Interest Rate Theory

Part III

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KTH

Interest Rate Theory

In this part of the course we will apply our theory to Interest Rate Markets specifically.

Bonds and Interest rates

The principal objects in an interest rate market are the zero coupon bonds:

Def A zero Coupon Bond (ZCB) with maturity T, called a T-bond, is a contract that pays out 1 SEK at time T.

We denote the price at time $t \leq T$ of a T-bond by $p(t,T)$. Note that $p(T,T) = 1$ (or there would be arbitrage opportunities). The function $T \mapsto p(t,T)$ is called the term structure at time t.

Consider now the following situation: Let $t < S < T$. Suppose that at time t we know that we will receive 1 SEK at time S. We may then trade as follows in the ZCB:

time	t	S	T
	$\begin{cases} \text{sell } 1 \ S\text{-bond} \\ \text{buy } \frac{p(t,S)}{p(t,T)} \ T\text{-bonds} \end{cases}$ This will cost nothing.	The S-bond matures; need to pay out 1 SEK. We use the 1 SEK we get to do this.	The T-bonds mature; we receive $\frac{p(t,S)}{p(t,T)}$.

We see from this that if we agreed at time t to borrow 1 SEK at time S and return it at time T, the fair interest rate should be such that it turns 1 SEK at time S into $\frac{p(t,S)}{p(t,T)}$ SEK at time T. We will quote interest rates as continuously compounded rates; hence, we define:

Def
- The continuously compounded forward rate for $[S,T]$ contracted at time $t<S$, denoted by $R(t;S,T)$, is defined by
$$e^{R(t;S,T)(T-S)} = \frac{p(t,S)}{p(t,T)}.$$
- The continuously compounded spot rate for $[S,T]$, denoted by $R(S,T)$, is defined by
$$e^{R(S,T)} = \frac{1}{p(S,T)}.$$

Assuming henceforth that the term structure is differentiable in T, we obtain:

$$R(t;S,T) = -\frac{\ln p(t,S) - \ln p(t,T)}{S-T} \xrightarrow[S \to T]{} -\frac{\partial}{\partial T} \ln p(t,T).$$

The limit can be interpret as riskless rate of interest contracted at t over an infinitesimal interval $[T, T+dT]$. It turns out that this quantity plays a major role; hence we define:

Def • The (instantaneous) forward rate with maturity T contracted at t, is given by

$$f(t,T) = -\frac{\partial}{\partial T} \ln p(t,T).$$

• The (instantaneous) short rate at time t is given by

$$r(t) = f(t,t).$$

Rem • Note that it is equivalent to specify $p(t,T)$ and $f(t,T)$; in particular:

$$\int_t^T f(t,u)\,du = -\int_t^T \frac{\partial}{\partial u} \ln p(t,u)\,du = -\left(\ln p(t,T) - \ln p(t,t)\right) = -\ln p(t,T)$$

$$\Rightarrow p(t,T) = e^{-\int_t^T f(t,u)\,du}.$$

• The short rate can be understood as an interest rate corresponding to the given ZCB-market. More precisely, defining a "bank account" by $B_t = e^{\int_0^t r(s)ds}$, the fact that we can trade ZCBs of all maturities implies that we effectively have the possibility to put money onto this "bank account". For $r(s)$ constant this is clear. For an heuristic argument in the general case, see the book. Below, for the specific models we consider, we will verify this link directly.

Interest rate derivatives are typically contracts depending on the short rate $r(t), t \leq T$; e.g. $X = \phi(r(T))$ or $X = \phi(\max_{t \leq T} r(t))$. Our aim is to construct a sensible model of the interest rate market so as to obtain fair prices of such contracts.

Short Rate Models

We will here focus on so-called short rate models which are based on directly modeling the short rate itself; the outline is to:

i) Specify a model of $r_t, t \geq 0$, (which may depend on some parameters);

ii) Define the associated bank account $B_t = e^{\int_0^t r_s ds}$;

iii) Specify prices of $p(t,T)$, $t \leq T$, which are internally consistent and consistent with i) and ii) so as to exclude arbitrage opportunities.

iv) Calibrate the model (i.e. fix the values of the parameters) to make sure it fits real market data.

v) Use the given model to price other interest rate derivatives.

