SABR MODEL CONVEXITY ADJUSTMENT FOR THE VALUATION OF AN ARITHMETIC AVERAGE RFR SWAP

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Abstract

A model independent convexity adjustment formula is derived for the valuation of a discrete arithmetic average interest rate swap that references an RFR index. The convexity adjustment is expressed as a symmetric quadratic swap on the canonical daily compounded RFR index. Our results are then specialised to the case of the SABR model for an RFR index, where we provide closed-form valuation of the convexity adjustment consistent with the volatility smile of market traded vanilla RFR caps and floors.

1 Introduction

Daily compounding of an overnight rate is fundamental to an interest rate benchmark regime centered on backward-looking RFR (Risk Free Rate) indices, for example SOFR (Secured Overnight Financing Rate) in the USD (United States Dollar) fixed income market. Daily compounding of an RFR index produces the correct economic value of an investment that accrues daily interest at the RFR rate over a term period. RFR-linked instruments have increased in importance as a result of LIBOR cessation, for example the replacement of LIBOR (London Interbank Offered Rate) with SOFR in the USD interest rate derivatives market.

Interest accrual at a simple *discrete arithmetic average* of an RFR index over a term period is more transparent, particularly to corporate end-users, than the theoretically correct method of daily compounding. Regulators have acknowledged the intuitive appeal of the arithmetic average convention, and have approved its use in client transactions. For example, the Alternative Reference Rates Committee (ARRC) in its guide for the use of SOFR states [1, Page 9]: "*The ARRC has recognized that either convention can be used and that the choice will depend on the specifics of the product, including trading and other conventions that may interact with the choice of interest accrual.*"

The arithmetic average convention introduces a convexity adjustment *relative* to the instrument that uses the canonical daily compounding convention. Any convexity adjustment for the arithmetic average product must be computed in a manner consistent with the instrument that does not require a convexity adjustment, so that valuation is arbitrage-free and portfolio risk can be consolidated.

The objective of this paper is to derive the convexity adjustment required for the valuation of a discrete arithmetic average RFR IRS (Interest Rate Swap). We first derive a model independent convexity adjustment that involves a *symmetric quadratic swap* on the canonical daily compounded RFR index. Takada [8] has derived the convexity adjustment for an arithmetic average IRS on an RFR index (USD Fed Funds) by the Hull-White model and by the Carr-Madan [2] replication formula. We go beyond Takada [8] to provide a *closed-form* formula for the convexity adjustment of an arithmetic average RFR IRS by use of the SABR model with time dependent coefficients [3] calibrated

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to market traded RFR Caps and Floors. Derivation of an analytic formula for the convexity adjustment is of significant practical benefit. An IRS can be decomposed into a portfolio of Swaplets. Each constituent of the portfolio requires computation of a *separate* convexity adjustment, which renders the replication method impractical. Use of the market standard SABR model facilitates consistency with the vanilla instruments, and leads to efficient consolidation of risk at a portfolio level.

2 Model setup

In this section, we introduce the core elements of our notation and model framework. The variable t will denote the current date expressed as an ACTUAL/ACTUAL year fraction offset, with the origin assigned at $t = 0$ for trade inception. Information available at t in our arbitrage-free economy will be modelled by the filtration $\mathcal{F}(t)$ [5, Section 7.3.1], where $\mathcal{F}(t) \supseteq \mathcal{F}(s)$ for $t \geq s$. Instruments within the scope of our analysis have full bilateral collateralisation with zero threshold and no minimum transfer amount, at the same RFR referenced in the payoff function for the relevant RFR IRS. Throughout this paper, unless otherwise stated, equality of two random variables is understood as $P - a.s.$ (probability almost surely).

A fixed verse float IRS can be decomposed into a portfolio of Swaplets, provided that (as in the case of RFR markets) fixed and float legs have the same payment frequency. Consequently, we will base our analysis on a single term period T_s to T_e, where $0 \leq T_s < T_e$. Business days d_i for the period $[T_s, T_e]$ will be arranged into the increasing sequence $\mathcal{T}$, where

$$\mathcal{T} : T_s = d_0 < d_1 < \cdots < d_N = T_e \tag{1}$$

for some $N \geq 1$ that is determined by the relevant business centre calendar(s), for example a SOFR IRS uses the US federal government bond market calendar. Denote by $\delta(d_{i-1}, d_i)$ the day count fraction for the period d_{i-1} to d_i, and by R_i the *realised* RFR for this period, where $i = 1, 2, \cdots, N$. The RFR for the period d_{i-1} to d_i will fix at $t = d_{i-1}$, be published at $t = d_i$ and contribute the value $\delta(d_{i-1}, d_i)R_i$ to the arithmetic average for the accrual period T_s to T_e. In the continuous-time limit, $\delta(d_{i-1}, d_i)R_i \to r(u)du$, where $r(u)$ is the short-rate for the infinitesimal period $[u, u + du]$.

An arithmetic average RFR IRS for the period $[T_s, T_e]$ will settle against the index $\mathcal{I}(R_a; T_e, T_e)$,

$$\mathcal{I}(R_a; T_e, T_e) \equiv \frac{1}{\delta(T_s, T_e)} \sum_{i=1}^{N} \delta(d_{i-1}, d_i) R_i. \tag{2}$$

Notation $\mathcal{I}(A; B, C)$ designates the value of an interest rate benchmark index A with $t = B$ the date the index is measurable and $t = C$ the index maturity date. For a backward-looking index $B = C$, while for a forward-looking index $B < C$. By contrast to (2), the index $\mathcal{I}(R_c; T_e, T_e)$ that involves the canonical daily compounding of an RFR over the term period $[T_s, T_e]$ is

$$\mathcal{I}(R_c; T_e, T_e) \equiv \frac{1}{\delta(T_s, T_e)} \left(\prod_{i=1}^{N} [1 + \delta(d_{i-1}, d_i) R_i] - 1 \right). \tag{3}$$

