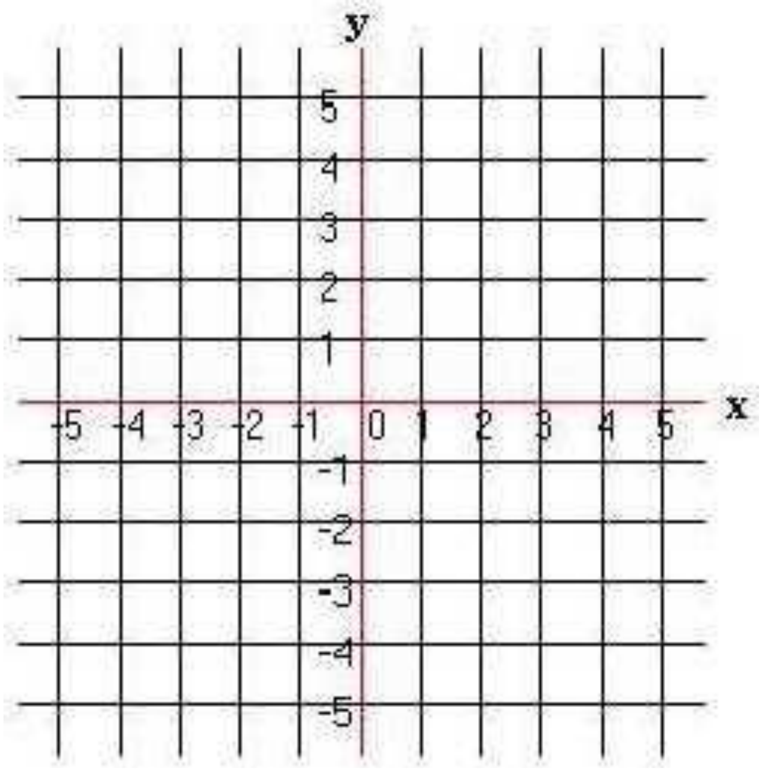
Inverses of Functions

Objective

- To be able to find inverses of functions.

Warm Up: Graph & give the domain & range.

$$f(x)=\begin{cases} x+5, & x<-3 \\ -2, & -3<x<1 \\ x-4, & x\geq 1 \end{cases}$$



Answer on Next Slide

Warm Up #3: Graph & give the domain & range.

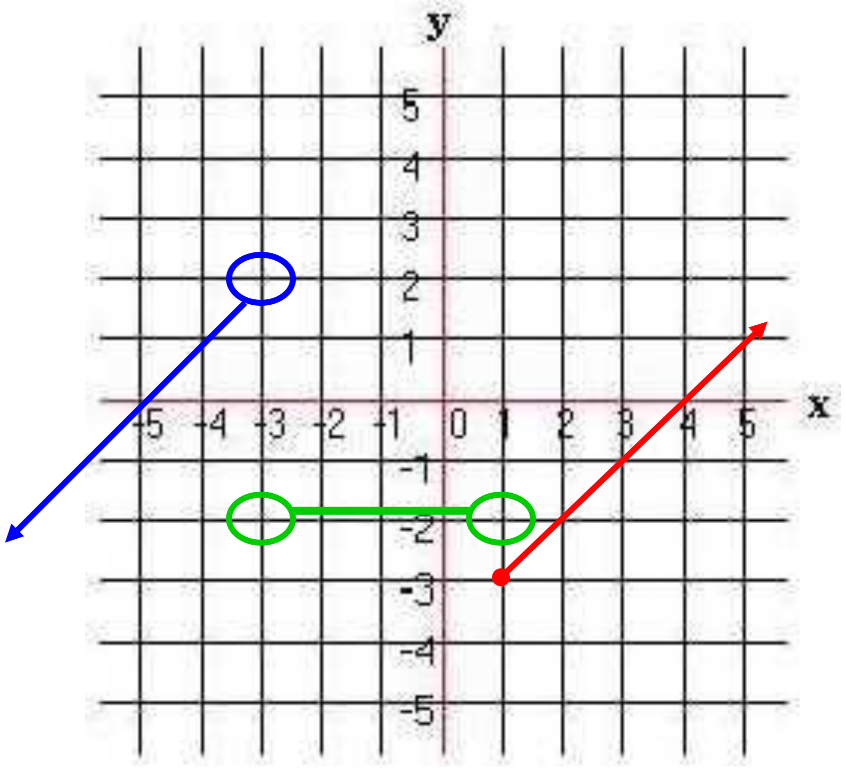
$$f(x)=\begin{cases} x+5, & x<-3 \\ -2, & -3<x<1 \\ x-4, & x\geq 1 \end{cases}$$

x	y
-3	2
-4	1
-5	0

x	y
1	-3
2	-2
3	-1
4	0

$\boxed{D:(-\infty,-3)\cup(-3,\infty)}$

$\boxed{R:(-\infty,-\infty)}$



- If f is a function from a to b, $f(a) = b$ we write $f: a \rightarrow b$. We say that f maps a to b
- If f is a function from x to y, $f(x) = y$ we write $f: x \rightarrow y$. We say that f maps x to y

