INDIA INSTITUTE OF TECHNOLOGY KHARAGPUR

Date ————, FN/AN, Time : 3 Hrs., Full Marks **50**, Dept. of Mathematics

No. of Students 62, End Autumn Semester Examination: 2011–12

Sub.No. MA40001/MA41007, Sub. Name: Functional Analysis, 4^{th} year

Instruction: Attempt any FIVE questions. All questions carries equal marks.

1(a) Let $B(X,Y)$ denotes the set of all bounded linear operators from a normed space X into a normed space Y. If Y is a Banach space, then prove that $(B(X,Y), \|.\|_{B(X,Y)})$ is a Banach space, where

$$\|T\|_{B(X,Y)} = \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Tx\|_Y}{\|x\|_X}$$

1(b) Find the dual basis of (e_1, e_2, e_3) for $\mathbb{R}^3$ where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

1(c) Find the dual space of $\mathbb{R}^n$.

2(a) If $(X, <,>)$ is a real inner product space in which $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ holds for all $x, y \in X$ then show show that $x \perp y$. Can the same conclusion be drawn if X is a complex inner product space. Justify your answer.

2(b) State and prove Schwarz inequality in case of inner product space.

2(c) Prove that a subspace of a separable Hilbert space is separable.

3(a) Let $T : X \to X$ be a bounded linear operator on a complex inner product space. If $< Tx, x >= 0$ for all $x \in X$ then show that $T = 0$. Show that this does not hold in the case of a real inner product space.

3(b) If Y is a closed subspace of a Hilbert space then prove that $Y = Y^{\perp\perp}$, where

$$Y^{\perp} = \{x \in H : x \perp Y\}.$$

3(c) Let X be the inner product space of all real valued continuous functions on $[0, 2\pi]$ with inner product defined by

$$< x, y >= \int_0^{2\pi} x(t)y(t)\, dt.$$

Show that (u_n) where $u_n(t) = \cos nt$ form an orthogonal sequence in X. Find its corresponding orthonormal sequence.

(P.T.O.)

4(a) If (e_k) is an orthonormal sequence in an inner product space X and $x \in X$ then show that $x - y$ with y given by

$$y = \sum_{k=1}^{n} \alpha_k e_k, \qquad \alpha_k = < x, e_k >$$

is orthogonal to the subspace $Y_n = \text{span}\,\{e_1, e_2, \cdots, e_n\}$.

4(b) State and prove Riesz's Theorem in regard to represent bounded linear functional on a Hilbert space.

4(c) On $\mathbb{C}^2$, let the operator $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by

$$Tx = (\xi_1 + i\xi_2,\ \xi_1 - i\xi_2),$$

where $x = (\xi_1, \xi_2)$. Find T^*. Further show that $T^*T = TT^* = 2I$.

5(a) Define Hilbert-adjoint operator T^* of a bounded linear operator $T : H_1 \to H_2$, where H_1 and H_2 are Hilbert spaces. Prove that T^* exists and it is unique. Also prove that T^* is a bounded linear operator with norm

$$\|T^*\| = \|T\|.$$

5(b) If (T_n) is a sequence of bounded linear operators on a Hilbert space and $T_n \to T$ then show that $T_n^* \to T^*$.

5(c) Let H be a Hilbert space and the operator $U : H \to H$ be unitary. Then prove that U^{-1} is unitary.

6(a) State and prove Banach fixed point theorem

6(b) Let $X = \{x \in \mathbb{R} : x \geq 1\} \subset \mathbb{R}$ and let the mapping $T : X \to X$ be defined by $Tx = \frac{x}{2} + \frac{1}{x}$. Using Banach fixed point theorem show that T has a unique fixed point in X.

6(c) Let $T : X \to X$ be a mapping on a complete metric space $X = (X, d)$ and suppose that T^m is a contraction on X for some positive integer m. Then show that T has a unique fixed point.