Project 1: Cavitation bubble

ME 303: Advanced Engineering Mathematics (Winter 2022)
Department of Mechanical and Mechatronics Engineering
University of Waterloo

Instructions

- Work in group(s) of five or less and submit a single report.
- At the end of the report, write a short and clear statement on the division of tasks among the team members.
- The project may be completed either in MATLAB or in Python. The required functions must be implemented into the code; that is to say, you may not use the canned functions in MATLAB or Python (e.g. derivation function, Runge-Kutta solver, Gaussian-Sedeil etc.)
- Larger code snippets and extra output should be placed in the appendix (if necessary). The complete scripts (*.m or *.py) should be submitted with the final report.
- The quality/efficiency of the programming will NOT be evaluated, but the codes should be fully functional and deliver the same results as in the report.

Report

- The report should be a self-contained document detailing the problem, the theoretical background and a contextualization of your solutions.
- No minimum page length is prescribed as long as the needed information is present. Conciseness (high information density) of the report is highly encouraged and credited.
- Proper citations and academic references are expected.

Submission details

- Due date: **11:59 pm on March 7th, 2021**. Reports submitted late will not be graded.
- The reports and code should be uploaded to the Dropbox on LEARN. Please use a consistent naming of the files: **ME303_2021S_Project_01_[LastName1]_[LastName2]_[LastName3]....pdf**.

Background

To tie-in the concepts/ideas/techniques seen in the first half of ME303 journey, we will turn our attention to underwater bubble dynamics, which links to the mysterious physical phenomenon of sonoluminescence, how a pistol shrimp hunts, fun party trick that breaks a beer bottle with bare hands, and what happens right after underwater (nuclear) explosion. Fascinating phenomenon occurs when a cavitation bubble in liquid collapses. At the end stage of the collapse, as the energy is concentrated into a point, the local temperature in the collapsing bubble can reach to (arguably) 10,000 K or higher for up to 50 pico seconds and visible light and powerful shock waves are emitted.

Some detailed physics of cavitation (i.e., sonoluminescence) are unsolved physical problems and, obviously, outside the scope of ME303. For project 1, we will be studying the governing equations behind the bubble oscillation.

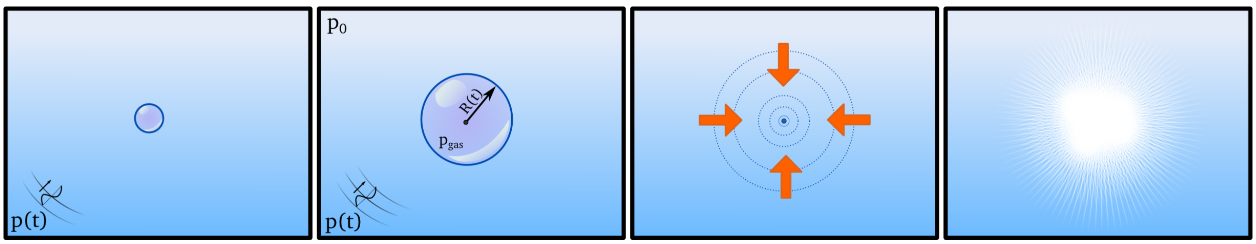

Figure 1: Stages of the bubble evolution leading to a sonoluminescent burst. Modified from Dake ©.

The radius of a bubble under a varying pressure field is defined by the Rayleigh-Plesset equation. This equation is derived using standard conservation laws under a number of simplifying assumptions:

$$\rho_l \left(R\ddot{R} + \frac{3}{2}\dot{R}^2 \right) = -P_0 - 4\mu\frac{\dot{R}}{R} - \frac{2\sigma}{R}. \tag{1}$$

In the above equation, $R(t)$ [m] represents the bubble radius. The other terms are: ρ_l $[kg \cdot m^{-3}]$ the density of the liquid, P_0 [Pa] the pressure difference of outside and inside the bubble, μ $[Pa \cdot s]$ molecular viscosity of the liquid and σ $[kg \cdot s^{-2}]$ the surface tension at the bubble-water interface. Some basic details about this equation can (even) be found on wikipedia page. For more contextual information on the bubble dynamic phenomena, please see the following sources:

[1] D. Lohse, Bubble puzzles, Physics Today 56(2), 36-41 (2003).

[2] W. Kreider, L. A. Crum, M. R. Bailey and O. A. Sapozhnikov, A reduced-order, single-bubble cavitation model with applications to therapeutic ultrasound, J Acoust Soc Am., 130(5), 3511-3530 (2011).

[3] S. Hilgenfeldt, M. P. Brenner, S. Grossmann and D. Lohse, Analysis of Rayleigh-Plesset dynamics for sonoluminescing bubbles, J. Fluid Mech., 365, 171-204, (2003).

