2.7 Application to Recurrence Relation Continued ...

Example 2.7.1. *1. Determine a generating function for the numbers $S(n,m),\ n,m \in \mathbb{Z}, n,m \geq 0$ that satisfy*

$$S(n,m) = mS(n-1,m) + S(n-1,m-1),\ (n,m) \neq (0,0) \quad \text{with} \tag{2.1}$$
$$S(0,0) = 1, S(n,0) = 0, \text{ for all } n > 0 \text{ and } S(0,m) = 0, \text{ for all } m > 0.$$

Hence or otherwise find a formula for the numbers $S(n,m)$.

Solution: *Define $G_m(y) = \sum\limits_{n \geq 0} S(n,m)y^n$. Then, for $m \geq 1$, Equation (2.1) gives*

$$\begin{aligned} G_m(y) &= \sum_{n\geq 0} S(n,m)y^n = \sum_{n \geq 0} \left(mS(n-1,m) + S(n-1,m-1)\right) y^n \\ &= m\sum_{n\geq 0} S(n-1,m)y^n + \sum_{n\geq 0} S(n-1,m-1)y^n \\ &= myG_m(y) + yG_{m-1}(y). \end{aligned}$$

Therefore, $G_m(y) = \dfrac{y}{1-my} G_{m-1}(y)$. Using initial conditions, $G_0(y) = 1$ and hence

$$G_m(y) = \frac{y^m}{(1-y)(1-2y)\cdots(1-my)} = y^m \sum_{k=1}^{m} \frac{\alpha_k}{1-ky}, \tag{2.2}$$

where $\alpha_k = \dfrac{(-1)^{m-k}k^m}{k!\,(m-k)!}$, for $1 \leq k \leq m$. Thus,

$$\begin{aligned} S(n,m) &= [y^n]\left(y^m \sum_{k=1}^{m} \frac{\alpha_k}{1-ky}\right) = \sum_{k=1}^{m} [y^{n-m}] \frac{\alpha_k}{1-ky} \\ &= \sum_{k=1}^{m} \alpha_k k^{n-m} = \sum_{k=1}^{m} \frac{(-1)^{m-k}k^n}{k!\,(m-k)!} \\ &= \frac{1}{m!}\sum_{k=1}^{m} (-1)^{m-k}k^n \binom{m}{k} = \frac{1}{m!}\sum_{k=1}^{m}(-1)^k(m-k)^n\binom{m}{k}. \end{aligned} \tag{2.3}$$

Therefore, $S(n,m) = \dfrac{1}{m!}\sum\limits_{k=1}^{m}(-1)^k(m-k)^n\binom{m}{k}$ and $m!\,S(n,m) = \sum\limits_{k=1}^{m}(-1)^k(m-k)^n\binom{m}{k}$. The above expression was already obtained earlier (see Equation (1.1) and Exercise 30). This identity is generally known as the STIRLING'S IDENTITY.

Observation:

(a) $H_n(x) = \sum\limits_{m\geq 0} S(n,m)x^m$ is not considered. But verify that

$$H_n(x) = (x + xD)^n \cdot 1 \quad \text{as} \quad H_0(x) = 1.$$

Therefore, $H_1(x) = x,\ H_2(x) = x + x^2, \cdots$. Hence, it is difficult to obtain a general formula for its coefficients. But it is helpful in showing that the numbers $S(n,m)$, *for fixed n, first increase and then decrease (commonly called* unimodal*). The same holds for the sequence of binomial coefficients $\{\binom{n}{m}, m = 0, 1, \ldots, n\}$.*

(b) *Since there is no restriction on the non-negative integers n and m, the expression Equation (2.3) is also valid for $n < m$. But, in this case, we know that $S(n, m) = 0$. Hence, verify that $\sum\limits_{k=1}^{m} \dfrac{(-1)^{m-k}\, k^{n-1}}{(k-1)!\,(m-k)!} = 0$, whenever $n < m$.*

