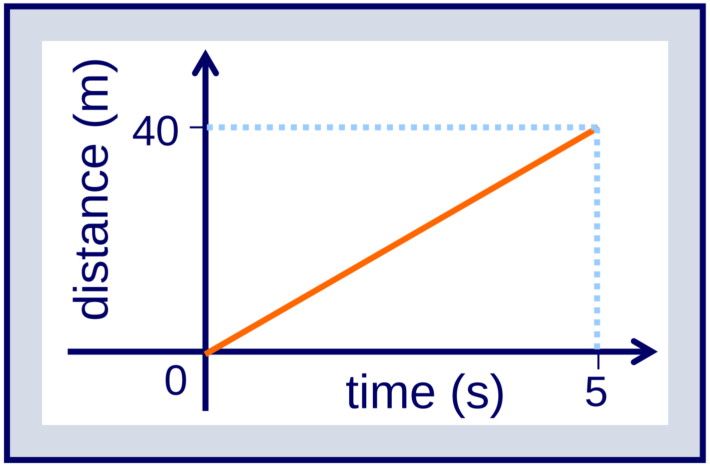
Derivative of a Function

Rates of change

This graph shows the distance that a car travels over a period of 5 seconds.



The gradient of the graph tells us the **rate** at which the distance changes with respect to time.

In other words, the gradient tells us the speed of the car.

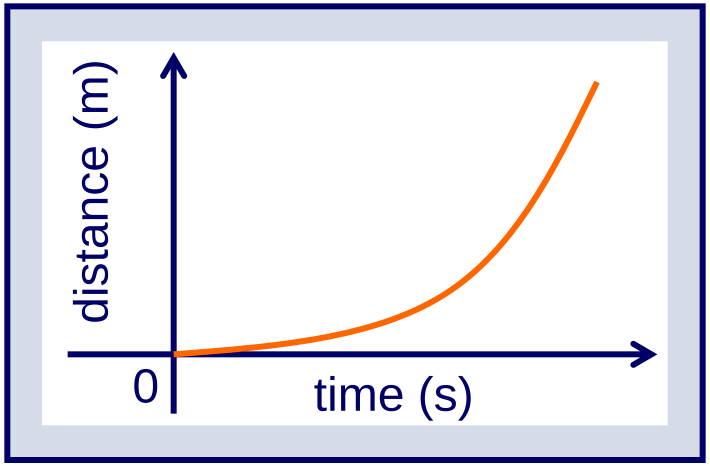
$$\text{gradient} = \frac{\text{change in distance}}{\text{change in time}} = \frac{40}{5} = 8 \text{ m/s}$$

The car in this example is travelling at a constant speed since the gradient is the same at every point on the graph.

Rates of change

In most situations, however, the speed will not be constant and the distance–time graph will be curved.

For example, this graph shows the distance–time graph as the car moves off from rest.



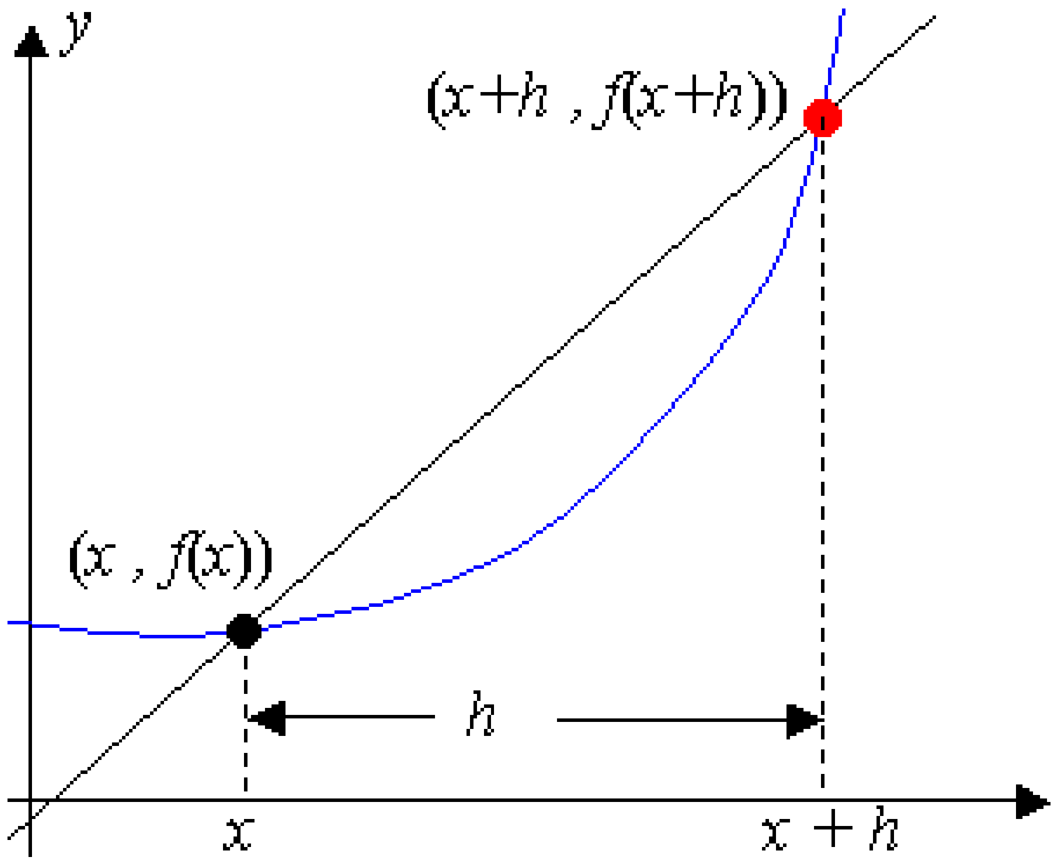
The speed of the car, and therefore the gradient, changes as you move along the curve.

