MA10002 Mathematics-II : Tutorial Sheet - 8

1. Evaluate $\iint x^2y^2\,dxdy$ over the circle $x^2+y^2 \le 1$.
2. Evaluate $\iint_R xy\,dxdy$, where R is the domain bounded by the x-axis, ordinate $x=2a$, and the curve $x^2=4ay$.
3. Evaluate $\iint \frac{r\,drd\theta}{\sqrt{a^2+r^2}}$ over loop of the lemniscates $r^2=a^2\cos\,2\theta$.
4. Evaluate $\iint r^3\,drd\theta$ over the area included between the circles $r=2a\cos\theta, r=2b\cos\theta$, where $b<a$.
5. Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dydx$ by changing to polar coordinates. Hence, deduce that $\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$.
6. Evaluate $\iint \sqrt{\frac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2+b^2x^2+a^2y^2}}\,dxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
7. Use the transformation $x+y=u$ and $y=uv$ to show that $\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}\,dydx=\frac{e-1}{2}$.
8. Changing the order of integration, find the value of the integral $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dydx$.
9. Evaluate the following integrals by changing the order of integration.
   (i) $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx$ (ii) $\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}dydx$ (iii) $\int_0^\infty\int_0^x xe^{\frac{-x^2}{y}}\,dydx$.
10. Find the area lying between the parabola $y^2=4ax$ and $x^2=4ay$.
11. Find the area of the cardioid $r=a(1+\cos\theta)$.
12. Find the volume contained between the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x}{a}$.
13. Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.
14. Find the area of the surface of the paraboloid $x^2+y^2=z$, which lies between the planes $z=0$ and $z=1$.
15. Find the area of the paraboloid $2z=\frac{x^2}{a}+\frac{y^2}{b}$ inside the cylinder $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
16. Evaluate $\iiint \frac{dxdydz}{(x+y+z+1)^3}$ over a terahedron bounded by coordinate planes and the plane $x+y+z=1$.
17. Evaluate the triple integral $\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}}dzdydx$.
18. Evaluate $I=\iiint\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dxdydz$ over the region $V=\{(x,y,z); x\ge 0, y\ge 0, z\ge 0, \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\}$.
19. Evaluate $I=\iiint_V (x^2+y^2+z^2)^m\,dxdydz, m>0$ over the region $V=\{(x,y,z); x^2+y^2+z^2\le 1\}$.
20. Find the volume of the portion cut off from the sphere $x^2+y^2+z^2=a^2$ by the cylinder $x^2+y^2=ax$.

Solutions (Assignment 8)

① Since $x^2+y^2 \le 1$, $x^2 \le 1$ and $y^2 \le 1-x^2$

or, $|x| \le 1$ and $|y| \le \sqrt{1-x^2}$

or, $-1 \le x \le 1$ and $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

The integrand $f(x,y) = x^2y^2$ is continuous over the region $R = \{(x,y) : -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$

$$\therefore \iint_R x^2y^2\,dx\,dy = \int_{-1}^{1}\left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2y^2\,dy\right]dx$$

$$= \int_{-1}^{1}\left[x^2\left\{\frac{y^3}{3}\right\}_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\right]dx = \int_{-1}^{1}\frac{2}{3}x^2(1-x^2)^{3/2}\,dx$$

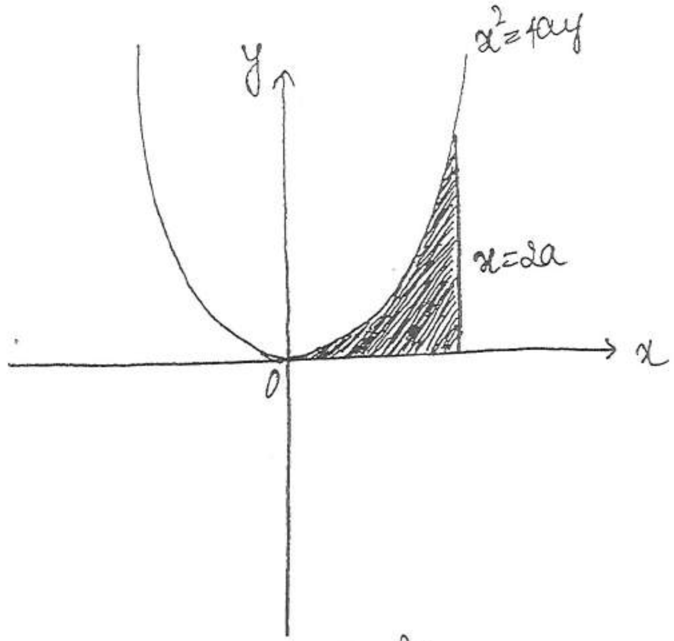
$$= \frac{4}{3}\int_0^1 x^2(1-x^2)^{3/2}\,dx, \text{ since integrand is even}$$

$$= \frac{4}{3}\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta, \text{ substituting } x = \sin\theta$$

$$= \frac{4}{3}\cdot\frac{1}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{24}$$

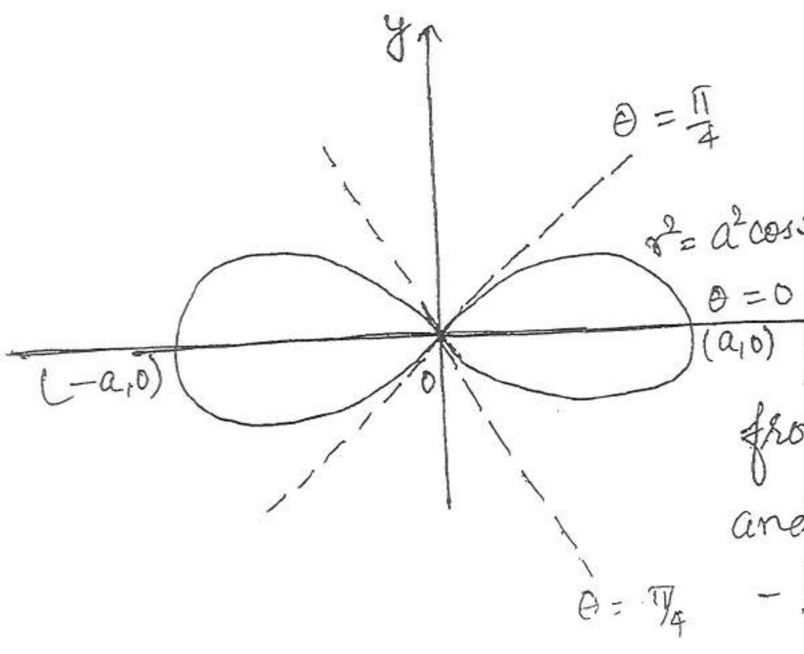
② The region of integration is
$R = \{(x,y) : 0 \le x \le 2a;\ 0 \le y \le \frac{x^2}{4a}\}$.

