The Conquest of Space: Space Exploration and Rocket Science

Technical part formulary and notes

Version 2017.1

Aerospace Engineering Department (UC3Mx)

Preface

The present notes correspond to the course "The Conquest of Space: space exploration and rocket science," imparted on UC3Mx by Manuel Sanjurjo, Mario Merino, Manuel Soler, Gonzalo Sánchez, David Morante, Filippo Cichocki, Daniel Pérez, Xin Chen, and Eduardo Ahedo from the UC3M Aerospace Engineering department.

The notes are provided as a companion material to the course, and are not intended as a substitute of a real book or the lessons in the course. They are conceived as a quick summary without extensive explanations, for which the learner must search elsewhere. The learner should have a working knowledge of fundamental mathematics (calculus and linear algebra) and physics (mechanics) to fully understand everything; if you don't, or if you feel a bit uncomfortable with these topics at the start the course, please search online or in your library for an introduction to them or ask in the forums.

 The latest version can be found on GitHub and on Mario Merino's personal webpage. If you detect any errata, or have any comments for its improvement, please contact Mario Merino at mario.merino@uc3m.es.

1 Astrodynamics: the two body problem

Position, velocity and acceleration of point particle In Cartesian coordinates, and calling $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ the unit vectors of the basis of our inertial reference frame, the position $\boldsymbol{r}$, velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ vectors of a point particle P are defined as:

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \tag{1.1}$$
$$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k} \tag{1.2}$$
$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} \tag{1.3}$$

The velocity is the derivative of the position, and the acceleration is the derivative of the velocity. Conversely, the position is the integral of the velocity, and the velocity is the integral of the acceleration:

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}; \qquad \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{a}; \tag{1.4}$$
$$\boldsymbol{r} = \int_0^t \boldsymbol{v}\mathrm{d}t + \boldsymbol{r}_0; \qquad \boldsymbol{v} = \int_0^t \boldsymbol{a}\mathrm{d}t + \boldsymbol{v}_0. \tag{1.5}$$

Newton's second law of mechanics states that the acceleration of a point particle states that the acceleration of the particle is proportional to the sum of all forces acting upon it. The constant of proportionality is its mass:

$$m\boldsymbol{a} = \sum \boldsymbol{F} \tag{1.6}$$

Thus, knowing the mass and the forces acting on a particle we may compute its acceleration. By integrating the acceleration twice we can find how the position of the particle changes in time, i.e., its *trajectory*. This is known as solving the direct problem of mechanics.

Newton's law of gravitation The force exerted upon a point particle of mass m by another of mass M located at the origin of coordinates is

$$\boldsymbol{F} = -G\frac{mM}{r^2}\boldsymbol{u}_r \tag{1.7}$$

where $\boldsymbol{u}_r = \boldsymbol{r}/r$ is the unit vector in the direction of $\boldsymbol{r}$, and $G = 6.67408 \cdot 10^{-11}$ $\mathrm{m^3kg^{-1}s^{-2}}$ is the universal gravitational constant.

The equation of motion of a point particle in the gravity field of a planet is then:

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -G\frac{M}{r^2}\boldsymbol{u} \tag{1.8}$$

Conservation of mechanical energy The mechanical energy of a point particle in the gravity field of a planet like the Earth is $E = E_{kin} + E_{pot}$, where

$$E_{kin} = \frac{1}{2}mv^2; \quad E_{pot} = -\frac{GmM}{r}. \tag{1.9}$$

The mechanical energy E is a conserved quantity of motion.

Demonstration: by denoting with $\cdot$ the scalar product, we dot-multiply Eq. (1.8) by $m\boldsymbol{v}$:

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{v} = -G\frac{mM}{r^2}\left(\frac{\boldsymbol{r}}{r}\right) \cdot \boldsymbol{v} \Rightarrow \frac{1}{2}m\frac{\mathrm{d}v^2}{\mathrm{d}t} = -G\frac{mM}{r^3}\boldsymbol{r} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = -G\frac{mM}{2r^3}\frac{\mathrm{d}r^2}{\mathrm{d}t} = GmM\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r}\right) \tag{1.10}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2 - \frac{GmM}{r}\right) = 0 \Rightarrow \frac{1}{2}mv^2 - \frac{GmM}{r} = E = \text{const.} \tag{1.11}$$

Conservation of angular momentum The angular momentum of a point particle about the origin is defined as

$$\boldsymbol{H} = m\boldsymbol{r} \times \boldsymbol{v}, \tag{1.12}$$

where $\times$ is the vector product. $\boldsymbol{H}$ is a vector that is conserved in the motion. Its magnitude can be written as $H = mr^2 \mathrm{d}\theta/\mathrm{d}t$, where θ is the polar angle of the motion of the particle.

