Rates of Change and Limits

Suppose you drive 200 miles, and it takes you 4 hours.

Then your average speed is: $200 \text{ mi} \div 4 \text{ hr} = 50 \dfrac{\text{mi}}{\text{hr}}$

$$\text{average speed} = \frac{\text{distance}}{\text{elapsed time}} = \frac{\Delta x}{\Delta t}$$

If you look at your speedometer during this trip, it might read 65 mph. This is your instantaneous speed.

→

A rock falls from a high cliff.

The position of the rock is given by: $y = 16t^2$

After 2 seconds: $y = 16 \cdot 2^2 = 64$

<u>average</u> speed: $V_{\mathrm{av}} = \dfrac{64 \text{ ft}}{2 \text{ sec}} = 32 \dfrac{\text{ft}}{\text{sec}}$

What is the *instantaneous* speed at 2 seconds?

→

$$V_{\text{instantaneous}} \approx \frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

for some very small change in t

where h = some very small change in t

→

$$V_{\text{instantaneous}} \approx \frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}$$

$$\left(16*(2+h)\wedge 2-64\right)\div h \,\Big|\, h=\{1,.1,.01,.001,.0001,.00001\}$$

We can see that the velocity approaches 64 ft/sec as h becomes very small.

We say that the velocity has a <u>limiting value</u> of 64 as <u>h approaches zero</u>.

(Note that h never actually becomes zero.)

h	$\frac{\Delta y}{\Delta t}$
0.1	65.6
.01	64.16
.001	64.016
.0001	64.0016
.00001	64.0002

The limit as h approaches zero:

$$\lim_{h \to 0} \frac{16(2+h)^2 - 16 \cdot 2^2}{h}$$

Since the 16 is unchanged as h approaches zero, we can factor 16 out.

$$16 \cdot \lim_{h \to 0} \frac{\left(4 + 4h + h^2\right) - 4}{h}$$

$$16 \cdot \lim_{h \to 0} \frac{\cancel{4} + 4\cancel{h} + h^{\cancel{2}} - \cancel{4}}{\cancel{h}}$$

$$16 \cdot \lim_{h \to 0} (4 + \cancel{h}^{\,0}) = 64$$

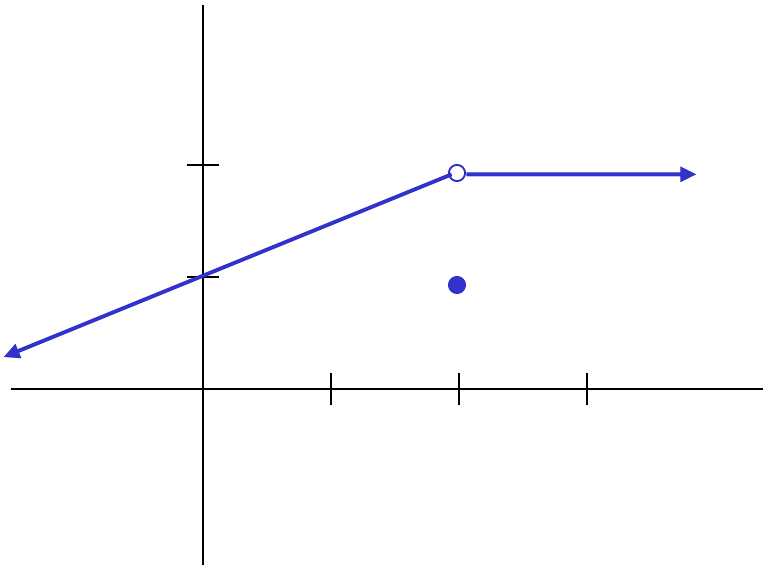
$\rightarrow$

Limit notation: $\lim\limits_{x \to c} f(x) = L$

"The limit of f of x as x approaches c is L."

