(1) $A = \begin{pmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 6 & 9 & -3 & 3 \end{pmatrix} \xrightarrow{R_3 - 3R_1} \begin{pmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\cancel{\because \det A = \begin{vmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 6 & 9 & -3 & 3 \end{vmatrix} = 3 \begin{vmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 2 & 3 & -1 & 1 \end{vmatrix} = 0}$, rank $A < 3$

$\begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} = -9 \neq 0$. Therefore Rank $A = 2$

(2) $A = \begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 2R_1}$

$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 2 & 6 & 0 & 4 & 2 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, R is a row reduced echelon matrix.

$\xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R$

(3) (i) $A = \begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow[R_3 - 5R_1]{R_2 - 3R_1}$

$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow[R_4 - 3R_2]{R_3 - 2R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

$\xrightarrow{R_{34}} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - 2R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = R,$

R is a row-echelon matrix.

And $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$ and $\det R = 0$ ∴ Rank of $R = 3$. Hence Rank $A = 3$

(4.) $A = \begin{pmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{pmatrix}$

Then, $\det A = 0$

i.e. $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} = 0$.

or, $\begin{vmatrix} 1+2x & x & x \\ 1+2x & 1 & x \\ 1+2x & x & 1 \end{vmatrix} = 0 \qquad C_1' = C_1 + C_2 + C_3$

or, $(1+2x)\begin{vmatrix} 1 & x & x \\ 1 & 1 & x \\ 1 & x & 1 \end{vmatrix} = 0$

or, $(1+2x)\begin{vmatrix} 1 & x & x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix} = 0 \qquad \begin{matrix} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{matrix}$

or, $(1+2x)(1-x)^2 = 0$

$\therefore \quad x = -1/2, 1, 1.$

(5.) α, β, γ are the roots of the equation $x^3 + qx + r = 0$
therefore, $\alpha + \beta + \gamma = 0$. [$\because$ Coefficient of x^2 is zero]

Now, $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix} \qquad C_1' = C_1 + C_2 + C_3$

$= (\alpha+\beta+\gamma)\begin{vmatrix} 1 & \beta & \gamma \\ 1 & \gamma & \alpha \\ 1 & \alpha & \beta \end{vmatrix} = 0.$ Therefore, rank of $A < 3$.

Now, $\begin{vmatrix} \alpha & \beta \\ \beta & \gamma \end{vmatrix} = \alpha\gamma - \beta^2$ | since, α, β, γ are in A.P. $2\beta = \gamma + \alpha$.

α, β, γ. $\beta - \alpha = \gamma - \beta$. $2\beta = \gamma + \alpha$.

$$= \alpha\gamma - \left(\frac{\gamma+\alpha}{2}\right)^2$$

$$= \frac{1}{4}\left[4\alpha\gamma - (\gamma+\alpha)^2\right]$$

$$= -\frac{1}{4}\left[(\gamma-\alpha)^2\right] \neq 0. \quad [\gamma \neq \alpha \text{ since the roots are distinct}]$$

$\therefore$ Rank of $A = 2$.

③ (ii) $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow[R_3 + R_2]{R_1 - R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = R.$

Now, $\det R = 0$ and $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

Therefore Rank of $R = 2$ and hence Rank of $A = 2$.

(iii) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2}$

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1/2 \\ 0 & 2 & -1 \end{pmatrix} \xrightarrow[R_3 - 2R_2]{R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} = R$

R is row-echelon matrix.

Now, $\det R = 0$, $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$, Therefore rank of $R = 2$

and hence Rank of $A = 2$.

6. (a)

$$x + y - z = 0$$
$$2x - 3y + z = 0$$
$$x - 4y + 2z = 0$$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 1 & -4 & 2 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 3 \end{pmatrix} \xrightarrow{R_2 + R_3} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

The reduced system of equations is

$$\left.\begin{aligned} x + y - z &= 0 \\ -5y + 3z &= 0 \end{aligned}\right\} \begin{aligned} &\text{No of equ}^n = 2 \\ &\text{No. of Var} = 3 \end{aligned}$$

So. No of equ^n < No of var. and Difference = 1 so put 1 var = c

, The system has infinitely many solutions.

let $z = c$, then $y = \frac{3}{5}c$, $x + \frac{3}{5}c - c = 0$

or, $x = \frac{2}{5}c$

Then, $\left(\frac{2}{5}c, \frac{3}{5}c, c\right)$ is the sol^n of the given system of equations, where c is an arbitrary constant.

