R.M.

MA 51002: Measure Theory and Integration
End Semester Examination (Spring 2016)
Time: 3 Hours, Full Marks: 50. Number of students = 85.

Answer **all** the ten problems. Numbers at the right hand side after each question denote marks. No clarification will be entertained during the examination.

(1) Let $(C(\Omega), \|.\|)$ be the set of all complex valued bounded continuous functions defined on Ω with the sup norm $\|.\|$. Define the metric $d(f,g) = \|f-g\|$. Prove that $C(\Omega)$ is a complete metric space. [5]

(2) In the following parts (a) and (b), determine whether the given $\mathcal{F}$ is an algebra, a σ-algebra, or neither.
(a) $\mathcal{F} = \{A \subset \mathbb{Z} : A \text{ is finite }\}$ (with $X = \mathbb{Z}$).
(b) $\mathcal{F} = \{A \subset \mathbb{Z} : \text{either} A \text{ or } A^c \text{ is finite }\}$ (with $X = \mathbb{Z}$). [2+3]

(3) (a) Show that if a set $E \subset \mathbb{R}$ has positive outer measure, then there is a bounded subset of E that also has positive outer measure.
(b) Show that each open set in $\mathbb{R}$ is an F_σ set, i.e. it can be written as countable union of closed sets. [2+3]

(4) Let $E_1, E_2, \ldots$ be a sequence of measurable sets. Show that if $m(E_n) < \frac{1}{2^n}$ then $\chi_{E_n} \to 0$ a.e.. Use this to prove the following: Let f be an integrable function over a set S. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ shuch that $\int_E |f| < \varepsilon$ whenever $E \subset S$ is a measurable set and $m(E) < \delta$. [2+3]

(5) If f, g are measurable functions, prove that fg is a measurable function. Give an example of a pointwise convergent sequence of bounded measurable functions $\{f_n\}$ such that the limit function f is bounded, but $\int_a^b f_n \nrightarrow \int_a^b f$. Explain why the Bounded Convergence Theorem is not applicable in this case. [2+3]

(6) Let f, g be two measurable functions on S. Show that the set $A = \{x : f(x) > g(x)\}$ is measurable. Suppose $m(A) > 0$. Show that there exists $\delta > 0$ such that $m(A_\delta) > 0$ where $A_\delta = \{x : f(x) - g(x) > \delta\}$. [2+3]

(7) State Lebesgue Dominated Convergence Theorem. Show that Dominated Convergence Theorem implies the Bounded Convergence Theorem. Give an example of a sequence $\{f_n\}$ which satisfies the conditions of the Dominated Convergence Theorem but doesn't satisfy the conditions of the Bounded Convergence Theorem. [1+2+2]

(8) State Fatou's Lemma. Give an example for which the inequality in Fatou's Lemma is strict. Deduce Bounded Convergence Theorem from Fatou's Lemma. [1+1+3]

(9) Let μ be Lebesgue measure on $[0,1]$. Let $g : [0,1] \to \mathbb{R}$ be Borel measurable, and set $g_n(x) = g\left(\frac{nx}{n+1}\right)$. Assume that g is bounded, and that g is continuous at x a.e.
(a) Show that $\int |g_n - g| \, d\mu \to 0$ as $n \to \infty$.
(b) Show by an example that this conclusion may fail if we drop the hypothesis that g is bounded. [3+2]

(10) Compute the following limits with proper justification:
(a) $\lim_{n\to\infty} \int_0^\infty (1 + x/n)^{-n} \sin(x/n) \, dx$
(b) $\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n} \, dx$ [3+2]