$\rightarrow$

The <u>limit</u> of a function refers to the value that the function <u>approaches</u>, <u>not</u> the actual value (if any).



$$\lim_{x \to 2} f(x) = 2$$

<u>not 1</u>

→

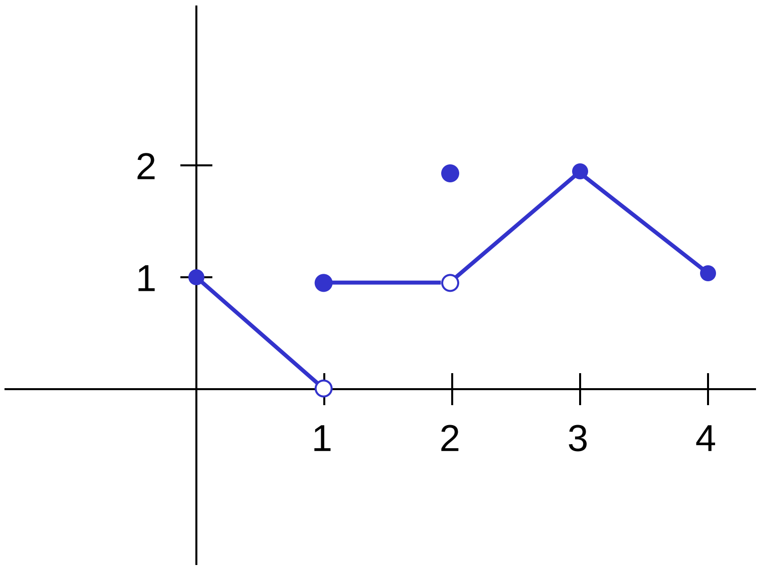
Properties of Limits:

Limits can be added, subtracted, multiplied, multiplied by a constant, divided, and raised to a power.

For a limit to exist, the function must approach the <u>same value</u> from both sides.

<u>One-sided limits</u> approach from either the left or right side only.

→



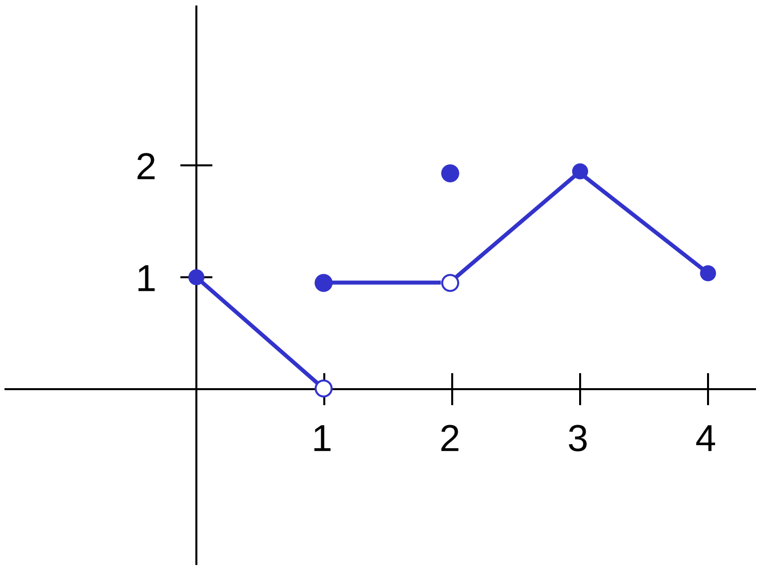
$$\lim_{x \to 1} f(x)$$ does not exist because the left and right hand limits do not match!

At x=1: $$\lim_{x \to 1^-} f(x) = 0$$ ← left hand limit

$$\lim_{x \to 1^+} f(x) = 1$$ ← right hand limit

$$f(1) = 1$$ ← value of the function

→



$$\lim_{x \to 2} f(x) = 1$$

because the left and right hand limits match.

