1.3 Strong Form of the Principle of Mathematical Induction

We are now ready to prove the strong form of the principle of mathematical induction.

Theorem 1.3.1 (Principle of Mathematical Induction: Strong Form)**.** *Let $P(n)$ be a statement about a positive integer n such that*

1. *$P(1)$ is true, and*

2. *$P(k+1)$ is true whenever one assume that $P(m)$ is true, for all m, $1 \leq m \leq k$.*

Then, $P(n)$ is true for all positive integer n.

Proof. Let $R(n)$ be the statement that "the statement $P(m)$ holds, for all positive integers m with $1 \leq m \leq n$". We prove that $R(n)$ holds, for all positive integers n, using the weak-form of mathematical induction. This will give us the required result as the statement "$R(n)$ holds true" clearly implies that "$P(n)$ also holds true".

As the first step of the induction hypothesis, we see that $R(1)$ holds true (already assumed in the hypothesis of the theorem). So, let us assume that $R(n)$ holds true. We need to prove that $R(n+1)$ holds true.

The assumption that $R(n)$ holds true is equivalent to the statement "$P(m)$ holds true, for all m, $1 \leq m \leq n$". Therefore, by Hypothesis 2, $P(n+1)$ holds true. That is, the statements "$R(n)$ holds true" and "$P(n+1)$ holds true" are equivalent to the statement "$P(m)$ holds true, for all m, $1 \leq m \leq n+1$". Hence, we have shown that $R(n+1)$ holds true. Therefore, we see that the result follows, using the weak-form of the principle of mathematical induction. ■

We state a corollary of the Theorem 1.3.1 without proof.

Corollary 1.3.2 (Principle of Mathematical Induction)**.** *Let $P(n)$ be a statement about a positive integer n such that for some fixed positive integer n_0,*

1. *$P(n_0)$ is true,*

2. *$P(k+1)$ is true whenever one assume that $P(m)$ is true, for all m, $n_0 \leq m \leq k$.*

Then $P(n)$ is true for all positive integer $n \geq n_0$.

Remark 1.3.3 (Pitfalls)**.** *Find the error in the following arguments:*

1. *If a set of n balls contains a green ball then all the balls in the set are green.*
   Solution: *If $n = 1$, we are done. So, let the result be true for any collection of n balls in which there is at least one green ball.*

   So, let us assume that we have a collection of $n+1$ balls that contains at least one green ball. From this collection, pick a collection of n balls that contains at least one green ball. Then by the induction hypothesis, this collection of n balls has all green balls.

Now, remove one ball from this collection and put the ball which was left out. Observe that the ball removed is green as by induction hypothesis all balls were green. Again, the new collection of n balls has at least one green ball and hence, by induction hypothesis, all the balls in this new collection are also green. Therefore, we see that all the $n+1$ balls are green. Hence the result follows by induction hypothesis.

2. *In any collection of n lines in a plane, no two of which are parallel, all the lines pass through a common point.*

   Solution: *If $n = 1, 2$ then the result is easily seen to be true. So, let the result be true for any collection of n lines, no two of which are parallel. That is, we assume that if we are given any collection of n lines which are pairwise non-parallel then they pass through a common point.*

   Now, let us consider a collection of $n+1$ lines in the plane. We are also given that no two lines in this collection are parallel. Let us denote these lines by $\ell_1, \ell_2, \ldots, \ell_{n+1}$. From this collection of lines, let us choose the subset $\ell_1, \ell_2, \ldots, \ell_n$, consisting of n lines. By induction hypothesis, all these lines pass through a common point, say P, the point of intersection of the lines ℓ_1 and ℓ_2. Now, consider the collection $\ell_1, \ell_2, \ldots, \ell_{n-1}, \ell_{n+1}$. This collection again consists of n non-parallel lines and hence by induction hypothesis, all these lines pass through a common point. This common point is P itself, as P is the point of intersection of the lines ℓ_1 and ℓ_2. Thus, by the principle of mathematical induction the proof of our statement is complete.

3. *Consider the polynomial $f(x) = x^2 - x + 41$. Check that for $1 \leq n \leq 40$, $f(n)$ is a prime number. Does this necessarily imply that $f(n)$ is prime for all positive integers n?* Check that $f(41) = 41^2$ and hence $f(41)$ is not a prime. Thus, the validity is being negated using the proof technique "disproving by counter-example".