ways to think about what, where we can start, how we can start, and so on. And this is where I said the Limited Scope Assumption procedures IP and CP will come very helpful. So suppose we start by the CP method. This is our conclusion A ⊃ (A ∨ B), . What do you want to assume? The obvious logical choice in your head should be what? That we start with A. Right? And if we can derive A ∨ B from that, then by CP we are going to get back A ⊃ (A ∨ B). Now from A, deriving A ∨ B is just child's play. You know already, that’s by Addition. So you assumed A, you derived A ∨ B by 1, Addition. Are we done? In a way. Do we need this A assumption anymore? No, we close it. So, this is the closure line and you get back A ⊃ (A ∨ B). How? 1 through 2 by CP. Got it?

So, this was a proof of tautology. Look at this. You have just shown that this…there exist a proof for this tautology, here is the proof. This was by C P.

Let’s try with IP, you can do the same. But you will see soon, you will find out that there are cases where the application of CP is going to be more efficient and there are cases where IP seems to be the real choice that you have available.

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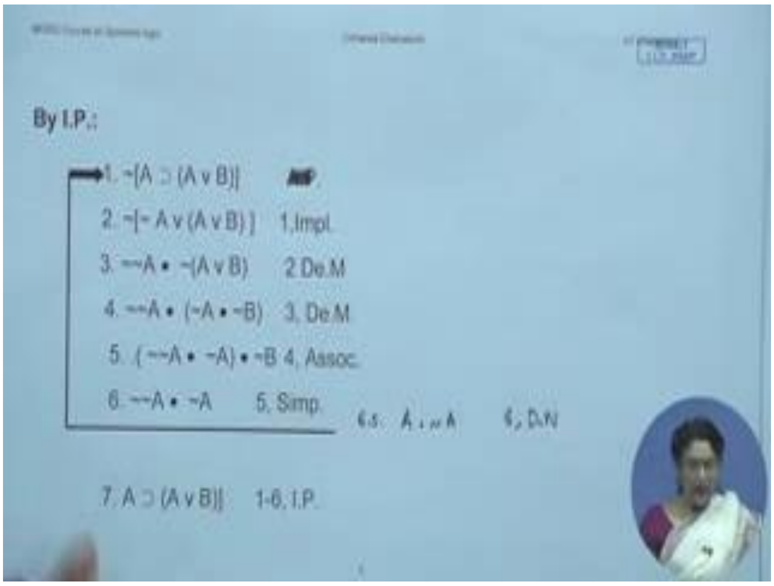

But let’s try because we are all beginners. So, we are going to try the same proof by IP this time. But remember what IP tells you. In case of IP where do you start? You can’t assume anythingbut the *negation of the conclusion*. In this case your conclusion is that tautology. So what you are entitled to assume is the negation of that conclusion. This is your starting point, if you are doing it by IP. What is your target? Let us remind ourselves the target is to derive an explicit contradiction, self contradiction of some sort. Only then we can have the original conclusion back. This is how the IP works. So, don’t get confused with CP and IP. Right now we are showing you how to do the proof with IP.

So, the rest is your strategy to see how you can derive a self contradiction of some sort. Take a look. This is where we have succeeded, in line number 6. So, we have worked with these rules and we are worked on several lines. Take a look, Implication, De Morgan, De Morgan, this is Association and this is Simplification. Ok? So, there you are, you have it here. If you want further, then you can also do a line number 6 from here you can do the 6.5 as A and ~A from line 6 by double negation. Right? That would be the classic self contradiction that you want to have. Once you have reached there, we can close it and say therefore, this conclusion follows 1 through 6 or 1 through 6.5 by IP, fine? So, there exists a proof in your system for a random, (Refer Time: 12:58) tautology. Did you see that?

