(1)

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

put

$x = X + h \quad , y = Y + k.$

$$Y' = \frac{a_1X + b_1Y + \overbrace{a_1h + b_1k + c_1}}{a_2X + b_2Y + \underbrace{a_2h + b_2k + c_2}}$$

If equation

$a_1h + b_1k + c_1 = 0$

$a_2h + b_2k + c_2 = 0$ } (*)

has soln.

then the above eqn is Homogeneous

If the eqn (*) have no Solution.

then

$a_1b_2 - a_2b_1 = 0$

and in this case either the

substitution

$u = a_1h + b_1k + c_1$, or

$u = a_2h + b_2k + c_2$

lead to a separate from Variable

(h, k)

$$Y' = \frac{a_1X + b_1Y}{a_2X + b_1X}$$

which is Homogeneous eqn.

$Y = uX$

$Y' = \cancel{u\,dx}\ \ u + x\frac{du}{dx}$

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$$y' = \frac{a_1 x + b_1 y}{a_2 x + b_2 y}$$

$$u + x\frac{du}{dx} = \frac{a_1 x + b_1 ux}{a_2 x + b_2 ux} = \frac{a_1 + b_1 u}{a_2 + b_2 u}$$

$$x\frac{du}{dx} = \frac{a_1 + b_1 u}{a_2 + b_2 u} - \frac{u}{1}$$

$$= \frac{\cancel{a_1 + b_1} u - u}{}$$

$$x\frac{du}{dx} = \frac{a_1 + b_1 u - u(a_2 + b_2 u)}{a_2 + b_2 u}$$

$$x\frac{du}{dx} = \frac{a_1 + b_1 u - u a_2 - b_2 u^2}{a_2 + b_2 u}$$

$$\int \frac{dx}{x} = \int \frac{a_2 + b_2 u}{a_1 + b_1 u - u(a_2 + b_2 u)} \, du$$

$$u = \frac{y}{x} \qquad Y = y - k$$
$$X = x - h$$

* Exact equations:

Suppose 1st order diffrensiable eqn $y' = f(x,y)$ is of the form.

$$y' = \frac{-M(x,y)}{N(x,y)}$$

or $$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

$$N(x,y)\,dy = -M(x,y)\,dx.$$

$$M(x,y)dx + N(x,y)dy = 0. \quad \text{— ①}$$

where M and N are real valued function defined for x and y on some rectangle R. The eqn ① is said to be exact on R. If there exists a function F having continuous first partial derivative. Such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ in R.

① Theorem: Suppose the equation

$$M(x,y)dx + N(x,y)dy = 0 \quad \text{— ①}$$

is exact in a rectangle R, and F is a real valued function such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ — ②

in R. Every diffrentiable function ϕ defined implicitly by a relation is

$F(x,y) = c$

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is solu^n of (1) and every solu^n of (1) where graph lies in R arise this way.

proof :- If eq^n (1) is exact in R. and F is function satisfied

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N$$

then

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0 \qquad \text{--- (3)}$$

$$dF = 0$$

taking integration

$$\int dF = \int 0$$

$$F(x, y) = c.$$

If ϕ is any solution on some interval I, then

$$\frac{\partial F}{\partial x}(x, \phi(x))\, dx + \frac{\partial F}{\partial y}(x, \phi(x))\, dy = \qquad \text{--- (4)}$$

$x \in I$

of $\phi(x) = F(x, \phi(x))$ the eq^n (4)

$\phi'(x) = 0$ at hence

$F(x, \phi(x)) = c \Rightarrow \phi$ is sol^n of (1)

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* Theorem: let M, N be two real valued functions which have continuous 1^{st} order partial derivatives on some Rectangle $R: |x - x_0| \le a$; $|y - y_0| \le b$; then the eq^n

$$M(x,y)dx + N(x,y)dy = 0$$

is exact iff and only iff — ①

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad - ②$$

⇒ Proof: Suppose eq^n ① is exact.
i.e ∃ $f(x,y) = c$ s.t.

$$\frac{\partial f}{\partial x} = M \quad \& \quad \frac{\partial f}{\partial y} = N$$

$\quad$ L (*) $\qquad\qquad$ L (**)

eq^n (*) diff w.r.t. y & eq^n (**) diff w.r.t. x.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \& \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

same

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Suppose $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is holds

We have to find a function f satisfied $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$

we note that if we has such a function then.

