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Graduation Project Report

M2 Mathematics for Finance and Data

Yield Curve Bootstrapping & Convexity Adjustment

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Abstract

This report explores various methods for constructing yield and forward curves, focusing on the implementation of bootstrapping techniques using C#. The project integrates different financial instruments such as swaps and futures and addresses the complexities of curve construction by offering multiple approaches. These approaches include various interpolation methods, choices of root-finding algorithms, and options for handling financial conventions, such as day-count standards and calendar considerations. The codebase is comprehensive, providing flexibility in curve construction through modular and customizable implementations.

In the second part of the project, we delve into the literature surrounding convexity adjustments between futures and forward rates, modeling interest rates as stochastic processes. We investigate two prominent approaches to derive the convexity adjustment term and implement them within our system. Through the exploration of popular rate dynamics, we derive exact expressions for the convexity adjustment and demonstrate their practical application in curve construction.

This project bridges theoretical finance and practical implementation, offering a robust toolset for yield and forward curve construction while contributing to the understanding of convexity adjustments in interest rate modeling.

Keywords : Yield curve, Bootstrapping, Interpolation, C#, Swaps, Future Contracts, Convexity Adjustment

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Contents

List of Figures

List of Tables

Acronyms

CME Chicago Mercantile Exchange.

ESTR Euro Short-Term Rate.

FRA Forward Rate Agreement.

IBOR Inter-bank Offered Rate.

IMM International Monetary Market.

IRS Interest Rate Swap.

JSON JavaScript Object Notation.

LIBOR London Inter-bank Offered Rate.

OIS Overnight Indexed Swap.

OOP Object-Oriented Programming.

RFR Risk-Free Rate.

SOFR Secured Overnight Financing Rate.

SONIA Sterling Overnight Index Average.

ZC Zero Coupon.

Chapter 1

Introduction

1.1 Motivation

In today's sophisticated financial markets, the importance of precise and reliable financial modeling cannot be overstated. Among the myriad of financial tools and metrics available, the yield curve remains a fundamental element, pivotal in the pricing of interest-bearing securities, risk management, and economic forecasting. The yield curve, which illustrates the relationship between interest rates and the maturities of debt instruments, serves as a critical benchmark for various financial operations. It enables investors to gauge future interest rate expectations, assists financial institutions in managing interest rate risk, and supports policymakers in analyzing economic conditions.

The yield curve's central role in financial analysis and decision-making underscores the need for accurate construction techniques. It is not merely a theoretical concept but a practical tool that influences real-world financial strategies and risk assessments. For instance, investors rely on the yield curve to determine the value of bonds and other fixed-income securities, guiding their investment decisions based on expected future interest rates. Financial institutions use the yield curve to manage risk, align assets and liabilities, and make strategic investment choices. Moreover, central banks and policymakers interpret the yield curve to make informed decisions about monetary policy and economic interventions.

Given the yield curve's significance, refining the methods used to construct it is crucial. Bootstrapping is a widely used technique for constructing yield curves, involving the extraction of zero-coupon yield curves from market data such as deposit rates, futures, and swaps. This method requires both theoretical understanding and practical implementation. As financial markets evolve and new instruments are introduced, enhancing bootstrapping techniques becomes necessary to accommodate these changes and ensure accurate yield curves. Improvements in interpolation methods, handling of various date conventions, and robustness in the face of market volatility are essential areas of focus.

Another critical aspect of yield curve construction is convexity adjustment. Convexity adjustments are necessary to reconcile theoretical models with actual market prices, particularly when dealing with futures and forward rates. These adjustments account for the curvature in the pricing of financial instruments and help align theoretical yield curves with observed market behavior. Accurate convexity adjustments are vital for producing reliable yield curves and improving the precision of financial models. Understanding and implementing these adjustments involves delving into complex mathematical concepts and stochastic processes, which can enhance the alignment of theoretical models with market realities.

The challenges in yield curve construction and convexity adjustment are multifaceted, encompassing interpolation methods, date conventions, and the inherent complexities of financial instruments. Addressing these challenges involves continuous refinement of techniques and methodologies. Advances in programming, mathematical modeling, and computational algorithms are pivotal in improving the accuracy and reliability of yield curves. By enhancing these techniques, financial professionals can better manage risk, optimize investment strategies, and make more informed decisions.

Looking ahead, the pursuit of improved yield curve construction methods and convexity adjustments has far-reaching implications. As financial markets continue to evolve, there is an ongoing need for innovative approaches to modeling and analysis. Emerging technologies, such as machine learning and advanced statistical methods, offer new opportunities to refine yield curve models and address the complexities of modern financial markets. Integrating these technologies with traditional financial modeling approaches has the potential to significantly enhance the accuracy and reliability of yield curves, providing valuable insights for investors, institutions, and policymakers.

In summary, the motivation to advance yield curve construction and convexity adjustment stems from the need for precision in financial modeling and decision-making. By addressing the challenges and refining the techniques involved, we can improve the accuracy of financial models and better manage the risks and opportunities in an increasingly complex financial environment. This ongoing effort is essential for supporting effective financial strategies and informed decision-making in today's dynamic financial landscape.

1.2 Project Framework

1.2.1 Objectives

The primary objective of this project is to develop a robust and well-architected C# application that effectively constructs the yield curve and other associated curves, such as discount and forward curves, using market data. The emphasis will be on creating high-quality code that adheres to the principles of object-oriented programming (OOP) and follows SOLID design principles, ensuring modularity, maintainability, and extensibility of the system. The implementation should address the various practical challenges faced by other practitioners, such as accurate bootstrapping, managing different date conventions, and handling potential numerical instabilities.

An integral part of this objective is to explore and compare different interpolation methods for yield curve construction. Since the choice of interpolation method can significantly affect the shape and accuracy of the yield curve, the project aims to implement and evaluate several techniques, including linear, linear on log, cubic, and quadratic interpolations. The goal is to identify the method that provides the best balance between accuracy and computational efficiency in various market conditions.

In addition to the primary focus on yield curve construction, a secondary objective of the project is to incorporate convexity adjustment into the system. Convexity adjustment is an essential factor in reconciling futures rates with forward rates, particularly when working with instruments like interest rate futures. The project will investigate different approaches to computing convexity adjustments and implement the most effective techniques to correct the futures rates extracted from market data, thereby refining the forward curve. The overall aim is to ensure that the final model captures the market dynamics with as much precision as possible, providing valuable insights for pricing and risk management.

1.2.2 Related Work/Literature Review

After the 2008 financial crisis, the use of multiple financial instruments to construct yield curves became crucial in accurately reflecting market conditions. This approach relies on determining the relationship between the prices of various instruments and the discount factor, a key element in curve construction. In the work by Awalee Consulting [2], the most significant instruments are identified, and their precise usage is detailed by expressing the mathematical relationships that link each instrument's market price to the discount factor. Instruments such as deposits, futures, Forward Rate Agreements (FRAs), Interest Rate Swaps (IRS), and Basis Swaps are utilized across different maturities, each contributing to the shape of the curve by incorporating the specific pricing dynamics of the market. The article provides a clear framework for applying these instruments in yield curve bootstrapping, ensuring that the curve accurately reflects current market conditions.

As for the interpolation used in the bootstrapping, the choice of interpolation methods in yield curve construction has been extensively studied, as interpolation directly impacts the smoothness and accuracy of the curve. In their seminal work, Hagan and West [4] delve into different interpolation techniques, such as piecewise linear interpolation on discount factors, cubic splines, and monotone convex interpolation. These methods aim to achieve a smooth and arbitrage-free curve. For instance, cubic spline interpolation ensures a smoother transition between curve segments, making it particularly useful for financial instruments sensitive to small interest rate changes. However, monotone convex interpolation is preferred in scenarios where ensuring no arbitrage is paramount, as it avoids unrealistic negative forward rates. The authors highlight the trade-offs between these methods in terms of complexity and computational efficiency, guiding practitioners in selecting the most appropriate technique for their specific use case.

The concept of convexity adjustment, a critical factor when converting futures rates to forward rates, is extensively covered in Hull's textbook [5]. The adjustment arises due to the mark-to-market nature of futures contracts, which introduces differences between futures and forward rates despite their convergence at maturity. Hull expands on this concept in his technical note [6], where he presents the convexity adjustment as a necessary correction for futures prices. Since futures contracts are settled daily and forward contracts are not, the volatility in interest rates affects the value of futures contracts differently than forward contracts. Hull provides a clear and actionable formula for this adjustment, which can be directly applied in financial models to account for the differences in pricing.

In addition to Hull's approach, another method of deriving the convexity adjustment is provided by Noel Vaillant [10]. Vaillant's method utilizes a martingale approach. He highlights that the convexity adjustment stems from the fundamental difference between futures and forward contracts due to the former's marking-to-market feature. By leveraging martingale techniques, Vaillant derives a theoretical formula for the adjustment, ensuring that the results are consistent with those obtained through other methodologies. Instead of modeling the short rate, Vaillant models the compounded rate and the futures rate, allowing for a more direct treatment of the relationship between these rates. This approach offers an alternative perspective, emphasizing probabilistic methods and providing a clear pathway for the practical implementation of convexity adjustments. The report also details how the derived formula can be approximated and implemented in a simple spreadsheet model, making it accessible for practitioners seeking a more intuitive approach to convexity adjustments.

1.2.3 Main Contributions of This Project

The main contributions of this work to yield curve construction and convexity adjustments are as follows:

- Implementation of OIS yield curve bootstrapping techniques in C#, including a thorough exploration of various interpolation methods and handling of date conventions.
- Integration of convexity adjustments into the yield curve construction process, with a comprehensive analysis of the theoretical insights.
- A comparative analysis of different methods used in the project, evaluating their effectiveness and accuracy in yield curve construction and the impact of convexity adjustments.

1.3 Outline

The thesis is organized into several key chapters to comprehensively address the topic of yield curve construction and convexity adjustments. In Chapter 1, we begin by providing the background and context for yield curve construction, outlining its significance in financial markets. We then define the objectives of the project, focusing on the implementation of yield curve bootstrapping and the exploration of convexity adjustments. The chapter concludes with a description of the scope and structure of the report, setting the stage for the subsequent discussions.

Chapter 2 delves into the techniques and methodologies used in yield curve construction. We start with an overview of yield curves and their various types, such as discount, yield, and forward curves. The chapter continues with a detailed discussion of the financial instruments involved in building yield curves, and covers interpolation methods, including linear, cubic, and quadratic approaches. We also address the challenges faced in yield curve construction, such as interpolation issues and date conventions.

Chapter 3 introduces the concept of convexity adjustment and its integration into yield curve construction. We present to existing approaches with which we worked to improve and apply for different stochastic models for the rates, deriving exact expressions for convexity adjustments, and discussing their practical implications. We conclude with an explanation of how these adjustments were incorporated into our yield curve construction implementation, focusing on the calibration of the used models.

In Chapter 4, we focus on the practical implementation of the discussed techniques using C#. This chapter covers the project architecture and design, detailing how the bootstrapping technique was implemented. We explain the handling of interpolation methods and data, discuss date and calendar conventions, and describe the testing and validation processes used to ensure the accuracy of the implementation, based on unit tests.

Chapter 5 presents the results and discussions of the project. We analyze the outcomes of the yield curve construction, evaluate the effects of incorporating convexity adjustments, and compare the various methods employed in the project.

Finally, we summarize the key findings of the thesis, discuss the limitations of the current work, and provide suggestions for future research and improvements.

Chapter 2

Foundations of Yield Curve Construction

To establish a robust foundation for the yield curve construction methods applied in this project, it is essential to delve into the underlying principles and instruments that inform these methodologies. We begin in Section 2.1 by introducing the mathematical background necessary for understanding yield curves, with a focus on zero coupon yields, forward rates, and their corresponding curves. In Section 2.2, we explore the financial instruments integral to curve construction, including deposits, futures, swaps, and basis swaps, clarifying their roles and applications. Section 2.3 is dedicated to discussing various interpolation methods, evaluating them against key criteria, and exploring techniques such as linear and spline-based interpolations. Lastly, Section 2.4 addresses the bootstrapping process, discussing the conceptual framework, the challenges it presents, and the practical considerations such as calendar conventions and market data handling.

