Solution Model for Assignment 3

Sol^n 9) Given $f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$

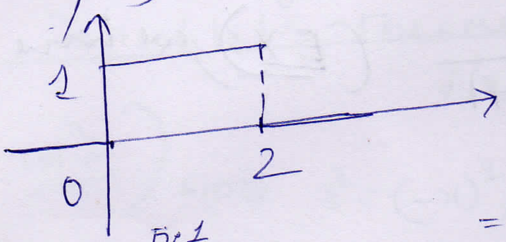

Fig 1

we have, $A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos(\alpha x)\, dx$

$$= \int_{-\infty}^{0} f(x) \cos \alpha x\, dx + \int_0^2 f(x) \cos \alpha x\, dx + \int_2^{\infty} f(x) \cos \alpha x\, dx$$

$$= \int_0^2 \cos \alpha x\, dx = \frac{\sin 2\alpha}{\alpha}$$

& $B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin(\alpha x)\, dx = \int_0^2 \sin(\alpha x)\, dx = \frac{1 - \cos(2\alpha)}{\alpha}$.

$\therefore$ the Fourier integral representation of the given f^n is

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\left(\frac{\sin 2\alpha}{\alpha} \right) \cos \alpha x + \left(\frac{1 - \cos 2\alpha}{\alpha} \right) \sin \alpha x \right] d\alpha$$

$$= \frac{2}{\pi} \int_0^{\infty} \left[\frac{\sin \alpha \cos[\alpha(x-1)]}{\alpha} \right] d\alpha \quad //$$

Sol^n 8) Given $\int_0^{\infty} f(x) \sin(\alpha x)\, dx = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1. \end{cases}$

Let $F_s(\alpha) = \int_0^{\infty} f(x) \sin(\alpha x)\, dx = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

Then $f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\alpha) \sin(\alpha x)\, d\alpha$

$$= \frac{2}{\pi} \int_0^1 (1-\alpha) \sin(\alpha x)\, d\alpha$$

$$= \frac{2(x - \sin x)}{\pi x^2}. \quad //$$

Sol^n 7) Given, $\int_0^{\infty} \frac{\cos(\lambda x)}{(\lambda^2+1)}\, d\lambda = \pi/2\, e^{-x}, \; x \geq 0.$

Let $f(x) = e^{-x}$ in the Fourier integral theorem

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos \lambda x \, d\lambda \int_0^\infty f(u) \cos \lambda u \, du$$

Then $\frac{2}{\pi}\int_0^\infty \cos \lambda x \, d\lambda \int_0^\infty e^{-u} \cos \lambda u \, du = e^{-x}$.

Since, $\int_0^\infty e^{-u} \cos \lambda u \, du = \frac{1}{\lambda^2+1}$ (EX), we have

$$\frac{2}{\pi}\int_0^\infty \frac{\cos \lambda x}{\lambda^2+1} d\lambda = e^{-x}$$

$$\Rightarrow \int_0^\infty \frac{\cos \lambda x}{\lambda^2+1} d\lambda = \frac{\pi}{2} e^{-x}. \quad //$$

Sol^{n}1) Given, $f(x) = |x|, \; -3 < x < 3$, $\quad 2l = 6 \Rightarrow l = 3$.

$f(x) = f(x+6)$.

The f^n $f(x) = |x|$ is continuous,

$\therefore a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx = \frac{1}{3}\int_{-3}^{3} |x|\, dx = 3$ (how?)

$$\boxed{|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx = \frac{1}{3}\int_{-3}^{3} |x| \cos\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{2}{3}\int_0^3 x \cos\left(\frac{n\pi x}{3}\right) dx = \frac{2}{3}\left[\frac{3x}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\right]_0^3 - \frac{2}{3}\int_0^3 \frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right) dx$$

$$\Rightarrow a_n = 0 + \frac{2}{n\pi}\left[\frac{3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)\right]_0^3$$

$$= \frac{6}{(n\pi)^2}\left[-1 + (-1)^n\right].$$

$$\therefore a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -12/(n\pi)^2, & \text{if } n \text{ is odd.} \end{cases}$$

$$\therefore a_{2k+1} = -\frac{12}{(2k+1)^2\pi^2}, \quad k = 0, 1, 2, \ldots$$

S^{ly}, $b_n = \frac{1}{l}\int_{-l}^{l} |x| \sin\left(\frac{n\pi x}{3}\right) dx = \frac{1}{3}\int_{-3}^{3} |x| \sin\left(\frac{n\pi x}{3}\right) dx$

$$= -\frac{1}{3}\int_{-3}^{0} x \sin\left(\frac{n\pi x}{3}\right) dx + \frac{1}{3}\int_0^3 x \sin\left(\frac{n\pi x}{3}\right) dx = 0, \forall n.$$

