Application of multiple integral.

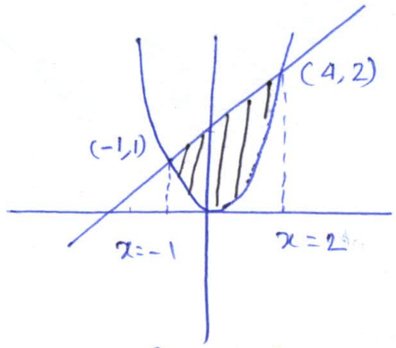
1. Evaluation of area.
2. Evaluation of surface area

$\Big\}$ double integrals.

Area.

$$\iint_{D_{xy}} dx\,dy = \text{area of } D_{xy}.$$

1. Find the area of the region bounded between the line $y=x+2$ & the parabola $y=x^2$.



$x+2=x^2$; or, $x^2-x-2=0$

or $(x-2)(x+1)=0$

$\therefore x=2,-1$.

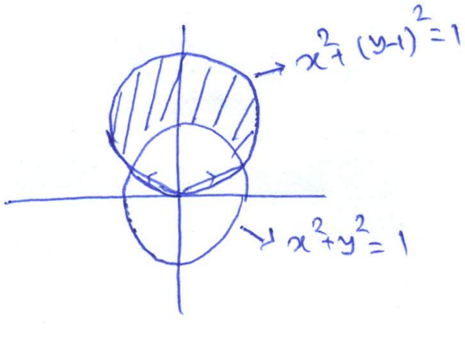
Soln. Area of $D = \iint_D dx\,dy$

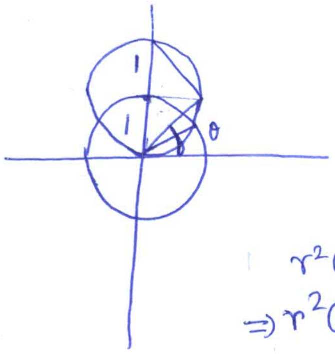
$$= \int_{x=-1}^{2}\int_{y=x^2}^{x+2} dy\,dx$$

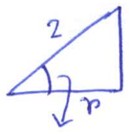
$$= \int_{x=-1}^{2} (x+2-x^2)\,dx = \frac{9}{2} \text{ sq units.}$$

2. Find the area of the region which lies inside the circle $x^2+(y-1)^2=1$, but outside the circle $x^2+y^2=1$.

Soln.







$\frac{r}{2} = \cos\left(\frac{\pi}{2} - \theta\right)$

$\Rightarrow r = 2\sin\theta$

$$r^2\cos^2\theta + (r\sin\theta - 1)^2 = 1$$
$$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta = 0$$
$$\Rightarrow r^2 - 2r\sin\theta$$
$$\Rightarrow r = 0, \quad r = 2\sin\theta$$

We have $\quad x^2 + (y-1)^2 = 1, \qquad x^2 + y^2 = 1$

$$\therefore (y-1)^2 = y^2 \Rightarrow 2y = 1 \Rightarrow y = \frac{1}{2}$$

also $\quad x^2 = 1 - y^2 = 1 - \frac{1}{4} = \frac{3}{4}, \quad x = \pm\frac{\sqrt{3}}{2}$

$$\theta_1 = \tan^{-1}\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\pi}{6} \qquad \theta_2 = \tan^{-1}\frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$I = \iint dx\,dy = \int_{\theta=\pi/6}^{5\pi/6}\int_{r=1}^{2\sin\theta} r\,dr\,d\theta$$

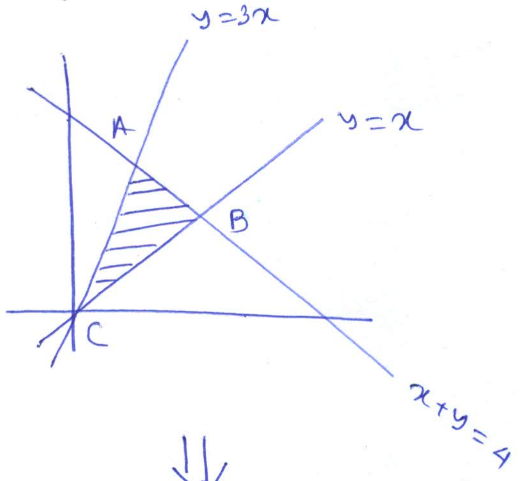
$$= \int_{\pi/6}^{5\pi/6} \frac{r^2}{2}\Bigg|_1^{2\sin\theta} d\theta = \int_{\pi/6}^{5\pi/6}\left(\frac{4\sin^2\theta - 1}{2}\right) d\theta$$

$$= \int_{\pi/6}^{5\pi/6}\left(\frac{1}{2} - \cos 2\theta\right) d\theta = \frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) - \frac{\sin 2\theta}{2}\Bigg|_{\pi/6}^{5\pi/6}$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

3. Find the area bounded by $y = x$; $y = 3x$; $y + x = 4$, using the transformations $2x = u - v$
$2y = u + v$.

Soln.