Demonstration: we cross-multiply Eq. (1.8) by $m\boldsymbol{r}$. The right-hand-side is identically zero thanks to the properties of the vector product:

$$m\boldsymbol{r} \times \boldsymbol{a} = -G\frac{mM}{r^2}\boldsymbol{r} \times \left(\frac{\boldsymbol{r}}{r}\right) = 0. \tag{1.13}$$

We now apply the following identity

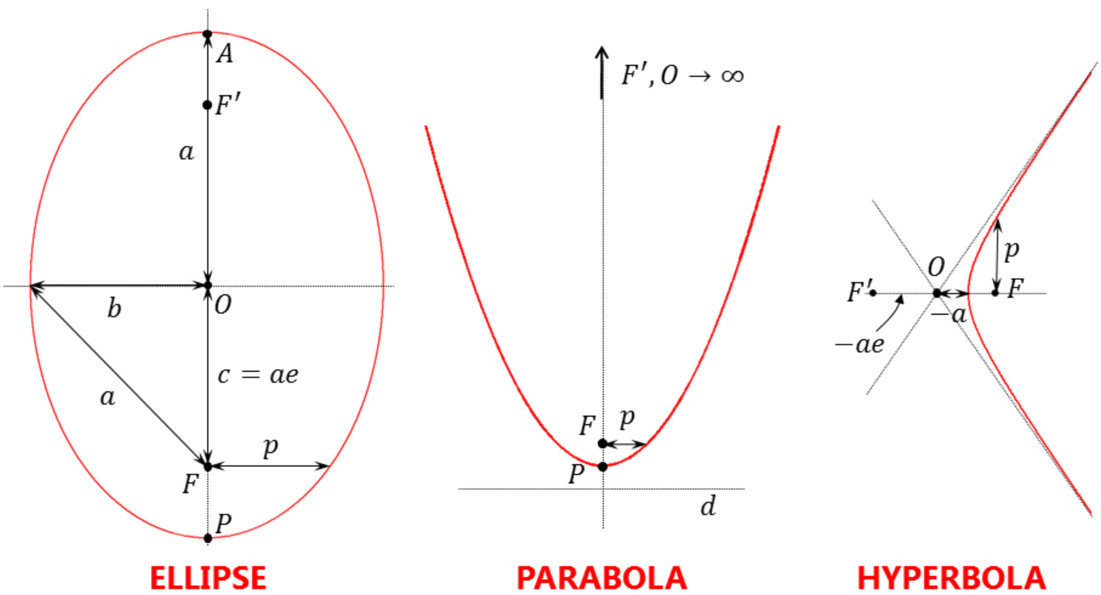
$$\frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{r} \times \boldsymbol{v}) = m\boldsymbol{v} \times \boldsymbol{v} + m\boldsymbol{r} \times \boldsymbol{a} = m\boldsymbol{r} \times \boldsymbol{a}, \tag{1.14}$$

and conclude that

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = 0 \Rightarrow \boldsymbol{H} = \text{const.} \tag{1.15}$$

Conic sections The trajectory of a point particle in the two body problem of astrodynamics is a conic section. Conic sections are characterized by their semi-major axis a and their eccentricity e. The conic section can be any of the following:

1. Ellipse ($a > 0$ and $e < 1$). The only closed orbit. When $e = 0$, the ellipse is a circle.
2. Parabola ($a \to \infty$ and $e = 1$). A particle in a parabolic trajectory will reach infinity with zero velocity. The semi-latus rectum $p = a(1 - e^2)$ is used instead of a in this case to avoid the indetermination.
3. Hyperbola ($a < 0$ and $e > 1$). A particle in a parabolic trajectory will reach infinity with non-zero velocity. Of the two branches of the hyperbola, the trajectory is only one of them.