Note now that if we specify a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a Brownian motion W, suppose that

$$dr_t = \alpha_t dt + \sigma_t dW_t, \qquad t \geq 0,$$

define

$$dB_t = B_t r_t dt, \qquad B_0 = 1,$$

and suppose that B_t is the only a-priori traded asset, then the prices $p(t,T)$ of ZCBs must be given by

$$p(t,T) = B_t \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_T} \Big| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \Big| \mathcal{F}_t\right], \tag{0}$$

for some equivalent martingale measure (EMM) $\mathbb{Q}$. Note also that since the market consisting of B_t only is free-of-arbitrage but not complete (cf. Lecture 7) there exists multiple EMMs $\mathbb{Q}$; we recall that we must fix one such EMM $\mathbb{Q}$ and use it for all bonds and derivatives to obtain internal consistency (so as to exclude arbitrage opportunities; cf. Lecture 7). Once we've fixed a $\mathbb{Q}$, the price of any interest rate derivative, e.g. $X = \phi(r_T)$, is given by

$$\Pi_t(X) = B_t \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{B_T} \Big| \mathcal{F}_t\right]. \tag{1}$$

<u>How to choose $\mathbb{Q}$?</u> To address this, let us first recall that any $\mathbb{Q} \sim \mathbb{P}$ is characterized by $\mathbb{Q}(A) = \mathbb{E}^{\mathbb{Q}}[1_A] = \mathbb{E}^{\mathbb{P}}[L_T 1_A]$, where

$$L_T = \mathcal{E}\left(\int_0^T \varphi_s dW_s\right) = \exp\left\{\int_0^T \varphi_s dW_s - \frac{1}{2}\int_0^T \varphi_s^2 ds\right\},$$

for some process φ_t, $t \leq T$, and by Girsanov,

$$\hat{W}_t = W_t - \int_0^t \varphi_s ds, \qquad t \leq T,$$

is a $\mathbb{Q}$-Brownian motion. Hence, it follows that prices are given by (1) where

$$dr_t = (\alpha_t + \sigma_t \varphi_t) dt + \sigma_t d\hat{W}_t, \tag{2}$$

for some process φ_t, $t \leq T$. The idea is now the following:

i) Postulate a model for r_t, $t \geq 0$;

ii) Use some data to obtain a good estimate for σ. There are different ways of doing this; we may e.g. find historical price data for r_t and calibrate to this. Indeed, historical price data for r_t follows the $\mathbb{P}$-dynamics (not the $\mathbb{Q}$-dynamics) but the volatility is the same.

iii) Compute the prices $p(0,T)$ using (1) & (2) (they will depend on $\alpha_t + \sigma_t \varphi_t$) and choose values of α_t, φ_t such that they fit today's prices in the market of zero coupon T-bonds. Note that we obtain an estimate of α_t, φ_t simultaneously.

In summary, the above procedure both calibrates the model and chooses $\mathbb{Q}$ so as to fit market data.

The Term Structure Equation

Suppose now that we have a model where $\alpha_t = \alpha(t,r_t)$ and $\sigma_t = \sigma(t,r_t)$. Writing $\lambda_t = -\varphi_t$ (recall the Market Price of Risk; cf. Lecture 7) and supposing that $\lambda_t = \lambda(t,r_t)$, we have

$$dr_t = \left(\alpha(t,r_t) - \lambda(t,r_t)\sigma(t,r_t)\right)dt + \sigma(t,r_t)\,d\hat{W}_t,$$

where $\hat{W}$ is a $\mathbb{Q}$-BM. For a simple claim $X = \phi(r_T)$ the price is then given by

$$\Pi_t(\phi(r_T)) = \mathbb{E}^{\mathbb{Q}}\left[e^{\int_t^T r_s ds}\phi(r_T)\,|\,\mathcal{F}_t\right].$$

It follows that $\Pi_t(\phi(r_T)) = F(t,r_t)$ for some function F; by use of Feynman-Kac (cf. Lecture 7) we obtain that F must solve the following PDE:

$$\left.\begin{aligned} F_t(t,r) + (\alpha - \lambda\sigma)(t,r)F_r(t,r) + \tfrac{1}{2}\sigma(t,r)^2 F_{rr}(t,r) - rF(t,r) &= 0 \\ F(T,r) &= \phi(r) \end{aligned}\right\}.$$

In particular, for ZCBs (i.e. for $p(t,T) = F(t,r_t)$) this PDE reduces to:

$$\left.\begin{aligned} F_t(t,r) + (\alpha - \lambda\sigma)(t,r)F_r(t,r) + \tfrac{1}{2}\sigma(t,r)^2 F_{rr}(t,r) - rF(t,r) &= 0 \\ F(T,r) &= 1 \end{aligned}\right\}; \qquad (3)$$

this equation is called the <u>term structure equation</u> (for it determines the term structure).

