4

Interest Rates

Interest rates are a factor in the valuation of virtually all derivatives and will feature prominently in much of the material that will be presented in the rest of this book. This chapter introduces a number of different types of interest rate. It deals with some fundamental issues concerned with the way interest rates are measured and analyzed. It explains the compounding frequency used to define an interest rate and the meaning of continuously compounded interest rates, which are used extensively in the analysis of derivatives. It covers zero rates, par yields, and yield curves, discusses bond pricing, and outlines a "bootstrap" procedure commonly used to calculate zero-coupon interest rates. It also covers forward rates and forward rate agreements and reviews different theories of the term structure of interest rates. Finally, it explains the use of duration and convexity measures to determine the sensitivity of bond prices to interest rate changes.

Chapter 6 will cover interest rate futures and show how the duration measure can be used when interest rate exposures are hedged. For ease of exposition, day count conventions will be ignored throughout this chapter. The nature of these conventions and their impact on calculations will be discussed in Chapters 6 and 7.

4.1 TYPES OF RATES

An interest rate in a particular situation defines the amount of money a borrower promises to pay the lender. For any given currency, many different types of interest rates are regularly quoted. These include mortgage rates, deposit rates, prime borrowing rates, and so on. The interest rate applicable in a situation depends on the credit risk. This is the risk that there will be a default by the borrower of funds, so that the interest and principal are not paid to the lender as promised. The higher the credit risk, the higher the interest rate that is promised by the borrower.

Interest rates are often expressed in basis points. One basis point is 0.01% per annum.

Treasury Rates

Treasury rates are the rates an investor earns on Treasury bills and Treasury bonds. These are the instruments used by a government to borrow in its own currency. Japanese Treasury rates are the rates at which the Japanese government borrows in

yen; U.S. Treasury rates are the rates at which the U.S. government borrows in U.S. dollars; and so on. It is usually assumed that there is no chance that a government will default on an obligation denominated in its own currency. Treasury rates are therefore regarded as risk-free in the sense that an investor who buys a Treasury bill or Treasury bond is certain that interest and principal payments will be made as promised.

LIBOR

LIBOR is short for *London Interbank Offered Rate*. It is an unsecured short-term borrowing rate between banks. LIBOR rates are quoted for a number of different currencies and borrowing periods. The borrowing periods range from one day to one year. LIBOR rates are used as reference rates for hundreds of trillions of dollars of transactions throughout the world. One popular derivatives transaction that uses LIBOR as a reference interest rate is an interest rate swap. (Interest rate swaps are introduced in Section 4.2 and discussed more fully in Chapter 7.) LIBOR rates are compiled by asking 18 global banks to provide quotes estimating the rate of interest at which they could borrow funds from other banks just prior to 11:00 a.m. (U.K. time). The highest four and the lowest four of the quotes for each currency/borrowing period combination are discarded and the remaining ones are averaged to determine the LIBOR fixings for a day. The banks submitting quotes typically have a AA credit rating.[1] LIBOR is therefore usually considered to be an estimate of the unsecured borrowing rate for a AA-rated bank.

Some traders working for banks have been investigated for attempting to manipulate LIBOR quotes. Why might they do this? Suppose that the payoff to a bank from a derivative depends on the LIBOR fixing on a particular day with the payoff increasing as the fixing increases. It is tempting for a trader to provide a high quote on that day and to try to persuade other banks to do the same. Tom Hayes was the first trader to be convicted of LIBOR manipulation. In August 2015, he was sentenced to 14 years (later reduced to 11 years) in prison by a court in the U.K. A problem with the system was that there was not enough interbank borrowing for banks to make accurate estimates of their borrowing rates for all the quotes that were required and some judgment was inevitably necessary. In an attempt to improve things, the number of different currencies has been reduced from 10 to 5 and the number of different borrowing periods has been reduced from 15 to 7. Also, regulatory oversight of the way the borrowing estimates are produced has been improved.

It is now recognized that LIBOR is a less-than-ideal reference rate for derivatives transactions because it is determined from estimates made by banks, not from market transactions. It is likely that the derivatives market will move to using other reference rates in the future.

Overnight Rates

Banks are required to maintain a certain amount of cash, known as a reserve, with the central bank. The reserve requirement for a bank at any time depends on its outstanding assets and liabilities. At the end of a day, some financial institutions typically have surplus funds in their accounts with the central bank while others have requirements for

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[1] The best credit rating given to a company by the rating agencies S&P and Fitch is AAA. The second best is AA. The corresponding credit ratings for Moody's are Aaa and Aa.

funds. This leads to borrowing and lending overnight. A broker usually matches borrowers and lenders. In the United States, the central bank is the Federal Reserve (often referred to as the Fed) and the overnight rate is called the *federal funds rate*. The weighted average of the rates in brokered transactions (with weights being determined by the size of the transaction) is termed the *effective federal funds rate*. This overnight rate is monitored by the Federal Reserve, which may intervene with its own transactions in an attempt to raise or lower it. Other countries have similar systems to the United States. For example, in the United Kingdom, the average of brokered overnight rates is termed the *sterling overnight index average* (SONIA), and in the eurozone, it is termed the *euro overnight index average* (EONIA).

Repo Rates

Unlike LIBOR and the overnight federal funds rate, repo rates are secured borrowing rates. In a repo (or repurchase agreement), a financial institution that owns securities agrees to sell the securities for a certain price and buy them back at a later time for a slightly higher price. The financial institution is obtaining a loan and the interest it pays is the difference between the price at which the securities are sold and the price at which they are repurchased. The interest rate is referred to as the *repo rate.*

If structured carefully, a repo involves very little credit risk. If the borrower does not honor the agreement, the lending company simply keeps the securities. If the lending company does not keep to its side of the agreement, the original owner of the securities keeps the cash provided by the lending company. The most common type of repo is an *overnight repo*, which may be rolled over day to day. However, longer-term arrangements, known as *term repos*, are sometimes used. Because it is a secured rate, a repo rate is generally slightly below the corresponding LIBOR or fed funds rate.

4.2 SWAP RATES

Swaps are discussed in Chapter 7, but it will be useful to provide a brief introduction to them here. The most common swap is an agreement where a LIBOR interest rate is exchanged for a fixed rate of interest for a period of time. For example, two parties could agree to exchange three-month LIBOR (with the rate being reset every three months) applied to a principal of \$100 million for a fixed rate of interest of 3% per annum applied to the same principal for five years. One party would pay LIBOR and receive the fixed rate of 3%; the other party would receive LIBOR and pay a fixed rate of 3%.

A swap is designed so that it has zero value initially. The fixed rate (3% in our example) is known as the swap rate. A bank can earn this five-year swap rate as follows:

1. Make a series of 20 three-month loans for \$100 million (at times 0, 3 months, 6 months, ..., 57 months) to AA-rated banks at LIBOR. The bank ensures that a borrowing bank is always rated AA at the beginning of the life of its three-month loan.

2. Enter into a swap where the three-month LIBOR rate of interest is paid and a fixed rate of interest (3% per annum in our example) is received.

Note that it would be more risky for a bank to lend money to a single AA-rated bank for

five years. A bank that is initially creditworthy is unlikely to default in three months, but over a five-year period its credit quality could well decline. It is better for the lending bank to have 20 three-month credit exposures than one five-year credit exposure. (The probability that a bank initially rated AA will default over five years is more than twenty times the probability that it will default in three months.)

An interest rate that is earned by renewing loans periodically in the way we have described is referred to as a *continually refreshed rate*. The swap rate in our example is a continually refreshed rate lasting five years and based on three-month LIBOR.

Overnight Indexed Swaps

An overnight indexed swap (OIS) is a swap where an agreed fixed rate for a period (e.g., one month or three months) is exchanged for the geometric average of the overnight rates during the period. The overnight rate is the one discussed in Section 4.1 (the effective federal funds rate in the United States or a similarly defined rate in another country). If during a certain period, a bank borrows funds at the overnight rate (rolling the interest and principal forward each day), the interest rate it pays for the period is the geometric average of the overnight interest rates. Similarly, if it lends money at the overnight interest rate every day (rolling the interest and principal forward each day), the interest it earns for the period is also the geometric average of the overnight interest rates. An OIS therefore allows overnight borrowing or lending for a period to be swapped for borrowing or lending at an agreed fixed rate for the period. The agreed fixed rate in an OIS is referred to as the *OIS rate*. If this fixed rate is greater than the geometric average of daily rates for the period, there is a payment from the fixed-rate payer to the floating-rate payer at the end of the period; otherwise, there is a payment from the floating-rate payer to the fixed-rate payer. For example, suppose that in a U.S. three-month OIS the notional principal is $100 million and the fixed rate (i.e., the OIS rate) is 3% per annum. If the geometric average of overnight effective federal funds rates during the three months proves to be 2.8% per annum, the fixed rate payer has to pay $0.25 \times (0.030 - 0.028) \times \$100,000,000$ or $50,000 to the floating rate payer. (This calculation does not take account of the impact of day count conventions, which will be discussed in later chapters.)