(b)

$$x + y - z = 0$$
$$2x + 4y - z = 0$$
$$3x + 2y + 2z = 0$$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 4 & -1 \\ 3 & 2 & 2 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 5 \end{pmatrix} \xrightarrow{R_3 + \frac{1}{2}R_2}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 5\frac{1}{2} \end{pmatrix}$$

$-1 + 2$

The reduced system of equations is

$$x + y - z = 0$$
$$2y + z = 0$$
$$\frac{11}{2} z = 0$$

The system of eqn's has many solutions.

let $z = c$, then $y = -c/2$, $x = c/2 + c = \frac{3}{2}c$

$\therefore$ $\left(\frac{3}{2}c, -\frac{c}{2}, c\right)$ is ~~a~~ the soln of the given system of equations where c is an arbitrary constant

(c)
$$x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 = 2$$
$$3x_1 - 9x_2 + 7x_3 - x_4 + 3x_5 = 7$$
$$2x_1 - 6x_2 + 7x_3 + 4x_4 - 5x_5 = 7$$

~~The~~ ~~co~~ The augmented matrix,

$$\tilde{A} = \begin{pmatrix} 1 & -3 & 2 & -1 & 2 & 2 \\ 3 & -9 & 7 & -1 & 3 & 7 \\ 2 & -6 & 7 & 4 & -5 & 7 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1}$$

$$\begin{pmatrix} 1 & -3 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 3 & 6 & -9 & 3 \end{pmatrix} \xrightarrow{R_3 - 3R_2} \begin{pmatrix} 1 & -3 & 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The system of eqn's is reduced to

$$x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 = 2$$
$$x_3 + 2x_4 - x_5 = 1$$

Let $x_2 = a$, $x_4 = b$, $x_5 = c$. where a, b, c are arbitrary constants, then $x_3 = 1 - 2b + 3c$, $x_1 = 3a + 5b - 8c$.

The general solution is

$$x_1 = 3a + 5b - 8c$$

$$x_2 = a, \quad x_3 = 1 - 2b + 3c, \quad x_4 = b, \quad x_5 = c.$$

(7) (a)

$$x - y + z = 1$$

$$x + 2y + 4z = a$$

$$x + 4y + 6z = a^2$$

The augmented matrix is

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & 4 & a \\ 1 & 4 & 6 & a^2 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 3 & 3 & a-1 \\ 0 & 5 & 5 & a^2-1 \end{pmatrix} \xrightarrow[\frac{1}{5}R_3]{\frac{1}{3}R_2}$$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & \frac{a-1}{3} \\ 0 & 1 & 1 & \frac{a^2-1}{5} \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & \frac{a-1}{3} \\ 0 & 0 & 0 & \frac{a^2-1}{5} - \frac{a-1}{3} \end{pmatrix}$$

The reduced system of equations is

$$x - y + z = 1$$

$$y + z = \frac{a-1}{3}$$

$$0 = \frac{a^2-1}{5} - \frac{a-1}{3}$$

for consistency of the above system of equations, $\frac{a^2-1}{5} - \frac{a-1}{3} = 0$

or, $3a^2 - 5a + 2 = 0$

or, $(a-1)(3a-2)=0$

or, $a=1, 2/3$

When $a=1$, system of equations takes the form

$$x-y+z=1$$
$$y+z=0$$

let $z=c$, then $y=-c$, $x=1-2c$

The general solution is $(1-2c, -c, c)$

ie. $c(-2,-1,1)+(1,0,0)$, $c\in\mathbb{R}$.

When $a=2/3$,

$$x-y+z=1$$
$$y+z=-\frac{1}{9}$$

let $z=c$, then $y=-\frac{1}{9}-c$, $x=\frac{8}{9}-2c$

The general solution is $\left(\frac{8}{9}-2c, -\frac{1}{9}-c, c\right)$

ie. $c(-2,-1,1)+\left(\frac{8}{9},-\frac{1}{9},0\right)$, $c\in\mathbb{R}$.