Martingale modeling and Affine Term Structure Models

To carry out the above procedure we note that we might equally well model the dynamics of $r_t, t \geq 0$, under $\mathbb{Q}$ directly (cf. (1) and (2)). This is the typical approach. That is, we postulate a model of the following form:

$$dr_t = \mu(t,r_t)\,dt + \sigma(t,r_t)\,d\hat{W}_t, \qquad t \geq 0, \qquad (4)$$

where $\hat{W}$ is a Brownian motion under the fixed pricing measure $\mathbb{Q}$, and where μ and σ are some parameter-dependent functions. The following are famous examples of such models:

Ex)

Vasicek	$dr_t = (b - ar_t)dt + \sigma d\hat{W}_t, \quad a>0;$
Cox-Ingersoll-Ross (CIR)	$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}\,d\hat{W}_t;$
Ho-Lee	$dr_t = \theta(t)dt + \sigma d\hat{W}_t;$
Hull-White (ext. Vasicek)	$dr_t = (\theta(t) - ar_t)dt + \sigma\, d\hat{W}_t;$
Hull-White (ext. CIR)	$dr_t = (\theta(t) - a(t)r_t)dt + \sigma(t)\sqrt{r_t}\,d\hat{W}_t, \quad a(t)>0.$

While it for an arbitrary model of the form (4) can be quite difficult to carry out the above calibration procedure, this turns out to be considerably simpler if the model belongs to the following class:

Def A model (of the form (4)) is said to possess an Affine Term Structure (ATS) if the associated term structure (computed from (4) and (0) (or (3))) is of the form

$$p(t,T) = e^{A(t,T) - B(t,T) r_t},$$

for some deterministic functions A and B.

Thm Suppose that μ and σ in (4) are given by

$$\begin{cases} \mu(t,r) = \alpha(t) r + \beta(t) \\ \sigma(t,r) = \sqrt{\gamma(t) r + \delta(t)} \end{cases}$$

for some (deterministic) functions $\alpha, \beta, \gamma, \delta$. Then the model has an ATS with $A(t,T)$ and $B(t,T)$ given as the solutions to

$$\left.\begin{aligned} B_t(t,T) + \alpha(t) B(t,T) - \tfrac{1}{2}\gamma(t) B(t,T)^2 &= -1 \\ B(T,T) &= 0 \end{aligned}\right\};$$

$$\begin{cases} A_t(t,T) = \beta(t) B(t,T) - \tfrac{1}{2}\delta(t) B(t,T)^2 \\ A(T,T) = 0 \end{cases}$$

In particular, $d\dfrac{p(t,T)}{B_t} = -\dfrac{p(t,T)}{B_t} B(t,T)\sqrt{\gamma(t) r_t + \delta(t)}\, d\hat{W}_t$.

Proof Fix $T > 0$. Let $F(t,r) = e^{A(t,T) - B(t,T) r}$, with A and B solutions to the above equations. Then, by Itô,

$$dF(t,r_t) = F(t,r_t)\Big(\{A_t(t,T) - B_t(t,T) r_t\} dt - B(t,T)\big(\mu(t,r_t) dt + \sigma(t,r_t) d\hat{W}_t\big) + \tfrac{1}{2} B(t,T)^2 \sigma(t,r_t)^2 dt\Big)$$

Moreover, $d\dfrac{F(t,r_t)}{B_t} = \dfrac{dF(t,r_t)}{B_t} - \dfrac{F(t,r_t)}{B_t^2} B_t r_t dt = \dfrac{F(t,r_t)}{B_t}\left(\dfrac{dF(t,r_t)}{F(t,r_t)} - r_t dt\right)$.

Hence,

$$d\frac{F(t,r_t)}{B_t} = \frac{F(t,r_t)}{B_t}\Big(\{A_t(t,T) - (1 + B_t(t,T)) r_t - B(t,T)\overbrace{\mu(t,r_t)}^{= \alpha(t) r_t + \beta(t)} + \tfrac{1}{2} B(t,T)^2 \overbrace{\sigma(t,r_t)^2}^{= \gamma(t) r_t + \delta(t)}\} dt - B(t,T)\sigma(t,r_t) d\hat{W}_t\Big) \quad (5)$$

$$= \frac{F(t,r_t)}{B_t}\Big(\{\underbrace{A_t(t,T) - \beta(t) B(t,T) + \tfrac{1}{2}\delta(t) B(t,T)^2}_{=0} - \underbrace{\big(1 + B_t(t,T) + \alpha(t) B(t,T) - \tfrac{1}{2}\gamma(t) B(t,T)^2\big)}_{=0} r_t\} dt$$

$$- B_t(t,T)\sqrt{\gamma(t) r_t + \delta(t)}\, d\hat{W}_t\Big)$$

$$= -\frac{F(t,r_t)}{B_t} B_t(t,T)\sqrt{\gamma(t) r_t + \delta(t)}\, d\hat{W}_t.$$