Hence, $f(x) = \frac{3}{2} - \frac{12}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n-1)^2} \cos\left[\frac{2n-1}{3}\pi x\right], 0 \le x \le 3.$

Here, $f(x)$ is continuous on $[0, 3]$, hence there is no discontinuity at the end points & no need to invoke Dirichlet's theorem.

$$f(x) = x^2, \quad f(x) = f(x+2\pi), \quad -\pi \le x \le \pi.$$

sol^{n}2) since $x^2 = (-x)^2$, $f(x)$ is an even f^n. Thus the Fourier series consists solely of even f^{ns} which means $b_n = 0, \forall n$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{3}\cdot\frac{1}{\pi}\left[x^3\right]_{-\pi}^{\pi} = \frac{2}{3}\pi^2$$

& $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\, dx \quad (n \ne 0)$

$$= \frac{1}{\pi}\left[\frac{x^2}{n}\sin(nx)\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{2x}{n}\sin(nx)\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{2x}{n^2}\cos(nx)\right]_{-\pi}^{\pi} - \frac{2}{\pi n}\int_{-\pi}^{\pi}\frac{\cos(nx)}{n}dx$$

$$= \frac{4}{n^2}(-1)^n$$

$$\therefore f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx), \quad -\pi \le x < \pi \quad \longrightarrow (1)$$

Putting $x = \pi$ in (1), we get

$$\pi^2 = \pi^2/3 + 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2/6$$

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sol^{n}3) Let $f(x) = H(x)$

$\therefore$ the given Fourier series is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}$$

Putting $x = \pi/2$, we get

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \pi/4,$$

a series attributed to the Scottish mathematician James Gregory (1638-1675)

Sol^n 4) Given, $t(\pi - t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{(2n-1)^3}, \quad 0 \le t \le \pi$

Parseval's theorem is

$$\int_{-\pi}^{\pi} [f(t)]^2 dt = \pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Now, the version for sine series is

$$\int_0^{\pi} [f(t)]^2 dt = \frac{\pi}{2} \sum_{n=1}^{\infty} b_n^2 \quad (\text{as } a_0 = a_n = 0),$$

$$L.H.S = \int_0^{\pi} t^2 (\pi - t)^2 dt = \frac{\pi^5}{30}$$

$$R.H.S = \frac{\pi}{2} \sum_{n=1}^{\infty} b_n^2 = \frac{32}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \boxed{\frac{\pi^6}{960}}$$

Also, we note that $\sum_{n=1}^{\infty} \frac{1}{n^6} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} + \frac{1}{2^6} \sum_{n=1}^{\infty} \frac{1}{n^6}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{64 \pi^6}{63 \times 960} = \boxed{\frac{\pi^6}{945}}$$

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5) Given, $f(t) = t^2 + t$, $-\pi \le t \le \pi$,
$f(t) = f(t+2\pi)$.

Given $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$

& $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$

we define $d_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,e^{int}\,dt = a_n + i b_n$

so that $d_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (t^2+t)\,e^{int}\,dt$

$$= \frac{1}{\pi}\left[\frac{t^2+t}{in}e^{int}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{2t+1}{in}\right)e^{int}\,dt$$

$$= \frac{1}{\pi}\left[\frac{t^2+t}{in}e^{int} - \frac{(2t+1)}{(in)^2}e^{int} \cdot \frac{2}{(in)^3}e^{int}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2+\pi}{in} - \frac{(\pi^2-\pi)}{in} + \frac{2\pi+1}{n^2} - \frac{(-2\pi+1)}{n^2}\right](-1)^n.$$

$$= (-1)^n\left(\frac{4}{n^2} + \frac{2}{in}\right) = (-1)^n\left(\frac{4}{n^2} - i\frac{2}{n}\right)$$

so, $a_n = \frac{4}{n^2}(-1)^n$, $b_n = \frac{2}{n}(-1)^{n+1}$

& $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+t)\,dt = \frac{2}{3}\pi^2$.