The region is bounded by $y=0$, $x=2a$ and the parabola $x^2 = 4ay$.



So,
$$\iint_R xy\,dx\,dy = \int_0^{2a}\left[\int_0^{x^2/4a} xy\,dy\right]dx$$
$$= \int_0^{2a} x\left\{\left[\frac{y^2}{2}\right]_0^{x^2/4a}\right\}dx = \int_0^{2a}\frac{x^5}{32a^2}\,dx$$
$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{32a^2}\left[\frac{64a^6}{6}\right] = \frac{a^4}{3}$$

(3)



From the figure of the curve, we note that in the region of integration, r varies from 0 to $a\sqrt{\cos 2\theta}$ and θ varies from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

Therefore,

$$\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}} = \int_{-\pi/4}^{\pi/4}\left[\int_0^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{a^2+r^2}}\,dr\right]d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[\frac{1}{2}\int_0^{a\sqrt{\cos 2\theta}} 2r\,(a^2+r^2)^{-1/2}\,dr\right]d\theta$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4}\left|\frac{(a^2+r^2)^{1/2}}{\frac{1}{2}}\right|_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} a\left[(1+\cos 2\theta)^{1/2}-1\right]d\theta$$

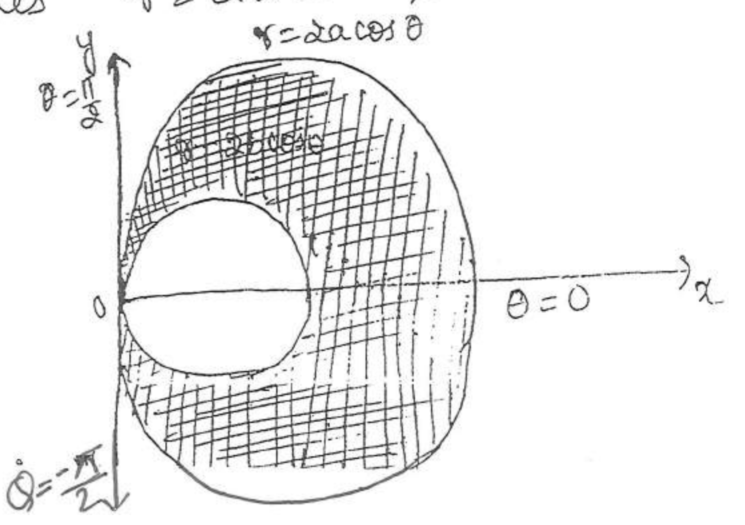
$$= a\int_{-\pi/4}^{\pi/4} a\left[(2\cos^2\theta)^{1/2}-1\right]d\theta$$

$$= a\int_{-\pi/4}^{\pi/4}(\sqrt{2}\cos\theta-1)\,d\theta$$

$$= 2a\int_0^{\pi/2}(\sqrt{2}\cos\theta-1)\,d\theta$$

$$= 2a\left|\sqrt{2}\sin\theta-\theta\right|_0^{\pi/4} = 2a\left(1-\frac{\pi}{4}\right)$$

(4) The region of integration between the given circles $r = 2a\cos\theta$, $r = 2b\cos\theta$ is shown in foll. fig



In the region of integration, r varies from $2b\cos\theta$ to $2a\cos\theta$, whereas θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Therefore,

$$\iint r^3\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \left[\int_{2b\cos\theta}^{2a\cos\theta} r^3\, dr \right] d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4} \right]_{2b\cos\theta}^{2a\cos\theta} d\theta$$

$$= \frac{1}{4} \int_{-\pi/2}^{\pi/2} \left[16a^4\cos^4\theta - 16b^4\cos^4\theta \right] d\theta$$

$$= 4 \int_{-\pi/2}^{\pi/2} (a^4 - b^4)\cos^4\theta\, d\theta$$

$$= 8 \int_{0}^{\pi/2} (a^4 - b^4)\cos^4\theta\, d\theta$$

$$= 8(a^4 - b^4) \cdot \frac{3\cdot 1}{4\cdot 2} \cdot \frac{\pi}{2} = \frac{3\pi}{2}(a^4 - b^4)$$

(5) In the given integral, both x and y vary from 0 to ∞. Hence, the region of integration is xy-plane. Changing to polar coordinates by substituting $x = r\cos\theta$ and $y = r\sin\theta$, we get $x^2+y^2=r^2$; and in the region of integration, r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$. Thus,

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dy\, dx = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^\infty r e^{-r^2}\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{\infty} e^{-t} \cdot \frac{dt}{2}\, d\theta, \quad r^2 = t$$

$$= \frac{1}{2}\int_0^{\pi/2} \left| \frac{e^{-t}}{-1} \right|_0^{\infty} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} (0-1)\, d\theta = \frac{\pi}{4} \quad \text{—— (1)}$$

On the other hand,

$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dy\, dx = \left[\int_0^{\infty} e^{-x^2} dx\right]\left[\int_0^{\infty} e^{-y^2} dy\right]$$

$$= \left[\int_0^{\infty} e^{-x^2} dx\right]^2$$

Thus, (1) implies

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

(6) Substituting $\frac{x}{a} = X$ and $\frac{y}{b} = Y$, the problem reduces to the evaluation of $\iint ab\sqrt{\frac{1-X^2-Y^2}{1+X^2+Y^2}}\, dx\, dy$ over the positive quadrant of the circle $X^2+Y^2=1$. Substituting $X = r\cos\theta$ and $Y = r\sin\theta$, we have $dX\, dY = r\, dr\, d\theta$. In the region of integration r ran from 0 to 1 and θ varies from 0 to $\frac{\pi}{2}$. Hence, the given integral reduces to

$$ab\int_0^{\pi/2}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\, dr\, d\theta$$

$$= ab\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\, dr \int_0^{\pi/2} d\theta$$

$$= \frac{ab\pi}{2} \int_0^{\pi/2} \sqrt{\frac{1-\sin t}{1+\sin t}} \cdot \frac{1}{2} \cos t \, dt, \quad r^2 = \sin t$$

$$= \frac{\pi ab}{4} \int_0^{\pi/2} \sqrt{\frac{1-\sin t}{1+\sin t}} \, \frac{\sqrt{1-\sin t}}{\sqrt{1-\sin t}} \cos t \, dt$$

$$= \frac{\pi ab}{4} \int_0^{\pi/2} \frac{1-\sin t}{\cos t} \cos t \, dt$$

$$= \frac{\pi ab}{4} \int_0^{\pi/2} (1-\sin t) \, dt$$

$$= \frac{\pi ab}{4} \left[t + \cos t \right]_0^{\pi/2} = \frac{\pi ab}{4} \left[\frac{\pi}{2} - 1 \right]$$

$$= \frac{\pi ab}{8} (\pi - 2).$$

(7) We have,

$$x = u - y = u - uv = u(1-v) \text{ and } y = uv$$

Therefore,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u(1-v) - (-uv) = u.$$

