I. ELECTROMAGNETIC FIELDS B (EMF B):

A. References

1. *Engineering Electromagnetics*, 5th Edition 1989, William H. Hayt, Jr: Tata McGraw Hill Edition
2. *Electromagnetic Fields*, 2nd Edition, 2014, R. Meenakumari, R. Subasri, New Age Int'l (NAI) Pub
3. *Electricity and Magnetism*, Prantosh Chakraborty, NAI Pub
4. *Elements of Electromagnetics*, N. O. Sadiku

B. Course Outline

1. Magnetostatic Fields: Biot-Savart and Ampere's law and Concepts
2. Magnetic Field Effects on Materials: Classification, Boundaries, H-Circuits
3. Energy and Electromagnetic Induction
4. Maxwell's Equations and EM Wave for Propagation

Summary:

II. Magnetostatic Fields: Free-space

Concepts

A magnetostatic field is produced when charges are moving with a constant velocity. Examples are magnetisation currents in permanent magnets or conduction currents in wires.

Main Laws

1. Biot-Savart's law
2. Ampere's circuital law

1. Biot-Savart's law

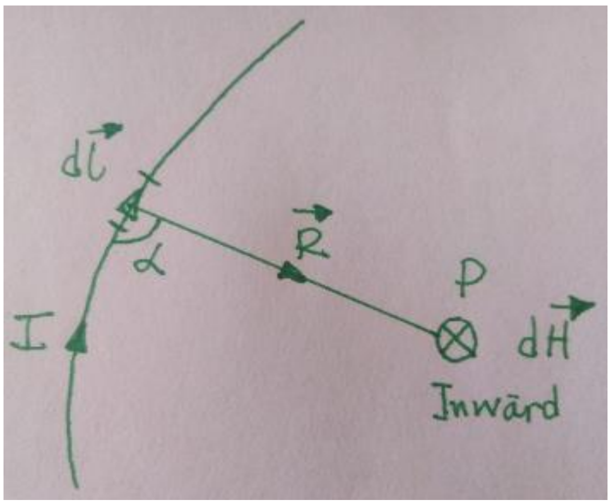


Fig. 1.1 Biot Savart Law

The magnetic field intensity, $d\vec{H}$ at a point, P due to current element $Id\vec{l}$ is

$$\boxed{d\vec{H} = \frac{Id\vec{l}\sin(\alpha)}{4\pi R^2}\quad \vec{a}_n \qquad [A/M]} \tag{1}$$

where α is the angle between the differential current element and the line joining the point P to the element, and since $\vec{a}_R = \frac{\vec{R}}{R}$:

$$d\vec{H} = \boxed{\frac{Id\vec{l} \times \vec{a}_R}{4\pi R^2}} = \boxed{\frac{Id\vec{l} \times \vec{R}}{4\pi R^3}} \tag{2}$$

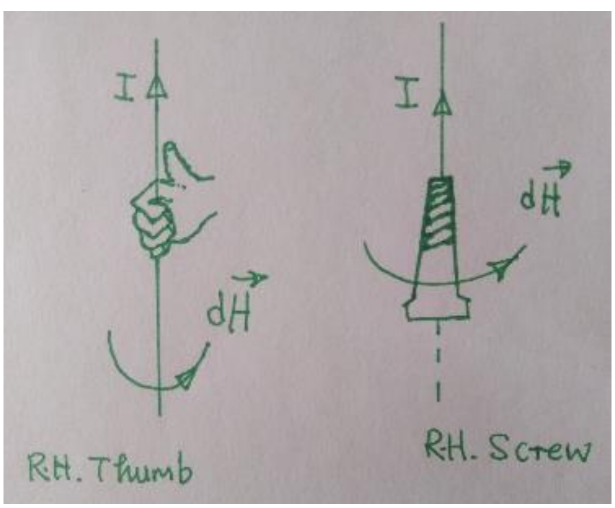


Fig. 1.2 Thumb and Screw Rules

The direction of $d\vec{H}$ is determined by the R-H rule where the R-H thumb points in the direction of current, the fingers encircle the wire in the direction of $d\vec{H}$.

Conventionally, the direction of magnetic field intensity H or I is represented by a small circle enclosing a dot, $\odot$, when out of page or enclosing a cross, $\otimes$, when into the page.

The following current distributions will be considered:

a) Line current
b) Surface current
c) Volume current

2. Ampere's circuital law

It states that the line integral of $\vec{H}$ about any closed path is exactly equal to the direct current enclosed by that path, such that:

$$\boxed{\oint \vec{H} \cdot d\vec{l} = I_{enc} = \int \vec{J} \cdot d\vec{S}} \tag{3}$$

where $\vec{J}$ is the current density
dS is the differential surface area of the path
The closed path is referred to as an Amperian path. By applying Stoke's law, the differential form of Ampere's circuital law is written as

$$\boxed{I_{enc} = \int \vec{J} \cdot d\vec{S} = \int (\bigtriangledown \times \vec{H}) \cdot d\vec{S}} \tag{4}$$

since $\bigtriangledown \times \vec{H} = \vec{J}$.

Magnetic Field Intensity, $\vec{H}$

Example: H around different shapes

Magnetic Field intensity around a straight conductor
a) Let a point P be in the neighbourhood of a straight conductor carrying current, I. Find the magnetic field intensity, $\vec{H}$ at P.
Solution

Biot-Savart's law:

$$d\vec{H} = \frac{Id\vec{l}\sin(\theta_3)}{4\pi R^2} \quad \vec{a}_n \tag{5}$$

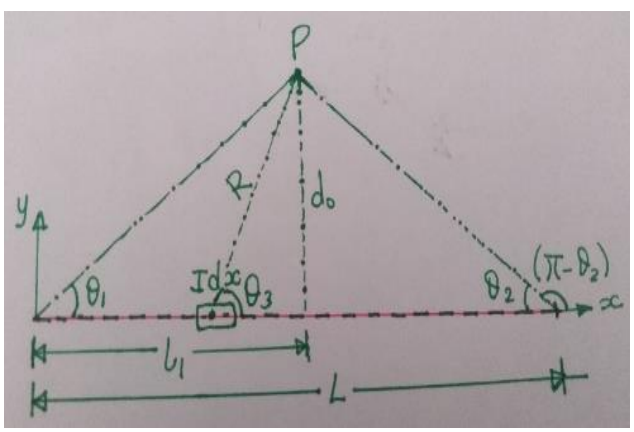


Fig. 1.3 H-Field around a straight wire

The magnetic field being along the $z-$direction.
We see from the figure:
$R^2 = (l_1 - x)^2 + d_0^2$
then

$$\begin{aligned} d\vec{H} &= \frac{Idx\sin(\theta_3)}{4\pi[(l_1 - x)^2 + d_0^2]} \quad \vec{a}_z \\ \vec{H} &= \int_{l=0}^{l=L} \frac{Idx\sin(\theta_3)}{4\pi[(l_1 - x)^2 + d_0^2]} \quad \vec{a}_z \end{aligned} \tag{6}$$

Integrate by substitution,

- $\tan\theta_3 = \frac{d_0}{l_1 - x}$
- $l_1 - x = d_0 \cot\theta_3$
- $dx = d_0 cosec^2(\theta)_3 d\theta$
- $(l_1 - x)^2 = d_0^2 \cot^2\theta_3$

Denominator: $R^2 = (l_1 - x)^2 + d_0^2 = d_0^2(1 + cot^2\theta_3)$
$1 + \cot^2(\theta) = cosec^2(\theta)$

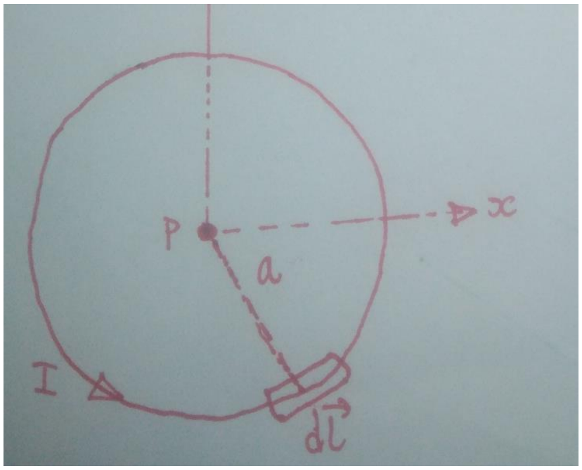
Change limits as θ_3 varies from θ_1 to $(\pi - \theta_2)$

$$\begin{aligned}\vec{H} &= \int_{\theta_1}^{\pi-\theta_2} \frac{I(d_0 cosec^2(\theta_3)d\theta)\sin(\theta_3)}{4\pi[d_0^2 cosec^2(\theta_3)]} \quad \vec{a}_z \\ &= \frac{I}{4\pi d_0}\int_{\theta_1}^{\pi-\theta_2} \sin\theta_3 d\theta \quad \vec{a}_z \\ &= \frac{I}{4\pi d_0}[-\cos\theta_3]_{\theta_1}^{\pi-\theta_2} \quad \vec{a}_z \\ &= \frac{I}{4\pi d_0}[\cos\theta_1 + \cos\theta_2] \quad \vec{a}_z\end{aligned} \tag{7}$$

(b) If the conductor is infinitely long, then both θ_1 and θ_2 approach zero. The $\vec{H}$ field is expressed as:

$$\begin{aligned}\vec{H} &= \frac{I}{4\pi d_0}[\cos\theta_1 + \cos\theta_2] \quad \vec{a}_z \\ &= \frac{I}{4\pi d_0}[\cos(0) + \cos(0)] \quad \vec{a}_z \\ &= \frac{I}{4\pi d_0}[2] \quad \vec{a}_z \\ &= \frac{I}{2\pi d_0} \quad \vec{a}_z\end{aligned} \tag{8}$$

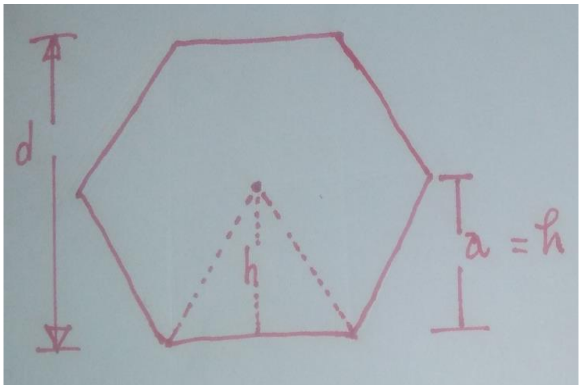
(c) Given a flexible wire, design a circular loop of wire carrying a current of I [A], with a radius a [m] by determining an expression for the magnetic field intensity $\vec{H}$ at the center of the circle.



$$\begin{aligned}d\vec{H} &= \frac{Id\vec{l} \times \vec{a}_R}{4\pi R^2}, \quad d\vec{l} = ad\phi\vec{a}_\phi, \quad \vec{R} = a\vec{a}_\rho \\ &= \frac{Iad\phi\vec{a}_\phi \times \vec{a}_\rho}{4\pi a^2} \\ \vec{H}_c &= \int_0^{2\pi} \frac{I}{4\pi a} d\phi\vec{a}_z \\ &= \frac{I}{2a}\vec{a}_z\end{aligned} \tag{9}$$

Fig. 1.4 H-Field at the center of a circle

(d)(i) Repeat the design by turning the wire into a regular hexagonal shape, where the distance between the opposite sides is equal to the diameter of the circle.
(ii) Hence, state the shape that gives more $\vec{H}$ at the center.