When we wish to distinguish a particular quantity as being either arithmetic average or daily compounded average, we will use the subscript a (arithmetic) or c (compounded). Continuous-time analogues of (2) and (3) will be denoted as $A_a(T_s, T_e)$ and $A_c(T_s, T_e)$, respectively, and are expressed in terms of the short-rate $r(u)$ as

$$A_a(T_s, T_e) = \frac{1}{\delta(T_s, T_e)} \int_{T_s}^{T_e} r(u) du, \quad A_c(T_s, T_e) = \frac{1}{\delta(T_s, T_e)} \left[e^{\int_{T_s}^{T_e} r(u) du} - 1 \right]. \tag{4}$$

Cash flows from an IRS that reference (2) or (3) do not usually occur at the end $t = T_e$ of the term period. Market convention is a payment lag (up to two business days) to facilitate smooth calculation and settlement of payments. We use the symbol λ to denote the market standard payment lag. The payoff function that we examine in this paper assumes that fixed and float legs have the same day counts, occurs at $t = T_e + \lambda$, and has the form

$$\pi(T_e + \lambda, K; \varpi) = \varpi\delta(T_s, T_e)\Big(\mathcal{I}(R_a; T_e, T_e) - K\Big). \tag{5}$$

The parameter K in (5) is the fixed coupon for the IRS. Symbol $\varpi = +1$ designates a *payer* (pay fixed) IRS, while $\varpi = -1$ signifies a *receiver* (receive fixed) IRS.

To accommodate the case when $T_s < t < T_e$, we let $n := \min\{i = 0, 1, \cdots, N : t < d_i\}$ denote the location of a given $t \in [0, T_e)$ within the tenor structure $\mathcal{T}$ of (1). In our notation, n is the number of known RFR fixings, for example $n = 0$ means that t is before the start of the averaging period at T_s, while $n = N$ means that all fixings are measurable. In terms of n, we find that (5) becomes

$$\pi(T_e + \lambda, K; \varpi) = \varpi\delta(T_s, T_e)\Big(\frac{1}{\delta(T_s, T_e)}\sum_{i=n+1}^{N}\delta(d_{i-1}, d_i)R_i - K_n\Big). \tag{6}$$

The summation term in (6) is stochastic, while the $\mathcal{F}(t)$-measurable random variable K_n is the deal strike K adjusted for the n known fixings at t according to the formula

$$K_n = K - \frac{1}{\delta(T_s, T_e)}\sum_{j=1}^{n}\delta(d_{j-1}, d_j)R_j. \tag{7}$$

We use the convention $\sum_a^b[\cdots] = 0$, when $a > b$ to preserve the boundary cases at $n = 0$ in (7) and $n = N$ in (6), respectively.

Equation (4) immediately implies that $A_a(T_s, T_e) = \ln[1 + \delta(T_s, T_e)A_c(T_s, T_e)]/\delta(T_s, T_e)$. We will now demonstrate that this formula is also valid in discrete-time for the market payoffs (2) and (3). In practice, $\delta(d_{i-1}, d_i) = 1/360$ or $1/365$ for business day intervals d_{i-1} to d_i that span weekdays with no holidays in between, while $\delta(d_{i-1}, d_i) = 3/360$ or $3/365$ for a period that spans a Friday that is a business day to a Monday that is also a business day; the factor 360 or 365 will be set by a known market convention. Irrespective of the market convention to compute $\delta(d_{i-1}, d_i)$, we have the strong inequality $\delta^2(d_{i-1}, d_i) \ll \delta(d_{i-1}, d_i)$. In terms of $\mathcal{O}$, the *Bachmann-Landau* order symbol, we obtain

$$e^{\delta(d_{i-1}, d_i)R_i} = 1 + \delta(d_{i-1}, d_i)R_i + \mathcal{O}\left(\delta^2(d_{i-1}, d_i)\right), \quad i = 1, \cdots, N, \tag{8}$$

an approximation that facilitates a connection between the discrete-time indices (2) and (3). The product of the sequence from $i = n + 1$ to $i = N$ when we treat (8) as an equality produces essentially the formula derived previously for the continuous case

$$\exp\left(\sum_{i=n+1}^{N}\delta(d_{i-1}, d_i)R_i\right) = \prod_{i=n+1}^{N}\left[1 + \delta(d_{i-1}, d_i)R_i\right], \tag{9}$$

but now connects discrete-time arithmetic average and daily compounding. Takada [8] first derived (9) in the case $n = 0$, and found it to be robust and accurate in a wide range of interest rate scenarios.

Define the backward-looking index $\mathcal{I}(R_n; T_e; T_e)$ as a generalisation of (3) through the formula

$$\mathcal{I}(R_n; T_e, T_e) := \begin{cases} \frac{1}{\delta(d_n, T_e)}\left(\prod_{i=n+1}^{N}\left[1 + \delta(d_{i-1}, d_i)R_i\right] - 1\right), & n = 0, 1, \cdots, N-1, \\ 0, \quad n = N. \end{cases} \tag{10}$$

Use of (9) and (10) to eliminate the summation term from (6) in favour of $\mathcal{I}(R_n; T_e, T_e)$ yields

$$\pi(T_e + \lambda, K; \varpi) = \varpi\delta(T_s, T_e)\Big(G\big(\mathcal{I}(R_n; T_e, T_e)\big) - K_n\Big), \tag{11}$$

where the deterministic function $G : \mathcal{D} = (-1/\delta(d_n, T_e), \infty) \mapsto \mathbb{R}$ is defined as

$$G(x) = \frac{1}{\delta(T_s, T_e)}\ln\big(1 + \delta(d_n, T_e)x\big). \tag{12}$$

The payoff function $\pi(T_e+\lambda, K; \varpi)$ as expressed by (11) and (12) is a *non-linear* function of $\mathcal{I}(R_n; T_e, T_e)$, which naturally leads to a convexity adjustment that we examine in the next section.