(b)
$$x+y+z=1$$
$$2x+3y-z=a+1$$
$$2x+y+5z=a^2+1.$$

The augmented matrix is,

$$\bar{A}=\begin{pmatrix}1&1&1&1\\2&3&-1&a+1\\2&1&5&a^2+1\end{pmatrix}\xrightarrow[R_3-2R_1]{R_2-2R_1}\begin{pmatrix}1&1&1&1\\0&1&-3&a-1\\0&-1&3&a^2-1\end{pmatrix}$$

$$\xrightarrow{R_3+R_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & a-1 \\ 0 & 0 & 0 & a^2+a-2 \end{pmatrix}$$

Therefore, the reduced system of equations is

$$x+y+z=1$$
$$0.x+y-3z=a-1$$
$$0.x+0.y+0.z=a^2+a-2$$

For the consistency of the above system of equations, $a^2+a-2=0$

or, $(a+2)(a-1)=0$

or, $a=-2,1$

For $\underline{a=1}$, $x+y+z=1$

$y-3z=0$

Take, $z=c$, $y=3c$, $x=1-4c$

The general solⁿ is $(1-4c, 3c, c)$

ie. $c(-4,3,1)+(1,0,0)$, $c\in\mathbb{R}$

for $\underline{a=-2}$, $x+y+z=1$

$y-3z=-3$

Take, $z=c$, $y=-3+3c$, $x=4-4c$.

$\therefore$ The general solⁿ is $(4-4c, -3+3c, c)$

ie. $c(-4,3,1)+(4,-3,0)$, $c\in\mathbb{R}$.

8. (a) $x+y+z=1$

$x+2y-z=b$

$5x+7y+az=b^2$

The system has a unique solution if the coefficient determinant be non-zero.

The coefficient determinant $= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & a \end{vmatrix} = a-1$

If $a-1 \neq 0$, ie. if $a \neq 1$, the system has only one solution.

If $a=1$, the system has either no solution or many solutions.

When $a=1$, the coefficient matrix of the system is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & 1 \end{pmatrix}$ and the augmented matrix of the system is $\tilde{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & b \\ 5 & 7 & 1 & b^2 \end{pmatrix}$

$\tilde{A} \xrightarrow[R_3-5R_1]{R_2-R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & b-1 \\ 0 & 2 & -4 & b^2-5 \end{pmatrix} \xrightarrow[R_3-2R_2]{R_1-R_2}$

$\begin{pmatrix} 1 & 0 & 3 & -b+2 \\ 0 & 1 & -2 & b-1 \\ 0 & 0 & 0 & b^2-2b-3 \end{pmatrix}$

If $b^2-2b-3=0$, then rank of $\tilde{A}$ = rank of A and therefore the system is consistent.

If $b^2-2b-3\neq 0$, then rank of $\tilde{A}=3$, rank of $A=2$ and since rank of $A\neq$ rank of $\tilde{A}$, the system is inconsistent.

Therefore, if $a=1$, $b\neq 1,-3$ the system has no solution; and if $a=1$, $b=-1$, or if $a=1$, $b=3$ the system has many solutions.

(b) $2x+3y+5z=9$

$7x+3y-2z=8$

$2x+3y+az=6$

The system has a unique solution if the coefficient determinant be non-zero.

The coefficient determinant $=\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & a \end{vmatrix}=15(5-a)$

If $a\neq 5$, the system has only one solution.

If $a=5$, the ~~coefficient~~ ~~matrix of~~ system has either no solution or many solutions.

When $a=5$, the coefficient matrix of the system is

$$A=\begin{pmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & a \end{pmatrix}$$

and the augmented matrix of the system is, $\tilde{A}=\begin{pmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & a & 6 \end{pmatrix}$

$$\tilde{A} = \begin{pmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & 6 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - \frac{7}{2}R_1} \begin{pmatrix} 2 & 3 & 5 & 9 \\ 0 & -\frac{15}{2} & -\frac{39}{2} & -\frac{47}{2} \\ 0 & 0 & 0 & 6-9 \end{pmatrix}$$

If $b-9=0$ i.e. $b=9$, then rank of $\tilde{A}$ = rank of A and therefore the system is consistent.

If $b-9 \neq 0$, then rank of $\tilde{A} = 3$, and rank of $A = 2$ and since rank $A \neq$ rank $\tilde{A}$, the system is inconsistent.

Therefore, if $a=5$, $b \neq 9$, the system of equations has no solutions and if $a=5$, $b=9$ the system of equations has infinitely many solutions.