2.1 Financial Background: Interest Rate Market

Interest rates play a crucial role in the global economy, influencing borrowing costs, investment decisions, and monetary policy. The **interest rate market** encompasses the broad array of instruments that are sensitive to changes in interest rates, including bonds, loans, and various derivatives. Within this market, benchmark interest rates are essential for pricing these instruments and managing risk.

Historically, **Interbank Offered Rates (IBORs)**, such as the London Interbank Offered Rate (LIBOR), were the primary reference rates used across global financial markets. IBORs served as benchmarks for trillions of dollars in financial products, from floating-rate loans and mortgages to derivatives like interest rate swaps. However, the 2008 financial crisis highlighted deep-seated vulnerabilities within this system, triggering a fundamental shift in how interest rate benchmarks are determined and utilized.

2.1.1 The 2008 Financial Crisis and the LIBOR Scandal

During the 2008 financial crisis, the reliability of IBORs came under intense scrutiny. LIBOR, for example, was based on submissions from a panel of banks estimating their borrowing costs in the unsecured interbank market. However, during the crisis, interbank lending virtually froze as banks became wary of each other's creditworthiness. This led to highly inaccurate rate submissions that did not reflect actual market conditions.

The situation worsened with the discovery of widespread manipulation of LIBOR by several major banks. Banks had been deliberately submitting false rates to benefit their trading positions or to appear financially healthier than they were. The resulting scandal severely undermined confidence in IBORs and led regulators to seek alternatives that were more transparent and based on actual market transactions.

2.1.2 The Shift to Risk-Free Rates (RFRs) and OIS

In response to the vulnerabilities exposed during the crisis, global regulators initiated a move away from IBORs towards **Risk-Free Rates (RFRs)**. Unlike IBORs, which included a credit risk component due to being based on unsecured interbank lending, RFRs are based on overnight borrowing rates that are considered to be virtually risk-free. These rates are typically derived from large volumes of actual transactions, making them more reliable and harder to manipulate.

One of the key developments in this transition has been the increased use of **Overnight Index Swaps (OIS)** as benchmarks. An OIS is an interest rate swap where the floating leg is tied to an overnight index, such as the Secured Overnight Financing Rate (SOFR) in the U.S., the Euro Short-Term Rate (ESTR) in the Eurozone, or the Sterling Overnight Index Average (SONIA) in the UK. Because these rates are based on overnight transactions, they are less exposed to credit risk and offer a purer measure of short-term interest rates.

2.1.3 Impact on Financial Markets

The shift from IBORs to OIS-based benchmarks represents a significant change in the structure of the financial markets. The new risk-free benchmarks provide a more accurate reflection of the cost of borrowing in the risk-free market and are expected to lead to greater stability in the pricing of financial products. However, the transition has not been without challenges. The vast number of financial contracts tied to IBORs has required extensive reworking, with market participants needing to adjust their systems, documentation, and pricing models.

This evolution has also affected the techniques used for interest rate curve construction, such as **bootstrapping**. In a post-IBOR world, bootstrapping OIS discounting curves has become the standard practice for pricing derivatives and other financial instruments, providing more consistent and market-aligned valuation.

The transformation of the interest rate market from IBORs to OIS-based rates is a direct consequence of the lessons learned from the 2008 financial crisis and ongoing efforts to strengthen the resilience and transparency of the global financial system.

2.2 Mathematical Background

The term structure of interest rates refers to the relationship between a bond's yield-to-maturity and its maturity, specifically for zero-coupon bonds. When pricing derivatives based on this curve in a continuous-time framework, it's logical to use continuously-compounded rates from the beginning.

2.2.1 Zero Coupon Yield

A zero coupon (ZC) bond with maturity T, also known as a T-bond, is a financial instrument that entitles its owner to a payment of P units of currency at time T. For simplicity, we assume $P = 1$. The price of the bond at time t is denoted by $P(t, T)$ and is referred to as the discount function. The zero coupon yield $y(t, T)$ is linked to $P(t, T)$ and the relation depends on it's type:

2.2.1.1 Simple yield

$$P(t,T) = \frac{1}{1 + y(t,T)(T-t)} \tag{2.1}$$

$$y(t,T) = -\frac{1 - P(t,T)}{(T-t)P(t,T)} \tag{2.2}$$

2.2.1.2 Compounded yield

$$P(t,T) = \frac{1}{(1 + y(t,T))^{T-t}} \tag{2.3}$$

$$y(t,T) = \left(\frac{1}{P(t,T}\right)^{\dfrac{1}{T-t}} - 1 \tag{2.4}$$

2.2.1.3 Continuous yield

$$P(t,T) = \exp(-y(t,T)(T-t)) \tag{2.5}$$

$$y(t,T) = -\frac{\ln P(t,T)}{T-t} \tag{2.6}$$

2.2.2 Zero Coupon Yield Curve

The zero coupon yield curve is defined as the function

$$T \mapsto y(t,T) \tag{2.7}$$

where t is fixed at the present time. The yield curve illustrates the yield of zero coupon bonds (ZCBs) with different maturities contracted at time t, assuming all bonds have the same creditworthiness and are preferably issued by the same issuer. It is evident that the discount function and the yield curve uniquely determine each other.

In typical markets, yield curves are upward-sloping, with long-term interest rates exceeding short-term ones. A downward-sloping curve is known as inverted, while a curve with one or more turning points is called mixed. Although it's commonly believed that mixed yield curves signal market instability or illiquidity, this isn't always accurate. The supply and demand dynamics of the instruments used to construct the curve can naturally lead to such shapes. In fact, a mixed yield curve can consistently appear in stable, liquid markets over long periods.

2.2.3 Forward rate

If we can secure a loan at a predetermined rate from time t to time t_1, and another loan from t_1 to t_2 at a rate fixed at time t, then effectively, we can borrow at a known rate from time t until t_2. This leads to the no-arbitrage condition:

$$P(t,t_1)P(t;t_1,t_2) = P(t,t_2), \tag{2.8}$$

where $P(t;t_1,t_2)$ represents the forward discount factor for the interval between t_1 and t_2. This factor must have this specific value at time t, based on the available information at that time, to avoid arbitrage opportunities.

As for the rate expression, it depends on the forward rate type: simple, compounded or continuous.

2.2.3.1 Simple forward

The forward rate applicable for the period from t_1 to t_2, denoted $f(0;t_1,t_2)$, satisfies the equation:

$$(1+y(t,t_1)(t_1-t))(1+f(t;t_1,t_2)(t_2-t_1)) = (1+y(t,t_2)(t_2-t)) \tag{2.9}$$

From this, the forward rate can be calculated as:

$$\begin{aligned} f(t;t_1,t_2) &= \frac{1}{t_2-t_1}\left(\frac{1+y(t,t_2)(t_2-t)}{1+y(t,t_1)(t_1-t)} - 1\right) \\ &= \frac{1}{t_2-t_1}\left(\frac{P(t,t_1)}{P(t,t_2)} - 1\right) \end{aligned} \tag{2.10}$$

2.2.3.2 Compounded forward

Now, $f(t;t_1,t_2)$, satisfies the equation:

$$(1+y(t,t_1))^{t_1-t}(1+f(t;t_1,t_2))^{t_2-t_1} = (1+y(t,t_2))^{t_2-t} \tag{2.11}$$

From this, the forward rate can be calculated as:

$$\begin{aligned} f(t;t_1,t_2) &= \left(\frac{(1+y(t,t_2))^{t_2-t}}{(1+y(t,t_1))^{t_1-t}}\right)^{\frac{1}{t_2-t_1}} - 1 \\ &= \left(\frac{P(t,t_1)}{P(t,t_2)}^{\frac{1}{t_2-t_1}}\right) - 1 \end{aligned} \tag{2.12}$$

2.2.3.3 Continuous forward

Here, $f(t;t_1,t_2)$, satisfies the equation:

$$\exp\left(-f(t;t_1,t_2)(t_2-t_1)\right) = P(t;t_1,t_2). \tag{2.13}$$

From this, the forward rate can be calculated as:

$$f(t;t_1,t_2) = -\frac{\ln P(t,t_2) - \ln P(t,t_1)}{t_2-t_1}, \tag{2.14}$$

It is evident that forward rates are positive if and only if the discount curve function is decreasing.

2.2.4 Forward Curve

The Forward Curve is defined as the function

$$t \mapsto f(0;t,t+K) \tag{2.15}$$

where K is called the tenor, representing the duration of the loan.

2.2.5 Instantaneous Forward Rate

The instantaneous forward rate $f(t,T)$, which represents the price at time t of a momentary loan at time T, i.e $f(t,T) = \lim_{\epsilon\to 0} f(t;T,T+\epsilon)$, can be derived from the zero coupon yield, for the continuously compounded rate, as follows:

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} \tag{2.16}$$

where t is fixed at the time of contracting. We can also write

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right) \tag{2.17}$$

2.3 Financial Instruments Used in Curve Construction

To get the discount factors and therefore the yield values for each maturity, a practitioner can pick an instrument from a wide range of instruments related to interest rates, if not a set of them, depending on the maturity or tenor. Before we present the most used instruments for constructing the yield curve, it is important to state that the price of an instrument or a contract is the sum of the actualised expected cash-flows, which we can write as

$$V(t) = \sum_{i=1}^{n} P(t,T_i)\mathbb{E}_t^{\mathbb{Q}_{T_i}}[C_i] \tag{2.18}$$

where $P(t,T_i)$ represents the discount factor, $\mathbb{E}_t^{\mathbb{Q}_{T_i}}$ denotes the expectation under the measure $\mathbb{Q}_{T_i}$, and C_j is the cash flow occurring at time T_i, which may depend on the spot

LIBOR rate $L_x(T_{i-1}, T_i)$, with $x = \delta(T_{i-1}, T_i)$ being the interest rate derivative tenor. The discount factor $P(t, T_i)$, presented in (2.2.1) is also defined by

$$P(t, T_i) = \mathbb{E}_t \left[\exp\left(-\int_t^{T_i} r(u)\, du \right) \right] \tag{2.19}$$

where $r(t)$ is the short rate.

2.3.1 Deposits

A deposit is a zero coupon contract where a counterparty A (lender) lends a nominal N at T_0 to a counterparty B (borrower), which at maturity T, pays the notional amount back to the lender plus an interest accrued over the period $[T_0, T]$ at a simply compounded rate $R_X(T_0, T)$ previously fixed. Finally, the deposit rate (of tenor $X = T - T_0$) for the period $[T_0, T]$ is given by:

$$R_X(T_0, T) = \frac{1}{\delta(T_0, T)} \left[\frac{1}{P(T_0, T)} - 1 \right] \tag{2.20}$$

2.3.2 OIS-based Futures

Interest rate futures are standardized derivative instruments that are marked to market daily, which minimizes their credit risk. These contracts are quoted in terms of price rather than in terms of rates. The futures price quoted for settlement on day T_i is given by:

$$P_X^{\text{Fut}}(t, T_{i-1}, T_i) = 100 - R_X^{\text{Fut}}(t, T_{i-1}, T_i) \tag{2.21}$$

where $R_X^{\text{Fut}}(t, T_{i-1}, T_i)$ represents the futures rate for the period from T_{i-1} to T_i, as observed at time t. To obtain the corresponding forward rate $F_X(t, T_{i-1}, T_i)$, a convexity adjustment is applied:

$$F_X(t, T_{i-1}, T_i) = R_X^{\text{Fut}}(t, T_{i-1}, T_i) - C_X^{\text{Fut}}(t, T_{i-1}, T_i) \tag{2.22}$$

The forward rate $F_X(t, T_{i-1}, T_i)$ is then used in constructing the short-to-medium term forward curve, depending on the type of compounding, presented in Section 2.3.3. In fact, using the simple forward rate, the expression would be

$$F_X(t, T_{i-1}, T_i) = \frac{1}{\delta(T_{i-1}, T_i)} \left[\frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right] \tag{2.23}$$

2.3.3 Swaps

Interest rate swaps are financial contracts where one party exchanges a series of floating payments for a series of fixed payments. Each series has its own dates of payments, as it can be shown in Figure (2.1)