[4] https://en.wikipedia.org/Rayleigh–Plesset equation

1. (*50 points*) **Bubble crunch in a water bottle**

By linearizing the Rayleigh-Plesset equation (1) and neglecting the viscous and the surface tension terms, a more tractable and simplified form of the equation can be obtained (see [3] for a similar, but not exactly the same, equation). The following equation is an linear approximation of the bubble radius under a constant external pressure due to the hydrodynamic and atmosphere pressure:

$$\ddot{R} + \lambda^2 (R - R_0) = -\frac{3}{2}\frac{P_0}{\rho_l R_0}, \tag{2}$$

where

$$\lambda^2 = \frac{3P_0}{\rho_l R_0^2}. \tag{3}$$

For this problem, we assume water as the working fluid ($\rho_l = 1000$ kg·m^{-3}). We assume an initial bubble radius of $R(0) = R_0 = 2$ mm and that the initial bubble radius change is null ($\dot{R}(0) = 0$ [m/s]).

(a) We assume the bubble appears in a (beer) bottle. The atmosphere pressure above the water level is $P_{atm} = 10^5$ [Pa] and the bubble in the (beer) bottle is 10 [cm] below the sufrace. We assume a vacumm inside the bubble meaning that the pressure inside the bubble is neglegiable, estimate P_0.

(b) Classify the ordinary differential equation and find the closed-form, general solution (please write-down all the steps). Use the ICs to find the particular solution and find the time (T^*) that the bubble takes to collapse for the first time (i.e., $R(T^*) = 0$). Note, in this case, the bubble may not be able to fully collapse to a singularity as you expected. This is not necessarily realistic as we have made way too many assumptions.

(c) Convert this second order ODE into two first-order ODEs.

(d) Solve the coupled set of ODEs using a fixed time step of your choice over the time interval of $[0, T^*]$ s. Use the following time advancement schemes:

- first-order, explicit Euler
- second-order, explicit Runge-Kutta (RK2)
- fourth-order, explicit Runge-Kutta (RK4)

Plot, on the same figure, all the numerical solutions along with the analytical solution. Explain and interpret your results.

(e) Quantify the maximum error over the specified temporal domain, with respect to the analytical solution, for all three time advancement schemes.

2. (*50 points*) **Under water explosion evolution**

The cavitation bubble in fact oscillates, meaning that after collapse, the bubble rebound and expand, and then collapse again after reaching to a maximum size. This process repeats, and emit 'periodical' sound waves that can travel far.

Consider a case that an underwater explosion happened at 1000 m below the sea level and it created a big bubble with maximum size of $R(t = 0) = 100$ m. Using the full Rayleigh–Plesset equation (1), try to attack the following problems with your choice of numerical schemes and grid spacing.

(a) Estimate how much energy (E_0 [J]) it takes to create such a bubble in deep sea, also convert E_0 in TNT equivalent.

(b) Try to numerically simulate the Bubble oscillation process using RK4 scheme with time step of your choice, by solving the full Rayleigh-Plesset equation (1). Plot $R(t)$ vs. t (show at least 5 collapse events), and comment on how the time between two neighboring collapses change over time.

(c) Show the grid convergence of your results.

(d) Vary the initial bubble size (e.g., R(0)=10 m, R(0)=1 m, R(0)=0.0001 m), and comment on how the initial bubble size (which is closely related to the explosion energy) affects the oscillation period and amplitude decay.

Note 1: The Rayleigh-Plesset equation becomes very stiff as $R \to 0$. Carefully compute the bubble evolution between the prescribed temporal bounds. If your scheme does not work well, try to use implicit and/or multi-step schemes with very small time steps.

Note 2: You can assume that right after the bubble collapses, the bubble rebound can be modeled by assuming that the surface of the bubble keep the same speed but move in the opposite direction.

Note 3: Clearly state any assumptions you have made.

Note 4: This paper (Alehossein and Qin, 2007) may help your project. You are supposed to be able to understand most of the numerical content of this paper well after the first half of the ME 303 studies.

Reference: Alehossein, H., & Qin, Z. (2007). Numerical analysis of Rayleigh–Plesset equation for cavitating water jets. International Journal for Numerical Methods in Engineering, 72(7), 780-807.

Bonus problem (up to 40% of the course):

In Chapter 3 of ME 303 we studied how to use the most fundamental/popular method (Euler/RK4) to solve ODE. They are easy to learn and program and works most of the time. If you can afford the computational cost, by reducing the grid spacing or time step, you can achieve very high accuracy. But, sometimes the methods we learned here can be slow and we may want/prefer a solver that is much faster with acceptable accuracy. The fast advancement of AI and machine learning gives us a choice. Check out the paper below (it is designed for PDEs but also works for ODEs):

Zongyi Li, Nikola Kovachki, Kamyar Azizzadenesheli, Burigede Liu, Kaushik Bhattacharya, Andrew Stuart, Anima Anandkumar (2020) "Fourier Neural Operator for Parametric Partial Differential Equations", arXiv:2010.08895v3 [cs.LG]. Link here.

Task for the bonus problem: solve R-P equation with the method/framework listed in the paper above.