OISs lasting 10 years or longer are now traded. An OIS lasting longer than one year is typically divided into three-month subperiods. At the end of each subperiod, the geometric average of the overnight rates during the subperiod is exchanged for the OIS rate.

The OIS rate is a continually refreshed overnight rate: it is the rate that can be earned by a financial institution when a series of overnight loans to other financial institutions are combined with a swap. The credit risk associated with a three-month OIS rate is the credit risk associated with a series of one-day loans, which is less than the credit risk associated with a single three-month loan.

4.3   THE RISK-FREE RATE

As we shall see, the usual approach to valuing derivatives involves setting up a riskless portfolio and arguing that the return on the portfolio should be the risk-free rate. The risk-free rate therefore plays a central role in derivatives pricing. It might be thought

that derivatives traders would use the rates on Treasury bills and Treasury bonds as risk-free rates. In fact they do not do this. This is because there are tax and regulatory factors that lead to Treasury rates being artificially low. For example:

1. Banks are not required to keep capital for investments in a Treasury instruments, but they are required to keep capital for other very low risk instruments.

2. In the United States, Treasury instruments are given favorable tax treatment compared with other very low risk instruments because the interest earned by investors is not taxed at the state level.

Prior to the credit crisis which started in 2007, LIBOR rates were regarded as risk-free rates. It was considered extremely unlikely that a AA-rated bank would default on a loan lasting 12 months or less. During the crisis, LIBOR rates soared and financial institutions realized that it was no longer reasonable to assume that they were risk-free rates. Since the crisis, OIS rates have been used as risk-free rates. As explained in Section 4.2, the OIS rate is a continually refreshed one-day rate. There is some chance that a creditworthy bank will default in one day, but this is considered sufficiently small to be ignored.

The three-month LIBOR–OIS spread is watched carefully by market participants as a measure of stress in financial markets. It is the amount by which three-month LIBOR exceeds the three-month OIS rate. It measures the difference between the credit risk in a three-month interbank loan and the credit risk in a series of one-day interbank loans. In normal market conditions, it is less than 15 basis points. However, it rose sharply during the credit crisis because banks became less willing to lend to each other for three-month periods. In October 2008, the spread spiked to an all time high of 364 basis points, but by the end of 2009 it had returned to more normal levels. Later it rose again as a result of concerns about the financial health of Greece and some other European countries.

4.4 MEASURING INTEREST RATES

A statement by a bank that the interest rate on one-year deposits is 10% per annum sounds straightforward and unambiguous. In fact, its precise meaning depends on the way the interest rate is measured.

If the interest rate is measured with annual compounding, the bank's statement that the interest rate is 10% means that $100 grows to

$$\$100 \times 1.1 = \$110$$

at the end of 1 year. When the interest rate is measured with semiannual compounding, it means that 5% is earned every 6 months, with the interest being reinvested. In this case, $100 grows to

$$\$100 \times 1.05 \times 1.05 = \$110.25$$

at the end of 1 year. When the interest rate is measured with quarterly compounding, the bank's statement means that 2.5% is earned every 3 months, with the interest being reinvested. The $100 then grows to

$$\$100 \times 1.025^4 = \$110.38$$

Table 4.1   Effect of the compounding frequency on the value of $100 at the end of 1 year when the interest rate is 10% per annum.

Compounding frequency	Value of $100 at end of year ($)
Annually ($m = 1$)	110.00
Semiannually ($m = 2$)	110.25
Quarterly ($m = 4$)	110.38
Monthly ($m = 12$)	110.47
Weekly ($m = 52$)	110.51
Daily ($m = 365$)	110.52

at the end of 1 year. Table 4.1 shows the effect of increasing the compounding frequency further.

The compounding frequency defines the units in which an interest rate is measured. A rate expressed with one compounding frequency can be converted into an equivalent rate with a different compounding frequency. For example, from Table 4.1 we see that 10.25% with annual compounding is equivalent to 10% with semiannual compounding. We can think of the difference between one compounding frequency and another to be analogous to the difference between kilometers and miles. They are two different units of measurement.

To generalize our results, suppose that an amount A is invested for n years at an interest rate of R per annum. If the rate is compounded once per annum, the terminal value of the investment is

$$A(1 + R)^n$$

If the rate is compounded m times per annum, the terminal value of the investment is

$$A\left(1 + \frac{R}{m}\right)^{mn} \tag{4.1}$$

When $m = 1$, the rate is sometimes referred to as the *equivalent annual interest rate*.

Continuous Compounding

The limit as the compounding frequency, m, tends to infinity is known as *continuous compounding*.[2] With continuous compounding, it can be shown that an amount A invested for n years at rate R grows to

$$Ae^{Rn} \tag{4.2}$$

where e is approximately 2.71828. The exponential function, e^x, is built into most calculators, so the computation of the expression in equation (4.2) presents no problems. In the example in Table 4.1, $A = 100$, $n = 1$, and $R = 0.1$, so that the value to which A

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[2] Actuaries sometimes refer to a continuously compounded rate as the *force of interest*.

grows with continuous compounding is

$$100e^{0.1} = \$110.52$$

This is (to two decimal places) the same as the value with daily compounding. For most practical purposes, continuous compounding can be thought of as being equivalent to daily compounding. Compounding a sum of money at a continuously compounded rate R for n years involves multiplying it by e^{Rn}. Discounting it at a continuously compounded rate R for n years involves multiplying by e^{-Rn}.

In this book, interest rates will be measured with continuous compounding except where stated otherwise. Readers used to working with interest rates that are measured with annual, semiannual, or some other compounding frequency may find this a little strange at first. However, continuously compounded interest rates are used to such a great extent in pricing derivatives that it makes sense to get used to working with them now.

Suppose that R_c is a rate of interest with continuous compounding and R_m is the equivalent rate with compounding m times per annum. From the results in equations (4.1) and (4.2), we have

$$Ae^{R_c n} = A\left(1 + \frac{R_m}{m}\right)^{mn}$$

or

$$e^{R_c} = \left(1 + \frac{R_m}{m}\right)^{m}$$

This means that

$$R_c = m \ln\left(1 + \frac{R_m}{m}\right) \tag{4.3}$$

and

$$R_m = m(e^{R_c/m} - 1) \tag{4.4}$$

These equations can be used to convert a rate with a compounding frequency of m times per annum to a continuously compounded rate and vice versa. The natural logarithm function $\ln x$, which is built into most calculators, is the inverse of the exponential function, so that, if $y = \ln x$, then $x = e^{y}$.

Example 4.1

Consider an interest rate that is quoted as 10% per annum with semiannual compounding. From equation (4.3) with $m = 2$ and $R_m = 0.1$, the equivalent rate with continuous compounding is

$$2 \ln\left(1 + \frac{0.1}{2}\right) = 0.09758$$

or 9.758% per annum.

Example 4.2

Suppose that a lender quotes the interest rate on loans as 8% per annum with continuous compounding, and that interest is actually paid quarterly. From equation (4.4) with $m = 4$ and $R_c = 0.08$, the equivalent rate with quarterly compounding is

$$4 \times (e^{0.08/4} - 1) = 0.0808$$

or 8.08% per annum. This means that on a $1,000 loan, interest payments of $20.20 would be required each quarter.

4.5   ZERO RATES

The n-year zero-coupon interest rate is the rate of interest earned on an investment that starts today and lasts for n years. All the interest and principal is realized at the end of n years. There are no intermediate payments. The n-year zero-coupon interest rate is sometimes also referred to as the n-year *spot rate*, the n-year *zero rate*, or just the n-year zero. Suppose a 5-year zero rate with continuous compounding is quoted as 5% per annum. This means that $100, if invested for 5 years, grows to

$$100 \times e^{0.05 \times 5} = 128.40$$

Most of the interest rates we observe directly in the market are not pure zero rates. Consider a 5-year risk-free bond that provides a 6% coupon. The price of this bond does not by itself determine the 5-year risk-free zero rate because some of the return on the bond is realized in the form of coupons prior to the end of year 5. Later in this chapter we will discuss how we can determine zero rates from the market prices of coupon-bearing instruments.

4.6   BOND PRICING

Most bonds pay coupons to the holder periodically. The bond's principal (which is also known as its par value or face value) is paid at the end of its life. The theoretical price of a bond can be calculated as the present value of all the cash flows that will be received by the owner of the bond. Sometimes bond traders use the same discount rate for all the cash flows underlying a bond, but a more accurate approach is to use a different zero rate for each cash flow.