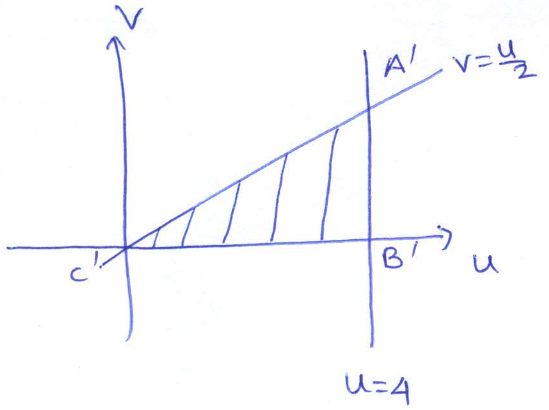
$$y = \frac{u+v}{2}$$
$$x = \frac{u-v}{2}$$

$x + y = 4 \rightarrow u = 4$

$y - x = 0 \rightarrow v = 0$

$y - 3x = 0 \rightarrow v = \frac{u}{2}$

$\Downarrow$



$y = 3x$

$$\Rightarrow \frac{u+v}{2} = \frac{3(u-v)}{2}$$

$\Rightarrow u = 2v$

Here $J = \frac{1}{2}$

$$\iint_{ABC} dx\,dy = \iint_{D_{uv}} |J|\,du\,dv = \frac{1}{2}\int_{u=0}^{4}\int_{v=0}^{u/2} dv\,du$$

$= 2$ sq units.

Surface area (SA).

1. Find the SA of some surface $z = f(x, y)$
2. Find the SA of $z = f_1(x, y)$ lying outside or inside $z = f_2(x, y)$.

Surface area of $z = f(x, y)$ lying over some region D_{xy}

$$SA = \iint_{D_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy \qquad z = f(x, y)$$

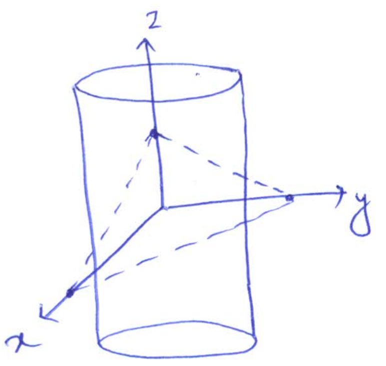
or as

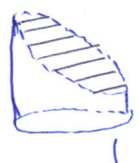
$$\iint_{D_{yz}} \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dy\, dz \qquad x = g(y, z)$$

or as

$$\iint_{D_{xz}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dx\, dz \qquad y = u(x, z)$$

Ex 1. Find the area of the part of the plane $\underbrace{x + 2y + z = 4}_{z = f(x,y)}$, which lies inside the ~~circle~~ cylinder $\underbrace{x^2 + y^2 = 1}_{D_{xy}}$.





$z = f(x, y)$
$= 4 - x - 2y$
$z_x = -1, \quad z_y = -2$

$$SA = \iint_{x^2 + y^2 \leq 1} \sqrt{1 + 1 + 4}\, dx\, dy$$

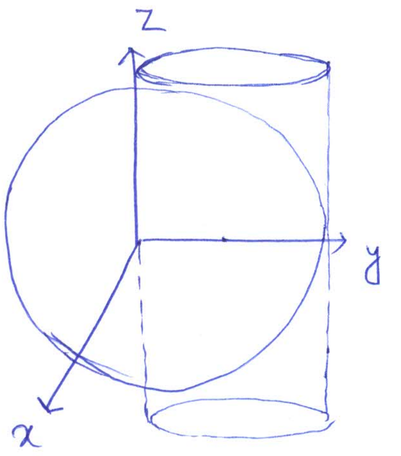
$$= \sqrt{6} \iint_{x^2 + y^2 \leq 1} dx\, dy = \sqrt{6}\pi.$$