Hence, $\dfrac{F(t,r_t)}{B_t}$ is a martingale under $\mathbb{Q}$; we obtain,

$$F(t,r_t) = B_t \mathbb{E}^{\mathbb{Q}}\left[\frac{F(T,r_T)}{B_T} \,\Big|\, \mathcal{F}_t\right] = B_t \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_T} \,\Big|\, \mathcal{F}_t\right] = p(t,T).$$

$\square$

Rem

- Note that the equation for $B(t,T)$ is a Ricatti-equation which does not involve $A(t,T)$; once we've solved it for $B(t,T)$ we can simply insert the expression for $B(t,T)$ into the equation for $A(t,T)$ and integrate to obtain $A(t,T)$.
- All the models listed in the example above are ATS-models.
- The result can also be proven by substituting the expression for $F(t,r)$ into the term-structure-equation (with $\alpha - \lambda\sigma = \mu$ etc).

Some standard models

Vasicek Recall that $dr_t = (b - ar_t)dt + d\hat{W}_t$;

the equations for $A(t,T)$ and $B(t,T)$ become

$$\left.\begin{aligned} B_t(t,T) - aB(t,T) &= -1 \\ B(T,T) &= 0 \end{aligned}\right\}$$

$$\left\{\begin{aligned} A_t(t,T) &= bB(t,T) - \sigma^2/2\, B(t,T)^2 \\ A(T,T) &= 0. \end{aligned}\right.$$

The eqn for $B(t,T)$ is a linear ODE with solution

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right).$$

Hence we get

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B(s,T)^2\,ds - b\int_t^T B(s,T)\,ds = \tfrac{1}{2}\sigma^2/a^2\int_t^T\left(1 - e^{-a(T-s)}\right)^2 ds - b/a\int_t^T\left(1 - e^{-a(T-s)}\right)ds$$

$$= \dots = \frac{(ab - \sigma^2/2)(B(t,T) - (T-t))}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a}.$$

To compute $f(t,T)$, note that for a function g with primitive fcn G, we have

$$\frac{\partial}{\partial T}\int_t^T g(T-s)\,ds = \frac{\partial}{\partial T}\left[-G(T-s)\right]_t^T = \frac{\partial}{\partial T}\left(G(T-t) - G(0)\right) = g(T-t).$$

Hence, we get $\quad f(t,T) = -\frac{\partial}{\partial T}\ln p(t,T) = \frac{\partial}{\partial T}\left\{-A(t,T) + B(t,T)r_t\right\}$

$$= \frac{\partial}{\partial T}\left\{b\int_t^T B(s,T)\,ds - \sigma^2/2\int_t^T B(s,T)^2\,ds + B(t,T)r_t\right\}$$

$$= bB(t,T) - \sigma^2/2\, B(t,T)^2 + \frac{\partial}{\partial T}B(t,T)r_t$$

$$= \frac{b}{a}\left(1 - e^{-a(T-t)}\right) - \tfrac{1}{2}\sigma^2/a^2\left(1 - e^{-a(T-t)}\right)^2 + e^{-a(T-t)}r_t;$$

we see from here that $f(t,t) = r_t$, $t \geq 0$.

Ho-Lee Recall that $dr_t = \theta(t)\,dt + \sigma d\hat{W}_t$.

Hence,
$$\left.\begin{array}{l} B_t(t,T) = -1 \\ B(T,T) = 0 \end{array}\right\}$$

$$\begin{cases} A_t(t,T) = \theta(t)B(t,T) - \frac{\sigma^2}{2}B(t,T)^2 \\ A(T,T) = 0 \end{cases}$$

We obtain
$$B(t,T) = T-t$$
$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2}{2}\int_t^T (T-s)^2\,ds = -\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2}{6}(T-t)^3.$$

$$\therefore\quad p(t,T) = \exp\left\{-\int_t^T \theta(s)(T-s)\,ds + \frac{\sigma^2}{6}(T-t)^3 - (T-t)r_t\right\}.$$

The task then is to choose $\theta(t)$ such that $p(0,T)$ matches real market prices for the zero coupon T-bonds — for all $T>0$; the fact that $\theta(t)$ depends on t implies that it is indeed possible to obtain such a perfect fit.

Hull-White (ext. Vasicek) A detailed study of this model is the focus of Project 2.

The Forward Measure

Consider a (martingale modeling) short-rate-model of the form (4) which posesses an affine-term-structure. We will now use change-of-numeraire techniques (cf. Lecture 10) to price interest-rate-derivatives using such a model.

Def For a fixed $T>0$, the T-forward measure, denoted by $\mathbb{Q}^T$, is the equivalent martingale measure corresponding to $\mathbb{Q}$ when using $p(t,T)$, $t\leq T$, as numeraire.