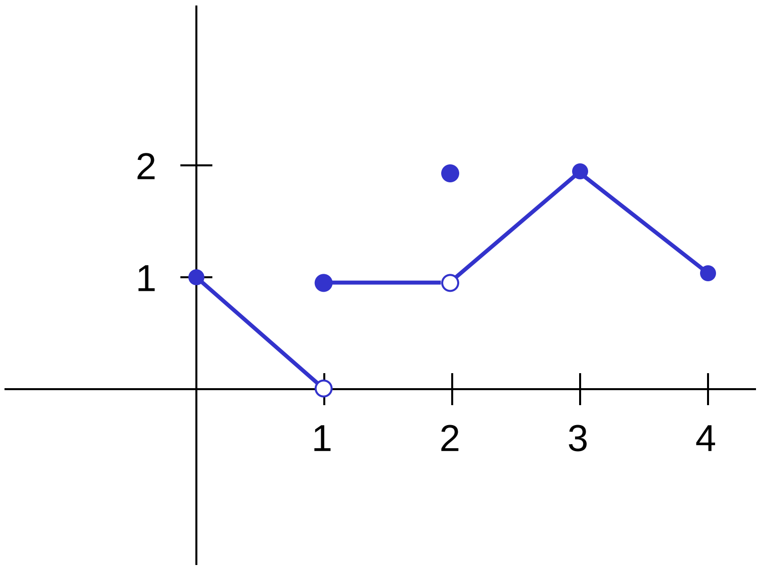
At x=2:

$$\lim_{x \to 2^-} f(x) = 1$$ left hand limit

$$\lim_{x \to 2^+} f(x) = 1$$ right hand limit

$$f(2) = 2$$ value of the function

→



$$\lim_{x \to 3} f(x) = 2$$

because the left and right hand limits match.

At x=3:

$$\lim_{x \to 3^-} f(x) = 2$$ ← left hand limit

$$\lim_{x \to 3^+} f(x) = 2$$ ← right hand limit

$$f(3) = 2$$ ← value of the function

→

One-Sided Limits

One-Sided Limits

- The limit of $f(x)$ as $x \to c$ does not exist when the function $f(x)$ approaches a different number from the left side of c than it approaches from the right side of c.

- This type of behavior can be described more concisely with the concept of a **one-sided limit.**

Limit from the left

- $\lim_{x \to c^-} f(x) = L_1$ or $f(x) \to L_1$ as $x \to c^-$

Limit from the right

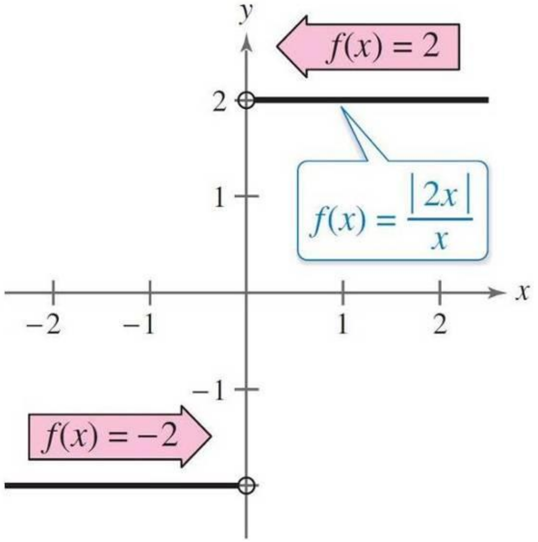
- $\lim_{x \to c^+} f(x) = L_2$ or $f(x) \to L_2$ as $x \to c^+$

Example – *Evaluating One-Sided Limits*

- Find the limit as $x \to 0$ for the left and the limit as $x \to 0$ from the right for

- $f(x) = \dfrac{|2x|}{x}$

- Solution:
- From the graph of f, shown in Figure, you can see that $f(x) = -2$ for all $x < 0$.



