INDETERMINATE FORMS

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0.\infty,\ 0^0,\ \infty^0,\ 1^\infty,\ \infty-\infty$$

L'HOSPITAL RULE: Let the function $f(x)$ and $\varphi(x)$ in $[a,b]$ satisfy the conditions of Cauchy theorem and vanish at the point $x=a$, i.e., $f(a)=\varphi(a)=0$. Then, if the ratio $\frac{f'(x)}{\varphi'(x)}$ has a limit as $x\to a$, there also exists $\lim_{x\to a}\frac{f(x)}{\varphi(x)}$

and $$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}$$

PROOF: Let $x\in[a,b]$ and $x\neq a$. Using Cauchy theorem

$$\frac{f(x)-f(a)}{\varphi(x)-\varphi(a)}=\frac{f'(\xi)}{\varphi'(\xi)} \quad \text{where } \xi\in(a,x)$$

Since $\varphi(a)=f(a)=0$, we have

$$\frac{f(x)}{\varphi(x)}=\frac{f'(\xi)}{\varphi'(\xi)}$$

Note that $x\to a$ implies $\xi\to a$ since $\xi\in(a,x)$. Then

$$\lim_{x\to a}\frac{f(x)}{\varphi(x)}=\lim_{\xi\to a}\frac{f'(\xi)}{\varphi'(\xi)}=\lim_{x\to a}\frac{f'(x)}{\varphi'(x)}$$

$\square$

REMARK 1: Theorem also holds for the case when the functions $f(x)$ and $\varphi(x)$ are not defined at $x=a$, but

$$\lim_{x\to a} f(x)=0 \quad \& \quad \lim_{x\to a} \varphi(x)=0$$

REMARK 2: If $f'(a)=\varphi'(a)=0$ and the derivatives $f'(x)$ and $\varphi'(x)$ satisfy the conditions that were imposed by the theorem on functions $f(x)$ and $\varphi(x)$, then applying the l'Hospital rule to the ratio $\frac{f'(x)}{\varphi'(x)}$, we have

$$\lim_{x\to a} \frac{f(x)}{\varphi(x)} = \lim_{x\to a} \frac{f'(x)}{\varphi'(x)} = \lim_{x\to a} \frac{f''(x)}{\varphi''(x)}$$

REMARK 3: The l'Hospital rule is also applicable if

$$\lim_{x\to \infty} f(x)=0 \quad \text{and} \quad \lim_{x\to \infty} \varphi(x)=0 .$$

EXAMPLE:

$$\lim_{x\to\infty} \frac{\sin\left(\frac{n}{x}\right)}{\frac{1}{x}} \qquad \left(\frac{0}{0}\right)$$

Applying l'Hospital rule:

$$\lim_{x\to\infty} \frac{\sin\left(\frac{n}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\cos\left(\frac{n}{x}\right)\left(-\frac{n}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$= n.$$

THEOREM: Suppose $f(x) = \infty$ and $g(x) = \infty$ as $x \to a$ (or as $x \to \pm\infty$). Then

$$\lim_{\substack{x \to a \\ (\text{or } x \to \pm\infty)}} \frac{f(x)}{g(x)} = \lim_{\substack{x \to a \\ (\text{or } x \to \pm\infty)}} \frac{f'(x)}{g'(x)}$$

provided the limit $\displaystyle\lim_{\substack{x \to a \\ (\text{or } x \to \pm\infty)}} \frac{f'(x)}{g'(x)}$ exists.

REMARK: If the limit $\displaystyle\lim_{x \to a} \frac{f'(x)}{g'(x)}$ does not exist, it does not mean that $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist.

EXAMPLE:

$$\lim_{x \to \infty} \frac{x + \sin x}{x} \quad \left(\frac{\infty}{\infty}\right)$$

Using l'Hospital rule:

$$\lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \frac{1 + \cos x}{1} \quad \text{does not exist.}$$

However,

$$\lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \left(1 + \frac{\sin x}{x}\right)$$

$$= 1$$

as $\displaystyle\lim_{x \to \infty} \frac{\sin x}{x} = 0$.

REMARK: Although l'Hospital rule can be applied to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ one of these may be better in a particular case.

we can change between these forms as

$$\frac{f}{g} = \frac{\left(\frac{1}{g}\right)}{\left(\frac{1}{f}\right)}$$

EXAMPLE:

$$\lim_{x \to 0+} x^n (\ln x)$$

If we set

$$\lim_{x \to 0+} \frac{x^n}{\left(\frac{1}{\ln x}\right)} \qquad \left(\frac{0}{0}\right)$$

we get difficulty getting derivative of $\left(\frac{1}{\ln x}\right)$ and in further calculations.