The pericenter and the apocenter are the points of the trajectory nearest and farthest away from the planet:

$$r_p = a(1-e); \quad r_a = a(1+e). \tag{1.16}$$

Observe that the definition of the apocenter only makes sense for elliptic orbits. Also, in the case of parabolic orbits, the pericenter can be computed simply as $r_p = p/2$.

At the pericenter and apocenter the position and velocity vectors are perpendicular to each other, i.e., $\boldsymbol{r} \cdot \boldsymbol{v} = 0$.

Vis-viva equation The constant E in the energy equation is related to the semi-major axis of the orbit:

$$\frac{1}{2}mv^2 - \frac{GmM}{r} = -\frac{GmM}{2a} \tag{1.17}$$

Demonstration: We particularize the mechanical energy equation at the pericenter and the apocenter and subtract the two expressions:

$$\frac{1}{2}mv_p^2 - \frac{GmM}{r_p} = E; \quad \frac{1}{2}mv_a^2 - \frac{GmM}{r_a} = E. \tag{1.18}$$

$$\frac{1}{2}m\left(v_p^2 - v_a^2\right) - GmM\left(\frac{1}{r_p} - \frac{1}{r_a}\right) = 0; \tag{1.19}$$

From the conservation of angular momentum, using $2a = r_a + r_p$, and since the position and velocity vectors are perpendicular at the pericenter and apocenter,

$$v_a = \frac{r_p}{2a - r_p}v_p \tag{1.20}$$

Substituting v_a in Eq. (1.19), and using again $2a = r_a + r_p$,

$$\frac{1}{2}mv_p^2\left(1 - \frac{r_p^2}{(2a - r_p)^2}\right) = GmM\left(\frac{1}{r_p} - \frac{1}{2a - r_p}\right) \Rightarrow v_p^2 = \frac{GM}{a}\left(\frac{2a}{r_p} - 1\right) \tag{1.21}$$

Finally, substituting in the first of Eqs. (1.18) we find $E = -GmM/(2a)$.

Clearly,

1. Particles in elliptic orbits have negative mechanical energy. The orbit is bounded (under no condition could the particle have $r > 2a$, as the velocity would become imaginary; in reality, the maximum r occurs at apocenter, $r_a = a(1 - e)$).
2. Particles in parabolic orbits have zero mechanical energy, as they reach zero velocity as $r \to \infty$.
3. Particles in hyperbolic orbits have positive mechanical energy, as they have a non-zero excess velocity at infinity.

Important velocities There are several important velocity quantities in the two body problem. Importantly, these velocities do not depend on the mass of the point particle, only on the mass of the planet M and the universal gravitational constant G:

- Circular velocity: $v_c = \sqrt{GM/r}$. This is the magnitude of the velocity that a particle must have at a radius r to be in circular orbit. Obtained from vis-viva equation, by imposing that the trajectory is a circle, $r = a$.
- Escape velocity: $v_e = \sqrt{2GM/r}$. Any particle with this velocity at a distance r from the origin is in parabolic orbit, and will reach infinity with zero velocity. Obtained from vis-viva equation by setting $a = \infty$.
- Velocity at pericenter, $v_p = \sqrt{GM\frac{1+e}{1-e}}$. Obtained from vis-viva equation, particularizing at $r_p = a(1 - e)$.
- Velocity at apocenter, $v_a = \sqrt{GM\frac{1-e}{1+e}}$. Obtained from vis-viva equation, particularizing at $r_a = a(1 + e)$. Observe that $v_p > v_a$ always.

Kepler laws Johannes Kepler stated his famous three laws of planetary motion before Newton established his mathematical theory of gravitation and motion:

1. *The orbit of a planet is an ellipse with the Sun at one focus.* We now know that, in general, according to the laws of motion and gravitation of Newton, the trajectory of a point particle (a planet) about the Sun can be an ellipse, a parabola, or a hyperbola (i.e. any conic section).

2. *The vector from the Sun to the planet sweeps out equal areas d during equal intervals of time.* The area sweep rate is $\mathrm{d}A/\mathrm{d}t = (1/2)r^2\mathrm{d}\theta/\mathrm{d}t = H/(2m)$, so this law is a direct consequence of the conservation of angular momentum.

