Tutorial Problems set-IV

Note: All these problems can be solved using the results of Chapter-4.

[**0.0.1**] ***Exercise*** A matrix $A \in \mathbb{M}_n(\mathbb{F})$ is said to be **nilpotent** if $A^k = 0_n$ for some positive integer k. Show that A is nilpotent if and only if the eigenvalues of A are 0.

Sol. We first assume that A is nilpotent. Then $A^k = 0_n$ for some positive integer k.

Let λ be an eigenvalue of A. Then λ^k is an eigenvalue of A^k. Since $A^k = 0_n$, then $\lambda^k = 0$. This implies $\lambda = 0$.

The eigenvalues of A are 0. Then the characteristic polynomial of A is x^n. Hence $A^n = 0_n$. Therefore A is nilpotent.

[**0.0.2**] ***Exercise*** Let A be nilpotent.

1. If $A \neq 0_n$, show that A is not diagonalizable.
2. What can you say about the minimal polynomial of A?

Sol. The eigenvalues of A are 0. Suppose A is diagonalizable. Then $P^{-1}AP = diag(0, \ldots, 0)$. This implies $A = 0_n$.

We know that the minimal polynomal of A divides the annihilating ploynomial. The characteristic polynomial of A is x^n. Then minimal polynomial is x^m where $m \leq n$.

[**0.0.3**] ***Exercise*** Let $A, B \in \mathbb{M}_n(\mathbb{C})$ and let $C = AB - BA$. Show that $I - C$ is not nilpotent.

Sol. The trace of C is zero. Then $trace(I - C) = n$. Hence $I - C$ is not nilpotent.

[**0.0.4**] ***Exercise*** What is the minimal polynomial of $A = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}$?

[**0.0.5**] ***Exercise*** Let $C = \begin{bmatrix} A & 0_n \\ 0_n & B \end{bmatrix}$ be a block diagonal matrix where $A, B \in \mathbb{M}_n(\mathbb{C})$. Prove that the minimal polynomial of C is the L.C.M (least common multiple) of the minimal polynomial of A and B.

Sol. Let $m_C(x), m_A(x)$ and $m_B(x)$ be the minimal polynomial of C, A and B, respectively. Let $P(x)$ be the L.C.M of $m_A(x)$ and $m_B(x)$.

$$m_C(C) = \begin{bmatrix} m_C(A) & 0 \\ 0 & m_C(B) \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} m_C(A) & 0 \\ 0 & m_C(B) \end{bmatrix}.$$

$m_C(A) = 0 = m_C(B)$. This implies $m_A(x)$ and $m_B(x)$ divides $m_C(x)$. Hence $P(x)$ divides $m_C(x)$.

$$P(C) = \begin{bmatrix} P(A) & 0 \\ 0 & P(B) \end{bmatrix}.$$

$$P(C) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This implies $m_C(x)$ divides $P(x)$. Hence $m_A(x) = P(x)$.

[**0.0.6**] ***Exercise*** Let $A \in \mathbb{M}_n(\mathbb{C})$. Show that $\{I, A, A^2, \ldots, A^n\}$ is a linearly dependent set in the vector space $\mathbb{M}_n(\mathbb{C})$.

Sol. If $A = 0_n$, then it is trivial. If $A \neq 0_n$, then using Cayley Hamilton theorem you can show that $\{I, A, A^2, \ldots, A^n\}$ is a linearly dependent.

[**0.0.7**] ***Exercise*** Let $A = uu^*$ where u is a non-zero column vector.

1. Show that the distinct eigenvalues of A are 0 and u^*u.
2. Show that u^*u is a simple eigenvalue of A.
3. Write down the $E_{(\lambda=0)}$ and $E_{(\lambda=u^*u)}$.
4. Compute the minimal polynomial of A.
5. Show that A is diagonalizable.

Sol. A is a Hermitian matrix. Then rank of A is the rank of u. Since u is non-zero, the rank of u is 1. Hence the rank of A is 1. Then 0 is an eigenvalue of A. The geometrix multiplicity of 0 is $n - rank(A) = n - 1$. Since A is Hermitian, then the algebraic multiplicity of 0 is $n - 1$. So it has one non-zero eigenvalue.

$$Au = uu^*u.$$

$Au = (u^*u)u$. This implies u^*u is an eigenvalue of A corresponding eigenvector u.

$$E_{(\lambda=u^*u)} = \text{LS}\, u \text{ and } E_{(\lambda=0)} = \{v \in \mathbb{C}^n : \langle v, u\rangle = 0\}.$$

Since A is Hermitian, A is diagonalizable and the minimal polynomial is $x(x - u^*u)$.