~~F(x,y)=~~

$F(x,y) - F(x_1,y_1) = F(x,y) - F(x_0,y_0) + F(x_0,y_0) - F(x_1,y_1)$

$$= \int_{x_0}^{x} \frac{\partial F}{\partial x}(s,y)\,ds + \int_{y_0}^{y} \frac{\partial F}{\partial y}(x,t)\,dt.$$

$$= \int_{x_0}^{x} M(s,y)\,ds + \int_{y_0}^{y} N(x,t)\,dt.$$

similarly

$F(x,y) - F(x_2,y_2) = F(x,y) - F(x_1,y_1) + F(x_1,y_1) - F(x_2,y_2)$

$$= \int_{y_0}^{y} \frac{\partial F}{\partial y}(x,t)\,dt + \int_{x_0}^{x} \frac{\partial F}{\partial x}(s,y)\,ds.$$

$$= \int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,y)\,ds.$$

Now we define F by

$$F(x,y) = \int_{x_0}^{x} M(s,y)\,dS + \int_{y_0}^{y} N(x,t)\,dt.$$

$\Rightarrow F(x,y) = 0$ at that.

$\frac{\partial F}{\partial x}(x,y) = M(x,y)$. For all $(x,y) \in R$.

If $F(x,y) = \int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,t)\,ds$

then

$$\frac{\partial F}{\partial y}(x,y) = N(x,y).$$

Ex] $\frac{dy}{dx} = \cancel{2x^2-2xy} \quad \frac{3x^2-2xy}{x^2-2y}$

→ $(3x^2-2xy)\,dx \cancel{+} (x^2-2y)\,dy = 0$

~~$N(x,y) = 3x^2-2xy$~~

~~$M(x,y) = x^2-2y$~~

$(3x^2-2xy)\,dx + (2y-x^2)\,dy = 0$

$M(x,y) = 3x^2-2xy$

$N(x,y) = 2y-x^2$

$$\frac{\partial M}{\partial y} = -2x$$

$$\frac{\partial N}{\partial x} = -2x.$$

which show that the eqn is exact.

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy.$$

$\frac{\partial F}{\partial x} = M$

$\frac{\partial F}{\partial y} = N =$

Integr

$F(x,y) =$

where f

$\frac{\partial f}{\partial y} =$

$-x^2 + f'(y) =$

$f'(y) =$

Integra

$f(y) =$

$F(x,y) =$

Thm: ① $M(x,y$

→ where

derivatives

A functio

continuou

is an in

iff and

$u\left(\frac{\partial M}{\partial y} - \right.$

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$\frac{\partial f}{\partial x} = M = 3x^2 - 2xy$

$\frac{\partial f}{\partial y} = N = 2y - x^2$

integrate $M = 3x^2 - 2xy$

$f(x,y) = x^3 - x^2y + f(y)$

∵ (because f is function of x, y).

where f is ~~ind~~ independent of x

$\frac{\partial f}{\partial y} = N$

$-x^2 + f'(y) = 2y - \cancel{x^2}$

$f'(y) = 2y$

Integrate we get

$\boxed{f(y) = y^2}$

$F(x,y) = x^3 - x^2y + y^2 = c$.

① $M(x,y)\,dx + N(x,y)\,dy = 0$ —①

→ where M, N are contineous partial derivatives on same Rectangle R.

A function $u(x,y)$ on R having contineous first partial derivative is an integrating factor of ① iff and only if

$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}$ on R

$$= \int_{x_0}^{x} \left[\frac{\partial}{\partial x} f(t, y(t)) \right] dt + f(x, y(x))$$

$\therefore$ $y' = 0 + f(x, y(x))$

$$\boxed{y' = f(x, y(x))}$$

* Thm : A function ϕ is a solution to initial value problem
$y' = f(x, y)$; $y(x_0) = y_0$ — ①
on an interval I if and only if is a solution of the integral eq

$$y = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \quad — ②$$

on I.

Proof: ent

IVP $\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$ — (1)

integral eqⁿ

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt. \quad \text{— (2)}$$

$\phi = \psi$

As a first approximation to a solution we consider the function ϕ_0 ~~denot~~ defined by:

$$\phi_0(x) = y_0$$

This function satisfies initial condition $\phi_0(x_0) = y_0$

But does not in general satisfy (2)

However, if we compute

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\,dt$$

we might expect that ϕ_1 is closer approximation to a solution then ϕ_0.