To illustrate this, consider the situation where zero rates, measured with continuous compounding, are as in Table 4.2. (We explain later how these can be calculated.) Suppose that a 2-year bond with a principal of $100 provides coupons at the rate of 6% per annum semiannually. To calculate the present value of the first coupon of $3, we discount it at 5.0% for 6 months; to calculate the present value of the second coupon of $3, we discount it at 5.8% for 1 year; and so on. Therefore, the theoretical price of the bond is

$$3e^{-0.05 \times 0.5} + 3e^{-0.058 \times 1.0} + 3e^{-0.064 \times 1.5} + 103e^{-0.068 \times 2.0} = 98.39$$

or $98.39. (DerivaGem can be used to calculate bond prices.)

Table 4.2   Zero rates.

Maturity (years)	Zero rate (% continuously compounded)
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

Bond Yield

A bond's yield is the single discount rate that, when applied to all cash flows, gives a bond price equal to its market price. Suppose that the theoretical price of the bond we have been considering, $98.39, is also its market value (i.e., the market's price of the bond is in exact agreement with the data in Table 4.2). If y is the yield on the bond, expressed with continuous compounding, it must be true that

$$3e^{-y \times 0.5} + 3e^{-y \times 1.0} + 3e^{-y \times 1.5} + 103e^{-y \times 2.0} = 98.39$$

This equation can be solved using an iterative ("trial and error") procedure to give $y = 6.76\%$.[3]

Par Yield

The *par yield* for a certain bond maturity is the coupon rate that causes the bond price to equal its par value. (The par value is the same as the principal value.) Usually the bond is assumed to provide semiannual coupons. Suppose that the coupon on a 2-year bond in our example is c per annum (or $\frac{1}{2}c$ per 6 months). Using the zero rates in Table 4.2, the value of the bond is equal to its par value of 100 when

$$\frac{c}{2}e^{-0.05 \times 0.5} + \frac{c}{2}e^{-0.058 \times 1.0} + \frac{c}{2}e^{-0.064 \times 1.5} + \left(100 + \frac{c}{2}\right)e^{-0.068 \times 2.0} = 100$$

This equation can be solved in a straightforward way to give $c = 6.87$. The 2-year par yield is therefore 6.87% per annum.

More generally, if d is the present value of $1 received at the maturity of the bond, A is the value of an annuity that pays one dollar on each coupon payment date, and m is the number of coupon payments per year, then the par yield c must satisfy

$$100 = A\frac{c}{m} + 100d$$

so that

$$c = \frac{(100 - 100d)m}{A}$$

In our example, $m = 2$, $d = e^{-0.068 \times 2} = 0.87284$, and

$$A = e^{-0.05 \times 0.5} + e^{-0.058 \times 1.0} + e^{-0.064 \times 1.5} + e^{-0.068 \times 2.0} = 3.70027$$

The formula confirms that the par yield is 6.87% per annum. A bond with this coupon and semiannual payments is worth par.

4.7 DETERMINING ZERO RATES

In this section we describe a procedure known as the *bootstrap method* which can be used to determine zero rates.

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[3] One way of solving nonlinear equations of the form $f(y) = 0$, such as this one, is to use the Newton–Raphson method. We start with an estimate y_0 of the solution and produce successively better estimates $y_1, y_2, y_3, \ldots$ using the formula $y_{i+1} = y_i - f(y_i)/f'(y_i)$, where $f'(y)$ denotes the derivative of f with respect to y.

Treasury Rates

We first show how Treasury zero rates can be calculated from Treasury bill and Treasury bond prices. Consider the data in Table 4.3 on the prices of five bonds. Because the first three bonds pay no coupons, the zero rates corresponding to the maturities of these bonds can easily be calculated. The 3-month bond has the effect of turning an investment of 99.6 into 100 in 3 months. The continuously compounded 3-month rate R is therefore given by solving

$$100 = 99.6e^{R \times 0.25}$$

It is 1.603% per annum. The 6-month continuously compounded rate is similarly given by solving

$$100 = 99.0e^{R \times 0.5}$$

It is 2.010% per annum. Similarly, the 1-year rate with continuous compounding is given by solving

$$100 = 97.8e^{R \times 1.0}$$

It is 2.225% per annum.

The fourth bond lasts 1.5 years. The cash flows it provides are as follows:

6 months: $2

1 year: $2

1.5 years: $102.

From our earlier calculations, we know that the discount rate for the payment at the end of 6 months is 2.010% and that the discount rate for the payment at the end of 1 year is 2.225%. We also know that the bond's price, $102.5, must equal the present value of all the payments received by the bondholder. Suppose the 1.5-year zero rate is denoted by R. It follows that

$$2e^{-0.02010 \times 0.5} + 2e^{-0.02225 \times 1.0} + 102e^{-R \times 1.5} = 102.5$$

This reduces to

$$e^{-1.5R} = 0.96631$$

or

$$R = -\frac{\ln(0.96631)}{1.5} = 0.02284$$

Table 4.3 Data for bootstrap method.

Bond principal ($)	Time to maturity (years)	Annual coupon* ($)	Bond price ($)	Bond yield** (%)
100	0.25	0	99.6	1.6064 (Q)
100	0.50	0	99.0	2.0202 (SA)
100	1.00	0	97.8	2.2495 (A)
100	1.50	4	102.5	2.2949 (SA)
100	2.00	5	105.0	2.4238 (SA)

*Half the stated coupon is assumed to be paid every 6 months. **Compounding frequency corresponds to payment frequency: Q = quarterly, SA = semiannual, A = annual.

Table 4.4 Continuously compounded zero
rates determined from data in Table 4.3.

| Maturity (years)	Zero rate (% continuously compounded)
0.25	1.603
0.50	2.010
1.00	2.225
1.50	2.284
2.00	2.416

The 1.5-year zero rate is therefore 2.284%. This is the only zero rate that is consistent with the 6-month rate, 1-year rate, and the data in Table 4.3.

The 2-year zero rate can be calculated similarly from the 6-month, 1-year, and 1.5-year zero rates, and the information on the last bond in Table 4.3. If R is the 2-year zero rate, then

$$2.5e^{-0.02010 \times 0.5} + 2.5e^{-0.02225 \times 1.0} + 2.5e^{-0.02284 \times 1.5} + 102.5e^{-R \times 2.0} = 105$$

This gives $R = 0.02416$, or 2.416%.

The rates we have calculated are summarized in Table 4.4. A chart showing the zero rate as a function of maturity is known as the *zero curve*. A common assumption is that the zero curve is linear between the points determined using the bootstrap method. (This means that the 1.25-year zero rate is $0.5 \times 2.225 + 0.5 \times 2.284 = 2.255\%$ in our example.) It is also usually assumed that the zero curve is horizontal prior to the first point and horizontal beyond the last point. Figure 4.1 shows the zero curve for our data using these assumptions. By using longer maturity bonds, the zero curve would be more accurately determined beyond 2 years.

Figure 4.1 Zero rates given by the bootstrap method.
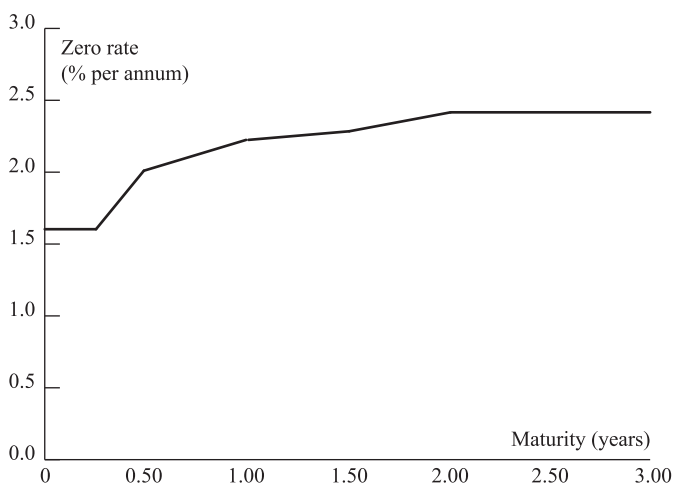
In practice, we do not usually have bonds with maturities equal to exactly 1.5 years, 2 years, 2.5 years, and so on. One approach is to interpolate between the bond price data before it is used to calculate the zero curve. For example, if it is known that a 2.3-year bond with a coupon of 6% sells for 108 and a 2.7-year bond with a coupon of 6.5% sells for 109, it might be assumed that a 2.5-year bond with a coupon of 6.25% would sell for 108.5. Another more general procedure, which we use below for the OIS example, is as follows. Define $t_1, t_2, \ldots, t_n$ as the maturities of the instruments whose prices are to be matched. Assume a piecewise linear curve with corners at these times. Use an iterative "trial and error" procedure to determine the rate at time t_1 that matches the price of the first instrument, then use a similar procedure to determine the rate at time t_2 that matches the price of the second instrument, and so on. For any trial rate, the rates used for coupons are determined by linear interpolation.

