Lecture-1. Thursday - 7.1.16

Linear Algebra

$\vec{F} = (F_x, F_y, F_z)$ $\vec{a} = (a_x, a_y, a_z)$

$$\vec{F} = m\vec{a}$$

$$\Rightarrow m\vec{a} = \vec{F} \quad (\vec{m}, \vec{F} \to \text{known})$$

$$\Rightarrow \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

$$\left.\begin{aligned} m a_x + 0.a_y + 0.a_z &= F_x \\ 0.a_x + m a_y + 0.a_z &= F_y \\ 0.a_x + 0.a_y + m a_z &= F_z \end{aligned}\right\} \text{system of linear equations.}$$

Gauss elimination method for solving system of equations

Ex 1.

E_1: $\overset{a_{11}=1}{x} + 2y - 3z = 4$

E_2: $\overset{a_{21}=1}{x} + 3y + z = 11$

E_3: $\overset{a_{31}=2}{2x} + 5y - 4z = 13$

E_4 $\overset{a_{41}=2}{2x} + 6y + 2z = 22$

$$\begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 1 \\ 2 & 5 & -4 \\ 2 & 6 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \\ 13 \\ 22 \end{pmatrix}$$

$$\underset{(4\times 3)}{A} \; \underset{(3\times 1)}{X} = B_{(4\times 1)}.$$

Step-1: Eliminate x from E_2, E_3, E_4.

$$E_i \to -a_{i1}E_1 + a_{11}E_i \; ; \; i=2,3,4$$

$$E_1 \xrightarrow{\text{unchanged}} x + 2y - 3z = 4$$

$$E_2 \xrightarrow{-1E_1 + 1\cdot E_2} \overset{a_{22}=1}{y} + 4z = 7$$

$$E_3 \xrightarrow{-2E_1 + 1\cdot E_3} \overset{a_{32}=1}{y} + 2z = 5$$

$$E_4 \xrightarrow{-2E_1 + 1\cdot E_4} \overset{a_{42}=2}{2y} + 8z = 14$$

Step-2. Keep 1st & 2nd eqnations unchanged.

$$E_i \to -a_{i2}E_2 + a_{22}E_i \; ; \; i=3,4.$$

$$E_1 \xrightarrow{\text{unchanged}} x + 2y - 3z = 4$$

$$E_2 \xrightarrow{\text{unchanged}} y + 4z = 7$$

$$E_3 \xrightarrow{-1\cdot E_2 + E_3} -2z = -2$$

$$E_4 \xrightarrow{-2\cdot E_2 + 1\cdot E_3} 0 = 0$$

Step-3. Back substitution.

$z=1$, $y = 7-4z = 3$, $x=1$

$(1,3,1) \to$ unicrne solution.

2. $2x + y - 2z + 3w = 1$

$3x + 2y - z + 2w = 4$

$3x + 3y + 3z - 3w = 5$

$E_1 \xrightarrow{\text{unchanged}} 2x + y - 2z + 3w = 1$

$E_2 \xrightarrow[-3E_1 + 2E_2]{-a_{21} \cdot E_1 + a_{11} E_2} y + 4z - 5w = 5$

$E_3 \xrightarrow[-3E_1 + 2E_3]{-a_{31} E_1 + a_{11} E_3} 3y + 12z - 15w = 7$

$E_1 \xrightarrow{\text{unchanged}} 2x + y - 2z + 3w = 1$

$E_2 \xrightarrow{\text{unchanged}} y + 4z - 5w = 5$

$E_3 \xrightarrow{-a_{32} E_2 + a_{22} E_3} 0 = -8.$

$\Rightarrow$ the system is inconsistent i.e. has no solution.

3. $x + 2y - 2z + 3w = 2 \quad (E_1)$

$2x + 4y - 3z + 4w = 5 \quad (E_2)$

$5x + 10y - 8z + 11w = 12 \quad (E_3).$

Step-1. eliminate x from E_2 & E_3

$E_1 \xrightarrow{\text{unchanged}} x + 2y - 2z + 3w = 2$

$E_2 \xrightarrow{-2E_1 + 1.E_2} z - 2w = 1$

$E_3 \xrightarrow{-5E_1 + 1 \cdot E_3} 2z - 4w = 2$

Step 2. eliminate z,

$$\begin{aligned} x + 2y - 2z + 3w &= 2 \\ z - 2w &= 1 \\ 0 &= 0. \end{aligned}$$