- If $g(x) = x^2 + 9x - 5$ we write

  $\boldsymbol{g: x \to x^2 + 9x - 5}$

- If $f(x) = 4x^2 - x$ we write

  $f: x \to 4x^2 - x$

- If $f(x) = 17$ we write

  $f: x \to 17$

- If $l(x) = \cos(x)$ we write

  $l : x \rightarrow \cos(x)$

- If $f(3x) = 4x^2$ we write

  $f: 3x \rightarrow 4x^2$

- If $f: x \rightarrow 1/x$ we write

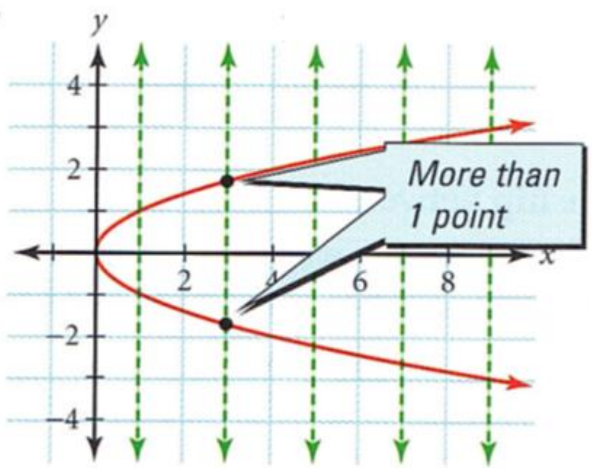
  $f(x) = 1/x$

Vertical Line Test

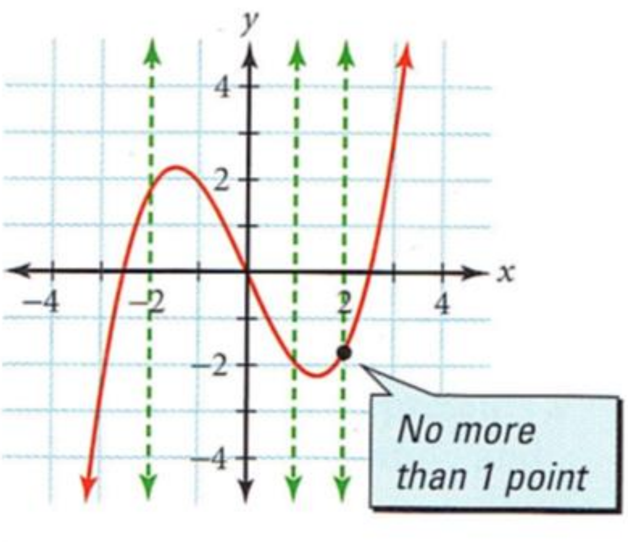
- Recall that a function passes the vertical line test.

Vertical Line Test

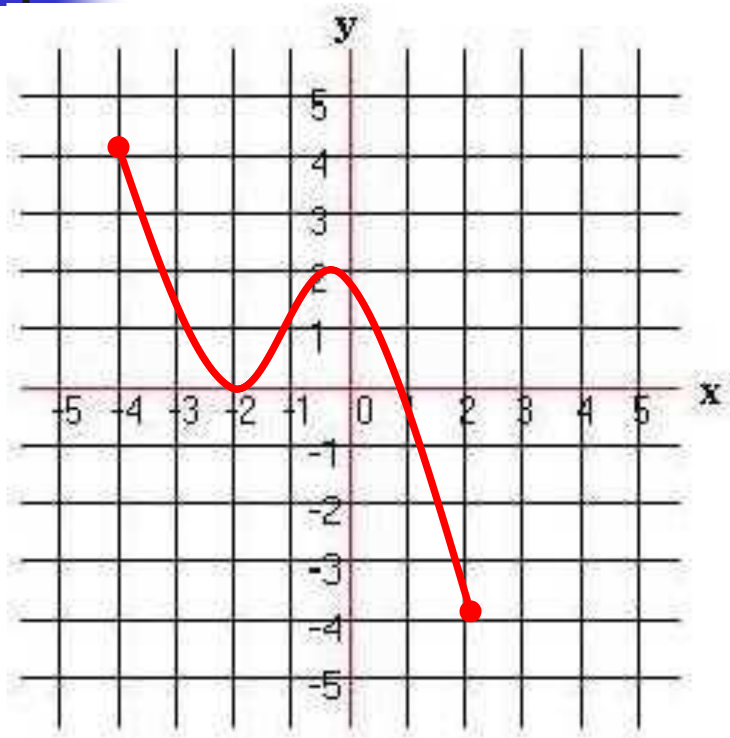
- Used to determine if a graph is a function.
- If a vertical line intersects the graph at more than one point, then the graph is **NOT** a function.



NOT a Function



Is it a function? Give the domain and range.

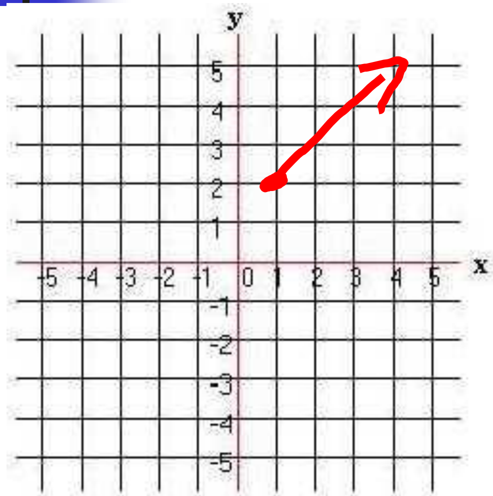


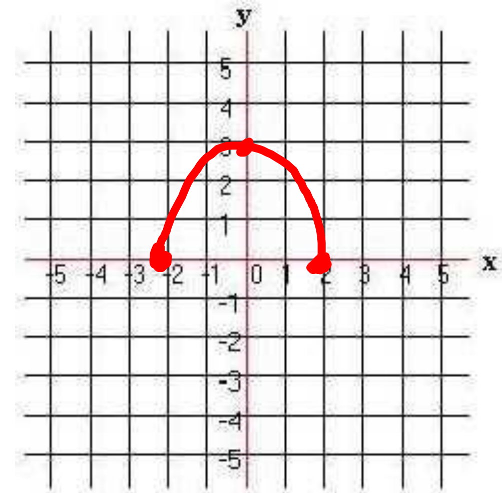
FUNCTION

$$Domain: [-4,2]$$

$$Range: [-4,4]$$

Give the Domain and Range.