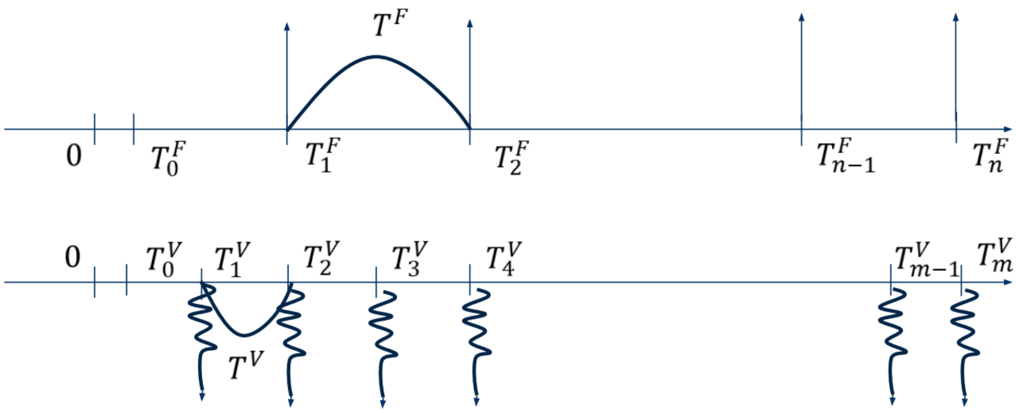

Figure 2.1: Swap Cash-flows

For a swap payer (where fixed payments are made), the value of the swap at time t is:

$$V_{\text{swap}}(t) = V_{\text{float}}(t) - V_{\text{fixed}}(t) \tag{2.24}$$

The value of the fixed leg at time t is

$$V_{\text{fixed}}(t) = NS \sum_{i=1}^{n} P(t, T_i^F)\delta(T_{i-1}^F, T_i^F) \tag{2.25}$$

where N is the notional amount, S is the fixed rate, and $\delta(T_{i-1}^F, T_i^F)$ represents the year fraction for the period between T_{i-1}^F and T_i^F.
The value of the floating leg at time t is:

$$V_{\text{float}}(t) = N \sum_{j=1}^{m} P(t, T_j^V)R_j(t)\delta(T_{j-1}^V, T_j^V) \tag{2.26}$$

where R_j is the variable interest rate, depending on the nature of the index related to the swap (BOR, OIS, ...).

At inception, for the swap to be fair, the fixed rate S is determined as:

$$S = \frac{\sum_{j=1}^{m} P(t, T_j^V)R_j(t)\delta(T_{j-1}^V, T_j^V)}{\sum_{i=1}^{n} P(t, T_i^F)\delta(T_{i-1}^F, T_i^F)} \tag{2.27}$$

To construct the forward rate curve, equation (2.27) is rearranged, and the forward rate term structure is given by:

$$\begin{aligned} R_m(t)P(t, T_m)\delta(T_{m-1}^V, T_m^V) = S &\sum_{i=1}^{n} P(t, T_i^F)\delta(T_{i-1}^F, T_i^F) \\ &- \sum_{j=1}^{m-1} P(t, T_j^V)R_j(t)\delta(T_{j-1}^V, T_j^V) \end{aligned} \tag{2.28}$$

This equation represents the bootstrapping process, from which forward rates can be calculated recursively.

For BOR index, it consists on a forward rate $R_j(t) = L_X(t, T^V_{j-1}, T_j)$. For OIS index, the rate is given by

$$R_j(t) = \frac{1}{\delta(T^V_{j-1}, T^V_j)} \left(\frac{P(t, T^V_{j-1})}{P(t, T^V_j)} - 1 \right) \tag{2.29}$$

which simplifies (2.27) to be

$$\begin{aligned} S &= \frac{\sum_{j=1}^m \left(P(t, T^V_{j-1}) - P(t, T^V_j)\right)}{\sum_{i=1}^n P(t, T^F_i)\delta(T^F_{i-1}, T^F_i)} \\ S &= \frac{P(t, T^V_0) - P(t, T^V_m)}{\sum_{i=1}^n P(t, T^F_i)\delta(T^F_{i-1}, T^F_i)} \end{aligned} \tag{2.30}$$

In general, we take $t = T^V_0 = 0$. This gives

$$S = \frac{1 - P(0, T^V_m)}{\sum_{i=1}^n P(0, T^F_i)\delta(T^F_{i-1}, T^F_i)} \tag{2.31}$$

2.3.4 Basis swaps

A basis swap is a type of interest rate swap where two parties exchange floating interest rate payments based on different reference rates with different payment frequencies, which we note S_j and T_i. In financial markets, basis swaps are quoted based on the difference between the par rates of these legs. For instance, the quotation for a basis swap between a 3-month (3M) and a 6-month (6M) leg is given by:

$$\Delta_{3M,6M}(t, T_i) = K_{6M}(t, T_i) - K_{3M}(t, T_i) \tag{2.32}$$

where $K_{6M}(t, T_i)$ and $K_{3M}(t, T_i)$ represent the par rates for swaps linked to the 6-month and 3-month Euribor rates, respectively.
The valuation of a basis swap follows a similar approach to that of a vanilla swap, with the key difference being that the fixed leg is replaced by a second floating leg with a different frequency:

$$\begin{aligned} V(t) = N &\sum_{j=1}^m P(t, S_j) \left[L_x(t, S_{j-1}, S_j) + \Delta\right] \tau(S_{j-1}, S_j) \\ - N &\sum_{i=1}^n P(t, T_i) L_{x'}(t, T_{i-1}, T_i) \tau(T_{i-1}, T_i) \end{aligned} \tag{2.33}$$

Here, $L_x(t, S_{j-1}, S_j)$ represents the floating rate for the shorter-frequency leg, while $L_{x'}(t, T_{i-1}, T_i)$ represents the floating rate for the longer-frequency leg. A spread Δ is added to the shorter-tenor leg to reflect the difference in frequency.

2.4 Interpolation Methods

For ease of understanding and simplicity, we will note from here, $y(t) := y(0, t)$, $f(t) := f(0, t)$ and $P(t) := P(0, t)$. Moreover for known values at time t_i, we can write $y(t_i) = y_i$,

$f(t_i) = f_i$ and $P(t_i) = P_i$. Looking back to (2.14), we can get for $t = 0$

$$f(0; t_1, t_2) = \frac{y_2 t_2 - y_1 t_1}{t_2 - t_1}. \tag{2.34}$$

And then we integrate (2.16) to get

$$\frac{y_i t_i - y_{i-1} t_{i-1}}{t_i - t_{i-1}} = \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} f(s) ds \tag{2.35}$$

and

$$y(t)t = y_{i-1} t_{i-1} + \int_{t_{i-1}}^{t_i} f(s) ds \tag{2.36}$$

This expression is a crucial interpolation formula as it gives the risk free function when having the forward function.

2.4.1 Criteria for Evaluating Interpolation Methods

When assessing the quality of a curve construction and its associated interpolation technique, several key criteria should be considered to ensure robustness and effectiveness. These criteria include:

- **Positivity and Continuity of Forward Rates:** Forward rates derived from the curve must be positive to prevent arbitrage opportunities. Additionally, these rates should be continuous to facilitate the accurate pricing of financial instruments that are sensitive to rate changes. Smoothness of the forward rates is also desirable, as it reduces abrupt changes in the curve that can lead to pricing anomalies and challenges in risk management.
- **Localness of the Interpolation Method:** The interpolation method used should exhibit local sensitivity, meaning that minor changes in the input data should not lead to significant or unintended variations in the curve. This ensures that the curve reacts in a predictable manner to adjustments in the data, thereby maintaining stability and reliability in practical applications.
- **Stability of Forward Rates:** It is crucial to evaluate the stability of the forward rates by analyzing how they respond to changes in the inputs. Specifically, one should assess the maximum change in the forward curve resulting from a small adjustment in the input parameters, measured in basis points. This criterion helps ensure that the curve remains stable and reliable under varying market conditions and input changes.
- **Localness of Hedging Instruments:** The assignment of delta risk in hedging strategies should be evaluated to determine whether the hedging instruments correspond to maturities that are close to the tenor of the position being hedged. This involves checking whether the hedges are applied in the vicinity of the relevant maturity dates or across different regions of the curve. Proper alignment ensures that the hedges are effective in mitigating risk without introducing significant basis risk.

By carefully considering these criteria, one can ensure that the curve construction and interpolation methods used are both practical and robust, providing a reliable foundation for pricing and risk management in financial applications.

2.4.2 Linear on Rates

Linear interpolation is one of the simplest methods to estimate the value of a function between two known points. For the interval $t_{i-1} < t < t_i$, the interpolation formula for the rate $y(t)$ is given by:

$$y(t) = \frac{t - t_{i-1}}{t_i - t_{i-1}} y_i + \frac{t_i - t}{t_i - t_{i-1}} y_{i-1} \tag{2.37}$$

Using equation (8), the forward rate $f(t)$ is then expressed as:

$$f(t) = \frac{2t - t_{i-1}}{t_i - t_{i-1}} y_i + \frac{t_i - 2t}{t_i - t_{i-1}} y_{i-1} \tag{2.38}$$

It is important to note that $f(t)$ is undefined at the points t_i because the function $y(t)t$ is not differentiable at these points. In the interpolation formula for rates, as time t approaches t_i, the contribution of y_{i-1} diminishes to zero—essentially, that rate has been "forgotten". However, this is not the case for the forward rate $f(t)$, leading to different left-hand and right-hand limits $f(t_i^+)$ and $f(t_i^-)$, which causes the forward rate to exhibit jumps at the interpolation nodes.

2.4.3 Linear on the Logarithm of Rates

Here, for values of t between t_{i-1} and t_i, the interpolation can be performed on the logarithm of the rates, which is given by:

$$\ln(y(t)) = \frac{t - t_{i-1}}{t_i - t_{i-1}} \ln(y_i) + \frac{t_i - t}{t_i - t_{i-1}} \ln(y_{i-1}) \tag{2.39}$$

Converting this back into a rate, the formula becomes:

$$y(t) = y_i^{\frac{t - t_{i-1}}{t_i - t_{i-1}}} \cdot y_{i-1}^{\frac{t_i - t}{t_i - t_{i-1}}} \tag{2.40}$$

While this method can be useful in certain situations, a significant drawback is that it doesn't accommodate negative interest rates, making it less flexible for some financial models. Additionally, similar to the linear on rates method, this approach results in forward rates that exhibit jumps at each node. This discontinuity can lead to undesirable behavior in the yield curve, such as a failure to consistently produce a decreasing Z function, which represents the present value of future cash flows.

2.4.4 Linear on Discount Factors

For values of t between t_{i-1} and t_i, the interpolation is performed on the discount factors, and the formula is given by:

$$P(t) = \frac{t - t_{i-1}}{t_i - t_{i-1}} P_i + \frac{t_i - t}{t_i - t_{i-1}} P_{i-1} \tag{2.41}$$

The corresponding interest rate can then be derived from this formula as:

$$y(t) = \frac{-1}{t} \ln \left[\frac{t - t_{i-1}}{t_i - t_{i-1}} e^{-y_i t_i} + \frac{t_i - t}{t_i - t_{i-1}} e^{-y_{i-1} t_{i-1}} \right] \tag{2.42}$$

This method ensures that the discount factors are interpolated linearly between known maturities. However, similar to other linear interpolation methods, it may result in a non-smooth forward curve with potential discontinuities, which could lead to unrealistic jumps in forward rates, particularly in scenarios with long-dated instruments.

2.4.5 Raw interpolation: linear on the log of discount factors

This is a straightforward method for constructing yield curves, corresponding to piecewise constant forward curves. This method is notably stable, easy to implement, and often serves as the initial approach when developing yield curve models. It is also useful for identifying errors in more advanced methods by comparing their results with those from the raw interpolation method.

By definition, raw interpolation involves constant instantaneous forward rates within each interval $t_{i-1} < t < t_i$. According to equation (2.35), this constant is the discrete forward rate for that interval, so the forward rate $f(t)$ can be expressed as:

$$f(t) = \frac{y_i t_i - y_{i-1} t_{i-1}}{t_i - t_{i-1}} \tag{2.43}$$

for $t_{i-1} < t < t_i$. Using equation (2.36), the spot rate $y(t)$ at any time t within the interval can be calculated as:

$$y(t) \cdot t = y_{i-1} t_{i-1} + \frac{(t - t_{i-1})(y_i t_i - y_{i-1} t_{i-1})}{t_i - t_{i-1}} \tag{2.44}$$

Rewriting the above expression with a common denominator $t_i - t_{i-1}$ and simplifying, the interpolation formula for the interval becomes:

$$y(t) \cdot t = \frac{t - t_{i-1}}{t_i - t_{i-1}} y_i t_i + \frac{t_i - t}{t_i - t_{i-1}} y_{i-1} t_{i-1} \tag{2.45}$$

This method is sometimes referred to as 'linear $y \cdot t$', as it represents linear interpolation on the points $y_i t_i$. Since $\pm y_i t_i$ is the logarithm of the capitalization/discount factors, the method is also aptly described as 'linear on the log of capitalization factors' or 'linear on the log of discount factors'.

