Assignment 9 (Mathematics II –MA10002)

(1) Find the gradient and the unit normal vector to the surface
(i) $x^2 + y - z = 4$ at the point $(2, 0, 0)$
(ii) $x^2 + 2y^2 + 3z^2 = 0$ at the point $(\sqrt{10}, 0, 0)$.

(2) Find the directional derivative of the following scalar valued functions
(i) $f(x, y) = e^x \cos y$ at $(0, \pi/4)$ in the direction of $(\hat{\mathbf{i}} + 3\hat{\mathbf{j}})/\sqrt{10}$
(ii) $f(x, y, z) = e^x + yz$ at $(1, 1, 1)$ in the direction of $\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$
(iii) $f(x, y, z) = \frac{1}{x^2+y^2+z^2}$ at $(2, 3, 1)$ in the direction of $\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$
(iv) $f(x, y) = \frac{y}{x^2+y^2}$ at $(0, 1)$ in the direction of a vector which makes an angle of $30°$ with the positive x-axis.

(3) If $r = |\mathbf{r}|$, where $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, then prove that
(i) $\nabla(\frac{1}{r}) = -\frac{\mathbf{r}}{r^3}$ (ii) $\nabla(\log(|\mathbf{r}|)) = \frac{\mathbf{r}}{r^2}$ (iii) $\nabla r^n = nr^{n-2}\mathbf{r}$.

(4) For any vector fields $\mathbf{F}, \mathbf{G}$, show that
(i) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
(ii) $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}(\mathbf{F}) - \mathbf{F} \cdot \text{curl}(\mathbf{G})$.

(5) Let $\mathbf{F} = 2xz^2\hat{\mathbf{i}} + \hat{\mathbf{j}} + xy^3z\hat{\mathbf{k}}$ and $f = x^2y$. Compute the following
(i) $\text{curl}(\mathbf{F})$ (ii) $\mathbf{F} \times \nabla f$ (iii) $\mathbf{F} \cdot (\nabla f)$.

(6) Evaluate the line integral $\int_C ydx + xdy$, where C is the path $(t^9, \sin^9(\pi t/2))$, $0 \le t \le 1$.

(7) Evaluate the line integral $\int_C x^2dx + xydy + dz$, where C is the curve $t\hat{\mathbf{i}} + t^2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ for $0 \le t \le 1$.

(8) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (x^2, xy)$ and C is the perimeter of the unit square joining the points $(0, 0), (1, 0), (1, 1), (0, 1)$ in the counter clockwise direction.

(9) Prove that a necessary and sufficient condition that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C is that $\nabla \times \mathbf{F} = \mathbf{0}$ identically.

(10) Check whether the line integral $\int_C (1 - \sin x \sin y)dx + (1 + \cos x \cos y)dy$ is independent of the path C joining the points $(\pi/4, \pi/4), (\pi/2, 0)$.

(11) If $\mathbf{F} = (4xy - 3x^2z^2, -2x^2, -2x^3z)$ then show $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the curve C joining to given points.

(12) Check whether $\mathbf{F}$ is a conservative vector field or not. If it is, find the potential function, where
(i) $\mathbf{F} = (2xy, x^2 + 2yz, y^2)$
(ii) $\mathbf{F} = (2xy + z^3, x^2, 3xz^2)$.

(13) Evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}dS$, where $\mathbf{F} = z\hat{\mathbf{i}} - x\hat{\mathbf{j}} + 3y^2z\hat{\mathbf{k}}$ and S is the surface of the cylinder $x^2+y^2 = 16$, included in the first octant between $z = 0, z = 5$.

(14) If $\mathbf{F} = (y, x - 2xz, -xy)$, then evaluate $\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}dS$, where S is the surface $x^2+y^2+z^2 = a^2$, above the xy-plane.

(15) Verify the Green's theorem for $\oint_C(xy+y^2)dx + x^2dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

(16) Using Green's theorem, evaluate $\oint_C ydx - xdy$, where C is the boundary of the square joining the points $(1,-1),(1,1),(-1,1),(-1,-1)$ in the counterclockwise direction.

(17) Verify the Gauss divergence theorem for $\mathbf{F} = (4xz, -y^2, yz)$ over the surface S of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

(18) Using Gauss divergence theorem, evaluate $\iint_S x^3\ dydz + x^2y\ dzdx + x^2z\ dxdy$, where S is the closed surface bounded by $x^2+y^2 = 4, z = 0, z = 3$.

(19) Using Stokes' theorem to evaluate the line integral $\int_C -y^3dx + x^3dy - z^3dz$, where C is the intersection of the cylinder $x^2+y^2 = 1$ and the plane $x+y+z = 1$ and the orientation of C corresponds to counterclockwise motion in the xy-plane.

(20) Verify the Stokes' theorem for $\mathbf{F} = (3x+3z, x+3y, 2y-3z)$, where S is the surface $6x+3y+4z = 12$ bounded by the coordinate planes and C is the boundary of it.