3.6 Group Action

Definition 3.6.1. *Let $(G, \cdot)$ be a group with identity e. Then G is said to act on a set X, if there exists an operator $\star : G \times X \longrightarrow X$ satisfying the following conditions:*

1. *$e \star x = x$, for all $x \in X$, and*
2. *$g \star (h \star x) = (g \cdot h) \star x$, for all $x \in X$ and $g, h \in G$.*

Remark 3.6.2. 1. *Let us assume that X consists of a set of points and let us suppose that the group G acts on X by moving the points. Then, Definition 3.6.1 can be interpreted as follows:*

   (a) *the first condition implies that the identity element of the group does not move any element of X. That is, the points in X remain fixed when they are acted upon by the identity element of G.*

   (b) *the second condition implies that if a point, say $x_0 \in X$, is first acted upon by an element $h \in G$ and then by an element $g \in G$ then the final position of x_0 is same as the position it would have reached if it was acted exactly once by the element $g \cdot h \in G$.*

2. *Fix an element $g \in G$. Then the set $\{g \star x : x \in X\} = X$.*
   For, otherwise, there exist $x, y \in X$ such that $g \star x = g \star y$. Then, by definition,

$$x = e \star x = (g^{-1} \cdot g) \star x = g^{-1} \star (g \star x) = g^{-1} \star (g \star y) = (g^{-1} \cdot g) \star y = e \star y = y.$$

   That is, g just permutes the elements of X. Or equivalently, each $g \in G$ gives rise to a one-one, onto function from X into itself.

3. *There may exist $g, h \in G$, with $g \neq h$ such that $g \star x = h \star x$, for all $x \in X$,.*

Before proceeding further with definitions and results related with group action, let us look at a few examples.

Example 3.6.3. 1. *Consider the dihedral group $D_6 = \{e, r, \ldots, r^5, f, rf, \ldots, r^5 f\}$, with $r^6 = e = f^2$ and $rf = fr^5$. Here, f stands for the vertical flip and r stands for counter clockwise rotation by an angle of $\frac{\pi}{3}$. Then D_6 acts on the labeled edges/vertices of a regular hexagon by permuting the labeling of the edges/vertices (see Figure 3.5).*

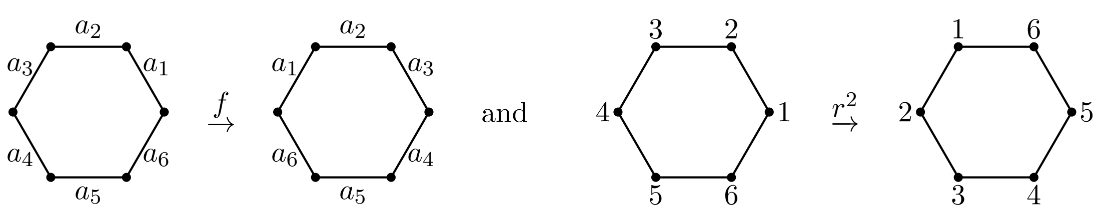

Figure 3.5: Action of f on labeled edges and of r^2 on labeled vertices of a regular hexagon.

2. *Let X denote the set of ways of coloring the vertices of a square with two colors, say, Red and Blue. Then X equals the set of all functions $h : \{1, 2, 3, 4\} \longrightarrow \{Red, Blue\}$, where the vertices south-west, south-east, north-east and north-west are respectively, labeled as $1, 2, 3$ and 4. Then, using Lemma 1.7.1, $|X| = 16$. The distinct colorings have been depicted in Figure 3.6, where R stands for the vertex colored "Red" and B stands for the vertex colored "Blue". For example, the figure labeled x_9 in Figure 3.6 corresponds to $h(1) = R = h(4)$ and $h(2) = B = h(3)$. Now, let us denote the permutation (1234) by r and the permutation $(12)(34)$ by f. Then the dihedral group $D_4 = \{e, r, r^2, r^3, f, rf, r^2f, r^3f\}$ acts on the set X. For example,*