Recall that since we're modeling the dynamics of r_t, $t\geq 0$, under $\mathbb{Q}$ directly we have effectively already fixed which pricing measure we're using; when changing numeraire we are therefore looking for the EMM in the new numeraire corresponding to that given $\mathbb{Q}$.

Given $\mathbb{Q}^T$ as defined above, we have that prices of interest-rate-derivatives, such as $X = \Phi(r_T)$ or $X = \Phi(\sup_{t\leq T} r_t)$, can be computed via either of the following formulae (cf. Lecture 10):

$$\Pi_t(X) = \underline{B_t \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{B_T}\Big|\mathcal{F}_t\right]} = p(t,T)\,\mathbb{E}^T\left[\frac{X}{\underbrace{p(T,T)}_{=1}}\Big|\mathcal{F}_t\right] = \underline{p(t,T)\,\mathbb{E}^T[X|\mathcal{F}_t]} \tag{6}$$

We note that the formula on the RHS is very neat; in particular, we do not need to explicitly compute $p(t,T)$ since we can observe its value in the market. To price a derivative of the form $X = \Phi(r_T)$ we need however the dynamics of r_t, $t\leq T$, under $\mathbb{Q}^T$; the next result provides this:

Thm Consider an ATS-model of the form (4). Then, the forward measure $\mathbb{Q}^T$ is given by $\mathbb{Q}^T(A) = \mathbb{E}^{\mathbb{Q}}[L_T \mathbb{1}_A]$, where

$$L_T = \mathcal{E}\left(-\int_0^T B(t,T)\sigma(t,r_t)\,d\hat{W}_t\right). \qquad (7)$$

In particular, $W_t^T := \hat{W}_t + \int_0^t B(s,T)\sigma(s,r_s)\,ds$ is a Brownian motion under $\mathbb{Q}^T$ and

$$dr_t = \left(\mu(t,r_t) - \sigma(t,r_t)^2 B(t,T)\right)dt + \sigma(t,r_t)\,dW_t^T, \qquad t \le T.$$

Proof Note that

$$B_0\,\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{B_T}\right] = p(0,T)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{B_0}{p(0,T)}\frac{p(T,T)}{B_T}\frac{X}{p(T,T)}\right] = p(0,T)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{p(T,T)/B_T}{p(0,T)/B_0}\,X\right];$$

comparing with (6) we see that we need $\mathbb{Q}^T$ to be defined by $\mathbb{Q}^T(A) = \mathbb{E}^{\mathbb{Q}}[L_T \mathbb{1}_A]$, with $L_T = \frac{p(T,T)/B_T}{p(0,T)/B_0}$. Now we know that we have an ATS, i.e. there are $A(t,T)$ and $B(t,T)$ s.t.

$$p(t,T) = e^{A(t,T) - B(t,T)r_t};$$

hence, we obtain by Itô (cf. (5)):

$$d\frac{p(t,T)}{B_t} = \frac{p(t,T)}{B_t}\left(\left\{A_t(t,T) - (B_t(t,T)+1)r_t - B(t,T)\mu(t,r_t) + \tfrac{1}{2}B(t,T)^2\sigma(t,r_t)^2\right\}dt - B(t,T)\sigma(t,r_t)\,d\hat{W}_t\right).$$

Now we know that $\frac{p(t,T)}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_T}\middle|\mathcal{F}_t\right]$ is a martingale under $\mathbb{Q}$; hence the drift term must vanish: i.e.

$$d\frac{p(t,T)}{B_t} = -\frac{p(t,T)}{B_t}B(t,T)\sigma(t,r_t)\,d\hat{W}_t \implies \frac{p(t,T)}{B_t} = \frac{p(0,T)}{B_0}\,\mathcal{E}\left(-\int_0^t B(s,T)\sigma(s,r_s)\,d\hat{W}_s\right),$$

which gives that $L_T = \frac{p(T,T)/B_T}{p(0,T)/B_0}$ is given by (7). The rest follows from Girsanov's thm $\square$

Rem $\mathbb{Q}$ and $\mathbb{Q}^T$ coincide, if and only if, r is deterministic.

Computing prices with the forward measure - examples

Ex Price $X = (r_T - r_0)^2$ using the Ho-Lee model.

In this case, $dr_t = \theta(t)dt + \sigma d\hat{W}_t = \theta(t)dt + \sigma(dW_t^T - (T-t)\sigma dt) = (\theta(t) - \sigma^2(T-t))dt + \sigma dW_t^T$.