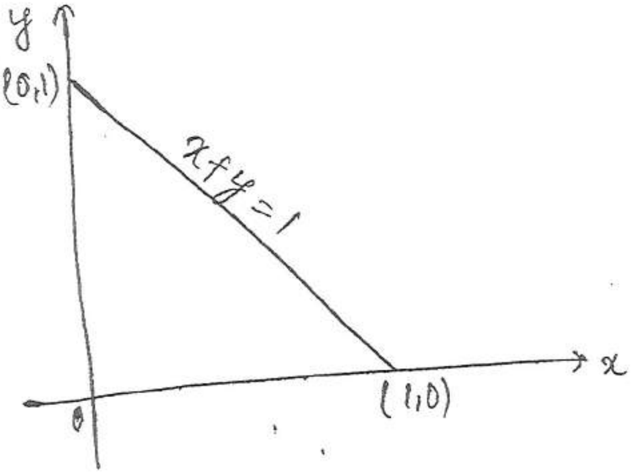
The Jacobian vanishes when $u=0$, that is, when $x = y = 0$, but not otherwise. Also, the origin $(0,0)$ corresponds to the whole line $u=0$ of the uv-plane so that the correspondence ceases to be one-to-o[ne]
In order to exclude $(0,0)$, we note that the given integral exists as the limit, when $h \to 0$ of the integral over the region is bounded by

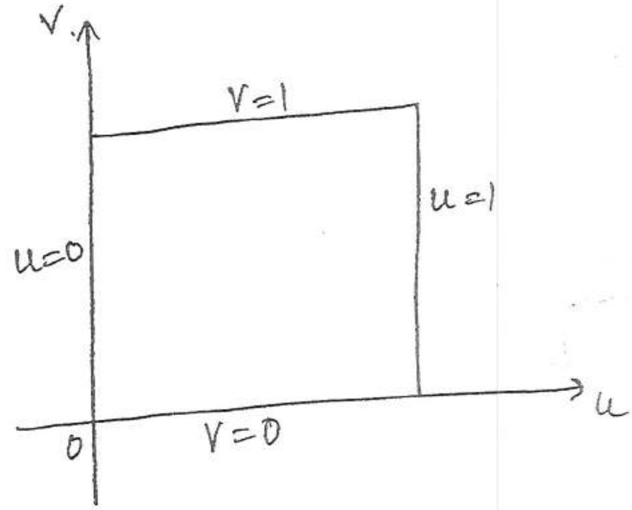
$x+y=1$, $x=h$ and $y=0$ where $h>0$

The transformed region is then bounded by the lines

$$u=1,\quad v=0,\quad \text{and} \quad u(1-v)=h$$

When $h \to 0$, the new region of the uv-plane ten as its limit, to the square bounded by the lines $u=1$, $v=1$, $u=0$ and $v=0$. Thus, the region of integration in xy- and uv-planes are as shown in the following figures:





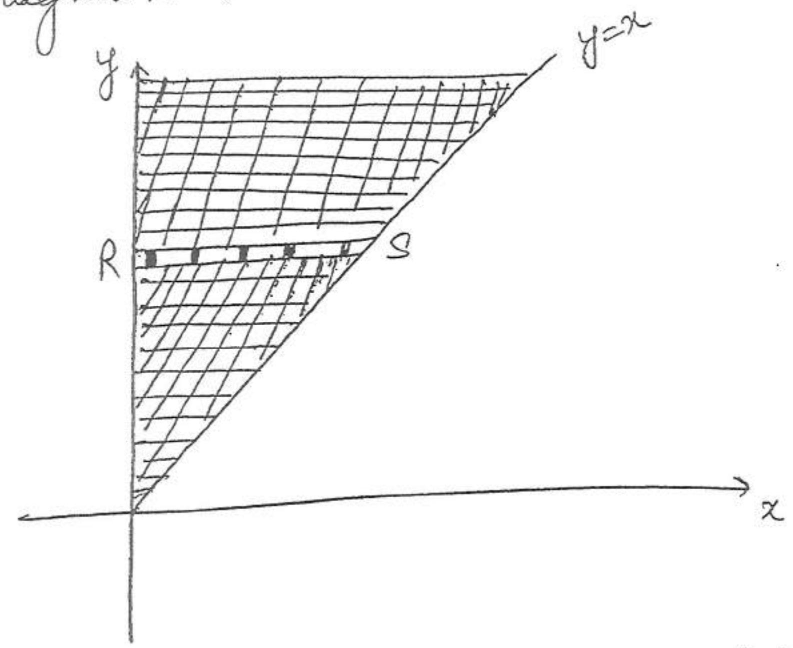
Therefore,

$$\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}}\, dy\, dx = \int_0^1 \int_0^1 e^{\frac{uv}{u}} . u\, du\, dv$$

$$= \int_0^1 e^v \int_0^1 u\, du\, dv = \int_0^1 e^v \left[\frac{u^2}{2}\right]_0^1 dv$$

$$= \frac{1}{2}\int_0^1 e^u\, dv = \frac{1}{2}\left[e^v\right]_0^1 = \frac{1}{2}(e-1)$$

(8) The region of integration is bounded by $x=0$ and $y=x$. The limits of x are from 0 to ∞ and those of y are from x to ∞. The region of integration is shown in the following figure.



On changing the order of integration, we first integrate the integrand, with respect to x, along a horizontal strip RS, which extends from $x=0$ to $x=y$. To cover the region of integration, we then integrate, with respect to y, from $y=0$ to $y=\infty$.