3 Model independent convexity adjustment

Define the $\mathcal{F}(t)$-measurable forward rate $R(t)$ for (10) through the conditional expectation

$$R(t) = \mathbb{E}^{T_e}\Big[\mathcal{I}(R_n; T_e, T_e)\big|\mathcal{F}(t)\Big] \tag{13}$$

in the $\mathbb{Q}^{T_e}$-forward measure, where the numéraire is the T_e-maturity riskless zero coupon bond $P(*; T_e)$. Three model independent properties characterise (13): 1) $R(t)$ is a $\mathbb{Q}^{T_e}$-martingale; 2) the convergence property $R(t) \to \mathcal{I}(R_n; T_e, T_e)\ P - a.s$ as $t \uparrow T_e$; 3) the replication formula [7, Equations (6)-(8)]

$$R(t) = \frac{1}{\delta(d_n, T_e)}\left(\frac{P(t; d_n)}{P(t; T_e)} - 1\right). \tag{14}$$

In (14), $\{P(t; T) : t \geq T\}$ represent deterministic discount factors calculated from a time t discount curve that is consistent with the Principal Alignment Interest of our IRS. Through use of the convergence property for $R(t)$, we can express (11) as

$$\pi(T_e + \lambda, K; \varpi) = \varpi\delta(T_s, T_e)\Big(G\big(R(T_e)\big) - K_n\Big). \tag{15}$$

Let $V(t)$ denote the time t value of the payoff defined by (15). By continuous time Arbitrage Pricing Theory [5, Chapter 7], we have the valuation formula

$$V(t) = P(t; T_e + \lambda)\mathbb{E}^{T_e+\lambda}\Big[\pi(T_e + \lambda, K; \varpi)\big|\mathcal{F}(t)\Big], \tag{16}$$

where conditional expectation in (16) is over the $\mathbb{Q}^{T_e+\lambda}$-forward measure with numéraire $P(*; T_e + \lambda)$. Forward measures $\mathbb{Q}^{T_e+\lambda}$ and $\mathbb{Q}^{T_e}$ are connected through the Radon-Nikodým derivative ζ, where [5, Theorem 7.39]

$$\zeta(t) = \frac{d\mathbb{Q}^{T_e+\lambda}}{d\mathbb{Q}^{T_e}}\Big|_{\mathcal{F}(t)} = \frac{P(t; T_e + \lambda)/P(0; T_e + \lambda)}{P(t; T_e)/P(0; T_e)}. \tag{17}$$

Conditional change of measure when (17) is introduced into (16) yields

$$V(t) = P(t; T_e + \lambda)\mathbb{E}^{T_e}\Big[\frac{\zeta(T_e)}{\zeta(t)}\pi(T_e + \lambda, K; \varpi)\big|\mathcal{F}(t)\Big]. \tag{18}$$

Expanding the expectation in (18) into a product of expectations, then neglecting the covariance term (because $\zeta(T_e)/\zeta(t)$ varies slowly) and finally using the fact that ζ is a $\mathbb{Q}^{T_e}$-martingale we obtain

$$V(t) = P(t; T_e + \lambda)\mathbb{E}^{T_e}\Big[\pi(T_e + \lambda, K; \varpi)\big|\mathcal{F}(t)\Big]. \tag{19}$$

Equation (19) is equivalent to the replacement of $P(T_e; T_e + \lambda)$ in (18) with its forward value at the current date t.

From use of (15), valuation formula (19) acquires the specialised form $V(t) \equiv V_a^{\mathrm{S}}(t, K; \varpi)$, where

$$V_a^{\mathrm{S}}(t, K; \varpi) = \varpi\delta(T_s, T_e)P(t; T_e + \lambda)\Big(\mathbb{E}^{T_e}\big[G\left(R(T_e)\right)\big|\mathcal{F}(t)\big] - K_n\Big). \tag{20}$$

To compute the expectation in (20), we first expand $G(R(T_e))$ in *distribution* (expectation in the $\mathbb{Q}^{T_e}$-forward measure) to second order around the forward rate (13) to obtain

$$\mathbb{E}^{T_e}\big[G\left(R(T_e)\right)\big|\mathcal{F}(t)\big] \simeq G\left(R(t)\right) + \frac{1}{2}G''\left(R(t)\right)\mathbb{E}^{T_e}\big[\left(R(T_e) - R(t)\right)^2\big|\mathcal{F}(t)\big], \tag{21}$$

where on account of (12) derivatives in (21) are computed from the formulae

$$G'(x) = \frac{\delta(d_n, T_e)}{\delta(T_s, T_e)}\Delta(x), \quad G''(x) = -\frac{\delta(d_n, T_e)^2}{\delta(T_s, T_e)}\Delta(x)^2, \quad \Delta(x) = \frac{1}{1 + \delta(d_n, T_e)x}. \tag{22}$$

Appendix A presents a justification for (21), henceforth we consider it an equality.

Let $V^{\mathrm{QS}}(t,\kappa)$ denote the time t forward value to payoff date T_e of a *symmetric quadratic swap* with fixed leg κ. The mathematical definition of $V^{\mathrm{QS}}(t,\kappa)$ is

$$V^{\mathrm{QS}}(t,\kappa) := \mathbb{E}^{T_e}\big[\,(R(T_e)-\kappa)^2\big|\mathcal{F}(t)\big]. \tag{23}$$

When we introduce (23) into (21), then use the resultant equation in (20), we obtain

$$V_a^{\mathrm{S}}(t,K;\varpi) = \varpi\delta(T_s,T_e)P(t;T_e+\lambda)\Big\{G(R(t)) - K_n + \tfrac{1}{2}G''\left(R(t)\right)V^{\mathrm{QS}}(t,R(t))\Big\}. \tag{24}$$

The valuation formula (24) is model independent. Quantities $G(R(t))$ and $G''\left(R(t)\right)$ can be computed from the calibrated yield curve at the current date t. In the next section, we assign model dynamics to the forward rate $R(t)$ and use these to compute $V^{\mathrm{QS}}(t,R(t))$ in closed-form.

