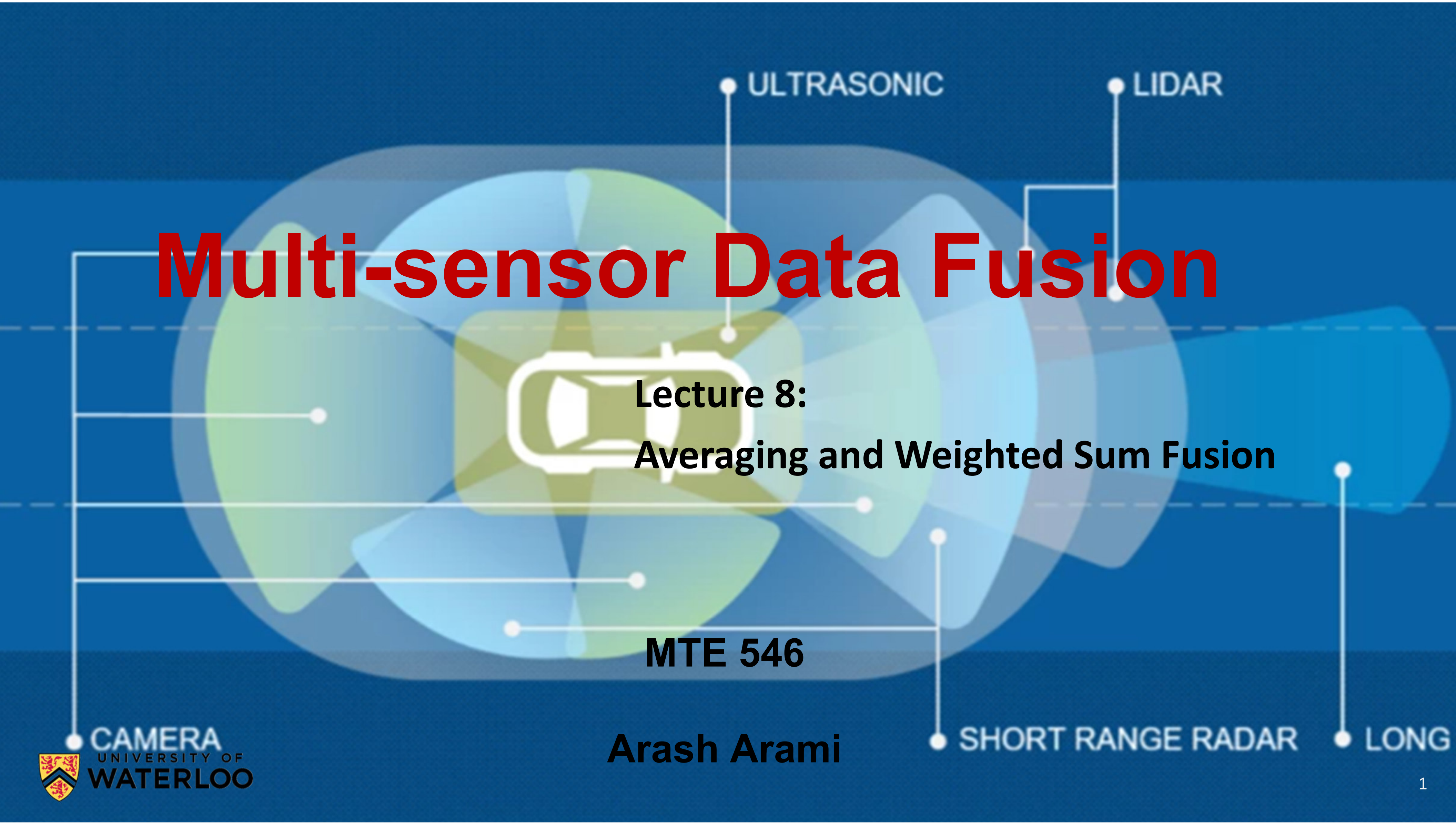
Multi-sensor Data Fusion



Lecture 8:

Averaging and Weighted Sum Fusion

MTE 546

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Multi-sensor measurements

- Let’s say we have *m* sensors y_i measuring states of a process x(t)

$y_1(t) = x(t) + \eta_1(t)$ $\eta_i(t) \sim N(0, \sigma^2)$

$y_2(t) = x(t) + \eta_2(t)$

…

$y_m(t) = x(t) + \eta_m(t)$

- How to reduce the noise and uncertainty?

