Date 24/07/2017

Lecture 3

Q) Find $\mathcal{L}(e^{at})$, $t \geq 0$,

a is a complex no.

If we let $a = i\omega$

$i = \sqrt{-1}$

$\boxed{\mathcal{L}(e^{i\omega t}) \text{ exist?}} = \dfrac{1}{s-(i\omega)}$

[$e^{i\omega}$ is complex valued, but it is both continuous & bounded, $\forall t$ so that its L.T certainly exists.]

A.M

$$\mathcal{L}(e^{i\omega t}) = \int_0^\infty e^{-st} \cdot e^{i\omega t}\, dt$$

$$= \int_0^\infty e^{t(i\omega - s)}\, dt = \left[\frac{e^{(i\omega - s)t}}{(i\omega - s)} \right]_0^\infty$$

$$= \frac{1}{s-(i\omega)}$$

$$= \frac{1}{(s-i\omega)} = \frac{s+i\omega}{(s-i\omega)(s+i\omega)}$$

$$= \frac{s+i\omega}{s^2+\omega^2}$$

$$\mathcal{L}(e^{i\omega t}) = \frac{s}{s^2+\omega^2} + i\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$\Rightarrow \mathcal{L}(\cos\omega t + i\sin\omega t)$$

$$= \mathcal{L}(\cos\omega t) + i\,\mathcal{L}(\sin\omega t)$$

[how??]

Equating real & imaginary parts of these two eqns we obtain

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}\sin(\omega t) = \omega/s^2+\omega^2$$

First shifting Theorem

(or, Translation on the s-axis)

