Tutorial Sheet - 11

SPRING 2017

MATHEMATICS-II (MA10002)

January 4, 2017

1. (a) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are constant vectors, then show that $\vec{r} = \vec{a}t^2 + \vec{b}t + \vec{c}$ is the path of a particle moving with constant acceleration.
   (b) Prove that a non-constant vector $\vec{u}$ has a constant length if and only if $\vec{u} \cdot \dfrac{d\vec{u}}{dt} = 0$.

2. Evaluate the following limits
   (a) $\lim_{t \to 1} \vec{r}(t)$, where $\vec{r}(t) = e^{t-1}\hat{i} + 4t\hat{j} + \dfrac{t-1}{t^2-1}\hat{k}$.
   (b) $\lim_{t \to 2} \vec{r}(t)$, where $\vec{r}(t) = \dfrac{1-e^{t+2}}{t^2+t+2}\hat{i} + \hat{j} + (t^2+6t)\hat{k}$.
   (c) $\lim_{t \to 1} \vec{r}(t)$, where $\vec{r}(t) = t^3\hat{i} + \dfrac{\sin(3t-3)}{t-1}\hat{j} + e^{2t}\hat{k}$.

3. Determine the vector equation for the line segment that starts at the point $P = (x_1, y_1, z_1)$ and ends at the point $Q = (x_2, y_2, z_2)$.

4. Find the gradient and the unit normal vector to the following surfaces
   (a) $x^2 + y - z = 4$ at the point $(2, 0, 0)$.
   (b) $x^2 + 2y^2 + 3z^2 = 0$ at the point $(\sqrt{10}, 0, 0)$.
   (c) $x^2y + 2xz = 4$ at the point $(2, -2, 3)$.

5. Find the directional derivatives of the following scalar valued functions
   (a) $f(x, y) = e^x \cos y$ at the point $(0, \frac{\pi}{4})$ in the direction of $(\hat{i} + 3\hat{j})/\sqrt{10}$.
   (b) $f(x, y, z) = e^x + yz$ at the point $(1, 1, 1)$ in the direction of $\hat{i} - \hat{j} + \hat{k}$.
   (c) $f(x, y, z) = \frac{1}{x^2+y^2+z^2}$ at the point $(2, 3, 1)$ in the direction of $\hat{i} + \hat{j} - 2\hat{k}$.

6. Find the directional derivative of the scalar valued function $f(x, y) = \dfrac{y}{x^2+y^2}$ at the point $(0, 1)$ in the direction of a vector which makes an angle of 30° with the positive x-axis.

7. (a) In what direction from the point $(1, 3, 2)$ the directional derivatives of $\phi = 2xz - y^2$ is maximum? What is the magnitude of this maximum?
   (b) Find the values of the constant a, b and c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at the point $(1, 2, -1)$ has maximum of magnitude 64 in the direction of the z-axis.

8. If $r = |\vec{r}|$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then prove that
   (a) $\nabla(\dfrac{1}{r}) = -\dfrac{\vec{r}}{r^3}$.

(b) $\nabla(\log(|\vec{r}|)) = \dfrac{\vec{r}}{r^2}$.

(c) $\nabla(r^n) = nr^{n-2}\vec{r}$.

9. Let $\vec{F} = 2xz^2\hat{i} + \hat{j} + xy^3z\hat{k}$ and $f = x^2y$. Then compute the following

   (a) $curl(\vec{F})$

   (b) $\vec{F} \times \nabla f$

   (c) $\vec{f} \cdot (\nabla f)$

10. For any two vector fields $\vec{F}$ and $\vec{G}$ show that

    (a) $\nabla \cdot (\nabla \times \vec{F}) = 0$

    (b) $div(\vec{F} \times \vec{G}) = curl(\vec{F}) \cdot \vec{G} - curl(\vec{G}) \cdot \vec{F}$

    (c) $\nabla \times (\nabla\vec{F}) = \vec{0}$

11. For all smooth scalar field f and g, show that

    (a) $\nabla(fg) = g\nabla f + f\nabla g$

    (b) $\nabla(\dfrac{f}{g}) = \dfrac{1}{g^2}(g\nabla f - f\nabla g)$

12. Check whether $\vec{F}$ is a conservative vector field or not. If it is, then find the corresponding potential function, where

    (a) $\vec{F} = (2xy, x^2 + 2yz, y^2)$

    (b) $\vec{F} = (2xy + z^3, x^2, 3xz^2)$

13. (a) Find the values of the constant a, b and c so that the vector $\vec{w} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$ becomes irrotational.
    (b) Determine the constant a so that the vector $\vec{v} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}$ is solenoidal.

14. Let $\vec{F} = yz^2\hat{i} + xy\hat{j} + yz\hat{k}$ be a vector field. Then what is the value of $div(curl(\vec{F}))$?

15. Let $\phi(x, y, x)) = x^2y - xe^z$, $P_0 = (2, -1, \pi)$ and $\vec{u} = \dfrac{1}{\sqrt{6}}(\hat{i} - 2\hat{j} + \hat{k})$. Then what is the rate of changes of $\phi(x, y, x))$ at the point P_0 in the direction of the vector $\vec{u}$?

16. Consider the surface $\phi(x, y, z) = z - \sqrt{x^2 + y^2} = 0$. Then find the normal vector and the tangent plane to the surface at the point $(1, 1, \sqrt{2})$.