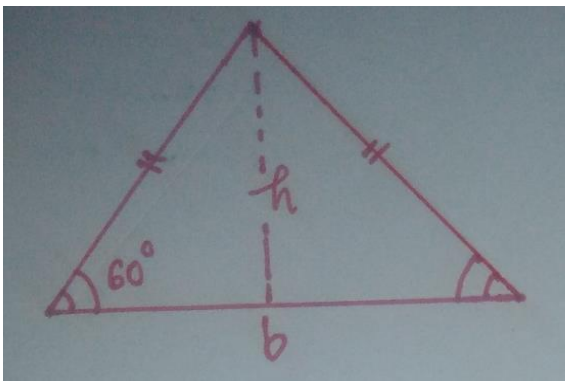
$$\vec{H} = \frac{I}{4\pi a}[\cos A + \cos B], \quad d = 2a, \quad A = B = 60^0$$
$$= \frac{I}{4\pi a}[2\cos 60^0] = \frac{I}{4\pi a} \tag{10}$$

For the six sides,

$$\vec{H}_{hx} = 6 \times \frac{I}{4\pi a} = \frac{1.5I}{\pi a}\vec{a}_z \tag{11}$$

Fig. 1.5 H-Field at the center of a hexagon

(e)(i) Determine the $H-$field at the center of an equilateral triangle with sides $b[m]$ and height, $h[m]$.



$$\vec{H} = \frac{I}{4\pi(h/3)}[\cos A + \cos B], \quad \tan(60^0) = \frac{h}{(b/2)}, \quad h = \frac{b\sqrt{3}}{2}$$
$$= \frac{I}{4\pi(\frac{b\sqrt{3}}{6})}[2\cos 30^0] = \frac{2I\cos 30^0}{4\pi(\frac{b\sqrt{3}}{6})} \tag{12}$$

For the 3 sides,

$$\vec{H}_{tr} = 3 \times \frac{2I\cos 30^0}{4\pi(\frac{b\sqrt{3}}{6})} = \frac{6I\cos 30^0}{4\pi(\frac{b\sqrt{3}}{6})}\vec{a}_z \tag{13}$$

Fig. 1.6 H-Field at the center of an equilateral

Example:

Consider a region with cylindrical symmetry where the conductivity is given by $\sigma = 1.5e^{-150\rho}k[S/m]$. An electric field is present in the region of $\vec{E} = 30\vec{z}\ [V/m]$.

a) Find the current density, $\vec{J}$: $(\vec{J} = \sigma\vec{E})$.

b) Find the total current crossing the surface $(\rho < \rho_0)$ and $z = 0$: $\quad (I = \int \vec{J} \cdot d\vec{s})$

$(ds = \rho d\rho d\phi)$

c) Calculate the magnetic field, $\vec{H}$: $\quad (\oint \vec{H} \cdot d\vec{l} = I_{encl})$

Magnetic Flux Density, $\vec{B}$

$\vec{B}$ (Wb/m^2) $or\ tesla$ is written as: $\vec{B} = \mu_0 \vec{H}$
where $\mu_0 = 4\pi \times 10^{-7}$ $[H/m]$ is the permeability of free-space,

the magnetic flux, ψ (Wb) through a surface, S is given as
$\psi = \int_S \vec{B} \cdot d\vec{S}$

Magnetic Scalar Potential, V_m

$\vec{H} = -\bigtriangledown V_m \rightarrow \bigtriangledown \times (-\bigtriangledown V_m) = \bigtriangledown \times \vec{H} = 0$ i.e. $(curl\ of\ a\ Gradient\ of\ \vec{F} = 0)$
$\bigtriangledown \times (\bigtriangledown \vec{F}) = 0$: Identity
But note that for Ampere's circuital law, $\bigtriangledown \times \vec{H} = \vec{J}$
Therefore, the magnetic field intensity can ONLY be related to scalar magnetic potential in the region where the current density is zero.
$\vec{H} = -\bigtriangledown V_m \quad (\vec{J} = 0)$

Gauss Law of Magnetostatics

• Conservation of magnetic flux
It is not possible to have isolated magnetic poles or charges, therefore the total flux through a closed surface in a magnetic field is ZERO.
$\oint_S \vec{B} \cdot d\vec{S} = 0$
The magnetic field is conservative.

By applying divergence theorem,

$\oint_S \vec{B} \cdot d\vec{S} = \int_V \bigtriangledown \cdot \vec{B} dV = 0$
$\downarrow\downarrow$
$\bigtriangledown \cdot \vec{B} = 0$

Deduction: The magnetic field has no source and it is solenoidal.

Example
Ex. 1
Determine if the vector field, **F** describes a magnetic field, given that $\mathbf{F}(x, y) = < x^2y, 2y - xy^2 >$.
Solution
For a magnetic field, $\bigtriangledown \cdot \mathbf{F} = 0$
$\bigtriangledown \cdot \mathbf{F}(x, y) = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y - xy^2)$
$= 2xy + 2 - 2xy$
$= 2$
$\downarrow\downarrow$
F doesn't model a magnetic field

Magnetic Forces
Magnetic forces may be experienced in the following ways:
a) A moving charged particle in a magnetic field, B
b) A current element in an external magnetic field
c) Between two current elements

Examples on Magnetic Forces

1. Force on a current Element

A charge dQ moving with velocity $\vec{u}$, which produces a convection current element $dQ\vec{u}$, is equivalent to a conduction current element $Id\vec{l}$, i.e.

$$Id\vec{l} = dQ\vec{u}$$
$$d\vec{F} = Id\vec{l} \times \vec{B}$$

For a closed path, the force is given by

$$\vec{F} = \oint_L Id\vec{l} \times \vec{B}$$

NOTE: The $\vec{B}$ field that exerts a force on $Id\vec{l}$ must be due to another element. e.g.

$$d\vec{F}_2 = I_1 d\vec{l} \times \vec{B}_2$$

$$d\vec{F}_{21} = I_2 d\vec{l}_2 \times \vec{B}_1$$

and by applying Biot-Savart's law for $d\vec{H}$, it is easily shown that:

$$d\vec{F}_{12} = I_1 d\vec{l}_1 \times \vec{B}_2 = I_1 d\vec{l}_1 \times \mu \int \frac{I_2 d\vec{l} \times \vec{a}_R}{4\pi R^2}$$

In the case of two loops of filamentary current, the net force on loop 1 due to loop 2 is written as:

$$d\vec{F}_{12} = \frac{\mu I_1 I_2}{4\pi} \oint_{c_1} \oint_{c_2} \frac{d\vec{l}_1 \times (d\vec{l}_2 \times \vec{a}_{12})}{R^2}$$

2. Force between two current Elements

Example

Consider two current elements in space carrying currents I_1 and I_2 in the same $x - direction$.

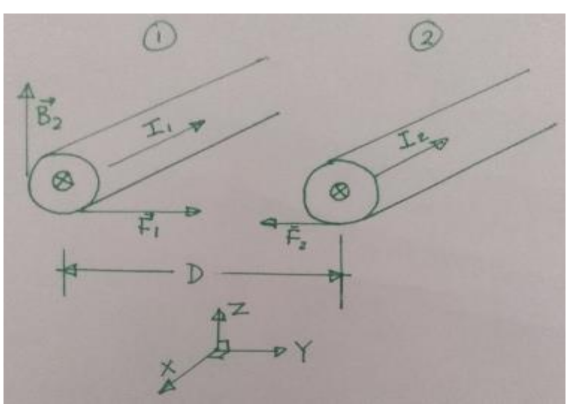


Fig. 1.7 Magnetic force due to two current elements

The magnetic field produced by a long conductor is given as $\vec{H} = \frac{I}{2\pi d}\vec{a}_z$, where d is the distance of any point P from the conductor.

$\downarrow\downarrow$
The field produced by conductor 1 at the location of conductor 2 is given as

$$\vec{H}_1 = \frac{I_1}{2\pi d}(-\vec{a}_z)$$
$$\vec{B}_1 = \frac{\mu_0 I_1}{2\pi d}(-\vec{a}_z)$$

Then conductor 2 is placed in the field of conductor 1, the force exerted on it is given as
$\vec{F}_1 = \vec{I}_2 l \times \vec{B}_1$

If the conductor is of length $l = 1[m]$

$\vec{F}_{12} = I_2(-a_x) \times B_1(-a_z)$

$\vec{F}_{12} = I_2 \cdot \frac{\mu_0 I_1}{2\pi d}(-a_y) = \frac{\mu_0 I_1 I_2}{2\pi d}(-a_y) \quad [N/M]$

Else: Place Conductor 1 in the B-field of 2
Show that
$\vec{F}_{21} = \frac{\mu_0 I_1 I_2}{2\pi d}(a_y) \quad [N/M]$

$\vec{F}_{21} = I_1(-a_x) \times B_2(a_z)$

$\vec{F}_{21} = I_1 \cdot \frac{\mu_0 I_2}{2\pi d}(a_y) = \frac{\mu_0 I_1 I_2}{2\pi d}(a_y) \quad [N/M]$

NOTE: Since we have $F_{12}(-a_y)$, and $F_{21}(a_y)$; there is a force of $Attraction$, when current is in the same direction. [C2 joins C1 and C1 joins C2].

NOTE
If the direction of only I_2 is reversed, the direction of force will reverse, thus a force of $Repulsion$.

$\vec{F}_{12'} = I_2(a_x) \times B_1(-a_z)$

$\vec{F}_{12'} = I_2 \cdot \frac{\mu_0 I_1}{2\pi d}(a_y) = \frac{\mu_0 I_1 I_2}{2\pi d}(a_y) \quad [N/M]$

Also:
$\vec{F}_{2'1} = I_1(-a_x) \times B_2(-a_z)$

$\vec{F}_{2'1} = I_1 \cdot \frac{\mu_0 I_2}{2\pi d}(-a_y) = \frac{\mu_0 I_1 I_2}{2\pi d}(-a_y) \quad [N/M]$

NOTE: Since we have $F_{12'}(a_y)$, and $F_{2'1}(-a_y)$; there is a force of $Repulsion$, when current is in the opposite direction.

