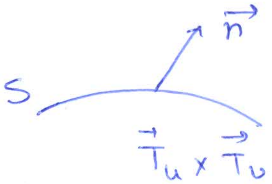
Surface integrals $\iint_S \vec{F}\cdot d\vec{S}$

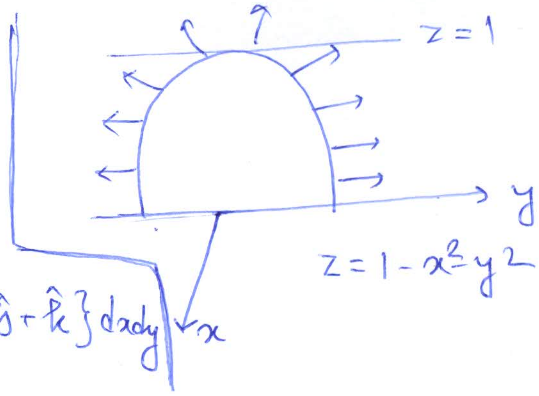
<u>Ex.</u> Suppose σ is the portion of the surface $z = 1 - x^2 - y^2$ above the xy plane. Let σ be oriented by outward normals. If $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$, evaluate $\iint_\sigma \vec{F}\cdot d\vec{S}$.

<u>Soln.</u> $I = \iint_\sigma \vec{F}\cdot d\vec{S} = \iint \vec{F}\cdot\vec{n}\, dS$



$$= \iint_{D_{uv}} \frac{\vec{F}\,(\vec{T}_u \times \vec{T}_v)}{\|\vec{T}_u \times \vec{T}_v\|} \|\vec{T}_u \times \vec{T}_v\|\, du\, dv.$$

$$= \iint_{D_{uv}} \vec{F}\cdot(\vec{T}_u \times \vec{T}_v)\, du\, dv = \iint \vec{F}\cdot\vec{\nabla}\phi\, dx\, dy.$$



$$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + \hat{k}$$

$$I = \iint_{D_{xy}:\, x^2+y^2 \le 1} \{x\hat{i} + y\hat{j} + (1 - x^2 - y^2)\hat{k}\}\cdot\{2x\hat{i} + 2y\hat{j} + \hat{k}\}\, dx\, dy$$

$$= \iint_{x^2+y^2 \le 1} (2x^2 + 2y^2 + 1 - x^2 - y^2)\, dx\, dy$$

$$= \iint_{x^2+y^2 \le 1} \{1 + (x^2 + y^2)\}\, dx\, dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (1 + r^2)\, r\, dr\, d\theta$$

$$= \frac{3\pi}{2}.$$

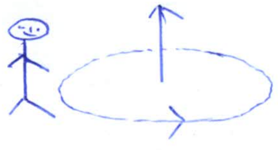
Stoke's theorem.

Consider an open surface S bounded by a closed curve C. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot d\vec{S}$$

$\vec{F} \rightarrow$ continuously differentiable vector function

Direction of C depends on orientation of S. If we imagine a man walking along a closed curve C, with this head in the general direction of the normals that orient S, then the man is walking along the '+ve' direction of C, if the surface S is on the man's left and in the '-ve' direction of C, if the surface S is in the man's right

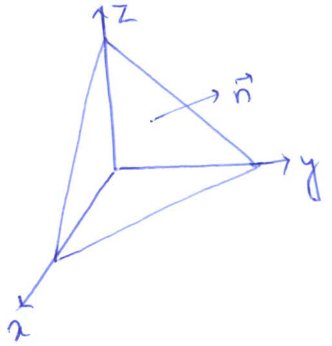


Ex. Verify Stoke's theorem $\vec{F} = (x-y)\hat{i} + (y-z)\hat{j} + (z-x)\hat{k}$.
where S : portion of the plane $x+y+z=1$ in the first octant.

Soln.

$\phi = x+y+z-1 \;,\; \vec{\nabla}\phi = \hat{i}+\hat{j}+\hat{k}$ ✓

or $\phi = 1-x-y-z \;,\; \vec{\nabla}\phi = -\hat{i}-\hat{j}-\hat{k}$ $\times$



$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \hat{i}+\hat{j}+\hat{k}$$

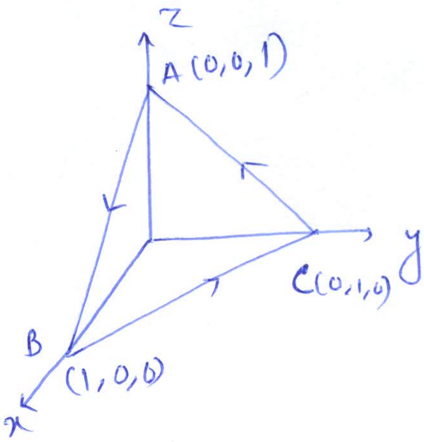
$$\iint \text{Curl}\,\vec{F}\cdot\vec{n}\,dS = \iint_{D_{xy}} \text{Curl}\,\vec{F}\cdot\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}\,|\vec{\nabla}\phi|\,dx\,dy$$