2. Bell Numbers: *For a positive integer n, the n^{th} Bell number, denoted $b(n)$, is the number of partitions of the set $\{1, 2, \ldots, n\}$. Therefore, by definition, $b(n) = \sum\limits_{m=1}^{n} S(n, m)$, for $n \geq 1$ and by convention (see Stirling Numbers), $b(0) = 1$. Thus, for $n \geq 1$,*

$$\begin{aligned} b(n) &= \sum_{m=1}^{n} S(n,m) = \sum_{m\geq 1} S(n,m) = \sum_{m\geq 1}\sum_{k=1}^{m} \frac{(-1)^{m-k}\, k^{n-1}}{(k-1)!\,(m-k)!} \\ &= \sum_{k\geq 1} \frac{k^n}{k!} \sum_{m\geq k} \frac{(-1)^{m-k}}{(m-k)!} = \frac{1}{e}\sum_{k\geq 1} \frac{k^n}{k!}. \end{aligned} \tag{2.4}$$

Note that Equation (2.4) is valid even for $n = 0$. Also, observe that $b(n)$ has terms of the form $\dfrac{k^n}{k!}$ and hence we compute its exponential generating function (see Exercise **??***.61). Thus, if $B(x) = \sum\limits_{n\geq 0} b(n)\dfrac{x^n}{n!}$ then*

$$\begin{aligned} B(x) &= 1 + \sum_{n\geq 1} b(n)\frac{x^n}{n!} = 1 + \sum_{n\geq 1}\left(\frac{1}{e}\sum_{k\geq 1}\frac{k^n}{k!}\right)\frac{x^n}{n!} \\ &= 1 + \frac{1}{e}\sum_{k\geq 1}\frac{1}{k!}\sum_{n\geq 1} k^n \frac{x^n}{n!} = 1 + \frac{1}{e}\sum_{k\geq 1}\frac{1}{k!}\sum_{n\geq 1}\frac{(kx)^n}{n!} \\ &= 1 + \frac{1}{e}\sum_{k\geq 1}\frac{1}{k!}\left(e^{kx} - 1\right) = 1 + \frac{1}{e}\sum_{k\geq 1}\left(\frac{(e^x)^k}{k!} - \frac{1}{k!}\right) \\ &= 1 + \frac{1}{e}\left(e^{e^x} - 1 - (e-1)\right) = e^{e^x - 1}. \end{aligned} \tag{2.5}$$

Recall that e^{e^x-1} is a valid formal power series (see Remark 2.4.5). Now, let us derive the recurrence relation for $b(n)$'s. Taking the natural logarithm on both the sides of Equation (2.5), one has $Ln\left(\sum\limits_{n\geq 0} b(n)\dfrac{x^n}{n!}\right) = e^x - 1$. Now, differentiation with respect to x gives $\dfrac{1}{\sum\limits_{n\geq 0} b(n)\dfrac{x^n}{n!}} \cdot \sum\limits_{n\geq 0} b(n)\dfrac{x^{n-1}}{(n-1)!} = e^x$. Therefore, after cross multiplication and a multiplication with x, implies

$$\sum_{n\geq 1} \frac{b(n)x^n}{(n-1)!} = xe^x \sum_{n\geq 0} b(n)\frac{x^n}{n!} = x\left(\sum_{m\geq 0}\frac{x^m}{m!}\right)\cdot\left(\sum_{n\geq 0} b(n)\frac{x^n}{n!}\right).$$

Thus,

$$\frac{b(n)}{(n-1)!} = [x^n] \sum_{n \geq 1} \frac{b(n)x^n}{(n-1)!} = [x^n]\, x \left(\sum_{m \geq 0} \frac{x^m}{m!} \right) \cdot \left(\sum_{n \geq 0} b(n) \frac{x^n}{n!} \right) = \sum_{m=0}^{n-1} \frac{1}{(n-1-m)!} \cdot \frac{b(m)}{m!}.$$

Hence, it follows that $b(n) = \sum\limits_{m=0}^{n-1} \binom{n-1}{m} b(m)$, *for* $n \geq 1$, *with* $b(0) = 1$.