$Domain: \ x \geq 1$

$Range: \ y \geq 2$

$Domain: -2 \leq x \leq 2$

$Range: 0 \leq y \leq 3$

Polynomial Functions

Definition of a Polynomial Function

Polynomial Function

A **polynomial function of degree *n*** is defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0,$$

for real numbers $a_n, a_{n-1}, \ldots, a_1$, and a_0, where $a_n \neq 0$ and n is a whole number.

Identifying Polynomial Functions

Circle only the polynomial functions.

$$f(x) = x^2 + 3x + 2$$

$$f(x) = \sqrt{x^5} + 3x^2 + 6$$

$$f(x) = \frac{3}{4}x^3 + 9x^2 + 10x$$

$$y = 2$$

$$y = \frac{3x^2 + 6x + 10}{2x}$$

EXAMPLE **Evaluating Polynomial Functions**

Let $f(x) = 4x^3 - 5x^2 + 7$. Find each value.

(b) $f(-3)$

$$
\begin{aligned}
f(x) &= 4x^3 - 5x^2 + 7 \\
f(\mathbf{-3}) &= 4 \cdot (\mathbf{-3})^3 - 5 \cdot (\mathbf{-3})^2 + 7 \\
&= 4 \cdot (-27) - 5 \cdot 9 + 7 \\
&= -108 - 45 + 7 \\
&= -146
\end{aligned}
$$

Adding and Subtracting Functions

Adding and Subtracting Functions

If $f(x)$ and $g(x)$ define functions, then

$$(f + g)(x) = f(x) + g(x) \quad \text{Sum function}$$

and $$(f - g)(x) = f(x) - g(x). \quad \text{Difference function}$$

In each case, the domain of the new function is the intersection of the domains of $f(x)$ and $g(x)$.

EXAMPLE Adding and Subtracting Functions

For the polynomial functions defined by

$$f(x) = 2x^2 - 3x + 4 \qquad \text{and} \qquad g(x) = x^2 + 9x - 5,$$

find **(a)** the sum and **(b)** the difference.

(a) $(f + g)(x) = f(x) + g(x)$ Use the definition.

$= (2x^2 - 3x + 4) + (x^2 + 9x - 5)$ Substitute.

$= 3x^2 + 6x - 1$ Add the polynomials.

(b) $(f - g)(x) = f(x) - g(x)$ Use the definition.

$= (2x^2 - 3x + 4) - (x^2 + 9x - 5)$ Substitute.

$= x^2 - 12x + 9$ Add.

EXAMPLE Adding and Subtracting Functions

For the polynomial functions defined by

$$f(x) = 4x^2 - x \qquad \text{and} \qquad g(x) = 3x,$$

find each of the following.

(a) $(f + g)(5)$

$(f + g)(5) = f(5) + g(5)$ Use the definition.

$= [4(5)^2 - 5] + 3(5)$ Substitute.

$= 110$

EXAMPLE **Adding and Subtracting Functions**

For the polynomial functions defined by

$f(x) = 4x^2 - x$ and $g(x) = 3x$,

find each of the following.

(a) $(f + g)(5)$

Alternatively, we could first find $(f + g)(x)$.

$(f + g)(x) = f(x) + g(x)$ Use the definition.

$= (4x^2 - x) + 3x$ Substitute.

$= 4x^2 + 2x$

EXAMPLE Adding and Subtracting Functions

For the polynomial functions defined by

$$f(x) = 4x^2 - x \qquad \text{and} \qquad g(x) = 3x,$$

find each of the following.

(b) $(f - g)(x)$ and $(f - g)(3)$

$$(f - g)(x) = f(x) - g(x) \qquad \text{Use the definition.}$$

$$= (4x^2 - x) - 3x \qquad \text{Substitute.}$$

$$= 4x^2 - 4x \qquad \text{Combine like terms.}$$

Then,

$$(f - g)(3) = 4(3)^2 - 4(3) = 24. \qquad \text{Substitute.}$$

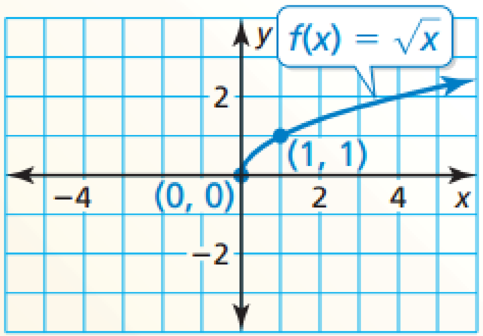
Confirm that $f(3) - g(3)$ gives the same result.

A power function of degree n is a function of the form

Core Concept

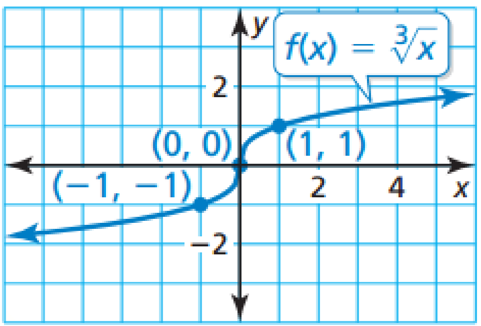
Parent Functions for Square Root and Cube Root Functions

The parent function for the family of square root functions is $f(x) = \sqrt{x}$.



Domain: $x \geq 0$, Range: $y \geq 0$

The parent function for the family of cube root functions is $f(x) = \sqrt[3]{x}$.



