The Conquest of Space: Space Exploration and Rocket Science

Technical part formulary and notes

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Aerospace Engineering Department (UC3Mx)

Preface

The present notes correspond to the course "The Conquest of Space: space exploration and rocket science," imparted on UC3Mx by Manuel Sanjurjo, Mario Merino, Manuel Soler, Gonzalo Sánchez, David Morante, Filippo Cichocki, Daniel Pérez, Xin Chen, and Eduardo Ahedo from the UC3M Aerospace Engineering department.

The notes are provided as a companion material to the course, and are not intended as a substitute of a real book or the lessons in the course. They are conceived as a quick summary without extensive explanations, for which the learner must search elsewhere. The learner should have a working knowledge of fundamental mathematics (calculus and linear algebra) and physics (mechanics) to fully understand everything; if you don't, or if you feel a bit uncomfortable with these topics at the start the course, please search online or in your library for an introduction to them or ask in the forums.

The document typeset in LaTeX and is released under the CC-BY-SA license v.4.0. The latest version can be found on GitHub and on Mario Merino's personal webpage. If you detect any errata, or have any comments for its improvement, please contact Mario Merino at mario.merino@uc3m.es.

1 Astrodynamics: the two body problem

Position, velocity and acceleration of point particle In Cartesian coordinates, and calling $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ the unit vectors of the basis of our inertial reference frame, the position $\boldsymbol{r}$, velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ vectors of a point particle P are defined as:

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \tag{1.1}$$
$$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k} \tag{1.2}$$
$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} \tag{1.3}$$

The velocity is the derivative of the position, and the acceleration is the derivative of the velocity. Conversely, the position is the integral of the velocity, and the velocity is the integral of the acceleration:

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v}; \qquad \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{a}; \tag{1.4}$$
$$\boldsymbol{r} = \int_0^t \boldsymbol{v}\mathrm{d}t + \boldsymbol{r}_0; \qquad \boldsymbol{v} = \int_0^t \boldsymbol{a}\mathrm{d}t + \boldsymbol{v}_0. \tag{1.5}$$

Newton's second law of mechanics states that the acceleration of a point particle states that the acceleration of the particle is proportional to the sum of all forces acting upon it. The constant of proportionality is its mass:

$$m\boldsymbol{a} = \sum \boldsymbol{F} \tag{1.6}$$

Thus, knowing the mass and the forces acting on a particle we may compute its acceleration. By integrating the acceleration twice we can find how the position of the particle changes in time, i.e., its *trajectory*. This is know as solving the direct problem of mechanics.

Newton's law of gravitation The force exerted upon a point particle of mass m by another of mass M located at the origin of coordinates is

$$\boldsymbol{F} = -G\frac{mM}{r^2}\boldsymbol{u}_r \tag{1.7}$$

where $\boldsymbol{u}_r = \boldsymbol{r}/r$ is the unit vector in the direction of $\boldsymbol{r}$, and $G = 6.67408 \cdot 10^{-11}$ $\mathrm{m^3kg^{-1}s^{-2}}$ is the universal gravitational constant.

The equation of motion of a point particle in the gravity field of a planet is then:

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -G\frac{M}{r^2}\boldsymbol{u} \tag{1.8}$$

Conservation of mechanical energy The mechanical energy of a point particle in the gravity field of a planet like the Earth is $E = E_{kin} + E_{pot}$, where

$$E_{kin} = \frac{1}{2}mv^2; \quad E_{pot} = -\frac{GmM}{r}. \tag{1.9}$$

The mechanical energy E is a conserved quantity of motion.

Demonstration: by denoting with $\cdot$ the scalar product, we dot-multiply Eq. (1.8) by $m\boldsymbol{v}$:

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\boldsymbol{v} = -G\frac{mM}{r^2}\left(\frac{\boldsymbol{r}}{r}\right)\cdot\boldsymbol{v} \Rightarrow \frac{1}{2}m\frac{\mathrm{d}v^2}{\mathrm{d}t} = -G\frac{mM}{r^3}\boldsymbol{r}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = -G\frac{mM}{2r^3}\frac{\mathrm{d}r^2}{\mathrm{d}t} = GmM\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r}\right) \tag{1.10}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2 - \frac{GmM}{r}\right) = 0 \Rightarrow \frac{1}{2}mv^2 - \frac{GmM}{r} = E = \text{const.} \tag{1.11}$$

Conservation of angular momentum The angular momentum of a point particle about the origin is defined as

$$\boldsymbol{H} = m\boldsymbol{r} \times \boldsymbol{v}, \tag{1.12}$$

where $\times$ is the vector product. $\boldsymbol{H}$ is a vector that is conserved in the motion. Its magnitude can be written as $H = mr^2 \mathrm{d}\theta/\mathrm{d}t$, where θ is the polar angle of the motion of the particle.

