Assignment 3

Note: For all coding problems, please post your codes to Github, those could be useful for your resume.

Problem 1. SDE and PDE

Given the stock process under risk-neutral measure:

$$dS/S = rdt + \sigma dW_t \tag{1}$$

And a derivative contract with maturity payoff $x(T)$.
Using the risk-neutral valuation, the value of this contract is the conditional expectation:

$$V(t, S) = E[e^{-r(T-t)}V(T)|S(t) = S] \tag{2}$$

Prove that, the value function of this contract, $V(t, S)$ satisfies:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{3}$$

$$V(T, S) = x(T) \tag{4}$$

According to Taylor's Formula

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\left(\frac{\partial^2 V}{\partial t^2}dt^2 + \frac{\partial^2 V}{\partial t\partial S}dt\,dS + \frac{\partial^2 V}{\partial S^2}dS^2\right) + O(dt^3, dS^3)$$

① $dt^2 \to 0$ (too small, so it can be ignored.)

② $dt\,dS = Srdt^2 + \sigma S\,dt\,dW_t \to 0$ (too small, so ignored)

③ $dS^2 = (Srdt + \sigma S dW_t)^2$

$$= S^2r^2\underset{\sim}{dt^2} + \sigma^2S^2dW_t^2 + 2r\sigma S^2\underline{dt\,dW_t}$$

$$= \sigma^2 S^2 dW_t^2$$

$$dW_t = W_{t+dt} - W_t \sim N(0, dt)$$

$$E(dW_t^2) = dt$$

Therefore, $dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2S^2dt$

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}(rSdt + \sigma S dW_t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2S^2dt$$

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}rSdt + \frac{\partial V}{\partial S}\sigma S dW_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2S^2dt$$

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2S^2\right)dt + \frac{\partial V}{\partial S}\sigma S dW_t$$

Because of risk-neutral, $E(e^{-rt}V)$ is a constant.

Then, $E(d(e^{-rt}V)) = 0$

$$d(e^{-rt}V) = e^{-rt}dV + Vde^{-rt}$$

$$= e^{-rt}dV + (-re^{-rt}Vdt)$$

$$= e^{-rt}(dV - rVdt)$$

(Substitute dV) $= e^{-rt}\left(dt\left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} - rV\right) + \frac{\partial V}{\partial S}\sigma S dW_t\right)$

Now, let's think about $E(e^{-rt}V) = 0$

$$E(e^{-rt}V) = e^{-rt}dt\left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} - rV\right) + \frac{\partial V}{\partial S}\sigma S\,E(dW_t)$$

$\left(dW_t \sim N(0, dt) \Rightarrow E(W_t) = 0\right)$

$$= e^{-rt}dt\left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} - rV\right) = 0$$

Finally, we can get:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} - rV = 0$$

$$V(T, S) = E(e^{-r(T-T)}V(T) | S(t) = S)$$

$$= E(V(T) | S(t) = S)$$

$$= x(T)$$

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Problem 2. Finite Difference Method

Research and developed a numerical method to solve the above PDE.
Reference: Finite Difference PDE
Code your algorithm in Python, to price both European/American call and put.

$0 \sim T$: divide into M subperiod. $\Delta t = \frac{T}{N}$

$0 \sim 3K$: divide into N pieces. $\Delta m = \frac{3K}{M}$

V_n^m = option value at time $m\Delta t$. with stock price $n\Delta m$.

$$\frac{\partial V}{\partial t} = \frac{V_n^m - V_n^{m-1}}{\Delta t}$$

$$\frac{\partial V}{\partial S} = \frac{V_{n+1}^m - V_{n-1}^m}{2\Delta S}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{V_{n+1}^m + V_{n-1}^m - 2V_n^m}{\Delta S^2}$$

$$\frac{V_n^m - V_n^{m-1}}{\Delta t} + rn\Delta S \times \frac{V_{n+1}^m - V_{n-1}^m}{2\Delta S} + \frac{1}{2}\sigma^2 n^2\Delta S^2 \times \frac{V_{n+1}^m + V_{n-1}^m - 2V_n^m}{\Delta S^2} = rV_n^m$$

$$V_n^m - V_n^{m-1} + \frac{1}{2}rn\Delta t(V_{n+1}^m - V_{n-1}^m) + \frac{1}{2}\sigma^2n^2\Delta t(V_{n+1}^m + V_{n-1}^m - 2V_n^m) = r\Delta t V_n^m$$

$$V_n^{m-1} = V_{n-1}^m\Big(\underbrace{\tfrac{1}{2}(\sigma^2n^2\Delta t - rn\Delta t)}_{d_n}\Big) + V_n^m\big(\underbrace{1 - r\Delta t - \sigma^2n^2\Delta t}_{l_n}\big) + V_{n+1}^m\Big(\underbrace{\tfrac{1}{2}(rn\Delta t + \sigma^2n^2\Delta t)}_{u_n}\Big)$$

S \ t	0	...	m-1	m	...	T
0	Z_0	...	Z_{m-1}	Z_m	...	Z_T
⋮						
3K	a_0	...	a_{m-1}	a_m	...	a_T

Known

$$\sigma^2n^2 = p_n \qquad rn = q_n$$

$$V_n^{m-1} = V_{n-1}^m\left(\tfrac{1}{2}(p_n - q_n)\Delta t\right) + V_n^m(1 - r\Delta t - p_n\Delta t) + V_{n+1}^m\left(\tfrac{1}{2}(p_n + q_n)\Delta t\right)$$