Domain and range: All real numbers

Exponential Functions

- A polynomial function has the basic form: $f(x) = x^3$
- An exponential function has the basic form: $f(x) = 3^x$
- An exponential function has the variable in the <u>exponent</u>, <u>not</u> in the <u>base</u>.
- General Form of an Exponential Function:

$$f(x) = N^x, N > 0$$

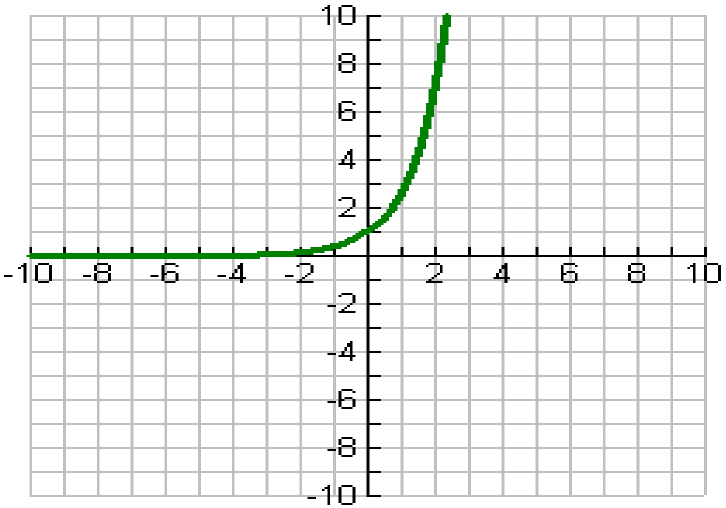
The Number e

A base often associated with exponential functions is:

$$e \approx 2.71828169$$

The Exponential Function

$$f(x) = e^x$$



Rational Functions

Rational functions are quotients of polynomial functions. This means that rational functions can be expressed as

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$. The **domain** of a rational function is the set of all real numbers except the x-values that make the denominator zero. For example, the domain of the rational function

This is $p(x)$.

$$f(x) = \frac{x^2 + 7x + 9}{x(x-2)(x+5)}$$

This is $q(x)$.

is the set of all real numbers except 0, 2, and -5.

Definition: Logarithmic Function

- For x > 0, b > 0 and b not equal to 1 toe logarithm of x with base b is defined by the following:

$$\log_b x = y \Leftrightarrow x = b^y$$

$$Sin = \frac{Opp\ \mathrm{Leg}}{Hyp}$$

$$Cos = \frac{Adj\ \mathrm{Leg}}{Hyp}$$

$$Tan = \frac{Opp\ \mathrm{Leg}}{Adj\ \mathrm{Leg}}$$

Inverse Trigonometric Function

- $y = \sin x$ has a unique inverse function called the **inverse sine function.** It is denoted by
- $y = \arcsin x$ or $y = \sin^{-1} x$.
- The notation $\sin^{-1} x$ is consistent with the inverse function notation $f^{-1}(x)$.

When we are trying to find a

side

we use sin, cos, or tan.

When we are trying to find an

angle

we use $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$.

Special Functions

Evaluating Piecewise Functions

- Piecewise functions are functions defined by **at least two equations**, each of which applies to a different part of the domain
- A piecewise function

$$f(x) := \begin{cases} x + 3 & x > 0 \\ 2 \cdot x - 1 & x \le 0 \end{cases}$$

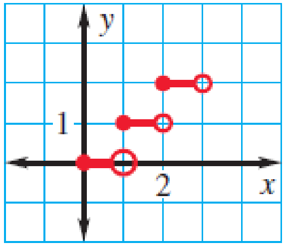
Step Functions

Looks like a flight of stairs

An example of a step function:

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } 1 \le x < 2 \\ 2, & \text{if } 2 \le x < 3 \end{cases}$$

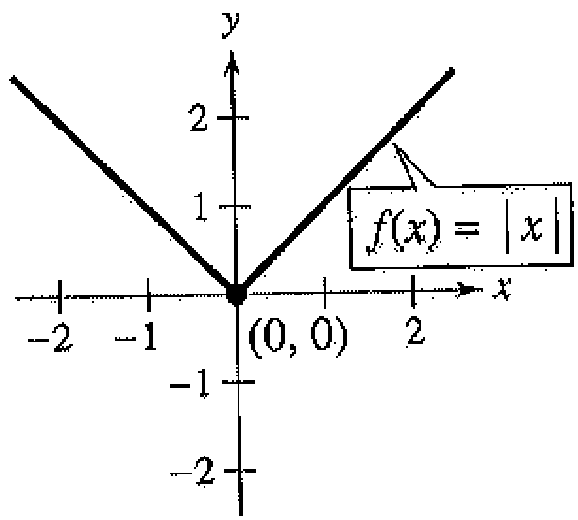
Graphically, the equation would look like this:



Absolute Value Function is a Piecewise Function

$$\textit{Graph of } f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$



Greatest Integer Function

The **greatest integer function,** usually written $\boldsymbol{f(x)} = [\![x]\!]$, is defined as follows:

$[\![x]\!]$ **denotes the largest integer that is less than or equal to $\boldsymbol{x}$.**

For example,

$[\![9]\!] = 9, \qquad [\![-3.8]\!] = -4, \qquad [\![5.7]\!] = 5.$

EXAMPLE **Graphing the Greatest Integer Function**

Graph $f(x) = [\![x]\!]$.