Example – *Solution*

- So, the limit from the left is

Limit from the left

- $$\lim_{x \to 0^-} \frac{|2x|}{x} = -2.$$

- Because $f(x) = 2$ for all $x > 0$, the limit from the right is

Limit from the right

- $$\lim_{x \to 0^+} \frac{|2x|}{x} = 2.$$

The Limit Process

Example

Set $f(x) = 4x + 5$ and take $c = 2$. As x approaches 2, $4x$ approaches 8 and $4x + 5$ approaches $8 + 5 = 13$. We conclude that

$$\lim_{x \to 2} f(x) = 13.$$

The Limit Process

Example

$$f(x)=\sqrt{1-x} \text{ and take } c=-8.$$

As x approaches -8, $1-x$ approaches 9 and $\sqrt{1-x}$ approaches 3. We conclude that

$$\lim_{x\to -8} f(x)=3$$

If for that same function we try to calculate

$$\lim_{x\to 2} f(x)$$

we run into a problem. The function $f(x)=\sqrt{1-x}$ is defined only for $x \le 1$. It is therefore not defined for x near 2, and the idea of taking the limit as x approaches 2 makes no sense at all:

$$\lim_{x\to 2} f(x) \quad \text{does not exist.}$$

The Limit Process

Example

$$\lim_{x \to 3} \frac{x^3 - 2x + 4}{x^2 + 1} = \frac{5}{2}.$$

First we work the numerator: as x approaches 3, x^3 approaches 27, $-2x$ approaches –6, and $x^3 - 2x + 4$ approaches 27 – 6 + 4 = 25. Now for the denominator: as x approaches 3, $x^2 + 1$ approaches 10. The quotient (it would seem) approaches 25/10 = 5/2.

Properties of Limits and Direct Substitution

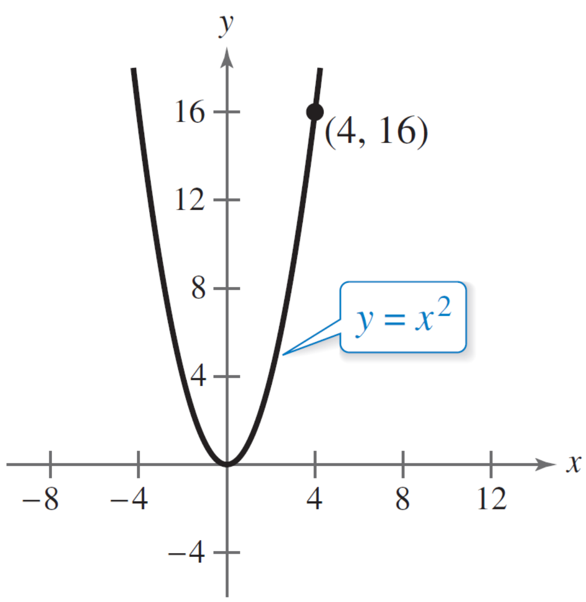
$$\lim_{x \to 4} x^2$$

- The algebraic solutions. To verify the limit in Example (a) numerically, for instance, create a table that shows values of x^2 for two sets of x-values—one set that approaches 4 from the left and one that approaches 4 from the right, as shown below.

x	3.9	3.99	3.999	4.0	4.001	4.01	4.1
x^2	15.2100	15.9201	15.9920	?	16.0080	16.0801	16.8100

Properties of Limits and Direct Substitution

- From the table, you can see that the limit as x approaches 4 is 16. To verify the limit graphically, sketch the graph of $y = x^2$. From the graph shown, you can determine that the limit as x approaches 4 is 16.



Properties of Limits and Direct Substitution

- The following summarizes the results of using direct substitution to evaluate limits of polynomial and rational functions.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

2. If r is a rational function $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Properties of Limits and Direct Substitution

- You have seen that sometimes the limit of $f(x)$ as $x \to c$ is simply f (c). In such cases, it is said that the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x \to c} f(x) = f(c)$$

Substitute *c* for *x*.

- There are many "well-behaved" functions, such as polynomial functions and rational functions with nonzero denominators, that have this property.

Properties of Limits and Direct Substitution

- Some of the basic ones are included in the following list.

Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$
4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, for n even and $c > 0$

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \qquad \text{and} \qquad \lim_{x \to c} g(x) = K$$

1. **Scalar multiple:** $\lim_{x \to c} [b f(x)] = bL$
2. **Sum or difference:** $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. **Product:** $\lim_{x \to c} [f(x) g(x)] = LK$
4. **Quotient:** $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$
5. **Power:** $\lim_{x \to c} [f(x)]^n = L^n$

Example – *Direct Substitution and Properties of Limits*

a. $\lim_{x \to 4} x^2$ Direct Substitution **a.** $\lim_{x \to 4} x^2 = (4)^2 = 16$

b. $\lim_{x \to 4} 5x$ Scalar Multiple Property **b.** $\lim_{x \to 4} 5x = 5 \lim_{x \to 4} x = 5(4) = 20$

c. $\lim_{x \to \pi} \frac{\tan x}{x}$ Quotient Property **c.** $\lim_{x \to \pi} \frac{\tan x}{x} = \frac{\lim_{x \to \pi} \tan x}{\lim_{x \to \pi} x} = \frac{0}{\pi} = 0$

d. $\lim_{x \to 9} \sqrt{x}$ Direct Substitution **d.** $\lim_{x \to 9} \sqrt{x} = \sqrt{9} = 3$

Example – *Direct Substitution and Properties of Limits*

e. $\lim_{x\to\pi}(x\cos x)$

Product Property

e. $\lim_{x\to\pi}(x\cos x) = \left(\lim_{x\to\pi} x\right)\left(\lim_{x\to\pi}\cos x\right)$

$= \pi(\cos\pi)$

$= -\pi$

f. $\lim_{x\to 3}(x+4)^2$

Sum and Power Properties

f. $\lim_{x\to 3}(x+4)^2 = \left[\left(\lim_{x\to 3} x\right) + \left(\lim_{x\to 3} 4\right)\right]^2$

$= (3+4)^2$

$= 7^2 = 49$

Properties of Limits and Direct Substitution

•The results of using direct substitution to evaluate limits of polynomial and rational functions are summarized as follows.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

2. If r is a rational function given by $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Dividing Out Technique

- We have studied several types of functions whose limits can be evaluated by direct substitution.

  In this section, you will study several techniques for evaluating limits of functions for which direct substitution fails.

  Suppose you were asked to find the following limit.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Dividing Out Technique

- Direct substitution fails because –3 is a zero of the denominator. By using a table, however, it appears that the limit of the function as *x* approaches –3 is –5.

x	-3.01	-3.001	-3.0001	-3	-2.9999	-2.999	-2.99
$\frac{x^2 + x - 6}{x + 3}$	-5.01	-5.001	-5.0001	?	-4.9999	-4.999	-4.99

Example – *Dividing Out Technique*

- Find the limit. $$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Factor numerator.

- Solution:

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3}$$

Example – *Solution*

- $$= \lim_{x \to -3} \frac{(x-2)\cancel{(x+3)}}{\cancel{x+3}}$$

Divide out common factor.

$$= \lim_{x \to -3}$$

Simplify.

- $$(x - 2)$$

Direct substitution

Simplify.

- $= -3 - 2$
- $= -5$

The Difference of Squares

Difference of Squares

$$x^2 - y^2 = (x + y)(x - y)$$

EXAMPLE

Factoring Differences of Squares

Factor polynomial.

$2n^2 - 50$

There is a common factor of 2.

$2n^2 - 50 = 2(n^2 - 25)$ Factor out the common factor.

$= 2(n + 5)(n - 5)$ Factor the difference of squares.

The Limit Process

Example

Set $f\left(x\right)=\dfrac{x^{2}-9}{x-3}$

and let c = 3. Note that the function f is not defined at 3: at 3, both numerator and denominator are 0. But that doesn't matter. For $x \neq 3$, and therefore *for all x near* 3,