To find the rate of change in speed we need to find the gradient of the curve.

The process of finding the rate at which one variable changes with respect to another is called **differentiation**.

In most situations this involves finding the gradient of a curve.

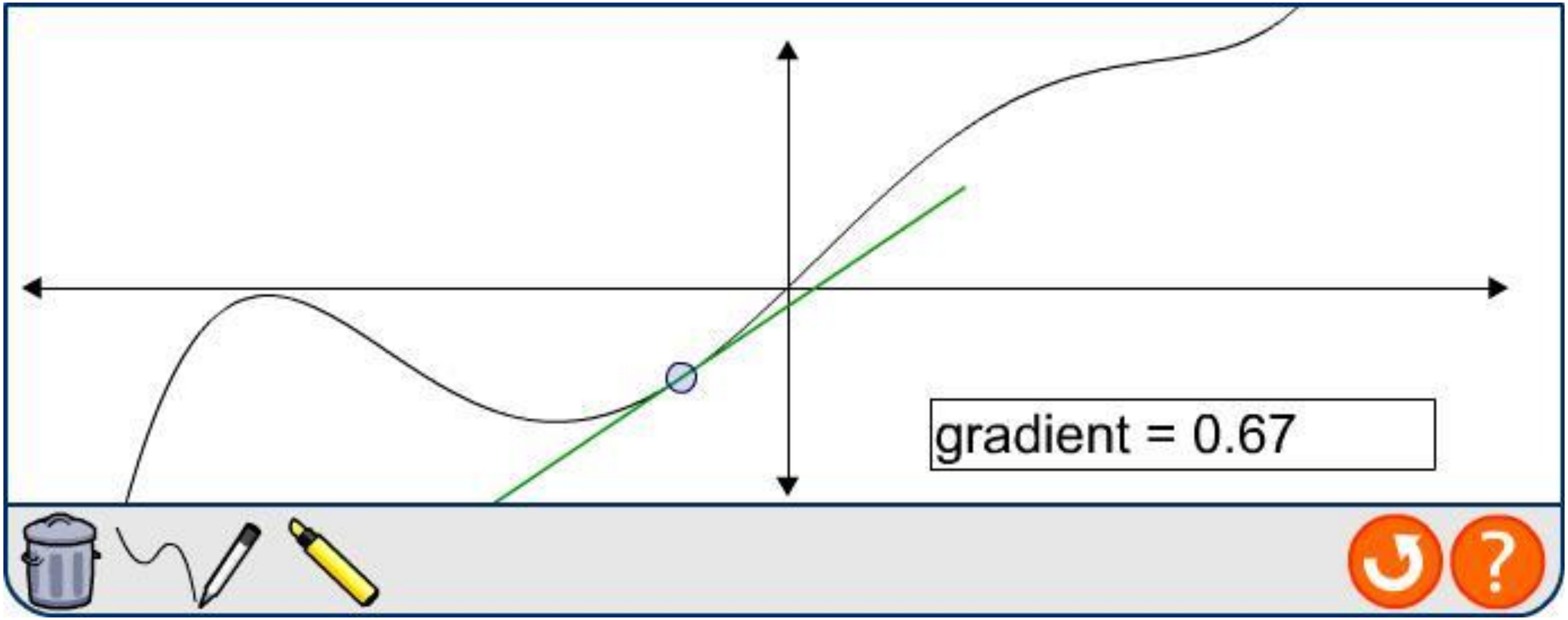
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



The gradient of a curve

The **gradient** of a curve at a point is given by the gradient of the tangent at that point.

