DERIVATIVE OF ANALYTIC FUNCTION

If $f(z)$ is analytic in a domain D, then its derivative at any point $z = z_0$ is given by

$$f^{(n)}(z_0) = \frac{\lfloor\underline{n}}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad , \; n = 1, 2, \ldots$$

where C is any simple closed curve in D enclosing the point z_0.

Proof: $n = 1$:

Using Cauchy-Integral formula we have

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

and $f(z_0 + \Delta z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0 - \Delta z_0}\, dz$

$$\Rightarrow \; f(z_0 + \Delta z_0) - f(z_0) = \frac{1}{2\pi i} \oint_C f(z) \left[\frac{1}{z - z_0 - \Delta z_0} - \frac{1}{z - z_0} \right] dz$$

$$= \frac{1}{2\pi i} \oint_C f(z) \cdot \frac{\Delta z_0}{(z - z_0 - \Delta z_0)(z - z_0)}\, dz$$

$$\Rightarrow \; \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z_0)(z - z_0)} \cdot dz$$

$$\Rightarrow \; \lim_{\Delta z_0 \to 0} \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0} = \lim_{\Delta z_0 \to 0} \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z_0)(z - z_0)}\, dz$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz$$

$$\Rightarrow \; \boxed{f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz}$$

Similarly one can prove results of higher order.

Since z_0 is arbitoary in D, the derivative of $f(z)$ of all orders are analytic in D if $f(z)$ is analytic in D.

CAUCHY - INEQUALITY: Let $f(z)$ be analytic inside and on a circle C of radius r and centre z_0 then

$$\left|f^{(n)}(z_0)\right| \le \frac{M \lfloor n}{r^n} \qquad n = 0, 1, 2 \dots$$

where M is a constant such that $|f(z)| \le M$.

Proof: By Cauchy integral formula:

$$f^{(n)}(z_0) = \frac{\lfloor n}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

$$\Rightarrow \left|f^{(n)}(z_0)\right| = \frac{\lfloor n}{2\pi} \left| \oint_C \frac{f(z)}{\underbrace{(z-z_0)}_{|z-z_0| = r}{}^{n+1}}\, dz \right|$$

$$\le \frac{\lfloor n}{\cancel{2\pi}} \cdot \frac{M}{r^{n+\cancel{1}}} \cdot \cancel{2\pi}\cancel{r}$$

$$= \frac{M \lfloor n}{r^n}.$$

MORERA'S THEOREM (Converse of Cauchy's integral theorem)

If $f(z)$ is continuous in a simply connected domain D and $\oint_C f(z)\, dz = 0$ for every closed path in D, then $f(z)$ is analytic in D.

Ex. Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4}dz$ $C: |z|=3.$

Sol: Let $f(z) = e^{2z}$ $z_0 = -1$ $n = 3.$

Cauchy integral formula:

$$f^{(3)}(z_0) = \frac{\underline{|3}}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}dz$$

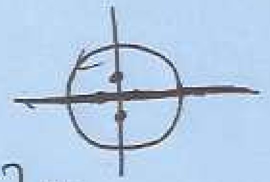
$$f'(z) = 2e^{2z} \Rightarrow f'(z_0) = 2e^{2\cdot(-1)} = 2e^{-2}$$

$$f''(-1) = 4e^{-2} \quad f^{(3)}(-1) = 8e^{-2}$$

$$\Rightarrow 8e^{-2} = \frac{6}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}dz$$

$$\Rightarrow \boxed{\oint_C \frac{e^{2z}}{(z+1)^4}dz = \frac{8\pi i e^{-2}}{3}}$$

Ex: Evaluate $\oint_C \frac{e^{zt}}{z^2+1}dz$ $C: |z|=3$



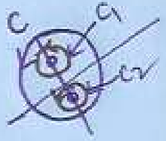
Sol: $\oint_C \frac{e^{zt}}{z^2+1}dz = \oint_C \frac{e^{zt}}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]dz$

$$= \frac{1}{2i}\left[\oint_C \frac{e^{zt}}{z-i}dz - \oint_C \frac{e^{zt}}{z+i}dz\right]$$

$$= \frac{1}{2i}\,2\pi i\left[e^{it} - e^{-it}\right]$$

$$= 2\pi i \sin t.$$

METHOD 2



$\oint_C \frac{e^{zt}}{z^2+1}dz = \oint_{C_1}\frac{e^{zt}}{z^2+1}dz + \oint_{C_2}\frac{e^{zt}}{z^2}$

$= \oint_{C_1}\frac{\frac{e^{zt}}{z+i}}{z-i}dz + \oint_{C_2}\frac{\frac{e^{zt}}{z-i}}{z+i}d$

$= 2\pi i\left[\frac{e^{it}}{2i} - \frac{e^{-it}}{2i}\right]$

$= 2\pi i \sin t$

Ex. Evaluate

$$\oint_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}\,dz \qquad C: |z|=1$$

Sol:

$$f(z) = \sin^6 z \quad z_0 = \frac{\pi}{6}, n=2$$

Cauchy integral formula.

$$f^{(2)}(\tfrac{\pi}{6}) = \frac{|2}{2\pi i}\oint_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^{2+1}}\,dz$$

$$\Rightarrow \oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{2\pi i}{2}\cdot f''(\tfrac{\pi}{6})$$

Note that $f'(z) = 6\sin^5 z\cos z$

$$f''(z) = 30\sin^4 z\cos^2 z + 6\sin^5 z(-\sin z)$$

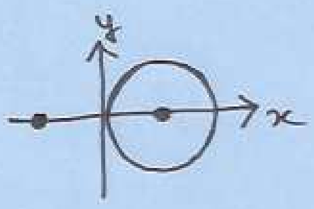
$$\Rightarrow f''(\tfrac{\pi}{6}) = 30\cdot\frac{1}{16}\cdot\frac{3}{4} - 6\cdot\frac{1}{32}\cdot\frac{1}{2}$$

$$= \frac{90-6}{64} = \frac{\cancel{84}^{21}}{\cancel{64}_{16}} = \frac{21}{16}$$

$$\Rightarrow \oint_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \pi i\cdot\frac{21}{16}.$$

Ex. Evaluate $\oint_C \frac{3z^2+z}{z^2-1}\,dz$ $\quad$ C: $|z-1|=1$

Singularities of integrand:



$$z^2-1=0 \Rightarrow z=\pm 1$$

Method I:

$$\oint_C \frac{3z^2+z}{(z+1)(z-1)}\,dz = \frac{1}{2}\oint_C \frac{3z^2+z}{z-1}\,dz + \frac{1}{2}\oint_C \frac{3z^2+z}{z+1}\cdot dz$$

$$= \frac{1}{2}\cdot 2\pi i\,(3+1) + 0$$

↑ Cauchy integral formula $\qquad$ ↑ Cauchy theorem

$$= 4\pi i$$

Method II:

$$\oint_C \frac{3z^2+z}{(z+1)(z-1)}\,dz = \oint_C \frac{\frac{3z^2+z}{z+1}}{(z-1)}\,dz$$

$$= 2\pi i\cdot\left(\frac{3+1}{2}\right)$$

$$= 4\pi i.$$