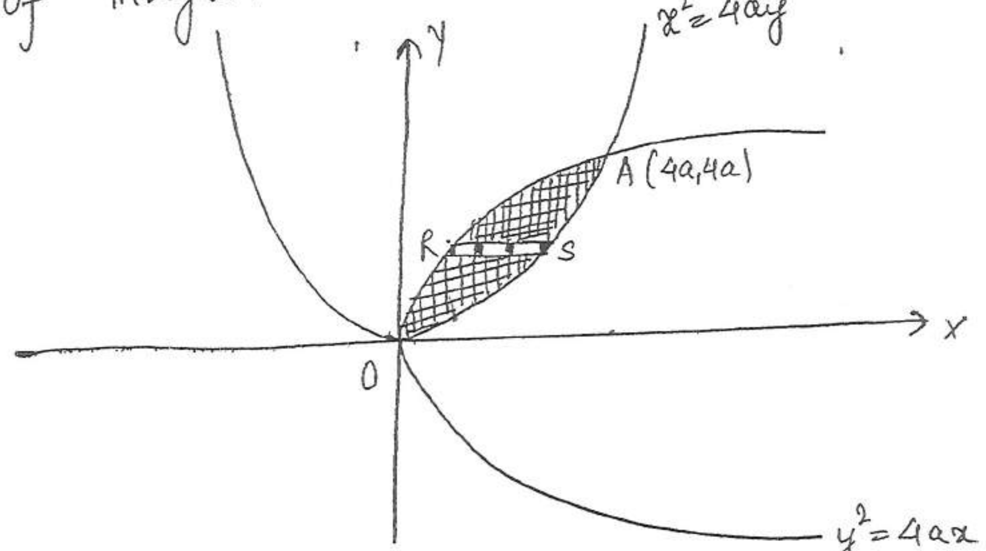
Thus,

$$I = \int_0^\infty \left[\int_0^y \frac{e^{-y}}{y}\,dx\right] dy = \int_0^\infty \frac{e^{-y}}{y}\,[x]_0^y\,dy$$

$$= \int_0^\infty e^{-y}\,dy = \left[-e^{-y}\right]_0^\infty = -\left[\frac{1}{e^y}\right]_0^\infty$$

$$= -(0-1) = 1.$$

9) (i) The given integral is $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$. The integration is first carried out with respect to y and then with respect to x. The region of integration is bounded by $x=0$, $x=4a$, and the parabolas $x^2=4ay$ and $y^2=4ax$. Thus, the region of integration is as shown in the following figure:



$x^2=4ay$
$y=\frac{x^2}{4a}$
$=2\sqrt{ay}$

The coordinates at the point of intersection of the parabolas are A(4a,4a)

On changing the order of integration, we first integrate the integrand, with respect to x, along the horizontal strip RS, which extends from $x=\frac{y^2}{4a}$ to $x=\sqrt{4ay}=2\sqrt{ay}$. To cover the region of integration, we then integrate with respect to y from $y=0$ to $y=4a$. Thus,

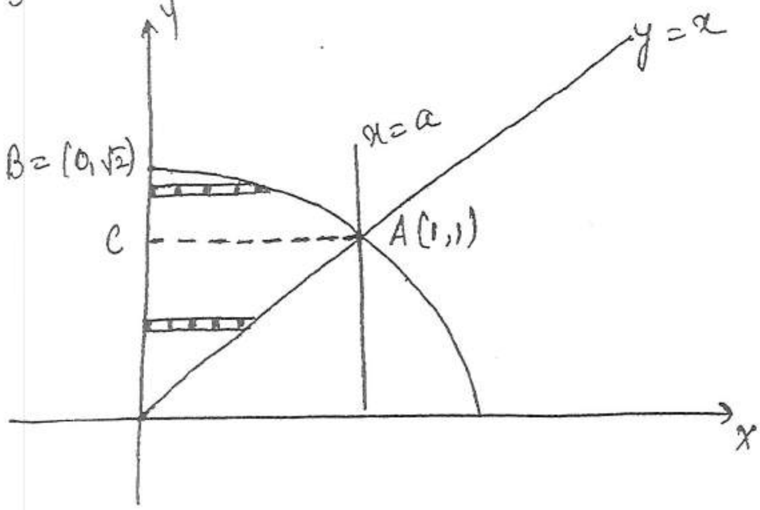
$$\int_0^{4a}\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\left[\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\right]dy = \int_0^{4a}[x]_{\frac{y^2}{4a}}^{2\sqrt{ay}}\,dy$$

$$= \int_0^{4a}\left[2\sqrt{ay}-\frac{y^2}{4a}\right]dy = \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2}-\frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{32a^2}{3}-\frac{16a^2}{3} = \frac{16a^2}{3}$$

(ii) For the given integral, the region of integration is bounded by $x=0$, $x=1$, $y=x$ and the circle $x^2+y^2=2$.

Thus, the region of integration is as shown in the following figure.



The point of intersection of the circle $x^2+y^2=2$ an $x=y$ is $A(1,1)$. Draw AC⊥ Thus, the region of integr is divided into two sub-regions ABCA and Δ A

On changing the order of integration, we first integrate wi respect to x, along the strips parallel to the x-axis.

In the subregion ABCA, the strip extends from $x=0$ to $x=\sqrt{2-y}$ To cover the subregion, we then integrate with respect to y fr $y=1$ to $y=\sqrt{2}$. Thus, the contribution to the integral due to this subregion is

$$I_1 = \int_1^{\sqrt{2}} \left[\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx \right] dy.$$

On the other hand, in the subregion ACO, the strip extends from $x=0$ to $x=y$. To cover this subregion, we then integrate with respect to y from $y=0$ to $y=1$. Thus, the contribution to the integral by this subregion is

$$I_2 = \int_0^1 \left[\int_0^y \frac{y}{\sqrt{x^2+y^2}}\, dx \right] dy$$

Hence, the given integral is equal to

$$I = I_1 + I_2 = \int_1^{\sqrt{2}} \left[\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx \right] dy + \int_0^1 \left[\int_0^y \frac{x}{\sqrt{x^2+y^2}}\, d$$

$$= \int_1 \left[(x^2+y^2)^{1/2} \right]_0 dy + \int_0^1 \left[(x^2+y^2)^{1/2} \right]_0 dy$$

