Hence the problem is to choose ξ & η so that (3) takes a simple form.

Case I: If $B^2 - 4AC > 0$ (Hyperbolic case)

We choose ξ & η so that $\bar{A} = \bar{C} = 0$, i.e.,

$$A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

& $$A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$$

The equation for η is the same as for ξ; therefore we need to solve only one equation.

Solve first equation we get:

$$\frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\Leftrightarrow \quad 2A\xi_x - \left(-B \pm \sqrt{B^2 - 4AC}\right)\xi_y = 0$$

In order to obtain a non-singular transformation we choose ξ to be a solution of

$$2A\xi_x - \left(-B - \sqrt{B^2 - 4AC}\right)\xi_y = 0 \qquad \text{— (3)}$$

and η to be a solution of

$$2A\eta_x - \left(-B + \sqrt{B^2 - 4AC}\right)\eta_y = 0 \qquad \text{— (4)}$$

Lagrange auxiliary equations for (3):

$$\frac{dx}{2A} = \frac{dy}{-\left(-B-\sqrt{B^2-4AC}\right)} = \frac{d\xi}{0} \quad \text{—— (5)}$$

Taking the first two fractions of (5) we get

$$\frac{dy}{dx} = -\frac{-B-\sqrt{B^2-4AC}}{2A}$$

or

$$\boxed{\frac{dy}{dx} = \frac{B+\sqrt{B^2-4AC}}{2A}} \quad \text{—— (6)}$$

The solution of (6) may be written as

$$\varphi_1(x,y) = c_1 \quad \text{—— (7)}$$

Taking last fraction of (5), we obtain

$$\xi = c_2$$

A solution of (3) may be written as

$$\xi = \varphi_1(x,y) \quad \text{—— (8)}$$

Similarly for (4), we get

$$\boxed{\frac{dy}{dx} = \frac{B-\sqrt{B^2-4AC}}{2A}} \quad \text{—— (9)}$$

& $\eta = \varphi_2(x,y)$ where $\varphi_2(x,y) =$ Constant

is the solution of (9).

Hence the transformation

$$\xi = \varphi_1(x,y) \quad \text{and} \quad \eta = \varphi_2(x,y)$$

will transform equation ① to a cannonical form

$$\omega_{\xi\eta} = \Phi(\xi, \eta, \omega, \omega_\xi, \omega_\eta).$$

The equations (8) & (9) are called characteristics equations of (1).

The solution of (8) or respectively (9) is called the characteristics of the equation (1).

Case II: The parabolic case $(B^2 - 4AC = 0)$

In this case there exists only one characteristic equation

$$\frac{dy}{dx} = \frac{B}{A}$$

(assuming A or C does not vanish together otherwise $A = C = 0 \Rightarrow B = 0$ suppose $A \neq 0$.)

In this case we obtain one transformation, say

$$\xi = \varphi(x,y) \qquad (\text{or } \eta = \varphi(x,y))$$

It follows that

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0.$$

It is easy to show that

$$\bar{B}^2 - 4\bar{A}\bar{C} = J^2(B^2 - 4AC) = 0$$

$\Rightarrow$ If $\bar{A}=0$ then $\bar{B}=0$

So the equation (1) reduces to

$$\omega_{\eta\eta} = \Phi(\xi,\eta,\omega,\omega_\xi,\omega_\eta)$$

for arbitrary values of $\eta(x,y)$ such that $J \neq 0$.

In practice one may choose $\eta = y$ for instance to have a nonsingular transformation

$$\xi = \psi(x,y)$$
$$\eta = y$$

$$J = \begin{vmatrix} \psi_x & \psi_y \\ 0 & 1 \end{vmatrix} = \psi_x \neq 0$$

(If $\psi_x = 0$ $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow B = 0$ parabolicity $\Rightarrow$ A or $C=0$ In that case original equation is already in canonical form)

<u>Case III</u>: (Similar to case I) elliptic case ($B^2-4AC<0$)

Since $B^2-4AC<0$, the elliptic equation has no real characteristic. Nevertheless we seek a transformation $\xi = \xi(x,y)$ & $\eta = \eta(x,y)$ which simplifies equation (1)

Proceeding in a similar fashion as in the case (I), we find ξ and η as complex conjugate.

