Symbolic Logic
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Lecture – 26
Completeness What It Is
Completeness: As a Desirable Property for a Formal Logic System
Is Propositional Logic Complete

Hello! Are we ready for our lesson? So, we are in module 26 today to go over this Formal Derivation system that we have learnt and then look into some interesting property in it. See, in the previous module, I have explained to you these nineteen rules, how to do the formal derivation out of that, right? So and we have done some examples together. But what today we are going discuss is an interesting property called *completeness*. What this property is, we will explain it also. But I will try to establish that *completeness* is a desirable property for a formal logic system. And then we will look into whether the propositional logic system, the proof system that we have learnt in the last one or two modules, is that *complete* in this sense? So that is on our agenda for this module on number 26.

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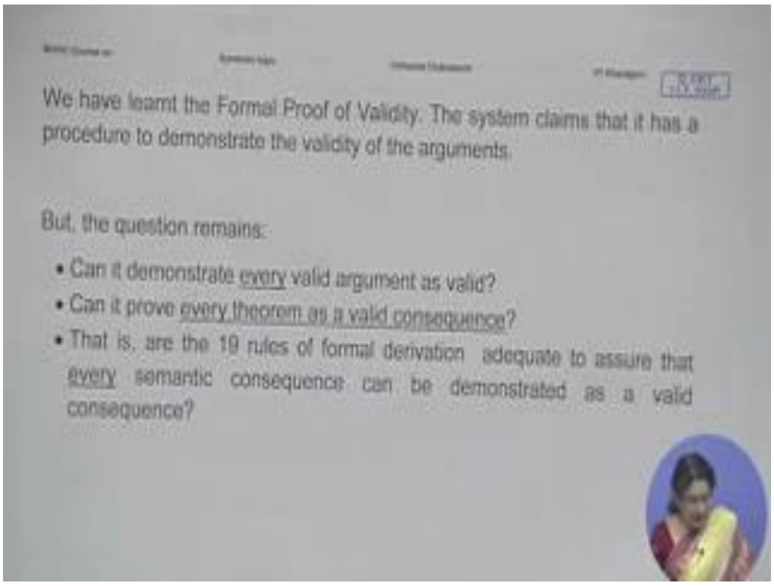

See, we have learnt this Formal Proof of Validity and which means that the system has made a claim, that we have the rules, the mechanisms, to demonstrate the validity of deductive arguments, right? Now that is not the only thing that we are concerned at this point. Given an argument, we know, in a way that this system has a mechanism to show how the conclusion can be derived from the premises. But there is a further question here. If we pick a random deductive argument and suppose it is valid also, can the proof system or the derivation system, is there a guarantee that it can demonstrate the validity of that argument? Did you understand what I just said? I said that we know that we have the rules, that we have all the mechanisms, and we know how to apply the rules. So we know in a way that given an argument, which is deductively valid, we will try to prove that, prove that the conclusion follows validly from the premises. That doesn't answer the question that we are asking next.

What we are asking next is that can we guarantee that *any*, any valid argument, which is deductively valid, our system is equipped to show, demonstrate its validity? So for every valid argument, can we guarantee that in our system there is a proof? Ok? So every conclusion that follows validly, that we can call as a theorem in our… in our system, is there a proof for it? Can we prove every theorem to be a valid consequence? In a way what we are asking is that we have so many rules in our rule base, the 19 Rules of Inference and of Equivalence or Replacement, are they adequate to give this guarantee that every semantic consequence can be demonstrated as a valid consequence?

Ok. If you still have not got it, then we will try to build upon this concept, but this is not just a question that here is an argument, show me the proof. It's deeper than that. What we are asking that: Ok, if you succeed in this case, in a given argument's case, that here is the conclusion and here is the proof. By that, can we then rest assured that if I give you *any* valid argument, deductive argument of course, that you will be able to demonstrate its validity in your system? *Any*, any is the operative point here.