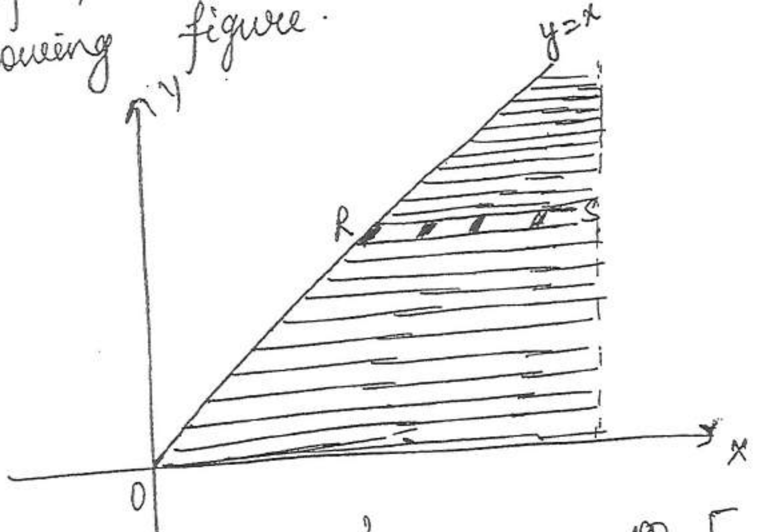
$$= \int_1^{\sqrt{2}} (\sqrt{2} - y)\, dy + \int_0^1 (\sqrt{2}y - y)\, dy$$

$$= \left[\sqrt{2}y - \frac{y^2}{2} \right]_1^{\sqrt{2}} + (\sqrt{2}-1) \left[\frac{y^2}{2} \right]_0^1$$

$$= \frac{2-\sqrt{2}}{2} = 1 - \frac{1}{\sqrt{2}}$$

(ii) The region of integration is bounded by the lines $x=0$, $x=\infty$, $y=0$, and $y=x$

Therefore, the region of integration is as shown in the following figure.



On changing the order of integration, we first integrate with respect to x and then, with respect to y.

$$\text{Thus, } \int_0^\infty \int_0^x x\, e^{-\frac{x^2}{y}}\, dy\, dx = \int_0^\infty \left[\int_y^\infty x\, e^{-\frac{x^2}{y}}\, dx \right] dy \quad \text{——} (1)$$

We first evaluate the inner integral. Substituting $x^2 = t$ we have $2x\,dx = dt$. When $x = y$, $t = y^2$ and when $x = \infty$, $t = \infty$

Therefore,
$$\int_y^\infty x\, e^{-x^2/y}\, dx = \frac{1}{2} \int_{y^2}^\infty e^{-\frac{t}{y}}\, dt = \frac{1}{2} \left[\frac{e^{-\frac{t}{y}}}{-\frac{1}{y}} \right]_{y^2}^\infty$$
$$= \frac{1}{2} y e^{-y}$$

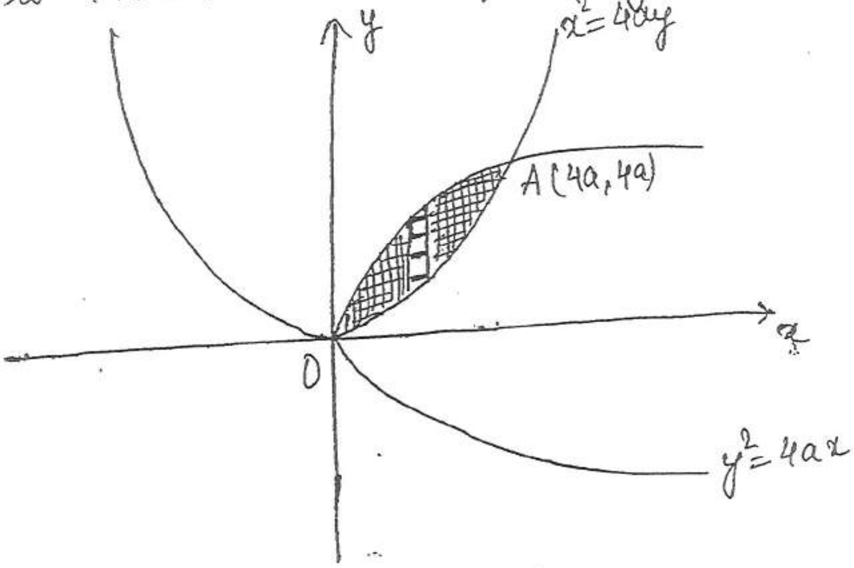
Therefore, (1) reduces to

$$\int_0^{\infty}\int_0^{x} x e^{-\frac{x}{y}}\,dy\,dx = \frac{1}{2}\int_0^{\infty} y e^{-y}\,dy$$

$$= \frac{1}{2}\left[\frac{y e^{-y}}{-1}\right]_0^{\infty} + \frac{1}{2}\int_0^{\infty} e^{-y}\,dy$$

$$= \frac{1}{2}\left[\frac{e^{-y}}{-1}\right]_0^{\infty} = \frac{1}{2}.$$

(10) Solving the equation of the given parabola, we have O(0, and A (4a, 4a) as the points of intersection. The region of integration is shown in the following figure.



Therefore, the required area is

$$A = \int_0^{4a}\left[\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\right] dx = \int_0^{4a} \left[y\right]_{\frac{x^2}{4a}}^{2\sqrt{ax}}$$

$$= \int_0^{4a}\left[2\sqrt{ax} - \frac{x^2}{4a}\right] dx$$

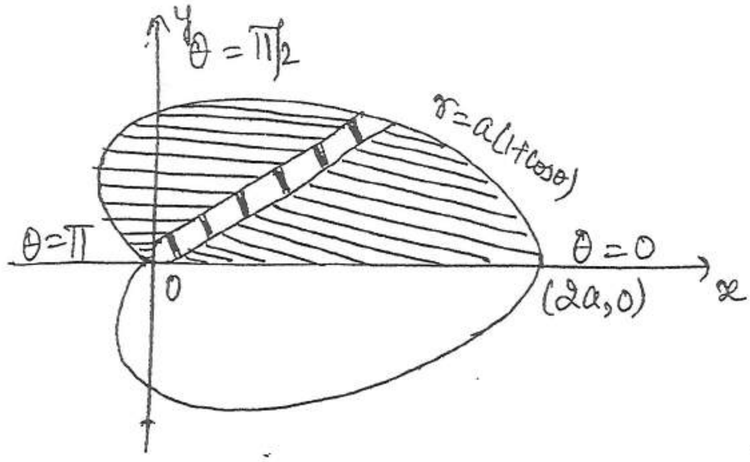
$$= 2\sqrt{a}\int_0^{4a}\sqrt{x}\,dx - \frac{1}{4a}\int_0^{4a} x^2\,dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\,(8a^{3/2}) - \frac{1}{12a}(64a^3) = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16}{3}a^2.$$

(1) (11) The curve passes through the origin and cuts the x-axis at $x = 2a$. Clearly, θ varies from 0 to π and r varies from 0 to $a(1+\cos\theta)$ in the upper half part of the integration region.