Ex 2. Find the SA of the part of the sphere $x^2+y^2+z^2=36$ inside the cylinder $x^2+y^2=6y$ and the above the xy plane.

$$x^2+(y-3)^2=3^2$$

Soln

$$z=\sqrt{36-x^2-y^2}=f(x,y) \qquad z_x=-\frac{x}{\sqrt{36-x^2-y^2}}$$

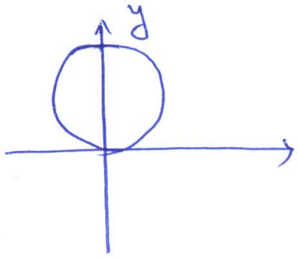


$$z_y=-\frac{y}{\sqrt{36-x^2-y^2}}$$

$$1+z_x^2+z_y^2=1+\frac{x^2}{36-x^2-y^2}+\frac{y^2}{36-x^2-y^2}$$

$$=\frac{36}{36-x^2-y^2}.$$

$$SA=\iint_{D_{xy}}\frac{\sqrt{36}}{\sqrt{36-x^2-y^2}}\,dx\,dy$$



$$=\int_{\theta=0}^{\pi}\int_{r=0}^{6\sin\theta}\frac{6}{\sqrt{36-r^2}}\,r\,dr\,d\theta$$

$$=\int_{\theta=0}^{\pi}6\sqrt{36-r^2}\,\Big|_{6\sin\theta}^{0}\,d\theta$$

$$=6\int_{\theta=0}^{\pi}\left(6-\sqrt{36-36\sin^2\theta}\right)d\theta$$

$$=6\times6\int_0^{\pi}\left(1-|\cos\theta|\right)d\theta \qquad \int_0^{\pi}\cos\theta\,d\theta=0.$$

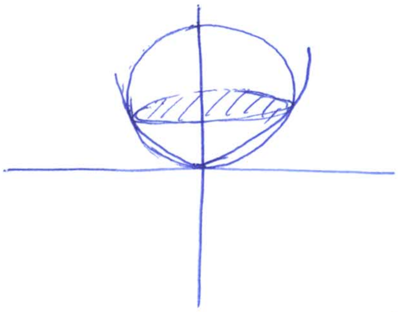
$$= 36\pi - 36\left[\int_0^{\pi/2} \cos\theta \, d\theta + \int_{\pi/2}^{\pi} -\cos\theta \, d\theta\right]$$

$$= 36\pi - 36\left[(1-0) - (0-1)\right]$$

$$= (36\pi - 72) \text{ sq units.}$$

<u>Ex 3.</u> Find SA of the sphere $x^2+y^2+z^2=4z$ inside the paraboloid $z=x^2+y^2$

<u>Soln.</u> $x^2+y^2+z^2=4z \Rightarrow x^2+y^2+(z-2)^2=2^2$



$$2x + 2z z_x - 4z_x = 0$$

$$z_x = \frac{2x}{4-2z} = -\frac{x}{z-2}$$

$$z_y = \frac{2y}{4-2z} = -\frac{y}{z-2}.$$

$$1+z_x^2+z_y^2 = \frac{x^2+y^2+(z-2)^2}{(z-2)^2} = \frac{4}{(z-2)^2}$$

$$SA = \iint \frac{2}{z-2}\,dx\,dy$$

$$= \iint \frac{2}{\sqrt{4-x^2-y^2}}$$

$$x^2+y^2+(z-2)^2 = 4$$

$$(z-2) = \sqrt{4-x^2-y^2}$$

D_{xy} is the intersection of $x^2+y^2+z^2=4z$ & $x^2+y^2=z$

or $z+z^2=4z$ or $z^2-3z=0 \Rightarrow z=0, 3$

$$x^2+y^2=3$$

$$SA = \iint_{x^2+y^2\le 3} \frac{2\,dx\,dy}{\sqrt{4-x^2-y^2}} = \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{3}} \frac{2r\,dr\,d\theta}{\sqrt{4-r^2}} = 4\pi \text{ sq units.}$$

Excercises.

1. Find the area of the annulus $x^2+y^2=1$ & $x^2+y^2=4$ lying in the 1st quadrant.

Hint : $$\int_{\theta=0}^{\pi/2}\int_{r=1}^{2} r\,dr\,d\theta$$

2. Find the SA of the part of the sphere $x^2+y^2+z^2=25$ between the planes $z=2$ & $z=4$.