1.9 Onto Functions and the Stirling Numbers of Second Kind

Before proceeding further, recall the definition of partition of a non-empty set into m parts given on Page 18.

Definition 1.9.1. *Let $|A| = n$. Then the number of partitions of the set A into m-parts is denoted by $S(n, m)$. The symbol $S(n, m)$ is called the* STIRLING NUMBER OF THE SECOND KIND.

Remark 1.9.2. *1. The following conventions will be used:*

$$S(n, m) = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n > 0, m = 0 \\ 0, & \text{if } n < m. \end{cases}.$$

2. If $n > m$ then a recursive method to compute the numbers $S(n, m)$ is given in Lemma 1.9.5. A formula for the numbers $S(n, m)$ is also given in Equation (1.2).

3. Consider the problem of DETERMINING THE NUMBER OF WAYS OF PUTTING m **distinguishable/distinct** BALLS INTO n **indistinguishable** BOXES WITH THE RESTRICTION THAT NO BOX IS EMPTY.

Let $M = \{a_1, a_2, \ldots, a_m\}$ be the set of m distinct balls. Then, we observe the following:

(a) Since the boxes are indistinguishable, we can assume that the number of balls in each of the boxes is in a non-increasing order.

(b) Let A_i, for $1 \leq i \leq n$, denote the set of balls in the i-th box. Then $|A_1| \geq |A_2| \geq \cdots \geq |A_n|$ and $\bigcup\limits_{i=1}^{n} A_i = M$.

(c) As each box is non-empty, each A_i is non-empty, for $1 \leq i \leq n$.

Thus, we see that we have obtained a partition of the set M, consisting of m elements, into n-parts, $A_1, A_2, \ldots, A_n$. Hence, the number of required ways is given by $S(m, n)$, the Stirling number of second kind.

We are now ready to look at the problem of counting the number of onto functions $f : M \longrightarrow N$. But to make the argument clear, we take an example.

Example 1.9.3. *Let $f : \{a, b, c, d, e\} \longrightarrow \{1, 2, 3\}$ be an onto function given by*

$$f(a) = f(b) = f(c) = 1, \; f(d) = 2 \text{ and } f(e) = 3.$$

Then this onto function, gives a partition $B_1 = \{a, b, c\}, B_2 = \{d\}$ and $B_3 = \{e\}$ of the set $\{a, b, c, d, e\}$ into 3-parts. Also, suppose that we are given a partition $A_1 = \{a, d\}$, $A_2 = \{b, e\}$

and $A_3 = \{c\}$ *of* $\{a, b, c, d, e\}$ *into* 3*-parts. Then, this partition gives rise to the following* $3!$ *onto functions from* $\{a, b, c, d, e\}$ *into* $\{1, 2, 3\}$*:*

$$\begin{array}{lll}
f_1(a) = f_1(d) = 1,\ f_1(b) = f_1(e) = 2, f_1(c) = 3, & i.e., & f_1(A_1) = 1, f_1(A_2) = 2, f_1(A_3) = 3 \\
f_2(a) = f_2(d) = 1,\ f_2(b) = f_2(e) = 3, f_2(c) = 2, & i.e., & f_2(A_1) = 1, f_2(A_2) = 3, f_2(A_3) = 2 \\
f_3(a) = f_3(d) = 2,\ f_3(b) = f_3(e) = 1, f_3(c) = 3, & i.e., & f_3(A_1) = 2, f_3(A_2) = 1, f_3(A_3) = 3 \\
f_4(a) = f_4(d) = 2,\ f_4(b) = f_4(e) = 3, f_4(c) = 1, & i.e., & f_4(A_1) = 2, f_4(A_2) = 3, f_4(A_3) = 1 \\
f_5(a) = f_5(d) = 3,\ f_5(b) = f_5(e) = 1, f_5(c) = 2, & i.e., & f_5(A_1) = 3, f_5(A_2) = 1, f_5(A_3) = 2 \\
f_6(a) = f_6(d) = 3,\ f_6(b) = f_6(e) = 2, f_6(c) = 1, & i.e., & f_6(A_1) = 3, f_6(A_2) = 2, f_6(A_3) = 1.
\end{array}$$

Lemma 1.9.4. *Let* M *and* N *be two finite sets with* $|M| = m$ *and* $|N| = n$*. Then the total number of onto functions* $f : M \longrightarrow N$ *is* $n!S(m, n)$*.*

Proof: By definition, "f is onto" implies that "for all $y \in N$ there exists $x \in M$ such that $f(x) = y$. Therefore, the number of onto functions $f : M \longrightarrow N$ is 0, whenever $m < n$. So, let us assume that $m \geq n$ and $N = \{b_1, b_2, \ldots, b_n\}$. Then, we observe the following:

1. Fix i, $1 \leq i \leq n$. Then $f^{-1}(b_i) = \{x \in M | f(x) = b_i\}$ is a non-empty set as f is an onto function.

2. $f^{-1}(b_i) \cap f^{-1}(b_j) = \emptyset$, whenever $1 \leq i \neq j \leq n$ as f is a function.