One of the key advantages of the raw interpolation method is that it inherently ensures positive instantaneous forward rates since each is equal to the discrete forward rate for the respective interval. However, it should be noted that the instantaneous forward rate is undefined at the points $t_1, t_2, \ldots, t_n$, and the function exhibits a jump at these points.

2.4.6 Quadratic splines

To construct a quadratic spline of a function x, we seek coefficients (a_i, b_i, c_i) for $1 \leq i \leq n - 1$. Given these coefficients, the function value at any point t is given by:

$$x(t) = a_i + b_i(t - t_i) + c_i(t - t_i)^2 \quad \text{for } t_i \leq t \leq t_{i+1} \tag{2.46}$$

The constraints are that the interpolating function must meet the given data (ensuring continuity) and the entire function must be differentiable. Thus, there are $3n-4$ constraints: $n-1$ left-hand function values to satisfy, $n-1$ right-hand function values to satisfy, and $n-2$ internal knots where differentiability must be ensured. However, there are $3n-3$ unknowns. With one degree of freedom remaining, it is logical to require that the left-hand derivative at t_n be zero, allowing the curve to be extrapolated with a horizontal asymptote.

Suppose we apply this method to the rates (so $x_i = y_i$). The forward curves produced are very similar to the piecewise linear forward curves—the curve can exhibit a 'zig-zag' appearance, and this zig-zag is influenced by the same parity of input considerations as before.

2.4.7 Cubic splines

This time we seek coefficients (a_i, b_i, c_i, d_i) for $1 \leq i \leq n-1$. Given these coefficients, the function value at any point t is given by:

$$x(t) = a_i + b_i(t-t_i) + c_i(t-t_i)^2 + d_i(t-t_i)^3 \quad \text{for } t_i \leq t \leq t_{i+1} \tag{2.47}$$

As before, we have $3n-4$ constraints, but this time there are $4n-4$ unknown coefficients. There are several possible ways to find the additional n constraints. Here are some methods:

- $x_i = y_i$. The function is required to be twice differentiable, which, as previously, adds another $n-2$ constraints. For the final two constraints, the function is required to be linear at the extremes (i.e., the second derivative of the interpolator at t_1 and t_n are zero). This is known as the natural cubic spline.
- $x_i = y_i$. The function is again required to be twice differentiable; for the final two constraints, the function is linear on the left and horizontal on the right. This is the so-called financial cubic spline by Adams [1].
- $x_i = y_i t_i$. The function is required to be twice differentiable; for the final two constraints, the function is linear on the right and quadratic on the left. This is the quadratic-natural spline proposed by McCulloch and Kochin [8].
- $x_i = y_i$. The values of b_i for $1 < i < n$ are chosen to be the slope at t_i of the quadratic that passes through (t_j, y_j) for $j = i-1, i, i+1$. The value of b_1 is chosen to be the slope at t_1 of the quadratic that passes through (t_j, y_j) for $j = 1, 2, 3$; the value of b_n is chosen likewise. This is the Bessel method by de Boor [3].
- $x_i = y_i t_i$. Again, Bessel interpolation.
- $x_i = y_i$. The monotone preserving cubic spline of Hyman [7]. The method specifies the values of b_i for $1 \leq i \leq n$, in a way that will be discussed in more detail shortly.

Significant problems can arise when using some of these methods. The spline is supposed to mitigate the problem of oscillation seen when fitting a single polynomial to a data set (as in the Lagrange polynomial); nevertheless, significant oscillatory behavior can still occur. Furthermore, the various types of clamping seen in some of the methods above (clamping refers to imposing conditions at the boundaries τ_1 or τ_n) can compromise the localness of the interpolator, sometimes severely.

2.5 Bootstrapping Idea and Challenges

2.5.1 Concept and Methodology

Bootstrapping is a sequential method used to construct a yield curve by deriving zero-coupon bond prices, or discount factors, from observable market data, typically from financial instruments like deposits, futures, and swaps. The essence of bootstrapping lies in the recursive nature of the process: to determine the price of a zero-coupon bond maturing at a specific time, we need to know the prices of all previously-maturing zero-coupon bonds. Each successive bond price is computed using the previously calculated values, ensuring consistency with market prices and financial conventions.

2.5.1.1 Using OIS Swaps Rates

We refer back to Section 2.3.3, and we rearrange the fixed rate expression. For simplicity, we take Equation 2.31 and we suppose that $m = n$ and $T_i^F = T_i^V, \forall i = 1 \dots n$, meaning that both fixed and floating payments happen on the same dates. We get

$$P(0, T_n) = \frac{P(0, T_0) - S \sum_{i=1}^{n-1} \delta(T_{i-1}, T_i) P(0, T_i)}{1 + \delta(T_{n-1}, T_n) S} \tag{2.48}$$

In fact, that fixed rate S is for the period $[T_0, T_n]$, so it is better to denote it $S(0, T_0, T_n)$. The idea here is to extract these rate for different maturities from the market and then for each $i = 1 \dots n$, we can compute the ZC bond's price for maturity T_i using

$$P(0, T_i) = \frac{P(0, T_0) - S(0, T_0, T_i) \sum_{j=1}^{i-1} \delta(T_{j-1}, T_j) P(0, T_j)}{1 + \delta(T_{i-1}, T_i) S(0, T_0, T_i)} \tag{2.49}$$

We can see that $P(0, T_i)$ depends only on the previous prices $(P(0, T_j))_{1 \leq j \leq i-1}$, which justifies the iterative method.

When working with this approach, we propose two ways to compute the prices:

2.5.1.1.1 Direct Solving Here we use Equation 2.49 as explained above. This works fine when we have all needed OIS rates $(S(0, T_0, T_i))_{1 \leq i \leq n}$. However, when one or more rate is missing, we can either interpolate that rate using the nearest present rates to have a complete set, or we employ the second way of computing the prices.

2.5.1.1.2 Root-Finding Solver Here, instead of determining the expression of the ZC bond's price from Equation 2.24, we compute the root of the swap function for a specific maturity

$$V_{\text{swap}}\left(P(0, T_i)\right) = 0 \tag{2.50}$$

Having the prices for previous maturities and the OIS rate for T_i, the only unknown is the ZC price. This is a more general approach and can deal with the problem stated before. In fact, if one OIS rate is missing, say $S(0, T_0, T_j)$, we proceed by not computing $P(0, T_j)$ and instead, we express it as a function, simply an interpolation, of $P(0, T_{j-1})$ and $P(0, T_{j+1})$ and perform the root-finding algorithm on the latter. Finally, we can go back and compute $P(0, T_j)$.

2.5.1.2 Using Futures Quotes

Going back to Section 2.3.2, we can easily see that the idea here is to extract futures' quotes from the market for different periods and compute the corresponding Forward rates. Then for each $i = 1 \ldots n$, we derive the ZC price $P(0, T_i)$ that satisfies Equation 2.14, i.e. we find the root of the following

$$F_X(t, T_{i-1}, T_i) - \frac{1}{\delta(T_{i-1}, T_i)} \left[\frac{P(t, T_{i-1})}{P(t, T_i)} - 1 \right] = 0 \tag{2.51}$$

Similarly to when using swaps, this also a general approach and can manage problems of missing data. However, the chance that futures quotes for an intermediate period will be missing is relatively lower than for the case of OIS swap rates.

2.5.1.3 Using both instruments

When constructing the yield curve or other related curves across a wide time horizon, such as from one day to 40 years, a significant challenge arises in ensuring adequate liquidity throughout the entire interval. Relying on a single type of financial instrument often proves insufficient due to limited market depth or availability. For instance, Futures and FRA quotes are typically liquid only up to two years, beyond which their availability diminishes.

To address this issue, a more robust approach involves dividing the full time horizon into segments, each corresponding to different maturity ranges. For each segment, the most appropriate and liquid financial instruments are selected. This could involve using short-term instruments such as FRAs and Futures for the shorter end of the curve, and transitioning to interest rate swaps for intermediate and longer-term exposures.

By adopting a segmented approach and tailoring the selection of instruments to the unique liquidity profiles and risk characteristics of each maturity range, it becomes possible to construct a more accurate and reliable yield curve that reflects market conditions across the full spectrum of time.

2.5.2 Calendar Conventions

We observe in the expressions presented in previous sections that one term is always present, which is the year fraction between two dates T_1 and T_2, denoted by $\delta(T_1, T_2)$. In practice, this term can vary depending on the approach and the convention chosen by the practitioner.

Different calendar conventions affect how the year fraction is calculated. For instance, the Actual/Actual convention uses the actual number of days in the period and the actual number of days in the year, while the 30/360 convention assumes each month has 30 days and each year has 360 days, simplifying calculations. Business day conventions determine how to adjust dates that fall on weekends or holidays, such as moving to the next business day or the previous one.

2.5.3 Handling Market Data

The challenge with the market data is that sometimes it can be sparse or unevenly distributed across maturities. In that case, practitioners often use interpolation techniques to estimate rates for intermediate maturities. Similarly, extrapolation is used to extend curves beyond the available data range. However, when there is significant sparseness in the data, especially between distant maturities, these methods can become inefficient or lead to less reliable results. For instance, if there are only a few data points available for very distant maturities, interpolation methods may struggle to accurately capture the curve's behavior in these regions. This can result in a curve that does not reflect true market conditions and may be overly influenced by the few available data points.

Chapter 3

Convexity Adjustment

In this chapter, we address convexity adjustment for interest rate derivatives. Section 3.1 focuses on the differences between futures and forward rate agreements (FRAs), deriving the necessary convexity adjustment using the Ho-Lee and Hull-White models. Section 3.2 applies the martingale approach to derive adjustments based on forward and futures rate dynamics. Section 3.3 introduces a modified martingale approach using short rate models, comparing its results with traditional methods. Finally, we discuss the calibration of Hull-White model in Section 3.5, as it represents an important task in the project.

The first two approaches, represent state-of-the-art methodologies in the field of convexity adjustment. However, the approach introduced in Section 3.3 is a novel contribution from our team, aimed at enhancing the accuracy and applicability of the second approach. This modified martingale method leverages short rate models in a more refined manner. All models and computations presented in this section are original work, developed and implemented by our team as part of this project.

3.1 Basic Approach by Hull

This is the most popular approach used by practitioners, for which they use just the final result found for a specific model. To understand it, we focus on the Eurodollar futures contract, and we compare it to the FRA. In fact, they are assumed to be the same for short maturities. However, differences become important for longer-dated contracts. In fact, let's take a Eurodollar futures contract on an interest rate for the period between times T_1 and T_2 and with an FRA for the same period. The Eurodollar futures contract is settled daily. The final settlement is at time T_1 and reflects the realized interest rate for the period between times T_1 and T_2. By contrast, the FRA is not settled daily and the final settlement reflecting the realized interest rate between times T_1 and T_2 is made at time T_2.
The convexity adjustment is introduced as follows:

$$\text{Forward rate} = \text{Futures rate} - \text{Convexity adjustment}$$

To get the expression of that term, we need to stochastically model the short rate, denoted by r_t. We consider two popular models: Ho-Lee model and Hull-White model, from which we can derive a particular case, Vasicek model.

