Differentiation Rules

The Product and Quotient Rules

The Product Rule

By analogy with the Sum and Difference Rules, one might be tempted to guess, that the derivative of a product is the product of the derivatives.

We can see, however, that this guess is wrong by looking at a particular example.

Let $f(x) = x$ and $g(x) = x^2$. Then the Power Rule gives $f'(x) = 1$ and $g'(x) = 2x$.

But $(fg)(x) = x^3$, so $(fg)'(x) = 3x^2$. Thus $(fg)' \neq f'g'$.

The Product Rule

So we have proved Equation 2, known as the Product Rule, for all differentiable functions *u* and *v*.

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Notice that this is not just the product of two derivatives.

This is sometimes memorized as: $d(uv) = u\ dv + v\ du$

$$\frac{d}{dx}\left[\left(x^2+3\right)\left(2x^3+5x\right)\right] = \left(x^2+3\right)\left(6x^2+5\right) + \left(2x^3+5x\right)\left(2x\right)$$

$$\frac{d}{dx}\left(2x^5+5x^3+6x^3+15x\right)$$

$$\frac{d}{dx}\left(2x^5+11x^3+15x\right)$$

$$6x^4+5x^2+18x^2+15+4x^4+10x^2$$

$$10x^4+33x^2+15$$

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Example

(a) If $f(x) = xe^x$, find $f'(x)$.

(b) Find the nth derivative, $f^{(n)}(x)$.

Solution:

(a) By the Product Rule, we have

$$f'(x) = \frac{d}{dx}(xe^x)$$

$$= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$

$$= xe^x + e^x \cdot 1 = (x + 1)e^x$$

The Quotient Rule

The Quotient Rule

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

In words, the Quotient Rule says that the *derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

or

$$d\left(\frac{u}{v}\right)=\frac{v\ du-u\ dv}{v^2}$$

$$\frac{d}{dx}\frac{2x^3+5x}{x^2+3}=\frac{\left(x^2+3\right)\left(6x^2+5\right)-\left(2x^3+5x\right)\left(2x\right)}{\left(x^2+3\right)^2}$$

Example

Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6)\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Higher Order Derivatives:

$$y' = \frac{dy}{dx}$$ is the first derivative of y with respect to x.

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$$ is the <u>second</u> derivative.

(y double prime)

$$y''' = \frac{dy''}{dx}$$ is the third derivative.

We will learn later what these higher order derivatives are used for.

$$y^{(4)} = \frac{d}{dx}y'''$$ is the fourth derivative.