(9.) $(3k-8)x + 3y + 3z = 0$

$3x + (3k-8)y + 3z = 0$

$3x + 3y + (3k-8)z = 0$

For the given system of equations to have a non-trivial solutions, the determinant of the coefficient matrix should be zero.

i.e. $$\begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} 3k-2 & 3 & 3 \\ 3k-2 & 3k-8 & 3 \\ 3k-2 & 3 & 3k-8 \end{vmatrix} = 0 \qquad [c_1' = c_1 + c_2 + c_3]$$

or, $(3K-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3K-8 & 3 \\ 1 & 3 & 3K-8 \end{vmatrix} = 0$

or, $(3K-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3K-11 & 0 \\ 0 & 0 & 3K-11 \end{vmatrix} = 0$ $\quad \left[\begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{array} \right]$

or, $(3K-2)(3K-11)^2 = 0$

or, $K = 2, \frac{11}{3}, \frac{11}{3}$

(10) $ax+by+cz=0$
$bx+cy+az=0$
$cx+ay+bz=0$

For the given system of equations to have non-trivial solution, the determinant of the coefficient matrix is zero.

ie. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

or, $\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$ $\quad [R_1' = R_1 + R_2 + R_3]$

or, $(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

or, $(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix} = 0$ $\quad \left[\begin{array}{l} C_2' = C_2 - C_1, \\ C_3' = C_3 - C_1 \end{array} \right]$

or, $(a+b+c)\left\{(c-b)(b-c)-(a-b)(a-c)\right\}=0$

or, $(a+b+c)\left(-a^2-b^2-c^2+ab+bc+ca\right)=0$

or, i.e. $a+b+c=0$ | or, $a^2+b^2+c^2=ab+bc+ca$

or, $a^2+b^2+c^2-ab-bc-ca=0$

or, $\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]=0$

or, $a=b,\ b=c,\ c=a$

or, $\overline{a=b=c}$

Hence the given system has a non-trivial solution if $a+b+c=0$ or $a=b=c$.

(10)

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$\Rightarrow \quad a(bc-a^2) - b(b^2-ac) + c(ab-c^2) = 0$

or. $\quad abc - a^3 - b^3 + abc + abc - c^3 = 0$

or $\quad 3abc - (a^3+b^3+c^3) = 0$

or, $\quad (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$

or, $\quad (a+b+c)\left\{\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\right\} = 0$

or, $\quad (a+b+c) = 0$

$\qquad$ or $\quad \frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right] = 0$

or, $\quad a+b+c=0 \quad$ or $\begin{cases} (a-b) = 0 \\ (b-c) = 0 \\ (c-a) = 0 \end{cases}$

$\therefore \quad a+b+c=0 \quad$ or $\quad a=b=c$

Proved.

(11)

$$A = \begin{pmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} 2-i & 3 & -1-3i \\ -5 & -i & 4+2i \end{pmatrix}$$

$$A^* = \bar{A}^T = \begin{pmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{pmatrix}$$

$$AA^* = \begin{pmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{pmatrix}_{2\times 3} \begin{pmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{pmatrix}_{3\times 2}$$

$$= \begin{pmatrix} 5+9+10 & -5(2+i)-3i+(4+2i)(-1+3i) \\ -5(2-i)+3i-(1+3i)(4-2i) & +25+1+20 \end{pmatrix}$$

$$= \begin{pmatrix} 24 & -20+2i \\ -20-2i & 46 \end{pmatrix}.$$

<u>Now</u>. $$(\overline{AA^*})^T = \begin{pmatrix} 24 & -20+2i \\ -20-2i & 46 \end{pmatrix}$$

$$= AA^*$$

Hence AA^* is Hermitian matrix.

(12) Express any matrix A as the sum of Hermitian matrix and skew-Hermitian matrix

as $$A = \frac{1}{2}[A + A^*] + \frac{1}{2}[A - A^*]$$

$\frac{1}{2}[A + A^*]$ is Hermitian part

$\frac{1}{2}[A - A^*]$ is skew-Hermitian part.

(13). Let A is a real and non symmetric matrix of order 3.

then let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$.

Now $$A - A^T = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} - \begin{pmatrix} a_1 & a_4 & a_7 \\ a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a_2 - a_4 & a_3 - a_7 \\ a_4 - a_2 & 0 & a_6 - a_8 \\ a_7 - a_3 & a_8 - a_6 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_1 & b_2 \\ -b_1 & 0 & b_3 \\ -b_2 & -b_3 & 0 \end{pmatrix}$$

where $b_1 = a_2 - a_4$, $b_2 = a_3 - a_7$, $b_3 = a_6 - a_8$,

Now, $(A - A^T)$ is skew symmetric matrix of odd order. Hence its determinant is 0.