3.1.1 Ho-Lee Model

In the traditional risk-neutral world, the risk-neutral process for the short rate here follows the dynamics

$$dr_t = \theta(t)dt + \sigma dW_t \tag{3.1}$$

where r_t is the instantaneous short rate, θ is a function of time, σ is a constant, and dW_t is a Brownian motion. We define $P(t,T)$ as the price at time t of the Zero-Coupon bond with maturity T. Since the model is affine, we know then that the price has the form:

$$P(t,T) = e^{A(t,T)-r_t B(t,T)} \tag{3.2}$$

where A and B solve the following system

$$\begin{aligned} &\partial_t B(t,T) = -1, \quad B(T,T) = 0 \\ &\partial_t A(t,T) = \theta(t)B(t,T) - \frac{1}{2}\sigma^2 B(t,T)^2, \quad A(T,T) = 0 \end{aligned} \tag{3.3}$$

This give us $B(t,T) = T - t$. Now, if we apply Itô's formula to the bond's price, $P(t,T) = F^T(t, r_t)$, i.e as a function of t and r_t, parameterized by T, we get

$$\begin{aligned} dP(t,T) &= (\partial_t A(t,T) + r_t)P(t,T)dt - (T-t)P(t,T)dr_t + \frac{1}{2}(T-t)^2 P(t,T)d < r, r >_t \\ &= \left(\theta(t)(T-t) - \frac{1}{2}\sigma^2(T-t)^2 + r_t - (T-t)\theta(t) + \frac{1}{2}\sigma^2(T-t)^2\right) P(t,T)dt \\ &- (T-t)\sigma P(t,T)dW_t \\ &= r_t P(t,T)dt - (T-t)\sigma P(t,T)dW_t \end{aligned} \tag{3.4}$$

Then, we define $f(t;T_1,T_2)$ to be the forward rate at time t for the period between T_1 and T_2, given by

$$f(t;T_1,T_2) = \frac{\ln[P(t,T_1)] - \ln[P(t,T_2)]}{T_2 - T_1}$$

Note that here, we choose the continuous compounded rate, presented in Section 2.3.3. Applying Itô's lemma for $\ln(P(t,T))$ gives

$$\begin{aligned} d\ln(P(t,T)) &= \frac{1}{P(t,T)}dP(t,T) - \frac{1}{2P(t,T)^2}d\langle P(\cdot,T), P(\cdot,T)\rangle_t \\ &= r_t dt - (T-t)\sigma dW_t - \frac{1}{2}(T-t)^2\sigma^2 dt \end{aligned}$$

Therefore, the dynamics of $f(t,T_1,T_2)$ can be written

$$\begin{aligned} df(t;T_1,T_2) &= \frac{d\ln[P(t,T_1)] - d\ln[P(t,T_2)]}{T_2 - T_1} \\ &= \frac{r_t dt - (T_1-t)\sigma dW_t - \frac{1}{2}(T_1-t)^2\sigma^2 dt - r_t dt + (T_2-t)\sigma dW_t + \frac{1}{2}(T_2-t)^2\sigma^2 dt}{T_2 - T_1} \\ &= \frac{(T_2-t)^2 - (T_1-t)^2}{2(T_2-T_1)}\sigma^2 dt + \sigma dW_t \end{aligned}$$

We can see that we have

$$\begin{aligned}\mathbb{E}\left[\int_0^{T_1} df(t,T_1,T_2)\right] &= \frac{\sigma^2}{2(T_2-T_1)}\int_0^{T_1}(T_2-t)^2-(T_1-t)^2dt \\ &= \frac{\sigma^2}{2(T_2-T_1)}\frac{T_2^3-(T_2-T_1)^3-T_1^3}{3} \\ &= \frac{1}{2}\sigma^2 T_1T_2\end{aligned}$$

In addition, we know that

- The forward rate equals the spot rate at time T_1, thus their expected values also at time T_1
- The expected value of the spot rate is the same as the futures rate in the traditional risk-neutral world

This means that we can write

$$\begin{aligned}\text{Convexity Adjustment} &= \text{Futures rate} - \text{Forward rate} \\ &= \mathbb{E}_{t=0}[f(T_1,T_1,T_2)] - \mathbb{E}_{t=0}[f(0,T_1,T_2)] \\ &= \mathbb{E}\left[\int_0^{T_1} df(t,T_1,T_2)\right]\end{aligned}$$

$$\text{Futures rate} = \mathbb{E}_{t=0}[f(T_1,T_1,T_2)] \quad \text{and} \quad \text{Forward rate} = \mathbb{E}_{t=0}[f(0,T_1,T_2)]$$

We finally get

$$\text{Convexity Adjustment} = \frac{1}{2}\sigma^2 T_1T_2 \tag{3.5}$$

This expression is highly preferred by practitioners for its straightforwardness and effectiveness in navigating real market conditions. Its simplicity combined with practical utility makes it an enduring choice in the field.

3.1.2 Hull-White

The Hull-White model assumes that the short rate, $r(t)$, follows a mean-reverting stochastic process, which can be written as:

$$dr_t = (\theta(t) - ar_t)\,dt + \sigma dW_t \tag{3.6}$$

where

- $\theta(t)$ is a time-dependent function chosen to fit the current term structure of interest rates. In fact, its expression is

$$\theta(t) = \frac{\delta f(0,t)}{\delta t} + af(0,t) + \frac{\sigma^2}{2a}\left(1-e^{-2at}\right) \tag{3.7}$$

- a is the mean-reversion rate, which controls how fast the short rate reverts to the long-term mean.

- σ is the volatility parameter, determining the randomness of the short rate.
- $dW(t)$ represents a Wiener process (Brownian motion).

This is also an affine model, but here A and B solve the following system

$$\begin{aligned} &\partial_t B(t,T) - aB(t,T) = -1, \quad B(T,T) = 0 \\ &\partial_t A(t,T) = \theta(t)B(t,T) - \frac{1}{2}\sigma^2 B(t,T)^2, \quad A(T,T) = 0 \end{aligned} \tag{3.8}$$

which gives us

$$B(t,T) = \frac{1-e^{-a(T-t)}}{a} \tag{3.9}$$

We now proceed like above, by applying Itô's formula to the bond's price

$$\begin{aligned} dP(t,T) &= (\partial_t A(t,T) - r_t \partial_t B(t,T))P(t,T)dt - B(t,T)P(t,T)dr_t + \frac{1}{2}B(t,T)^2 P(t,T)d\langle r,r\rangle_t \\ &= \left(\theta(t)B(t,T) - \frac{1}{2}\sigma^2 B(t,T)^2 + e^{-a(T-t)}r_t + B(t,T)\theta(t) - aB(t,T)r_t + \frac{1}{2}\sigma^2 B(t,T)^2\right) \\ &\quad P(t,T)dt - B(t,T)\sigma P(t,T)dW_t \\ &= \left(e^{-a(T-t)} + aB(t,T)\right) r_t P(t,T)dt - B(t,T)\sigma P(t,T)dW_t \\ &= r_t P(t,T)dt - B(t,T)\sigma P(t,T)dW_t \end{aligned} \tag{3.10}$$

Then, we proceed with Itô's formula for $\ln(P(t,T))$

$$\begin{aligned} d\ln(P(t,T)) &= \frac{1}{P(t,T)}dP(t,T) - \frac{1}{2P(t,T)^2}d\langle P(\cdot,T),P(\cdot,T)\rangle_t \\ &= r_t dt - B(t,T)\sigma dW_t - \frac{1}{2}B(t,T)^2\sigma^2 dt \end{aligned}$$

which gives

$$\begin{aligned} df(t,T_1,T_2) &= \frac{d\ln[P(t,T_1)] - d\ln[P(t,T_2)]}{T_2 - T_1} \\ &= \frac{B(t,T_2)^2 - B(t,T_1)^2}{2(T_2-T_1)}\sigma^2 dt + \frac{B(t,T_2) - B(t,T_1)}{T_2 - T_1}\sigma dW_t \\ &= \frac{e^{-2a(T_2-t)} - e^{-2a(T_1-t)} + 2e^{-a(T_1-t)} - 2e^{-a(T_2-t)}}{2a^2(T_2-T_1)}\sigma^2 dt + \frac{e^{-a(T_1-t)} - e^{-a(T_2-t)}}{a(T_2-T_1)}\sigma dW_t \end{aligned}$$

As explained above, to get the convexity adjustment, we need to compute the expected change of the forward rate between time 0 and time T_1

$$\begin{aligned} \text{Convexity Adjustment} &= \mathbb{E}\left[\int_0^{T_1} df(t,T_1,T_2)\right] \\ &= \frac{\sigma^2}{2a^2(T_2-T_1)}\int_0^{T_1}\left(e^{-2a(T_2-t)} - e^{-2a(T_1-t)} + 2e^{-a(T_1-t)} - 2e^{-a(T_2-t)}\right)dt \\ &= \frac{\sigma^2}{4a^3(T_2-T_1)}\Big(e^{-2a(T_2-T_1)} - e^{-2aT_2} - 1 + e^{-2aT_1} + 4 - 4e^{-aT_1} \\ &\quad - 4e^{-a(T_2-T_1)} + 4e^{-aT_2}\Big) \\ &= \frac{B(T_1,T_2)}{T_2-T_1}\left[B(T_1,T_2)(1-e^{-2aT_1}) + 2aB(0,T_1)^2\right]\frac{\sigma^2}{4a} \end{aligned}$$

In contrast to Ho-Lee model, this model results in a more complex expression and incorporates an additional parameter, a. Notably, as a approaches zero, the expression simplifies to the first one. Thus, for very small values of a, using the initial model remains a practical and effective approach. In fact, a Taylor series expansion of $a \mapsto e^{-ta}$ shows that

$$\lim_{a\to 0} \frac{1-e^{-ta}}{a} = t \tag{3.11}$$

and therefore

$$\lim_{a\to 0} B(T_1,T_2) = T_2 - T_1 \tag{3.12}$$

which gives

$$\begin{aligned} \lim_{a\to 0} \text{Convexity Adjustment} &= \sigma^2 \left((T_2 - T_1)\frac{2T_1}{4} + \frac{1}{2}T_1^2 \right) \\ &= \frac{1}{2}\sigma^2 T_1 T_2 \end{aligned} \tag{3.13}$$

3.2 Martingale Approach

Although the objective is the same as the approach described before, here we add few assumptions and we use some knowledge about the forward contract. For simplicity, we note $L_t = f(t;T_1,T_2)$, $V_t = P(t,T_2)$, $\alpha = \delta(T_1,T_2)$. In addition, we define the futures rate at time t for the period between T_1 and T_2 by F_t and the value at time t of the forward contract struck at a rate K by Π_t.

Both rates L_t and F_t converge at time T_1 to the then prevailing money market rate for the period between T_1 and T_2, so we have $L_{T_1} = F_{T_1}$, as well as

$$\Pi_{T_1} = \alpha V_{T_1}(L_{T_1} - K) = \alpha V_{T_1}(F_{T_1} - K)$$

As for time $t = 0$, we have

$$\Pi_0 = \alpha V_0 (L_0 - K)$$

and the goal is to show that we can write

$$\Pi_0 = g(V_0, F_0)$$

for some appropriate function g, so that we get the relationship between F_0 and L_0.
To get that, the approach based on creating a portfolio, with initial amount of cash $v_0 = g(V_0,F_0)$, a continuous trading strategy $\theta = (\theta_t)$ in the futures contract, and reinvesting the cash in the discount bond V_t. The process $\pi = (\pi_t)$ describing the portfolio's value, satisfy the following SDE:

$$d\pi_t = \theta_t dF_t + \frac{\pi_t}{V_t} dV_t, \quad \pi_0 = v_0$$

We define the process $C = (C_t)$ by

$$C_t := \exp\left(\int_0^t \frac{1}{F_s V_s} d\langle F, V\rangle_s \right)$$

Then, we can prove that

$$g(V_0, F_0) \coloneqq v_0 = \alpha V_0(C_{T_1} F_0 - K) \tag{3.14}$$

and we conclude that

$$L_0 = C_{T_1} F_0$$

The proof of 3.14 is really extensive, and can be found in [10]. Therefore, we move to the next step, which is writing an explicit formulation of C_T, starting with some chosen diffusions for F and V. We fix the dynamics of F to be

$$dF_t = \mu(t)F_t dt + \sigma_F(t)F_t dW_t^1 \tag{3.15}$$

where W^1 is a Brownian motion under the historical probability. As for V, we choose different dynamics for the continuous yield $R_t = y(t, T_2)$ and then use the definition of V given by

$$V_t \coloneqq \exp\left(-(T_2 - t)R_t\right) \tag{3.16}$$

3.2.1 Ho-Lee Model

The continuously compounded spot rate R_t follows the SDE

$$dR_t = \theta(t)dt + \sigma_R dW_t^2$$

and we assume that W^1 and W^2 have deterministic correlation $\rho(t)$. We can then write

$$dW_t^2 = \rho(t)dW_t^1 + \sqrt{1-\rho^2(t)}dZ_t$$

where Z is a Brownian motion, independent of W^1. Applying Itô's formula to 3.16 gives

$$\begin{aligned} dV_t &= R_t V_t dt - (T_2 - t)V_t dR_t + \frac{1}{2}(T_2 - t)^2 V_t \langle R \rangle_t \\ &= R_t V_t dt - (T_2 - t)\theta(t)V_t dt - (T_2 - t)\sigma_R V_t dW_t^2 + \frac{1}{2}(T_2 - t)^2 \sigma_R^2 V_t dt \\ &= \left(R_t - (T_2 - t)\theta(t) + \frac{1}{2}(T_2 - t)^2\sigma_R^2\right) V_t dt - (T_2 - t)\sigma_R V_t dW_t^2 \end{aligned} \tag{3.17}$$