3. *The square of the orbital period of the planet is proportional to the cube of the semi-major axis of its ellipse, i.e.* $T^2 \propto a^3$. This can easily be proven with the explanation above, taking into account that the area of an ellipse is $A = \pi ab$, where $b = a\sqrt{1-e^2}$ is the semi-minor axis.

2 Rocket motion

Action-reaction principle: thrust force As the third law of Newton states, for every action there is an equal but opposite action elsewhere. By ejecting propellant, a rocket exploits this law to create a thrust force on the vehicle equal to the momentum expelled from the rocket per unit time:

$$F = \dot{m}c, \tag{2.1}$$

where $\dot{m} = -\mathrm{d}m/\mathrm{d}t$ is the mass flow rate at which propellant is being ejected, and c the exhaust velocity of the propellant leaving the rocket. The velocity c is also known as **specific impulse**, I_{sp}. Due to historical reasons, the I_{sp} is sometimes expressed in seconds instead of in velocity units, dividing c by $g_0 = 9.81$ m/s^2, the gravity acceleration on the ground:

$$I_{sp}^{v} = c \text{ (in velocity units);} \quad I_{sp}\,[\mathrm{s}] = \frac{c\,[\mathrm{m/s}]}{g_0\,[\mathrm{m/s}^2]} \text{ (in seconds).} \tag{2.2}$$

Tsiolkovsky's rocket equation The equation of motion of a rocket system of mass m and velocity v with thrust $F = \dot{m}c$ is:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = \dot{m}c = -\frac{\mathrm{d}m}{\mathrm{d}t}c \tag{2.3}$$

integrating this equation between two instants of time t_1 and t_2, between which the rocket velocity changes from v_1 to v_2, and the rocket mass changes from m_1 to m_2, requires some knowledge of differential equations. The result is:

$$\frac{\mathrm{d}v}{c} = \frac{\mathrm{d}m}{m} \Rightarrow \frac{v_2 - v_1}{c} = \ln\left(\frac{m_1}{m_2}\right). \tag{2.4}$$

In this equation, "ln" is the natural logarithm. We define the quantity $\Delta v = v_2 - v_1$, pronounced *delta-vee*, as the velocity increase obtained thanks to the rocket, which has used a mass of propellant equal to $m_2 - m_1$. Each space mission has a well-defined "cost" in terms of the necessary Δv to accomplish it.

This equation can be inverted using the exponential function "exp" into

$$\frac{m_2}{m_1} = \exp\left(-\frac{\Delta v}{c}\right) \equiv \exp\left(-\frac{\Delta v}{I_{sp}g_0}\right). \tag{2.5}$$

This equation shows that unless the Δv of the mission that we want to carry out is smaller or comparable to the $I_{sp}g_0$ of our rocket technology, the final mass m_2 will be tiny compared to the initial mass m_1 of the rocket. Typically, sending a small satellite into space requires a large rocket launcher for this reason.

Rocket staging Tsiolkovsky's rocket equation includes in m_2 not only the payload m_{pay} of the rocket (e.g. the satellite we want to put into orbit), but also any inert mass in the system such as structural mass of the rocket itself, m_{strut}. Given the adverse scaling of the rocket size, anything that can reduce inert mass is desirable.

Staging consists in releasing structural mass as soon as it is no longer needed (e.g. depleted propellant tanks, used-up rockets, etc). This way, the remaining of the trip is done with a lower inert mass, improving the overall mass performance of the rocket system. In effect, this is the same as stacking several rockets on top of each other, and igniting them in series.

To solve staging problems, it is useful to write down the initial mass of the i-th rocket stage as

$$m_{0,i} = m_{fuel,i} + m_{struct,i} + m_{pay,i}, \tag{2.6}$$

and note that the payload of the i-th stage is actually the next stage, $(i + 1)$. The advantages of staging are great for systems with up to 3–4 stages, but beyond that, the reduction of the initial system mass is very limited.

Impulsive maneuvers Rocket maneuvers allow to change the trajectory (and thus the orbit) of a spacecraft. Chemical rockets provide large thrust levels for short periods of time; hence, we can simplify the analysis of chemical rocket maneuvers by imagining that the spacecraft changes instantaneously its velocity vector $\boldsymbol{v}$ when the maneuver takes place. This approximation is known as the *impulsive maneuver model*. Using vector operations, velocity changes from $\boldsymbol{v}_1$ to $\boldsymbol{v}_2 = \boldsymbol{v}_1 + \Delta\boldsymbol{v}$, while the position vector $\boldsymbol{r}$ remains unchanged during the instantaneous rocket firing.