(T)



The required area is given by

$$A = 2\int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r\,dr\right]d\theta = 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2 d\theta = 4a^2\int_0^{\pi}\left(\cos^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta$$

$$= 8a^2\int_0^{\pi/2}\cos^4\phi\,d\phi, \quad \theta = 2\phi$$

$$= 8a^2 \cdot \frac{3}{4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi a^2}{2}.$$

(12) (12) The equation of the given elliptical cylinder is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a}$$

Substituting $\frac{x}{a} = r\cos\theta$ and $\frac{y}{b} = r\sin\theta$, this equation yields

$$r^2 = r\cos\theta. \text{ or } r = \cos\theta$$

The required volume is given by

$$V = 4\iint c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\; dy\, dx$$

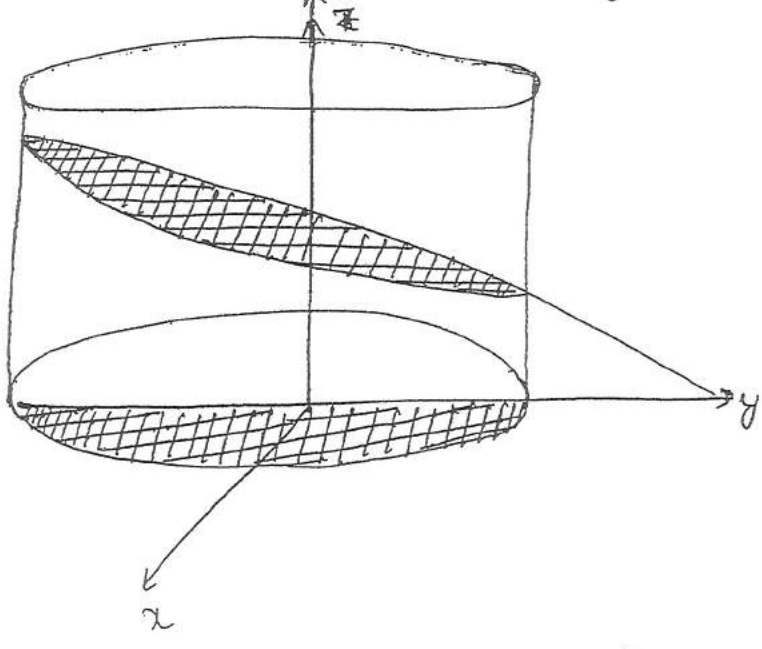
$$= 4abc\int_0^{\pi/2}\int_0^{\cos\theta}\sqrt{1-r^2}\; r\, dr\, d\theta$$

$$= -\frac{4abc}{2}\int_0^{\pi/2}\left[\frac{(1-r^2)^{3/2}}{3/2}\right]_0^{\cos\theta} d\theta$$

$$= -\frac{4abc}{2}\int_0^{\pi/2}(\sin^3\theta - 1)\, d\theta$$

$$= -\frac{4abc}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right] = \frac{2}{9}abc\,(3\pi-4)$$

(13) To find the required volume, $z = 4-y$ is to be integrated over the circle $x^2+y^2=4$ in the xy-pl



To cover the area (half of the circle) in the xy-pla
x varies from 0 to $\sqrt{4-y^2}$ and y varies from -2 to 2

Thus,

$$V = 2\int_{-2}^{2}\left[\int_{0}^{\sqrt{4-y^2}} z\,dx\right]dy$$

$$= 2\int_{-2}^{2}\left[\int_{0}^{\sqrt{4-y^2}} (4-y)\,dx\right]dy$$

$$= 2\int_{-2}^{2} (4-y)\,[x]_0^{\sqrt{4-y^2}}\,dy$$

$$= 2\int_{-2}^{2} (4-y)\sqrt{4-y^2}\,dy$$

$$= 2\left[4\int_{-2}^{2}\sqrt{4-y^2}\,dy - \int_{-2}^{2} y\sqrt{4-y^2}\,dy\right]$$

$$= 8\int_{-2}^{2}\sqrt{4-y^2}\,dy, \text{ second integrand being odd}$$

$$= 16\int_{0}^{2}\sqrt{4-y^2}\,dy, \text{ because of even integrand}$$

$$= 16\left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right]_0^2$$

$$= 16\,(2\sin^{-1}1) = \frac{32\pi}{2} = 16\pi$$

(4) The required surface area is given by

$$S = \iint \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\;dx\,dy$$

But, $\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 2y$

Therefore,

$$S = \iint \sqrt{1+4(x^2+y^2)}\ dx\, dy$$

$$= \iint \sqrt{1+4r^2}\ r\ dr\, d\theta \quad \text{(changing to polar coordinates)}$$

To find the limits, we see that the projection on the plane $z=1$ is the circle $x^2+y^2=1$ or $r^2=1$ and the circle lies between $\theta=0$ and $\theta=2\pi$. Hence,

$$S = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\ r\ d\theta\, dr$$

$$= \frac{1}{8}\int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\ 8r\ d\theta\, dr$$

$$= \frac{1}{8}\int_0^{2\pi}\left[\frac{(1+4r^2)^{3/2}}{3/2}\right]_0^1 d\theta$$