Then we get

$$d\langle F, V\rangle_t = -(T_2 - t)\sigma_R \sigma_F(t)\rho(t)F_t V_t dt$$

Finally, we find the convexity adjustment

$$C_{T_1} = \exp\left(\sigma_R \int_0^{T_1} -(T_2 - t)\sigma_F(t)\rho(t)dt\right) \tag{3.18}$$

We can simply this more by taking $\sigma_F(t) = \sigma_F$ and $\rho(t) = \rho$ to be constant. We have then

$$C_{T_1} = \exp\left(-\frac{T_1(2T_2 - T_1)}{2}\sigma_R\sigma_F\rho\right) \tag{3.19}$$

We see that the expression depend only on the diffusion terms and the relationship between them. We can also judge that we have a rather simple and straightforward expression. However, in practice, this will more difficult than the first approach, since we would be required to calibrate 2 dynamics that are correlated.

3.2.2 Hull-White Model

The continuously compounded spot rate R_t follows the SDE

$$dR_t = (\theta(t) - aR_t)\, dt + \sigma_R dW_t^2$$

If we follow the same steps as above, we can see that we will have the same expression for the convexity adjustment as in 3.18, since the drift term does not influence the cross-variation process. In fact, unlike for the first approach where we use the affine terms to get the $P(t,T)$, here Itô's formula is straightforward and therefore the drift term will be eliminated.

Observing that, when working with continuously compounded rate R_t, only the diffusion term matters, we tried to modify the approach by setting the stochastic dynamics for the shirt rate r_t instead of R_t.

3.3 Modified Martingale Approach

The idea of this approach is based on the previous one with just one change on the variable on which we will focus.

3.3.1 Ho-Lee Model

We take r_t to solve

$$dr_t = \theta(t)dt + \sigma_R dW_t^2$$

and we use the result of 3.4 to get directly the dynamics of $V_t = P(t, T_2)$

$$dV_t = r_t V_t dt - (T_2 - t)\sigma_R V_t dW_t^2$$

We observe that we still have the same diffusion term for V_t as in the basic martingale approach, in equation 3.17, thus we will have the same expression for the convexity adjustment as in 3.18. So, taking the volatilities and the correlation to be constant again, we get

$$C_{T_1} = \exp\left(-\frac{T_1(2T_2 - T_1)}{2}\sigma_R\sigma_F\rho\right) \tag{3.20}$$

3.3.2 Hull-White Model

Here r solves

$$dr_t = (\theta(t) - ar_t))\, dt + \sigma_R dW_t^2$$

Similarly, the result of 3.10 to get

$$V_t = r_t V_t dt - B(t, T_2)\sigma_R V_t dW_t^2$$

where $B(t,T) = \dfrac{1 - e^{-a(T-t)}}{a}$. Finally, the convexity adjustment will be:

$$C_{T_1} = \exp\left(\sigma_R \int_0^{T_1} -B(t, T_2)\sigma_F(t)\rho(t)dt\right) \tag{3.21}$$

If we take $\sigma_F(t) = \sigma_F$ and $\rho(t) = \rho$ constants, we get

$$\begin{aligned} C_{T_1} &= \exp\left(\frac{\sigma_R\sigma_F\rho}{a}\int_0^{T_1}\left(e^{-a(T_2-t)}-1\right)dt\right) \\ &= \exp\left(\frac{\sigma_R\sigma_F\rho}{a^2}\left(e^{-a(T_2-T_1)}-e^{-aT_2}-aT_1\right)\right) \end{aligned} \tag{3.22}$$

We find then, that with the modified approach, the expression of the convexity adjustment change when we change the dynamics of the short rate.

3.4 Recap of found expressions

In order to present our findings and compare between them, we must first standardize the definition of the convexity adjustment. Therefore, we pick the first one, stating

$$\text{Convexity adjustment} = \text{Futures rate} - \text{Forward rate}$$

Then, for the second and third approaches, where we have

$$\text{Forward rate} = C_{T_1}\text{Futures rate} \tag{3.23}$$

and we get

$$\text{Convexity adjustment} = F_0\left(1 - C_{T_1}\right) \tag{3.24}$$

Now, we can showcase our results in Table 3.1, noting $B(t,T) = \dfrac{1-e^{-a(T-t)}}{a}$.

Approach	Model	Convexity Adjustment
Basic	HL	$\frac{1}{2}\sigma^2 T_1 T_2$
	HW	$\frac{B(T_1,T_2)}{T_2-T_1}\left[B(T_1,T_2)(1-e^{-aT_1})+2aB(0,T_1)^2\right]\frac{\sigma^2}{4a}$
Martingale	HL	$F_0\left(1-\exp\left(-\frac{T_1(2T_2-T_1)}{2}\sigma_R\sigma_F\rho\right)\right)$
	HW	$F_0\left(1-\exp\left(-\frac{T_1(2T_2-T_1)}{2}\sigma_R\sigma_F\rho\right)\right)$
Modified Martingale	HL	$F_0\left(1-\exp\left(-\frac{T_1(2T_2-T_1)}{2}\sigma_R\sigma_F\rho\right)\right)$
	HW	$F_0\left(1-\exp\left(\frac{\sigma_R\sigma_F\rho}{a^2}\left(e^{-a(T_2-T_1)}-e^{-aT_2}-aT_1\right)\right)\right)$

Table 3.1: Convexity Adjustment expressions

Moreover, we can see that there are some similarities. In fact, a Taylor series expansion of the exponential function shows that

$$\left(1-\exp\left(-\frac{T_1(2T_2-T_1)}{2}\sigma_R\sigma_F\rho\right)\right) \approx \frac{T_1(2T_2-T_1)}{2}\sigma_R\sigma_F\rho \tag{3.25}$$

Then, for $\sigma_R \approx \sigma_F$, $\rho \approx 1$ and $T_2 \approx T_1$, we get

$$\frac{T_1(2T_2 - T_1)}{2}\sigma_R\sigma_F\rho \approx \frac{1}{2}\sigma^2 T_1 T_2 \tag{3.26}$$

which is the first found expression of the convexity adjustment.

3.5 Model Calibration

For this task, we considered working with the Hull-White model as a start and then extend to other models. The aim is to approximate the parameters of the model, presented in 3.6. However, seeing the expressions determined for the convexity adjustment in the previous sections, we observe that the important parameters are a and σ, whereas $\theta(t)$ is not present and we might not focus on it in the calibration process.

The calibration of the Hull-White model involves estimating the parameters a (the speed of mean reversion) and σ (the volatility of the short rate process) so that the model accurately reflects observed market prices for interest rate derivatives. In this section, we detail the calibration procedure, including the optimization techniques used to determine the best-fit parameters.

3.5.1 Calibration Objective

The primary objective of calibration is to minimize the discrepancy between the model prices and the market prices of interest rate derivatives. For the Hull-White model, this discrepancy can be quantified using the following objective function:

$$J(a, \sigma) = \frac{1}{N}\sum_{i=1}^{N}\left(P_{m,i} - P_{m,i}^{\text{model}}(a, \sigma)\right)^2, \tag{3.27}$$

where $P_{m,i}$ represents the market price of the i-th derivative, and $P_{m,i}^{\text{model}}(a, \sigma)$ denotes the model price of the same derivative given the parameters a and σ. The objective is to find the parameter values a and σ that minimize this objective function, using market prices of various instruments such as caplets and swaptions.

3.5.2 Model Price Calculation

In the calibration of the Hull-White model, the model prices $P_{m,i}^{\text{model}}(a, \sigma)$ for various interest rate derivatives are crucial. Typically, caplets and swaptions are used in the calibration process as they are more sensitive to the parameters a and σ.

For example, consider pricing a European caplet under the Hull-White model. The price of a caplet can be derived using the closed-form solution available in the Hull-White framework, which accounts for the dynamics of the short rate process. The caplet price is given by:

$$C(t) = P(t, T)\left[\Phi(d_1) - K\Phi(d_2)\right],$$

where $P(t,T)$ is the bond price, K is the strike price, and Φ denotes the cumulative distribution function of the standard normal distribution. The terms d_1 and d_2 are defined as:

$$d_1 = \frac{\log\left(\frac{P(t,T)}{K}\right) + \frac{1}{2}\sigma^2}{\sigma}, \quad d_2 = d_1 - \sigma.$$

Here, σ depends on the parameters of the Hull-White model, particularly the volatility parameter, and incorporates the impact of the mean reversion rate a.

Similarly, for swaptions, the model price can be derived using the Jamshidian decomposition or other closed-form pricing formulas applicable in the Hull-White framework.

3.5.3 Optimization Technique

To estimate the parameters a and σ, we use an optimization technique such as the Nelder-Mead simplex algorithm or the Levenberg-Marquardt algorithm. These methods iteratively adjust the parameters to minimize the objective function $J(a,\sigma)$. The Nelder-Mead algorithm, for instance, is a direct search method that does not require derivatives and is suitable for optimizing non-differentiable objective functions.

The algorithm proceeds as follows:

- **Initialization**: Start with an initial guess for a and σ.
- **Iteration**: Evaluate the objective function $J(a,\sigma)$ at the current parameter values.
- **Update**: Adjust the parameters according to the optimization algorithm's rules to reduce $J(a,\sigma)$.
- **Convergence Check**: Repeat the iteration until the change in $J(a,\sigma)$ is below a specified tolerance level.

3.5.4 Implementation and Results

The calibration is performed using market data for interest rate derivatives. The choice of initial values for a and σ and the selection of the optimization algorithm can significantly impact the results. It is crucial to validate the calibrated parameters by comparing the model prices with out-of-sample market prices.

For example, if the calibration yields $a = 0.1$ and $\sigma = 0.015$, we check the model's pricing of caplets and swaptions to ensure they align closely with observed market prices. This validation process helps confirm the accuracy and reliability of the calibrated model parameters.

3.5.5 Conclusion

In summary, calibrating the Hull-White model involves estimating the parameters a and σ to match the model prices of interest rate derivatives with market prices. By minimizing the objective function and using appropriate optimization techniques, we obtain parameter values that reflect current market conditions. This calibration process provides a foundation for applying the Hull-White model to various financial applications and further extending to more complex interest rate models.

Chapter 4

Implementation in C#

4.1 Existing Code

The implementation builds upon and extends the quantitative library developed by Awalee. The library provides a robust foundation, offering essential mathematical and financial tools, which have been invaluable for this project. In particular, this library includes key classes and functions that address the following

- **Mathematical Classes**: The library features a range of mathematical tools, including classes for various distribution functions with capabilities for evaluating the functions and their first derivatives. Additionally, a Newton solver is available, which is critical for solving non-linear equations in curve construction and optimization problems.
- **Time and Date Management Classes**: Managing time and dates accurately is crucial in financial modeling, and the library provides comprehensive classes for this purpose. These include classes for defining calendars, handling different date conventions (such as 30/360 or actual/365 day counts), and working with periods of time (e.g., calculating time differences between dates, adding periods to dates).
- **Pricing Engines**: Awalee's library contains pricing engines that can calculate the price of derivatives using closed-form solutions, particularly for the Black-Scholes model.

While the library provided a strong foundation, it required significant enhancements to meet the full scope of this project. Several important features were missing, necessitating the development of additional code. These additions included new functions for more sophisticated financial calculations, expanded date management methods to handle complex calendar and day-count conventions, and other utilities that were essential for accurate yield curve construction.