Further Example on Magnetic Force on materials
(ii) Determine the force between two parallel circular coaxial coils of radius R $[m]$, which are a small distance, d $[m]$ apart in free space and carry currents I_1 and I_2, respectively. Each of the coils has a single turn only. **(xxx marks)**
Solution

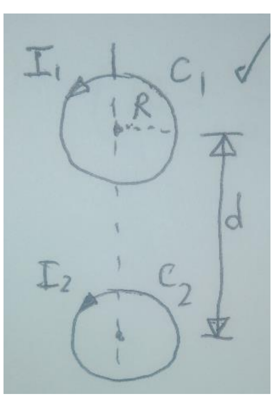


Fig. 1.8 Force on Two single turn Coils

Treat the coils as parallel wires carrying current

$$F_D = \mu_0 \frac{I_1 I_2}{2\pi d} \tag{14}$$

The net force around the diameter is given as, $2\pi R(F_D)$

$$F = (2\pi R)(\mu_0 \frac{I_1 I_2}{2\pi d}) = (\mu_0 I_1 I_2)(\frac{R}{d}) \tag{15}$$

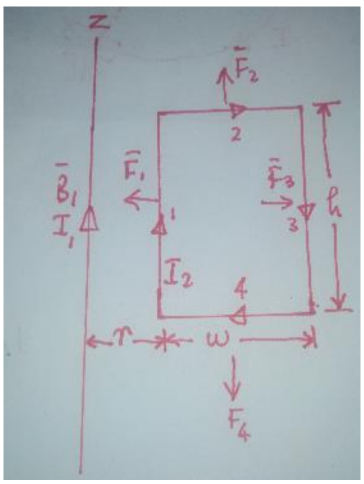
Lorentz' Equation of Force

$$\boxed{F = Q(\vec{E} + \vec{u} \times \vec{B})}$$

(i) Explain why a magnetic force produced by a charge moving at a constant velocity cannot exert a force on the charge.

The force is perpendicular to both $\vec{u}$ and $\vec{B}$ and cannot perform work to increase the kinetic energy of the charge.

(ii) Consider a rectangular loop carrying a current I_2 which is placed parallel to an infinitely long filamentary wire carring current I_1. The closest side of the loop is r[m] away from the wire and it is w[m] wide and h[m] high. Derive and explain the forces around the wire.



For infinitely long wire:
$\vec{B}_1 = \frac{\mu_1 I_1}{2\pi r}\vec{a}_\phi$

$\vec{F}_1 = I_2 \int d\vec{l}_2 \times \vec{B}_1 = I_2 \int_{z=0}^{h} dz\vec{a}_z \times \frac{\mu_0 I_1}{2\pi r}\vec{a}_\phi$
$= -\frac{\mu_0 I_1 I_2 h}{2\pi r}\vec{a}_\rho$ (attractive)

$F_l = (F_1 + F_2 + F_3 + F_4)$
Similarly,
$\vec{F}_3 = \frac{\mu_0 I_1 I_2 h}{2\pi(r+w)}\vec{a}_\rho$ (repulsive)

$\vec{F}_2 = I_2 \int_{r=r}^{(r+w)} dr\vec{a}_\rho \times \frac{\mu_0 I_1}{2\pi r}\vec{a}_\phi$

$= \frac{\mu_0 I_1 I_2}{2\pi} \ln\left(\frac{r+w}{r}\right)\vec{a}_z$ (parallel)

Fig. 1.9 Force on a loop

$\vec{F}_4 = -\vec{F}_2$

$\vec{F}_l = \vec{F}_1 + \vec{F}_3 = \frac{\mu_0 I_1 I_2 h}{2\pi}\left[\frac{1}{r} - \frac{1}{(r+w)}\right] \ (-\vec{a}_\rho)$

Further examples on $\vec{H}$

1. A two wire transmission line circuit consists of two parallel conductors 1 and 2, which are $1[cm]$ in diameter and spaced $1[m]$ apart. They carry current of $I_1 = +80[A]$ and $I_2 = -80[A]$, respectively. Determine:
i) Magnetic field intensity, $\vec{H}$ at each surface of the conductors,
ii) $\vec{H}$ at the midway between the conductors

2. Given a vector field, $\vec{F}(x, y, z) = (e^x\vec{i} - 6yz\vec{j} + y^2z\vec{k})$, find the divergence at a point, $P(0, -2, 2)$. Comment on the significance of the result.

3. Determine the current density that is associated with the magnetic field given by, $\vec{H} = (2\vec{i} - 3\vec{j} - 4x\vec{k})$ $[A/m]$

Suggested approaches

1. A two wire transmission line circuit consists of two parallel conductors 1 and 2, which are $1[cm]$ in diameter and spaced $1[m]$ apart. They carry current of $I_1 = +80[A]$ and $I_2 = -80[A]$, respectively. Determine:
i) Magnetic field intensity, $\vec{H}$ at each surface of the conductors,

ii) $\vec{H}$ at the midway between the conductors

Solution 1.

Sketch the fields: they are directed upwards at a point P in the middle
$\vec{H}_1 = \frac{I_1}{2\pi r_1}\vec{a}_z$
$\vec{H}_2 = \frac{I_2}{2\pi r_2}\vec{a}_z$
$\vec{H} = (\vec{H}_1 + \vec{H}_2)$
i) The field at each surface
$\vec{H} = \frac{I}{2\pi}[\frac{1}{r_1} + \frac{1}{r_2}]\vec{a}_z$
$\vec{H} = \frac{I}{2\pi}[\frac{1}{0.5/100} + \frac{1}{1}]\vec{a}_z$
$\vec{H} = \frac{8}{\pi}[kA/m]$

ii) Field at midway between 1 and 2
$r_1 = r_2 = 0.5[m]$
$\vec{H} = \frac{I}{2\pi}[\frac{1}{r_1} + \frac{1}{r_2}]\vec{a}_z$
$\vec{H} = \frac{I}{2\pi}[\frac{1}{0.5} + \frac{1}{0.5}]\vec{a}_z$
$\vec{H} = \frac{160}{\pi}[A/m]$

2. Given a vector field, $\vec{F}(x, y, z) = e^x\vec{i} - 6yz\vec{j} + y^2z\vec{k}$, find the divergence at a point, $P(0, -2, 2)$. Comment on the significance of the result.

Solution 2.

$\bigtriangledown \cdot \vec{F}(x, y, z) = \frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(-6yz) + \frac{\partial}{\partial z}(y^2z)$
$= e^x - 6z + y^2$
$\bigtriangledown \cdot \vec{F}(0, -2, 2) = e^0 - 6(2) + (-2^2)$
$= 1 - 12 + 4 = -7$
Inward flow of charge density at the point.

3. Determine the current density that is associated with the magnetic field given by, $\vec{H} = (2\vec{i} - 3\vec{j} - 4x\vec{k})\ [A/m]$

Solution 3.

$\vec{J} = (\bigtriangledown \times \vec{H})$
Determinant:

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 & -3 & -4x \end{array}$$

$= [\frac{\partial}{\partial y}(-4x) - \frac{\partial}{\partial z}(-3)]\vec{i} - [\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial z}(2)]\vec{j} + [\frac{\partial}{\partial x}(-3) - \frac{\partial}{\partial y}(2)]\vec{k}$

$\vec{J} = (\bigtriangledown \times \vec{H}) = +4\vec{j}\ [A/M^2]$

Further Example on $\vec{B}$

a) Three wires sit at the corners of a square, all carrying currents of $I_n[A]$ into the page, where n denotes the nth corner. Sketch and calculate the magnitude of the magnetic field at the other corner of the square, point P located at the bottom right corner, if the length of each side of the square is R $[cm]$.

b) Three wires sit at the corners of a square, all carrying currents of $4[A]$ into the page. Calculate the magnitude of the magnetic field at the other corner of the square, point P located at the bottom right corner, if the length of each side of the square is $0.5[cm]$.

c) Three wires sit at the corners of a square, where two wires carry currents of $4[A]$ into the page, while the current in the wire at the northeast corner carries equal magnitude out of the page. Sketch and calculate the magnitude of the magnetic field at the other corner of the square, point P located at the bottom right corner, if the length of each side of the square is $0.5[cm]$.

Summary:

III. Effects of Magnetic Fields on Material Media

A. *Magnetization on Materials*

Magnetic Dipole

A magnetic field produced by a small current loop or bar magnet in a distant vicinity.
Magnetic dipole moment, $\vec{m}$ is given as:
$\vec{m} = I\pi a^2 \vec{a}_n$
$\vec{a}_n$ is the direction of $\vec{m}$, which is given by RH thumb/screw rule.

Magnetization

It refers to the magnetic dipole moment per unit volume, where $\vec{M}$ is the magnetic polarisation density of the medium.
In a material medium, $\vec{M} \neq 0$,
$\vec{B} = \mu_0(\vec{H} + \vec{M})$
For linear material, $\vec{M}$ depends linearly on $\vec{H}$
$\vec{M} = \chi_m \vec{H}$
, where,
$\chi_m \rightarrow$ magnetic susceptibility of the medium
It refers to the measure of the sensitivity of a material to magnetic field.
$\vec{B} = \mu_0(1 + \chi_m)\vec{H} = \mu\vec{H}$
$\vec{B} = \mu_0\mu_r\vec{H}$
$\mu_r = (1 + \chi_m)$
In magnetic screening, shields are made up of very strong magnetic materials with infinite permeability such as steel and iron: $\vec{B} = \mu_0\mu_r(\vec{H} + \vec{M})$

Magnetic materials are classified as follows:
- Non-magnetic materials: $\chi_m = 0 \;\; (\mu_r = 1)$: e.g. free-space, air
- Dia-magnetic materials: small $-$ve $\chi_m = 0 \;\; (\mu_r \leq 1)$: e.g. Lead, Copper, Na^+Cl
- Para-magnetic materials: small $+$ve $\chi_m = 0 \;\; (\mu_r \geq 1)$: e.g. $\mu_r = 10$, Tungsten, K
- Ferro-magnetic materials: very large $+$ve $\chi_m = 0 \;\; (\mu_r >> 1)$: e.g. Iron, Cobalt, Nickel

At low temperatures, the ferro-magnetic materials are non-linear. As a result, $\vec{B} = \mu_0\mu_r\vec{H}$ does not hold since $\mu_r \propto B$.
Also, super-conductors are diamagnetic and contain very little magnetic fields.

(Negative magnetic permeability is considered when a material, in response to an imposed magnetic field, forms a magnetic dipole that is in the opposite direction to the imposed field.)