So, this is how the proof of tautology is going to work with our added procedures and with our additional power that we have of deriving. This is how we are going to work with. It will take a little bit of a practice to see where there is an opportunity to get the result quickly by application of CP and where there is a quicker way to get the result by IP. You don't have to do both. You do not have to use both the procedures to give a proof of tautology. Choose any one of them. The whole point is that this tautology can be proven in our system and this is the way to do it. There exists at least one derivation that I can show. So, any which way that we can show is good enough.

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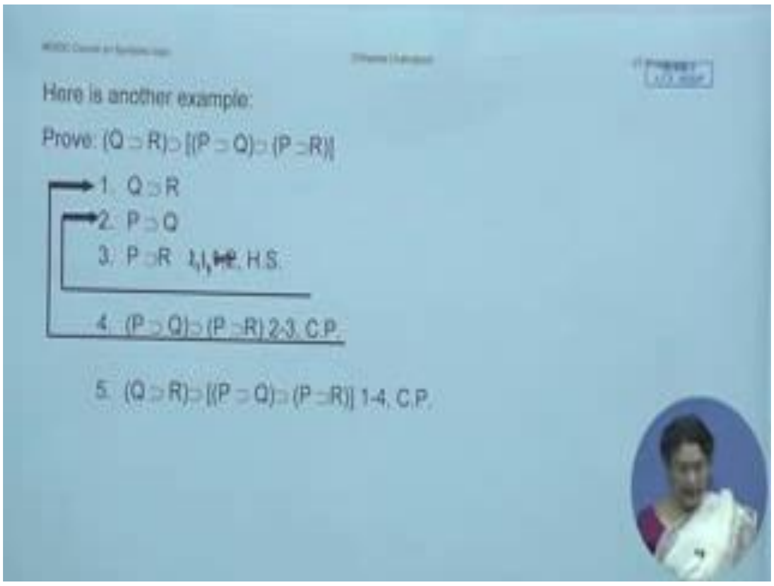

Let's see another example now and here is the tautology. So, (Q ⊃ R) ⊃ [(P ⊃ Q) ⊃ (P ⊃ R)], that's a tautology. Now how do I do, what do I do, what would be my premise? Etcetera. You have number of choices here. So many horseshoes. So, that horseshoes might tell you that this is a prime case of application of CP, Conditional Proof. Because it seems like we can assume and then when we discharge we are going to get horseshoes back. So, that would be my overall strategy that we are going to use the CP and not IP. IP also will give you the result, but it may not be the shortest, it may not be the most efficient way to do that.

If we are choosing CP here, what can we assume? You can assume this, you can assume Q, you can assume Q ⊃ R and so on and so forth. But probably the fastest way you can do this is to first assume Q ⊃ R, and then assume P ⊃ Q. Why, because I can see how you can immediately get P ⊃ R. Right? So, that is my next step from 1 and 2. We have HS, sorry this should be 2 and 1 by HS. That ordering is important because you are going to have P ⊃ R. So, 2, 1 H S will give you this line P ⊃ R. Once you have that now it's time to let go of the assumptions. You are going to get back in the first slot. This is LIFO, so last in first out. And this is 2 to 3 CP, and this is 1 through 4, CP. So, you are going to have the tautology back.

So, this shows again that there is a proof in the system for this tautology here. So, overall then what we have learnt is the how to back up the completeness claim, how with this new proof procedures now we can venture out to any given tautology. In fact you we do not even have to know which tautology you are going to get next. But we are confident that with this we can show it as a proven consequence. There will be a derivation standing which will show the tautology as its valid consequence.

And same goes for any other argument it does not have to be tautology, but any true consequence is now provable in our propositional logic. I have already mentioned that (Refer Time: 16:54) Bernays has actually proved it theoretically to show why this completes the system. Alright? There we end the discussion on the formal derivation or Formal Proof of Validity. From next module onwards we are going to look into the logic of classes and so on. But this is how far will go with the formal derivation technique, ok? Thanks.

Thank you very much and keep up the good work.