   (a) *x_1 and x_{16} are mapped to itself under the action of every element of D_4. That is, $g \star x_1 = x_1$ and $g \star x_{16} = x_{16}$, for all $g \in G$.*

   (b) *$r \star x_2 = x_5$ and $f \star x_2 = x_3$.*

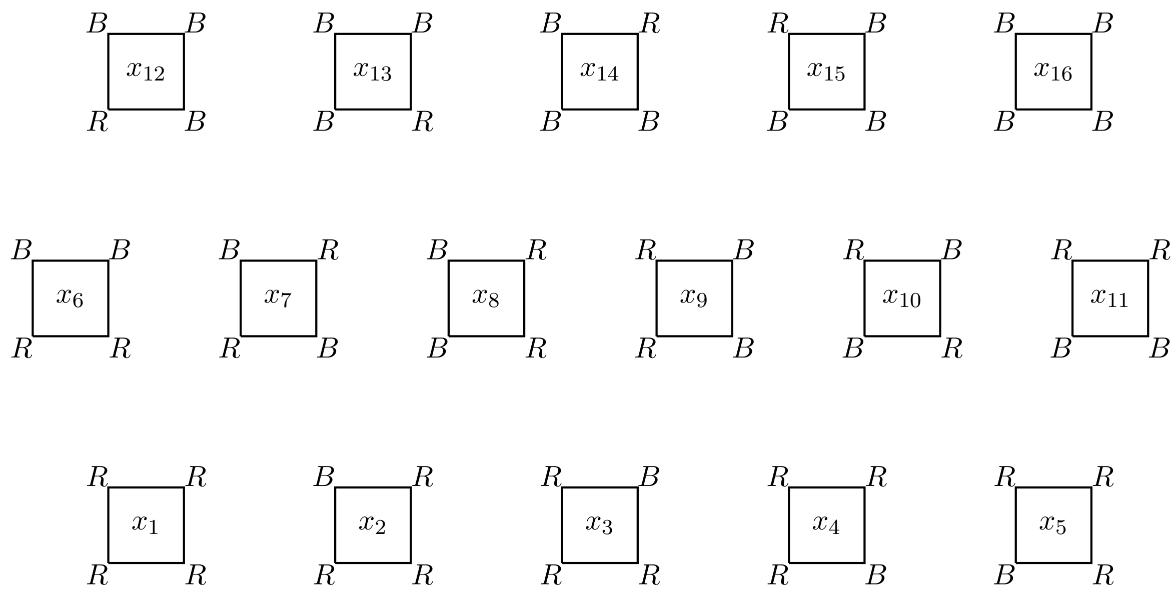

Figure 3.6: Coloring the vertices of a square.

There are three important sets associated with a group action. We first define them and then try to understand them using an example.

Definition 3.6.4. *Let G act on a set X. Then*

1. *for each fixed $x \in X$, $\mathcal{O}(x) = \{g \star x : \ g \in G\} \subset X$ is called the* Orbit *of x.*

2. *for each fixed $x \in X$, $G_x = \{g \in G : g \star x = x\} \subset G$ is called the* Stabilizer *of x in G.*

3. *for each fixed $g \in G$, $F_g = \{x \in X : g \cdot x = x\} \subset X$ is called the* Fix *of g.*

Let us now understand the above definitions using the following example.

Example 3.6.5. *Consider the set X given in Example 3.6.3.2. Then using the depiction of the set X in Figure 3.6, we have*

$$\mathcal{O}(x_2) = \{x_2, x_3, x_4, x_5\}, \; G_{x_2} = \{e, rf\}, \quad and \;\; F_{rf} = \{x_1, x_2, x_4, x_7, x_{10}, x_{13}, x_{15}, x_{16}\}.$$

The readers should compute the different sets by taking other examples to understand the above defined sets.

We now state a few results associated with the above definitions. The proofs are omitted as they can be easily verified.