The Hohmann transfer explained in the course is an example of application of the impulsive maneuver model.

3 Space Environment

Isothermal atmosphere model Earth's atmosphere is composed of many different layers. Temperature varies in a non-trivial way, inverting its gradient several times. Pressure, however, decreases monotonically as we ascend in the atmosphere. Modeling the atmospheric pressure and density in the upper layers of the atmosphere is important to determine air drag on low-Earth-orbit satellites.

We can pose a very simple model for pressure if we assume that temperature is constant in the atmosphere $T = \text{const}$. This is only an approximation of course, and more advanced models exist that take temperature variation with altitude into account. We call this model the "isothermal atmosphere" model.

Consider a small cylindrical column of air of height $\mathrm{d}h$, and call A the area of its bottom and upper surfaces. If air density is ρ at this height, the weight of this column of air is simply $\rho g A \mathrm{d}h$, where g is the gravity acceleration.

Since this column is in equilibrium (i.e., it is not falling or rising), this weight must be compensated by the pressure difference $\mathrm{d}p$ between its bottom and upper surfaces. Then, we can write the following force balance equation:

$$Ap - A(p + \mathrm{d}p) = \rho g A \mathrm{d}h \Rightarrow \mathrm{d}p = -\rho g \mathrm{d}h \tag{3.1}$$

Using the ideal gas law $p = \rho RT/M$, where $R = 8.314$ J/(K·mol) is the universal gas constant and M is the gas molecular mass (0.029 kg/mol for air), we can substitute ρ and integrate this equation to obtain an exponential evolution of pressure with height:

$$\frac{\mathrm{d}p}{p} = -\frac{gM}{RT}\mathrm{d}h \Rightarrow p = p_0 \exp\left(\frac{-h}{h_0}\right) \tag{3.2}$$

where p_0 is the pressure at $h = 0$ and $h_0 = RT/(gM)$.

Solar radiation The Sun emits a tremendous amount of electromagnetic radiation at all frequencies and in all directions. At the Earth, we receive about

$$S_\oplus = 1366 \text{ W/m}^2 \tag{3.3}$$

of radiation. Naturally, the power per unit area (i.e. the *irradiance*) increases as we get closer to the Sun, and decreases as we get away from it.

Due to conservation of energy, the irradiance $S(r)$ integrated over a the surface of a sphere of radius r from the Sun is a constant independent of r:

$$4\pi r^2 S(r) = 4\pi r_\oplus S_\oplus \Rightarrow S(r) = S_\oplus \frac{r_\oplus^2}{r^2} \tag{3.4}$$

where $r_\oplus = 150$ million km (1 astronomical unit) is roughly the distance between the Earth and the Sun. This formula is useful to compute the solar irradiance anywhere in the solar system, if we know the distance to the Sun, r.

Solar irradiance affects how much electric power we can generate with solar arrays, and it defines how hot our spacecraft will get when illuminated. It also affects how much solar radiation pressure our spacecraft will feel.

4 Space Systems I

Breakdown of a space system A space system consists of several segments working together, where at least one of them is spaceborne. The common segments of a space system are:

1. The space segment: this is the spacecraft (or the various spacecraft) that are in orbit
2. The ground segment: the control centers and ground stations used to track, monitor, and command the space segment
3. The launcher segment: the launch vehicles used to set the space segment in orbit.

Within the space segment, each spacecraft can be decomposed into the **payload** (i.e., the instruments, transponders, etc that the client wants to have in orbit) and the **spacecraft platform or bus** (consisting of all the subsystems needed to support the operation of the payload and keep the satellite alive).