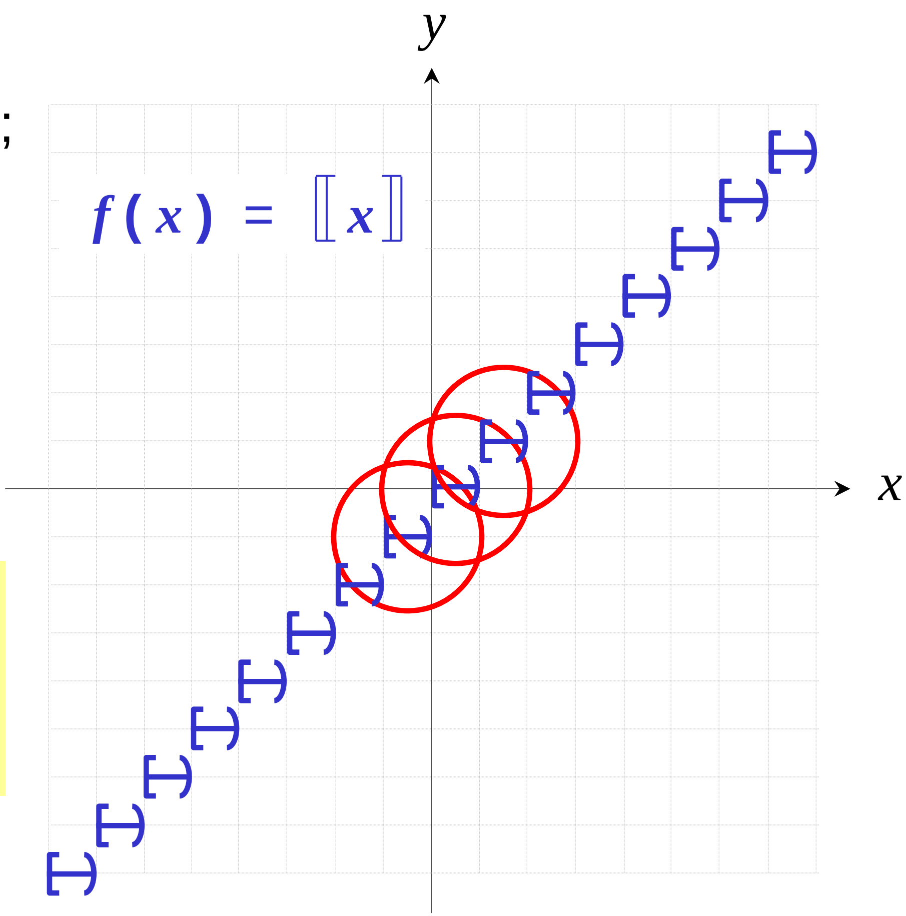
For $[\![x]\!]$,

if $-1 \le x < 0$, then $[\![x]\!] = -1$;

if $0 \le x < 1$, then $[\![x]\!] = 0$;

if $1 \le x < 2$, then $[\![x]\!] = 1$,

and so on.



The appearance of the graph is the reason that this function is called a **step function.**

Composition of Functions

Composition of Functions

If f and g are functions, then the **composite function**, or **composition**, of g and f is defined by

$$(g \circ f)(x) = g(f(x))$$

for all x in the domain of f such that $f(x)$ is in the domain of g.

EXAMPLE **Evaluating a Composite Function**

Let $f(x) = 3x^2 + 5$ and $g(x) = x - 7$. Find $(f \circ g)(2)$.

$(f \circ g)(2)$	$= f(g(\mathbf{2}))$	Definition
	$= f(\mathbf{2} - 7)$	Use the rule for $g(x)$; $g(2) = 2 - 7$.
	$= f(\mathbf{-5})$	Subtract.
	$= 3(\mathbf{-5})^2 + 5$	Use the rule for $f(x)$; $f(-5) = 3(-5)^2 + 5$.
	$= 80$	

EXAMPLE **Evaluating a Composite Function**

Let $f(x) = 3x^2 + 5$ and $g(x) = x - 7$. Now find $(g \circ f)(2)$.

$(g \circ f)(2)$	$= g(f(\mathbf{2}))$	Definition
	$= g(3(\mathbf{2})^2 + 5)$	Use the rule for $f(x)$; $f(2) = 3(2)^2 + 5$.
	$= g(\mathbf{17})$	Square, multiply, and then add.
	$= \mathbf{17} - 7$	Use the rule for $g(x)$; $g(17) = 17 - 7$.
	$= 10$	

Additional Graphs of Functions; Composition

EXAMPLE Evaluating a Composite Function

Let $f(x) = 3x^2 + 5$ and $g(x) = x - 7$. Notice that $(f \circ g)(2) \neq (g \circ f)(2)$.

$$\begin{aligned}(f \circ g)(2) &= f[g(2)] \\ &= f(2 - 7) \\ &= f(-5) \\ &= 3(-5)^2 + 5 \\ &= 80\end{aligned}$$

$$\begin{aligned}(g \circ f)(2) &= g[f(2)] \\ &= g(3(2)^2 + 5) \\ &= g(17) \\ &= 17 - 7 \\ &= 10\end{aligned}$$

In general, $(f \circ g)(2) \neq (g \circ f)(2)$.

EXAMPLE **Finding Composite Functions**

Let $f(x) = 5x + 1$ and $g(x) = x^2 - 4$. Find each of the following.

(a) $(f \circ g)(-3)$

$$(f \circ g)(\mathbf{-3}) = f\,[\,g(\mathbf{-3})\,]$$

$$= f\,((\mathbf{-3})^2 - 4) \qquad g(x) = x^2 - 4$$

$$= f(\mathbf{5})$$

$$= 5(\mathbf{5}) + 1 \qquad f(x) = 5x + 1$$

$$= 26$$

EXAMPLE **Finding Composite Functions**

Let $f(x) = 5x + 1$ and $g(x) = x^2 - 4$. Find each of the following.

(b) $(f \circ g)(n)$

$$(f \circ g)(n) = f[g(n)]$$

$$= f(n^2 - 4) \qquad g(x) = x^2 - 4$$

$$= 5(n^2 - 4) + 1 \qquad f(x) = 5x + 1$$

$$= 5n^2 - 19$$

EXAMPLE Finding Composite Functions

Let $f(x) = 5x + 1$ and $g(x) = x^2 - 4$. Find each of the following.