[**0.0.8**] ***Exercise*** The characteristic polynomial of a matrix $A \in \mathbb{M}_5(\mathbb{R})$ is given by $x^5 + \alpha x^4 + \beta x^3$, where α and β are non-zero real numbers. What are the possible values of the rank of A?

Sol. $x^5 + \alpha x^4 + \beta x^3 = x^3(x^2 + \alpha x + \beta)$. Since $\beta \neq 0$, then 0 is not the root of $(x^2 + \alpha x + \beta)$. Then 0 is an eigenvalue of A with algebraic multiplicity 3. So geometric multiplicity of 0 is at most 3.

We know that $rank(A) = n - gm(0)$. So the possible values of the rank of A are $n-1$, $n-2$ and $n-3$.

[**0.0.9**] ***Exercise*** Write down all the eigenvalues (along with their multiplicities) of the matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ where $a_{ij} = 1$ for all $1 \leq i, j \leq n$.

Sol. $A = xx^*$ where $x = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Using Exercise 0.0.7, 0 and $x^*x = n$ are distinct eigenvalues with algebraic multiplicities $n-1$ and 1, respectively.

[**0.0.10**] ***Exercise*** Let $A \in \mathbb{M}_3(\mathbb{C})$ be a matrix such that $A^2 = A$ (idempotent matrix). Then prove that A is diagonalizable.

Sol. Since $A^2 = I$, then $x^2 - x$ is an annihilating polynomial of A. We know that the minimal polynomial of A divides $x^2 - 1$. Then all the possibilities of the minimal polynomial of A are following.

i) x.

ii) $x - 1$.

iii) $x(x-1)$.

If x is the minimal polynomial of A, then $A = 0_n$. Hence A is diagonalizable.

If $x - 1$ is the minimal polynomial of A, then $A = I$. Hence A is diagonalizable.

If $x(x-1)$ is the minimal polynomial of A, A is diagonalizable because $x(x-1)$ is the product distinct linear factors.

[**0.0.11**] ***Exercise*** Let $A \in \mathbb{M}_3(\mathbb{C})$ be a matrix such that $A^3 = I$. Then prove that A is diagonalizable.

Sol. Since $A^3 = I$, then $x^3 - 1$ is an annihilating polynomial of A. Then $x^3 - 1 = (x-1)(x-\alpha)(x-\alpha^2)$.

We know that the minimal polynomial of A divides $x^3 - 1$. Then all the possibilities of the minimal polynomial of A are following.

i) $x - 1$.

ii) $x - \alpha$.

iii) $(x - \alpha^2)$.

iv) $(x - 1)(x - \alpha)$.

v) $(x - 1)(x - \alpha^2)$.

vi) $(x - \alpha)(x - \alpha^2)$.

vii) $(x - 1)(x - \alpha)(x - \alpha^2)$.

For each case, you can show that A is diagonalizable.

[**0.0.12**] ***Exercise*** Let $A \in \mathbb{M}_3(\mathbb{C})$ be a matrix such that $A^2 = I$ (involutory matrix). Then prove that A is diagonalizable.

Sol. Since $A^2 = I$, then $x^2 - 1$ is an annihilating polynomial of A. We know that the minimal polynomial of A divides $x^2 - 1$. Then all the possibilities of the minimal polynomial of A are following.

i) $x - 1$.

ii) $x + 1$.

iii) $(x - 1)(x + 1)$.

If $x - 1$ is the minimal polynomial of A, then $A = I$. Hence A is diagonalizable.

If $x + 1$ is the minimal polynomial of A, then $A = -I$. Hence A is diagonalizable.

If $(x - 1)(x + 1)$ is the minimal polynomial of A, A is diagonalizable because $(x - 1)(x + 1)$ is the product distinct linear factors.

[**0.0.13**] ***Exercise*** Find the minimal polynomial of the following matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

[**0.0.14**] ***Exercise*** Let $A \in \mathbb{M}_n(\mathbb{C})$. Then show that the following are equivalent.

1. A is diagonalizable.

2. $P(A)$ is nilpotent $\implies$ $P(A) = 0_n$ for any polynomial P with complex co-efficient.

Sol.

(1) $\implies$ (2). We first assume that A is diagonalizable. To prove $P(A)$ is nilpotent $\implies$ $P(A) = 0_n$ for any polynomial P with complex co-efficient.

Let $P(x)$ be a polynomial such that $P(A)$ is nilpotent. To show that $P(A) = 0_n$. Since A is diagonalizable, we have a non-singular matrix S such that $S^{-1}AS = D$. Then $P(S^{-1}AS) = P(D)$ this implies $S^{-1}P(A)S = P(D)$. Here $P(D)$ is a diagonal matrix. This says that $P(A)$ is diagonalizable.