Hence, $\Pi_0((r_T - r_0)^2) = p(0,T)\,\mathbb{E}^T\left[\left(\int_0^T \theta(t) - \sigma^2(T-t)\,dt + \sigma W_T^T\right)^2\right]$

$$\overset{W_T^T \overset{\mathbb{Q}^T}{\sim} \mathcal{N}(0,T)}{=} p(0,T)\left(\left(\int_0^T \theta(t) - \sigma^2(T-t)\,dt\right)^2 + \sigma^2 T\right)$$

Ex Price $\phi(r_T)$ using the Vasicek model.

In this case,

$$\begin{aligned} dr_t &= (b - ar_t)dt + \sigma d\hat{W}_t \\ &= (b - ar_t)dt + \sigma(dW_t^T - B(t,T)\sigma\,dt) \\ &= (b - \sigma^2 B(t,T) - ar_t)dt + \sigma dW_t^T, \qquad \text{where } B(t,T) = \tfrac{1}{a}\left(1 - e^{-a(T-t)}\right). \end{aligned}$$

To solve this SDE for r_t, let $X_t = r_t e^{at}$. Then, by Itô:

$$dX_t = e^{at}\left(\cancel{a r_t dt} + (b - \sigma^2 B(t,T) - \cancel{a r_t})dt + \sigma dW_t^T\right)$$

$$\Rightarrow \quad X_t = X_0 + \int_0^t e^{as}\left(b - \sigma^2 B(s,T)\right)ds + \sigma \int_0^t e^{as} dW_s^T$$

$$\Rightarrow \quad \underline{r_t} = r_0 e^{-at} + \underbrace{\int_0^t e^{-a(t-s)}\left(b - \sigma^2 B(s,T)\right)ds}_{=b\int_0^t e^{-a(t-s)}ds - \frac{\sigma^2}{a}\int_0^t e^{-a(t-s)}\left(1 - e^{-a(t-s)}\right)ds \,=\, \frac{b}{a} e^{-a(t-s)}\Big|_0^t + \frac{1}{2}\frac{\sigma^2}{a^2}\left(1 - e^{-a(t-s)}\right)^2\Big|_0^t \,=\, \frac{b}{a}\left(1 - e^{-at}\right) - \frac{1}{2}\frac{\sigma^2}{a^2}\left(1 - e^{-at}\right)^2} + \sigma \int_0^t e^{-a(t-s)} dW_s^T$$

$$= \underbrace{r_0 e^{-at} + \frac{b}{a}\left(1 - e^{-at}\right) - \frac{1}{2}\frac{\sigma^2}{a^2}\left(1 - e^{-at}\right)^2}_{= f(0,t) \ \text{(cf. ex. above)}} + \sigma \int_0^t e^{-a(t-s)} dW_s^T$$

$$= f(0,t) + \sigma \int_0^t e^{-a(t-s)} dW_s^T \overset{\mathbb{Q}^T}{\sim} \mathcal{N}\Big(f(0,t), \sigma^2 \underbrace{\int_0^t e^{-2a(t-s)} ds}_{= \frac{1}{2a} e^{-2a(t-s)}\big|_0^t = \frac{1}{2a}\left(1 - e^{-2at}\right)}\Big) = \underline{\mathcal{N}\left(f(0,t), \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)\right)},$$

where we used that for a deterministic integrand $\alpha(t)$ and BM W, $\int_0^t \alpha(s)\,dW_s \sim \mathcal{N}\left(0, \int_0^t \alpha(s)^2 ds\right)$.

In consequence,

$$\Pi_0\left(\phi(r_T)\right) = p(0,T)\, \mathbb{E}^T\left\{\phi(r_T)\right\} = p(0,T) \int_{-\infty}^{\infty} \phi(x) \frac{1}{\sqrt{\pi \frac{\sigma^2}{a}\left(1 - e^{-2aT}\right)}} e^{-\frac{(x - f(0,T))^2}{\frac{\sigma^2}{a}\left(1 - e^{-2aT}\right)}} dx.$$

In the above computation we obtained that $\mathbb{E}^T\{r_T\} = f(0,T)$ which simplified the expressions. One may ask whether this is a general fact? It is indeed; we will verify this below.

Ex) Price the bond option $\underline{X = \left(p(T_1,T_2) - K\right)_+}$, where $T_1 < T_2$, using the <u>Hull-White</u> model.