4 Specialisation to the SABR model

The Carr-Madan replication formula [2] can be used to compute (23), but with two significant disadvantages: 1) separate computations of $V^{\mathrm{QS}}(t,R(t))$ are required for each Swaplet, which makes valuation and risk scenarios computationally expensive; 2) replication requires vanilla options on R at strikes significantly far from the ATM (At the Money) level, and consequently introduces illiquid high-strike Vega risk. Analogous comments also apply to (20). We present a closed-form analytic formula for $V^{\mathrm{QS}}(t,R(t))$ that avoids the implementation difficulties of the replication methodology.

Let $t \le u \le T_e$ for a given t, where u will now act as a temporal variable, and the index n will be known. The forward rate $R(u)$ will diffuse from its initial state $R(t)$ through the dimensionless SABR model with a *time dependent coefficient* [3],

$$dR(u) = \varepsilon\psi(u)\sigma(u)C(R(u))dW(u), \tag{25}$$

$$d\sigma(u) = \varepsilon\nu\sigma(u)dZ(u), \quad \sigma(t)=\alpha. \tag{26}$$

In (25) and (26), ε is a small positive scaling parameter, and W and Z are Brownian motions in the $\mathbb{Q}^{T_e}$-forward measure, with quadratic covariation $d[W,Z](u) = \rho du$ for $-1 \le \rho \le 1$. The local volatility backbone is $C(x) = (x+\theta)^\beta$ with constant elasticity of variance (CEV) parameter $0 \le \beta \le 1$, and offset $\theta \ge 0$ that allows the model to achieve negative forward rates for $\beta > 0$ by a boundary at $R(u) = -\theta$. When $\beta = 1$ and $\rho > 0$, the SABR process is only a local martingale [6, Theorem 1], therefore this combination should be excluded. The volatility-of-volatility ν controls the convexity of the volatility smile, while α is the initial level of the stochastic volatility σ. In the limit $u \uparrow T_e$, R will be completely deterministic, and hence have zero volatility. The role of $\psi(u)$ in (25) is to smoothly attenuate the volatility of R to zero as u passes from t to T_e. Following [9], we use

$$\psi(u) = \min\left\{1, \Big(\frac{T_e-u}{T_e-T_s}\Big)^q\right\} \tag{27}$$

for our volatility decay function. The positive parameter q in (27) controls the speed of the volatility decay, and allows for concave ($0 < q < 1$), convex ($q > 1$) or linear ($q = 1$) decay profiles.

The system of coupled stochastic differential equations (25), (26) is a special case of the dynamic SABR model [3]. In [3], the authors derive *effective* SABR parameters $\widehat{\alpha}$, $\widehat{\rho}$ and $\widehat{\nu}$ for (25), (26) that can be used in the implied volatility formula for the standard *constant* coefficient SABR model, for example see [3, Equation 1.4a]. CEV parameter β and offset parameter θ do not change. The effective SABR parameters are adroitly computed in [3] by a combination of singular perturbation expansion in ε and effective media analysis to ensure that the reduced (marginal in R) terminal probability density of (25), (26) matches to $\mathcal{O}(\varepsilon^2)$ the corresponding density for the constant coefficient version of this model. Formulae for the computation of the effective SABR parameters in the distinguished limit $\varepsilon = 1$ of (25), (26) have been derived in [9] by use of [3, Equations 3.2a-e]. Appendix B contains a summary of the formulae in [9], suitably adapted to make our paper self-contained.

Given effective SABR parameters $\widehat{\alpha}$, $\widehat{\rho}$ and $\widehat{\nu}$ as determined for the particular value of n by either (35) or (36) (see Appendix B) then $V^{\mathrm{QS}}(t, R(t))$ in (24) is computed analytically in terms of the ATM implied *normal* volatility $\sigma_N(R(t))$ for the forward rate $R(t)$ as [4]

$$V^{\mathrm{QS}}(t, R(t)) = s_Q^2 \left(T_e - t\right), \tag{28}$$

where s_Q is the implied quadratic swap volatility:

$$s_Q = \sigma_N\left(R(t)\right)\left\{1 + \left(\frac{1}{24}\frac{\beta\left(11\beta - 4\right)\widehat{\alpha}^2}{\left(R(t)+\theta\right)^{2-2\beta}} + \frac{3}{4}\frac{\widehat{\rho}\,\widehat{\nu}\,\widehat{\alpha}\,\beta}{\left(R(t)+\theta\right)^{1-\beta}} + \frac{1}{24}\left(4 + 3\widehat{\rho}^2\right)\widehat{\nu}^2\right)\left(T_e - t\right)\right\}.$$

In practice, $\widehat{\alpha}$, $\widehat{\rho}$, $\widehat{\nu}$ and $\sigma_N(R(t))$ will be computed from parameters that are intrinsic to the SABR model (25), (26) that has been calibrated to market traded Caplets and Floorlets. Volatility decay parameter q is usually specified exogenously. Equation (28) closes the valuation formula (24) for a discrete arithmetic average RFR IRS. An alternative approach to the model used in this paper can be found in [7], which models all forward rates jointly and can produce smooth behaviour within the accrual period.