or, First Translation Theorem

or, Replacement of s by $(s-a)$ in the transform

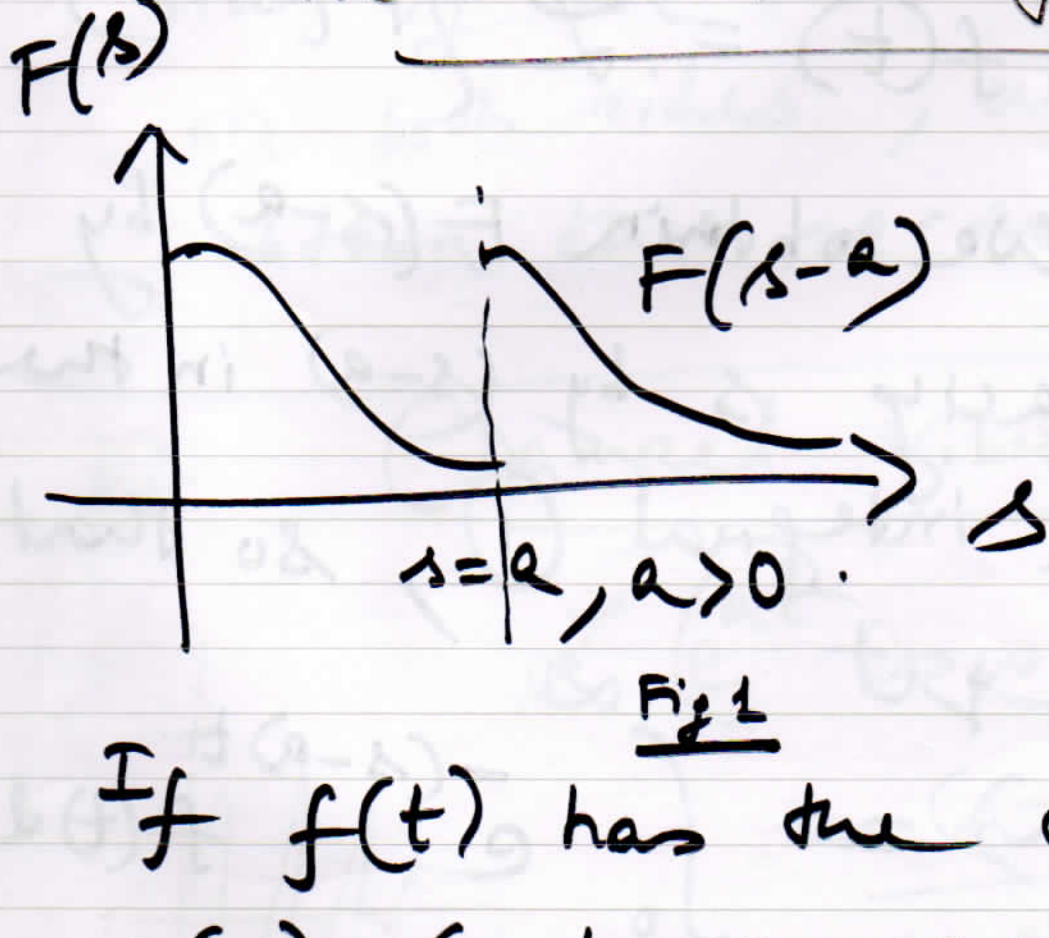

Fig 1

If $f(t)$ has the transform $F(s)$ (where $s > k$), then $e^{at} f(t)$ has the transform $F(s-a)$, (where $(s-a) > k$).

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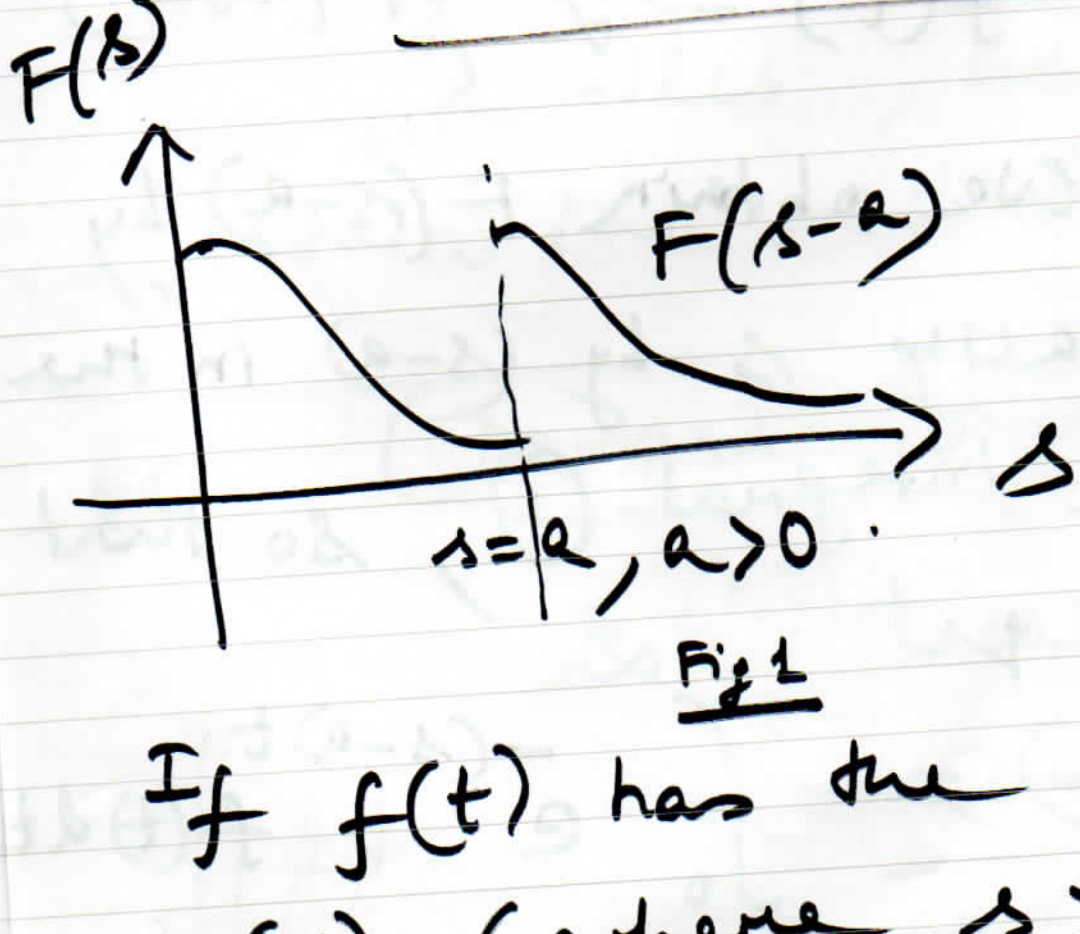


