POWER SERIES:

A series of the form

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots = \sum_{n=0}^{\infty} a_n (z-a)^n$$

is called power series in $(z-a)$.

For every power series $\sum_{n=0}^{\infty} a_n (z-a)^n$, there exists a non-negative real number r such that for every $|z-a|<r$, the series is absolutely convergent and $|z-a|>r$, the series is not convergent.
The number r is called the radius of convergence of the power series and the circle $|z-a| = r$ is called the circle of convergence.

No general statement can be made about the convergence of a power series on the circle of convergence.

We write $r=\infty$ if the series converges for all z & $r=0$ if the series converges only for $z=a$.

The radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

Can be calculated as:

$$r = \frac{1}{\lim_{n\to\infty} \left|\left(\frac{a_{n+1}}{a_n}\right)\right|} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

$$= \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$$

Example:

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{\lfloor n}$

radius of convergence

$$r = \frac{1}{\lim_{n\to\infty}\left(\frac{1}{n+1}\right)} = \infty$$

Hence ∞ is the radius of convergence.

TAYLOR'S THEOREM :

Let $f(z)$ be analytic inside and on a simple closed curve C. Let z_0 and z_0+h be two points inside C. Then

$$f(z_0+h) = f(z_0) + hf'(z_0) + \frac{h^2}{\lfloor 2}f''(z_0) + \cdots + \frac{h^n}{\lfloor n}f^{(n)}(z_0) + \cdots$$

OR

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots + \frac{f^{(n)}(z_0)}{\lfloor n}(z-z_0)^n + \cdots$$

If $z_0 = 0$, then the Taylor's series is called Maclaurin's series.

Here $f^{(n)}(z_0)$ can be calculated as:

$$f^{(n)}(z_0) = \frac{\lfloor n}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

with counterclockwise integration around a simple closed path C that contains z_0 in it.

Example: Find Taylor's series about $z=0$ of the function

$$f(z) = \frac{1}{1-z}$$

$f'(z) = \frac{1}{(1-z)^2}$ $\quad f'(0) = 1$

$f''(z) = \frac{2}{(1-z)^3}$ $\quad f''(0) = 2$

$f'''(z) = \frac{1\cdot 2\cdot 3}{(1-z)^4}$ $\quad f'''(0) = \lfloor 3$

$\vdots$

$$f^{(n)}(0) = \lfloor\underline{n}$$

$$f(z) = 1 + z + z^2 + \cdots + z^n + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} z^n$$

Radius of convergence:

$$r = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{1} = 1$$

Hence the series converges for all z : $|z|<1$

__Example:__ Taylor's series of $\dfrac{z+2}{1-z^2}$ about $z_0=0$.

$$\frac{z+2}{1-z^2} = \frac{3}{2}\cdot\frac{1}{1-z} + \frac{1}{2}\frac{1}{1+z}$$

$$= \frac{3}{2}\left[\sum_{n=0}^{\infty} z^n\right] + \frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n z^n\right]$$

valid in $|z|<1$

$$= 2 + z + 2z^2 + z^3 + \cdots$$

Example: Maclaurin's series of

$$\frac{1}{z+3i}$$

$$\frac{1}{z+3i} = \frac{1}{3i} \frac{1}{\left(1+\frac{z}{3i}\right)}$$

$$= \frac{1}{3i} \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(3i)^n} \qquad \text{valid in } \left|\frac{z}{3i}\right| < 1$$

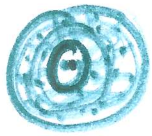
$$\text{or } |z| < 3$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(3i)^{n+1}}$$

$$= -\frac{i}{3} + \frac{1}{9} z + \frac{i}{27} z^2 - \frac{1}{81} z^3 + \cdots$$

In many applications we encounter functions that are not analytic at some points, or in some region of the complex plain and consequently Taylor's series cannot be employed in the neighboured of such points. However another series representation, called Laurent's series, can frequently be found in which both positive and negative powers of $(z-z_0)$ exists. Such series is valid for functions that are analytic in and on a circular annulus $R_1 \leq |z-z_0| \leq R_2$.

