**Example:** Find the general solution of the boundary value problem

$$y'' = y + x \qquad x \in [0,1] \qquad \text{--- (1)}$$

$$y(0) = 0 \qquad y(1) = 0 \qquad \text{--- (2)}$$

with the shooting method. Use Runge-Kutta method of second order to solve the IVP with $h = 0.2$.

**Sol:** We set-up the two IVPs:

$$u'' = u + x; \qquad u(0) = 0 \qquad u'(0) = 0 \qquad \text{--- (3)}$$

$$v'' = v + x; \qquad v(0) = 0 \qquad v'(0) = 1 \qquad \text{--- (4)}$$

and write the general solution of the BVP as

$$y(x) = \theta u(x) + (1-\theta) v(x)$$

and determine $\theta$ so that

$$y(1) = 0 .$$

Both the equations (3) & (4) can be converted into the following type of system of equations:

$$\begin{bmatrix} p(x) \\ q(x) \end{bmatrix}' = \begin{bmatrix} q(x) \\ p(x) + x \end{bmatrix} \qquad \text{--- (5)}$$

$p(x) = y$
$q(x) = y'$

Applying the RK second order method to (5):

$$\bar{K}_1 = \begin{bmatrix} f_1(x_n, p_n, q_n) \\ f_2(x_n, p_n, q_n) \end{bmatrix} = \begin{bmatrix} q_n \\ p_n + x_n \end{bmatrix}$$

$$\bar{K}_2 = \begin{bmatrix} f_1(x_n + h, p_n + \bar{K}_1^{(1)} h, q_n + \bar{K}_1^{(2)} h) \\ f_2(x_n + h, p_n + \bar{K}_1^{(1)} h, q_n + K_1^{(2)} h) \end{bmatrix} = \begin{bmatrix} q_n + h(p_n + x_n) \\ x_n + h + p_n + q_n h \end{bmatrix}$$

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n \\ q_n \end{bmatrix} + \frac{h}{2}\left[\bar{K}_1 + \bar{K}_2\right]$$

$$= \begin{bmatrix} p_n \\ q_n \end{bmatrix} + \frac{h}{2}\begin{bmatrix} 2q_n + h(p_n + x_n) \\ 2p_n + 2x_n + h + hq_n \end{bmatrix}$$

$$= \begin{bmatrix} p_n\left(1+\frac{h^2}{2}\right) + hq_n \\ hp_n + q_n\left(1+\frac{h^2}{2}\right) \end{bmatrix} + \begin{bmatrix} \frac{h^2}{2}x_n \\ h\,x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{h^2}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} 1.02\,p_n + 0.2\,q_n \\ 0.2\,p_n + 1.02\,q_n \end{bmatrix} + \begin{bmatrix} 0.02\,x_n \\ 0.2\,x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} 1.02 & 0.2 \\ 0.2 & 1.02 \end{bmatrix}\begin{bmatrix} p_n \\ q_n \end{bmatrix} + \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix}x_n + \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}$$

For the system (3): $p_n = u_n$, $q_n = u'_n$ $p_0 = 0$, $q_0 = 0$

At: $x_1 = 0.2$:

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}$$

At $x_2 = 0.4$:

$$\begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1.02 & 0.2 \\ 0.2 & 1.02 \end{bmatrix}\begin{bmatrix} 0 \\ 0.02 \end{bmatrix} + \begin{bmatrix} 0.02 \\ 0.2 \end{bmatrix}\begin{bmatrix} 0.2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}$$

$$= \begin{bmatrix} 0.008 \\ 0.0804 \end{bmatrix}$$

At $x_3 = 0.6$:

$$\begin{bmatrix} p_3 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0.03224 \\ 0.183604 \end{bmatrix}$$

At $x = 0.8$:

$$\begin{bmatrix} p_4 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0.081606 \\ 0.333728 \end{bmatrix}$$

At $x = 1$:

$$\begin{bmatrix} p_5 \\ q_5 \end{bmatrix} = \begin{bmatrix} \boxed{0.165984} \\ 0.536724 \end{bmatrix} = u(1)$$

Similarly for the system 4:

$$p_n = v_n, \quad q_n = v_n' \quad p_0 = 0 \quad q_0 = 1.$$

$$\begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1.04 \end{bmatrix} \qquad \begin{bmatrix} p_3 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0.66448 \\ 1.367216 \end{bmatrix}$$

$$\begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0.416 \\ 1.1608 \end{bmatrix} \qquad \begin{bmatrix} p_4 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0.963213 \\ 1.667456 \end{bmatrix}$$

$$\begin{bmatrix} p_5 \\ q_5 \end{bmatrix} = \begin{bmatrix} \boxed{1.331968} \\ 2.073448 \end{bmatrix} - v(1)$$

Determination of $\theta$:

$$y(x) = \theta\, u(x) + (1-\theta)\, v(x)$$

$$y(1) = 0 \Rightarrow \quad 0 = \theta\,(0.165984) + (1-\theta)\,(1.331968)$$

$$\Rightarrow \theta = 1.142355$$

Hence we get:

$$\boxed{y(x) = 1.142355\, u(x) - 0.142355\, v(x)}$$

So,

$y(0.2) \simeq -0.0284710$ $\qquad$ $y(0.4) \simeq -0.0500808$

$y(0.6) \simeq -0.0577625$ $\qquad$ $y(0.8) \simeq -0.0577625$

$y(1.0) = 0.$

Example: Using shooting method, solve the mixed boundary value problem

$$y'' = y - 4xe^{x} \qquad 0 < x < 1 \quad \text{——} (1)$$

$$y(0) - y'(0) = -1$$
$$y(1) + y'(1) = -e$$

Use the Taylor's series method

$$y_{j+1} = y_j + h y_j' + \frac{h^2}{2} y_j'' + \frac{h^3}{6} y_j'''$$
$$y_{j+1}' = y_j' + h y_j'' + \frac{h^2}{2} y_j'''$$

Assume $h = 0.25$. compare numerical result with the exact solution $y(x) = x(1-x)e^{x}$.

<u>Solution:</u>

We solve (1) with the two initial values:

a) $y(0) = 0$ & $y'(0) = 1$

b) $y(0) = 1$ & $y'(0) = 2$

If we call the solution $u$ & $v$ of the equation (1) associated with the ICs a) & b) respectively.

Then we can write

$$y(x) = \theta u(x) + (1-\theta) v(x)$$

The parameter $\theta$ can be calculated so that the given BC $y(1) + y'(1) = -e$ is satisfied.