Fig 1

If $f(t)$ has the transform $F(s)$ (where $s>k$), then $e^{at} f(t)$ has the transform $F(s-a)$, (where $(s-a)>k$).

In formulas,

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

or, if we take the inverse on both sides,

$$e^{at} f(t) = \mathcal{L}^{-1}\{F(s-a)\}$$

prm:- We obtain $F(s-a)$ by replacing s by $(s-a)$ in the given integral (1), so that we get

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\left[e^{at} f(t)\right] dt$$

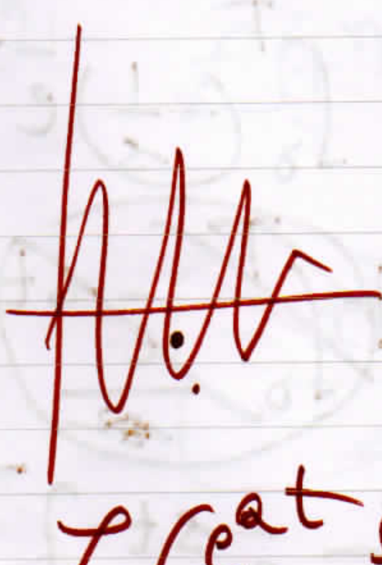
$$= \mathcal{L}\left[e^{at} f(t)\right]$$

If $F(s)$ exists (i.e., is finite) for $s > k$, then our first integral exists for $(s-a) > k$.

Now, if we take inverse on both sides, we will obtain the second formula.

EX (Damped Vibrations)

$$\mathcal{L}\left(e^{at}\cos\omega t\right)$$



$$= \frac{(s-a)}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\left(e^{at}\sin\omega t\right) = \frac{\omega}{(s-a)^2+\omega^2}.$$

For negative a, these $f(t)$ are damped vibrations.

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Q) Find the Laplace Transform of the function $f(t) = t$

Sol^n :- Using the def^n of Laplace transform,

$$\mathcal{L}(t) = \lim_{T \to \infty} \int_0^T t\, e^{-st}\, dt$$

Now, we have that

$$\int_0^T t\, e^{-st}\, dt = \left[\frac{-t\, e^{-st}}{s} \right]_0^T - \int_0^T \left(-\frac{1}{s}\right) e^{-st}\, dt$$

$$= -\frac{T e^{-sT}}{s} + \cancel{\int_0^T \frac{e^{-st}}{s^2}}$$

$$= -\frac{T}{s} e^{-sT} + \left[-\frac{1}{s^2} e^{-st} \right]_0^T$$

$$-\frac{1}{s^2} e^{-sT} + \frac{1}{s^2}$$

$$\therefore \mathcal{L}(t) = \lim_{T\to\infty} \left[-\frac{T/s\, e^{-sT}}{} - \frac{1}{s^2} e^{-sT} + \frac{1}{s^2} \right]$$

$$\boxed{\mathcal{L}(t) = \frac{1}{s^2}.}$$

we can generalise this result :

Cor:- $\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$, n is a positive integer.