5 Analysis of results

The convexity adjustment in (24) is the term that involves the symmetric quadratic swap. Fair value $V_a^{\mathrm{S}}(t, K^*; \varpi) = 0$ occurs at the convexity adjusted fixed rate

$$K^* = G(R(t)) + \tfrac{1}{2}G''\left(R(t)\right)V^{\mathrm{QS}}(t, R(t)).$$

From (22), we find that $G''(x) < 0$ and hence the sign of the convexity adjustment for an arithmetic average RFR IRS is *negative*. Jensen's inequality in conditional expectation form [5, Section 3.2] can be used to confirm our results. Start from (20) for a generic n, then apply Jensen's inequality for *concave* functions and the fact that $R(T_e)$ is a $\mathbb{Q}^{T_e}$-martingale to yield

$$\mathbb{E}^{T_e}\left[G\left(R(T_e)\right)\middle|\mathcal{F}(t)\right] \leq G\big(\mathbb{E}^{T_e}\left[R(T_e)\middle|\mathcal{F}(t)\right]\big) = G\big(R(t)\big).$$

Convexity is negative because each RFR fixing is paid *without* daily compounding. A pay fixed position in an arithmetic average RFR IRS would require compensation in the form of a fixed rate that is *lower* than the value $G(R(t))$ predicted from the yield curve by computation of the forward rate $R(t)$. A payer (receiver) average rate RFR IRS produces a short (long) Vega position.

We now examine how material the convexity adjustment is for a fixed verse float *payer* ($\varpi = +1$) arithmetic average RFR IRS. Numerical results that we present use $t = 0$, and hence $n = 0$, for *each* Swaplet. For an RFR IRS, fixed and float legs have the same frequency and payment dates to eliminate any credit risk. Let $0 \leq T_0 < T_1 < \cdots < T_M$ be the nodes for the fixed and float legs, where M will be a parameter that we will vary in our analysis. In terms of our previous notation, we identify $T_s = T_0, T_1, \cdots, T_{M-1}$ and $T_e = T_1, T_2, \cdots, T_M$. Day count fraction $\delta(T_s, T_e) = \delta(T_{i-1}, T_i)$. From (24), we find that the discounted value, $\Gamma(0)$, of the *total* convexity adjustment at trade inception is

$$\Gamma(0) = \tfrac{1}{2}\sum_{i=1}^{M}\delta(T_{i-1}, T_i)P(0; T_i + \lambda)G''\left(F_i(0)\right)V^{\mathrm{QS}}(0, F_i(0)), \tag{29}$$

where from (22) the formula for $G''\left(F_i(0)\right)$ is

$$G''\left(F_i(0)\right) = -\frac{\delta(T_{i-1}, T_i)}{\left[1 + \delta(T_{i-1}, T_i)F_i(0)\right]^2},$$

and the forward rate $F_i(0) = R(0)$ for the accrual period T_{i-1} to T_i is computed, see (14), as

$$F_i(0) = \frac{1}{\delta(T_{i-1}, T_i)}\left(\frac{P(0; T_{i-1})}{P(0; T_i)} - 1\right).$$

Market practitioners will often view (29) in units of the fixed leg annuity $A(0)$ to yield a convexity $\Gamma_{\text{bps}}(0)$ expressed in basis points (bps) running as

$$\Gamma_{\text{bps}}(0) = 10^4 \cdot \frac{\Gamma(0)}{A(0)}, \quad A(0) := \sum_{i=1}^{M} \delta(T_{i-1}, T_i) P(0; T_i + \lambda). \tag{30}$$

From the formulae for the effective SABR parameters, see Appendix B, $\Gamma_{\text{bps}}(0)$ is dependent on the two time scales $\tau_s = T_s - t$ and $\tau_e = T_e - t$. Since $\tau_s = \tau_e - (T_e - T_s)$, the two time scales are the length $T_e - T_s$ of the averaging period and the length τ_e of time to the cash flow date for a given Swaplet. The length of the averaging period is determined by the frequency of the IRS, for example an annual IRS will have an averaging period that is four times the length of a quarterly IRS. In Figure 1, we examine the relative importance of the two time scales on the convexity adjustment defined by (30). Various combinations for the length of the averaging period and IRS maturity are presented in Figure 1. Our test scenario uses a flat interest rate curve with discount factor $P(0; T) = \exp(0.03T)$ for $T \geq 0$. Payment lag is set to $\lambda = 0$. SABR parameters are chosen as $\alpha = 50$bps, $\beta = 0$, $\rho = -47\%$ and $\nu^2 T_i = 0.9$ for each averaging period T_{i-1} to T_i, where $i = 1, \cdots, M$. Three factors have influenced the rationale behind our choice of SABR parameters: 1) produce a realistic volatility smile that has convexity at short maturities (often observed in the market); 2) the choice $\beta = 0$ provides a transparent link between α and $\sigma_N(R(0))$ because to good accuracy $\sigma_N(R(0)) \simeq \alpha$ for the normal ($\beta = 0$) SABR model; 3) ensure that we remain within the region of validity for the SABR expansion, which necessitates $\nu^2 T_i < 1$.

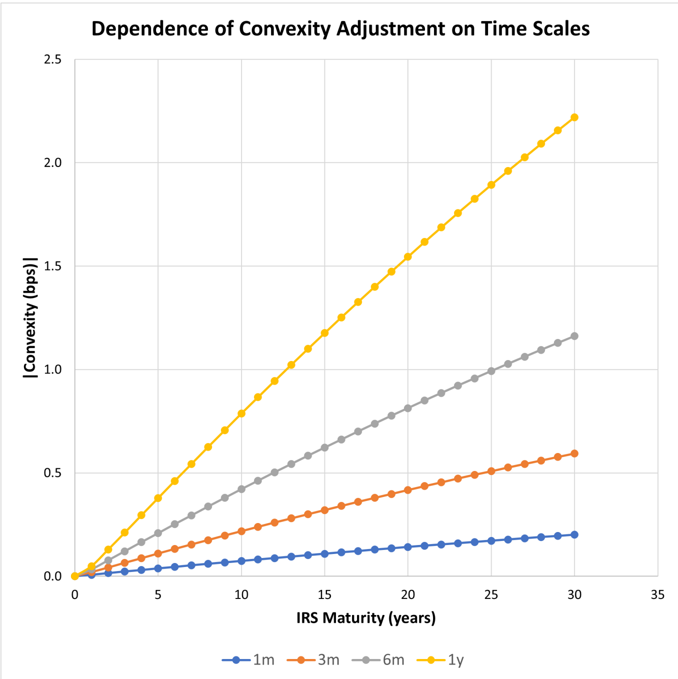

Figure 1: Scenario with $P(0; T) = \exp(0.03T)$, $\alpha = 50$bps, $\beta = 0$, $\rho = -47\%$ and $\nu^2 T_i = 0.9$.