A more sophisticated approach is to use polynomial or exponential functions, rather than linear functions, for the zero curve between times t_i and t_{i+1} for all i. The functions are chosen so that they price the bonds correctly and so that the gradient of the zero curve does not change at any of the t_i. This is referred to as using a *spline function* for the zero curve.

OIS Rates

A similar bootstrap approach to that just described can be used to determine OIS zero rates. As explained in Section 4.3, OIS rates are the risk-free rates used by traders to value derivatives. An overnight indexed swap (OIS) involves exchanging a fixed rate for a floating rate. The floating rate is calculated by assuming that someone invests at the (very low risk) overnight rate, reinvesting the proceeds each day.

An OIS with a maturity of 12 months or less typically involves the fixed rate being exchanged for the floating rate just once. The (fixed) OIS rate that is exchanged for floating is therefore already a zero rate. It can therefore be treated like the three-month, six-month, and 12-month Treasury rates in Table 4.4. When an OIS has a maturity longer than 12 months, payments are usually exchanged every three months. The OIS rate can then be treated as the rate on a par yield bond.

Suppose that the fixed OIS rates that can be exchanged for floating in the market are those in Table 4.5. There is a single exchange at maturity for the first four swaps and exchanges take place every three months for the two-year and five-year swaps. The compounding frequency with which swap rates are expressed in the market reflects the

Table 4.5 OIS Rates and the calculation of the OIS zero curve.

OIS maturity	OIS rate	Compounding frequency for OIS rate	Zero rate (cont. comp.)
1 month	1.8%	Monthly	1.7987%
3 months	2.0%	Quarterly	1.9950%
6 months	2.2%	Semiannually	2.1880%
12 months	2.5%	Annually	2.4693%
2 years	3.0%	Quarterly	2.9994%
5 years	4.0%	Quarterly	4.0401%

Figure 4.2 OIS zero rates in Table 4.5.
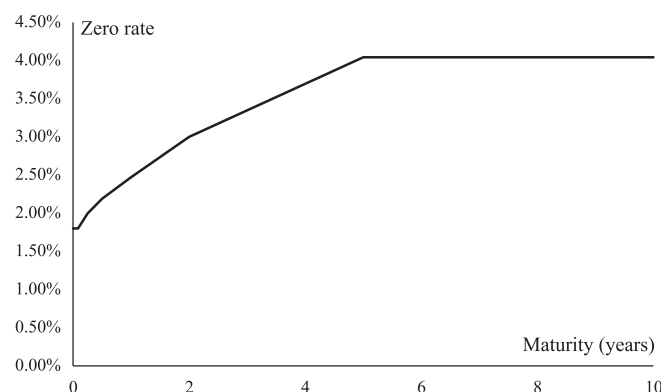
frequency of payments. The third column shows this. The one-month rate in Table 4.3 is expressed with monthly compounding; the three-month rate is expressed with quarterly compounding; and so on. We use continuous compounding for the zero rates in the final column of Table 4.5. The one-month, three-month, six-month, and 12-month OIS zero rates are the OIS rates in column 2, adjusted for the compounding frequency difference. To determine the other rates, a two-year bond with a principal of $100 paying a coupon of 3% per year ($0.75 every three months) is assumed to be worth $100; a five-year bond with a principal of $100 paying a coupon of 4% ($1 every three months) is also assumed to be worth $100; and so on. The zero rates as a function of maturity are assumed to be linear between maturities as indicated in Figure 4.2. An iterative trial and error procedure is used to determine the positions of the corners.

The calculations in Table 4.5 can be carried out by DerivaGem. As in the case of the Treasury instruments considered in Table 4.3, the calculations do not take account of complications created by statutory holidays and day count conventions.

4.8 FORWARD RATES

Forward interest rates are the rates of interest implied by current zero rates for periods of time in the future. To illustrate how they are calculated, we suppose that zero rates are as shown in the second column of Table 4.6. The rates are assumed to be continuously compounded. Thus, the 3% per annum rate for 1 year means that, in return for an investment of $100 today, an amount $100e^{0.03 \times 1} = \$103.05$ is received in 1 year; the 4% per annum rate for 2 years means that, in return for an investment of $100 today, an amount $100e^{0.04 \times 2} = \$108.33$ is received in 2 years; and so on.

The forward interest rate in Table 4.6 for year 2 is 5% per annum. This is the rate of interest that is implied by the zero rates for the period of time between the end of the first year and the end of the second year. It can be calculated from the 1-year zero interest rate of 3% per annum and the 2-year zero interest rate of 4% per annum. It is the rate of interest for year 2 that, when combined with 3% per annum for year 1, gives 4% overall for the 2 years. To show that the correct answer is 5% per annum, suppose

Table 4.6   Calculation of forward rates.

Year (n)	Zero rate for an n-year investment (% per annum)	Forward rate for nth year (% per annum)
1	3.0	
2	4.0	5.0
3	4.6	5.8
4	5.0	6.2
5	5.3	6.5

that \$100 is invested. A rate of 3% for the first year and 5% for the second year gives

$$100e^{0.03 \times 1} e^{0.05 \times 1} = \$108.33$$

at the end of the second year. A rate of 4% per annum for 2 years gives

$$100e^{0.04 \times 2}$$

which is also \$108.33. This example illustrates the general result that when interest rates are continuously compounded and rates in successive time periods are combined, the overall equivalent rate is simply the average rate during the whole period. In our example, 3% for the first year and 5% for the second year average to 4% over the 2 years. The result is only approximately true when the rates are not continuously compounded.

The forward rate for year 3 is the rate of interest that is implied by a 4% per annum 2-year zero rate and a 4.6% per annum 3-year zero rate. It is 5.8% per annum. The reason is that an investment for 2 years at 4% per annum combined with an investment for one year at 5.8% per annum gives an overall average return for the three years of 4.6% per annum. The other forward rates can be calculated similarly and are shown in the third column of the table. In general, if R_1 and R_2 are the zero rates for maturities T_1 and T_2, respectively, and R_F is the forward interest rate for the period of time between T_1 and T_2, then

$$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1} \tag{4.5}$$

To illustrate this formula, consider the calculation of the year-4 forward rate from the data in Table 4.6: $T_1 = 3$, $T_2 = 4$, $R_1 = 0.046$, and $R_2 = 0.05$, and the formula gives $R_F = 0.062$.

Equation (4.5) can be written as

$$R_F = R_2 + (R_2 - R_1)\frac{T_1}{T_2 - T_1} \tag{4.6}$$

This shows that, if the zero curve is upward sloping between T_1 and T_2 so that $R_2 > R_1$, then $R_F > R_2$ (i.e., the forward rate for a period of time ending at T_2 is greater than the T_2 zero rate). Similarly, if the zero curve is downward sloping with $R_2 < R_1$, then $R_F < R_2$ (i.e., the forward rate is less than the T_2 zero rate). Taking limits as T_2

> **Business Snapshot 4.1**   Orange County's Yield Curve Plays
>
> Suppose a large investor can borrow or lend at the rates given in Table 4.5 and thinks that 1-year interest rates will not change much over the next 5 years. The investor can borrow 1-year funds and invest for 5-years. The 1-year borrowings can be rolled over for further 1-year periods at the end of the first, second, third, and fourth years. If interest rates do stay about the same, this strategy will yield a profit of about 2.3% per year, because interest will be received at 5.3% and paid at 3%. This type of trading strategy is known as a *yield curve play*. The investor is speculating that rates in the future will be quite different from the forward rates observed in the market today. (In our example, forward rates observed in the market today for future 1-year periods are 5%, 5.8%, 6.2%, and 6.5%.)
>
>   Robert Citron, the Treasurer at Orange County, used yield curve plays similar to the one we have just described very successfully in 1992 and 1993. The profit from Mr. Citron's trades became an important contributor to Orange County's budget and he was re-elected. (No one listened to his opponent in the election, who said his trading strategy was too risky.)
>
>   In 1994 Mr. Citron expanded his yield curve plays. He invested heavily in *inverse floaters*. These pay a rate of interest equal to a fixed rate of interest minus a floating rate. He also leveraged his position by borrowing in the repo market. If short-term interest rates had remained the same or declined he would have continued to do well. As it happened, interest rates rose sharply during 1994. On December 1, 1994, Orange County announced that its investment portfolio had lost $1.5 billion and several days later it filed for bankruptcy protection.

approaches T_1 in equation (4.6) and letting the common value of the two be T, we obtain

$$R_F = R + T\frac{\partial R}{\partial T}$$

where R is the zero rate for a maturity of T. The value of R_F obtained in this way is known as the *instantaneous forward rate* for a maturity of T. This is the forward rate that is applicable to a very short future time period that begins at time T. Define $P(0, T)$ as the price of a zero-coupon bond maturing at time T. Because $P(0, T) = e^{-RT}$, the equation for the instantaneous forward rate can also be written as

$$R_F = -\frac{\partial}{\partial T}\ln P(0, T)$$

   If a large financial institution can borrow or lend at the rates in Table 4.5, it can lock in the forward rates. For example, it can borrow $100 at 3% for 1 year and invest the money at 4% for 2 years, the result is a cash outflow of $100e^{0.03 \times 1} = \$103.05$ at the end of year 1 and an inflow of $100e^{0.04 \times 2} = \$108.33$ at the end of year 2. Since $108.33 = 103.05e^{0.05}$, a return equal to the forward rate (5%) is earned on $103.05 during the second year. Alternatively, it can borrow $100 for 4 years at 5% and invest it for 3 years at 4.6%. The result is a cash inflow of $100e^{0.046 \times 3} = \$114.80$ at the end of the third year and a cash outflow of $100e^{0.05 \times 4} = \$122.14$ at the end of the fourth year. Since $122.14 = 114.80e^{0.062}$, money is being borrowed for the fourth year at the forward rate of 6.2%.

