Cauchy-Euler Equations

A linear differential equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X \tag{8}$$

or

$$(x^n D^n + a_1 D^{n-1} + \cdots + a_n)y = X \tag{9}$$

is called Euler-Cauchy equation.

Working Rule: To solve equation (8) we change the variable from x to z by putting $x = e^z$ i.e. $z = \ln(x)$.

$$z = \ln(x) \Rightarrow \frac{dz}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \Rightarrow \boxed{x \frac{dy}{dx} = \frac{dy}{dz}}$$

We define a new operator

$$x \frac{d}{dx} \equiv \frac{d}{dz} \equiv D_1$$

Again

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dz} \right) = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d}{dx} \left(\frac{dy}{dz} \right) = -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x^2} \frac{d^2 y}{dz^2}$$

$$\Rightarrow x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} = D_1(D_1 - 1)y$$

Thus we have the following formulas for $D \equiv \frac{d}{dx}$ and $D_1 \equiv \frac{d}{dz}$

$$xD = D_1$$
$$x^2 D^2 = D_1(D_1 - 1)$$
$$x^3 D^3 = D_1(D_1 - 1)(D_1 - 2)$$
$$\vdots$$
$$x^n D^n = D_1(D_1 - 1)(D_1 - 2) \ldots (D_1 - n + 1)$$

Substituting these operator relations in the equation (9), we obtain a linear differential equation with constant coefficient

$$f(D_1)y = Z, \quad \text{where Z becomes a function of z only}$$

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Example 1.

$$(x^2 D^2 - xD + 2)y = x \ln x \qquad (10)$$

Let $x = e^z$ so that $z = \ln x$ and $D_1 \equiv \frac{d}{dz}$ then the equation (10) becomes

$$[D_1(D_1 - 1) - D_1 + 2]\, y = ze^z$$

Auxiliary equation $m^2 - 2m + 2 = 0$ and its roots are $m = 1 \pm i$
Hence

C.F. $= e^z\, [c_1 \cos(z) + c_2 \sin(z)] = x\, [c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))]$

$$\text{P.I.} = \frac{1}{D_1^2 - 2D_1 + 2} ze^z = e^z \frac{1}{(D_1 + 1)^2 - 2(D_1 + 1) + 2} z$$
$$= e^z \frac{1}{D_1^2 + 1} z = e^z (1 + D_1^2)^{-1} z = e^z z = x \ln(x)$$

General solution

$$\boxed{y = x\, [c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))] + x \ln(x)}$$

Equations reducible to Euler-Cauchy form There can be several forms of equation which can be reduced to Euler-Cauchy form

Example 1: Solve
$$\frac{d^3 y}{dx^3} - \frac{4}{x}\frac{d^2 y}{dx^2} + \frac{5}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = 1$$

Solution: $y = c_1 x^2 + c_2 x^{(5+\sqrt{21})/2} + c_3 x^{(5-\sqrt{21})/2} - x^3/5$

Example 2:
$$2x^2 y \frac{d^2 y}{dx^2} + 4y^2 = x^2 \left(\frac{dy}{dx}\right)^2 + 2xy \frac{dy}{dx}$$

Hint: $y = z^2$

Solution:

$$y = z^2 \Rightarrow \frac{dy}{dx} = 2z \frac{dz}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\left(\frac{dz}{dx}\right)^2 + 2z\frac{d^2z}{dx^2}$$

Substituting these values in the differential equation we get

$$x^2\frac{d^2z}{dx^2} - x\frac{dz}{dx} + z = 0$$

or

$$[x^2 D^2 - xD + 1]z = 0$$

Substitute $x = e^t \Leftrightarrow \ln x = t$

$$\Rightarrow x\frac{dz}{dx} = \frac{dz}{dt} \Rightarrow xD \equiv D_1$$

Similarly

$$x^2 D^2 = D_1(D_1 - 1)$$

Then the equation becomes

$$[D_1^2 - 2D_1 + 1]z = 0 \Rightarrow z = [c_1 + c_2 t]e^t$$

$$\Rightarrow z = [c_1 + c_2 \ln(x)]x$$

$$\Rightarrow \boxed{y = (c_1 + c_2 \ln(x))^2 x^2}$$

Example 3: A differential equation of the form

$$(a + bx)^n\frac{d^n y}{dx^n} + a_1(a + bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(a + bx)\frac{dy}{dx} + a_n y = X$$

can be reduced to Euler-Cauchy equation by putting

$$a + bx = v \Rightarrow \frac{dv}{dx} = b$$

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = b\frac{dy}{dv}$$

Again

$$\frac{d^2y}{dx^2} = b^2\frac{d^2y}{dv^2} \text{ or in general } \frac{d^n y}{dx^n} = b^n\frac{d^n y}{dv^n}$$

Substituting these derivatives in the equation, we get

$$v^n \frac{d^n y}{dv^n} + \frac{a_1}{b} v^{n-1} \frac{d^{n-1}y}{dv^{n-1}} + \cdots + \frac{a_{n-1}}{b^{n-1}} v \frac{dy}{dv} + \frac{a_n}{b^n} y = \frac{X}{b^n}$$

which is an standard Euler-Cauchy equation.

Example 4: solve

$$(1+x)^2 \frac{d^2 y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos \ln(1+x)$$

Solution: Let $(1+x) = v \Rightarrow \frac{dv}{dx} = 1$.
Hence $\frac{dy}{dx} = \frac{dy}{dv}$ and $\frac{d^2 y}{dx^2} = \frac{d^2 y}{dv^2}$ and the differential equation becomes

$$v^2 \frac{d^2 y}{dv^2} + v \frac{dy}{dv} + y = 4\cos \ln v$$

Put $v = e^z \Rightarrow \ln(v) = z$ and let $D_1 \equiv \frac{d}{dz}$

$$[D_1(D_1 - 1) + D_1 + 1] y = 4\cos z$$

$$\left(D_1^2 + 1\right) y = 4\cos z$$

$$\text{C.F.} = c_1 \cos(z) + c_2 \sin(z) = c_1 \cos(\ln v) + c_2 \sin(\ln v)$$
$$= c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+x))$$

$$\text{P.I.} = 2z \sin z = 2\ln(v) \sin(\ln(v)) = 2\ln(1+x) \sin(\ln(1+x)).$$

The general solution

$$\boxed{y = c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+x)) + 2\ln(1+x) \sin(\ln(1+x)).}$$