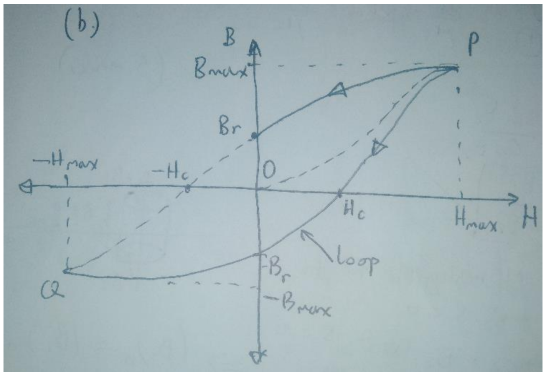


Fig. 2.1 B-H Curves

$B-H$ curve provides the relationship between the $\vec{B}$ and $\vec{H}$ fields. The relationship is non-linear for ferro-magnetic materials.

• There is no $\vec{B}$ and $\vec{H}$ fields when the ferro-magnetic material is unmagnetized.
• The initial magnetization curve, OP is produced as $\vec{H}$ increases due to current from 0 to $\vec{H}_{max}$.
• Saturation occurs at point P.
• When $\vec{H}$ is then decreased, $\vec{B}$ lags $\vec{H}$. The lag is referred to as $hysteresis$.
• A permanent flux density, $\vec{B}_r$ remains even when $\vec{H}$ is reduced to zero. The retentivity field, $\vec{B}_r$ is a percentage of $\vec{H}_{max}$
• When the direction of current is reversed, $\vec{H}$ increases negatively and $\vec{B}$ becomes zero as $\vec{H}$ becomes $\vec{H}_c$, the $coersive\ field\ intensity$.
• Further icrease in H in the negative direction to reach Q and and a reverse in its direction to reach P gives a closed curve called $hysteresis\ loop.$

Application Areas

• The area of the hysteresis loop gives the energy loss per unit volume during one cycle of the periodic magnetization of the ferromagnetic material. Energy is lost in the form of heat. E.g. A tall and narrow loop implies a limited loss.
• Magnetic storage: Rectangular hysteresis loops are utilized in computers for magnetic storage.
• The shape of the hysteresis loop is applied in the design of Motors, Transformers and Generators

Soft materials: These materials have $narrow$ and $smaller$ hysteresis areas and find applications in $transformers$ and $motors$ since there is low energy losses through heat. They possess low coercivity, H_c and low retentivity, β_r.

Hard materials: These materials have $wide$ and $bigger$ hysteresis areas and find applications in $credit\ cards$, $computer\ disk\ drives$ and $audio\ recording$. They possess higher coercivity, H_c which makes it hard to erase the memory.

B. *Boundary Conditions for Magnetic Fields*

State and Demonstrate the Conditions

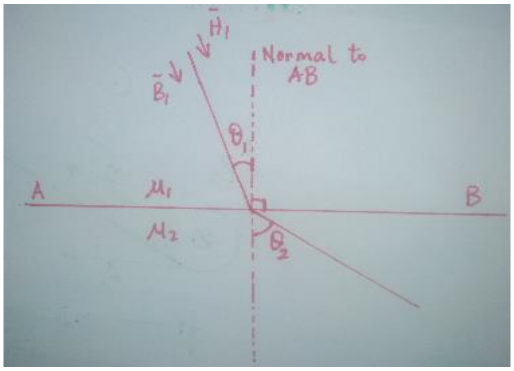


Fig. 2.2 Magnetic Boundary

Magnetic boundary conditions are the conditions that $\vec{H}$ and $\vec{B}$ must satisfy between two different media. Gauss' law for magnetic fields: $\oint \vec{B} \cdot d\vec{S} = 0$, and Ampere's Circuital law: $\vec{H} \cdot d\vec{l} = I$ are used for deriving the boundary conditions.

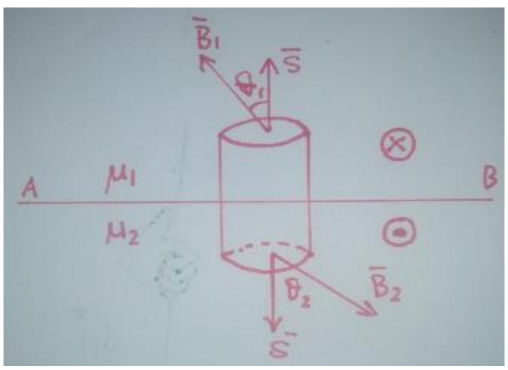


Fig. 2.3 Pill Box

a) Consider a pillbox shape of a negligible height

$\int B.dS = B_1.S + B_2.S = 0$

$-B_1 \cos(\theta_1) \cdot S + B_2 \cos(\theta_2) \cdot S = 0 \rightarrow$ note field direction change: $\otimes, \odot$

$B_1 \cos(\theta_1) = B_2 \cos(\theta_2)$

$(B_1)_n = (B_2)_n$

Condition i) Normal component of magnetic induction is continuous across the boundary

b) Consider a differential length of negligible height

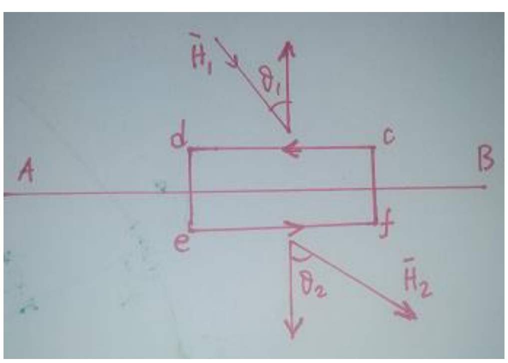


Fig. 2.4 Rectangular Loop

$\int H \cdot dl = H_1 dl + H_2 dl = 0 \rightarrow$ the path does not enclose any current

$0 = \int_c^d \vec{H}_1 \cdot d\vec{l} + \int_e^f \vec{H}_2 \cdot d\vec{l} + \int_d^e \vec{H}_{XY} \cdot d\vec{l} + \int_f^c \vec{H}_{XY} \cdot d\vec{l}$

$-H_1 \sin(\theta_1) \cdot l + H_2 \sin(\theta_2) \cdot l = 0 \rightarrow$ note field direction change ($c \rightarrow d$ and $e \rightarrow f$)

$H_1 \sin(\theta_1) = H_2 \sin(\theta_2)$

$(H_1)_t = (H_2)_t$

Condition ii) Tangential component of magnetic field is continuous across the boundary

Derive the Law:

$B_1 \cos(\theta_1) = B_2 \cos(\theta_2)$

$H_1 \sin(\theta_1) = H_2 \sin(\theta_2)$

By dividing:

$\frac{1}{\mu_1}\frac{\sin(\theta_1)}{\cos(\theta_1)} = \frac{1}{\mu_2}\frac{\sin(\theta_2)}{\cos(\theta_2)}$

$\downarrow\downarrow$

$$\boxed{\frac{\mu_1}{\mu_2} = \frac{\tan(\theta_1)}{\tan(\theta_2)}}$$

Deduce the Law:

If $\mu_1 > \mu_2; \quad \theta_1 > \theta_2$, then in passing from a medium of higher to one of lower permeability, the lines of magnetic induction bend towards the normal.(where there are no surface currents)

C. *Inductance and Energy Stored in Magnetic Fields*

Inductor

It refers to a conductor that is designed in a unique shape in order to store magnetic energy.
Examples:
Toroids, rectangular solenoids, coaxial transmission lines, parallel-wire transmission lines

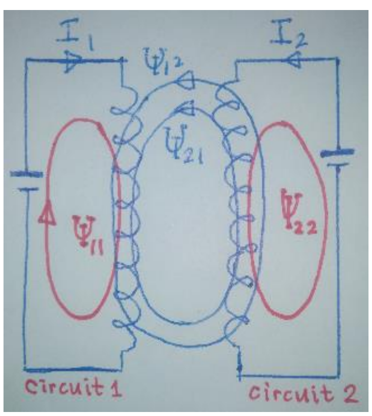
A closed conducting path carrying current I, produces a magnetic field $\vec{B}$, which causes a flux $\Psi = \int_S \vec{B} \cdot d\vec{S}$ to pass through each turn of the circuit.
For N identical turns, the flux linkage λ is given as:
$\lambda = N\Psi$
For a linear medium that sorrounds the circuit, the flux linkage is proportional to the current that produces it and is given as:
$\lambda = kI = LI$
$L \rightarrow$ inductance of the circuit, which is the constant of proportionality.

Inductance

It refers to the ratio of the magnetic flux linkage λ to the current through the conductor.
$L = \frac{\lambda}{I} = \frac{N\Psi}{I} \quad [Wb/A]\ or\ [H]$

Self-inductance occurs when the flux linkages are produced by the inductor itself.

Mutual-inductance occurs when there is flux linkage produced in a circuit due to current in another circuit.
Given, $\Psi_{11}, \Psi_{12}, \Psi_{21}, \Psi_{22}$
$\Psi_{11} \rightarrow$ self-inductance in circuit 1
$\Psi_{12} = \int_{S_1} \vec{B}_2 \cdot d\vec{S} \rightarrow$ flux through circuit 1 due to I_2 in circuit 2
$\Psi_{21} = \int_{S_2} \vec{B}_1 \cdot d\vec{S} \rightarrow$flux through circuit 2 due to I_1 in circuit 1
$\Psi_{22} \rightarrow$ self-inductance in circuit 2



$\downarrow\downarrow$
$M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1\Psi_{12}}{I_2}$
$M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2\Psi_{21}}{I_1}$
Self-inductance of circuits 1 and 2
$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1\Psi_1}{I_1}$
$L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2\Psi_2}{I_2}$
where
$\Psi_1 = (\Psi_{11} + \Psi_{12})$
$\Psi_2 = (\Psi_{22} + \Psi_{21})$

Fig. 2.5 Mutual Inductance

Example

A solenoid with N_1 turns and core of $r_1[cm]$ of length $40[cm]$ is placed in concentric position within another coil of N_2 turns, $r_2[cm]$ in radius of core and of equal length as the first. Assume no leakages and fringing.
i) Sketch the implementation
ii) Find the mutual inductance of the second solenoid due to flux in the first solenoid
iii) Find the self inductance in the first circuit: $L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1\Psi_1}{I_1},\ \Psi_{11} = (\Psi_{11} + \Psi_{21})$

Solution

Mutual inductance of second solenoid due to flux in the first solenoid i.e. ϕ_{21}

$$\begin{aligned}
M_{21} &= \frac{N_2\Psi_{21}}{I_1} \\
\Psi_{11} &= A_1 \cdot \frac{\mu_0 N_1 I_1}{l_1}, \quad A_1 = \pi r_1^2 \\
\Psi_{21} &= \Psi_{11} \\
M_{21} &= \frac{N_2[4\pi \times 10^{-7} \times N_1 I_1 \times (\pi r_1^2)]}{0.4 I_1}
\end{aligned} \tag{16}$$

Energy Stored In Magnetic Fields

solenoid

- thin wire wound into a coil that produces a magnetic field when an electric current is passed through it

toroid

- donut-shaped coil closely wound around that is one continuous wire

The magnetic field strength inside a solenoid is $B = \mu_0 n I$ (inside a solenoid)
where n is the number of loops per unit length of the solenoid. The field inside is very uniform in magnitude and direction.