Learning objectives

You will be able to perform

- Fusion of multi-sensors using average and weighted sum

describe

- What can be gained from those simple fusion operations
- Compute the variance of estimation

Definition

Multi-sensor averaging

- Let's say we have *m* sensors y_i measuring states of a process x(t)

$y_1(t) = x(t) + \eta_1(t)$ $\eta_i(t) \sim N(0, \sigma^2)$: independent identically distributed

… $y_m(t) = x(t) + \eta_m(t)$ i.i.d. $Cov(\eta_i, \eta_j) = 0$

Average operation:

$$y_{ave}(t) = \frac{1}{m} \sum_{i=1}^{m} y_i(t) = \frac{1}{m} \sum_{i=1}^{m} \left(x(t) + \eta_i(t) \right)$$

$$= \frac{1}{m} \left(m\,x(t) + \sum \eta_i(t) \right) = x(t) + \frac{1}{m} \sum \eta_i(t)$$

Important

Multi-sensor averaging

- Let's say we have *m* sensors y_i measuring states of a process x(t)

$y_1(t) = x(t) + \eta_1(t)$ $\qquad\qquad$ $\eta_i(t) \sim N(0, \sigma^2)$

variance of measurement due to the noise

… $y_m(t) = x(t) + \eta_m(t)$

Average operation:

$var(y_i(t)) = \underbrace{Var(x(t))}_{\sigma_x^2} + \underbrace{Var(\eta_i(t))}_{\sigma^2} + \cancel{2Cov(x,\eta)} \; \emptyset$

information

uncertainty

$var(y_{ave}(t_0)) =$

Review from statistics or stochastic process

Review of variance

- $Var(x) = \sigma^2 = \frac{(x-\mu)^T(x-\mu)}{N-1}$
- $Var(ax) = \frac{(ax-a\mu)^T(ax-a\mu)}{N-1} = a^2\sigma^2$
- $Var(x+y) = Var(x) + Var(y) + 2cov(x,y)$
- $Var(a) = 0$
- $cov(x,y) = \frac{(x-\mu_x)^T(y-\mu_y)}{N-1}$
- $cov(x,a) = 0$
- $cov(ax,by) = ab\ cov(x,y)$
- If x and y are independent random variables then $cov(x,y) = 0$

Important

Multi-sensor averaging

- Let's say we have *m* sensors y_i measuring states of a process x(t)

$y_1(t) = x(t) + \eta_1(t)$ $\eta_i(t) \sim N(0, \sigma^2)$

variance of measurement due to the noise

… $y_m(t) = x(t) + \eta_m(t)$

Average operation:

$var(y_i(t)) =$ $Var(x(t)) + Var(\eta_i(t)) + \cancel{2Cov(x, \eta)} \rightarrow 0$

$var(y_{ave}(t_0)) =$ $Var\left(\frac{1}{m}\sum x(t_0) + \eta_i(t_0)\right) = \left(\frac{1}{m}\right)^2 Var\left(\sum x(t_0) + \eta_i(t_0)\right)$

Constant

$= \frac{1}{m^2} Var\left(\sum \eta_i(t_0)\right) = \frac{1}{m^2}\left(m\, Var(\eta_i(t_0))\right)$

$= \frac{1}{m}\sigma^2$

Example

Multi-sensor weighted sum

- Let’s now assume that the noise characteristics of *m* sensors are different

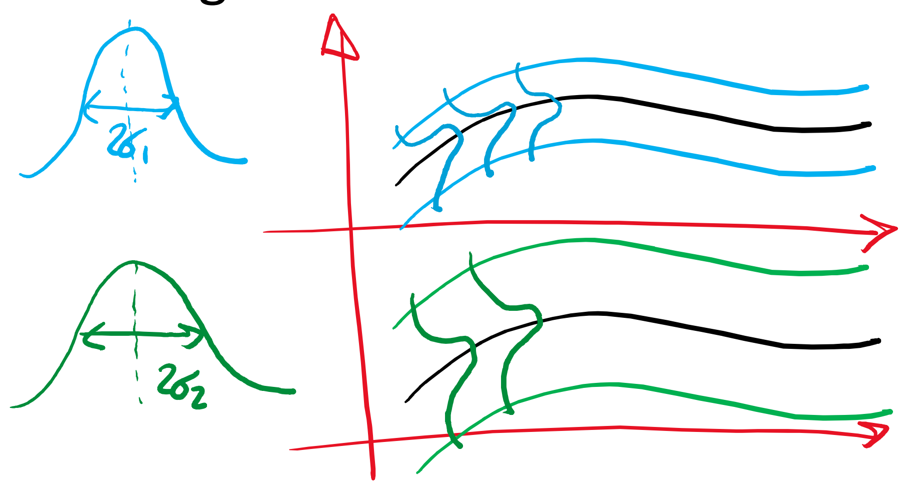
Sensor i uncertainty

$$y_i(t) = x(t) + \eta_i(t) \qquad \eta_i(t) \sim N(0, \sigma_i^2)$$

We can fuse their information considering their uncertainties

Example a two-sensor case

We can weigh them based on their certainty or invers of their $\underbrace{\text{variance}}_{\text{measurement noise}}$