Hence rank $(A - A^T) < 3$.

Now consider the minor

$$\begin{vmatrix} 0 & b_1 \\ -b_1 & 0 \end{vmatrix} = b_1^2, \quad \begin{vmatrix} 0 & b_3 \\ -b_3 & 0 \end{vmatrix} = b_3^2, \quad \begin{vmatrix} 0 & b_2 \\ -b_2 & 0 \end{vmatrix} = b_2^2$$

Now if rank $(A - A^T) = 1$ then.

$b_1^2 = 0,\ b_2^2 = 0,\ b_3^2 = 0$

$\Rightarrow b_1 = b_2 = b_3 = 0$

Hence A become a symmetric matrix which is a contradiction to the fact A is non-symmetric

Hence rank $(A - A^T) = 2$ Proved.

(4) Consider the product

$$A^t(A+B)B^t$$
$$= (A^tA + A^tB)B^t$$
$$= (I + A^tB)B^t \quad [\because A^tA = I]$$
$$= B^t + A^tBB^t$$
$$= [B^t + A^t] \quad [\because BB^t = I]$$
$$= (A+B)^t$$

Now taking det both side we get,

$$\det\{A^t(A+B)B^t\} = \det(A+B)^t$$
$$= \det(A+B)$$

or, $\det A^t . \det(A+B) \det B^t = \det(A+B)$

or. $\det A . \det(A+B) . \det B = \det(A+B)$

or. $\det A . \det(A+B)(-\det A) = \det(A+B)$

$[\because \det A + \det B = 0]$

or, $\det[A+B)\{1 + (\det A)^2]\} = 0$

Since A is orthogonal matrix

$|\det A| = 1$

$2 \det(A+B) = 0 \Rightarrow \det(A+B) = 0$ Pf.

(18) Same as (17).

(16) Since A is skew-Hermitian matrix $A = -\bar{A}^T$.
Now it is given that $(I+A)$ is non-singular
hence $(I+A)^{-1}$ exist.

Now. $(I+A)^{-1}(I-A)\{\overline{(I+A)^{-1}(I-A)}\}^T$

$$= (I+A)^{-1}(I-A)\{(\overline{I-A})^T(I+\bar{A})^{-T}\}$$

$$= (I+A)^{-1}(I-A)(I-\bar{A}^T)(I+\bar{A}^T)^{-1}$$

$$= (I+A)^{-1}\{I+A\bar{A}^T-A-\bar{A}^T\}\{I+\bar{A}^T\}^{-1}$$

$$= (I+A)^{-1}\{I+A\bar{A}^T\}\{I+\bar{A}^T\}^{-1}$$

$$= \{(I+A)^{-1}+(I+A)^{-1}A\bar{A}^T\}\{I+\bar{A}^T\}^{-1}$$

$$= \{(I+\bar{A}^T)(I+A)\}^{-1}+(I+A)^{-1}A\bar{A}^T(I+\bar{A}^T)^{-1}$$

$$= \{I+A\bar{A}^T\}^{-1}+(I+A)^{-1}A\bar{A}^T(I+\bar{A}^T)^{-1}$$

$$= \{(I-A^2)^{-1}\}+(I+A)^{-1}(-A^2)(I-A)^{-1}$$

$$= \{(I-A)(I+A)\}^{-1}+(I+A)^{-1}(-A^2)(I-A)^{-1}$$

$$= (I+A)^{-1}(I-A)^{-1}+(I+A)^{-1}(-A^2)(I-A)^{-1}$$

$$= (I+A)^{-1}\{(I-A)^{-1}-A^2(I-A)^{-1}\}$$

$$= (I+A)^{-1}(I-A^2)(I-A)^{-1}$$

$$= \cancel{(I+A)^{-1}}\cancel{(I+A)}\cancel{(I-A)}\cancel{(I-A)^{-1}} = I \text{ Pf}$$