The magnetic field strength inside a toroid is $B = \frac{\mu_o N I}{2\pi r}$ (within the toroid),
where N is the number of windings. The field inside a toroid is not uniform and varies with the distance as $1/r$.

a) Consider a toroidal solenoid and show that energy density stored in a magnetic field is given as $\boxed{E_v = \frac{1}{2}\frac{B^2}{\mu_0}}$

1) Self-inductance of an infinitely long solenoid

The field inside the toroid is given as:
$H = \frac{NI}{l_m}$ $(H = \frac{I}{2\pi d})$
$l_m \rightarrow$ mean circumference
$N \rightarrow$ number of turns
$a\ [m] \rightarrow$ radius of core
$A = \pi a^2\ [m^2] \rightarrow$ area
$\Psi = B \times A = (\frac{\mu N I \times A}{l_m})$

Flux linkage,
$\lambda = N\Psi = N \cdot (\frac{\mu N I \times A}{l_m}) = (\frac{\mu N^2 I \times A}{l_m})$

Inductance, L of infinitely long solenoid is given as:
$L = \frac{\lambda}{I} = (\frac{\mu N^2 \times A}{l_m})$ $[H]$

When a current flowing through an inductance L changes an e.m.f. is induced in it to oppose the change in the current and the applied voltage overcomes the induced e.m.f., if the change in current is to be maintained. Let the current increase from 0 to $I[A]$, in t seconds, then the work done in establishing the current in the inductive circuit is given as:

$E = P \times t$
$E = -\int_0^t e.i\ dt$
$e \rightarrow$ induced e.m.f
$i \rightarrow$ instantaneous current
But,

$$\boxed{e = -L\frac{di}{dt}}$$

$E = -\int_0^t e.i\ dt = -\int_0^t (-L\frac{di}{dt})\ i\ dt = \int_{i=0}^{I} Li\ di = \frac{1}{2}LI^2$
$\downarrow\downarrow$
Stored energy of the magnetic field. It is equivalent to the energy released when the current is brought down to zero again.
$\downarrow\downarrow$
$E = \frac{1}{2}LI^2 = \frac{1}{2}(\frac{\mu N^2 A}{l_m})I^2 = \frac{1}{2}\mu(\frac{N^2I^2}{l_m^2})(Al_m)$
since $H = \frac{NI}{l_m}$
$\downarrow\downarrow$
$E = \frac{1}{2}\mu H^2(Al_m)$
Energy volume density:
$E_v = \frac{E}{Al_m} = \frac{1}{2}\mu H^2$

In terms of B:

$$\boxed{E_v = \frac{1}{2}\mu_0 H^2 = \mu_0\frac{1}{2}\frac{B^2}{\mu_0^2} = \frac{1}{2}\frac{B^2}{\mu_0}}$$

b) Show that the lifting power of an electromagnet with cross-sectional area, A, when the south pole is moved $dx[m]$ at a velocity, $\vec{u}$, is given as: $\boxed{P_m = \frac{1}{2}\frac{B^2}{\mu_0} \times A\vec{u}}$

Increase in stored energy is given as:
$E = E_v \times Volume$
$= \frac{1}{2}\frac{B^2}{\mu} \times (Adx)$
Work done, W: $W = F \times dx$
$\downarrow\downarrow$
$W = F \times dx = \frac{1}{2}\frac{B^2}{\mu} \times (Adx)$
$\downarrow\downarrow$
$F = \frac{1}{2}\frac{B^2}{\mu} \times (A)$
$\downarrow\downarrow$
Power, P_m = Force $\times$ Velocity
$\downarrow\downarrow$

$$\boxed{P_m = \frac{1}{2}\frac{B^2}{\mu_0} \times A\vec{u}}$$

Exercise

Derive the inductance of a rectangular toroid of sides h and $(r_2 - r_1)$ with N turns of coil.
$\left(l = \frac{\lambda}{I} = \frac{\mu N^2 h}{2\pi}\ln\frac{r_2}{r_1}\right)$

Summary:

Materials are classified as paramagnetic, diamagnetic, or ferromagnetic, depending on how they behave in an applied magnetic field.

Paramagnetic materials have partial alignment of their magnetic dipoles with an applied magnetic field. This is a positive magnetic susceptibility. Only a surface current remains, creating a solenoid-like magnetic field. Their magnetic dipoles align partially in the same direction as the applied magnetic field; when this field is removed, the material is unmagnetized

Diamagnetic materials exhibit induced dipoles opposite to an applied magnetic field. This is a negative magnetic susceptibility.

Ferromagnetic materials have groups of dipoles, called domains, which align with the applied magnetic field. However, when the field is removed, the ferromagnetic material remains magnetized, unlike paramagnetic materials. This magnetization of the material versus the applied field effect is called hysteresis.

IV. Time Varying Electromagnetic Fields

- Electrostatic Fields: Static electric charges, Q
- Magnetoststic Fields: Motion of electric charges with uniform velocity, $I = \frac{dQ}{dt}$
- Time-Varying Fields: Accelerated charges, $\frac{dI}{dt} = \frac{d^2Q}{dt^2} \to$ waves

A static magnetic field produces no flow of current.
A time varying field produces an induced voltage ($e.m.f.$), in a closed circuit, which causes a flow of current.

• An electrostatic field, E_e is conservative and cannot maintain a steady current in a closed circuit:
$\oint_L E_e \cdot dl = IR = 0$
• An $e.m.f.$ produced field, E_f is non-conservative and maintains a steady current

From Faraday's law,
Let $V_{e.m.f.} = -\frac{d\Psi}{dt}$, where $\Psi = \int \vec{B} \cdot d\vec{s}$
$= \oint_L \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{s}$
$\downarrow\downarrow$
Both electric and magnetic fields are present and inter-related.

Faraday's law
Induced emf is created in a closed loop due to a change in magnetic flux through the loop
Induced emf
Short-lived voltage generated by a conductor or coil moving in a magnetic field
Magnetic flux
Measurement of the amount of magnetic field lines through a given area

A. *Faraday's Law of EM Induction*

The e.m.f., $V_{e.m.f.}[V]$ in any closed circuit is equal to the time rate of change of magnetic flux linkage by the circuit. i.e.
$V_{e.m.f.} = \frac{d\lambda}{dt} = -N\frac{d\Psi}{dt}$

B. *Lenz's Law*

The induced magnetic field produced by the induced current will oppose the original magnetic field. This is because the induced voltage acts in such a way as to oppose the flux producing it.

• Electric fields can be produced from electric charges or $e.m.f.$ produced fields.
• EMF produced fields convert non-electric energy into electrical energy and they include: Batteries, Fuel Cells, Generators

C. *Maxwell's Equations*

State Maxwell's equations for time-varying electromagnetic fields in integral form and explain their physical significance.

- The dot product of the field densities and the cross product of the field intensities.
- The cross product of electric field produces time-varying magnetic field
- The cross product of magnetic field produces time-varying electric field

Point Form	Integral Form	Significance
$\bigtriangledown \cdot \vec{D}_s = \rho_v$	$\oint \vec{D}_s \cdot d\vec{s} = \int \rho_v \cdot dv$	Gauss's law, Electrostatics
$\bigtriangledown \cdot \vec{B}_s = 0$	$\oint \vec{B}_s \cdot d\vec{s} = 0$	Gauss's law, Magnetostatics
$\bigtriangledown \times \vec{E}_s = -\frac{\partial \vec{B}}{dt}$	$\oint \vec{E}_s \cdot d\vec{l} = -\frac{\partial}{dt} \int \vec{B}_s \cdot d\vec{s}$	Faraday's law
$\bigtriangledown \times \vec{H}_s = J_s + \frac{\partial \vec{D}}{dt}$	$\oint \vec{H}_s \cdot d\vec{l} = \int (J_s + \frac{\partial \vec{D}}{dt}) \cdot d\vec{s}$	Ampere's circuital law

D. *Electromagnetic Potential*

Start from Maxwell's third EM equation and show that the potential gradient of time-varying fields is given as

$$\boxed{\vec{E} = -\bigtriangledown V - \frac{\partial \vec{A}}{\partial t}}$$

• It is desired to deduce the relationship between electric scalar potential, V and vector magnetic potential, $\vec{A}$. From the 3^{rd} law:

$\bigtriangledown \times \vec{E}_s = -\frac{\partial \vec{B}}{dt}$

As $\boxed{\vec{B} = \bigtriangledown \times \vec{A}}$

$\downarrow\downarrow$

$\bigtriangledown \times \vec{E}_s = -\frac{\partial}{dt}(\bigtriangledown \times \vec{A})$

$\bigtriangledown \times \vec{E}_s + \frac{\partial}{dt}(\bigtriangledown \times \vec{A}) = 0$

$\bigtriangledown \times (\vec{E}_s + \frac{\partial \vec{A}}{dt}) = 0$

From the vector identity $\boxed{\bigtriangledown \times \bigtriangledown \Phi = 0}$

Let $\Phi \rightarrow$ potential V,

$\bigtriangledown \Phi = \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\bigtriangledown V$

$$\boxed{\vec{E} = -\bigtriangledown V - \frac{\partial \vec{A}}{\partial t}}$$

E. *Variation of Magnetic Flux*

The variation of the magnetic flux may happen in the following ways:

1. Stationary loop in a time-varying B-field
2. A time-varying loop in a stationary B-field
3. A time-varing loop area in a time-varying B-field

CASE A: Stationary loop in a time-varying B-field

In this case:

$V_{emf} = \oint_L \vec{E} \cdot d\vec{l} = \int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$

This is referred to as Transformer e.m.f

Applying Stoke's theorem: (The Curl integral function over a surface equal the dot-product in length)

$\oint_s (\bigtriangledown \times \vec{E}) \cdot d\vec{s} = \int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$

$\downarrow\downarrow$

$\boxed{\bigtriangledown \times \vec{E}_s = -\frac{\partial \vec{B}}{dt}}$ $\rightarrow\rightarrow$ Time varying B-field is not conservative

CASE B: Moving Loop in a Static B-field

Force: $\vec{F} = Q\vec{E}$
For a charge moving with a uniform velocity, $\vec{u}$, the magnetic field, $\vec{B}$ is given as $\vec{F} = Q\vec{u} \times \vec{B}$
$\downarrow\downarrow$
$\vec{E}_m = \frac{\vec{F}}{Q} = \vec{u} \times \vec{B}$
$V_{emf} = \oint_l \vec{E}_m \cdot d\vec{l} = \oint_l (\vec{u} \times \vec{B}) \cdot d\vec{l}$