Note first that

$$\Pi_0\left(\left(p(T_1,T_2) - K\right)_+\right) = \mathbb{E}^{\mathbb{Q}}\left[\frac{p(T_1,T_2) - K}{B_{T_1}} \mathbb{1}_{\{p(T_1,T_2) \geq K\}}\right]$$

$$= \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\frac{p(T_1,T_2)}{B_{T_1}} \mathbb{1}_{\{p(T_1,T_2) \geq K\}}\right]} - K\, \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{T_1}} \mathbb{1}_{\{p(T_1,T_2) \geq K\}}\right]$$

$$\underset{\frac{p(t,T)}{B_t} \text{ a } \mathbb{Q}\text{-mtg}}{=} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{p(T_2,T_2)}{B_{T_2}} \Big| \mathcal{F}_{T_1}\right] \mathbb{1}_{\{p(T_1,T_2) \geq K\}}\right] \overset{p(T_1,T_2)\ \mathcal{F}_{T_1}\text{-meas.}}{=} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{p(T_2,T_2)}{B_{T_2}} \mathbb{1}_{\{p(T_1,T_2) \geq K\}} \Big| \mathcal{F}_{T_1}\right]\right]$$

cf. page 32 lecture 4

$$\underset{\text{Tower prop.}}{=} \mathbb{E}^{\mathbb{Q}}\left[\frac{p(T_2,T_2)}{B_{T_2}} \mathbb{1}_{\{p(T_1,T_2) \geq K\}}\right] \underset{p(T_2,T_2)=1}{=} \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{T_2}} \mathbb{1}_{\{p(T_1,T_2) \geq K\}}\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{T_2}} \mathbb{1}_{\{p(T_1,T_2) \geq K\}}\right] - K\, \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{T_1}} \mathbb{1}_{\{p(T_1,T_2) \geq K\}}\right]$$

$$= p(0,T_2)\, \mathbb{E}^{T_2}\left[\mathbb{1}_{\{p(T_1,T_2) \geq K\}}\right] - K\, p(0,T_1)\, \mathbb{E}^{T_1}\left[\mathbb{1}_{\{p(T_1,T_2) \geq K\}}\right]$$

$$= p(0,T_2)\, \mathbb{Q}^{T_2}\left(p(T_1,T_2) \geq K\right) - K\, p(0,T_1)\, \mathbb{Q}^{T_1}\left(p(T_1,T_2) \geq K\right).$$

Recall now that $p(t,T) = e^{A(t,T) - B(t,T) r_t}$, where $B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$.

Hence, $\frac{p(t,T_1)}{p(t,T_2)} = e^{A(t,T_1) - A(t,T_2) - (B(t,T_1) - B(t,T_2)) r_t}$; since $dr_t = (\theta(t) - a r_t)dt + \sigma d\hat{W}_t$,

$$d\,\frac{p(t,T_1)}{p(t,T_2)} = \frac{p(t,T_1)}{p(t,T_2)}\left(\{\dots\}dt - (B(t,T_1) - B(t,T_2))\sigma\, d\tilde{W}_t\right) = \frac{p(t,T_1)}{p(t,T_2)}(B(t,T_1) - B(t,T_2))\sigma\, dW_t^{T_2}$$

$$\Rightarrow \frac{p(t,T_1)}{p(t,T_2)} = \frac{p(0,T_1)}{p(0,T_2)} \exp\left\{-\sigma\int_0^t B(s,T_1) - B(s,T_2)\, dW_s^{T_2} - \frac{\sigma^2}{2}\int_0^t (B(s,T_1) - B(s,T_2))^2 ds\right\}$$

$$\ln\left(\frac{p(t,T_1)}{p(t,T_2)} \Big/ \frac{p(0,T_1)}{p(0,T_2)}\right) \sim^{\mathbb{Q}^{T_2}} N\left(-\frac{\sigma^2}{2}\int_0^t (B(s,T_1) - B(s,T_2))^2 ds,\ \sigma^2\int_0^t (B(s,T_1) - B(s,T_2))^2 ds\right) = \underline{N\left(-\tfrac{1}{2}\Sigma_t^2, \Sigma_t^2\right)}, \quad t \le T_1,$$

where $$\underline{\Sigma_t^2} = \sigma^2\int_0^t (B(s,T_1) - B(s,T_2))^2 ds = \frac{\sigma^2}{a^2}\int_0^t \left(e^{-a(T_2-s)} - e^{-a(T_1-s)}\right)^2 ds = \frac{\sigma^2}{a^2}\left(e^{-aT_2} - e^{-aT_1}\right)^2 \int_0^t e^{2as} ds$$

$$= \frac{\sigma^2}{a^2}\left(e^{-aT_2} - e^{-aT_1}\right)^2 \frac{1}{2a} e^{2as}\Big|_0^t = \underline{\frac{\sigma^2}{2a^3}\left(e^{-aT_2} - e^{-aT_1}\right)^2\left(e^{2at} - 1\right)}.$$

$$\Longrightarrow \ln\left(\frac{p(T_1,T_1)}{p(T_1,T_2)} \Big/ \frac{p(0,T_1)}{p(0,T_2)}\right) \sim^{\mathbb{Q}^{T_2}} N\left(-\tfrac{1}{2}\Sigma^2, \Sigma^2\right), \text{ where } \Sigma^2 = \frac{\sigma^2}{2a^3}\left(e^{-aT_2} - e^{-aT_1}\right)^2\left(e^{2aT_1} - 1\right).$$