Therefore $P(A)$ is nilpotent and diagonalizable implies $P(A) = 0_n$.

(2) $\implies$ (1). $P(A)$ is nilpotent $\implies$ $P(A) = 0_n$ for any polynomial P with complex co-efficient. To prove A is diagonalizable.

Let Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A with algebraic multiplicity $m_1, \ldots, m_k$, respectively.

Let $m = \max\{m_1, \ldots, m_k\}$. Let $P(x) = \prod_{i=1}^{k}(x - \lambda_i)$. Then $P(x)^m = P_A(x)q(x)$. Therefore $(P(A))^k = P_A(A)q(A)$. We know that $P_A(A) = 0_n$. Then $(P(A))^m = 0_n$. Therefore $P(A)$ is nilpotent and this implies $P(A) = 0$.

Hence $P(x)$ is the minimal polynomial of A which product of distinct linear factors. Then A is diagonalizable.

[**0.0.15**] ***Exercise*** Let $A, B \in \mathbb{M}_n(\mathbb{C})$.

1. If $AX - XB = 0_n$, then show that $P(A)X - XP(B) = 0_n$ for any polynomial P.

2. If A and B do not have common eigenvalues, then show that $AX - XB = 0_n \implies X = 0_n$.

Sol.

1. Let $P(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0$. We have

   $AX - XB = 0_n$

   $AX = XB$

$A^2X = AXB$ (multiplying both side by A)

$A^2X = XBB$ (use $AX = XB$)

$A^2X = XB^2$

Continuing this process we $A^kX = XB^k$ for each positive integer k.

$P(A) = a_kA^k + a_{k-1}A^{k-1} + \cdots + a_0I.$

$P(A)X = (a_kA^k + a_{k-1}A^{k-1} + \cdots + a_0I)X.$

$P(A)X = a_kA^kX + a_{k-1}A^{k-1}X + \cdots + a_0IX.$

$P(A)X = Xa_kB^k + Xa_{k-1}B^{k-1} + \cdots + Xa_0I.$

$P(A)X = XP(B).$

2. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A with algebraic multiplicity $m_1, \ldots, m_k$, respectively. Then $P_A(x) = \prod_{i=1}^{k}(x - \lambda_i)^{m_i}$.

   Using above result we have $P_A(A)X = XP_A(B)$. Using Cayley Hamilton theorem we have $P_A(A) = 0_n$. Then $XP_A(B) = 0_n$. We mow show that $P_A(B)$ is non-singular. $P_A(B) = \prod_{i=1}^{k}(B - \lambda_iI)^{m_i}$.

   Take $(B - \lambda I)^{m_i}$, the determinant of $\mathrm{DET}(B - \lambda_iI) \neq 0$, otherwise λ_i is an eigenvalue of A which is not possible. Hence $\mathrm{DET}(P_A(B)) = \prod_{i=1}^{k}(\mathrm{DET}(B - \lambda_iI))^{m_i}$. We have seen that $\mathrm{DET}(B - \lambda_iI) \neq 0$ for $i = 1, \ldots, k$. Then $\mathrm{DET}(P_A(B)) \neq 0$. This implies $P_A(B)$ is invertible.

   $XP_A(B) = 0_n$

   $X = 0_n$ (multiplying both side by the inverse of $P_A(B)$).

[0.0.16] ***Exercise*** Let $A, B \in \mathbb{M}_n(\mathbb{C})$ such that $A = AB - BA$. Let v be an eigenvector of B with eigenvalue λ

1. Prove that either Av is zero or an eigenvector of B.

2. Prove that there exists a natural k such that $A^k v = 0_n$.

Sol.

1. $A = AB - BA$.

   $Av = ABv - BAv$ (multiplying both side by v)

   $Av = \lambda Av - BAv$ $(Bv = \lambda v)$

   $BAv = (\lambda - 1)Av$. This implies either $Av = 0$ or Av is an eigenvector of B corresponding to the eigenvalue $\lambda - 1$.

2. $Av = \lambda Av - BAv$

   $A^2v = \lambda A^2v - ABAv$ (multiplying both side by A)

   $A^2v = \lambda A^2v - (A + BA)Av$ (replace AB with $A + BA$)

   $A^2v = \lambda A^2v - A^2v - BA^2v$.

   $BA^2v = (\lambda - 2)A^2v$. This implies either $A^2v = 0$ or A^2v is an eigenvector of B corresponding to the eigenvalue $\lambda - 2$.

   Continuing same process we have $BA^m v = (\lambda - k)A^m v$ where k is any positive integers. Since B is a matrix of size n, B has exactly n eigenvalue. Then there exists k such that $A^k v$ is not an eigenvector of B. Hence $A^k v = 0$.