This is referred to as Motional e.m.f: Motors, Generators, Alternators

For a conductor of length $l[m]$:

$$\boxed{V_{emf} = \int \vec{E}_m \cdot d\vec{l} = uBl \sin\theta}$$

where, θ is the angle between the direction/point of motion and the magnetic field orientation.
By applying Stoke's theorem:
$\oint_s (\bigtriangledown \times \vec{E}_m) \cdot d\vec{s} = \int_s \bigtriangledown \times (\vec{u} \times \vec{B}) \cdot d\vec{s}$
$\downarrow\downarrow$

$$\boxed{\bigtriangledown \times \vec{E}_m = \bigtriangledown \times (\vec{u} \times \vec{B})}$$

Example
A straight conductor of length $l = 17$ $[cm]$ moves perpendicularly to its axis at a velocity of 35 $[m/s]$ in a uniform magnetic field of flux density 0.76 $[T]$. Determine the $e.m.f$ induced given that the direction of motion is:
a) normal to the field
b) parallel to the field
c) at an angle of 42^0 to the orientation of the field

Solution

$$\boxed{V = uBl \sin(\theta)}$$

Case (a) : $\theta = 90^0$: $V = (35 \times 0.76 \times 0.17 \times \sin[90]) \rightarrow$ maximum
Case (b) : $\theta = 0^0$
Case (c) : $\theta = 42^0$

CASE C: Moving Loop in a Time-Varying B-field

In this case, both motional and transformer emf will exist.
$V_{emf} = \oint_l \vec{E} \cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_l (\vec{u} \times \vec{B}) \cdot d\vec{l}$

By applying Stoke's theorem:

$$\boxed{\bigtriangledown \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \bigtriangledown \times (\vec{u} \times \vec{B})}$$

Displacement Current

Starting from Ampere's Circuital law for Electrostatics, $\bigtriangledown \times \vec{H} = \vec{J}$, show that the modified form of the law for time-varying field is given as $\boxed{\bigtriangledown \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$

The divergence of a Curl of any vector field is zero, i.e. $\bigtriangledown \cdot (\bigtriangledown \times \vec{H}) = \bigtriangledown \cdot \vec{J} = 0$
But,
from the equation of continuity:
$\bigtriangledown \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \neq 0$
$\downarrow\downarrow$ It implies existence of displacement current
$\bigtriangledown \times \vec{H} = (\vec{J} + \vec{J_d})$
$\bigtriangledown \cdot (\bigtriangledown \times \vec{H}) = 0 = \bigtriangledown \cdot (\vec{J} + \vec{J_d}) = \bigtriangledown \cdot \vec{J} + \bigtriangledown \cdot \vec{J_d}$
$\downarrow\downarrow$
$\bigtriangledown \cdot \vec{J_d} = - \bigtriangledown \cdot \vec{J} = -(-\frac{\partial \rho_v}{\partial t})$
Gauss's Law: $(\bigtriangledown \cdot \vec{D} = \rho_v)$
$\bigtriangledown \cdot \vec{J_d} = \frac{\partial}{\partial t}(\bigtriangledown \cdot \vec{D}) = \bigtriangledown \cdot \frac{\partial \vec{D}}{\partial t}$
$\downarrow\downarrow$
$\vec{J_d} = \frac{\partial \vec{D}}{\partial t}$

$\boxed{\bigtriangledown \times \vec{H} = (\vec{J} + \vec{J_d}) = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$ EM wave propagation is ONLY possible due to $\vec{J_d}$

Show that the displacement current in the dielectric of a parallel plate capacitor is equal to the conduction current in the leads.

Capaitance: $C = \frac{\epsilon A}{d}$

i) Conduction current, $Q = CV$

$i_c dt = C dV$
$i_c = C\frac{dV}{dt}$

$$\boxed{i_c = (\frac{\epsilon A}{d})\frac{dV}{dt}}$$

ii) Electric Field in the dielectric

$E = \frac{V}{d}$
$D = \epsilon E = \epsilon \frac{V}{d}$
Derivative: $\frac{\partial D}{\partial t} = \frac{\epsilon}{d}\frac{dV}{dt} = J_d$

iii) Displacement current

$i_d = \int_s \vec{J_d} \cdot d\vec{s}$
$= \int_s \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$
$= \int_s \frac{\epsilon}{d}\frac{dV}{dt} \cdot d\vec{s} = \frac{\epsilon}{d}\frac{dV}{dt} \cdot \int_s d\vec{s} = \frac{\epsilon}{d}\frac{dV}{dt} \cdot Area, A$

$$\boxed{i_d = (\frac{\epsilon A}{d})\frac{dV}{dt}}$$

Applications of EM induction

- Tablet
- Credit Card
- Hybrid electric vehicle
- Transcranial Stimulation

For a tablet, there are tiny wires running across the length and width of the screen. The pen has a tiny magnetic field coming from the tip. As the tip brushes across the screen, a changing magnetic field is felt in the wires which translates into an induced EMF that is converted into the line you just drew.

Magnetic stripe on the back of your personal credit card as used at the grocery store or the ATM machine. This works on the same principle as the audio or video tape, in which a playback head reads personal information from your card.

A tablet with a specially designed pen to write with is another application of magnetic induction.

For EVs the motor can act as a generator whenever the car is braking, taking advantage of the back emf produced. This extra emf can be newly acquired stored energy in the car's battery, prolonging the life of the battery.

For TMS, a rapidly varying and very localized magnetic field is placed close to certain sites identified in the brain to excite certain electrical activities in treating hullicinations and depression.

Summary:

V. Electromagnetic Wave Propagation

Waves are a means of transporting energy or conveying information.

Examples: TV signals, RADAR, Light rays

EM Energy has the following Characteristics:

a) Travel at a given velocity
b) Assume properties of waves
c) Radiate outwards from a source

Concepts:

1) From Maxwell's two Curl equations, in the case of free-space, obtain a three dimensional wave equation in either $\vec{E}$ or $\vec{H}$, hence wave velocity in free-space.
2) Determine the intrinsic impedance of plane waves in good conductors, $\sigma \approx \infty$
3) Explain the significance of Skin Depth of propagation at different frequencies
4) Determine the average wave power in terms of Poynting's Vector:
a) $\vec{p} = \vec{E} \times \vec{H}$
b) $\int_s \vec{p} \cdot d\vec{s}$

A. 3D WAVE Equation

Consider a perfect dielectric region free of charges ($\rho_v = 0$), for which μ and ϵ are constants.
Maxwell's equations are modified as follows:
Modified Point Form
$\bigtriangledown \cdot \vec{D} = \rho_v = 0$, (charge-free)
$\bigtriangledown \cdot \vec{B} = 0$
$\bigtriangledown \times \vec{E} = -\frac{\partial \vec{B}}{dt} = -\mu \frac{\partial \vec{H}}{dt}$
$\bigtriangledown \times \vec{H} = (J_s + \frac{\partial \vec{D}}{dt}) = \frac{\partial \vec{D}}{dt} = \epsilon \frac{\partial \vec{E}}{dt}$ (J_s=0, dielectric)
Taking a curl on both sides
$\bigtriangledown \times (\bigtriangledown \times \vec{H}) = \epsilon(\bigtriangledown \times \frac{\partial \vec{E}}{dt})$: $\boxed{\bigtriangledown \times (\bigtriangledown \times \vec{H}) = \bigtriangledown(\bigtriangledown \cdot \vec{H}) - \bigtriangledown^2 \vec{H}}$

$\bigtriangledown(\bigtriangledown \cdot \vec{H}) - \bigtriangledown^2 \vec{H} = \epsilon \frac{\partial}{\partial t}(\bigtriangledown \times \vec{E})$ but, $\boxed{\bigtriangledown \cdot \vec{H} = 0, \quad \bigtriangledown \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}}$
$\downdownarrows$
$-\bigtriangledown^2 \vec{H} = \epsilon \frac{\partial}{\partial t}(-\mu \frac{\partial \vec{H}}{\partial t}) = -\mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2}$
$\downdownarrows$

$$\boxed{\bigtriangledown^2 \vec{H} = \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2}} \rightarrow 3D \text{ WAVE Equation} \rightarrow \boxed{\bigtriangledown^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$$

B. 1D WAVE Equation

If the wave is propagating in $z-$ direction only, $1D$ Wave Equation is written as:

$$\boxed{\frac{\partial^2 \vec{H}}{\partial^2 z} = \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2}} \rightarrow 1D \text{ WAVE Equation} \rightarrow \boxed{\frac{\partial^2 \vec{E}}{\partial^2 z} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$$

C. *WAVE Velocity*

$$\boxed{\bigtriangledown^2 \vec{H} = \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2}} \rightarrow \frac{\partial^2 \vec{H}}{\partial t^2} = \frac{1}{\mu\epsilon} \bigtriangledown^2 \vec{H}: \quad \text{Let } u = \sqrt{\frac{1}{\mu\epsilon}}$$

$\frac{\partial^2 \vec{H}}{\partial t^2} = u^2 \bigtriangledown^2 \vec{H} \rightarrow \quad 3D$ Wave Equation

Wave Velocity:
$u_0 = \frac{1}{\sqrt{\mu_0\epsilon_0}}$;
$u_0 = \frac{1}{\sqrt{4\pi\times 10^{-7} \times \frac{10^{-9}}{36\pi}}} = 3\times 10^8 \quad [m/s]$

D. *Wave Propagation*

The imaginary part of a positive going wave can be written as follows:
$\vec{E}^+ = A\sin(\omega t - \beta z)$

where
A: is the amplitude of the wave, $[m]$
$(\omega t - \beta z)$: the phase of the wave in radians
ω: angular frequency, $[rad/s]$
β: wave number or phase constant, $[rad/m]$

The wave travels a distance of $\lambda[m]$ in $T[s]$ at a speed of $u[m/s]$:
$\lambda = u \times T$
$T = \frac{1}{f}: \quad f \rightarrow$ frequency in hertz.
$u = f\lambda$
$\omega = 2\pi f, \quad \beta = \frac{\omega}{u} = \frac{2\pi f}{u} = \frac{2\pi}{(\frac{u}{f})=\lambda}$
$\downarrow$
$\boxed{\beta = \frac{2\pi}{\lambda}}$: a phase change of 2π radians for every wavelength of distance travelled.