If a large investor thinks that rates in the future will be different from today's forward rates, there are many trading strategies that the investor will find attractive (see Business Snapshot 4.1). One of these involves entering into a contract known as a *forward rate agreement*. We will now discuss how this contract works and how it is valued.

4.9 FORWARD RATE AGREEMENTS

A forward rate agreement (FRA) is an over-the-counter contract designed to fix the interest rate that will apply to either borrowing or lending a certain principal amount during a specified future time period. When an FRA is first negotiated the specified interest rate usually equals the forward rate. The contract then has zero value.

Most FRAs are based on LIBOR. A trader who will borrow a certain principal amount at LIBOR for a future period can enter into an FRA where for the specified time period LIBOR will be received on the principal amount and a predetermined fixed rate will be paid on the principal amount. This converts the uncertain floating LIBOR rate to a fixed rate. If LIBOR proves to be greater (less) than the fixed rate the payoff from the FRA is positive (negative).

A trader who will earn interest at LIBOR for a future time period can similarly lock in the rate by entering into an FRA where LIBOR is paid and a fixed rate is received. Because interest is paid in arrears on loans, the payoff from an FRA is due at the end of the specified time period. Usually however, the contract is settled at the beginning of the period by a payment of the present value of the payoff.

Example 4.3

Suppose that a company enters into an FRA that is designed to ensure it will receive a fixed rate of 4% on a principal of $100 million for a 3-month period starting in 3 years. The FRA is an exchange where LIBOR is paid and 4% is received for the 3-month period. If 3-month LIBOR proves to be 4.5% for the 3-month period, the cash flow to the lender will be

$$100{,}000{,}000 \times (0.04 - 0.045) \times 0.25 = -\$125{,}000$$

at the 3.25-year point. This is equivalent to a cash flow equal to the present value of −$125,000 at the 3-year point. The cash flow to the party on the opposite side of the transaction will be +$125,000 at the 3.25-year point or the present value of this at the 3-year point. (All interest rates in this example are expressed with quarterly compounding.)

Suppose company X has agreed to lend money at LIBOR to company Y for the period of time between T_1 and T_2 and enters into an FRA to fix the rate of interest it will receive. Define:

R_K: The rate of interest agreed to in the FRA

R_F: The forward LIBOR interest rate for the period between times T_1 and T_2, calculated today[4]

---

[4] The determination of LIBOR forward rates is discussed in Chapter 7.

R_M: The actual LIBOR interest rate observed in the market at time T_1 for the period between times T_1 and T_2

L: The principal underlying the contract.

We assume that the rates R_K, R_F, and R_M are all measured with a compounding frequency reflecting the length of the period they apply to. This means that, if $T_2 - T_1 = 0.5$, they are expressed with semiannual compounding; if $T_2 - T_1 = 0.25$, they are expressed with quarterly compounding; and so on. (This assumption corresponds to the usual market practices for FRAs.)

Normally company X would earn R_M from the LIBOR loan. The FRA means that it will earn R_K. The extra interest rate (which may be negative) that it earns as a result of entering into the FRA is $R_K - R_M$. The interest rate is set at time T_1 and paid at time T_2. The extra interest rate therefore leads to a cash flow to company X at time T_2 of

$$L(R_K - R_M)(T_2 - T_1) \qquad (4.7)$$

Similarly company Y's cash flow at time T_2 (which can be negative) is

$$L(R_M - R_K)(T_2 - T_1) \qquad (4.8)$$

Equations (4.7) and (4.8) make it clear that the FRA is an agreement where company X will receive interest on the principal between T_1 and T_2 at the fixed rate of R_K and pay interest at the realized LIBOR rate of R_M. Company Y will pay interest on the principal between T_1 and T_2 at the fixed rate of R_K and receive interest at R_M. This interpretation will be important when we come to consider swaps in Chapter 7.

As mentioned, FRAs are usually settled at time T_1 rather than T_2. The payoff must then be discounted from time T_2 to T_1. For company X the payoff is the present value at time T_1 of

$$L(R_K - R_M)(T_2 - T_1)$$

received at time T_2, and for company Y the payoff is the present value at time T_1 of

$$L(R_M - R_K)(T_2 - T_1)$$

received at time T_2. The discount rate should be the risk-free rate at time T_1 for the period between T_1 and T_2.

Valuation

To value an FRA, we first note that it is always worth zero when $R_K = R_F$ and usually the contract is designed so that $R_K = R_F$ at time zero. As time passes, R_K (the agreed fixed rate) remains the same but R_F is likely to change. The contract therefore no longer has a value of zero.

To determine the value of an FRA after it has been initiated, we compare two FRAs. The first promises that the current LIBOR forward rate R_F will be received on a principal of L between times T_1 and T_2; the second promises that R_K will be received on the same principal between the same two dates. In both cases the realized LIBOR, R_M, is paid. The two contracts are the same except for the interest payments received at time T_2. In the case of the first contract the interest is $L R_F (T_2 - T_1)$; in the case of the second contract it is $L R_K (T_2 - T_1)$. The excess of the value of the second contract over the first is, therefore,

the present value of the difference between these interest payment, or

$$L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2}$$

where R_2 is the continuously compounded riskless zero rate for a maturity T_2.[5] Because the value of the first FRA (where R_F is received) is zero, the value of the second FRA (where R_K is received) is

$$V_{FRA} = L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2} \qquad (4.9)$$

Similarly, the value of an FRA which promises that an interest rate of R_K will be paid on borrowings of L between T_1 and T_2 is

$$V_{FRA} = L(R_F - R_K)(T_2 - T_1)e^{-R_2 T_2} \qquad (4.10)$$

By comparing equations (4.7) and (4.9) (or equations (4.8) and (4.10)), we see that an FRA can be valued if we:

1. Calculate the payoff on the assumption that forward rates are realized (that is, on the assumption that $R_M = R_F$)

2. Discount this payoff at the risk-free rate

We shall use this result when we come to value swaps (which are porfolios of FRAs) in Chapter 7.

Example 4.4

Suppose that the forward LIBOR rate for the period between time 1.5 years and time 2 years in the future is 5% (with semiannual compounding) and that some time ago a company entered into an FRA where it will receive 5.8% (with semi-annual compounding) and pay LIBOR on a principal of $100 million for the period. The 2-year risk-free rate is 4% (with continuous compounding). From equation (4.9), the value of the FRA is

$$100,000,000 \times (0.058 - 0.050) \times 0.5e^{-0.04 \times 2} = \$369,200$$

4.10 DURATION

The *duration* of a bond, as its name implies, is a measure of how long the holder of the bond has to wait before receiving the present value of the cash payments. A zero-coupon bond that lasts n years has a duration of n years. However, a coupon-bearing bond lasting n years has a duration of less than n years, because the holder receives some of the cash payments prior to year n.

Suppose that a bond provides the holder with cash flows c_i at time t_i ($1 \leqslant i \leqslant n$). The bond price B and bond yield y (continuously compounded) are related by

$$B = \sum_{i=1}^{n} c_i e^{-y t_i} \qquad (4.11)$$

---

[5] Note that R_K, R_M, and R_F are expressed with a compounding frequency corresponding to $T_2 - T_1$, whereas R_2 is expressed with continuous compounding.

The duration of the bond, D, is defined as

$$D = \frac{\sum_{i=1}^{n} t_i c_i e^{-y t_i}}{B} \tag{4.12}$$

This can be written

$$D = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-y t_i}}{B} \right]$$

The term in square brackets is the ratio of the present value of the cash flow at time t_i to the bond price. The bond price is the present value of all payments. The duration is therefore a weighted average of the times when payments are made, with the weight applied to time t_i being equal to the proportion of the bond's total present value provided by the cash flow at time t_i. The sum of the weights is 1.0. Note that, for the purposes of the definition of duration, all discounting is done at the bond yield rate of interest, y. (We do not use a different zero rate for each cash flow in the way described in Section 4.6.)