(c) $(g \circ f)(n)$

$$(g \circ f)(n) = g[f(n)]$$

$$= g(5n + 1) \qquad f(x) = 5x + 1$$

$$= (5n + 1)^2 - 4 \qquad g(x) = x^2 - 4$$

$$= 25n^2 + 10n + 1 - 4$$

$$= 25n^2 + 10n - 3$$

Intro to Inverses

- A function describes the relationship between 2 variables, applying a rule to an input that generates exactly one output.
- For such relationships, we are often compelled to “reverse” or “undo” the rule.

Progress of Inverses Throughout Math

- Learned Addition and then its inverse operation Subtraction.
- Learned Multiplication and then its inverse operation Division.
- Learning Perfect Squares connects with extracting Square Roots
- **Basically – Inverses are a second operation that reverses the first one!**

Inverse of a relation

- The inverse of the ordered pairs (**x**, **y**) is the set of all ordered pairs (**y**, **x**).

- The Domain of the function is the range of the inverse and the Range of the function is the Domain of the inverse.

- Symbol: $\boxed{f^{-1}(x)}$

In other words, switch the x's and y's!

Example: {(1,2), (2, 4), (3, 6), (4, 8)}

Inverse:

$$\{(2,1),(4,2),(6,3),(8,4)\}$$

To find an inverse…

- Switch the *x*'s and *y*'s.
- Solve for *y*.
- Change to functional notation.