1) Wave Propagation in LOSSY Dielectric: In this medium, $\sigma \neq 0$, and it is partially conducting. Power is lost due to such conduction.
We consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge-free, $\rho_v = 0$
From Maxwell's 3^{rd} equation, $\boxed{\bigtriangledown \times \vec{E} = -j\omega\vec{B}}$

Taking a Curl on both sides:
$\bigtriangledown \times (\bigtriangledown \times \vec{E}) = \bigtriangledown \times (-j\omega\vec{B}) = \bigtriangledown \times (-j\omega\mu\vec{H}) = -j\omega\mu(\bigtriangledown \times \vec{H})$

From 4^{th} Equation: $\bigtriangledown \times \vec{H} = (\vec{J} + j\omega\vec{D}) = (\sigma\vec{E} + j\omega\epsilon\vec{E}) = (\sigma + j\omega\epsilon)\vec{E}$
$\downarrow\downarrow$
$\bigtriangledown \times (\bigtriangledown \times \vec{E}) = -j\omega\mu(\bigtriangledown \times \vec{H}) = -j\omega\mu(\sigma + j\omega\epsilon)\vec{E}$
$\downarrow\downarrow$
$\bigtriangledown \times (\bigtriangledown \times \vec{E}) = -\bigtriangledown^2 \vec{E} = -j\omega\mu(\sigma + j\omega\epsilon)\vec{E} \quad$ since $\boxed{\bigtriangledown \cdot \vec{E} = 0}$ from

$\boxed{\bigtriangledown \times (\bigtriangledown \times \vec{E}) = \bigtriangledown(\bigtriangledown \cdot \vec{E}) - \bigtriangledown^2 \vec{E}}$
Let $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$, then:
$\downarrow\downarrow$
$\boxed{\bigtriangledown^2 \vec{E} = \gamma^2 \vec{E}} \quad , \quad \boxed{\bigtriangledown^2 \vec{H} = \gamma^2 \vec{H}}$: Helmholtz's/ vector wave equations

COMPONENTS OF A COMPLEX PROPAGATION CONSTANT

γ is a complex quantity defined as follows:
1.) $\gamma = \alpha + j\beta$
2.) $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) = j\omega\mu\sigma - \omega^2\mu\epsilon = \omega\mu(j\sigma - \omega\epsilon)$
3.) $\gamma^2 = (\alpha + j\beta)^2 = \alpha^2 - \beta^2 + 2j\alpha\beta$

Equating real and imaginary parts of (2) and (3):
4.) $\alpha^2 - \beta^2 = -\omega^2\mu\epsilon; \quad -\alpha^2 + \beta^2 = \omega^2\mu\epsilon$
$2\alpha\beta = \omega\mu\sigma$
From (3):
5.) $|\gamma^2| = \sqrt{(\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2} = (\alpha^2 + \beta^2)$
From (2):
6.) $|\gamma^2| = \omega\mu\sqrt{\sigma^2 + \omega^2\epsilon^2}$

Equating (5) and (6):
7.) $(\alpha^2 + \beta^2) = \omega\mu\sqrt{\sigma^2 + \omega^2\epsilon^2}$

Add (4) to (7):
$2\beta^2 = \omega^2\mu\epsilon + \omega\mu\sqrt{\sigma^2 + \omega^2\epsilon^2}$
$\downarrow\downarrow$
8.) $\beta = \omega[\sqrt{\frac{\mu\epsilon}{2}(1 + \sqrt{(\frac{\sigma^2}{\omega^2\epsilon^2} + 1)})}]$
$\alpha = \omega[\sqrt{\frac{\mu\epsilon}{2}(-1 + \sqrt{(\frac{\sigma^2}{\omega^2\epsilon^2} + 1)})}]$

2) Plane Waves in Good Conductors: A GOOD perfect conductor has $\sigma >> \omega\epsilon$. Hence $\sigma \approx \infty, \epsilon = \epsilon_0\epsilon_r, \mu = \mu_0\mu_r$
Substituting in equation (8),
$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{2\pi f\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}$
$u = \frac{\omega}{\beta} = \frac{2\pi f}{\sqrt{\pi f\mu\sigma}} = \sqrt{\frac{(2\pi f)^2}{\pi f\mu\sigma}} = \sqrt{\frac{2\omega}{\mu\sigma}}$
Also
$u = \frac{\omega}{\beta} \rightarrow \lambda = \frac{u}{f} = \frac{\omega}{\beta f} = \frac{2\pi f}{\beta f}$
$\downarrow\downarrow$

$$\boxed{\lambda = \frac{2\pi}{\beta}}$$

Intrinsic Impedance:

Considering $P = VI$:
From,
$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) = j\omega\mu\sigma - \omega^2\mu\epsilon$

$$\boxed{\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}}$$

As $\sigma >> \omega\epsilon$, $j\omega\epsilon$ can be neglected
$\eta = \sqrt{\frac{j\omega\mu}{\sigma}}$
$|\eta| = \sqrt{\frac{\omega\mu}{\sigma}}$
The magnetic field can be expressed in terms of an electric field such that,

$$\boxed{\vec{H} = \frac{E_0}{|\eta|}e^{-\alpha z}\cos(\omega t - \beta z - \theta_n)\vec{a}_y}$$

Example
In a free-space medium, the electric field is given as $E(z,t) = A\sin(\omega t - \beta z)\vec{a}_y$ $[V/m]$. Obtain the expression for the magnetic field and the propagation constant, γ at a frequency of $f = 32$ $[MHz]$.
Solution
$E(z,t) = A\sin(\omega t - \beta z)\vec{a}_y$ $[V/m]$
Let the wave propagate in the z-direction
It follows that $\vec{H}$ is along $(-a_x)$ direction i.e. $(\vec{a}_y \times \vec{a}_x) = -\vec{a}_z$ and $(\vec{a}_y \times -\vec{a}_x) = \vec{a}_z$

$\vec{H}(z,t) = \frac{E_0}{|\eta|}\cos(\omega t - \beta z)(-\vec{a}_x)$ $[A/m]$
$E_0 = A, \quad \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$; $\sigma \approx 0$; $\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi$
$\vec{H}(z,t) = \frac{A}{120\pi}\cos(\omega t - \beta z)(-\vec{a}_x)$ $[A/m]$

For propagation constant: $\gamma = (\alpha + j\beta)$:
In free-space: $\alpha = 0$; no attenuation, $\gamma = j\beta$

$\beta = \frac{\omega}{u} = \frac{2\pi f}{u} = (\frac{2\pi\times 32\times 10^6}{3\times 10^8)}$ $[rad/m]$

Example

A dielectric medium has a dielectric constant of $\epsilon_r = 2.3$. Its conduction current equals displacement current at a 5G frequency of 3.5 $[GHz]$. Find the conductivity of the medium.

Solution

$J_c = \sigma E, \quad J_d = \omega D = \omega \epsilon E$
$I_d = I_c$, IFF, $\quad \sigma E = \omega \epsilon E$
$\sigma = \omega \epsilon$
$\sigma = 2\pi f \epsilon_r \epsilon_0$
$\sigma = (2\pi \times 3.5 \times 10^9 \times 2.3 \times 8.85 \times 10^{-12})$

E. *SKIN Depth*

It refers to the distance, δ through which a wave amplitude decreases to a factor $e^{-1}(37\%)$ in a conducting medium.
While a radio wave is propagating in a uniform conducting medium, 63% of current density flows within the skin depth. The amplitudes are attenuated by a factor $e^{-\alpha z}$. This depth of penetration (skin depth, δ) is given as:

$$\delta = \sqrt{\frac{1}{\pi f \mu \sigma}}$$

where
f: the frequency of the source/ carrier frequency, (Hz)
μ: magnetic permeability of tissue, (H/m)
σ: conductivity of the medium, (S/m)

Note:
$E_0 e^{-\alpha \delta} = E_0 e^{-1} \rightarrow \delta = \frac{1}{\alpha}, \quad \alpha = \sqrt{\pi f \mu \sigma}; \quad \delta = \sqrt{\frac{1}{\pi f \mu \sigma}} = \sqrt{\frac{2}{\mu \sigma \omega}}$

<u>Skin Effect and Skin Resistance</u>

Propagation of waves in the skin depth results in skin effect, which leads to surface/skin resistance, $R_s[\Omega/m^2]$.
Skin resistance: Is the real part of the intrinsic impedance, η, of a good conductor.

Given: $\boxed{\eta = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\epsilon}}}$

$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^0$: $\quad$ in good conductors, $\sigma >> j\omega\epsilon$
$\alpha^2 = \pi f \mu \sigma \quad \rightarrow \quad \mu = \frac{\alpha^2}{\pi f \sigma}$
$\eta = \sqrt{\frac{\omega(\frac{\alpha^2}{\pi f \sigma})}{\sigma}}\angle 45^0 = \sqrt{\frac{\omega(\frac{\alpha^2}{\pi f})}{\sigma^2}}\angle 45^0 = \sqrt{2}\frac{\alpha}{\sigma}\angle 45^0$
$\eta = \frac{\sqrt{2}}{\sigma\delta}\angle 45^0 = \frac{\sqrt{2}}{\sigma\delta}(\cos 45^0 + j \sin 45^0)$

$\eta = (\frac{1+j}{\sigma\delta})$
$\downarrow\downarrow$
$\boxed{R_s = \dfrac{1}{\sigma\delta}}$

<u>APPLICATION in EM shields</u>

EM shields are made of conducting sheets that must be thicker than the skin depth. The skin resistance increases with frequency, and therefore the effective area of wire decreases. To solve this problem in high frequency applications, use wire of many strands to reduce the skin effect.

F. *Average Power and Poynting's Vector*

State Poynting's theorem and the significance of Poynting's Vector

The rate at which EM waves transport energy can be estimated from Maxwell's equations.

Poynting's Theorem:

The net power flowing out of a given volume, V, is equal to the time rate of decrease in the energy stored within the volume, minus conduction losses.

Poynting's Theorem: Significance

• $\vec{p} = (\vec{E} \times \vec{H}) \quad [w/m^2]$: Refers to the instantaneous power density vector assocoated with the EM field at a point. The power $\vec{p}$ is normal to $\vec{E}$ and $\vec{H}$.
• $\int_s \vec{p}.d\vec{s}$: The integration of $\vec{p}$ over a closed surface implies the net power flowing out of that surface.
• There exists the rate of decrease of energy in both magnetic and electric fields.
• There is ohmic power dissipated.