When a small change Δy in the yield is considered, it is approximately true that

$$\Delta B = \frac{dB}{dy} \Delta y \tag{4.13}$$

From equation (4.11), this becomes

$$\Delta B = -\Delta y \sum_{i=1}^{n} c_i t_i e^{-y t_i} \tag{4.14}$$

(Note that there is a negative relationship between B and y. When bond yields increase, bond prices decrease. When bond yields decrease, bond prices increase.) From equations (4.12) and (4.14), the key duration relationship is obtained:

$$\Delta B = -BD \, \Delta y \tag{4.15}$$

This can be written

$$\frac{\Delta B}{B} = -D \, \Delta y \tag{4.16}$$

Equation (4.16) is an approximate relationship between percentage changes in a bond price and changes in its yield. It is easy to use and is the reason why duration, first suggested by Frederick Macaulay in 1938, has become such a popular measure.

Consider a 3-year 10% coupon bond with a face value of $100. Suppose that the yield on the bond is 12% per annum with continuous compounding. This means that $y = 0.12$. Coupon payments of $5 are made every 6 months. Table 4.7 shows the calculations necessary to determine the bond's duration. The present values of the bond's cash flows, using the yield as the discount rate, are shown in column 3 (e.g., the present value of the first cash flow is $5e^{-0.12 \times 0.5} = 4.709$). The sum of the numbers in column 3 gives the bond's price as 94.213. The weights are calculated by dividing the numbers in column 3 by 94.213. The sum of the numbers in column 5 gives the duration as 2.653 years.

DV01 is the price change from a 1-basis-point increase in all rates. Gamma is the change in DV01 from a 1-basis-point increase in all rates. The following example investigates the accuracy of the duration relationship in equation (4.15).

Table 4.7   Calculation of duration.

Time (years)	Cash flow ($)	Present value	Weight	Time × weight
0.5	5	4.709	0.050	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333
Total:	130	94.213	1.000	2.653

Example 4.5

For the bond in Table 4.7, the bond price, B, is 94.213 and the duration, D, is 2.653, so that equation (4.15) gives

$$\Delta B = -94.213 \times 2.653 \times \Delta y$$

or

$$\Delta B = -249.95 \times \Delta y$$

When the yield on the bond increases by 10 basis points (= 0.1%), $\Delta y = +0.001$. The duration relationship predicts that $\Delta B = -249.95 \times 0.001 = -0.250$, so that the bond price goes down to $94.213 - 0.250 = 93.963$. How accurate is this? Valuing the bond in terms of its yield in the usual way, we find that, when the bond yield increases by 10 basis points to 12.1%, the bond price is

$$5e^{-0.121 \times 0.5} + 5e^{-0.121 \times 1.0} + 5e^{-0.121 \times 1.5} + 5e^{-0.121 \times 2.0}$$
$$+5e^{-0.121 \times 2.5} + 105e^{-0.121 \times 3.0} = 93.963$$

which is (to three decimal places) the same as that predicted by the duration relationship.

Modified Duration

The preceding analysis is based on the assumption that y is expressed with continuous compounding. If y is expressed with annual compounding, it can be shown that the approximate relationship in equation (4.15) becomes

$$\Delta B = -\frac{BD\,\Delta y}{1 + y}$$

More generally, if y is expressed with a compounding frequency of m times per year, then

$$\Delta B = -\frac{BD\,\Delta y}{1 + y/m}$$

A variable D^*, defined by

$$D^* = \frac{D}{1 + y/m}$$

is sometimes referred to as the bond's *modified duration*. It allows the duration relationship to be simplified to

$$\Delta B = -BD^* \Delta y \qquad (4.17)$$

when y is expressed with a compounding frequency of m times per year. The following example investigates the accuracy of the modified duration relationship.

Example 4.6

The bond in Table 4.6 has a price of 94.213 and a duration of 2.653. The yield, expressed with semiannual compounding is 12.3673%. The modified duration, D^*, is given by

$$D^* = \frac{2.653}{1 + 0.123673/2} = 2.499$$

From equation (4.17),

$$\Delta B = -94.213 \times 2.4985 \times \Delta y$$

or

$$\Delta B = -235.39 \times \Delta y$$

When the yield (semiannually compounded) increases by 10 basis points ($= 0.1\%$), we have $\Delta y = +0.001$. The duration relationship predicts that we expect ΔB to be $-235.39 \times 0.001 = -0.235$, so that the bond price goes down to $94.213 - 0.235 = 93.978$. How accurate is this? An exact calculation similar to that in the previous example shows that, when the bond yield (semiannually compounded) increases by 10 basis points to 12.4673%, the bond price becomes 93.978. This shows that the modified duration calculation gives good accuracy for small yield changes.

Another term that is sometimes used is *dollar duration*. This is the product of modified duration and bond price, so that $\Delta B = -D_\$ \Delta y$, where $D_\$$ is dollar duration.

Bond Portfolios

The duration, D, of a bond portfolio can be defined as a weighted average of the durations of the individual bonds in the portfolio, with the weights being proportional to the bond prices. Equations (4.15) to (4.17) then apply, with B being defined as the value of the bond portfolio. They estimate the change in the value of the bond portfolio for a small change Δy in the yields of all the bonds.

It is important to realize that, when duration is used for bond portfolios, there is an implicit assumption that the yields of all bonds will change by approximately the same amount. When the bonds have widely differing maturities, this happens only when there is a parallel shift in the zero-coupon yield curve. We should therefore interpret equations (4.15) to (4.17) as providing estimates of the impact on the price of a bond portfolio of a small parallel shift, Δy, in the zero curve.

By choosing a portfolio so that the duration of assets equals the duration of liabilities (i.e., the net duration is zero), a financial institution eliminates its exposure to small parallel shifts in the yield curve. But it is still exposed to shifts that are either large or nonparallel.

4.11 CONVEXITY

The duration relationship applies only to small changes in yields. This is illustrated in Figure 4.3, which shows the relationship between the percentage change in value and change in yield for two bond portfolios having the same duration. The gradients of the two curves are the same at the origin. This means that both bond portfolios change in value by the same percentage for small yield changes and is consistent with equation (4.16). For large yield changes, the portfolios behave differently. Portfolio X has more curvature in its relationship with yields than portfolio Y. A factor known as *convexity* measures this curvature and can be used to improve the relationship in equation (4.16).

A measure of convexity is

$$C = \frac{1}{B} \frac{d^2 B}{dy^2} = \frac{\sum_{i=1}^{n} c_i t_i^2 e^{-y t_i}}{B}$$

From Taylor series expansions, we obtain a more accurate expression than equation (4.13), given by

$$\Delta B = \frac{dB}{dy} \Delta y + \frac{1}{2} \frac{d^2 B}{dy^2} \Delta y^2 \qquad\qquad \textbf{(4.18)}$$
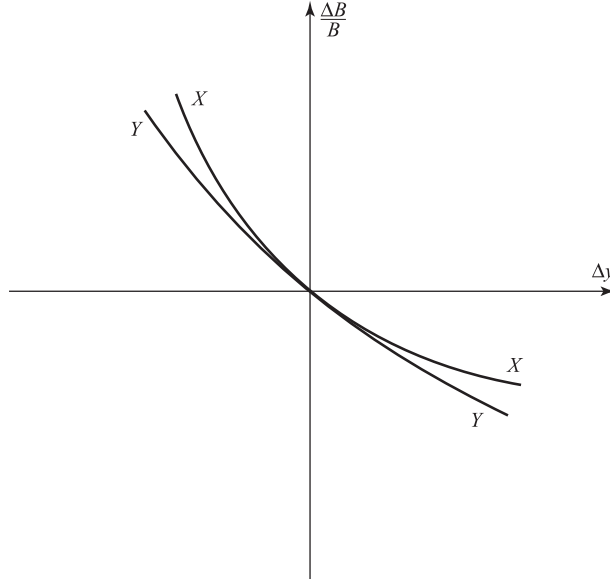
This leads to

$$\frac{\Delta B}{B} = -D\,\Delta y + \frac{1}{2} C (\Delta y)^2$$

For a portfolio with a particular duration, the convexity of a bond portfolio tends to be greatest when the portfolio provides payments evenly over a long period of time. It is

Figure 4.3 Two bond portfolios with the same duration.

least when the payments are concentrated around one particular point in time. By choosing a portfolio of assets and liabilities with a net duration of zero and a net convexity of zero, a financial institution can make itself immune to relatively large parallel shifts in the zero curve. However, it is still exposed to nonparallel shifts.

4.12 THEORIES OF THE TERM STRUCTURE OF INTEREST RATES

It is natural to ask what determines the shape of the zero curve. Why is it sometimes downward sloping, sometimes upward sloping, and sometimes partly upward sloping and partly downward sloping? A number of different theories have been proposed. The simplest is *expectations theory*, which conjectures that long-term interest rates should reflect expected future short-term interest rates. More precisely, it argues that a forward interest rate corresponding to a certain future period is equal to the expected future zero interest rate for that period. Another idea, *market segmentation theory*, conjectures that there need be no relationship between short-, medium-, and long-term interest rates. Under the theory, a major investor such as a large pension fund or an insurance company invests in bonds of a certain maturity and does not readily switch from one maturity to another. The short-term interest rate is determined by supply and demand in the short-term bond market; the medium-term interest rate is determined by supply and demand in the medium-term bond market; and so on.