Sol^n:- $\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st}\, dt$

$$= \underbrace{\left[-\frac{t^n e^{-st}}{s} \right]_0^\infty}_{(=0)} + \int_0^\infty \frac{n t^{n-1} e^{-st}}{s}\, dt$$

$$= \frac{n}{s} \mathcal{L}(t^{n-1}).$$

If we put $n=2$, in this recurrence relation we obtain

$$\mathcal{L}(t^2) = \frac{2}{s}\mathcal{L}(t) = \frac{2}{s^3}$$

If we assume

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$$

then

$$\mathcal{L}(t^{n+1}) = \frac{(n+1)}{s}\mathcal{L}(t^n)$$

$$= \frac{(n+1)}{s}\cdot\frac{n!}{s^{n+1}}$$

$$= \frac{(n+1)!}{s^{n+2}}.$$

This establishes that

$$\boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$$

Again,

$\mathcal{L}(t^a)$, a is an positive real no.

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} \cdot t^a \, dt$$

$$= \int_0^\infty e^{-x} \cdot \left(\frac{x}{s}\right)^a \frac{dx}{s}$$

$\left[\begin{array}{l} \text{Let } st = x. \\ \therefore s\,dt = dx \\ \Rightarrow dt = dx/s \end{array} \right.$

$x =$

$$= \frac{\Gamma(a+1)}{s^{a+1}}$$

$$= \frac{1}{s^{a+1}} \int_0^\infty e^{-x} \cdot x^{(a+1)-1} \, dx,$$

$\therefore$

$s > 0$

$$= \frac{\Gamma(a+1)}{s^{a+1}} .$$

EX/ From first shifting theorem, we obtain

$$\mathcal{L}\left(t^n e^{at}\right) = \frac{n!}{(s-a)^{n+1}}$$

If, $$\mathcal{L}\left(t\, e^{at}\right) = \frac{1}{(s-a)^2}.$$

Table

Some f^{ns} $f(t)$ & their L. Ts $\mathcal{L}\{f(t)\}$

	$f(t)$	$\mathcal{L}\{f(t)\}$		$f(t)$	$\mathcal{L}\{f(t)\}$
1.	1	$1/s$	7.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
2.	t	$1/s^2$	8.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
3.	t^2	$2!/s^3$	9.	$\cosh at$	$\frac{s}{s^2-a^2}$
4.	t^n $(n=0,1,\dots)$	$\frac{n!}{s^{n+1}}$	10.	$\sinh at$	$\frac{a}{s^2-a^2}$
5.	t^a (a is positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	11.	$e^{at}\cos \omega t$	$\frac{(s-a)}{(s-a)^2+\omega^2}$
6.	e^{at}	$\frac{1}{(s-a)}$	12.	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

Q) Find the L.T of $\mathcal{L}(te^{at})$ & deduce the value of $\mathcal{L}(t^n e^{at})$, where a is a real constant & n is a positive integer.

Note :- we now consider the L.T of other trigonometric f^ns.

$\mathcal{L}(\tan t)$

$\int_0^\infty e^{-st} \tan t \, dt.$

$\Rightarrow |e^{-st} \tan t| \to \infty$ as $t \to \pi/2$.

$\mathcal{L}(\cot t)$

$|e^{-st} \cot t| \to \infty$ as $t \to 0$

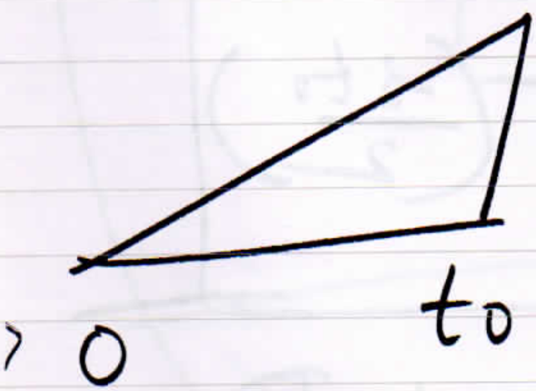
$\mathcal{L}(\operatorname{cosec} t) = ?$

$\mathcal{L}(\sec t) = ?$ D. not exist.