4.2 Extension of the Library

In order to achieve our goal of constructing the yield curve, we focused on the the bootstrapping technique and the interpolation methods. However, several other classes and method were required to be implemented because of their importance and usefulness to the project, such as

- Deriving the IMM date for a certain month and year, which represents the maturity of the futures contract
- Mathematical functions: we started with basic functions such as inverse, exponential, and linear, then moved to more complex ones like piece-wise defined functions and composite functions. We also added new operator to be able to sum and multiply this new objects. Another important detail was to introduce for each function the first derivative, which will be useful for the Newton-Raphson method
- Mathematical tools such as points and intervals, useful when dealing with function defined on a specified interval
- Computing the duration of period depending on the start date and the calendar convention

4.3 Main Code for Bootstrapping and Interpolation

Building on the extensions to the library described in the previous section, our main focus shifted to implementing the core features required for yield curve construction. These implementations included critical components necessary for achieving accurate financial modeling. The details of our work are presented in the following points:

- Instruments classes and the extraction of the data from JSON files by parsing the contents of the file. For future quotes, the date needed to be manipulated to get the rates and the start and end dates of the contract (IMM dates)
- Implementation of the bootstrapping techniques explained in Section 2.5.1. These approaches can be manipulated by the user to account for several choices such as
  - The interpolation method to use
  - The set of instruments (it could be one set) to work with
  - Whether use direct solving or Newton-Raphson method to compute the discount factors/yield values
  - In the case of Newton-Raphson solver, the choice of the variable (Discount factor or yield), as well as the convectional parameters (target, first guess and tolerance)
  - Data: whether fill the missing values of the data, if it is the case, or not (after defining how the date frequency of the required data)
  - Date: The pricing date, the calendar convention, and the periodicity of the bootstrapped values (before interpolating), and thus the expected frequency of the data
  - The approach and model for computing the convexity adjustment
- Swap and Futures pricers, when given the yield curve, used to check the validity of our results

- Classes for the parameters of the project, which can be read also from a JSON file
- Convexity Adjustment implementation and the update of futures quotes to get the forward rates

The codebase for these tasks was developed from scratch, adhering to modern programming principles of OOP. It is publicly available on GitHub [9], providing full access to the implementation and ensuring transparency for further development and collaboration.

A key aspect of the implementation is the flexibility of the interpolation methods, which were designed to be both detailed and generic. This ensures that the framework can easily accommodate new interpolation techniques as they become necessary. The implementation allows for various types of financial instruments to be priced with precision, while maintaining a clear structure that supports future extensions.

Additionally, market data and important project parameters—such as the choice of interpolation methods, bootstrapping approaches, Newton solver parameters, and strategies for handling missing data—are all stored in easily accessible JSON files. This design choice allows for seamless integration and user-driven customization, empowering users to adjust configurations without needing to modify the core code. By externalizing these parameters, the system becomes more systematic and adheres to SOLID principles, particularly the Open/Closed Principle, where the core logic remains intact while being open to extensions.

The extraction and parsing of data from JSON files was also designed with future extensibility in mind. While the current implementation focuses on a specific set of instruments, the framework is flexible enough to accommodate additional financial instruments with minimal effort. By structuring the data extraction and parsing logic generically, the system can be extended to support new instruments by simply updating the data structure or implementing new instrument-specific logic. This makes the system highly adaptable and ready to handle an expanding range of financial products, further solidifying its role as a comprehensive tool for financial modeling.

One final point that we focused on in our code, is the unit tests. Starting with basic ones to ensure the validity of simple implementation such as the date manipulation functions, evaluating functions and their derivatives, moving to applying the Newton solver for some chosen functions. Then, the main tests focus on constructing the discount curve and using it to reprice the instrument used in that construction and verify that we have the same values. In fact, we implemented that test to try all the combinations of the parameters and choices that we described above.

Chapter 5

Results and Discussion

5.1 Market Data

In order to perform the bootstrapping technique, the first step is to extract data from the market. In our case, we decided to work with two instruments:

1. IRS indexed on Secured Overnight Financing Rate (SOFR) which is a reference rate for an OIS. Table 5.1 presents the found values as of the date 02/05/2024, extracted from the website of Chatham Financial, an independent financial risk management advisory and technology firm.

Maturity	**Swap rate**
1 Year	5.181%
2 Year	4.816%
3 Year	4.582%
5 Year	4.340%
7 Year	4.250%
10 Year	4.203%
15 Year	4.197%
30 Year	3.959%

Table 5.1: Swap rates data

We can observe that, although the maturity dates extend up to 30 years, there are noticeable gaps in the available rates. Specifically, several maturities are not represented in the dataset, which results in incomplete coverage across the full range of maturities.

The declining swap rates over time suggest that the market anticipates lower future interest rates. This trend indicates expectations of easing monetary policy or slower long-term economic growth. It could also reflect a decrease in risk premiums for long-term investments or changes in market liquidity and supply-demand dynamics.

Overall, these lower rates imply a market outlook of more accommodative monetary conditions and potentially weaker future economic activity.

2. ESTR Futures quotes, presented in Table 5.1, were extracted from the CME group website.

Start Month	Futures quote
June 2024	96.26%
September 2024	96.5625%
December 2024	96.81%
March 2025	97.025%
June 2025	97.195%
September 2025	97.3175%
December 2025	97.4%
March 2026	97.4975%
June 2026	4.250%

Table 5.2: ESTR Futures quotes data

For the futures quotes, in contrast, we observe a different pattern: the maturities extend only up to 2 years into the future. However, the dataset is well-populated within this range, with values available for each quarter from the present day up to the 2-year horizon. This comprehensive coverage ensures that all relevant time periods within this short-term window are represented.

5.2 Initial Results

Since our work takes into account several parameters that can affect the results, we start by observing the curves for a single set of parameters (using swap rates, interpolated data, linear interpolation on the yield, direct solving). As said before, the main curves to construct, due to their importance and usefulness, are the discount curve (Figure 5.1), the yield curve (Figure 5.2) and the forward curves for different tenors (Figure 5.3).

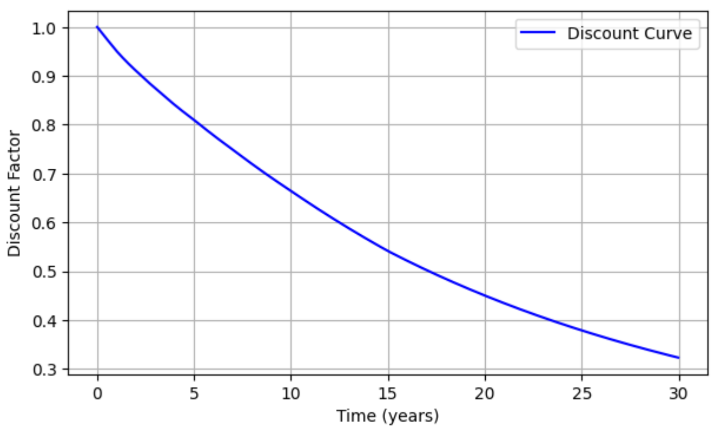

Figure 5.1: Discount Curve using Swaps

The discount curve starts at a factor of 1 for the present time (0 years), meaning that cash flows today are valued at face value. As we look further into the future, the discount factor decreases, indicating that cash flows in the future are worth less in today's terms.

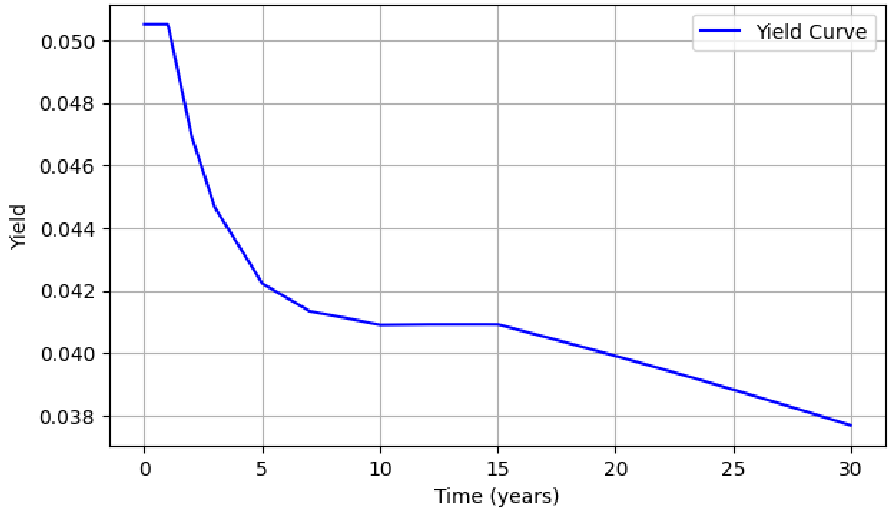

Figure 5.2: Yield Curve

This yield curve is inverted, meaning that short-term yields are higher than long-term yields. An inverted yield curve is often interpreted as a sign of market expectations of declining interest rates, which can indicate concerns about future economic slowdowns or recessions.

Additionally, the yield curve is piece-wise linear, meaning that it is constructed using segments of straight lines connecting different points on the curve. This indicates that yields are interpolated linearly between the observed data points, giving the curve its distinct shape with straight line segments.

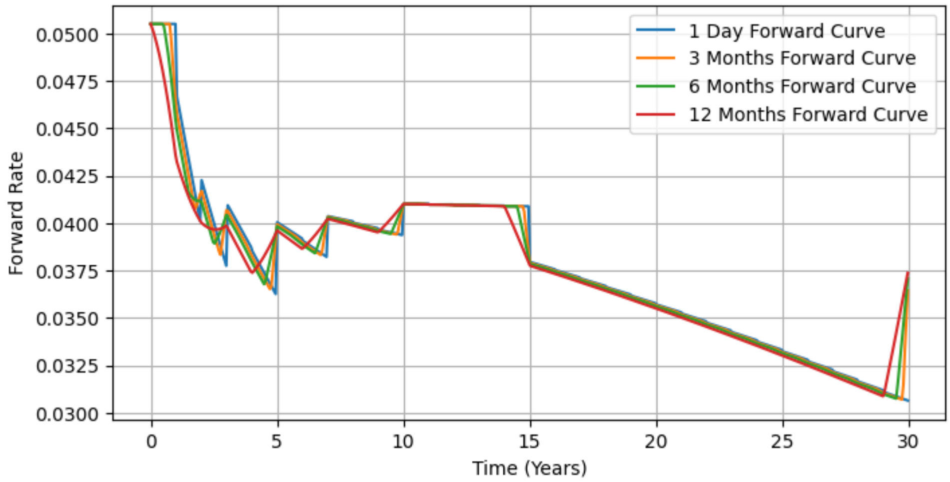

Figure 5.3: Forward Curves

Here, these curves presents some challenges as there are some big variations in the rates. In fact, looking back at Equation 2.14, we see that the for each value, we compute the difference of two values of the ZC price. Moreover, these curves are someway an approximation of the instantaneous forward rate defined in Equation 2.16. In this case, we take the yield curve to be piece-wise linear. So, if we have for an interval $[t_i, t_{i+1}]$

$$y(0,t) = at + b, \tag{5.1}$$

Then,

$$\begin{aligned} f(0,t) &= -\frac{\partial \ln P(0,t)}{\partial T} \\ &= -\frac{\partial \ln \left(e^{-y(0,t)t}\right)}{\partial T} \\ &= y(0,t) + \frac{\partial y(0,t)}{\partial t} t \\ &= 2at + b \end{aligned} \tag{5.2}$$

The instantaneous forward rate would be also piece-wise linear with slopes equal to the double of the slopes of the yield. This can be more observed if we plot the 1-Day forward curve, which represents an approximation of the instantaneous forward rate, and the yield curve together

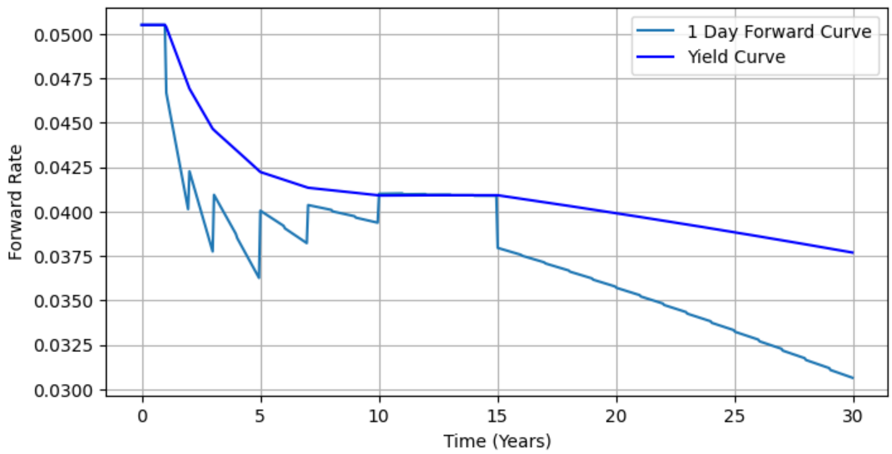

Figure 5.4: Yield Curve and 1D Forward Curve

Moving the unit tests, we use the computed rates to reprice the swap rates using Equation 2.31 and then we compare it to the market data in Figure 5.5. We find that the error is of order 10^{-11}.