The offset parameter θ is redundant when $\beta = 0$. Volatility decay parameter is set to $q = 1$. Day count fractions match the frequency of the relevant IRS, for example $\delta(T_{i-1}, T_i) = 0.25$ for a quarterly IRS. Figure 1, which displays $|\Gamma_{\text{bps}}(0)|$, reveals that the length of the averaging period is the dominant time scale in the convexity adjustment. When ratios of the series presented in Figure 1 are constructed to the 1y (year) series, we find that $\Gamma_{\text{bps}}(0)$ has the linear scaling property 1y/1m $\simeq 12$, 1y/3m $\simeq 4$ and 1y/6m $\simeq 2$. In the SOFR fixed verse float IRS market, bid-ask spreads for 5y to 30y maturities typically trade in the range 0.2bps-0.5bps. Results in Figure 1–particularly for the market standard case of 1y frequency–demonstrate that the convexity *is* significant, and can reach multiples of the bid-ask spread in the SOFR market. The uniformity in sign of the convexity adjustment at the Swaplet level precludes any cancellation of the convexity.

Sensitivity of the convexity adjustment (30) to the SABR parameters α, β ρ and ν is presented in Figure 2. Results in Figure 2 are produced with the base settings from Figure 1. We display results for the *change* in the value of (30), computed as perturbed scenario minus base scenario. Each SABR parameter is set to its base value before a perturbation. When a negative sensitivity is displayed, this indicates that the convexity increases in magnitude (becomes more negative) in the perturbed scenario, and the opposite when a positive sensitivity is displayed. First, we observe that an increase in the SABR parameter α causes the convexity to increase. An increase in α causes the entire volatility smile to shift *upwards*, but with no change in the shape of the smile. The affect of an upwards movement of the smile is to increase the implied volatility at each strike, most notably for the ATM strike. From (28) it is immediate that an increase in the ATM implied normal volatility will directly cause the convexity to increase, and therefore α is a material risk factor. The standard method to calibrate the SABR model is to set β exogenously, then calibrate the remaining SABR parameters α, ρ and ν to match the volatility smile. A consequence of this approach is that the calibrated SABR parameters depend implicitly on β, hence they would have to be re-keyed for a sufficiently large change in β. To avoid the situation where we introduce cross-effects into our scenario analysis through changes in α, ρ or ν, we pick a small perturbation in β, so we can isolate the affect of this parameter. With good accuracy [4, Equation 2.24b] $\sigma_N(R(0)) \simeq \alpha R(0)^\beta$, and hence because $0 < \alpha R(0) < 1$ we have $\partial\sigma_N(R(0))/\partial\beta < 0$, the reverse impact of the α perturbation as shown in Figure 2. Numerical values calculated for the sensitivity to the SABR parameter ρ are approximately two orders of magnitude smaller than the sensitivity to the other SABR parameters, and therefore ρ is not a significant risk factor; this is the reason why sensitivity to ρ displays as zero in Figure 2 because it is small relative to the other sensitivities. We note that although β and ρ both control the ATM skew, the SABR parameter β induces a material impact on the convexity adjustment because it affects the ATM implied normal volatility, which is the key volatility input into the convexity adjustment formula. Sensitivity to the SABR parameter ν is material. When the wings of the volatility smile either steepen or flatten, then the sensitivity to the SABR parameter ν can be of similar size to the bid-ask spread of the vanilla IRS. Sensitivity to the SABR parameter ν implies that a discrete arithmetic average RFR IRS shares some characteristics of a Constant Maturity Swap (CMS): sensitivity to the wings of the volatility smile, and non-local Vega risk, both a result of the symmetric quadratic swap.

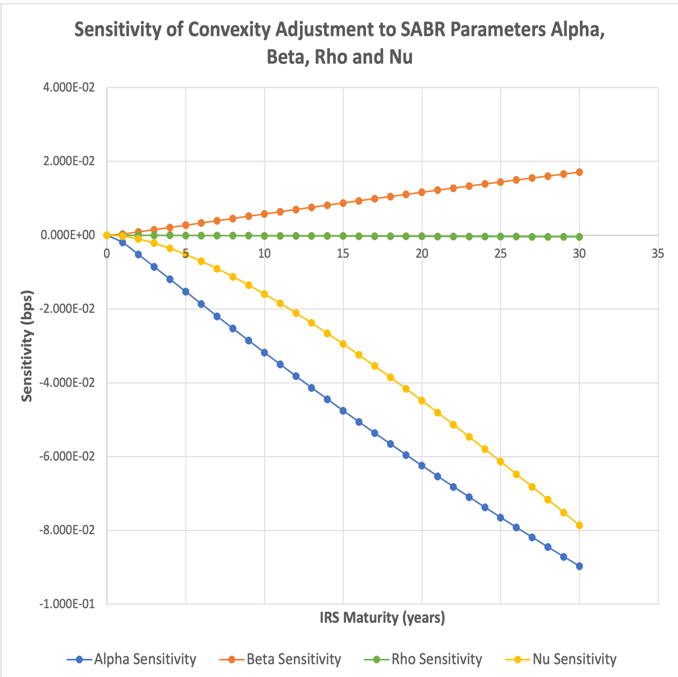

Figure 2: Sensitivity of $\Gamma_{\text{bps}}(0)$ in the scenarios $\alpha \to \alpha+1\text{bps}$, $\beta \to \beta+10\text{bps}$, $\rho \to \rho+1\%$, $\nu \to \nu+1\%$.