Demonstration: we cross-multiply Eq. (1.8) by $m\boldsymbol{r}$. The right-hand-side is identically zero thanks to the properties of the vector product:

$$m\boldsymbol{r} \times \boldsymbol{a} = -G\frac{mM}{r^2}\boldsymbol{r} \times \left(\frac{\boldsymbol{r}}{r}\right) = 0. \tag{1.13}$$

We now apply the following identity

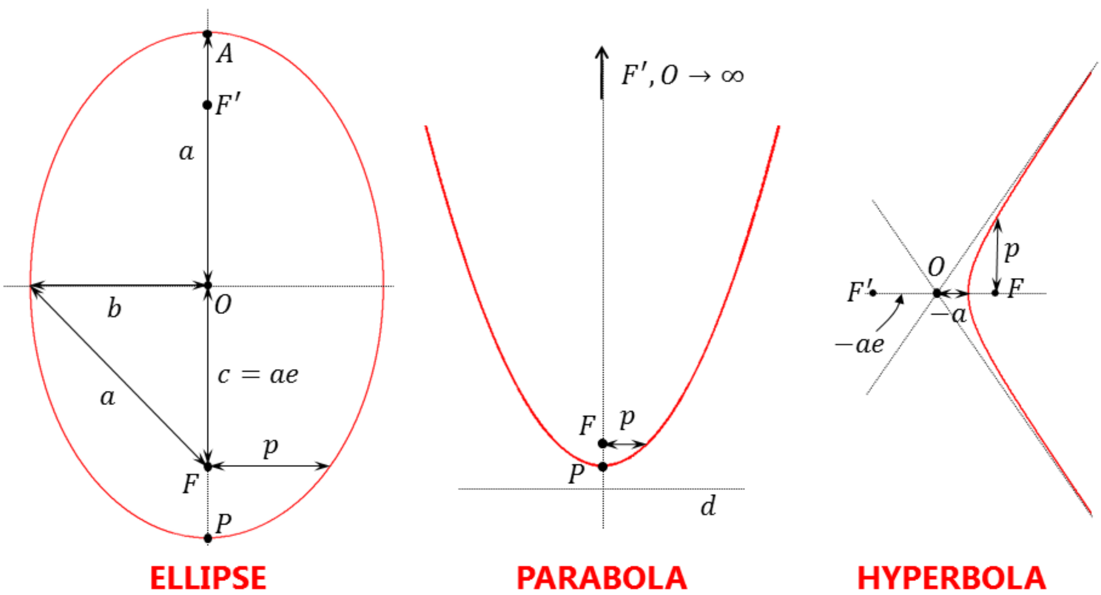
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\boldsymbol{r} \times \boldsymbol{v}\right) = m\boldsymbol{v} \times \boldsymbol{v} + m\boldsymbol{r} \times \boldsymbol{a} = m\boldsymbol{r} \times \boldsymbol{a}, \tag{1.14}$$

and conclude that

$$\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t} = 0 \Rightarrow \boldsymbol{H} = \text{const}. \tag{1.15}$$

Conic sections The trajectory of a point particle in the two body problem of astrodynamics is a conic section. Conic sections are characterized by their semi-major axis a and their eccentricity e. The conic section can be any of the following:

1. Ellipse ($a > 0$ and $e < 1$). The only closed orbit. When $e = 0$, the ellipse is a circle.
2. Parabola ($a \to \infty$ and $e = 1$). A particle in a parabolic trajectory will reach infinity with zero velocity. The semi-latus rectum $p = a(1 - e^2)$ is used instead of a in this case to avoid the indetermination.
3. Hyperbola ($a < 0$ and $e > 1$). A particle in a parabolic trajectory will reach infinity with non-zero velocity. Of the two branches of the hyperbola, the trajectory is only one of them.



