Forwards & Futures

We will now consider two standard types of contracts - namely, so called forwards and futures. As with all contracts they allow you to hedge (or speculate) on the evolving price of a risky asset. However, compared to what we've seen before, they involve a different "payment schem

Forwards

We consider a general multi-period (discrete) market model as set up before (cf. Lecture 5). We suppose that it is free of arbitrage and complete (i.e. there is a unique $\mathbb{Q} \sim \mathbb{P}$ such that $\tilde{S}_t$ is a martingale under $\mathbb{Q}$) and denote the fair price at time t of a claim X by $\Pi_t(X)$ recall that it is given by $\Pi_t(X) = S^0_t \, \mathbb{E}^{\mathbb{Q}}[X/S^0_T \mid \mathcal{F}_t]$.

One typical agreement is thus that at time t, the buyer pays the price $\Pi_t(X)$ to the seller and at time T the buyer receives X from the seller. In a forward contract written on X one agrees instead on a price which is to be paid at time T (although decided on at time t). No transactions are to take place at time t. Hence, we define as follows:

Def The forward price at time t of a claim X to be delivered at time T, denoted by $f(t;T,X)$, is the $\mathcal{F}_t$-measurable random variable for which

$$\Pi_t\big(X - f(t;T,X)\big) = 0.$$

Prop The forward price $f(t;T,X)$ of a claim X is given by

$$f(t;T,X) = \frac{\Pi_t(X)}{p(t,T)},$$

where $p(t,T) := \Pi_t(1)$.

Proof We have that

$$\Pi_t(X - f(t;T,X)) = S^0_t \, \mathbb{E}^{\Phi}\left[\frac{X - f(t;T,X)}{S^0_T} \Big| \mathcal{F}_t\right]$$

$$= S^0_t \, \mathbb{E}^{\Phi}\left(\frac{X}{S^0_T} \Big| \mathcal{F}_t\right) - S^0_t f(t;T,X) \, \mathbb{E}^{\Phi}\left(\frac{1}{S^0_T} \Big| \mathcal{F}_t\right)$$

$$= \Pi_t(X) - f(t;T,X)\, p(t,T),$$

where we used that $f(t;T,X)$ is $\mathcal{F}_t$-meas. The result follows by the definition of $f(t;T,X)$. $\square$

Ex) The forward price of the underlying asset itself is given by $f(t;T,S^i_T) = \Pi_t(S^i_T)/p(t,T) = S^i_t/p(t,T)$, $i=1,\dots,D$.

Rem Fix $t<T$ and a claim X. Let $t<u<T$. It is important to distinguish between the following two prices:

- The forward price $f(u;T,X)$ which is to be paid at time T to the seller of a forward contract entered at time u.
- The price at time u of a forward contract entered at time t (with time of delivery T). This price is given by

$$\Pi_u(X - f(t;T,X)) = \Pi_u(X) - p(u,T)\, f(t;T,X).$$

Futures

Yet a different type of contract that allows you to hedge (or speculate) using the price movements of an underlying risky asset, is the futures contract.

Similarly to a forward contract, it costs nothing to enter. However, in contrast, you are here obliged/allowed to pay/receive the debts/profit over time. That is, expected profits/losses are regulated on a running basis and not postponed until the terminal date. Specifically, a futures contract written on an underlying claim X with maturity T is governed by the futures price $F(t;T,X)$ and if the contract is entered at time $t=0$ it involves the following payments:

- $t=0$: contract entered - no payments
- $t=1,\dots,T$: the difference $\Delta F(t;T,X) := F(t;T,X) - F(t-1;T,X)$ is payed. (it can be negative).
- $t=T$: $F(T;T,X)$ is payed and X received.

We consider the same market model as above, the futures price is then defined as follows:

Def Given a claim X, a futures price process is an adapted process $F(t;T,X)$ such that $F(T;T,X) = X$ and at each $t<T$, the value of all the upcoming payments (from $t+1$ onwards) equals zero.

Rem • From the definition it follows that the exchange of $F(T;T,X)$ for X at time T has no value and can be omitted.

• From the definition it follows that the cost of buying (i.e. entering) into a futures contract at any time $t<T$ (after $\Delta F(t;T,X)$ has been settled) is zero.

Prop Suppose that $(S_t^0)_{t=0,\dots,T}$ is predictable (i.e. S_t^0 is $\mathcal{F}_{t-1}$-meas.). Then given a claim X, its future price process is given by

$$F(t;T,X) = \mathbb{E}^{Q}\{X \mid \mathcal{F}_t\}, \quad t=0,\dots,T.$$

Proof Note first that $F(t;T,X)$ thus defined satisfies the definition; indeed, clearly $F(T;T,X) = \mathbb{E}^{Q}\{X\mid\mathcal{F}_T\} = X$ and for $t<T$,

$$\text{"value at time } t \text{ of all upcoming payments"} = \sum_{i=t+1}^{T} \text{"value at time } t \text{ of } \Delta F(i;T,X) \text{ being paid at time } i\text{"}$$

$$= \sum_{i=t+1}^{T} S_t^0\, \mathbb{E}^{Q}\left\{\frac{F(i;T,X)-F(i-1;T,X)}{S_i^0}\,\middle|\,\mathcal{F}_t\right\}$$

$$= \sum_{i=t+1}^{T} S_t^0\, \mathbb{E}^{Q}\Big\{\frac{1}{S_i^0}\underbrace{\mathbb{E}^{Q}\{F(i;T,X)-F(i-1;T,X)\mid\mathcal{F}_{i-1}\}}_{=0}\,\Big|\,\mathcal{F}_t\Big\}=0.$$

