Partial Differential Equations (MA20103)

Assignment – 3

Second order PDE

Q1. Solve the following second order homogenous PDE with constant coefficients (symbols have usual meanings, say, $r=\dfrac{\partial z}{\partial x};s=\dfrac{\partial^2 z}{\partial x\partial y};t=\dfrac{\partial z}{\partial y}$)

(i) $25r-40s+16t=0$

(ii) $r+(a+b)s+abt=xy$

(iii) $r-t=x-y$

(iv) $r+t+2s=xy$

(v) $2r-3s-2t=0$

(vi) $r-4s+4t=0$

(vii) $r+3s+2t=2x+3y$

(viii) $r-s-2t=(y-1)e^x$

(ix) $r-5s+4t=\sin(4x+y)$

(x) $r+t=\cos mx\cos ny$

Q2. Classify and reduce the following equations in to canonical form

(i) $\dfrac{\partial^2 z}{\partial x^2}-x^2\dfrac{\partial^2 z}{\partial y^2}=0$ (also find general solution)

(ii) $\dfrac{\partial^2 z}{\partial x^2}+x^2\dfrac{\partial^2 z}{\partial y^2}=0$ (also find general solution)

(iii) $16\dfrac{\partial^2 z}{\partial x^2}-y^{10}\dfrac{\partial^2 z}{\partial y^2}=5y^9\dfrac{\partial z}{\partial y}$

(iv) $y^2\dfrac{\partial^2 z}{\partial x^2}-2xy\dfrac{\partial^2 z}{\partial x\partial y}+x^2\dfrac{\partial^2 z}{\partial y^2}=\dfrac{y^2}{x}\dfrac{\partial z}{\partial x}+\dfrac{x^2}{y}\dfrac{\partial z}{\partial y}$

(v) $(y-1)\dfrac{\partial^2 z}{\partial x^2}-\left(y^2-1\right)\dfrac{\partial^2 z}{\partial x\partial y}+y(y-1)\dfrac{\partial^2 z}{\partial y^2}+\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}=2ye^{2x}(1-y)^3$

Q3. Show that the solution of the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

satisfying the conditions

(i) $u \to 0$ as $t \to \infty, \forall\ x$

(ii) $u=0$ for $x=0$ and $x=a\ \forall\ t>0$

(ii) $u=x$ when $t=0$ and $0<x<a$ is

$$u(x,\mathrm{t})=\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin(x)\exp\left[-\left(\frac{n\pi}{a}\right)^2 t\right]$$

Q4. Solve $\nabla^2 u=0$

subject to $u(x,0)=0, u(x,a)=0, u(x,y)\to 0$ as $x\to\infty$ where $x\geq 0$ and $0\leq y\leq a$

Q5. Solve the 2 dimensional Laplace equation in polar co-ordinates r and θ in the region $0\leq r\leq a, 0\leq\theta\leq 2\pi$ subject to

(i) u remains finite as $r\to 0$

(ii) $u=\sum_n c_n\cos(n\theta)$ on $r=a$

Q6. A tightly stretched string with fixed end point $x=0$ and $x=l$ is initially in a position given by $u=u_0\sin^3\frac{\pi x}{l}$. If it is released from rest from this position, show that the displacement is given by

$$u(x,t)=\frac{u_0}{4}\left(3\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l}-\cos\frac{3\pi ct}{l}\sin\frac{3\pi x}{l}\right)$$