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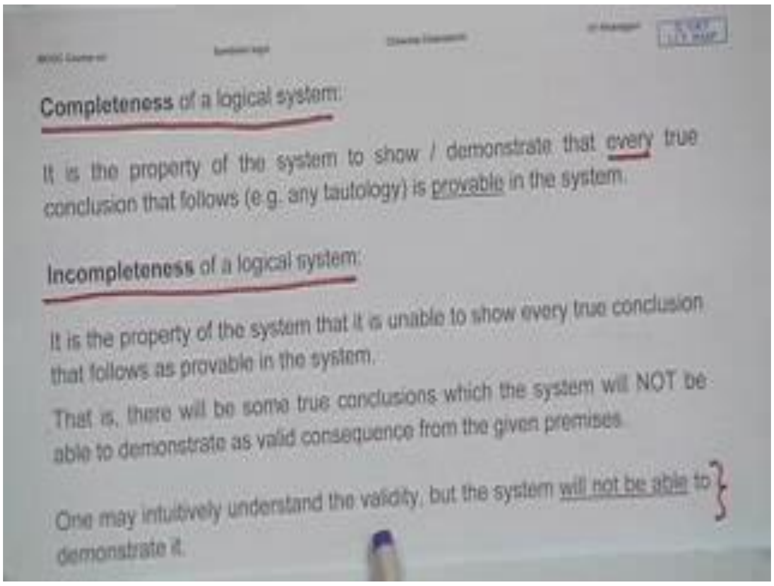

So, any given argument. Is there a proof for it in the system?

Why are we discussing this? As I told you that there is this property called completeness. Now, completeness is a curious property of a logical system, by which the system is able to *show*, 'show' as in *demonstrate*, that *every* true conclusion that is expressible in it, ok? including the tautologies, every true conclusion is *provable* in the system. So if there is something that is expressible in the system as true, as a true conclusion, there exists a proof in the system for it. That is the quality that we will call *completeness*. The system is *complete* in this sense.

What would be *incompleteness* of a logical system? Well, its just opposite of that property, where the system is *unable*, there is an *inability* in the system, to show every true conclusion that follows as provable in the system. So it can't show every true conclusion as a proven conclusion. Get it? So in that case what will happen? That there may be some conclusions that you know are true, and that they follow validly, we will see some examples soon, but they are valid and you know that they are true intuitively that is clear. But what is *not available* is a proof or a derivation how they can be derived from the given premises. And, that is something, a property that we would call *incompleteness* of a logical system. So this *failure to demonstrate* even when you know that something is valid, you know, this gap, that is what *incompleteness* is.

Now if I leave this in front of view and ask you which one is desirable in a logical system? You know, the system which claims that it has a system of proof, that it has a Formal Derivation system. Which one is desirable?

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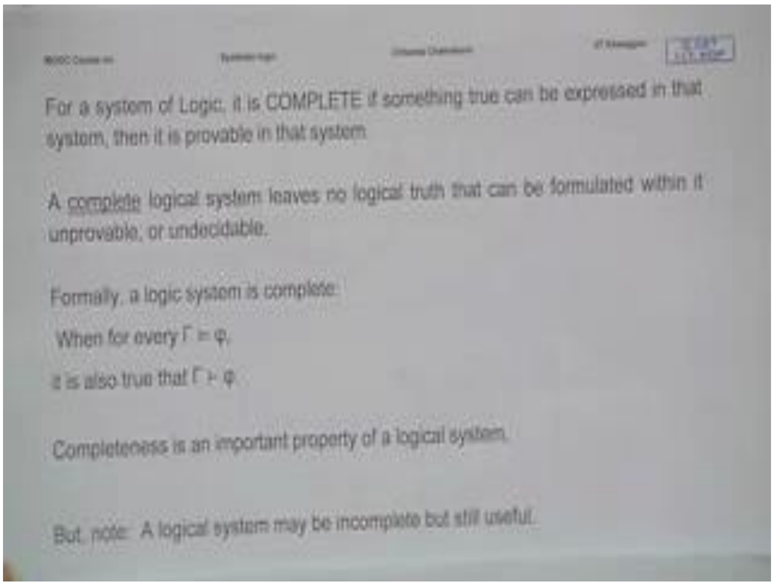

Obviously you will say: Completeness. So the proof system has to be complete, comprehensive enough to cover every true conclusion that follows. Isn't it? Whereas incompleteness is somewhat questionable property.

