Chapter 1

Properties of Integers and Basic Counting

We will use the following notation throughout these notes.

1. The empty set, denoted $\emptyset$, is the set that has no element.
2. $\mathbb{N} := \{0, 1, 2, \ldots\}$, the set of Natural numbers;
3. $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of Integers;
4. $\mathbb{Q} := \{\frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0\}$, the set of Rational numbers;
5. $\mathbb{R} :=$ the set of Real numbers; and
6. $\mathbb{C} :=$ the set of Complex numbers.

For the sake of convenience, we have assumed that the integer 0, is also a natural number. This chapter will be devoted to understanding set theory, relations, functions and the principle of mathematical induction. We start with basic set theory.

1.1 Basic Set Theory

We have already seen examples of sets, such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ at the beginning of this chapter. For example, one can also look at the following sets.

Example 1.1.1. *1.* $\{1, 3, 5, 7, \ldots\}$, *the set of odd natural numbers.*

2. $\{0, 2, 4, 6, \ldots\}$, *the set of even natural numbers.*

3. $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$, *the set of odd integers.*

4. $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$, *the set of even integers.*

5. $\{0, 1, 2, \ldots, 10\}$.

6. $\{1, 2, \ldots, 10\}$.

7. $\mathbb{Q}^+ = \{x \in \mathbb{Q} : \ x > 0\}$, *the set of positive rational numbers.*

8. $\mathbb{R}^+ = \{x \in \mathbb{R} : \ x > 0\}$, *the set of positive real numbers.*

9. $\mathbb{Q}^* = \{x \in \mathbb{Q} : \ x \neq 0\}$, *the set of non-zero rational numbers.*

10. $\mathbb{R}^* = \{x \in \mathbb{R} : \ x \neq 0\}$, *the set of non-zero real numbers.*

We observe that the sets that appear in Example 1.1.1 have been obtained by picking certain elements from the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. These sets are example of what are called "subsets of a set", which we define next. We also define certain operations on sets.

Definition 1.1.2 (Subset, Complement, Union, Intersection)**.** *1. Let A be a set. If B is a set such that each element of B is also an element of the set A, then B is said to be a subset of the set A, denoted $B \subseteq A$.*

2. Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$, denoted $A = B$.

3. Let A be a subset of a set Ω. Then the complement of A in Ω, denoted A', is a set that contains every element of Ω that is not an element of A. Specifically, $A' = \{x \in \Omega : x \notin A\}$.

4. Let A and B be two subsets of a set Ω. Then their

(a) union, denoted $A \cup B$, is the set that exactly contains all the elements of A and all the elements of B. To be more precise, $A \cup B = \{x \in \Omega : x \in A \text{ or } x \in B\}$.

(b) intersection, denoted $A \cap B$, is the set that exactly contains those elements of A that are also elements of B. To be more precise, $A \cap B = \{x \in \Omega : x \in A \text{ and } x \in B\}$.

Example 1.1.3. *1. Let A be a set. Then $A \subseteq A$.*

2. *The empty set is a subset of every set.*

3. *Observe that* $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

4. *As mentioned earlier, all examples that appear in Example 1.1.1 are subsets of one or more sets from* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ *and* $\mathbb{C}$.

5. *Let* A *be the set of odd integers and* B *be the set of even integers. Then* $A \cap B = \emptyset$ *and* $A \cup B = \mathbb{Z}$. *Thus, it also follows that the complement of* A, *in* $\mathbb{Z}$, *equals* B *and vice-versa.*

6. *Let* $A = \{\{b, c\}, \{\{b\}, \{c\}\}\}$ *and* $B = \{a, b, c\}$ *be subsets of a set* Ω. *Then* $A \cap B = \emptyset$ *and* $A \cup B = \{a, b, c, \{b, c\}, \{\{b\}, \{c\}\}\ \}$.

Definition 1.1.4 (Cardinality). *A set* A *is said to have finite cardinality, denoted* $|A|$, *if the number of distinct elements in* A *is finite, else the set* A *is said to have infinite cardinality.*

Example 1.1.5. 1. *The cardinality of the empty set equals* 0. *That is,* $|\emptyset| = 0$.

2. *Fix a positive integer* n *and consider the set* $A = \{1, 2, \ldots, n\}$. *Then* $|A| = n$.

3. *Let* $S = \{2x \in \mathbb{Z} : x \in \mathbb{Z}\}$. *Then* S *is the set of even integers and it's cardinality is infinite.*

4. *Let* $A = \{a_1, a_2, \ldots, a_m\}$ *and* $B = \{b_1, b_2, \ldots, b_n\}$ *be two finite subsets of a set* Ω, *with* $|A| = m$ *and* $B| = n$. *Also, assume that* $A \cap B = \emptyset$. *Then, by definition it follows that*
$$A \cup B = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n\}$$
and hence $|A \cup B| = |A| + |B|$.

5. *Let* $A = \{a_1, a_2, \ldots, a_m\}$ *and* $B = \{b_1, b_2, \ldots, b_n\}$ *be two finite subsets of a set* Ω. *Then* $|A \cup B| = |A| + |B| - |A \cap B|$. *Observe that Example 1.1.5.4 is a particular case of this result, when* $A \cap B = \emptyset$.

6. *Let* $A = \{\{a_1\}, \{a_2\}, \ldots, \{a_m\}\}$ *be a collection of singletons of a set* Ω. *Now choose an element* $a \in \Omega$ *such that* $a \neq a_i$, *for any* $i, 1 \leq i \leq n$. *Then verify that the set* $B = \{S \cup \{a\} : S \in A\}$ *equals* $\{\{a, a_1\}, \{a, a_2\}, \ldots, \{a, a_m\}\}$. *Also, observe that* $A \cap B = \emptyset$ *and* $|B| = |A|$.

Definition 1.1.6 (Power Set). *Let* A *be a subset of a set* Ω. *Then the set that contains all subsets of* A *is called the power set of* A *and is denoted by* $\mathcal{P}(A)$ *or* 2^A.

Example 1.1.7. 1. *Let* $A = \emptyset$. *Then* $\mathcal{P}(\emptyset) = \{\emptyset, A\} = \{\emptyset\}$.

2. *Let* $A = \{\emptyset\}$. *Then* $\mathcal{P}(A) = \{\emptyset, A\} = \{\emptyset, \{\emptyset\}\}$.

3. *Let* $A = \{a, b, c\}$. *Then* $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

4. *Let* $A = \{\{b, c\}, \{\{b\}, \{c\}\}\}$. *Then* $\mathcal{P}(A) = \{\emptyset, \{\{b, c\}\}, \{\{\{b\}, \{c\}\}\}, \{\{b, c\}, \{\{b\}, \{c\}\}\}\ \}$.