$$\frac{x^{2}-9}{x-3}=\frac{\left(x-3\right)\left(x+3\right)}{x-3}=x+3$$

Therefore, if x is close to 3, then $\dfrac{x^{2}-9}{x-3}=x+3$

is close to 3 + 3 = 6. We conclude that

$$\lim_{x\to 3}\frac{x^{2}-9}{x-3}=\lim_{x\to 3}\left(x+3\right)=6$$

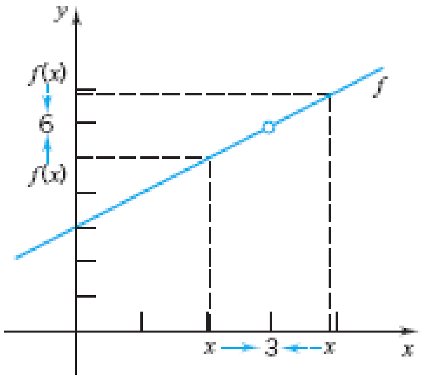


Figure 2.1.6

Special Factoring

Factoring Summary

Special Types of Factoring ***(Memorize)***

Difference of Squares $x^2 - y^2 = (x + y)(x - y)$

Perfect Square Trinomial $x^2 + 2xy + y^2 = (x + y)^2$

$x^2 - 2xy + y^2 = (x - y)^2$

Difference of Cubes $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Sum of Cubes $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Factoring Sum or Difference of Cubes

If you have a sum or difference of cubes such as $a^3 + b^3$ or $a^3 - b^3$, you can factor by using the following patterns.

Sum of Two Cubes

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Difference of Two Cubes

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Example

Factor $x^3 + 343$.

Note: This is a binomial. Are the first and last terms cubed?

$$\sqrt[3]{x^3} = x \qquad \sqrt[3]{343} = 7$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$x^3 + 343 = (x)^3 + (7)^3$

$= (x + 7)(x^2 - 7x + 49)$

EXAMPLE **Factoring Sums of Cubes**

Factor each polynomial. Recall, $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

(a) $n^3 + 8 = n^3 + 2^3$

$= (n + 2)(n^2 - 2n + 2^2)$

$= (n + 2)(n^2 - 2n + 4)$

(b) $64v^3 + 27g^3 = (4v)^3 + (3g)^3$

$= (4v + 3g)\,[\,(4v)^2 - (4v)(3g) + (3g)^2\,]$

$= (4v + 3g)\,(16v^2 - 12gv + 9g^2)$

EXAMPLE **Factoring Difference of Cubes**

Factor each polynomial. Recall, $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

(a) $a^3 - 125 = a^3 - 5^3$

$= (a - 5)(a^2 + 5a + 5^2)$

$= (a - 5)(a^2 + 5a + 25)$

EXAMPLE **Factoring Difference of Cubes**

Factor each polynomial. Recall, $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

(b) $8g^3 - h^3 = (2g)^3 - h^3$

$= (2g - h)[(2g)^2 + (2g)(h) + h^2]$

$= (2g - h)(4g^2 + 2gh + h^2)$

Example

Factor $64a^4 – 27a$

$= a(64a^3 – 27)$

Note: Binomial. Is the first and last terms cubes?

$= a((4a)^3 – (3)^3)$

Note: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$= a(4a - 3)(16a^2 + 12a + 9)$

Example

The Limit Process

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = 12.$$

The function $f(x) = \dfrac{x^3 - 8}{x - 2}$ is undefined at $x = 2$. But, as we said before, that doesn't matter. For all $x \neq 2$,

$$\frac{x^3 - 8}{x - 2} = \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} = x^2 + 2x + 4.$$

Therefore,

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} (x^2 + 2x + 4) = 12.$$

The Limit Process

Example

If $f(x) = \begin{cases} 3x - 4, & x \neq 0 \\ 10, & x = 0, \end{cases}$ then $\lim\limits_{x \to 0} f(x) = -4.$

It does not matter that f(0) = 10. For x ≠ 0, and thus for all x near 0,

$f(x) = 3x - 4$ and therefore $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} (3x - 4) = -4.$

Dividing Out Technique

- The dividing out technique should be applied only when direct substitution produces 0 in both the numerator *and* the denominator.

- An expression such as $\frac{0}{0}$ has no meaning as a real number.

- It is called an **indeterminate form** because you cannot, from the form alone, determine the limit.