Find Inverse: $\boxed{f(x) = 8x - 1}$

$$f(x) = 8x - 1$$

$$y = 8x - 1$$

$$x = 8y - 1$$

$$8y = x + 1$$

$$y = \frac{x+1}{8}$$

$$f^{-1}(x) = \frac{x+1}{8}$$

Find Inverse: $f(x) = 8x - 2$

$$f(x) = 8x - 2$$

$$y = 8x - 2$$

$$x = 8y - 2$$

$$8y = x + 2$$

$$y = \frac{x+2}{8}$$

$$f^{-1} = \frac{x+2}{8}$$

Find Inverse: $f(x) = \frac{3x+1}{2}$

$$f(x) = \frac{3x+1}{2}$$

$$y = \frac{3x+1}{2}$$

$$x = \frac{3y+1}{2}$$

$$3y+1 = 2x$$

$$3y = 2x-1$$

$$y = \frac{2x-1}{3}$$

$$f^{-1} = \frac{2x-1}{3}$$

Find Inverse: $\boxed{f(x) = x^2 + 4}$

$$f(x) = x^2 + 4$$

$$y = x^2 + 4$$

$$x = y^2 + 4$$

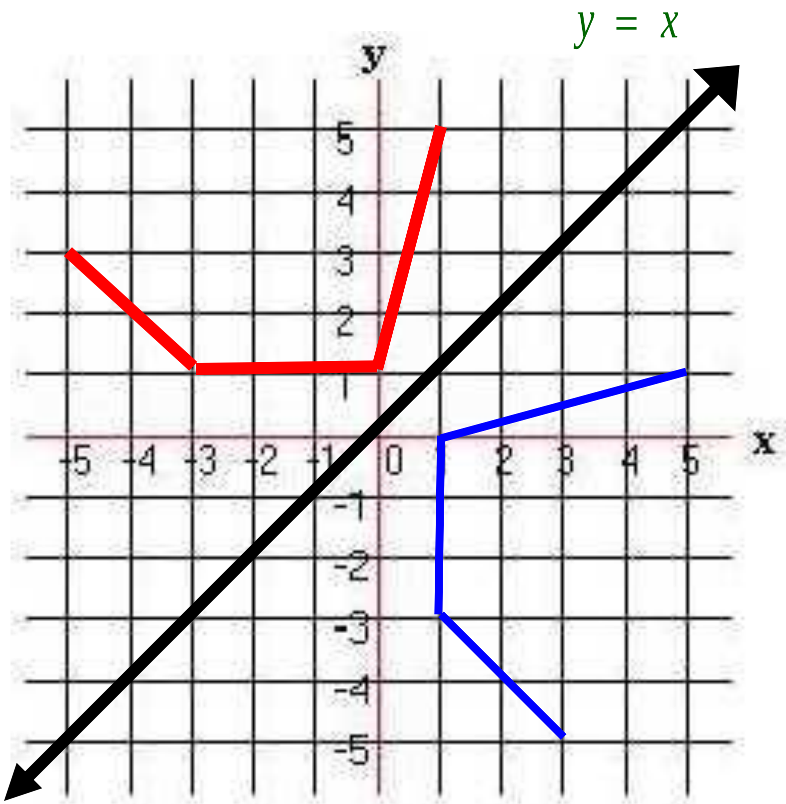
$$y^2 = x - 4$$

$$y = \sqrt{x - 4}$$

$$f^{-1}(x) = \pm\sqrt{x - 4}$$

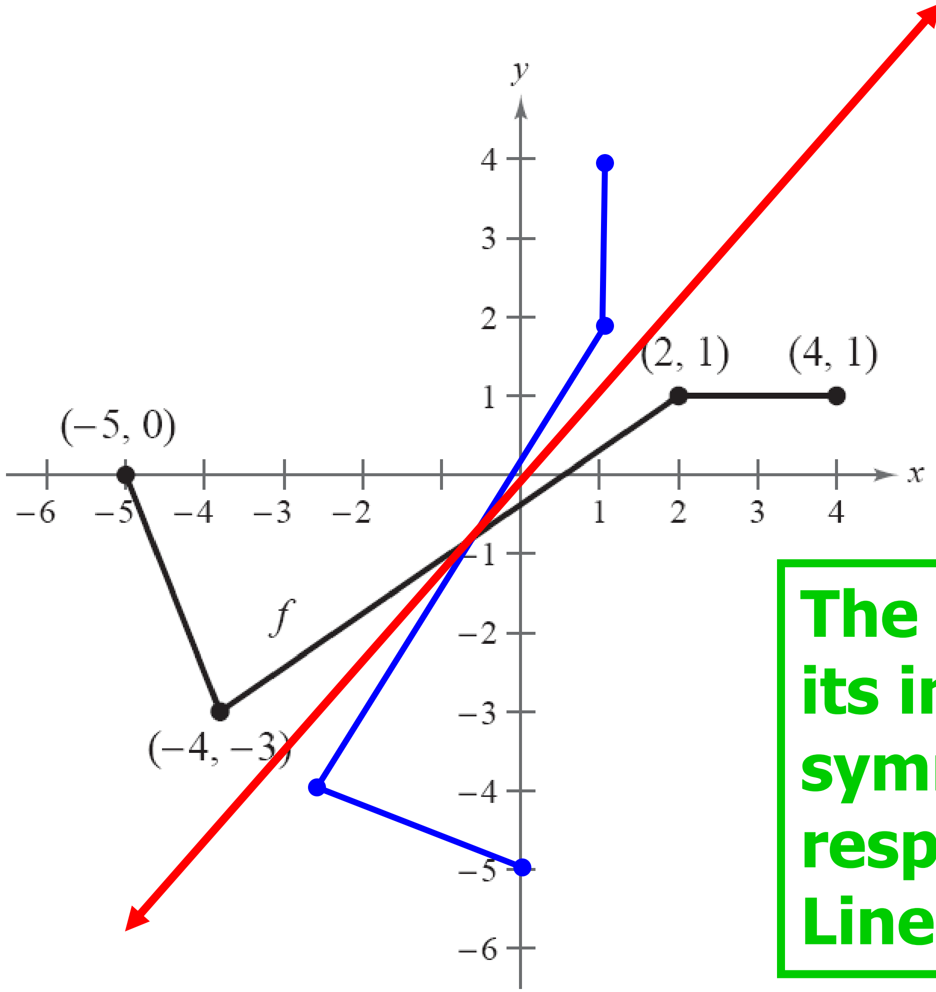
Draw the inverse. Compare to the line y = x. What do you notice?

$(-5,3)$
$(-4,2)$
$(-3,1)$
$(-2,1)$
$(-1,1)$
$(0,1)$
$(1,5)$



$(3,-5)$
$(2,-4)$
$(1,-3)$
$(1,-2)$
$(1,-1)$
$(1,0)$
$(5,1)$

Graph the inverse of the following:



x	y
0	-5
-3	-4
1	2
1	4

The function and its inverse are symmetric with respect to the Line y = x.

Things to note..

- The domain of $f^{-1}(x)$ is the range of *f(x).*

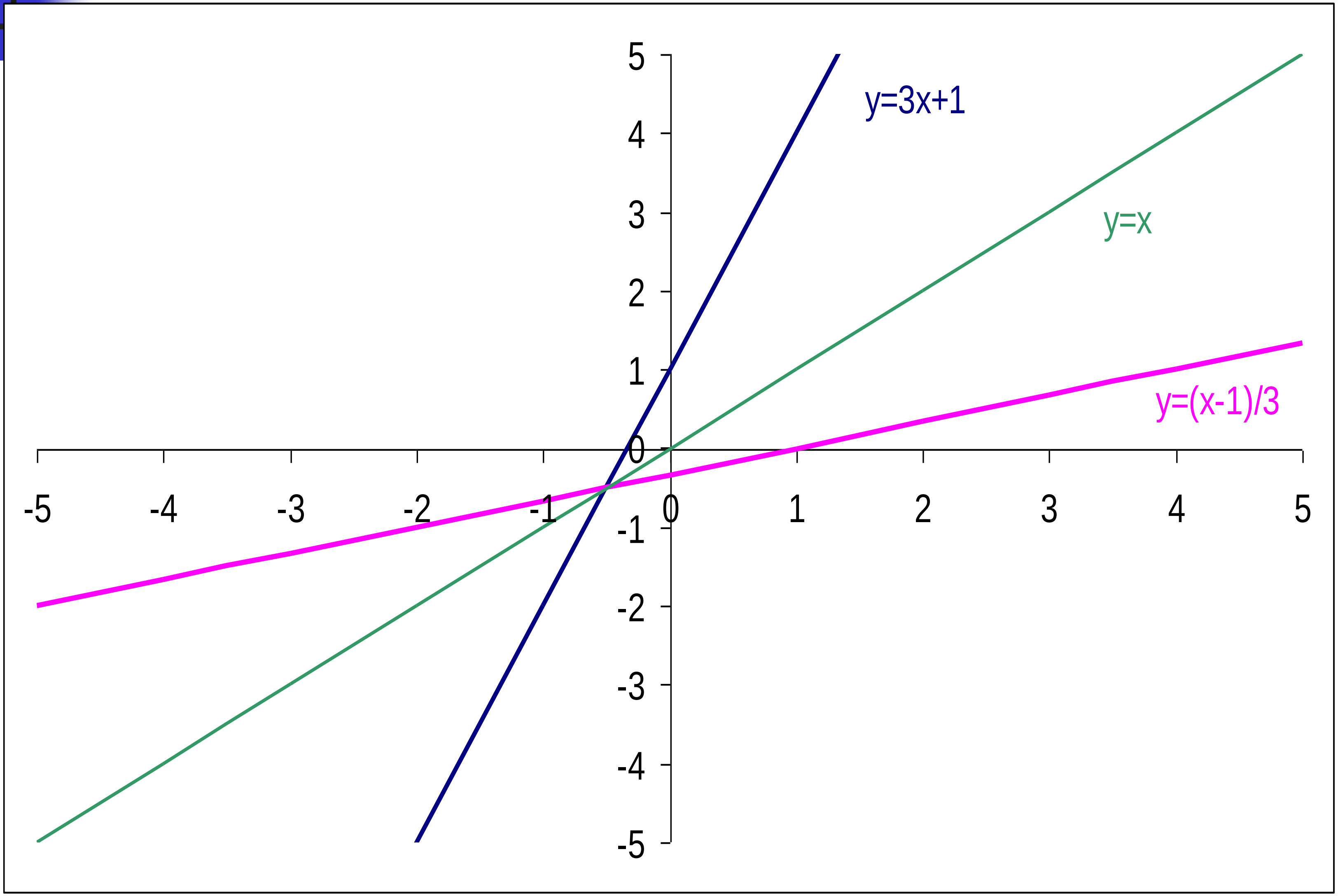
- The graph of an inverse function can be found by reflecting a function in the line *y=x.*