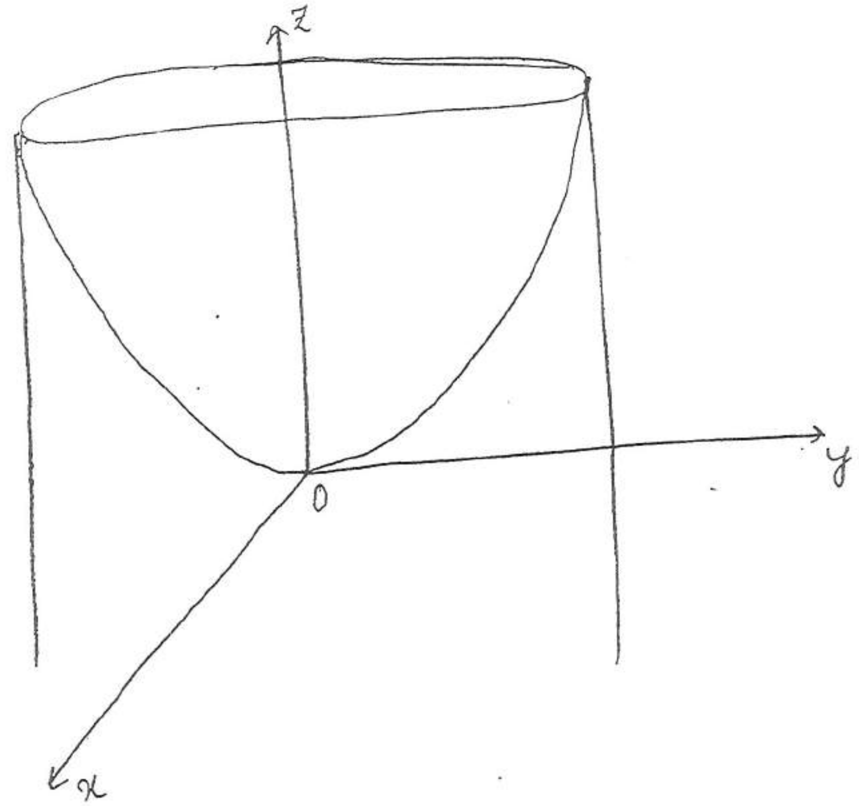
$$= \frac{1}{12}\int_0^{2\pi}(5\sqrt{5}-1)\ d\theta$$

$$= \frac{5\sqrt{5}-1}{12}\left[\theta\right]_0^{2\pi} = \frac{\pi}{6}(5\sqrt{5}-1)$$

(15) The required area is

$$S = 4\iint \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ dx\, dy.$$

where the integration extends over the positive octant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



We have $\frac{\partial z}{\partial x} = \frac{x}{a}$ and $\frac{\partial z}{\partial y} = \frac{y}{b}$. Therefore,

$$S = 4 \iint \left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{1/2} dx\, dy$$

$$= 4ab \iint (1 + \xi^2 + \eta^2)\, d\xi\, d\eta$$

$x = a\xi$, $y = b\eta$ so that $\xi^2 + \eta^2 = 1$

$$= 4ab \int_0^{\pi/2} \int_0^1 (1 + r^2)\, r\, dr\, d\theta$$

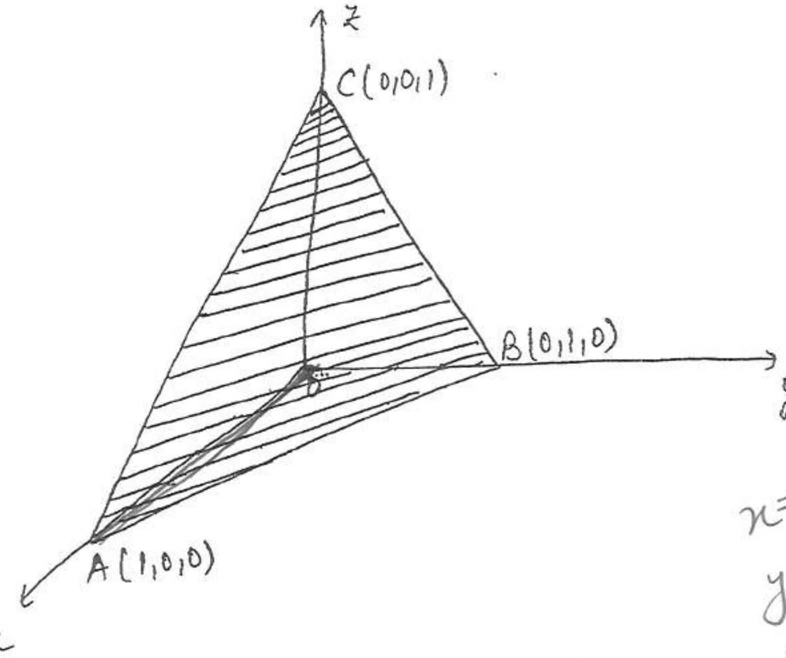
$\xi = r\cos\theta$, $\eta = r\sin\theta$

$$= \frac{2}{3} \pi ab \left(2^{3/2} - 1\right)$$

(16) The region of integration is bounded by the coordinate planes $x=0$, $y=0$, and $z=0$ and the plane $x+y+z=1$. Thus,

$$R = \{(x,y,z);\ x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1\}$$

$$= \{(x,y,z);\ 0 \leq x \leq 1,\ 0 \leq y \leq 1-x;\ 0 \leq z \leq 1-x-y\}$$



$x+y+z=1$
$1-x$

$x=0,1$
$y=0$ to $1-x$
$z=0$ to $1-x-y$

Therefore,

$$\iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3}$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} (x+y+z+1)^{-3}\, dz \right] dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2} \right]_0^{1-x-y} dy\, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left[\frac{1}{(x+y+1)^2} - \frac{1}{4} \right] dy\, dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{-1}{\ldots} - \frac{y}{4} \right]^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left(-\frac{1}{2} - \frac{1-x}{4} + \frac{1}{x+1}\right) dx$$

$$= \frac{1}{2}\int_0^1 \left[-\frac{3}{4} + \frac{x}{4} + \frac{1}{x+1}\right] dx$$

$$= \frac{1}{2}\left[-\frac{3x}{4} + \frac{x^2}{8} + \log(x+1)\right]_0^1$$

$$= \frac{1}{2}\left[-\frac{3}{4} + \frac{1}{8} + \log 2\right] = \frac{1}{2}\log 2 - \frac{5}{16}$$

7) $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{(a^2-x^2-y^2)-z^2}}\, dz\right] dy\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{a^2-x^2-y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\, dx$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} [\sin^{-1} 1]\, dy\right] dx$$

$$= \frac{\pi}{2}\int_0^a [y]_0^{\sqrt{a^2-x^2}} dx = \frac{\pi}{2}\int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \frac{\pi}{2}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{\pi}{2}\left[0 + \frac{a^2}{2}\sin^{-1} 1 - 0\right] = \frac{\pi a^2}{4}\sin^{-1} 1$$

$$= \frac{\pi a^2}{4} \times \frac{\pi}{2} = \frac{\pi^2 a^2}{8}$$

10) 18. Substituting $\frac{x}{a} = X$, $\frac{y}{b} = Y$, $\frac{z}{c} = Z$ so that