The theory that is most appealing is *liquidity preference theory*. The basic assumption underlying the theory is that investors prefer to preserve their liquidity and invest funds for short periods of time. Borrowers, on the other hand, usually prefer to borrow at fixed rates for long periods of time. This leads to a situation in which forward rates are greater than expected future zero rates. The theory is also consistent with the empirical result that yield curves tend to be upward sloping more often than they are downward sloping.

The Management of Net Interest Income

To understand liquidity preference theory, it is useful to consider the interest rate risk faced by banks when they take deposits and make loans. The *net interest income* of the bank is the excess of the interest received over the interest paid and needs to be carefully managed.

Consider a simple situation where a bank offers consumers a one-year and a five-year deposit rate as well as a one-year and five-year mortgage rate. The rates are shown in Table 4.8. We make the simplifying assumption that the expected one-year interest rate for future time periods to equal the one-year rates prevailing in the market today. Loosely speaking this means that the market considers interest rate increases to be just as likely as interest rate decreases. As a result, the rates in Table 4.8 are "fair" in that

Table 4.8   Example of rates offered by a bank to its customers.

Maturity (*years*)	*Deposit rate*	*Mortgage rate*
1	3%	6%
5	3%	6%

they reflect the market's expectations (i.e., they correspond to expectations theory). Investing money for one year and reinvesting for four further one-year periods give the same expected overall return as a single five-year investment. Similarly, borrowing money for one year and refinancing each year for the next four years leads to the same expected financing costs as a single five-year loan.

Suppose you have money to deposit and agree with the prevailing view that interest rate increases are just as likely as interest rate decreases. Would you choose to deposit your money for one year at 3% per annum or for five years at 3% per annum? The chances are that you would choose one year because this gives you more financial flexibility. It ties up your funds for a shorter period of time.

Now suppose that you want a mortgage. Again you agree with the prevailing view that interest rate increases are just as likely as interest rate decreases. Would you choose a one-year mortgage at 6% or a five-year mortgage at 6%? The chances are that you would choose a five-year mortgage because it fixes your borrowing rate for the next five years and subjects you to less refinancing risk.

When the bank posts the rates shown in Table 4.8, it is likely to find that the majority of its depositors opt for one-year deposits and the majority of its borrowers opt for five-year mortgages. This creates an asset/liability mismatch for the bank and subjects it to risks. There is no problem if interest rates fall. The bank will find itself financing the five-year 6% loans with deposits that cost less than 3% in the future and net interest income will increase. However, if rates rise, the deposits that are financing these 6% loans will cost more than 3% in the future and net interest income will decline. A 3% rise in interest rates would reduce the net interest income to zero.

It is the job of the asset/liability management group to ensure that the maturities of the assets on which interest is earned and the maturities of the liabilities on which interest is paid are matched. One way it can do this is by increasing the five-year rate on both deposits and mortgages. For example, it could move to the situation in Table 4.9 where the five-year deposit rate is 4% and the five-year mortgage rate 7%. This would make five-year deposits relatively more attractive and one-year mortgages relatively more attractive. Some customers who chose one-year deposits when the rates were as in Table 4.8 will switch to five-year deposits in the Table 4.9 situation. Some customers who chose five-year mortgages when the rates were as in Table 4.8 will choose one-year mortgages. This may lead to the maturities of assets and liabilities being matched. If there is still an imbalance with depositors tending to choose a one-year maturity and borrowers a five-year maturity, five-year deposit and mortgage rates could be increased even further. Eventually the imbalance will disappear.

The net result of all banks behaving in the way we have just described is liquidity preference theory. Long-term rates tend to be higher than those that would be predicted by expected future short-term rates. The yield curve is upward sloping most

Table 4.9   Five-year rates are increased in an attempt to match maturities of assets and liabilities.

Maturity (years)	Deposit rate	Mortgage rate
1	3%	6%
5	4%	7%

> **Business Snapshot 4.2** Liquidity and the 2007–2009 Financial Crisis
>
> During the credit crisis that started in July 2007 there was a "flight to quality," where financial institutions and investors looked for safe investments and were less inclined than before to take credit risks. Financial institutions that relied on short-term funding experienced liquidity problems. One example is Northern Rock in the United Kingdom, which financed much of its mortgage portfolio with wholesale deposits, some lasting only 3 months. Starting in September 2007, the depositors became nervous and refused to roll over the funding they were providing to Northern Rock, i.e., at the end of a 3-month period they would refuse to deposit their funds for a further 3-month period. As a result, Northern Rock was unable to finance its assets. It was taken over by the U.K. government in early 2008. In the United States, financial institutions such as Bear Stearns and Lehman Brothers experienced similar liquidity problems because they had chosen to fund part of their operations with short-term funds.

of the time. It is downward sloping only when the market expects a steep decline in short-term rates.

Many banks now have sophisticated systems for monitoring the decisions being made by customers so that, when they detect small differences between the maturities of the assets and liabilities being chosen by customers they can fine tune the rates they offer. Sometimes derivatives such as interest rate swaps are also used to manage their exposure. The result of all this is that net interest income is usually very stable. This has not always been the case. In the United States, the failure of Savings and Loan companies in the 1980s and the failure of Continental Illinois in 1984 were to a large extent a result of the fact that they did not match the maturities of assets and liabilities. Both failures proved to be very expensive for U.S. taxpayers.

Liquidity

In addition to creating problems in the way that has been described, a portfolio where maturities are mismatched can lead to liquidity problems. Consider a financial institution that funds 5-year fixed rate loans with wholesale deposits that last only 3 months. It might recognize its exposure to rising interest rates and hedge its interest rate risk. (One way of doing this is by using interest rate swaps, as mentioned earlier.) However, it still has a liquidity risk. Wholesale depositors may, for some reason, lose confidence in the financial institution and refuse to continue to provide the financial institution with short-term funding. The financial institution, even if it has adequate equity capital, will then experience a severe liquidity problem that could lead to its downfall. As described in Business Snapshot 4.2, these types of liquidity problems were the root cause of some of the failures of financial institutions during the crisis that started in 2007.

SUMMARY

The compounding frequency used for an interest rate defines the units in which it is measured. The difference between an annually compounded rate and a quarterly

compounded rate is analogous to the difference between a distance measured in miles and a distance measured in kilometers. Traders frequently use continuous compounding when analyzing the value of options and more complex derivatives.

Many different types of interest rates are quoted in financial markets and calculated by analysts. The n-year zero or spot rate is the rate applicable to an investment lasting for n years when all of the return is realized at the end. The par yield on a bond of a certain maturity is the coupon rate that causes the bond to sell for its par value. Forward rates are the rates applicable to future periods of time implied by today's zero rates.

The method most commonly used to calculate zero rates is known as the bootstrap method. It involves starting with short-term instruments and moving progressively to longer-term instruments, making sure that the zero rates calculated at each stage are consistent with the prices of the instruments. It is used daily by trading desks to calculate a variety of zero curves.

A forward rate agreement (FRA) is an over-the-counter agreement where an interest rate (usually LIBOR) will be exchanged for a specified interest rate with both rates being applied to a predetermined principal over a predetermined period. An FRA can be valued by assuming that forward rates are realized and discounting the resulting payoff.

An important concept in interest rate markets is *duration*. Duration measures the sensitivity of the value of a bond portfolio to a small parallel shift in the zero-coupon yield curve. Specifically,

$$\Delta B = -BD\,\Delta y$$

where B is the value of the bond portfolio, D is the duration of the portfolio, Δy is the size of a small parallel shift in the zero curve, and ΔB is the resultant effect on the value of the bond portfolio.

Liquidity preference theory can be used to explain the interest rate term structures that are observed in practice. The theory argues that most entities like to borrow long and lend short. To match the maturities of borrowers and lenders, it is necessary for financial institutions to raise long-term rates so that forward interest rates are higher than expected future spot interest rates.

FURTHER READING

Fabozzi, F. J. *Bond Markets, Analysis, and Strategies*, 8th edn. Upper Saddle River, NJ: Pearson, 2012.

Grinblatt, M., and F. A. Longstaff. "Financial Innovation and the Role of Derivatives Securities: An Empirical Analysis of the Treasury Strips Program," *Journal of Finance*, 55, 3 (2000): 1415–36.

Jorion, P. *Big Bets Gone Bad: Derivatives and Bankruptcy in Orange County*. New York: Academic Press, 1995.

Stigum, M., and A. Crescenzi. *Money Markets*, 4th edn. New York: McGraw Hill, 2007.