3. $\bigcup\limits_{i=1}^{n} f^{-1}(b_i) = M$ as the domain of f is M.

Therefore, if we write $A_i = f^{-1}(b_i)$, for $1 \leq i \leq n$, then $A_1, A_2, \ldots, A_n$ gives a partition of M into n-parts. Also, for $1 \leq i \leq n$ and $x \in A_i$, we note that $f(x) = b_i$. That is, for $1 \leq i \leq n$, $|f(A_i)| = |\{b_i\}| = 1$.

Conversely, each onto function $f : M \longrightarrow N$ is completely determined by

- a partition, say $A_1, A_2, \ldots, A_n$, of M into $n = |N|$ parts, and

- a one-to-one function $g : \{A_1, A_2, \ldots, A_n\} \longrightarrow N$, where $f(x) = b_i$, whenever $x \in A_j$ and $g(A_j) = b_i$.

Hence,

$$\begin{aligned}
\big|\{f : M \longrightarrow N : f \text{ is onto}\}\big| &= \big|\{g : \{A_1, A_2, \ldots, A_n\} \longrightarrow N : g \text{ is one-to-one}\}\big| \\
&\qquad \times \big|\text{Partition of } M \text{ into } n\text{-parts}\big| \\
&= n!\ S(m, n). \qquad (1.1)
\end{aligned}$$

Lemma 1.9.5. *Let* m *and* n *be two positive integers and let* $\ell = \min\{m, n\}$*. Then*

$$n^m = \sum_{k=1}^{\ell} \binom{n}{k} k! S(m, k). \qquad (1.2)$$

Proof: Let M and N be two sets with $|M| = m$ and $|N| = n$ and let A denote the set of all functions $f: M \longrightarrow N$. We compute $|A|$ using two different methods to get Equation (1.2).

The first method uses Lemma 1.7.1 to give $|A| = n^m$. The second method uses the idea of onto functions. Let $f_0 : M \longrightarrow N$ be any function and let $K = f_0(M) = \{f_0(x) : x \in M\} \subset N$. Then, using f_0, we define a function $g: M \longrightarrow K$, by $g(x) = f_0(x)$, for all $x \in M$. Then clearly g is an onto function with $|K| = k$ for some $k, 1 \le k \le \ell = \min\{m, n\}$. Thus, $A = \bigcup\limits_{k=1}^{\ell} A_k$, where $A_k = \{f : M \longrightarrow N \mid |f(M)| = k\}$, for $1 \le k \le \ell$. Note that $A_k \cap A_j = \emptyset$, whenever $1 \le j \ne k \le \ell$. Now, using Lemma 1.8.1, a subset of N of size k can be selected in $\binom{n}{k}$ ways. Thus, for $1 \le k \le \ell$

$$|A_k| = \Big|\{K : K \subset N, |K| = k\}\Big| \times \Big|\{f : M \longrightarrow K \mid f \text{is onto}\}\Big| = \binom{n}{k} k! S(m,k).$$

Therefore,

$$|A| = \left|\bigcup_{k=1}^{\ell} A_i\right| = \sum_{k=1}^{\ell} |A_k| = \sum_{k=1}^{\ell} \binom{n}{k} k! S(m,k).$$

■

Remark 1.9.6. *1. The numbers $S(m,k)$ can be recursively calculated using Equation (1.2).*

(a) For example, taking $n \ge 1$ and substituting $m = 1$ in Equation (1.2) gives

$$n = n^1 = \sum_{k=1}^{1} \binom{n}{k} k! S(1,k) = n \cdot 1! \cdot S(1,1).$$

Thus, $S(1,1) = 1$. Now, using $n = 1$ and $m \ge 2$ in Equation (1.2) gives

$$1 = 1^m = \sum_{k=1}^{1} \binom{1}{k} k! S(m,k) = 1 \cdot 1! \cdot S(m,1).$$

Hence, the above two calculations implies that $S(m,1) = 1$ for all $m \ge 1$.

(b) Use this to verify that $S(5,2) = 15,\ S(5,3) = 25, ; S(5,4) = 10,\ S(5,5) = 1$.

2. The problem of COUNTING THE TOTAL NUMBER OF ONTO FUNCTIONS $f: M \longrightarrow N$, WITH $|M| = m$ AND $|N| = n$ *is similar to the problem of* DETERMINING THE NUMBER OF WAYS TO PUT m **distinguishable/distinct** BALLS INTO n **distinguishable/distinct** BOXES WITH THE RESTRICTION THAT NO BOX IS EMPTY.

Example 1.9.7. *Determine the number of ways to seat 4 couples in a row if each couple seats together.*

Solution: *A couple can be thought of as one cohesive group (they are to be seated together). So, the 4 cohesive groups can be arranged in 4! ways. But a couple can sit either as "wife and husband" or "husband and wife". So, the total number of arrangements is $2^4\ 4!$.*