Now this is our understanding of the completeness and incompleteness. But I am going to put, add some more nuances into these concepts. So if it is a complete system in logic, then something true that is expressible in it, is provable. This is what we said. Expressible as in syntactically expressible, you can see that it has to be a valid consequence, but provability means that you have to show by process, the actual step by step process that there exists a derivation for it.

What happens in a complete logic system is that it leaves no logical truth within it unprovable or undecidable. And that's a really, really applaudable quality in a logical system. I mean it something praiseworthy. (Refer Time: 08:26) That some no logical truth is unprovable or undecidable in the system. That is what completeness ensures. So that symbolically we can say that, you know, for any Γ (gamma), set of statements, if α

(alpha) happens to be the semantic consequence, then it's also true that there is a proof for it, there is a derivation that shows how α follows from that set of premises.

So completeness, I am trying to emphasize upon, is an extremely important property for a logic system. Having said that I will add that there may be, I mean, incomplete logic systems which are still useful. Useful as in the sense that it will work for you in certain cases quite well. But the problem is that you don't know when it's going to stop functioning. The incomplete logical system what will it do? It will continue to give you proofs for some arguments. What you don't know is the next one, next valid argument that you have in hand, whether your system is actually able to show that. Next time. So that guarantee will not be there in the incompleteness. So in that way it's useful, but it is a limited usefulness, whereas what is preferred by all means is the complete logical system. So, completeness is the very important property.

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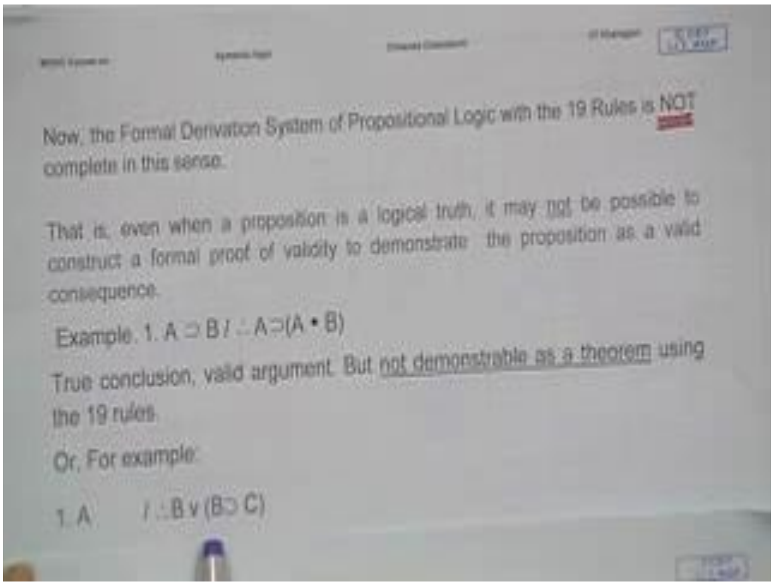

Let's now come back to our formal derivation system, the one that we have learnt and that we are trying to master. Now with the 19 rules, let me just make this point very very clear that what you have learnt with the 19 rules, it is not complete in this sense. That is, though the number of the rules is quite high, you have 19 of them, but they cannot demonstrate the validity of every single true conclusion that follows in the system. You will not believe this, but I will try to demonstrate that. I am not going into the entire mathematical proof of it, but I will try to sort of conceptually explain it to you. So what will happen and I am going to show you with examples, is that, a proposition may be a logical truth, but with the 19 rules, you may not be able to construct the formal proof of derivation for it in the system.