Proposition 3.6.6. *Let G act on a set X.*

1. *Then for each fixed $x \in X$, the set G_x is a subgroup of G.*
2. *Define a relation, denoted $\sim$, on the set X, by $x \sim y$ if there exists $g \in G$, such that $g \star x = y$. Then prove that $\sim$ defines an equivalence relation on the set X. Furthermore, the equivalence class containing $x \in X$ equals $\mathcal{O}(x) = \{g \star x : \; g \in G\} \subset X$.*
3. *Fix $x \in X$ and let $t \in \mathcal{O}(x)$. Then $\mathcal{O}(x) = \mathcal{O}(t)$. Moreover, if $g \star x = t$ then $G_x = g^{-1} G_t g$.*

Let G act on a set X. Then Proposition 3.6.6 helps us to relate the distinct orbits of X under the action of G with the cosets of G. This is stated and proved as the next result.

Theorem 3.6.7. *Let a group G act on a set X. Then for each fixed $x \in X$, there is a one-to-one correspondence between the elements of $\mathcal{O}(x)$ and the set of all left cosets of G_x in G. In particular,*

$$|\mathcal{O}(x)| = [G \; : \; G_x] = \text{ the number of left cosets of } G_x \text{ in } G.$$

Moreover, if G is a finite group then $|G| = |\mathcal{O}(x)| \cdot |G_x|$, for all $x \in X$.

Proof. Let S be the set of distinct left cosets of G_x in G. Then $S = \{gG_x : \; g \in G\}$ and $|S| = [G : \; G_x]$. Consider the map $\tau : \; S \longrightarrow \mathcal{O}(x)$ by $\tau(gG_x) = g \star x$. Let us first check that this map is well-defined.

So, suppose that the left cosets gG_x and hG_x are equal. That is, $gG_x = hG_x$. Then, using Theorem 3.4.5 and the definition of group action, one obtains the following sequence of assertions:

$$gG_x = hG_x \Longleftrightarrow h^{-1}g \in G_x \Longleftrightarrow (h^{-1}g) \star x = x \Longleftrightarrow h^{-1} \star (g \star x) = x \Longleftrightarrow g \star x = h \star x.$$

Thus, by definition of the map τ, one has $gG_x = hG_x \Longleftrightarrow \tau(gG_x) = \tau(hG_x)$. Hence, τ is not only well-defined but also one-one.

To show τ is onto, note that for each $y \in \mathcal{O}(x)$, there exists an $h \in G$, such that $h \star x = y$. Also, for this choice of $h \in G$, the coset $hG_x \in S$. Therefore, for this choice of $h \in G$, $\tau(hG_x) = h \star x = y$ holds. Hence, τ is onto.

Therefore, we have shown that τ gives a one-to-one correspondence between $\mathcal{O}(x)$ and the set S. This completes the proof of the first part. The other part follows by observing that by definition ; $[G : G_x] = \dfrac{|G|}{|G_x|}$, for each subgroup G_x of G whenever $|G|$ is finite. ■

The following lemmas are immediate consequences of Proposition 3.6.6 and Theorem 3.6.7. We give the proof for the sake of completeness.

Lemma 3.6.8. *Let G be a finite group acting on a set X. Then, for each $y \in X$,*

$$\sum_{x \in \mathcal{O}(y)} |G_x| = |G|.$$

Proof. Recall that, for each $x \in \mathcal{O}(y)$, $|\mathcal{O}(x)| = |\mathcal{O}(y)|$. Hence, using Theorem 3.6.7, one has $|G| = |G_x| \cdot |\mathcal{O}(x)|$, for all $x \in X$. Therefore,

$$\sum_{x \in \mathcal{O}(y)} |G_x| = \sum_{x \in \mathcal{O}(y)} \frac{|G|}{|\mathcal{O}(x)|} = \sum_{x \in \mathcal{O}(y)} \frac{|G|}{|\mathcal{O}(y)|} = \frac{|G|}{|\mathcal{O}(y)|} \sum_{x \in \mathcal{O}(y)} 1 = \frac{|G|}{|\mathcal{O}(y)|} |\mathcal{O}(y)| = |G|.$$ ■