Dividing Out Technique

- When you try to evaluate a limit of a rational function by direct substitution and encounter this form, you can conclude that the numerator and denominator must have a common factor.

- After factoring and dividing out, you should try direct substitution again.

Rationalizing Technique

Rationalizing Technique

- Another way to find the limits of some functions is first to rationalize the numerator of the function. This is called the **rationalizing technique.**

- Rationalizing the numerator means multiplying the numerator and denominator by the conjugate of the numerator.

Rationalizing Technique

- Find the limit.

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$$

- Solution:
- By direct substitution, you obtain the indeterminate form $\frac{0}{0}$
- .

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} = \frac{\sqrt{0+1} - 1}{0}$$

$$= \frac{0}{0}$$

Indeterminate form

Example – *Solution*

- In this case, you can rewrite the fraction by rationalizing the numerator.

$$\frac{\sqrt{x+1}-1}{x} = \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$$

$$= \frac{(x+1)-1}{x\left(\sqrt{x+1}+1\right)}$$ Multiply.

$$= \frac{x}{x\left(\sqrt{x+1}+1\right)}$$ Simplify.

Solution

$$= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1}+1\right)}$$

Divide out common factor.

$$= \frac{1}{\sqrt{x+1}+1}$$

Simplify.

- $$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x+1}+1} \quad , x \neq 0$$

- Now you can evaluate the limit by direct substitution.

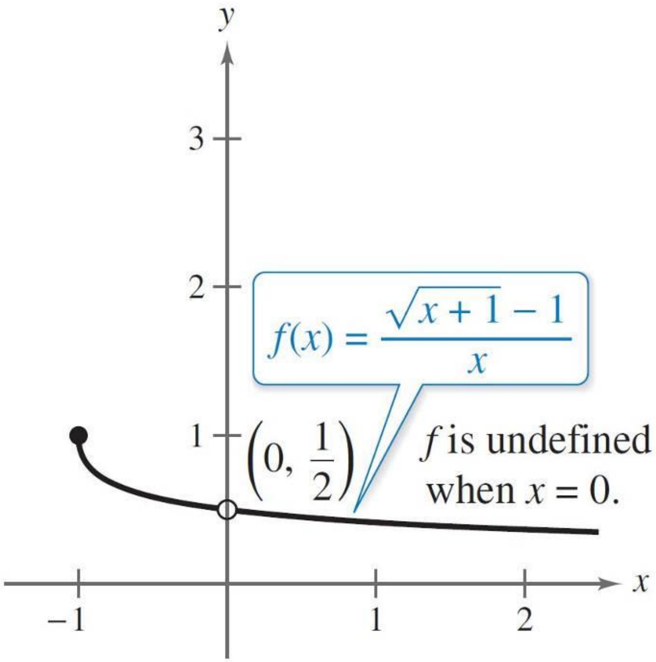
$$= \frac{1}{\sqrt{0+1}+1}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Solution

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881



A Limit from Calculus

A Limit from Calculus

- A Limit from Calculus, you will study an important type of limit from calculus—the limit of a *difference quotient*.

Example – *Evaluating a Limit from Calculus*

- For the function given by $f(x) = x^2 - 1$, find

$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$

- $$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{[(3 + h)^2 - 1] - [(3)^2 - 1]}{h}$$

- Direct substitution produces an indeterminate form.

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 1 - 9 + 1}{h}$$

Example – *Solution*

- $$= \lim_{h \to 0} \frac{6h + h^2}{h}$$

$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{6h + h^2}{h}$$

- By factoring and dividing out, you obtain the following.

$$= \lim_{h \to 0} \frac{h(6 + h)}{h}$$

Example – *Solution*

- $= \lim_{h\to 0} (6 + h)$
- $= 6 + 0$
- $= 6$
- So, the limit is 6.

The Sandwich Theorem:

If $g(x) \leq f(x) \leq h(x)$ for all $x \neq c$ in some interval about c and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} f(x) = L$.

