INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR

Date.................................FN / AN Time : 3 Hrs. Full Marks 50 No. of Students 55

Autumn / ~~Spring Semester~~, 2011-11. Deptt. Mathematics Sub. No. MA 40001 & MA 41007

4th Yr. ~~B.Tech. (H) / B.Arch. (H)~~ / M.Sc. (Int.) Sub. Name Functional Analysis

& 1st Yr. (2 Yr.) M.Sc. Maths

Instruction

Attempt Any FIVE Questions.

1(a) Let X and Y be metric spaces and $T : X \to Y$ a continuous mapping. Prove that the image of a compact subset M of X under T is compact.

(b) Let Z be a vector space of all matrices $A = (a_{ij})$ of order $m \times n$ with fixed m and n under usual addition and scalar multiplication of matrices. Define analogous norms of $\|\cdot\|_1$, $\|\cdot\|_2$ & $\|\cdot\|_\infty$ (defined below for R^n) for the space Z where

$$\|x\|_1 = \sum_{i=1}^{n} |\xi_i| \;;\; \|x\|_2 = \left(\sum_{i=1}^{\infty} |\xi_i|^2\right)^{1/2},\; \|x\|_\infty = \max_{1 \le i \le n} |\xi_i| \quad \text{for}$$

$x = (\xi_1, \xi_2, \ldots, \xi_n) \in R^n$. Are these norms equivalent.

(c) Show that in a Banach space, an absolutely convergent series is convergent.

2(a) Show that: a finite dimensional vector space is algebraically reflexive.

(b) Show that the dual space of the space c_0 is l_1.

(c) Let $M \neq \phi$ be any subset of a normed space X. The annihilator M^α of M is defined to be the set of all bounded linear functionals on X which are zero everywhere on M. Show that M^α is vector subspace of X', dual of and is closed. What are X^α and $\{0\}^\alpha$?

3(a) State and prove polarization identity in order to 'rediscover' the inner product from the corresponding norm.

(b) Let Y be a subspace of a Hilbert space H. Then

(P.T.O.)

if H is separable, then prove that Y is separable.

(c) State and prove Projection theorem (in case of Hilbert space)

4 (a) Prove that: For any subset $M \neq \Phi$ of a Hilbert space H, the span of M is dense in H if and only if $M^{\perp} = \{0\}$.

(b) If (e_K) is an orthogonal sequence each having norm 1 in an IPS X, and $x \in X$, show that $x - y$ with y given by $y_1 = \sum_{K=1}^{n} \alpha_K e_K$, $\alpha_K = \langle x, e_K \rangle$ is orthogonal to the subspace $Y_n = \text{Span}\{e_1, e_2, \ldots, e_n\}$.

(c) Show that any linear functional f on R^3 can be represented by a dot product:

$f(x) = x \cdot z = x_1 z_1 + x_2 z_2 + x_3 z_3$ where $x = (x_1, x_2, x_3) \in R^3$, $z = (z_1, z_2, z_3) \in R^3$.

5 (a) If z is any fixed element of an IPS X, Show that $f(x) = \langle x, z \rangle$ defines a bounded linear functional f on X, of norm $\|z\|$.

(b) Let $T : H_1 \to H_2$ be a bounded linear operator from Hilbert space H_1 to Hilbert space H_2. Then prove that

$\|T^* T\| = \|T T^*\| = \|T\|^2$

(c) Let (T_n) be a sequence of bounded self-adjoint linear operators $T_n : H \to H$ on a Hilbert space H. Suppose $T_n \xrightarrow{H} T$ i.e. $\|T_n - T\|_H \to 0$. Show that the limit operator T is a bounded self-adjoint linear operator on H.

6 (a) State and prove Hahn-Banach theorem (in case of Normed spaces).

(b) Show that the column vectors of a unitary matrix constitute an orthonormal set with respect to the inner product on $\mathbb{C}^n$

(c) If $f(x) = f(y)$ for every bounded linear functional f on a normed space X, show that $x = y$.

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