By the initial value Theorem we have

$$\underset{t\to 0}{Lt}\ 3e^{-2t} = \underset{s\to\infty}{Lt}\ sF(s)$$

$$= \underset{s\to\infty}{Lt}\left(\frac{3s}{s+2}\right)$$

$$\Rightarrow 3 = 3.$$ (how?)

By the Final-Value Theorem

$$\underset{t\to\infty}{Lt}\ 3e^{-2t} = \underset{s\to 0}{Lt}\ sF(s)$$

$$= \underset{s\to 0}{Lt}\left(\frac{3s}{s+2}\right)$$

$$\Rightarrow 0 = 0$$ (how?)

H.W ★★★★★

Ex1/ Demonstrate the initial & final value theorems using the function

$$f(t) = e^{-t}.$$

Expand e^{-t} as a power series, evaluate term by term & confirm the legitimacy of term by term evaluation.

Hint:- $\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t}\}$

$$= \frac{1}{(s+1)}$$

Note:— Since the improper integral converges independently of the value of s & all limits exist, it is therefore correct to assume that the order of the two processes (taking the limit & performing the integral) can be exchanged.

Suppose the f^n $f(t)$ can be expressed as a power series as follows:

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots$$

If we assume that the L.T of $f(t)$ exists, i.e., $f(t)$ is of exponential order & is piece-wise cont. If further we assume that the power series for $f(t)$ is ~~absolutely~~ absolutely & uniformly convergent then the L.T can be applied term by term.

$$f(t) = \sum_{n=0}^{\infty} a_n t^n$$

$$\mathcal{L}\{f(t)\} = F(s)$$
$$= \mathcal{L}\{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots\}$$

$$= a_0 \mathcal{L}\{1\} + a_1 \mathcal{L}\{t\} + a_2 \mathcal{L}\{t^2\} + \dots + \mathcal{L}\{t^n\} + \dots,$$

provided the transformed series is <u>convergent</u>.

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

the R-H-S becomes

$$a_0/s + a_1/s^2 + \frac{2a_2}{s^3} + \dots + \frac{n!\, a_n}{s^{n+1}} + \dots$$

Hence,

$$F(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^2} + \dots + \frac{n!\, a_n}{s^{n+1}} + \dots$$

Sol^n Ex1/ $\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$

$\therefore \lim_{t\to 0} f(t) = f(0) = e^{-0} = 1.$

$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left(\frac{s}{s+1}\right) = \lim_{s\to\infty}\frac{1}{(1+1/s)} = \frac{1}{(1+0)} = 1.$

This confirms the Initial value Theorem.
The Final value theorem is also confirmed as follows:-

$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} e^{-t} = 0.$

$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left(\frac{s}{s+1}\right) = 0.$

The power expansion for e^{-t} is —

$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots + (-1)^n\frac{t^n}{n!} + \dots$

$\therefore \mathcal{L}[e^{-t}] = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} - \dots + \frac{(-1)^n}{s^{n+1}}$

$= \frac{1}{s}\left[1 - \frac{1}{s} + \frac{1}{s^2} - \dots + \frac{(-1)^n}{s^n} + \dots\right]$

$= \frac{1}{s}\left(1+\frac{1}{s}\right)^{-1} = \frac{1}{s(1+1/s)} = \frac{1}{(s+1)}$

Hence, the term by term evaluation of the power series expansion for e^{-t} gives the right answer. This is not a proof of the series expansion method of course, merely a verification that the method gives the right answer in this instance.