So, it is better to form as

$$\lim_{x \to 0+} \frac{\ln x}{\frac{1}{x^n}} = \lim_{x \to 0} \frac{\frac{1}{x}}{\left(-\frac{n}{x^{n+1}}\right)}$$

$$= 0.$$

REMARK: The forms

$$0^{\infty}, \quad \infty \cdot \infty, \quad \infty + \infty, \quad \infty^{\infty} \quad \text{or} \quad \infty^{-\infty}$$

are not indeterminate forms and l'Hospital rule is not applicable. Note that

$0^{\infty} = 0$, $\quad \infty \cdot \infty = \infty$, $\quad \infty + \infty = \infty$, $\quad \infty^{\infty} = \infty$, $\quad \infty^{-\infty} = 0$.

Indeterminate form $0 \cdot \infty$

Suppose $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$ (or $\pm\infty$)
then $f(x) \cdot g(x)$ as $x \to a$ is undefined.

In this case rewrite

$$f(x) \cdot g(x) = \frac{f(x)}{\left(\frac{1}{g(x)}\right)} \quad \text{or} \quad \frac{g(x)}{\left(\frac{1}{f(x)}\right)}$$

and apply l'Hospital rule.

Indeterminate form $\infty - \infty$:

$$f(x) - g(x) = \frac{\left(\frac{1}{g(x)} - \frac{1}{f(x)}\right)}{\frac{1}{f(x)\,g(x)}} \qquad \left(\frac{0}{0}\right)$$

Indeterminate form of the type $0^0, \infty^0, 1^\infty$

$$\lim_{x \to a} f(x)^{g(x)}$$

$f(x) \to 0$, $g(x) \to 0$
OR $f(x) \to \infty$, $g(x) \to 0$
OR $f(x) \to 1$, $g(x) \to \infty$

consider

$$y(x) = f(x)^{g(x)}$$

$$\ln y(x) = g(x) \ln f(x) \qquad (0 \cdot \infty)$$

$$\lim_{x \to a} \ln y = A \text{ (say)}$$

Then $\ln\left(\lim_{x \to a} y\right) = A \Rightarrow \lim_{x \to a} y = e^A$.

<u>EXAMPLE:</u>

$$\lim_{x\to 0} x^x$$

Let $\quad y = x^x$

$$\ln y = x \ln x$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} x \ln x$$

$$= \lim_{x\to 0} \frac{\ln x}{\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to 0} \frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)} = 0$$

$$\Rightarrow \ln\left(\lim_{x\to 0} y\right) = 0 \Rightarrow \lim_{x\to 0} y = 1.$$

<u>EXAMPLE:</u> $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$ $\quad (\infty - \infty$ form$)$

$$= \lim_{x\to 0}\left(\frac{\sin^2 x - x^2}{x^2 \sin^2 x}\right) \qquad \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0}\left(\frac{x^2}{\sin^2 x}\right) \cdot \lim_{x\to 0}\left(\frac{\sin^2 x - x^2}{x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{2x}{2\sin x\cos x}\right) \lim_{x\to 0}\left(\frac{2\sin x\cos x - 2x}{4x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{2}{\cos 2x \cdot 2}\right) \lim_{x\to 0}\left(\frac{2\cos 2x - 2}{12x^2}\right)$$

$$= 1 \cdot \lim_{x\to 0} \frac{-2\sin 2x}{12x} = \lim_{x\to 0} \frac{-\cos 2x \cdot 2}{6}$$

$$= -\frac{1}{3}.$$

$\square$

Example:

$$\lim_{x\to\infty}\left(x+\frac{1}{\ln\left(1-\frac{1}{x}\right)}\right)=\lim_{x\to\infty}\left(\frac{x\ln\left(1-\frac{1}{x}\right)+1}{\ln\left(1-\frac{1}{x}\right)}\right)$$

↑ $(\infty-\infty)$ form as $\frac{1}{\ln\left(1-\frac{1}{x}\right)}\to-\infty$

as

$$\lim_{x\to\infty}\left[x\ln\left(1-\frac{1}{x}\right)+1\right]=1+\lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\overset{\text{l'Hosp.}}{=}\quad 1+\lim_{x\to\infty}\frac{\frac{x}{x-1}\left(+\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$=1+\lim_{x\to\infty}\left(-1-\frac{1}{x-1}\right)$$

$$=0$$

Then

$$\lim_{x\to\infty}\frac{x\ln\left(1-\frac{1}{x}\right)+1}{\ln\left(1-\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{\ln\left(1-\frac{1}{x}\right)+\frac{x^2}{x-1}\left(\frac{1}{x^2}\right)}{\frac{x}{x-1}\left(\frac{1}{x^2}\right)}\quad\left(\frac{0}{0}\right.$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x(x-1)}-\frac{1}{(x-1)^2}}{-\frac{1}{x^2(x-1)^2}(2x-1)}=\lim_{x\to\infty}\frac{\left[(x-1)-x\right]x}{-(2x-1)}$$

$$=\lim_{x\to\infty}\left(\frac{x}{2x-1}\right)=\lim_{x\to\infty}\frac{1}{2}\qquad\square$$

$$=\frac{1}{2}.$$

$\square$