$dx = a\,dX$, $dy = b\,dY$, $dz = c\,dZ$, and hence

$dx\,dy\,dz = abc\,dX\,dY\,dZ$. Therefore,

$$I = abc \iiint (1 - X^2 - Y^2 - Z^2)^{1/2}\, dx\,dy\,dz \quad \text{over the region}$$

$$V' = \{(X, Y, Z);\ X \geq 0,\ Y \geq 0,\ Z \geq 0,\ X^2 + Y^2 + Z^2 \leq 1\}$$

Using spherical polar coordinates

$X = r\sin\theta\cos\phi$, $Y = r\sin\theta\sin\phi$, and $Z = r\cos\theta$

the region of integration becomes

$$V'' = \{(r, \theta, \phi);\ 0 \leq r \leq 1,\ 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq \phi \leq \tfrac{\pi}{2}\}.$$

Hence,

$$I = abc \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} (1 - r^2)^{1/2} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= abc \int_0^1 r^2 (1 - r^2)^{1/2} \int_0^{\pi/2} \sin\theta \left[\int_0^{\pi/2} d\phi \right] d\theta\, dr$$

$$= abc \int_0^1 r^2 (1 - r^2)^{1/2} \int_0^{\pi/2} \sin\theta \, [\phi]_0^{\pi/2}\, d\theta\, dr$$

$$= \frac{abc\pi}{2} \int_0^1 r^2 (1 - r^2)^{1/2} \left[\int_0^{\pi/2} \sin\theta\, d\theta \right] dr$$

$$= \frac{abc\pi}{2} \int_0^1 r^2 (1 - r^2)^{1/2} \left[-\cos\theta \right]_0^{\pi/2} dr$$

$$= \frac{abc\pi}{2} \int_0^1 r^2 (1 - r^2)^{1/2} dr \quad \text{——— (1)}$$

But, substituting $r = \sin t$ so that $dr = \cos t\, dt$, we have

$$\int_0^1 r^2(1-r^2)^{1/2}dr = \int_0^{\pi/2} \sin^2 t\sqrt{1-\sin^2 t}\,\cos t\,dt$$
$$= \int_0^{\pi/2} \sin^2 t\cos^2 t\,dt = \frac{1}{4\times 2}\times\frac{\pi}{2} = \frac{\pi}{16}.$$

Hence, (1) reduces to.
$$I = \frac{\pi abc}{2}\left(\frac{\pi}{16}\right) = \frac{\pi^2 abc}{32}.$$

(2) The given region of integration is
$$V = \{(x,y,z);\ x^2+y^2+z^2 \le 1\}$$

Changing to spherical polar coordinates by substituting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$

We get, $x^2+y^2+z^2 = r^2$ and $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

Therefore, the region of integration reduces to
$$V' = \{(r,\theta,\phi);\ 0\le r\le 1,\ 0\le\theta\le\pi,\ 0\le\phi\le 2\pi\}.$$

Hence, $I = \int_0^1\int_0^\pi\int_0^{2\pi} r^{2m+2}\sin\theta\,dr\,d\theta\,d\phi$

$$= \int_0^1 r^{2m+2}\int_0^\pi \sin\theta\left[\int_0^{2\pi} d\phi\right]d\theta\,dr$$

$$= \int_0^1 r^{2m+2}\int_0^\pi \sin\theta\,[\phi]_0^{2\pi}\,d\theta\,dr$$

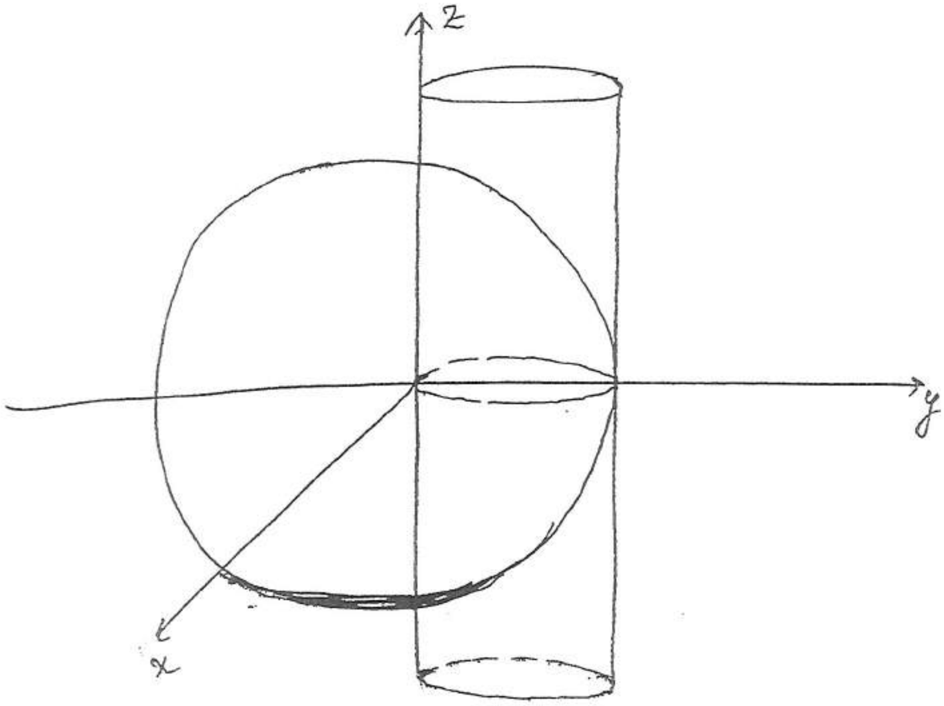
$$= 2\pi\int_0^1 r^{2m+2}\left[\int_0^\pi \sin\theta\,d\theta\right]dr$$

$$= 2\pi\int_0^1 r^{2m+2}\left[-\cos\theta\right]_0^\pi\,dr$$

$$= 4\pi \int_0^1 r^{2m+2}\, dr = 4\pi \left[\frac{r^{2m+3}}{2m+3}\right]_0^1$$

$$= \frac{4\pi}{2m+3}.$$

(20)



The required volume is

$$V = 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{ax-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} dx\,dy\,dz$$

$$= 4\int_0^a \int_0^{\sqrt{ax-x^2}} (a^2-x^2-y^2)^{1/2}\, dy\, dx$$

$$= 4\int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\; r\, dr\, d\theta$$ (changing to polar coordinates)

$$= \frac{4}{2}\int_0^{\pi/2}\int_0^{a\cos\theta} 2r\sqrt{a^2-r^2}\; dr\, d\theta$$

$$= 2\int_0^{\pi/2}\left[\frac{-(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}\left[-a^3\sin^3\theta + a^3\right] d\theta$$

$$= \frac{4}{3}a^3\left[-\frac{2}{3}+\frac{\pi}{2}\right] = \frac{2}{3}a^3\left(\pi - \frac{4}{3}\right).$$