Conversely, suppose that $F(t;T,X)$ satisfies the definition. Define

$$I_t = \sum_{i=1}^{t} \frac{\Delta F(i;T,X)}{S_i^0}.$$

Then, "value of all upcoming payments at time t" $= S_t^0\,\mathbb{E}^{Q}\{I_T - I_t \mid \mathcal{F}_t\} = 0$

Hence, I_t is a Q-mtg. Since

$$F(t;T,X) = \sum_{i=1}^{t} S_i^0 (I_i - I_{i-1}) + F(0;T,X)$$

it follows that also $F(t;T,X)$ is a Q-mtg. Hence,

$$F(t;T,X) = \mathbb{E}^{Q}\{F(T;T,X)\mid\mathcal{F}_t\} = \mathbb{E}^{Q}\{X\mid\mathcal{F}_t\}. \qquad \square$$

Prop If $(S^0_t)_{t=0,\dots,T}$ is deterministic, then the forward and future price processes coincide; that is

$$f(t;T,X) = F(t;T,X) = \mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{F}_t].$$

Proof If (S^0_t) is deterministic, then

$$f(t;T,X) = \frac{\Pi_t(X)}{p(t,T)} = \frac{S^0_t \, \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S^0_T} \mid \mathcal{F}_t\right]}{S^0_t \, \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{S^0_T} \mid \mathcal{F}_t\right]} = \frac{\mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{F}_t]}{\mathbb{E}^{\mathbb{Q}}[1 \mid \mathcal{F}_t]} = \mathbb{E}^{\mathbb{Q}}[X \mid \mathcal{F}_t].$$

□

Rem • In reality, futures are very common. In particular when it comes to hedging/speculating on the prices of eg. oil, meat and corn, people typically use futures rather than actually buying the assets themselves.

Black's formula for options on futures

We now consider the Black Scholes model as set up before. We recall that this is a continuous-time model; however, one can show that also in continuous time, forward and future prices coincide when the values of the riskless asset are deterministic (as they are for the Black Scholes model); moreover, the definition of a forward contract is exactly as for the discrete case.

We then have the following well-known formula:

Prop (Black's -76 formula): Consider a European Call Option with strike K and maturity T written on a futures contract on the underlying asset S with delivery $T_1 > T$; that is:

$$\text{payoff at } T: \quad X = \left(F(T;T_1,S_{T_1}) - K\right)_+ .$$

Then the price at time $t<T$ is given by

$$\Pi_t(X) = e^{-r(T-t)}\left(F(t;T_1,S_{T_1})\Phi(d_1(t)) - K\Phi(d_2(t))\right),$$

where Φ is the CDF for the $N(0,1)$-distribution and

$$\begin{cases} d_1(t) = \dfrac{1}{\sigma\sqrt{T-t}}\left(\ln \dfrac{F(t;T_1,S_{T_1})}{K} + \sigma^2/2\,(T-t)\right) \\ d_2(t) = \dfrac{1}{\sigma\sqrt{T-t}}\left(\ln \dfrac{F(t;T_1,S_{T_1})}{K} - \sigma^2/2\,(T-t)\right). \end{cases}$$

Proof The forward, and thus futures, price satisfies

$$\Pi_t\left(S_{T_1} - F(t;T_1,S_{T_1})\right) = 0 \implies F(t;T_1,S_{T_1}) = \frac{\Pi_t(S_{T_1})}{\Pi_t(1)} = \frac{S_t}{e^{-r(T_1-t)}}.$$

Hence,

$$X = \left(e^{r(T_1-T)}S_T - K\right)_+ = e^{r(T_1-T)}\left(S_T - e^{-r(T_1-T)}K\right)_+ ;$$

we may thus see this as $e^{r(T_1-T)}$ "usual" call options with strike $e^{-r(T_1-T)}K$ and apply BS-formula to those. We get

$$\Pi_t(X) = e^{r(T_1-T)}\left(S_t\Phi(d_1(t,S_t)) - e^{-r(T-t)}e^{-r(T_1-T)}K\Phi(d_2(t,S_t))\right)$$

$$= e^{-r(T-t)}\left(F(t;T_1,S_{T_1})\Phi(d_1(t,S_t)) - K\Phi(d_2(t,S_t))\right),$$

where

$$\begin{cases} d_1(t,S_t) = \dfrac{1}{\sigma\sqrt{T-t}}\left(\overbrace{\ln\dfrac{S_t}{e^{-r(T_1-T)}K}}^{=\ln e^{-r(T-t)}F(t;T_1,S_{T_1})/K = -r(T-t) + \ln F(t;T_1,S_{T_1})/K} + (r+\sigma^2/2)(T-t)\right) = \dfrac{1}{\sigma\sqrt{T-t}}\left(\ln\dfrac{F(t;T_1,S_{T_1})}{K} + \sigma^2/2\,(T-t)\right) \\ d_2(t,S_t) = d_1(t,S_t) - \sigma\sqrt{T-t} \end{cases}$$

$\square$