The final scenario analysis examines the dependence of (30) on the volatility decay parameter q. Figure 3 displays our results when we use the same initial test configuration as above, and allow the parameter q to vary. We observe in Figure 3 a small sensitivity to q. The explanation for the sensitivity is that changes in q trigger changes in the effective SABR parameter $\widehat{\nu}$ since it is parametrised in terms of q. As observed in the scenario for Figure 2, there is sensitivity to the SABR parameter ν, and hence to $\widehat{\nu}$. Changes in q cascade through the effective SABR parameters to produce a sensitivity in the convexity adjustment $\Gamma_{\text{bps}}(0)$.

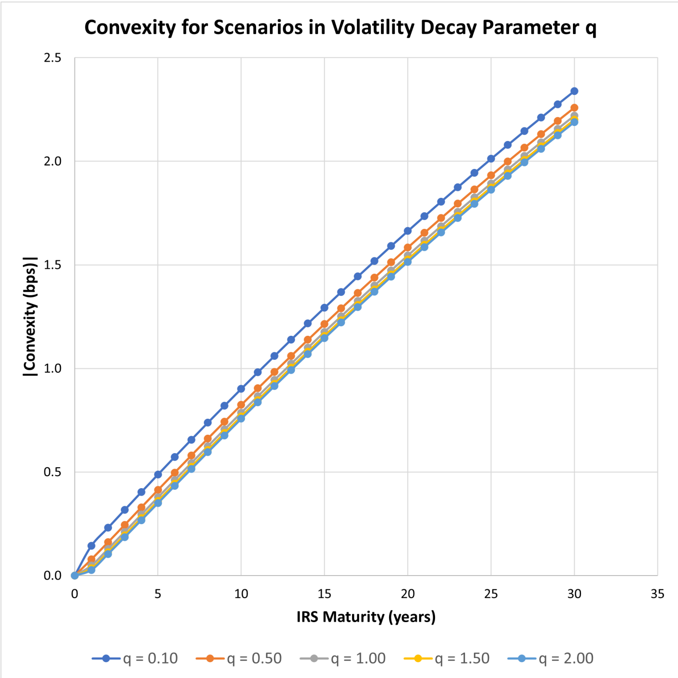

Figure 3: Dependence of $\Gamma_{\text{bps}}(0)$ on the volatility decay parameter q.

6 Conclusion

In this paper, we developed a model independent convexity adjustment formula for the valuation of a discrete arithmetic average RFR IRS. A concrete realisation of our results was presented by specialisation to the case of the market standard SABR model. Our analysis reveals that convexity is material for a discrete arithmetic average RFR IRS, and that the valuation and risk management of this product has characteristics in common with a CMS.

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Appendix A

In this appendix, we show that (21) is an accurate approximation uniformly in n. For t in (21), let $t \leq u \leq T_e$. By Itô's formula [5, Theorem 4.3.1] applied to $G(R(u))$, we have the semi-martingale

$$G(R(T_e)) = G(R(t)) + \int_t^{T_e} G'(R(u))dR(u) + \frac{1}{2}\int_t^{T_e} G''(R(u))d[R,R](u), \tag{31}$$

where $[R, R](u)$ denotes quadratic variation [5, Section 3.4.2] of the Itô process for $R(u)$. Applying the operator $\mathbb{E}^{T_e}[*|\mathcal{F}(t)]$ throughout (31), then using that R is a $\mathbb{Q}^{T_e}$-martingale, we obtain

$$\mathbb{E}^{T_e}\big[G\left(R(T_e)\right)\big|\mathcal{F}(t)\big] = G(R(t)) + \frac{1}{2}\mathbb{E}^{T_e}\Big[\int_t^{T_e} G''(R(u))d[R,R](u)\big|\mathcal{F}(t)\Big]. \tag{32}$$

RFR markets have the convention of annual fixed leg frequency, hence $0 \leq \delta(d_n, T_e) \leq 1$, uniformly in n. Consequently, we can infer from (22) that over a wide range of interest rate scenarios, $G''(R(u))$ will vary slowly from its value at $u = t$, and therefore we can *freeze* this stochastic variable at its time t-value to obtain for $t \leq u \leq T_e$ the approximation $G''(R(u)) \simeq G''(R(t))$. This approximation is beneficial because it makes $G''(R(u))$ $\mathcal{F}(t)$-measurable, and permits simplification of (32) to

$$\mathbb{E}^{T_e}\big[G\left(R(T_e)\right)\big|\mathcal{F}(t)\big] \simeq G(R(t)) + \frac{1}{2}G''(R(t))\mathbb{E}^{T_e}\Big[\int_t^{T_e} d[R,R](u)\big|\mathcal{F}(t)\Big]. \tag{33}$$

Itô isometry [5, Lemma 4.12] combined with the Martingale Representation Theorem [5, Theorem 5.49] allow us to express (33) in the equivalent form

$$\mathbb{E}^{T_e}\big[G\left(R(T_e)\right)\big|\mathcal{F}(t)\big] \simeq G(R(t)) + \frac{1}{2}G''(R(t))\mathbb{E}^{T_e}\Big[\Big(\int_t^{T_e} dR(u)\Big)^2\big|\mathcal{F}(t)\Big].$$

The integral term can now be computed to arrive immediately at our stated objective (21).