Similarly,

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f(t, \phi_1(t))\,dt.$$

In fact, if we continue the process and define successively

$$\phi_0(x) = y_0$$

$$\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_n(t))\,dt \qquad n = 0, 1, 2, \ldots \quad (3)$$

we might expect on taking limit as $n \to \infty$ i.e

$$\phi_n(x) \to \phi(x) \quad (\text{say})$$

where ϕ satisfy

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt.$$

Then ϕ will be solution we call the function $\phi_0, \phi_1 \ldots$ defined by (3) successive approximation to a sol^n of (2) hence (1)

eg] $\begin{cases} y' = xy \\ y(0) = 1 \end{cases}$ IVP.

→ First find integral eq^n.

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt.$$

$$\boxed{y(x) = 1 + \int_0^x t\,y(t)\,dt.}$$

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$$\phi_0(x) = y_0 = 1$$

$$\phi_1(x) = y_0 + \int_0^x t\,\phi_0(t)\,dt = 1 + \int_0^x 1 \cdot t\,dt$$

$$= 1 + \frac{x^2}{2}$$

$$\phi_2(x) = 1 + \int_0^x t\left(1 + \frac{t^2}{2}\right)dt$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{8}$$

$\vdots$

$$\phi_n(x) = 1 + \frac{x^2}{2} + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{x^2}{2}\right)^3 + \frac{1}{n!}\left(\frac{x^2}{2}\right)^n$$

~~If we take~~ $\boxed{y = \frac{x^2}{2}}$

$$\lim_{n \to \infty} \phi_n(x) \to e^{x^2/2} = \phi(x)$$

$$\phi(x) = e^{\frac{x^2}{2}}$$

$$\phi(0) = 1$$

$$\phi'(x) = x e^{x^2/2} = x \cdot \phi(x)$$

$$\boxed{y' = xy}$$

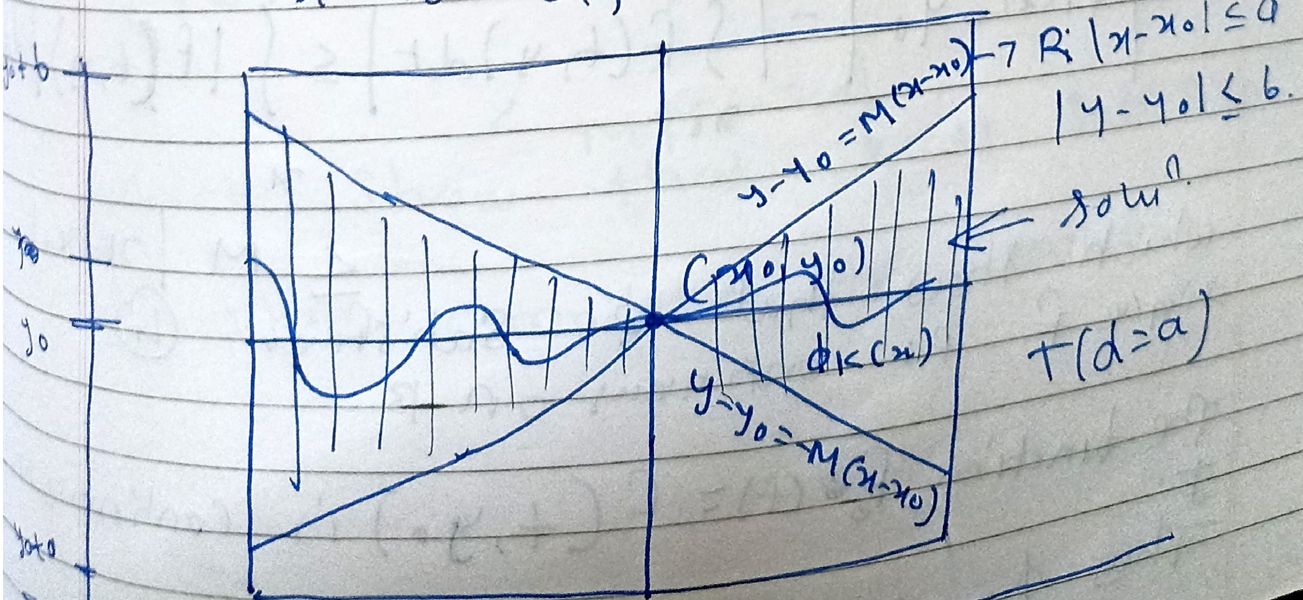
Thm: (3) The successive approximation ϕ_k defined exacts on continuous function as
$I \quad |x-x_0| \le \alpha = \min\left\{a, \frac{b}{M}\right\}$