  Check this by plotting $y = 3x + 1$ and

  $$y = \frac{x-1}{3}$$

Take a look

Reflecting..



Find the inverse of the function.

$$f(x) = \sqrt{x}$$

$$x = \sqrt{y}$$

$$x^2 = \left(\sqrt{y}\right)^2$$

$$y = x^2$$

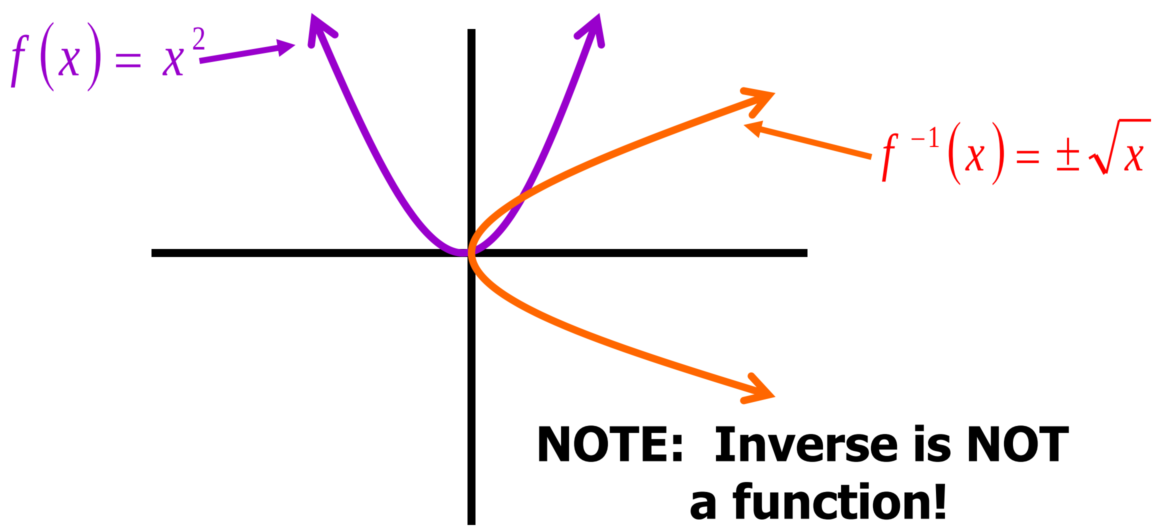
Is the inverse also a function? Let's look at the graphs.

$$If \quad f(x) = x^2,$$

$$x = y^2$$

$$\sqrt{x} = \sqrt{y^2}$$

$$y = \pm\sqrt{x}$$ Inverse



NOTE: Inverse is NOT a function!

Is it an Inverse?

A function can only have an inverse if it is one-to-one.

You can use the horizontal line test on graphical representations to see if the function is one-to-one.

Composition and Inverses

- If *f* and *g* are functions and

$$\boxed{(f \circ g)(x) = (g \circ f)(x) = x,}$$

then *f* and *g* are inverses of one another.

!!!!!!!!!!!!!!!!!!!!!!

Example: Show that the following are inverses of each other.

$$f(x) = 7x - 2 \; and \; g(x) = \frac{1}{7}x + \frac{2}{7}$$

$$\begin{aligned}(f \circ g)(x) &= 7\left(\frac{1}{7}x + \frac{2}{7}\right) - 2 \\ &= x + 2 - 2 \\ &= x\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= \frac{1}{7}(7x - 2) + \frac{2}{7} \\ &= x - \frac{2}{7} + \frac{2}{7} \\ &= x\end{aligned}$$

The composition of each both produce a value of x; Therefore, they are inverses of each other.

Are f & g inverses?

$$f(x) = x^3 - 4$$
$$g(x) = \sqrt[3]{x+4}$$

$$(f \circ g)(x) = \left(\sqrt[3]{x+4}\right)^3 - 4$$
$$= x + 4 - 4$$
$$= x$$

$$(g \circ f)(x) = \sqrt[3]{x^3 - 4 + 4}$$
$$= \sqrt[3]{x^3}$$
$$= x$$

YES!

You Try....

- Show that

$$\boxed{f(x) = 4x - 3 \; and \; g(x) = \frac{1}{4}x + \frac{3}{4}}$$

- are inverses of each other.

$$(f \circ g)(x) = (g \circ f)(x) = x$$

Therefore, they ARE inverses of each other.

Are f & g inverses?

$$f(x) = 3x - 2$$

$$g(x) = \frac{x+2}{3}$$

$$(f \circ g)(x) = 3\left(\frac{x+2}{3}\right) - 2$$

$$= x + 2 - 2$$

$$= x$$

$$(g \circ f)(x) = \frac{3x - 2 + 2}{3}$$

$$= \frac{3x}{3}$$

$$= x$$

YES!