$(\boldsymbol{\sigma}$: bps, $\boldsymbol{\kappa}$: %)	$\boldsymbol{T_s = 0, T_e = 1}$	$\boldsymbol{T_s = 9, T_e = 10}$	$\boldsymbol{T_s = 29, T_e = 30}$
(25, 1)	1.030E-04	2.652E-03	6.912E-03
(25, 2)	1.030E-04	2.418E-03	5.360E-03
(25, 3)	1.020E-04	2.211E-03	4.270E-03
(50, 1)	4.140E-04	1.061E-02	2.765E-02
(50, 2)	4.100E-04	9.672E-03	2.144E-02
(50, 3)	4.070E-04	8.844E-03	1.708E-02
(100, 1)	1.654E-03	4.244E-02	1.107E-01
(100, 2)	1.642E-03	3.870E-02	8.581E-02
(100, 3)	1.630E-03	3.538E-02	6.834E-02

Table 1: Percentage error $100|(\widetilde{\gamma}(0) - \gamma(0))/\gamma(0)|$ for various scenarios in the Hull-White model.

To check the accuracy of (21), we set $t = 0$, work in continuous-time, and model the short rate $r(u)$ in the $\mathbb{Q}^{T_e}$-forward measure with a single-factor Hull-White model in the form [8, Section 4]

$$r(u) = f(0;u) + H(u;u) + Z(u), \quad dZ(u) = -\left[\sigma\xi(u;T_e) + \kappa Z(u)\right]du + \sigma dW^{T_e}(u). \tag{34}$$

In (34), σ, κ are, respectively, the volatility and speed of mean reversion for the short rate, $f(0;u)$ is the initial instantaneous forward rate curve, the stochastic driver $Z(u)$ evolves from $Z(0) = 0$ and

$$H(u;u) = \int_0^u \sigma(s;u)\xi(s;u)ds, \quad \sigma(s;u) = \sigma e^{-\kappa(u-s)}, \quad \xi(s;u) = \int_s^u \sigma(s;y)dy.$$

With the model dynamics (34), we can compute the *exact* convexity adjustment, $\gamma(0)$, in closed-form by reference to (20) with $G(R(T_e)) = A_a(T_s, T_e)$ from (4) to obtain

$$\gamma(0) = \frac{M}{\delta(T_s,T_e)}, \quad M := -\frac{\sigma^2}{2\kappa^3}\left[\kappa(T_e - T_s) + e^{-\kappa T_e}(e^{\kappa T_s} + e^{-\kappa T_s}) - \tfrac{1}{2}(e^{-2\kappa T_e} + e^{-2\kappa T_s}) - 1\right].$$

Within the framework of (34), we can also compute the *approximate* convexity adjustment, $\widetilde{\gamma}(0)$, from (24) in closed-form. To compute the value of the symmetric quadratic swap from the model dynamics (34), we use $R(T_e) = A_c(T_s, T_e)$ from (4). We obtain the formula for $\widetilde{\gamma}(0)$ as

$$\widetilde{\gamma}(0) = -\frac{(1 + e^{2(M+V^2)} - 2e^{M+\frac{1}{2}V^2})}{2\delta(T_s,T_e)}, \; V^2 := \frac{\sigma^2}{\kappa^3}\left[\kappa(T_e - T_s) + e^{-\kappa(T_e-T_s)} - \tfrac{1}{2}(e^{-\kappa T_e} - e^{-\kappa T_s})^2 - 1\right].$$

Results in Table 1 demonstrate that the method based on the symmetric quadratic swap to compute the convexity adjustment for an arithmetic average RFR IRS is very accurate.

Appendix B

We present a summary here of the formulae [9] for the computation of the effective SABR parameters $\widehat{\alpha}$, $\widehat{\rho}$ and $\widehat{\nu}$. Our presentation uses the ACTUAL/ACTUAL date interval lengths $\tau_s = T_s - t$ and $\tau_e = T_e - t$ from the current date t to, respectively, the start and end dates of the interest rate accrual (averaging) period. Two cases are distinguished, each characterised by the position of t relative to T_s.

■ Case 1: $0 \leq t < T_s$.

Effective SABR parameters $\widehat{\alpha}$, $\widehat{\rho}$ and $\widehat{\nu}$ for $n = 0$ are computed as [9, Theorem 4.2],

$$\widehat{\rho} = \rho \frac{(3\tau^2 + 2q\tau_s^2 + \tau_e^2)}{\sqrt{\gamma}(6q+4)}, \quad \widehat{\nu}^2 = \nu^2 \gamma \frac{(2q+1)}{\tau^3 \tau_e}, \quad \widehat{\alpha}^2 = \frac{\alpha^2}{(2q+1)} \frac{\tau}{\tau_e} e^{\frac{1}{2} H \tau_e}, \tag{35}$$

where:

$$\tau = 2q\tau_s + \tau_e, \quad H = \nu^2 \frac{(\tau^2 + 2q\tau_s^2 + \tau_e^2)}{2(q+1)\tau\tau_e} - \widehat{\nu}^2;$$

$$\gamma = \tau \frac{(2\tau^3 + \tau_e^3 + 2q[2q-1]\tau_s^3 + 6q\tau_s^2\tau_e)}{(4q+3)(2q+1)} + 3q\rho^2(\tau_e - \tau_s)^2 \frac{(3\tau^2 - \tau_e^2 + 5q\tau_s^2 + 4\tau_s\tau_e)}{(4q+3)(3q+2)^2}.$$

■ Case 2: $T_s \leq t < T_e$.

Effective SABR parameters $\widehat{\alpha}$, $\widehat{\rho}$ and $\widehat{\nu}$ for $n = 1, \cdots, N-1$ are computed as [9, Theorem 4.1],

$$\widehat{\rho} = \frac{2\rho}{\sqrt{\zeta}(3q+2)}, \quad \widehat{\nu}^2 = (2q+1)\nu^2\zeta, \quad \widehat{\alpha}^2 = \frac{\alpha^2}{(2q+1)} \left(\frac{\tau_e}{\tau_e - \tau_s}\right)^{2q} e^{\frac{1}{2}(\frac{\nu^2}{q+1} - \widehat{\nu}^2)\tau_s}, \tag{36}$$

where:

$$\zeta = \frac{3}{(4q+3)} \left(\frac{1}{2q+1} + \rho^2 \frac{2q}{(3q+2)^2} \right).$$

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