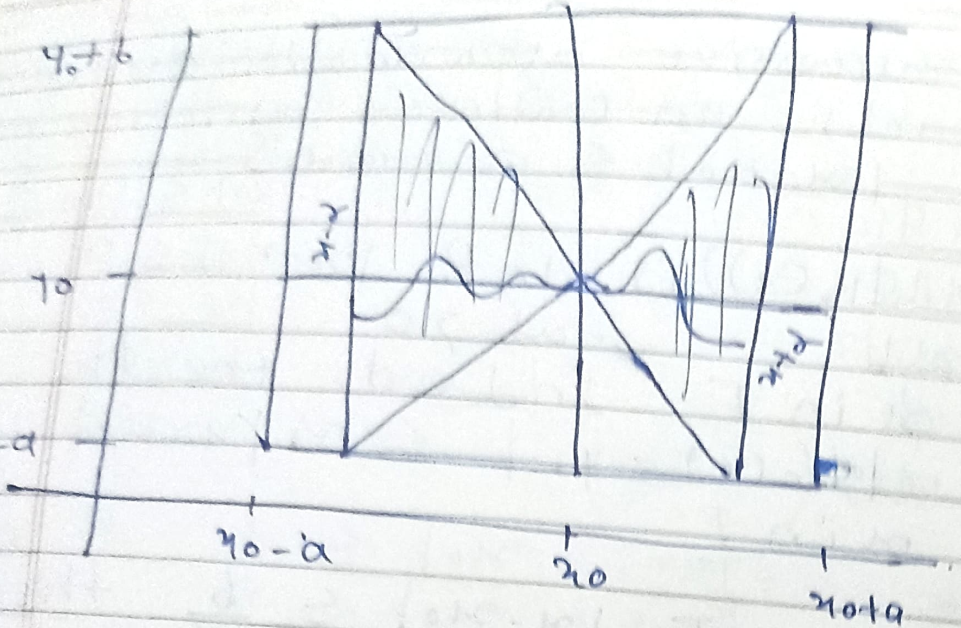
and $(x, \phi_k(x))$ is in R $|x-x_0| \le a$
$|y-y_0| \le b$ $\quad ab > 0$
for all x in I. Indeed, the ϕ_k satisfy $|\phi_k(x) - y_0| \le M|x-x_0|$
for all x in I.

since for $x \in I$, $|x-x_0| \le \frac{b}{M}$ then

$|\phi_k(x) - y_0| \le b$
for all x in I, which should that the point $(x, \phi_k(x))$ are in R.
for x in I. The graph of each $\phi_k(x)$ lies in the region T in R bounded by two lines
$y - y_0 = M(x-x_0)$, $y - y_0 = -M(x-x_0)$

and
$x - x_0 = \alpha$, $\quad x - x_0 = -\alpha$.





$T(\alpha = b/M)$

Clearly $\phi_k(x)$ ~~exw~~ exacts on I as a continuous function and satisfies solution

$$|\phi_k(x) - y_0| \le M|x - x_0| \quad \text{--- (4)}$$

* with $k=0$. now

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, y)\, dt$$

and Here.

$$\left|\phi_1(x) - y_0\right| = \left|\int_{x_0}^{x} f(t,y)\,dt\right| \le \int_{x}^{x} |f(t,y)|\,dt$$

$$\le M|x - x_0|$$

which show that ϕ_1 satisfies (4) since f is continuous on R.

The function $f_0(t) = f(t, y_0)$ is continuous on I.

Thus ϕ_1 which is given by.

$$\phi_1(x) = y_0 + \int_{x_0}^{x} F_0(t)\,dt.$$ is Contineous on I

now

Assume that the result hold for $\phi_0, \phi_1 \ldots \phi_k$ we prove that it for ϕ_{k+1}

we know that $(t, \phi_k(t))$ is in R for t in I then the function F_k given by

$$F_k(t) = f(t, \phi_k(t))$$

exacts for t in I. It is contineous on I since f i Contineous on R, and Hence ϕ_k is Contineous of I.

Therfore ϕ_{k+1} which is defined by

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} F_k(t)\,dt.$$

exacts as a Contineous function on I

morever

$$|\phi_k(x) - y_0| \leq \int_{x_0}^{x} |F_i(t)|\,dt.$$

$$\leq M|x - x_0|$$

which Show that ϕ_{k+1} Satisfies (4)

This complete the proof.