$y, w \to$ free variable.

let $w = 1$, $y = 0$ $\Rightarrow z = 3$

$x = 2 + 2z - 3w = 5$

$(5, 0, 3, 1) \to$ a solution of the system

let $y = b$, $w = d$

$z = 1 + 2w = 1 + 2d$

$$\begin{aligned} x &= 2 - 2y + 2z - 3w \\ &= 2 - 2b + 2(1 + 2d) - 3d \\ &= 4 - 2b + d \end{aligned}$$

$(x, y, z, w) = (4 - 2b + d, b, 1 + 2d, d)$.

$$\begin{aligned} x + 2y - 2z + 3w &= 2 \\ -2w + z &= 1 \end{aligned}$$

n unknowns are there, if there are r equations in the reduced system. then number of free variables $= n - r$.

In the above example $n = 4$

$r = 2$

$\therefore$ number of free variables $= 4 - 2 = 2$.

system of equations

- Consistent (there exists a solution)
  - unique solution
  - infinitely many solutions.
- inconsistent (no solution).

When there are more no. of unknowns than the equations, then it is never possible to get a unique solution. The system will either have no solution or have infinitely many solutions.

Ex 1.

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 1 \\ 2 & 5 & -4 \\ 2 & 6 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \\ 13 \\ 22 \end{pmatrix}$$

After Gauss elimination

$$\begin{aligned} x + 2y - 3z &= 4 \\ y + 4z &= 7 \\ -2z &= -2 \end{aligned} \qquad \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -2 \end{pmatrix}$$

Row operations

1. $R_i \longleftrightarrow R_j$
2. $R_i \longrightarrow kR_i$
3. $R_j \longrightarrow k'R_j + R_i$

$R_j \longrightarrow k'R_j + kR_i$

$$\begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 1 \\ 2 & 5 & -4 \\ 2 & 6 & 2 \end{pmatrix} \xrightarrow[R_4 \to -2R_1+R_4]{\substack{R_2 \to -R_1+R_2 \\ R_3 \to -2R_1+R_3}} \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{pmatrix}$$

$$\xrightarrow[R_4 \to -2R_2+R_4]{R_3 \to -R_2+R_3} \begin{pmatrix} \boxed{1} & 2 & -3 \\ 0 & \boxed{1} & 4 \\ 0 & 0 & \boxed{-2} \\ 0 & 0 & 0 \end{pmatrix}$$ echelon matrix.

Echelon matrix: A matrix is an echelon matrix if no. of zeros proceeding the 1st nonzero element of each row increases row by row, until we arrive at the zero row, if there be any.

Distinguished element: The first non zero element in each row is the distinguished element of that row. In the previous example $1, 1, -2$ are the distinguished elements.

An echelon matrix is said to be row reduced
(i) if all the distinguished elements are 1.
(ii) the distinguished elements in its respective columns is the only non-zero elements.

$$\begin{pmatrix} 0 & \boxed{1} & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix} \rightarrow$$ row reduced echelon matrix.

Rank of a matrix.

Rank of a matrix A = no. of non-zero rows in echelon form.

Ex 1.

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

rank (A) = 3

Ex 2.

$$A = \begin{pmatrix} 2 & 1 & -2 & 3 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank (A) = 2

Ex 3.

$$A = \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank (A) = 2.

Augmented matrix.

Ex 1.

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

rank (A|B) = 3

Ex-2.

$$(A|B) = \begin{pmatrix} 2 & 1 & -2 & -3 & 1 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & 0 & 0 & 0 & -8 \end{pmatrix}$$

rank (A|B) = 3

Ex-3.

$$\begin{pmatrix} 1 & 2 & -2 & 3 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$ rank (A|B) = 2.

System of equations

- inconsistent
  Rank(A) $\neq$ Rank(A|B)
- Consistent
  Rank(A) = Rank(A|B) = r (say)
  - $r = n$
    (unique solution)
  - $r < n$
    (many solution)

<u>Homogeneous system of equations.</u>

$$\left.\begin{aligned} x + 2y - 3z &= 0 \\ x + 4y - 6z &= 0 \\ 3x - 2y + 5z &= 0 \end{aligned}\right\}$$

A homogeneous system is always consistent, since (0, 0, 0) is a solution.

Either it will have unique solution i.e. the zero solution, or it will have many solutions.

<u>Theorem</u> In a homogeneous system of linear equations, if there are more unknowns than equations, then the system will has a non zero solution.