$$= \iint_{D_{xy}} \text{Curl}\,\vec{F}\cdot\vec{\nabla}\phi\,dx\,dy$$

$$= \iint_{x+y\leq 1} (\hat{i}+\hat{j}+\hat{k})\cdot(\hat{i}+\hat{j}+\hat{k})\,dx\,dy$$

$$= \frac{3}{2}.$$

$$\oint_C \vec{F}\cdot d\vec{r}$$



$$= \int_A^B + \int_B^C + \int_C^A$$

AB: $\frac{x-0}{1-0} = \frac{y-0}{0-0} = \frac{z-1}{0-1} = t$; $x=t$, $y=0$, $z=1-t$
$dx=dt$, $dy=0$, $dz=-dt$

BC: $\frac{x-1}{0-1} = \frac{y-0}{1-0} = \frac{z-0}{0-0} = t$; $x=1-t$, $y=t$, $z=0$
$dx=-dt$, $dy=dt$, $dz=0$

CA: $\frac{x-0}{0-0} = \frac{y-1}{0-1} = \frac{z-0}{1-0} = t$; $x=0$, $y=1-t$, $z=t$
$dx=0$, $dy=-dt$, $dz=dt$.

$$\int_A^B (x-\cancel{y})^0 dx + (y-z)\,\cancel{dy}^0 + (z-x)\,dz$$

$$= \int_{t=0}^{1} t\,dt + (1-t-t)\,dt = \int_0^1 (1-t)\,dt = \frac{1}{2}$$

$$\int_B^C = \frac{1}{2}, \quad \int_C^A = \frac{1}{2} \quad \therefore \oint_C \vec{F}\cdot d\vec{r} = \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 3/2.$$

Ex. Evaluate $\oint \vec{F}\cdot d\vec{r}$ using stoke's theorem, where

$C : \cap$ of $z = x^2+y^2$ & the plane $z = y$

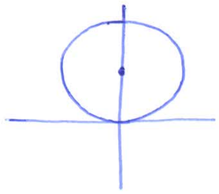
$\vec{F} = xy\hat{i} + x^2\hat{j} + z^2\hat{k}$.

Soln. # Always write the statement of the theorem to compute actually $\iint \text{Curl}\,\vec{F}\cdot\vec{n}\,ds$.

$\text{Curl}\,\vec{F} = x\hat{k}$

$x^2+y^2 = y$

$D_{xy} \Rightarrow x^2+(y-\frac{1}{2})^2 = (\frac{1}{2})^2$



$\phi = x^2+y^2-z$; $\vec{\nabla}\phi = 2x\hat{i}+2y\hat{j}-\hat{k}$

$\vec{n}\cdot ds = \vec{\nabla}\phi\,dx\,dy$

$$\iint_{D_{xy}} \text{Curl}\,\vec{F}\cdot\vec{\nabla}\phi\,dx\,dy = 0$$

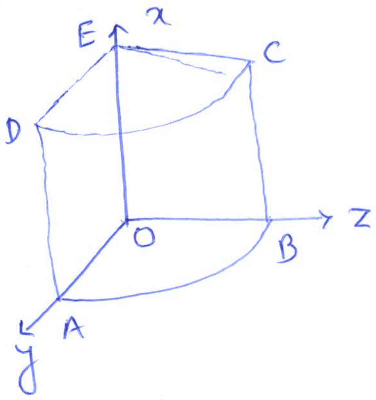
Gauss Divergence Theorem

$S \to$ closed surface enclosing volume V.

$\vec{F} \to$ continuously differentiable vector function.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \mathrm{div}\,\vec{F}\, dV$$

Ex. Verify the gauss divergence theorem for $\vec{F} = 2x^2y\,\hat{i} - y^2\,\hat{j} + 4z^2x\,\hat{k}$, taken over the region in the 1st octant bounded by the cylinder $y^2+z^2 = 9$ & the plane $x = 2$.