Practice Questions (Answers in Solutions Manual)

4.1. A bank quotes an interest rate of 7% per annum with quarterly compounding. What is the equivalent rate with (a) continuous compounding and (b) annual compounding?

4.2. Explain how LIBOR is determined.

4.3. The 6-month and 1-year zero rates are both 5% per annum. For a bond that has a life of 18 months and pays a coupon of 4% per annum (with semiannual payments and one having just been made), the yield is 5.2% per annum. What is the bond's price? What is the 18-month zero rate? All rates are quoted with semiannual compounding.

4.4. An investor receives $1,100 in one year in return for an investment of $1,000 now. Calculate the percentage return per annum with:
   (a) Annual compounding
   (b) Semiannual compounding
   (c) Monthly compounding
   (d) Continuous compounding.

4.5. Suppose that risk-free zero interest rates with continuous compounding are as follows:

Maturity (*months*)	*Rate* (*% per annum*)
3	3.0
6	3.2
9	3.4
12	3.5
15	3.6
18	3.7

Calculate forward interest rates for the second, third, fourth, fifth, and sixth quarters.

4.6. Assuming that risk-free zero rates are as in Problem 4.5, what is the value of an FRA where the holder will pay LIBOR and receive 4.5% (quarterly compounded) for a three-month period starting in one year on a principal of $1,000,000? The forward LIBOR rate for the three-month period is 5% (quarterly compounded).

4.7. The term structure of interest rates is upward-sloping. Put the following in order of magnitude:
   (a) The 5-year zero rate
   (b) The yield on a 5-year coupon-bearing bond
   (c) The forward rate corresponding to the period between 4.75 and 5 years in the future.
   What is the answer when the term structure of interest rates is downward-sloping?

4.8. What does duration tell you about the sensitivity of a bond portfolio to interest rates. What are the limitations of the duration measure?

4.9. What rate of interest with continuous compounding is equivalent to 8% per annum with monthly compounding?

4.10. A deposit account pays 4% per annum with continuous compounding, but interest is actually paid quarterly. How much interest will be paid each quarter on a $10,000 deposit?

4.11. Suppose that 6-month, 12-month, 18-month, 24-month, and 30-month zero rates are, respectively, 4%, 4.2%, 4.4%, 4.6%, and 4.8% per annum, with continuous compounding. Estimate the cash price of a bond with a face value of 100 that will mature in 30 months and pay a coupon of 4% per annum semiannually.

4.12. A 3-year bond provides a coupon of 8% semiannually and has a cash price of 104. What is the bond's yield?

4.13. Suppose that the 6-month, 12-month, 18-month, and 24-month zero rates are 5%, 6%, 6.5%, and 7%, respectively. What is the 2-year par yield?

4.14. Suppose that risk-free zero interest rates with continuous compounding are as follows:

Maturity (years)	Rate (% per annum)
1	2.0
2	3.0
3	3.7
4	4.2
5	4.5

Calculate forward interest rates for the second, third, fourth, and fifth years.

4.15. Use the risk-free rates in Problem 4.14 to value an FRA where you will pay 5% (annually compounded) and receive LIBOR for the third year on $1 million. The forward LIBOR rate (annually compounded) for the third year is 5.5%.

4.16. A 10-year 8% coupon bond currently sells for $90. A 10-year 4% coupon bond currently sells for $80. What is the 10-year zero rate? (*Hint*: Consider taking a long position in two of the 4% coupon bonds and a short position in one of the 8% coupon bonds.)

4.17. Explain carefully why liquidity preference theory is consistent with the observation that the term structure of interest rates tends to be upward-sloping more often than it is downward-sloping.

4.18. "When the zero curve is upward-sloping, the zero rate for a particular maturity is greater than the par yield for that maturity. When the zero curve is downward-sloping the reverse is true." Explain why this is so.

4.19. Why are U.S. Treasury rates significantly lower than other rates that are close to risk-free?

4.20. Why does a loan in the repo market involve very little credit risk?

4.21. Explain why an FRA is equivalent to the exchange of a floating rate of interest for a fixed rate of interest.

4.22. A 5-year bond with a yield of 7% (continuously compounded) pays an 8% coupon at the end of each year.
(a) What is the bond's price?
(b) What is the bond's duration?
(c) Use the duration to calculate the effect on the bond's price of a 0.2% decrease in its yield.
(d) Recalculate the bond's price on the basis of a 6.8% per annum yield and verify that the result is in agreement with your answer to (c).

4.23. The cash prices of 6-month and 1-year Treasury bills are 94.0 and 89.0. A 1.5-year Treasury bond that will pay coupons of $4 every 6 months currently sells for $94.84. A 2-year Treasury bond that will pay coupons of $5 every 6 months currently sells for $97.12. Calculate the 6-month, 1-year, 1.5-year, and 2-year Treasury zero rates.

4.24. "An interest rate swap where 6-month LIBOR is exchanged for a fixed rate of 5% on a principal of $100 million for 5 years involves a known cash flow and a portfolio of nine FRAs." Explain this statement.

Further Questions

4.25. When compounded annually an interest rate is 11%. What is the rate when expressed with (a) semiannual compounding, (b) quarterly compounding, (c) monthly compounding, (d) weekly compounding, and (e) daily compounding.

4.26. The table below gives Treasury zero rates and cash flows on a Treasury bond. Zero rates are continuously compounded.
   (a) What is the bond's theoretical price?
   (b) What is the bond's yield assuming it sells for its theoretical price?

Maturity (years)	Zero rate	Coupon payment	Principal
0.5	2.0%	$20	
1.0	2.3%	$20	
1.5	2.7%	$20	
2.0	3.2%	$20	$1,000

4.27. A 5-year bond provides a coupon of 5% per annum payable semiannually. Its price is 104. What is the bond's yield? You may find Excel's Solver useful.

4.28. Suppose that 3-month, 6-month, 12-month, 2-year, and 3-year OIS rates are 2.0%, 2.5%, 3.2%, 4.5%, and 5%, respectively. The 3-month, 6-month, and 12-month OISs involve a single exchange at maturity; the 2-year and 3-year OISs involve quarterly exchanges. The compounding frequencies used for expressing the rates correspond to the frequency of exchanges. Calculate the OIS zero rates using continuous compounding. Interpolate between continuously compounded rates linearly to determine rates between 6 months and 12 months, between 12 months and 2 years, and between 2 years and 3 years. You may find Excel's Solver useful.

4.29. An interest rate is quoted as 5% per annum with semiannual compounding. What is the equivalent rate with (a) annual compounding, (b) monthly compounding, and (c) continuous compounding.

4.30. The 6-month, 12-month, 18-month, and 24-month zero rates are 4%, 4.5%, 4.75%, and 5%, with semiannual compounding.
   (a) What are the rates with continuous compounding?
   (b) What is the forward rate for the 6-month period beginning in 18 months?
   (c) What is the two-year par yield?

4.31. Suppose the risk-free rates are as in Problem 4.30. What is the value of an FRA where the holder pays LIBOR and receives 7% (semiannually compounded) for a six-month period beginning in 18 months? The current forward rate for this period is 6% (semiannually compounded) and the principal is $10 million.

4.32. The following table gives the prices of bonds:

Bond principal ($)	Time to maturity (years)	Annual coupon* ($)	Bond price ($)
100	0.50	0.0	98
100	1.00	0.0	95
100	1.50	6.2	101
100	2.00	8.0	104

* Half the stated coupon is assumed to be paid every six months.

(a) Calculate zero rates for maturities of 6 months, 12 months, 18 months, and 24 months.

(b) What are the forward rates for the following periods: 6 months to 12 months, 12 months to 18 months, and 18 months to 24 months?

(c) What are the 6-month, 12-month, 18-month, and 24-month par yields for bonds that provide semiannual coupon payments?

(d) Estimate the price and yield of a 2-year bond providing a semiannual coupon of 7% per annum.

4.33. Portfolio A consists of a 1-year zero-coupon bond with a face value of $2,000 and a 10-year zero-coupon bond with a face value of $6,000. Portfolio B consists of a 5.95-year zero-coupon bond with a face value of $5,000. The current yield on all bonds is 10% per annum.

(a) Show that both portfolios have the same duration.

(b) Show that the percentage changes in the values of the two portfolios for a 0.1% per annum increase in yields are the same.

(c) What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?

4.34. Verify that DerivaGem agrees with the price of the bond in Section 4.6. Test how well DV01 predicts the effect of a 1-basis-point increase in all rates. Estimate the duration of the bond from DV01. Use DV01 and Gamma to predict the effect of a 200-basis-point increase in all rates. Use Gamma to estimate the bond's convexity. (*Hint*: In DerivaGem, DV01 is dB/dy, where B is the price of the bond and y is its yield measured in basis points, and Gamma is d^2B/dy^2, where y is measured in percent.)