As in case I we can arrive at

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = \Phi(\xi, \eta, w, w_\xi, w_\eta)$$ where η & ξ are complex conjugate.

To get a real cannonical form we make further transformation

$$\alpha = \frac{1}{2}(\xi + \eta) \qquad \beta = \frac{i}{2}(-\xi + \eta)$$

$$w(\xi, \eta) = w(\xi(\alpha, \beta), \eta(\alpha, \beta)) = \bar{w}(\alpha, \beta)$$

$$\frac{\partial w}{\partial \xi} = \frac{\partial \bar{w}}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial \xi} + \frac{\partial \bar{w}}{\partial \beta} \cdot \frac{\partial \beta}{\partial \xi}$$

$$= \frac{\partial \bar{w}}{\partial \alpha} \cdot \left(\frac{1}{2}\right) + \frac{\partial \bar{w}}{\partial \beta}\left(-\frac{i}{2}\right)$$

$$\frac{\partial^2 w}{\partial \eta \partial \xi} = \frac{1}{2}\left[\frac{\partial^2 \bar{w}}{\partial \alpha^2} \cdot \frac{\partial \alpha}{\partial \eta} + \cancel{\frac{\partial^2 \bar{w}}{\partial \alpha \partial \beta}} \cdot \frac{\partial \beta}{\partial \eta}\right] - \frac{i}{2}\left[\frac{\partial^2 \bar{w}}{\partial \alpha \partial \beta} \frac{\partial \alpha}{\partial \eta} + \frac{\partial^2 \bar{w}}{\partial \beta^2} \frac{\partial \beta}{\partial \eta}\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2 \bar{w}}{\partial \alpha^2} \cdot \frac{1}{2} + \cancel{\frac{\partial^2 \bar{w}}{\partial \alpha \partial \beta}}\left(\frac{i}{2}\right)\right] - \frac{i}{2}\left[\frac{1}{2} \cancel{\frac{\partial^2 \bar{w}}{\partial \alpha \partial \beta}} + \frac{i}{2} \frac{\partial^2 \bar{w}}{\partial \beta^2}\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2 \bar{w}}{\partial \alpha^2} + \frac{\partial^2 \bar{w}}{\partial \beta^2}\right]$$

So the desired canonical form is

$$\boxed{\frac{\partial^2 \bar{w}}{\partial \alpha^2} + \frac{\partial^2 \bar{w}}{\partial \beta^2} = \psi(\alpha, \beta, \bar{w}, \bar{w}_\alpha, \bar{w}_\beta)}$$

Ex: Find the canonical form of

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0$$

and solve it.

Sol:

$$A = 3 \quad B = 10 \quad C = 3$$

$$B^2 - 4AC = 100 - 36 = 64 > 0.$$

The given PDE is of hyperbolic type.

The corresponding characteristic equations are

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{10 + \sqrt{100 - 36}}{2 \cdot 3} = \frac{10 + 8}{6} = 3$$

$$\& \quad \frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{10 - 8}{6} = \frac{1}{3}$$

Characteristics are $y - 3x = c_1$

$\& \quad y - \frac{x}{3} = c_2$

To find the canonical form we take the following transformation

$$\xi = y - 3x \quad \& \quad \eta = y - x/3.$$

Let $u(x,y) = w(\xi, \eta)$

$$u_x = w_\xi \cdot \xi_x + w_\eta \cdot \eta_x$$

$$= w_\xi (-3) + w_\eta \left(-\tfrac{1}{3}\right)$$

$$u_{xx} = -3\left[w_{\xi\xi} \cdot (-3) + w_{\xi\eta}\left(-\tfrac{1}{3}\right)\right] - \tfrac{1}{3}\left[w_{\eta\xi}(-3) + w_{\eta\eta}\left(-\tfrac{1}{3}\right)\right]$$

$$= \ldots$$

$$= \ldots \quad 9w_{\xi\xi} + 2w_{\xi\eta} + \tfrac{1}{9}w_{\eta\eta}$$

$$u_{xy} = -3\left[w_{\xi\xi}\xi_y + w_{\xi\eta}\cdot\eta_y\right] - \frac{1}{3}\left[w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y\right]$$