(Electrical engineering)

Note :- Find the L.T of the f^n $f(t)$,

$$f(t) = \begin{cases} t, & 0 \le t < t_0 \\ 2t_0 - t, & t_0 \le t \le 2t_0 \\ 0, & t > 2t_0 \end{cases}$$



$$\int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{t_0} t \cdot e^{-st}\,dt + \int_{t_0}^{2t_0} (2t_0 - t)\, e^{-st}\,dt$$

$$= \left[-\tfrac{t}{s} e^{-st}\right]_0^{t_0} + \int_0^{t_0} \tfrac{1}{s} e^{-st}\,dt + \left[\frac{(-2t_0 - t)}{s} e^{-st}\right]_{t_0}^{2t_0}$$

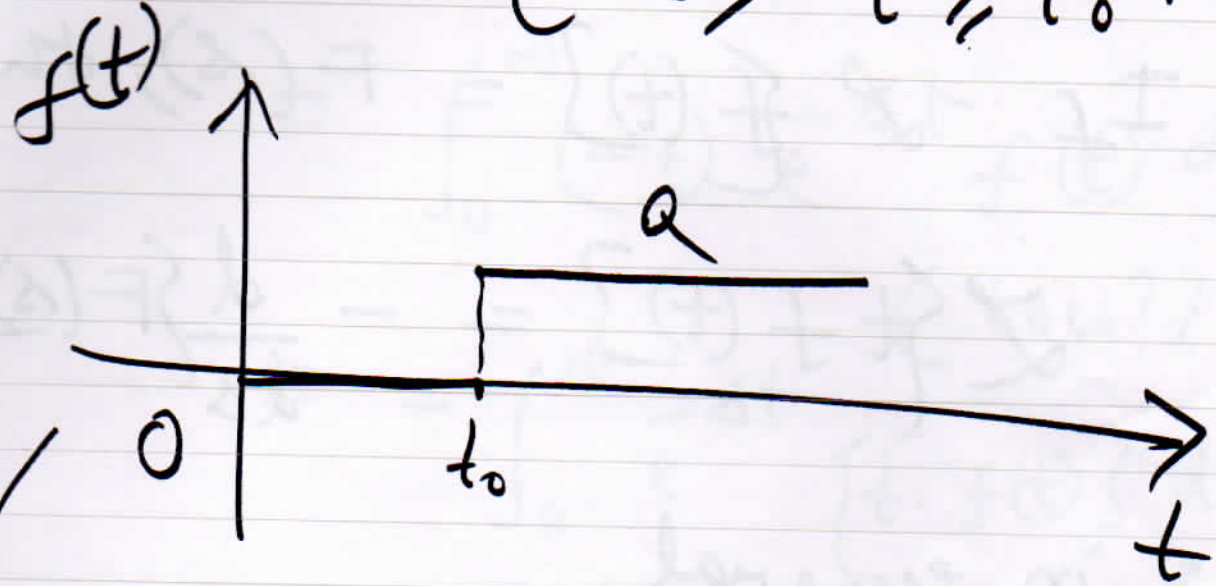
$$= \frac{1}{s^2}\left[1 - 2e^{-st_0} + 2e^{-2st_0}\right]$$

$$= \frac{1}{s^2}\left[1 - e^{-st_0}\right]^2$$

$$= \boxed{\frac{4}{s^2} e^{-st_0} \sinh^2\left(\frac{st_0}{2}\right)}$$

Q) Determine the L.T of the step function f(t), defined by

$$f(t) = \begin{cases} 0, & 0 \le t \le t_0 \\ a, & t \geqslant t_0 . \end{cases}$$



Q) $\mathcal{L} f(t)$,

$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \geqslant 3 . \end{cases}$$

$$\mathcal{L}\{f(t)\} = a\int_{t_0}^{\infty} e^{-st}\, dt \quad = \lim_{N \to \infty} \left[\frac{a}{s} e^{-st} \right]_{t_0}^{N}$$

$$= \boxed{\frac{a}{s} e^{-s t_0}}$$