Soln. portion of the cylinder is bounded by ABCD (Curved surface) S_1, AOED S_2, OBCE S_3, CED S_4, OAB S_5

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} + \iint_{S_2} \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S} + \iint_{S_4} \vec{F}\cdot d\vec{S} + \iint_{S_5} \vec{F}\cdot d\vec{S}$$

S_2 (xy plane): $-\hat{k}\,dx\,dy$

S_3: $-\hat{j}\,dz\,dx$

S_4: $\hat{i}\,dy\,dz$

S_5: $-\hat{i}\,dy\,dz$.

$$I_1 = \iint (2x^2y\,\hat{i} - y^2\,\hat{j} + 4z^2x\,\hat{k}).d\vec{S}$$

S: surface of

$y^2+z^2=9$

The cylindrical surface $T_u \times T_v$

$y^2+z^2=9, \quad y = 3\cos\theta, \quad z = 3\sin\theta$

can be parametrically represented as

$x = x, \quad y = 3\cos\theta, \quad z = 3\sin\theta\,; \qquad 0 \leq x \leq 2$

$0 \leq \theta \leq \frac{\pi}{2}$

$$I_1 = \int_{x=0}^{2}\int_{\theta=0}^{\pi/2} (2x^2.3\cos\theta\,\hat{i} - 9\cos^2\theta\,\hat{j} + 4.9.\sin^2\theta\, x\hat{k}).(3\cos\theta\,\hat{j} + 3\sin\theta\,\hat{k})\,dx\,d\theta$$

$$\vec{T}_x \times \vec{T}_\theta = 3\cos\theta\,\hat{j} + 3\sin\theta\,\hat{k}$$

$$\vec{T}_x \rightarrow \left(\frac{\partial x}{\partial x}, \frac{\partial y}{\partial x}, \frac{\partial z}{\partial x}\right)$$

$$\vec{T}_\theta \rightarrow \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\right)$$

$$I_1 = \int_{x=0}^{2}\int_{\theta=0}^{\pi/2} (-27\cos^3\theta + 108x\sin^3\theta)\,dx\,d\theta \qquad = 108$$

$$\beta(m,n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta.$$

$$\iint_{S_2: AOED} \vec{F}.\,d\vec{S} = \iint (2x^2y\,\hat{i} - y^2\hat{j} + 4z^2x\,\hat{k}).(-\hat{k}\,dx\,dy)$$

$$= -\iint 4\underbrace{z^2}_{=0}x\,dx\,dy \qquad \because \text{On AOED } z=0$$

$$= 0$$

$$\iint_{S_3:\,OBCE} \vec{F}\cdot d\vec{S} = \iint \vec{F}\cdot(-\hat{\jmath})\, dz\, dx = \iint \underbrace{y^2}_{=0}\, dz\, dx$$

$$= 0$$

$$\iint_{AOB} \vec{F}\cdot d\vec{S} = 0 \quad \text{(check)}$$

Compute $\displaystyle\iint_{CDE} \vec{F}\cdot d\vec{S} = \iint (2x^2y)_{x=2}\, dy\, dz$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{3} 8y\, dy\, dz = 72.$$

$$\therefore \iint_S \vec{F}\cdot d\vec{S} = 108 + 72 = 180.$$

$$\iiint_V \operatorname{div}\vec{F}\cdot dV = \iiint_V (4xy - 2y + 8zx)\, dx\, dy\, dz$$

$$= 180.$$

Stoke's theorem.

Green's theorem in space.

$$\iint_S \text{Curl}\,\vec{F}\cdot\vec{n}\,dS = \oint \vec{F}\cdot d\vec{r}$$

In 2D

$$\vec{F} = (P, Q, 0) \qquad P \equiv P(x,y),\ Q = Q(x,y)$$

$$\vec{F}\cdot d\vec{r} = (P, Q, 0)\cdot(dx, dy, dz)$$

$$= P\,dx + Q\,dy$$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}\cdot\hat{k}\,dx\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy.$$

$$\oint \vec{F}\cdot d\vec{r} = \oint P\,dx + Q\,dy$$

Green's theorem in plane is a special case of stoke's theorem. Hence the latter is often called Green's theorem in space.

Gauss divergence theorem

$$\iiint_V \operatorname{div}\vec{F}\cdot dv = \iint_S \vec{F}\cdot d\vec{s}$$

$\vec{F} = (F_1, F_2, F_3)$

$$\therefore \iint_S F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz \quad \text{——} (1)$$

Take $F_1 = \frac{x}{3}, \quad F_2 = \frac{y}{3}, \quad F_3 = z/3$

R.H.S of (1) $= \iiint_V dx\, dy\, dz =$ volume of the region V

L.H.S of (1) $= \frac{1}{3}\iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$

$\therefore$ The surface integral

$$\frac{1}{3}\iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$$

represents the volume of the region enclosed by the surface S.