$$= -3\left[w_{\xi\xi}\cdot 1 + w_{\xi\eta}\cdot 1\right] - \frac{1}{3}\left[w_{\eta\xi}\cdot 1 + w_{\eta\eta}\cdot 1\right]$$

$$= -3w_{\xi\xi} - \frac{10}{3}w_{\xi\eta} - \frac{1}{3}w_{\eta\eta}$$

$$u_y = \left(w_\xi \xi_y + w_\eta \eta_y\right)$$

$$= w_\xi + w_\eta$$

$$u_{yy} = \left(w_{\xi\xi} + w_{\xi\eta}\right) + \left(w_{\eta\xi} + w_{\eta\eta}\right)$$

$$= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}.$$

Substituting in the given PDE.

$$3\left(\cancel{9w_{\xi\xi}} + 2w_{\xi\eta} + \cancel{\frac{1}{9}w_{\eta\eta}}\right) + 10\left(\cancel{-3w_{\xi\xi}} - \frac{10}{3}w_{\xi\eta} - \cancel{\frac{1}{3}w_{\eta\eta}}\right) + 3\left(\cancel{w_{\xi\xi}} + 2w_{\xi\eta} + \cancel{w_{\eta\eta}}\right) = 0$$

$$\Rightarrow \left(6 - \frac{100}{3} + 6\right)w_{\xi\eta} = 0$$

$$\Rightarrow \boxed{w_{\xi\eta} = 0} \rightarrow \text{desired canonical form.}$$

on integration, we get

$$w_\xi = \Phi_1(\xi)$$

Again integrating

$$w = \int \Phi_1(\xi)\,d\xi + \psi(\eta) \Rightarrow w(\xi,\eta) = \Phi(\xi) + \psi(\eta)$$

or

$$u(x,y) = \Phi(y-3x) + \psi(y-\tfrac{x}{3})$$

Ex: Reduce the equation $u_{xx} + x^2 u_{yy} = 0$ to a canonical form.

Sol:

$$A = 1. \quad B = 0 \quad C = x^2$$

$$B^2 - 4AC = -4x^2 < 0$$

Hence the given PDE is elliptic.

The characteristic equations are

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{0 + \sqrt{-4x^2}}{2} = +ix$$

& $$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -ix.$$

Integration gives:

$$y = \frac{ix^2}{2} - ic_1 \Rightarrow iy + \frac{x^2}{2} = c_1$$

& $$y = -\frac{ix^2}{2} + ic_1 \Rightarrow -iy + \frac{x^2}{2} = c_2$$

Hence $\xi = \frac{1}{2}x^2 + iy$, $\eta = \frac{1}{2}x^2 - iy$

Introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2} \qquad \beta = \frac{i}{2}(-\xi + \eta)$$

$$\Rightarrow \alpha = \frac{x^2}{2} \qquad \beta = -\frac{i}{2} \cdot 2iy = y$$

Take. $u(x,y) = \omega(\alpha, \beta)$ and subst. in given PDE we get

$$\boxed{\omega_{\alpha\alpha} + \omega_{\beta\beta} = -\frac{\omega_\alpha}{2\alpha}}$$

Ex: Reduce the equation

$$y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} = \frac{y^2}{x} u_x + \frac{x^2}{y} u_y$$

to a canonical form and solve it.

Sol: $A = y^2$
$B = -2xy$
$C = x^2$

So $B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$ (parabolic type)

Characteristic equation

$$\frac{dy}{dx} = \frac{B}{2A} = -\frac{2xy}{2y^2} = -\frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx} + \frac{x}{y} = 0$$

$$\Rightarrow y^2 = -x^2 + c_1$$

$$\Rightarrow y^2 + x^2 = c_1 \quad \text{therefore } \xi = x^2 + y^2$$

Let us choose $\eta = x^2 - y^2$

$$J = \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} = -8xy$$

Subst. $u(x,y) = \omega(\xi, \eta)$ with $\xi = x^2 + y^2$ & $\eta = x^2 - y^2$

we get $\omega_{\eta\eta} = 0$

$$\Rightarrow \omega_\eta = f(\xi)$$

$$\Rightarrow \omega = f(\xi)\eta + g(\xi)$$

$$\Rightarrow u(x,y) = (x^2 - y^2) f(x^2 + y^2) + g(x^2 + y^2)$$

$\square$.