(17) $(I+A)^{-1}(I-A)\left\{\overline{(I+A)^{-1}(I-A)}\right\}^T = I$

or, $(I+A)^{-1}(I-A)\,(\overline{I-A})^T\left(\left(\overline{(I+A)^{-1}}\right)\right)^T = I$

or, $(I+A)^{-1}(I-A)\,(I-\bar{A}^T)\,(I+\bar{A}^T)^{-1} = I$

or, $(I+A)^{-1}(I-A)\,(I-\bar{A}^T) = (I+\bar{A}^T)$

or, $(I-A)(I-\bar{A}^T) = (I+A)(I+\bar{A}^T)$

or, $\cancel{I}-A-\bar{A}^T+\cancel{A\bar{A}^T} = \cancel{I}+A+\bar{A}^T+\cancel{A\bar{A}^T}$

or, $2(A+\bar{A}^T) = 0$

$\Rightarrow A = -\bar{A}^T$

Hence A is skew-Hermitian.

(8) Since A is unitary matrix

$\therefore A\bar{A}^T = \bar{A}^T A = I.$

Let $B = (I+A)^{-1}(I-A)$.

$\therefore \bar{B}^T = \left(\left(\overline{(I+A)^{-1}(I-A)}\right)\right)^T$

$= \left[(I+\bar{A})^{-1}(I-\bar{A})\right]^T$

$= (I-\bar{A}^T)(I+\bar{A}^T)^{-1}$

$= (A\bar{A}^T-\bar{A}^T)(A\bar{A}^T+\bar{A}^T)^{-1}$

$= (A-I)\bar{A}^T(\bar{A}^T)^{-1}(A+I)^{-1}$

$= (A-I)(A+I)^{-1} = -(I-A)(I+A)^{-1}$

Now $A^2 - I = (A+I)(A-I) = (A-I)(A+I)$

$\therefore \quad (A+I)(A-I) = (A-I)(A+I)$

or $\quad (A-I) = (A+I)^{-1}(A-I)(A+I)$

or, $\quad (A-I)(A+I)^{-1} = (A+I)^{-1}(A-I)$.

Hence

$$\bar{B}^T = (A+I)^{-1}(A-I)$$
$$= -(I+A)^{-1}(I-A)$$
$$= -B$$

Hence B is skew-Hermitian matrix.

(18) Let $A = \begin{pmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{pmatrix}$, $(\bar{A})^T = \begin{pmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{pmatrix}$

Now. $A\bar{A}^T = \begin{pmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{pmatrix} \begin{pmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{pmatrix}$

$$= \begin{pmatrix} \alpha^2+\gamma^2+\beta^2+\delta^2 & \cancel{(\alpha+i\gamma)(\beta-i\delta)-(\alpha+i\gamma)(\beta+i\delta)} \\ \cancel{(\beta+i\delta)(\alpha-i\gamma)-(\alpha-i\gamma)(\beta+i\delta)} & \beta^2+\delta^2+\alpha^2+\gamma^2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^2+\beta^2+\gamma^2+\delta^2 & 0 \\ 0 & \alpha^2+\beta^2+\gamma^2+\delta^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad [\text{Since } A \text{ is unitary}]$$

Hence $\alpha^2+\beta^2+\gamma^2+\delta^2 = 1$ pf

20. $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$, where $a = e^{\frac{2i\pi}{3}}$.

Now: $M\bar{M} = 3I$ we have to show.

$$\bar{M} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \bar{a}^2 & \bar{a} \\ 1 & \bar{a} & \bar{a}^2 \end{bmatrix}$$

Now. $M\bar{M} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \bar{a}^2 & \bar{a} \\ 1 & \bar{a} & \bar{a}^2 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 1+\bar{a}^2+\bar{a} & 1+\bar{a}+\bar{a}^2 \\ 1+a+a^2 & 1+a^2\bar{a}^2+a\bar{a} & 1+a^2\bar{a}+a\bar{a}^2 \\ 1+a+a^2 & 1+a\bar{a}^2+a^2\bar{a} & 1+a\bar{a}+a^2\bar{a}^2 \end{bmatrix}$$

Now. $\bar{a} = e^{-\frac{2i\pi}{3}}$, $\bar{a}^2 = e^{-\frac{4i\pi}{3}}$.

$a^3 = 1$, Hence: a is the cube root of unity

$\therefore\ 1+a+a^2 = 0$

$\bar{a}^3 = 1$ Hence $1+\bar{a}+\bar{a}^2 = 0$.

$$\therefore M\bar{M} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I$$

Hence $\frac{\bar{M}}{3} = M^{-1}$ (Proved).