Project 2: Heat equations

ME 303: Advanced Engineering Mathematics (Winter 2022)
Department of Mechanical and Mechatronics Engineering
University of Waterloo

Instructions

- Work in group(s) of 1 – 5 and submit a single report.
- At the end of the report, write a short and clear statement on the division of tasks among the team members.
- The project may be completed either in MATLAB or in Python.
- Larger code snippets and extra output should be placed in the appendix. The complete scripts (*.m or *.py) should be submitted with the final report.
- The quality/efficiency of the programming will NOT be evaluated, but the codes should be fully functional and deliver the same results as in the report.

Report

- The report should be a self-contained document detailing the problem, the theoretical background and a contextualization of your solutions.
- No minimum page length is prescribed as long as the needed information is present. Conciseness (high information density) of the report is highly encouraged and credited.
- Proper citations and academic references are expected.

Submission details

- Due date: **11:59 pm on Mar. 28th 2022**. Reports submitted late will not be evaluated.
- The reports, codes, and videos (if applicable) should be uploaded to the Dropbox on LEARN. Please use a consistent naming of the files: for example:
  ME303_2022W_Project_02_[LastName1]_[LastName2].pdf.
  ME303_2022W_Project_02_LastName1_LastName2_name_of_the_code.m.
- No paper reports will be accepted. This will save more than 1000 pages of printed paper.

Background

Heat equation and wave equation are two of the most important topics in PDE. Heuristically, you can think of their positions in the field of PDE are similar to the 2nd order linear ODE (which models a mass-spring-damping system) in the realm of ODE. The beauty of these equations lays in two aspects:
i) they are simple enough that there are **analytical** solutions to them in simple domains;
ii) they are complex enough to model many important physical, engineering, sociological, and economical processes.
Personal opinion: a such elegant balance does not often appear in life.

By just understanding the fundamental concepts and analytical and numerical techniques of PDEs, you are armed for many topics of future studies. In addition, I hope by finishing this project, you can more or less feel and appreciate the beauty of differential equations.

The techniques needed in this project can be found in my lecture notes or in any basic text books about PDEs for example: Kreyszig (2009).

Reference: Kreyszig, Erwin. "Advanced Engineering Mathematics, 10th Eddition." (2009).

1. (*50 points*) **1D heat equation in spherical coordinate**

You are asked to design a cooker to boil eggs. This cooker simply keeps the water in the pot boiling ($T = 100\,^{\circ}C$) to heat up eggs. We assume i) eggs are perfectly spherical with radius R, ii) the 'material' of an egg is homogeneous, meaning that the shell, white, and yolk have the same thermal conductivity. Let $T(r,t)$ be the temperature inside an egg, we assume the criteria of an fully cooked egg is 'keeping $T(r,t) \geq 80\,^{\circ}C, r \in [0, R)$ for longer than 10 sec [1].

(a) Model the process using a well defined PDE and corresponding BCs and ICs.

(b) By solving the PDE you proposed in (a), (see Note 1), **estimate** the time it takes for this cooker to cook a quail egg, chicken egg, and ostrich egg. Use proper tables and/or plots to show (off) your results.

(c) Is there any more economic way in terms of energy consumption to cook the eggs? Relate your engineering suggestions with

Note 1: Look up online for any physical properties you need and list them in the report.

Note 2: Solve this problem numerically (with scheme and grid spacing of your choice).

2. (*30 points*) **1D heat equation**

Solve the following 1D heat equation numerically AND analytically. For numerical solution, write out the time advancement scheme for your solver. For analytical solution, write out your solution procedure explicitly and the final results.

$$\begin{aligned} &PDE: T_t = 2T_{xx}, \qquad x \in (0,1), t \in (0,\infty) \\ &BCs: T(x=0,t) = 0,\ T(x=1,t) = 2, \\ &IC: T(x,t=0) = \cos(\pi x) \end{aligned} \tag{1}$$

(a) Compare your numerical and analytical solutions by plotting them at $t = 0.1$ and $t = 1$, and $t = 10$.

(b) Investigate and discuses how the choice of your grid spacing for numerical solution and terms you keep in analytical solutions affect the accuracy of your solutions.

3. (*20 points*) **2D heat equation**

Consider the process of a cooling-down flat bread, which is governed by 2D heat equations with all Dirichlet BCs:

$$\begin{aligned} &PDE: T_t = \nabla^2 T, \qquad x \in (0,1), y \in (0,1), t \in (0,\infty) \\ &BCs: T(x=0,t) = 0, T(x=1,t) = 0, T(y=0,t) = \sin(\pi x), T(y=1,t) = \cos(2\pi x) - 1 \\ &IC: T(x,y,0) = \sin(\pi x)\sin(4\pi y), \end{aligned} \tag{2}$$

or with mixed BCs.

$$\begin{aligned} &PDE: T_t = \nabla^2 T, \qquad x \in (0,1), y \in (0,1), t \in (0,\infty) \\ &BCs: \nabla T(x=0,t) \cdot \boldsymbol{n} = 0, \nabla T(x=1,t) \cdot \boldsymbol{n} = 2, T(y=0,t) = 1, T(y=1,t) = -1 \\ &IC: T(x,y,0) = \sin(4\pi x)\cos(4\pi y). \end{aligned} \tag{3}$$

(a) Explain the physical meaning of the above two setups (problem 2 and 3), and numerically solve them.

[1] Disclaimer: The instruction team made up this cooking criteria. We don't take any responsibility for potential risks, including but not limited to bio/fire hazards caused by the practice of the cooking criteria

(b) Visualize your solution with proper techniques.
Note: a demo code will be provided.