A function $f(z)$ analytic in an annulus



$$R_1 \leq |z - z_0| \leq R_2$$

may be represented by the expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

where

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

C is any simple closed curve in the region of analyticity enclosing the inner boundary

$$|z - z_0| = R_1$$

The coefficient of the term $\frac{1}{z - z_0}$, which is c_{-1} in our notation is called the residue of the function $f(z)$. The negative powers of the Laurent's series are referred to as the principal part of $f(z)$.

REMARK: Suppose $f(z)$ is analytic everywhere inside the circle $|z - z_0| = R_1$. Then by Cauchy's theorem $c_n = 0$ for $n \leq -1$. In this case Laurent's series reduces to the Taylor's series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

Example: Expand $\frac{1}{1-z}$

a) in nonnegative powers of z

b) in negative powers of z

Sol:

a) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z|<1)$$

b) $$\frac{1}{1-z} = \frac{1}{-z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} \qquad \text{for } \left|\frac{1}{z}\right|<1$$

or $|z|>1$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Example: Find all Taylor and Laurent series of

$$f(z) = \frac{-2z+3}{z^2-3z+2} \quad \text{with center } 0.$$

Sol:

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}$$

$$-\frac{1}{z-2} = \frac{1}{2\left(1-\frac{z}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}} z^n$$

valid in $|z|<2$

$$-\frac{1}{z-2} = -\frac{1}{z\left(1-\frac{2}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n}$$

valid in $\frac{2}{|z|}<1$

or $|z|>2$.

I: For $|z|<1$:

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n$$

II: For $1 < |z| < 2$:

$$f(z) = -\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n$$

$$= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n$$

III: For $|z|>2$:

$$f(z) = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

$$= -\sum_{n=0}^{\infty} (2^n+1) \frac{1}{z^{n+1}}$$

EXAMPLE: Find the Laurent series of $f(z) = \frac{1}{1+z}$ for $|z|>1$.

$$\frac{1}{1+z} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \frac{1}{z}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n\right]$$

EXAMPLE: Find Laurent series

$$f(z) = \frac{1}{(z-1)(z-2)} \quad \text{for } 1<|z|<2$$

$$= -\frac{1}{z-1} + \frac{1}{z-2}$$

$$= \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$= -\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)$$

$$\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

where $c_n = \begin{cases} -1 & n \le -1 \\ \frac{1}{2^{n+1}} & n \ge 0 \end{cases}$

EXAMPLE:

$$f(z) = \frac{z}{z^2-3z+2} = \frac{z}{(z-1)(z-2)}$$

Expand in powers of z in the region

a) $|z|<1$ b) $1<|z|<2$ c) $|z|>2$

a)

$$f(z) = \frac{-1}{z-1} + \frac{2}{z-2}$$

$$= \frac{1}{1-z} + \frac{2}{-2\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{1-z} - \frac{1}{1-\frac{z}{2}}$$

$$= \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} \frac{z^n}{2^n}$$

$$= \sum_{n=0}^{\infty} \left(1-\frac{1}{2^n}\right) z^n$$

b)

$$f(z) = \frac{-1}{z\left(1-\frac{1}{z}\right)} - \sum_{n=0}^{\infty} \frac{z^n}{2^n}$$

$$= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{z^n}{2^n}$$

c)

$$f(z) = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \frac{2}{z} \cdot \frac{1}{\left(1-\frac{2}{z}\right)}$$

$$= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{2^{n+1}}{z^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(2^{n+1}-1)}{z^{n+1}}$$

EXAMPLE: Find the Laurent's series of the function

$$f(z) = \frac{1}{z^3(1-z)}$$

about $z=1$ in the region $0<|z-1|<1$

$$f(z) = \frac{-1}{(z-1)\left[1+(z-1)\right]^3}$$

$$= -\frac{1}{(z-1)} \cdot \left[1 - 3(z-1) + \frac{3\cdot 4}{\lfloor 2}(z-1)^2 - \frac{3\cdot 4\cdot 5}{\lfloor 3}(z-1)^3 + \cdots\right]$$

$$= -\frac{1}{(z-1)} + 3 - 6(z-1) + 10(z-1)^2 - \cdots$$

$$(1+z)^{\alpha} = 1 + \alpha z + \frac{\alpha(\alpha-1)}{\lfloor 2} z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{\lfloor 3} z^3 + \cdots$$

$|z| < 1$

α is a negative integer