For example, look at this. It's a very simple argument. The premise says, if A then B or $A \supset B$, the conclusion says if A then of course A and B, $A \supset (A \bullet B)$. Ok? Now intuitively, you should understand why this has to be valid. Those of you who want to try out more, why don't you do the truth table of this to see for yourself why this has to be valid? I am going to explain only conceptually so that everybody sort of gets the point. See what we are saying? We are saying that $A \supset B$ is true, if A then B, fine? Now from A, remember A follows. From *any* given proposition, itself will follow, right? So from A, A will follow; if A then A. That much we know for sure. As for B, that is already stated in the premise. If A then B and that is true, right? So given A, B follows. So, when we captured that, that given A, A and B follows, it cant be false. That's my conceptual explanation to you. For those who want to do further, you can do the truth table test, you can do the truth tree test which I have taught you earlier. Anyway to see why if the premises true, the conclusion *cannot* be false, *cannot* be false. Which means, this has to be a valid argument.

But unfortunately there is no proof of it possible with the 19 rules. The 19 rules have their limitations, right? I mean they work in a certain way; they don't work in other ways. With those you cannot demonstrate how from this, you can come here unfortunately. You can try, you can try, you don't have to take my word for it, you can try. But let me also ensure you that you will not be able to do that. Because the rules are such that they cannot, after a certain point you cannot jump to a state, which will land you here exactly. So you may try as I said, I mean, try Implication, try De morgan anything that you want, Exportation, Distribution anything that you want, all the rules. But there is no proof with the 19 rules, with the said nineteen rules.

Here is another example this one. Now if you did not understand my point here and you are still racking your brain why we cannot have a proof of this, how is it possible that we cannot have a proof, see this. Premise is A, from this what? $B \vee (B \supset C)$. See it? And my claim is that with the 19 rules from this premise, you cannot have a step by step formal derivation that will land you here. There is simply no mechanism there. Again you don't have to take my word, but you try it and you will see. Conceptually I will give my explanation, why this is valid. Try it yourself and you will find that $B \vee (B \supset C)$ is a tautology. So it is always true, if it is always true it doesn't matter what value A is. If A is true well and good, if A is false even then it doesn't matter. The argument would be valid, right? Why? Because the conclusion is always true.

Therefore, this is a valid consequence, in a trivial sense it is a valid consequence. And remember if it is a logical truth that is expressible in our system, if our system was complete, there should have been a proof of it, right? But unfortunately you will not be able to show it.

Now what is the solution? So what we have just said is that the formal derivation system, as it stands with the 19 rules, it is not complete. Then what? I mean are we going to leave it at that? Or just live with that complaint that it is not complete? No. what we are going to do is to sort of work on it.

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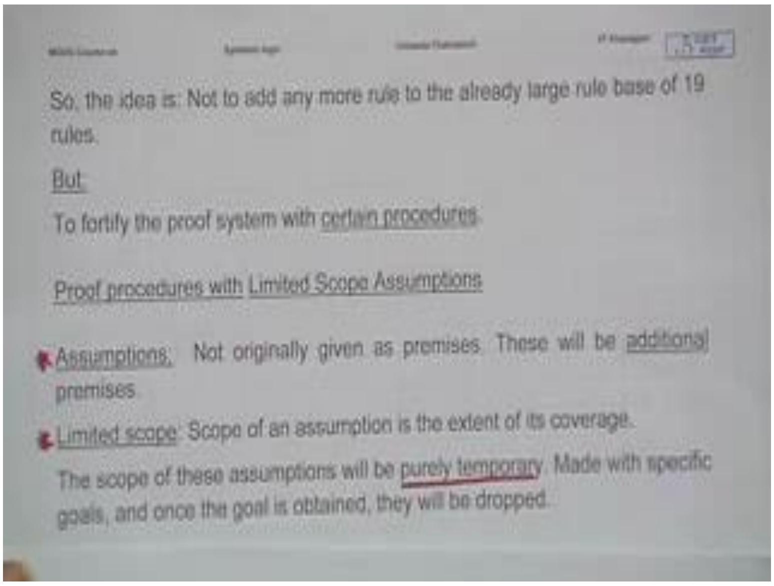

So what we will do is *not to add more rules*. Because we already have 19 rules, even then we have found that the rule base is not complete, it does not give us that guarantee. So we are not going to add any more rules to it. Instead, what we are going to do is to further strengthen the system with some procedures, some additional procedures will be added. These will be proof procedures, which will call *proof procedures with limited scope assumptions*. Limited Scope assumptions.