Look at how the gradient changes as we move along a curve:



Vocab/Formulas

- **Derivative:**
  – Slope of a curve of y = f(x) at a point where x=a as

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

  – When it exists, this limit is called the **derivative of f at a**. f'(x) is the **derivative of the function**.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **Differentiable:** a function is **differentiable at a point** when the f'(x) exists
  – If the function is differentiable at every point in the domain, it is a **differentiable function**

The gradient function

So the gradient of the tangent to the curve $y = x^2$ at the general point (x, y) is $2x$.

$2x$ is often called the **gradient function** or the **derived function** of $y = x^2$.

If the curve is written using function notation as $y = f(x)$, then the derived function can be written as $f'(x)$.

So, if: $f(x) = x^2$

Then: $f'(x) = 2x$

This notation is useful if we want to find the gradient of $f(x)$ at a particular point.

For example, the gradient of $f(x) = x^2$ at the point (5, 25) is:

$$f'(5) = 2 \times 5 = 10$$

Notation

- **Derivative:**
  - $f'(x)$
  - y'
  - $\dfrac{dy}{dx}$
  - $\dfrac{d}{dx}f(x)$
  - $\dfrac{df}{dx}$

If we continued the process of differentiating from first principles we would obtain the following results:

y	x	x^2	x^3	x^4	x^5	x^6
$\frac{dy}{dx}$	1	$2x$	$3x^2$	$4x^3$	$5x^4$	$6x^5$

What pattern do you notice?

In general:

$$\text{If } y = x^n \text{ then } \frac{dy}{dx} = nx^{n-1}$$

and when x^n is preceded by a constant multiplier a we have:

$$\text{If } y = ax^n \text{ then } \frac{dy}{dx} = anx^{n-1}$$

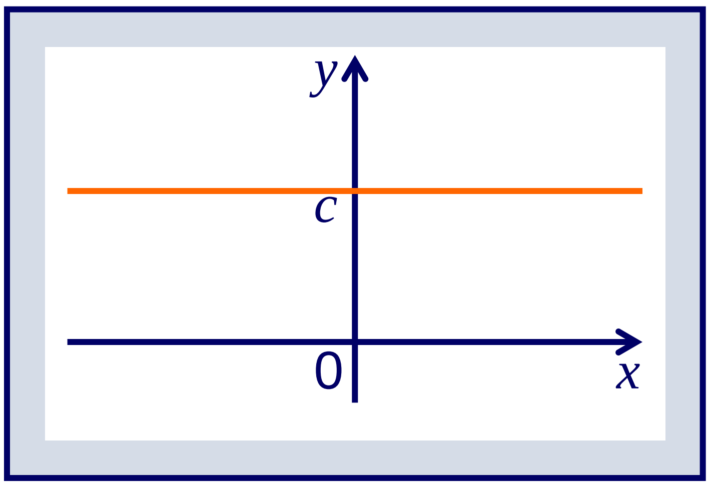
Differentiation

For example, if $y = 2x^4$

$$\frac{dy}{dx} = 2 \times 4x^{4-1} = 8x^3$$

Suppose we have a function of the form $y = c$, where c is a constant number.

This corresponds to a horizontal line through the point (0, c).



The gradient of this line is always equal to 0.

$\therefore$ If $y = c$

$$\frac{dy}{dx} = 0$$

Differentiation

Find the gradient of the curve $y = 3x^4$ at the point (–2, 48).

Differentiating: $$\frac{dy}{dx} = 12x^3$$

At the point (–2, 48) x = –2 so:

$$\frac{dy}{dx} = 12(-2)^3$$

$$= 12 \times -8$$

$$= -96$$

The gradient of the curve $y = 3x^4$ at the point (–2, 48) is –96.

Differentiating polynomials

Suppose we want to differentiate a polynomial function. For example:

Differentiate $y = x^4 + 3x^2 - 5x + 2$ with respect to x.

We can differentiate each term in the function to give $\frac{dy}{dx}$.

So: $$\frac{dy}{dx} = 4x^3 + 6x - 5$$

In general, if y is made up of the sum or difference of any given number of functions, its derivative will be made up of the derivative of each of the functions.