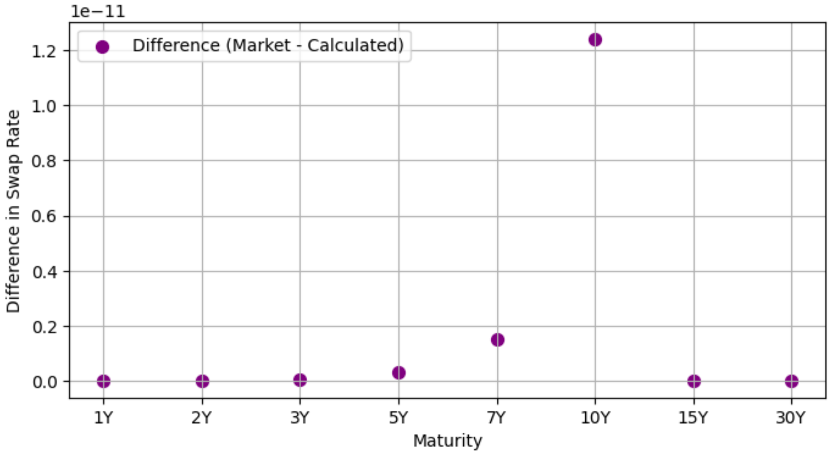

Figure 5.5: Difference between Market and Calculated Swap Rates

Similarly, we reprice the quotes of the futures contracts and compare it the market data in Figure 5.6 and we find that the error is of order 10^{-14}.

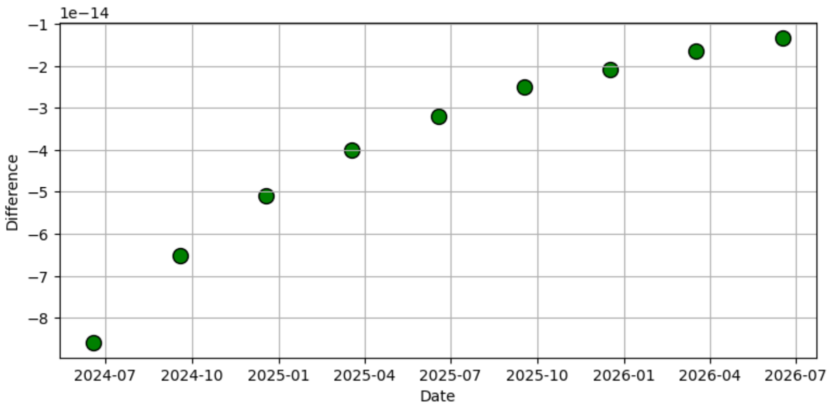

Figure 5.6: Difference between Market and Calculated Futures Rates

5.3 Comparing the Choices

One choice we pick, when we run the construction of the curves, is how to use the data when it has missing values. For example, if we suppose that the swap rates should be present for each year, then with the data presented in Table 5.1, we could proceed in two ways:

- Linearly interpolate the missing values of the swap rate
- Use only the market data (raw data) to bootstrap and then interpolate the desired variable (yield or discount factor)

Therefore, we present in Figure 5.7 the yield curves for both these approaches.

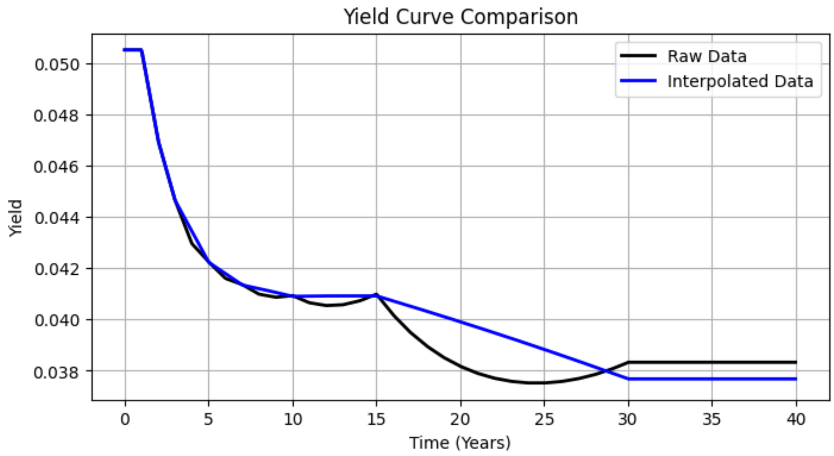

Figure 5.7: Yield Curves for Different Data Manipulations

Observing the figure, we find that it would be more interesting to plot the difference between these curves. Figure 5.8 presents then that difference.

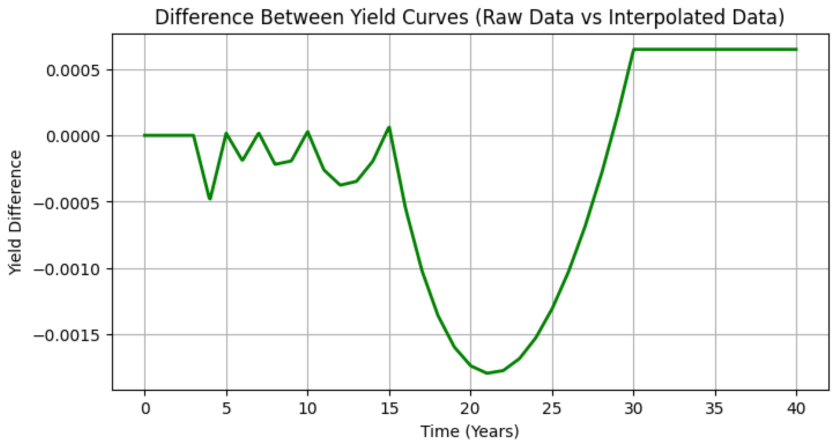

Figure 5.8: Difference in Yield Curves: Raw Data vs Interpolated Data

We find that, throughout the 30 years span, the difference is bounded by approximately 0.0018, which is 18 basis points (bps). The bound is actually high, and we should pick the blue curve because it does not have oscillations like the black one.

In fact, as the interval where no data is present, the difference between the two methods will get bigger, particularly for the values in the middle of the interval, because they are the farthest to any present data value.

One other choice that is worth checking is the variable on which we perform the Newton-Raphson algorithm. In our work, we considered solving the roots of the instrument's contract in terms of the discount factor or the yield. Figure 5.9 shows the difference between both approaches.

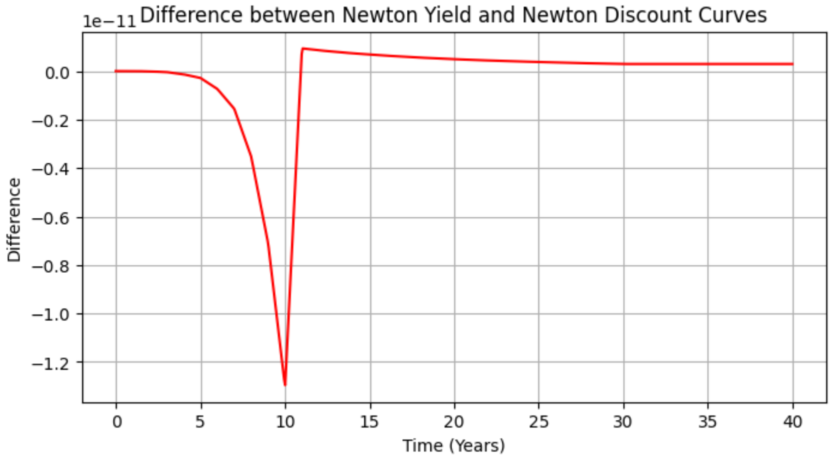

Figure 5.9: Difference in Yield Curves: Newton on Yield vs Discount

We find that the difference is significantly low, which shows that even here we can't favor a choice on the other one.

General Conclusion and Future Perspectives

Summary

The project's primary goal was to construct accurate yield curves, discount curves, and forward rates across various tenors. We achieved this by developing a robust C# solution that extended the company's existing library, which initially provided limited functionality for mathematical operations and date manipulations. Our development involved comprehensive steps including extracting market data, handling missing values through linear interpolation, and performing bootstrapping specifically for OIS-related instruments such as swaps and futures. The system was designed with significant flexibility, allowing various parameters to be configured through a user-defined JSON file. This adaptability facilitated extensive customization, enabling adjustments to bootstrapping techniques, interpolation methods, and numerical solver based on user requirements.

Our implementation effectively constructed yield curves that accurately repriced the market data used in unit tests, demonstrating both precision and reliability. Detailed comparisons of different parameter settings and methods were conducted to assess their impact on results.

In our theoretical research on convexity adjustment, we identified two notable approaches in the literature and introduced a third approach by manipulating one of the existing methods. We derived and solved the mathematical expressions for convexity adjustment using both the Ho-Lee and Hull-White models. While this theoretical exploration provided valuable insights, the practical implementation and calibration of these models are still pending.

Despite these successes, several limitations were encountered. The project focused exclusively on OIS-related instruments and employed only linear interpolation methods. Additionally, while our theoretical work on convexity adjustments was promising, it remains incomplete, with practical implementation and model calibration still required. Furthermore, the code's design, while efficient and aligned with SOLID principles, may need additional refinements to handle more complex scenarios and broader market conditions.

Future Perspectives

Looking ahead, there are several key areas for further development:

1. **Expansion to Other Instruments and Multi-Curve Framework:** The current work was limited to OIS-related instruments. Future efforts should include BOR instruments, such as 3-month EURIBOR swaps, and RFRs. The importance of this expansion is underscored by the need for a multi-curve framework, which became evident following the 2008 financial crisis. The mono-curve approach, which uses a single yield curve to discount all cash flows, has limitations in capturing the complexities of today's financial markets. A multi-curve framework accommodates different yield curves for different tenors and instruments, reflecting the varying risk premiums and liquidity conditions across different market segments. This approach provides a more accurate representation of market conditions and risk, enhancing the robustness and applicability of the yield curve construction.

2. **Advanced Interpolation Methods:** Although linear and linear-on-log interpolation methods were implemented, additional interpolation techniques such as quadratic and cubic splines were not fully explored due to time constraints. Implementing and testing these methods could improve the accuracy and flexibility of the yield curve construction. Advanced interpolation techniques can provide smoother and more accurate curves, particularly in regions with sparse data or where the linear methods fall short.

3. **Convexity Adjustment Implementation:** We have made progress in designing and theoretically exploring convexity adjustment methods. The next steps involve implementing these adjustments, calibrating the models with appropriate parameters, and evaluating their practical effectiveness in refining yield curves. Practical

implementation will include integrating these adjustments into the existing framework and testing their impact on the accuracy and reliability of the yield curves.

4. **Further Code Optimization:** While the current implementation adheres to SOLID principles and is efficient, additional optimization may be necessary to address more complex market conditions and to improve computational performance. Enhancing the efficiency of the code will be crucial for handling large datasets and performing real-time calculations, which are increasingly important in dynamic financial markets.

Overall, these future developments will not only enhance the accuracy and applicability of the yield curve models but also contribute to a deeper understanding of convexity adjustments and interpolation methods in financial modeling.

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