Show that: $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

The maximum value of sine is 1

The minimum value of sine is -1

So: $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$

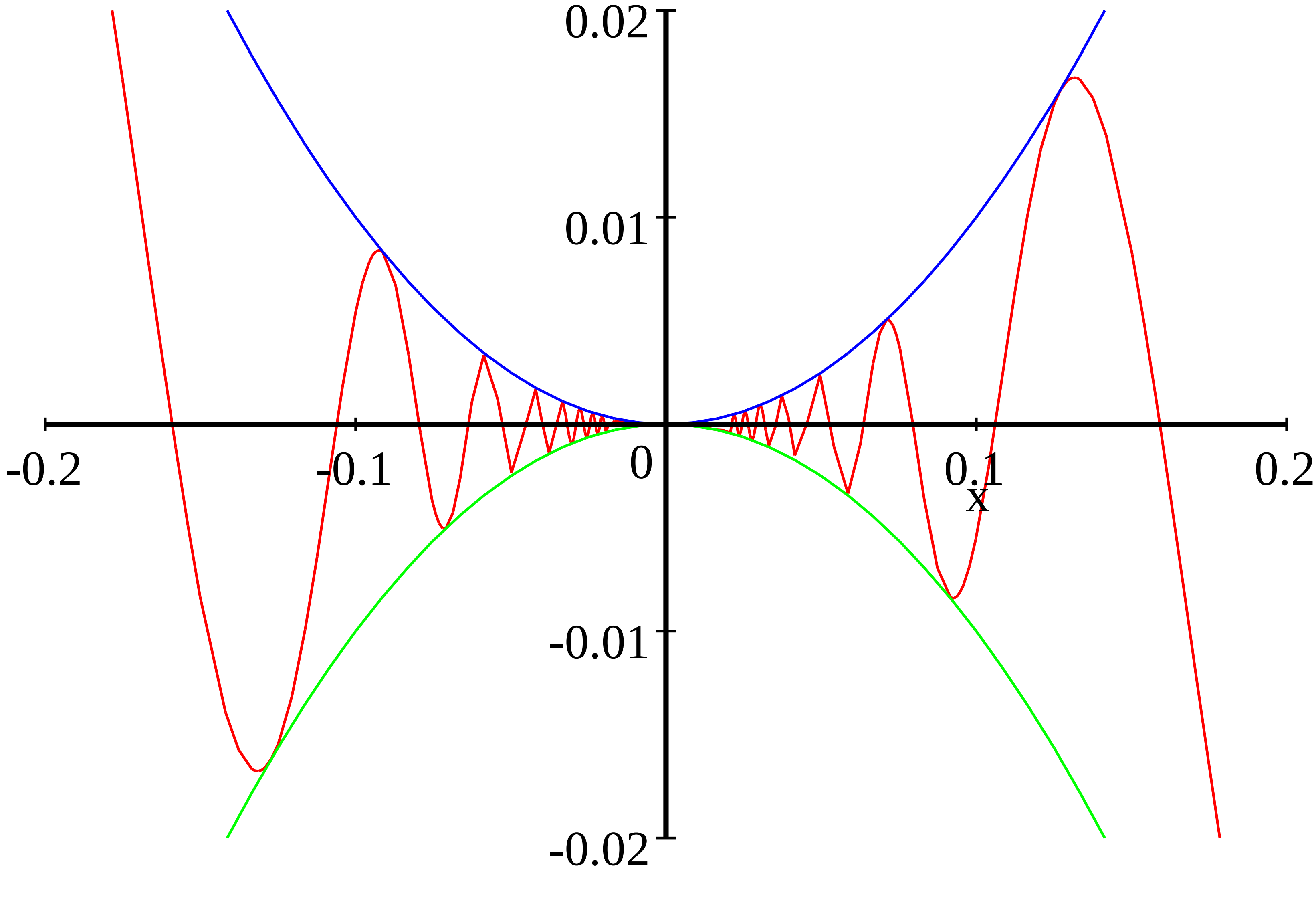
→

$$\lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$0 \le \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \le 0$$

By the sandwich theorem: $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

$\rightarrow$



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