There are several spacecraft platform subsystems:

1. Power subsystem: in charge of producing, accumulating, distributing electric power to the different loads.
2. Telecommunications: uplink (telecommand) and downlink (telemetry and mission data) require of a communication subsystem to work.
3. Attitude determination and control subsystem: this subsystem determines the current orientation (attitude) of the spacecraft using sensors, and then changes it to the desired orientation using actuators like momentum wheels.
4. Propulsion subsystem: needed if the satellite must perform propulsive maneuvers. Every now and then, small corrective maneuvers are required to maintain the same operational orbit, or to desaturate the ADCS.
5. Data handling: the on board computer must process the incoming commands from the ground segment, monitor and control all the other subsystems, produce the telemetry that will be sent to the ground, and store and sometimes process the mission data until it can be delivered to a ground station.
6. Structural subsystems: the structure is in charge of maintaining the geometrical distribution of the spacecraft, and enduring the stresses that occur during launch.
7. Thermal control subsystem: this subsystem must deal with the changing environmental conditions (sunlight/eclipse, different spacecraft orientations with respect to the Sun), and maintain the temperature of each component within the allowable ranges at all times.
8. Environmental control and life support: In case of manned missions, it is necessary to provide air, water, and food to the crew, keep temperature and humidity at reasonable levels, and to dispose or recycle the generated waste.

Telecommunications: link budget equation When trying to reach the ground from a spacecraft, the signal in our communication must be sufficiently focused and have enough power for the ground antenna to be able to receive it and tell the signal apart from the background electromagnetic noise.

If we have determined that the minimum energy per bit of information for a successful communication link is E_b, we can write down the link budget equation as follows:

$$E_b \leqslant \frac{P L_t L_r L_a L_s G_t G_r}{R} \tag{4.1}$$

where:

- P is the power, i.e. energy per unit time, spent by the spacecraft in sending the signal

- L_t are any efficiency losses in the transmitter. E.g., if only 80% of the input power eventually becomes the signal power, then $L_t = 0.8$.
- L_r contemplates any similar losses at the receiving end of the link.
- L_a are any losses that may exist due to the propagation of the signal through the atmosphere. The atmosphere may damp or absorb part of the signal power, meaning that only a fraction of the power makes it to the ground. The absorption depends on the wavelength of the signal, λ.
- L_s are the free space losses, which scale with the transmission distance d. Since the emitted power at the satellite expands outwards roughly spherically, the power per unit area decreases as $1/d^2$. L_s is formally defined as $L_s = \lambda^2/(4\pi d)^2$.
- G_t is the gain of the transmitting antenna, i.e., how well the antenna concentrates the radiated power in the desired direction, rather than emitting in all directions. It is defined as $G_t = 4\pi A_t/\lambda^2$, where A_t is the effective area of the antenna.
- G_r is the gain of the receiving antenna, i.e., how well the antenna selectively listens only to the desired direction, rather than all directions. It is defined as $G_r = 4\pi A_r/\lambda^2$, where A_r is the effective area of the antenna.
- R is the data rate of the transmission, i.e., how many bits per unit time are being sent.

The product PL_tG_t is known as the *effective isotropic radiated power* (EIRP).

Rotational dynamics: attitude determination and control The rotational dynamics of the spacecraft in orbit are governed by the angular momentum equation—the rotational analogous of the (linear) Newton's second law.

$$\frac{\mathrm{d}\boldsymbol{H}_G}{\mathrm{d}t} = \boldsymbol{M}_G \tag{4.2}$$

where $\boldsymbol{H}_G$ is the total angular momentum of the spacecraft about its center of mass G, and $\boldsymbol{M}_G = \sum(\boldsymbol{r} - \boldsymbol{r}_G) \times \boldsymbol{F}$ the moment of all forces about the center of mass.

While this equation may seem complicated, the important aspect to be understood in this course is that in the absence of external moments of forces (i.e., if $\boldsymbol{M}_G = 0$), the total angular momentum of the spacecraft is conserved. This means that we cannot change its state of rotation, unless the spacecraft is composed of two or more bodies and we transmit angular momentum from one body to another, or unless we use propulsion to change the total angular momentum.

For instance, imagine a cubic spacecraft floating in empty space, with a wheel in its interior. If the cube is not rotating initially, we can make it rotate about the axis of the wheel if we spin up or down the wheel. Observe that during the whole maneuver, the total angular momentum (sum of the angular momentum of the cube and the wheel) remain constant.

This mechanism is used in the ADCS to control the orientation and rotation state of the spacecraft in the presence of perturbations that create a moment of forces. However, when the angular momentum stored in the wheels is too large, the rotational speed of the wheels may reach its maximum, and we need to use the propulsion system to *desaturate* the wheels and start over.