The pericenter and the apocenter are the points of the trajectory nearest and farthest away from the planet:

$$r_p = a(1 - e); \quad r_a = a(1 + e). \tag{1.16}$$

Observe that the definition of the apocenter only makes sense for elliptic orbits. Also, in the case of parabolic orbits, the pericenter can be computed simply as $r_p = p/2$.

At the pericenter and apocenter the position and velocity vectors are perpendicular to each other, i.e., $\boldsymbol{r} \cdot \boldsymbol{v} = 0$.

Vis-viva equation The constant E in the energy equation is related to the semi-major axis of the orbit:

$$\frac{1}{2}mv^2 - \frac{GmM}{r} = -\frac{GmM}{2a} \tag{1.17}$$

Demonstration: We particularize the mechanical energy equation at the pericenter and the apocenter and subtract the two expressions:

$$\frac{1}{2}mv_p^2 - \frac{GmM}{r_p} = E; \quad \frac{1}{2}mv_a^2 - \frac{GmM}{r_a} = E. \tag{1.18}$$

$$\frac{1}{2}m\left(v_p^2 - v_a^2\right) - GmM\left(\frac{1}{r_p} - \frac{1}{r_a}\right) = 0; \tag{1.19}$$

From the conservation of angular momentum, using $2a = r_a + r_p$, and since the position and velocity vectors are perpendicular at the pericenter and apocenter,

$$v_a = \frac{r_p}{2a - r_p}v_p \tag{1.20}$$

Substituting v_a in Eq. (1.19), and using again $2a = r_a + r_p$,

$$\frac{1}{2}mv_p^2\left(1 - \frac{r_p^2}{(2a - r_p)^2}\right) = GmM\left(\frac{1}{r_p} - \frac{1}{2a - r_p}\right) \Rightarrow v_p^2 = \frac{GM}{a}\left(\frac{2a}{r_p} - 1\right) \tag{1.21}$$

Finally, substituting in the first of Eqs. (1.18) we find $E = -GmM/(2a)$.

Clearly,

1. Particles in elliptic orbits have negative mechanical energy. The orbit is bounded (under no condition could the particle have $r > 2a$, as the velocity would become imaginary; in reality, the maximum r occurs at apocenter, $r_a = a(1 - e)$).
2. Particles in parabolic orbits have zero mechanical energy, as they reach zero velocity as $r \to \infty$.
3. Particles in hyperbolic orbits have positive mechanical energy, as they have a non-zero excess velocity at infinity.

Important velocities There are several important velocity quantities in the two body problem. Importantly, these velocities do not depend on the mass of the point particle, only on the mass of the planet M and the universal gravitational constant G:

- Circular velocity: $v_c = \sqrt{GM/r}$. This is the magnitude of the velocity that a particle must have at a radius r to be in circular orbit. Obtained from vis-viva equation, by imposing that the trajectory is a circle, $r = a$.
- Escape velocity: $v_e = \sqrt{2GM/r}$. Any particle with this velocity at a distance r from the origin is in parabolic orbit, and will reach infinity with zero velocity. Obtained from vis-viva equation by setting $a = \infty$.
- Velocity at pericenter, $v_p = \sqrt{GM\frac{1+e}{1-e}}$. Obtained from vis-viva equation, particularizing at $r_p = a(1 - e)$.
- Velocity at apocenter, $v_a = \sqrt{GM\frac{1-e}{1+e}}$. Obtained from vis-viva equation, particularizing at $r_a = a(1 + e)$. Observe that $v_p > v_a$ always.

Kepler laws Johannes Kepler stated his famous three laws of planetary motion before Newton established his mathematical theory of gravitation and motion:

1. *The orbit of a planet is an ellipse with the Sun at one focus.* We now know that, in general, according to the laws of motion and gravitation of Newton, the trajectory of a point particle (a planet) about the Sun can be an ellipse, a parabola, or a hyperbola (i.e. any conic section).

2. *The vector from the Sun to the planet sweeps out equal areas d during equal intervals of time.* The area sweep rate is $\mathrm{d}A/\mathrm{d}t = (1/2)r^2\mathrm{d}\theta/\mathrm{d}t = H/(2m)$, so this law is a direct consequence of the conservation of angular momentum.

3. *The square of the orbital period of the planet is proportional to the cube of the semi-major axis of its ellipse, i.e.* $T^2 \propto a^3$. This can easily be proven with the explanation above, taking into account that the area of an ellipse is $A = \pi ab$, where $b = a\sqrt{1-e^2}$ is the semi-minor axis.