If $y = f(x) \pm g(x)$ then $\frac{dy}{dx} = f'(x) \pm g'(x)$

Find the point on $y = 4x^2 - x - 5$ where the gradient is 15.

$$\frac{dy}{dx} = 8x - 1$$

$\frac{dy}{dx} = 15$ when $8x - 1 = 15$

$$8x = 16$$

$$x = 2$$

We now substitute this value into the equation $y = 4x^2 - x - 5$ to find the value of y when $x = 2$

$$y = 4(2)^2 - 2 - 5$$

$$= 9$$

The gradient of $y = 4x^2 - x - 5$ is 15 at the point (2, 9).

Differentiating polynomials

In some cases, a function will have to be written as separate terms containing powers of x before differentiating.

For example:

Given that $f(x) = (2x - 3)(x^2 - 5)$ find $f'(x)$.

Expanding: $f(x) = 2x^3 - 3x^2 - 10x + 15$

$$f'(x) = 6x^2 - 6x - 10$$

Find the gradient of $f(x)$ at the point $(-3, -36)$.

When $x = -3$ we have $f'(-3) = 6(-3)^2 - 6(-3) - 10$

$$= 54 + 18 - 10$$

$$= 62$$

Differentiating polynomials

Differentiate $y = \dfrac{3x^5 + 4x^2 - 8x}{2x}$ with respect to x.

$$y = \frac{3x^5}{2x} + \frac{4x^2}{2x} - \frac{8x}{2x}$$

$$y = \tfrac{3}{2}x^4 + 2x - 4$$

We can now differentiate:

$$\frac{dy}{dx} = \tfrac{12}{2}x^3 + 2$$

$$= 6x^3 + 2$$

$$= 2(3x^3 + 1)$$

Differentiating ax^n for all rational n

$$\text{If } y = ax^n \text{ then } \frac{dy}{dx} = anx^{n-1}$$

This is true for all negative or fractional values of n.

For example:

Differentiate $y = \dfrac{2}{x}$ with respect to x.

Start by writing this as $y = 2x^{-1}$

So:

$$\frac{dy}{dx} = -1 \times 2x^{-1-1}$$

$$= -2x^{-2}$$

$$= \frac{-2}{x^2}$$

Differentiating ax^n for all rational n

Differentiate $y = 4\sqrt{x}$ with respect to x.

Start by writing this as $y = 4x^{\frac{1}{2}}$

So:

$$\frac{dy}{dx} = \tfrac{1}{2} \times 4x^{\frac{1}{2}-1}$$

$$= 2x^{-\frac{1}{2}}$$

$$= \frac{2}{\sqrt{x}}$$

Remember, in some cases, a function will have to be written as separate terms containing powers of x before differentiating.

Differentiating ax^n for all rational n

Given that $f(x) = (1+\sqrt{x})^2$ find $f'(x)$.

$$f(x) = (1+\sqrt{x})(1+\sqrt{x})$$

$$= 1 + 2\sqrt{x} + x$$

$$= 1 + 2x^{\frac{1}{2}} + x$$

$$f'(x) = x^{-\frac{1}{2}} + 1$$

$$= \frac{1}{\sqrt{x}} + 1$$

Second order derivatives

Contents

Second order derivatives

Differentiating a function $y = f(x)$ gives us the derivative

$$\frac{dy}{dx} \quad \text{or} \quad f'(x)$$

Differentiating the function a second time gives us the **second order derivative**. This can be written as