In the same way we obtain $\ln\left(\frac{p(T_1,T_2)}{p(T_1,T_1)} \Big/ \frac{p(0,T_2)}{p(0,T_1)}\right) \sim^{\mathbb{Q}^{T_1}} N\left(-\tfrac{1}{2}\Sigma^2, \Sigma^2\right)$.

$$\text{Hence, } \mathbb{Q}^{T_1}(p(T_1,T_2) \ge K) = \mathbb{Q}^{T_1}\left(\underbrace{\frac{\ln\left(\frac{p(T_1,T_2)}{p(T_1,T_1)} \Big/ \frac{p(0,T_2)}{p(0,T_1)}\right) + \frac{1}{2}\Sigma^2}{\Sigma}}_{\sim N(0,1)} \ge \frac{\ln\left(K \Big/ \frac{p(0,T_2)}{p(0,T_1)}\right) + \frac{1}{2}\Sigma^2}{\Sigma}\right)$$

$$= \Phi\Bigg(\underbrace{\frac{\ln \frac{1}{K} \Big/ \frac{p(0,T_1)}{p(0,T_2)} - \frac{1}{2}\Sigma^2}{\Sigma}}_{=:\, d_2}\Bigg).$$

$$\text{Moreover, } \mathbb{Q}^{T_2}(p(T_1,T_2) \ge K) = \mathbb{Q}^{T_2}\left(\frac{p(T_1,T_1)}{p(T_1,T_2)} \le \frac{1}{K}\right)$$

$$= \mathbb{Q}^{T_2}\Bigg(\underbrace{\frac{\ln \frac{p(T_1,T_1)}{p(T_1,T_2)} \Big/ \frac{p(0,T_1)}{p(0,T_2)} + \frac{1}{2}\Sigma^2}{\Sigma}}_{\sim N(0,1)} \le \underbrace{\frac{\ln\left(\frac{1}{K} \Big/ \frac{p(0,T_1)}{p(0,T_2)}\right) + \frac{1}{2}\Sigma^2}{\Sigma}}_{=:\, d_1}\Bigg) = \Phi(d_1).$$

Putting things together, we get

$$\Pi_0\left((p(T_1,T_2) - K)_+\right) = p(0,T_2)\, \mathbb{Q}^{T_2}(p(T_1,T_2) \ge K) - K p(0,T_1)\, \mathbb{Q}^{T_1}(p(T_1,T_2) \ge K)$$

$$= p(0,T_2)\,\Phi(d_1) - K p(0,T_1)\,\Phi(d_2).$$

The martingale property of $f(t,T)$ under $\mathbb{Q}^T$

We have observed various instances of the fact that

$$f(t,t) = r_t \quad \text{and} \quad \mathbb{E}^T[r_T] = f(0,T).$$

That these facts holds more generally is a consequence of the following result:

Thm Consider a short-rate-model of the form (4). It then holds that

$$f(t,T) = \mathbb{E}^T\{r_T \mid \mathcal{F}_t\}, \qquad t \le T.$$

In particular, $f(t,T)$, $t \le T$, is a $\mathbb{Q}^T$-martingale and $f(t,t) = r_t$.

Proof (Sketch)

$$\Pi_t(r_T) = B_t \mathbb{E}^{\mathbb{Q}}\left\{\frac{r_T}{B_T} \mid \mathcal{F}_t\right\} = p(t,T)\, \mathbb{E}^T\{r_T \mid \mathcal{F}_t\}.$$

Hence,
$$\begin{aligned}
\mathbb{E}^T\{r_T \mid \mathcal{F}_t\} &= \frac{1}{p(t,T)} \mathbb{E}^{\mathbb{Q}}\left\{r_T e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\right\} \\
&= \frac{-1}{p(t,T)} \mathbb{E}^{\mathbb{Q}}\left\{\frac{\partial}{\partial T} e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\right\} \\
&= \frac{-1}{p(t,T)} \frac{\partial}{\partial T} \underbrace{\mathbb{E}^{\mathbb{Q}}\left\{e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\right\}}_{= B_t \mathbb{E}^{\mathbb{Q}}\{1/B_T \mid \mathcal{F}_t\} = p(t,T)} \\
&= -\frac{\frac{\partial}{\partial T} p(t,T)}{p(t,T)} = -\frac{\partial}{\partial T} \ln p(t,T). \qquad \square
\end{aligned}$$

The result verifies that it is indeed the instantaneous short rate we've been modeling; it is also very useful for computing prices (cf. the Vasicek ex. above).