Derivation:
$\bigtriangledown \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$
$\bigtriangledown \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}$
Taking a dot product with $\vec{E}$:
(1.)
$\vec{E} \cdot (\bigtriangledown \times \vec{H}) = \sigma \vec{E} \cdot \vec{E} + \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = (\sigma E^2 + \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t})$
From the identity: $\boxed{\bigtriangledown \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\bigtriangledown \times \vec{A}) - \vec{A} \cdot (\bigtriangledown \times \vec{B})}$

Let $\vec{A} = \vec{H}$ and $\vec{B} = \vec{E}$
(2.)
$\bigtriangledown \cdot (\vec{H} \times \vec{E}) = \vec{E} \cdot (\bigtriangledown \times \vec{H}) - \vec{H} \cdot (\bigtriangledown \times \vec{E})$
$\vec{E} \cdot (\bigtriangledown \times \vec{H}) = \bigtriangledown \cdot (\vec{H} \times \vec{E}) + \vec{H} \cdot (\bigtriangledown \times \vec{E})$
From (1.) and (2.):
$(\sigma E^2 + \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}) = \bigtriangledown \cdot (\vec{H} \times \vec{E}) + \vec{H} \cdot (\bigtriangledown \times \vec{E}) = \bigtriangledown \cdot (\vec{H} \times \vec{E}) + \vec{H} \cdot (-\mu \frac{\partial \vec{H}}{\partial t})$
$\downarrow\downarrow$
$\bigtriangledown \cdot (\vec{H} \times \vec{E}) = (\sigma E^2 + \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}) + \mu \vec{H} \cdot (\frac{\partial \vec{H}}{\partial t})$

By interchanging $\vec{E}$ and $\vec{H}$: $\boxed{(\frac{\partial H^2}{\partial t}) = \frac{\partial (\vec{H} \cdot \vec{H})}{\partial t} = 2\vec{H} \cdot \frac{\partial \vec{H}}{\partial t} \rightarrow \vec{H} \cdot (\frac{\partial \vec{H}}{\partial t}) = \frac{1}{2}(\frac{\partial H^2}{\partial t})}$

$\bigtriangledown \cdot (\vec{E} \times \vec{H}) = -\sigma E^2 - \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} - \mu \vec{H} \cdot (\frac{\partial \vec{H}}{\partial t})$
$= -\sigma E^2 - \frac{1}{2} \epsilon \frac{\partial E^2}{\partial t} - \frac{1}{2} \mu (\frac{\partial H^2}{\partial t})$

Taking the volume integral:
$\int_v \bigtriangledown \cdot (\vec{E} \times \vec{H}) \cdot dv = -\frac{\partial}{\partial t} \int_v \frac{1}{2} [\epsilon E^2 + \mu H^2] - \int_v \sigma E^2 \cdot dv$

Applying divergence theorem:

$$\boxed{\oint_s \bigtriangledown \cdot (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\frac{\partial}{\partial t} \int_v \frac{1}{2} [\epsilon E^2 + \mu H^2] - \int_v \sigma E^2 \cdot dv}$$

where
$-\frac{\partial}{\partial t} \int_v \frac{1}{2} [\epsilon \frac{\partial E^2}{\partial t} + \mu (\frac{\partial H^2}{\partial t})] \rightarrow$: Decrease in energy in both electric and magnetic fields

$-\int_v \sigma E^2 \cdot dv \rightarrow$: decrease in Ohmic power

Example

A very long and straight wire is oriented along the z axis and carries a current, $I \quad [A]$. Use the concept of Poynting's vector to show that the total current entering a unit length of wire of resistance, R is given as $p = I^2 R \quad [W]$.

Solution

The magnetic field intensity at the surface due to an infinite wire of radius, r is given as:

$$\boxed{\vec{H} = \frac{I}{2\pi r}\vec{a}_\phi}$$

Also:
$\vec{J} = \sigma\vec{E}$

$$\boxed{\vec{E} = \frac{\vec{J}}{\sigma}\ \vec{a}_z = \frac{\vec{I}}{A\sigma}\ \vec{a}_z}$$

Poynting's Vector:
$\vec{p} = \vec{E} \times \vec{H}$
$\vec{p} = \frac{I}{A\sigma}\ \vec{a}_z \times \frac{I}{2\pi r}\vec{a}_\phi = \frac{I}{A\sigma}\ \frac{I}{2\pi r}(\vec{a}_z \times \vec{a}_\phi) = \ \frac{I^2}{2\pi r A\sigma}(-\vec{a}_\rho)$

Total Power: Cylindrical surface
$P = \oint\oint \vec{p}\cdot d\vec{s}$
$= \oint\oint \frac{I^2}{2\pi r A\sigma}(-\vec{a}_\rho)\cdot r d\phi dz \vec{a}_\rho$
$= \int_{z=0}^{l}\int_{\phi=0}^{2\pi} \frac{I^2}{2\pi A\sigma}\cdot d\phi dz$
$= \frac{I^2}{2\pi A\sigma} \times 2\pi \times l$

$= \frac{I^2 l}{A\sigma} = I^2 R \quad [W]$ since $R = \frac{l}{A\sigma} = \frac{\rho l}{A}; \quad \rho \to$ resistivity

Example

The electric field of an EM field is described by $\vec{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\vec{a}_x$ and the angle between the fields is θ_η. Use Poynting's vector to show that the average power is given as $\boxed{\vec{p}_{av} = \frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos(\theta_\eta)\cdot\vec{a}_z \quad [W/m^2]}$.

Solution

Since the E field is in the $\vec{a}_x$ direction and the wave propagates in the z-direction, the magnetic field is in the y-direction.

- $\vec{H}(z,t) = \frac{E_0}{\eta}e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\vec{a}_y$
- $\vec{p}(z,t) = \frac{E_0^2}{\eta}e^{-2\alpha z}\cos(\omega t - \beta z)\cos(\omega t - \beta z - \theta_\eta)\vec{a}_z$
- $\boxed{\vec{p}_{av}(z) = \frac{1}{T}\int \vec{p}(z,t)dt}$
- $\vec{p}_{av}(z) = \frac{1}{T}\frac{E_0^2}{2\eta}e^{-2\alpha z}\left[\int_0^T \cos(\theta_\eta)dt + \int_0^T \cos(2\omega t - 2\beta z - \theta_\eta)dt\right]$
- $\vec{p}_{av}(z) = \frac{1}{T}\frac{E_0^2}{2\eta}e^{-2\alpha z}\left[\cos(\theta_\eta)\cdot T + \frac{\sin(4\pi - 2\beta z - \theta_\eta)}{2\omega} - \frac{\sin(0 - 2\beta z - \theta_\eta)}{2\omega}\right] : \quad T = \frac{2\pi}{\omega}$

$$\boxed{\vec{p}_{av} = \frac{E_0^2}{2|\eta|}e^{-2\alpha z}\cos(\theta_\eta)\cdot\vec{a}_z \quad [W/m^2]}.$$

Further Examples

Example

Determine if the vector field, **F** describes a magnetic field, given that $\mathbf{F}(x,y) =< x^2y, 2y - xy^2 >$.

Solution

For a magnetic field, $\bigtriangledown \cdot \mathbf{F} = 0$

$\bigtriangledown \cdot \mathbf{F}(x,y) = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y - xy^2)$

$= 2xy + 2 - 2xy$

$= 2$

$\downarrow\downarrow$

F doesn't model a magnetic field

Example

Determine the existence of flow of electric charges at a point, $(0, 2, -1)$ in the vector field, $\mathbf{F}(x,y,z,) = e^x\vec{i} + 2yz\vec{j} - yz^3\vec{k}$.

Solution

Determine the divergence at the point: $\bigtriangledown \cdot \mathbf{F}$

$\bigtriangledown \cdot \mathbf{F}(x,y) = \frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(yz^3)$

$= e^x + 2z - 3yz^2$

$\downarrow\downarrow$

$\bigtriangledown \cdot \mathbf{F}(0,2,-1) = e^0 + 2(-1) - (3 \cdot 2 \cdot (-1)^2) = 1 - 2 - 6 = -7$

There is an inward flow of charge density at the point.

VI. MAGNETIC CIRCUITS

Example of Devices that exploit Magnetic circuits

Transformers, Generators, Motors, Relays, Toroids

In the magnetic field, $\bigtriangledown \cdot \vec{F} = 0$

A. *ANALOGY of Magnetic Circuits and Electric Circuits*

TABLE I
SUMMARY

Electric	**Magnetic**
Resistance, R	Reluctance,$\mathcal{R}$
Conductance, $G = \frac{1}{R}$	Permeance, $P = \frac{1}{\mathcal{R}}$
Conductivity, σ	Permeability,μ
Current, I	Flux,$\Psi = \int B \cdot ds$
Current Density, $J = \sigma E$	Flux Density,$B = \mu H$
Field Intensity, $\vec{E}$	Field Intensity,$\vec{H}$
Kirchoff's Law: $\sum I = 0$; $\sum V = \sum RI$	$\sum \Psi = 0$; $\sum F$=$\sum \mathcal{R} \cdot \Psi$
Ohm's Law: $V = E \cdot l = IR$	Magnetomotive Force, $F_m = \Psi\mathcal{R} = NI$

B. *Similarities*

Kirchoff's current and voltage laws can be applied to magnetic circuits, to nodes and loops. Example with n elements:

a) Series

$\Psi_1 = \Psi_2 = \Psi_3 = \cdots = \Psi_n$

$F = F_1 + F_2 + \cdots + F_n$

b) Parallel

$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \cdots + \Psi_n$

$F = F_1 = F_2 = \cdots = F_n$

C. *Differences*

a) Current, I flows in electric circuits, whereas magnetic flux, Ψ does not flow.

b) Conductivity, σ does not change with current density, $J = \sigma E$ in electric fields, whereas permeability, μ varies with flux density, $\vec{B}$, in the magnetic field.

Example

For an industrial application, a steel material in the form of a toroid of diameter $D = 22$ $[cm]$ and a circular cross-sectional radius of $r = 1$ $[cm]$ is used to produce a flux of 0.55 $[mWb]$ through $N = 250$ turns of coil. If steel has relative permeability of $\mu_r = 1000$, find the current applied to coil.

Solution

KEY: $\boxed{\mathcal{R} = \dfrac{l}{\mu A}}$

Length of the toroid: $\updownarrow = \pi D = \pi \times 22 \times 10^{-2} =$
Area, $A = \pi \times r^2 = \pi \times [1 \times 10^{-2}]^2 =$
Reluctance, $\mathcal{R} = \frac{l}{\mu A}$; $\quad \mu = \mu_0 \mu_r$
Magnetomotive Force, $F = NI = \Psi \mathcal{R}$
$\downarrow\downarrow$
$I = \frac{\Psi \mathcal{R}}{N} =? \ [A]$

Quizz: Repeat the design if only a supply of 5 $[A]$ is available to produce the same amount of flux. Don't change the dimensions of the toroid.

Example

A ferrite ring with a cross-sectional area of 3 $[cm^2]$ and a mean circumference of $30[cm]$ is wound with 280 turns of wire and carrying a current of 0.75 $[A]$. Calculate the flux in the ring if the relative permeability is given as, $\mu_r = 1500$.

Solution

KEY: $\boxed{\mathcal{R} = \dfrac{l}{\mu A}}$

$\mathcal{R} = \frac{l}{\mu A} = \frac{30 \times 10^{-2}}{4\pi \times 10^{-7} \times 1500 \times 3 \times 10^{-4}} =$

$F = NI = 280 \times 0.75 =$

$\Psi = \frac{NI}{\mathcal{R}} =? \ [Wb]$