∴ the given Fourier series is

$$f(t) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}(-1)^n\left(\frac{4\cos(nt)}{n^2} - \frac{2\sin(nt)}{n}\right),$$

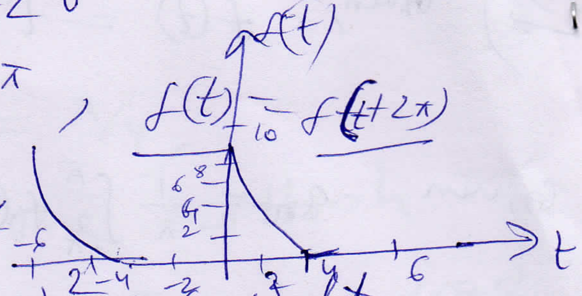
$-\pi < t < \pi$

∴ the complex Fourier series is

$$f(t) = \sum_{t=-\infty}^{\infty} d_n e^{int} = (-1)^n\sum_{t=-\infty}^{\infty}\left(\frac{4}{n^2} + \frac{2}{in}\right)e^{int},\ d_{-n} = a_n - ib_n,\ d_0 = \frac{1}{2}a_0,$$

$n = 1, 2, \ldots$

Sol^n 6) let $f(t) = \begin{cases} \pi^2, & -\pi < t < 0 \\ (t-\pi)^2, & 0 \le t < \pi \end{cases}$, $f(t) = f(t+2\pi)$



The sine series is given by

$a_n = 0$, $b_n = \frac{2}{\pi}\int_0^{\pi}(t-\pi)^2 \sin(nt)\,dt$

$\therefore f(t) = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)t}{(2k+1)} + 2\pi\sum_{n=1}^{\infty}\frac{\sin(nt)}{n}$.

The cosine series is given by

$b_n = 0$, $a_n = \frac{2}{\pi}\int_0^{\pi}(t-\pi)^2\cos(nt)\,dt$

$\therefore f(t) = -\pi^2/3 + 4\sum_{n=1}^{\infty}\frac{\cos(nt)}{n^2}$

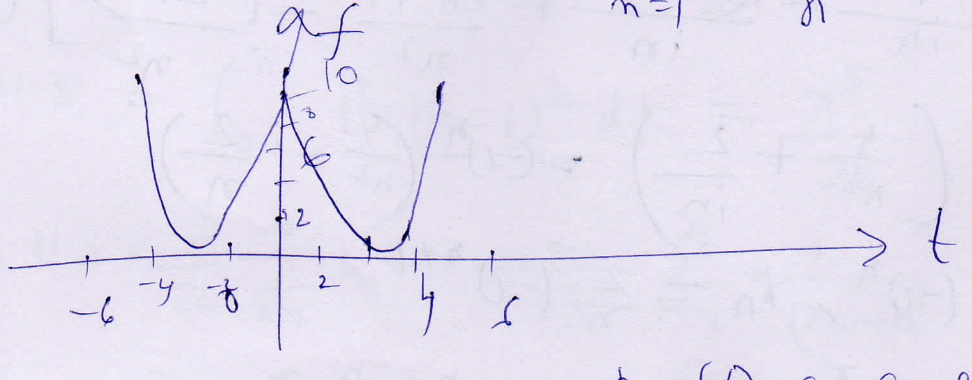
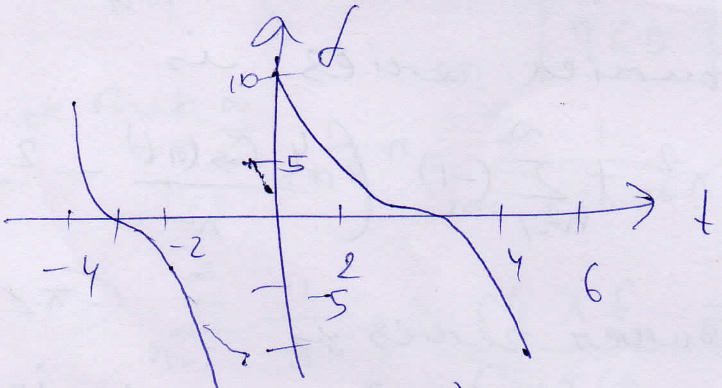


Fig 2.1 — The f^n $f(t)$ as an even f^n.



The f^n $f(t)$ as an odd f^n.