$$\frac{d^2y}{dx^2} \quad \text{or} \quad f''(x)$$

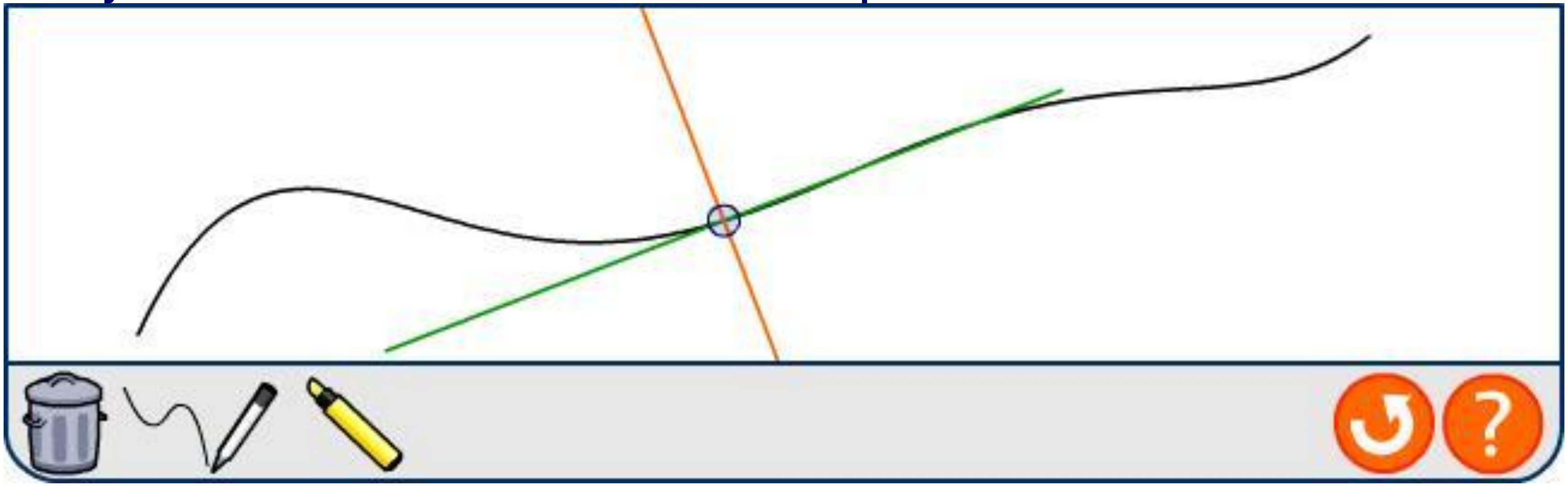
The second order derivative gives us the rate of change of the gradient of a function.

We can think of it as the gradient of the gradient.

Contents

Tangents and normals

Remember, the **tangent** to a curve at a point is a straight line that just touches the curve at that point.



The **normal** to a curve at a point is a straight line that is perpendicular to the tangent at that point.

We can use differentiation to find the equation of the tangent or the normal to a curve at a given point. For example:

Find the equation of the tangent and the normal to the curve $y = x^2 - 5x + 8$ at the point P(3, 2).

$$y = x^2 - 5x + 8$$

$$\frac{dy}{dx} = 2x - 5$$

At the point P(3, 2) $x = 3$ so:

$$\frac{dy}{dx} = 2(3) - 5 = 1$$

The gradient of the tangent at P is therefore 1.

Using $y - y_1 = m(x - x_1)$, give the equation of the tangent at the point P(3, 2):

$$y - 2 = x - 3$$

$$y = x - 1$$

Tangents and normals

The normal to the curve at the point P(3, 2) is perpendicular to the tangent at that point.

The gradient of the tangent at P is 1 and so the gradient of the normal is –1.

Using $y - y_1 = m(x - x_1)$ give the equation of the tangent at the point P(3, 2):

$$y - 2 = -(x - 3)$$

$$y + x - 5 = 0$$

Examination-style questions

Contents

Examination-style question

The curve $f(x) = x^2 + 2x - 8$ cuts the x-axes at the points A($-a$, 0) and B(b, 0) where a and b are positive integers.

a) Find the coordinates of points A and B.

b) Find the gradient function of the curve.

c) Find the equations of the normals to the curve at points A and B.

d) Given that these normals intersect at the point C, find the coordinates of C.

a) The equation of the curve can be written in factorized form as $y = (x + 4)(x - 2)$ so the coordinates of A are (–4, 0) and the coordinates of B are (2, 0).

Examination-style question

b) $f(x) = x^2 + 2x - 8 \Rightarrow f'(x) = 2x + 2$

c) At (–4, 0) the gradient of the curve is –8 + 2 = –6

$\therefore$ The gradient of the normal at (–4, 0) is $\frac{1}{6}$.

Using $y - y_1 = m(x - x_1)$, the equation of the normal at (–4, 0) is:

$$y - 0 = \frac{1}{6}(x + 4)$$

$$6y = x + 4 \qquad ①$$

At (2, 0) the gradient of the curve is 4 + 2 = 6

$\therefore$ The gradient of the normal at (2, 0) is $-\frac{1}{6}$.

Using $y - y_1 = m(x - x_1)$, the equation of the normal at (2, 0) is:

$$y - 0 = -\frac{1}{6}(x - 2)$$

$$6y = 2 - x \qquad ②$$

Examination-style question

d) Equating ① and ② :

$$x + 4 = 2 - x$$

$$2x = -2$$

$$x = -1$$

When $x = -1$, $y = \frac{1}{2}$. So the coordinates of C are $(-1, \frac{1}{2})$.