Example

Multi-sensor weighted sum

$y_i(t) = x(t) + \eta_i(t)$ $\qquad\qquad$ $\eta_i(t) \sim N(0, \sigma_i^2)$

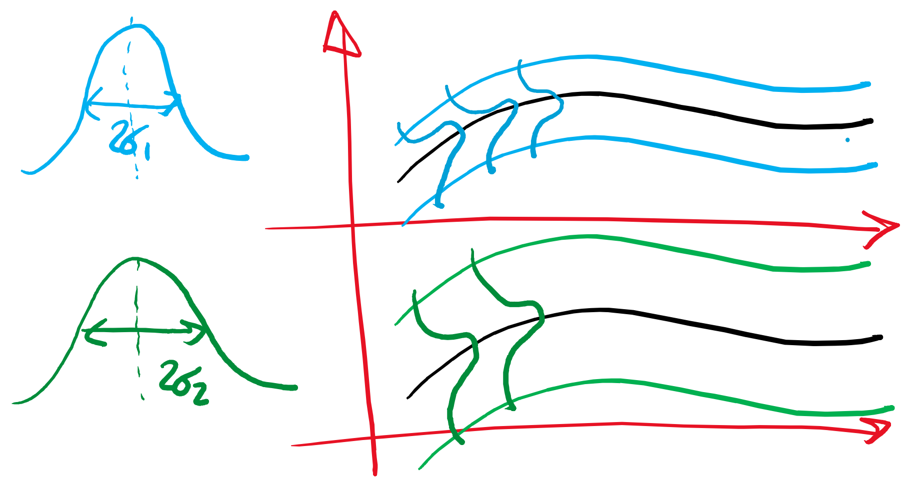
We can fuse their information considering their uncertainties

Example a two-sensor case

$$\hat{y}_{ws} = \frac{w_1 y_1 + w_2 y_2}{w_1 + w_2}$$

$$w_1 = \frac{1}{\sigma_1^2} \qquad w_2 = \frac{1}{\sigma_2^2}$$

$$\hat{y}_{ws} = \frac{\sigma_2^2 y_1 + \sigma_1^2 y_2}{\sigma_1^2 + \sigma_2^2}$$



Example

Multi-sensor weighted sum

$y_i(t) = x(t) + \eta_i(t)$ $\qquad\qquad$ $\eta_i(t) \sim N(0, \sigma_i^2)$

We can fuse their information considering their uncertainties

Example a two-sensor case

$$\hat{y}_{ws} = \frac{\sigma_2^2 y_1 + \sigma_1^2 y_2}{\sigma_1^2 + \sigma_2^2} \qquad\qquad \underbrace{E}_{\text{expected value}}(\hat{y}_{ws}) = x$$

$$\operatorname{var}(\hat{y}_{ws}) = \operatorname{var}\left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} y_1 + \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} y_2\right)$$

Example

Multi-sensor weighted sum

$$y_1(t) = x(t) + \eta_1(t)$$
$$y_2(t) = x(t) + \eta_2(t)$$

$$\eta_i(t) \sim N(0, \sigma_i^2)$$

$$\hat{y}_{ws} = \frac{\sigma_2^2 y_1 + \sigma_1^2 y_2}{\sigma_1^2 + \sigma_2^2}$$

$$\mathrm{Var}(\hat{y}_{ws}) = \mathrm{var}\left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} y_1 + \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} y_2\right)$$

let's assume $x = \text{Constant}$ $\longrightarrow$ $\mathrm{Var}(\hat{y}_{ws}) = \mathrm{Var}\left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\eta_1 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\eta_2\right)$

$$\mathrm{Var}(\hat{y}_{ws}) = \left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2 \sigma_1^2 + \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 \sigma_2^2$$

Important

Multi-sensor weighted sum

$$y_1(t) = x(t) + \eta_1(t)$$
$$y_2(t) = x(t) + \eta_2(t)$$

$$\eta_i(t) \sim N(0, \sigma_i^2)$$

$$\hat{y}_{ws} = \frac{\sigma_2^2 y_1 + \sigma_1^2 y_2}{\sigma_1^2 + \sigma_2^2}$$

let's assume $x = \text{Constant}$

$$\text{Var}(\hat{y}_{ws}) = \left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2 \sigma_1^2 + \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 \sigma_2^2$$

$\sigma_1^2 = 1 \qquad \sigma_2^2 = 2$

$$\text{Var}(\hat{y}_{ws}) = \left(\frac{2}{3}\right)^2 \times 1 + \left(\frac{1}{3}\right)^2 \times 2 = \frac{4+2}{9} = \underbrace{\frac{2}{3} < 1}_{\text{fusing is better than the best sensor}}$$

Important

Extension to m sensors

$$y_i(t) = x(t) + \eta_i(t) \qquad \eta_i(t) \sim N(0, \sigma_i^2)$$

$$i \in \{1,2,\ldots,m\}$$

Weighted sum fusion based on inverse of variances

$$\hat{y}_{ws} = \frac{\frac{y_1}{\sigma_1^2} + \frac{y_2}{\sigma_2^2} + \cdots + \frac{y_m}{\sigma_m^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \cdots + \frac{1}{\sigma_m^2}}$$

Is it always better to fuse?

- In many cases, similar to the mathematical proof, fusing more sensors results in lower variance of error
- Practically, sometimes, you can get lower estimation variance if stick with the best sensor

Summary

- Averaging over multiple sensors
- Weighted sum fusion
- Variance of error

Next session

- State space modeling review
- Bayesian filtering

Any question?

https://piazza.com/uwaterloo.ca/winter2024/mte546