Now when I say assumptions what we mean is not exactly the given premises. So assumptions are not originally given. Originally given are the premises; the specifications that you start with. Assumptions are, you are saying let's assume this to be true. So they are not part of the premises, but can be *added* to the premise base. Listen what I said. They can be *added*. So they are additional premises, which you are assuming to be true. That kind of assumptions addition these procedures will allow.

Second I said, limited scope, what does it mean? Well, the every assumption has a *scope*, meaning, how far does it cover? The extent of its coverage. So the scope of an assumption, remember an assumption falls by default within its own scope. It covers itself plus it covers some more. So, the scope of an assumption is where it starts and where it ends. What we are dealing with in these procedures would be limited scope assumptions; meaning, they will be *temporary*, purely temporary with some limited scope so that it does not go all the way from the beginning of the proof till the end. So the last line of the proof will be free of all assumptions. Ok? And they are temporary in the sense that you are starting the assumptions with some goals in mind. As soon as that goal is reached, the assumption will be dropped or discharged.

So limited scope assumptions, we are going to see this in our next modules. I am just going to introduce the names

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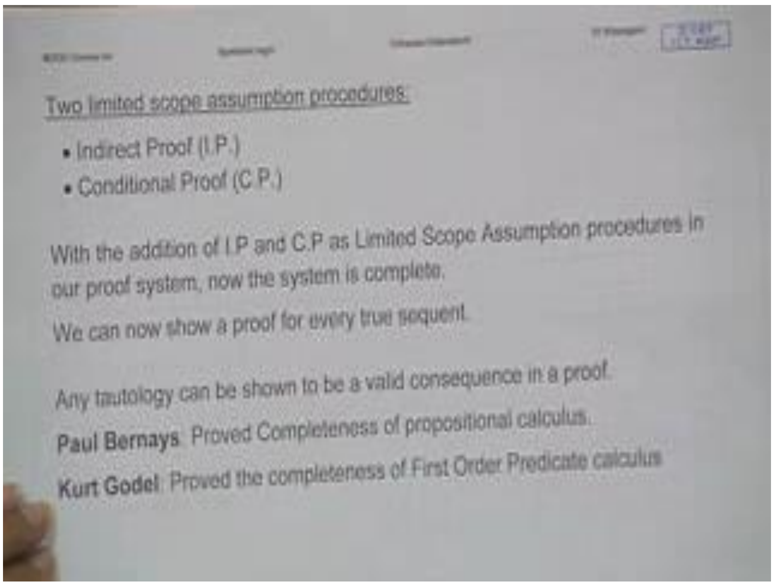

so that you know what is coming up. We will look into two limited scope assumptions in the next modules. One is called the Indirect Proof which will call IP, and the other one is called Conditional Proof or CP. We are going to refer to it as CP. So and you will see after we have discussed this and after we have shown you that with the addition of these two proof procedures, our proof system will then stand as *complete*. Then given any kind of true conclusion we'll be ready to come up with a proof. So this is what will be achieved and if you give us any kind of tautology, any true conclusion that follows we'll be able to provide a proof for it.

Now so far I have said it and I have not really gone into the intricacies of the theoretical proof or showing you the mathematical proof, but I will refer to you like this. That if you are interested, you can look up Paul Bernays. He is the one who proved the completeness of Propositional Calculus. So don't take my word for it, there exists a proof, mathematical proof that shows why after this insertions, these additional proof procedures, the propositional calculus will be complete. And Kurt Gödel, some of you may have heard the name, Gödel. So Gödel proved the completeness of the First Order Predicate calculus. We have not gone to First Order Predicate calculus yet, We are here at the Propositional Logic level and this is the person who actually proved why and how Propositional Calculus is now *complete.*

So, with that I am going to end this module and in the next module as I said we will pick up these Limited Scope Assumption procedures and explain it to you, alright? So looking forward to see in you again in the next module.

Thank you very much.