Consider the general multi-step method or k-step method

$$u_{j+1} = \sum_{i=1}^{k} a_i u_{j-i+1} + h \sum_{i=0}^{k} b_i u'_{j-i+1}$$

Determination of a_i's & b_i's :

For a linear multistep method of order p, we have

$$\tau_{j+1} = y(t_{j+1}) - \sum_{i=1}^{k} a_i y(t_{j-i+1}) - h \sum_{i=0}^{k} b_i y'(t_{j-i+1})$$

$$= C_{p+1} h^{p+1} y^{(p+1)}(t_j) + O(h^{p+2})$$

Not that the coefficients a_i's and b_i's are independent of $y(t)$. These coefficients can be determined by choosing $y(t) = e^t$. Substituting $y(t) = e^t$ in the above equation to get

$$\left[e^{t_{j+1}} - a_1 e^{t_j} - a_2 e^{t_{j-1}} - \cdots - a_k e^{t_{j-k+1}}\right]$$
$$- h\left[b_0 e^{t_{j+1}} + b_1 e^{t_j} + \cdots + b_k e^{t_{j-k+1}}\right] = O(h^{p+1})$$

$$\Rightarrow \left[e^{t_{j-k+1}+kh} - a_1 e^{t_{j-k+1}+(k-1)h} - \cdots - a_k e^{t_{j-k+1}}\right]$$
$$- h\left[b_0 e^{t_{j-k+1}+kh} + b_1 e^{\cdots} + \cdots + b_k e^{t_{j-k+1}}\right] = O(h^{p+1})$$

$$\Rightarrow \left\{\left[e^{kh} - a_1 e^{(k-1)h} - a_2 e^{(k-2)h} - \cdots - a_k\right] - h\left[b_0 e^{kh} + b_1 e^{(k-1)h} + \cdots + b_k\right]\right\} e^{t_{j-k+1}} = O(h^{p+1})$$

$$\Rightarrow \left[\rho(e^h) - h\sigma(e^h)\right] e^{t_{j-k+1}} = C_{p+1} h^{p+1} e^{t_j} + O(h^{p+2})$$

$$\Rightarrow \quad \rho(e^h) - h\sigma(e^h) = \bar{C}_{p+1} h^{p+1} + O(h^{p+2}) \qquad — ①$$

Setting $e^h = \xi$, we have $h = \ln(\xi)$

Not that, as $h \to 0$, $\xi \to 1$. We rewrite ① in powers of $(\xi - 1)$. We have

$$\ln \xi = \ln\left[(\xi-1)+1\right] = (\xi-1) - \frac{1}{2}(\xi-1)^2 + \cdots\cdots$$

$$h^{p+1} = [\ln \xi]^{p+1} = (\xi-1)^{p+1} + O(\xi-1)^{p+2}.$$

Now equation ① can be rewritten as

$$\rho(\xi) - \ln(\xi)\,\sigma(\xi) = \bar{C}_{p+1}(\xi-1)^{p+1} + O(\xi-1)^{p+2} \qquad — ②$$

If $\sigma(\xi)$ is given then this equation (2) can be used to determine $\rho(\xi)$. We expand $\ln \xi$ and $\sigma(\xi)$ in powers of $(\xi-1)$, simplify and retain the terms of required order.

For implicit method, $\rho(\xi)$ & $\sigma(\xi)$ are of same order, whereas, for explicit method $\rho(\xi)$ is one degree higher than the $\sigma(\xi)$.

If $\rho(\xi)$ is given, then the equation (2) can be rewritten as,

$$\frac{\rho(\xi)}{\ln(\xi)} - \sigma(\xi) = \bar{C}_{p+1}(\xi-1)^{p} + O(\xi-1)^{p+1} \qquad — ③$$

We now expand $\rho(\xi)$ & $\ln(\xi)$ in powers of $(\xi-1)$ simplify and retain the terms of required order.

Adams-Bashforth Methods (Explicit)

$$\rho(\xi) = \xi^{K-1}(\xi-1) \quad \text{and} \quad \sigma(\xi) \text{ is of degree } K-1$$

For $K=2$:

$$\rho(\xi) = \xi(\xi-1) = (\xi-1+1)(\xi-1)$$
$$= (\xi-1)^2+(\xi-1)$$

Now consider

$$\frac{\rho(\xi)}{\ln\xi} = \frac{(\xi-1)^2+(\xi-1)}{\ln[1+(\xi-1)]} = \frac{(\xi-1)^2+(\xi-1)}{(\xi-1)-\frac{1}{2}(\xi-1)^2+\cdots}$$

$$= [1+(\xi-1)]\left[1-\frac{1}{2}(\xi-1)+\cdots\right]^{-1}$$

$$= [1+(\xi-1)]\left[1+\frac{1}{2}(\xi-1)+\cdots\right]$$

$$= \underbrace{\left[1+\frac{3}{2}(\xi-1)\right]}_{\sigma(\xi)} + O(\xi-1)^2$$

Thus we have

$$\sigma(\xi) = \frac{3}{2}\xi - \frac{1}{2}$$

The numerical method becomes:

$$\rho(E)u_{j-1} - h\sigma(E)u'_{j-1} = 0$$

$$\Rightarrow (E^2-E)u_{j-1} - h\cdot\frac{1}{2}[3E-1]u'_{j-1} = 0$$

$$\Rightarrow \boxed{u_{j+1} = u_j + \frac{h}{2}(3u'_j - u'_{j-1})}$$

ORDER OF THE METHOD is 2.