空間資訊平差處理與分析
Adjustment and Analysis of Spatial Information
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Homework No.8
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1 Abstract

This report's main topic is "Unified LSQ Adjustment", Involving one questions in a simple 2-D case with 3 certain situations. The problems are similar to Homework No. 7 (case I and II), but now with LSQ techniques.

The matlab code will attach in $3 Appendix and upload to github.

Contents

2 Statement of Problem

Repeat Homework No.7 (case I and II), but now using the following two LSQ techniques:

A. Case I: A unified LSQ.

B. Case II: A unified LSQ with constraints.

Statement of Homework No. 7
We are interested in the coordinates of points D and E. Four distances are measured: $s1 = 12.41, s2 = 8.06, s3 = 3.87$, and $s4 = 8.83$. All are assumed uncorrelated and of equal precision. Points $A(0,4), B(4,0)$, and $C(8,0)$ are known (errorless) coordinates. Consider the following three cases:

I. No more information than above

II. In addition to above, points B, D, and E lie on a straight line.

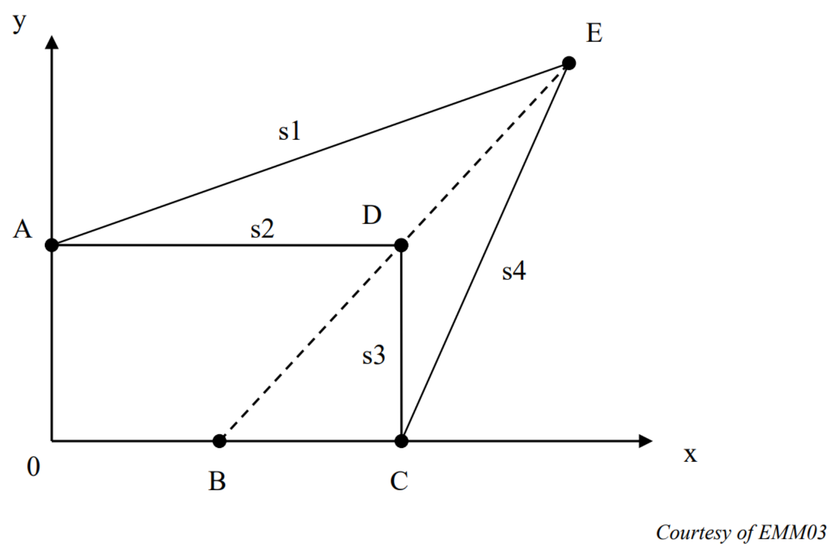

Figure 1: Position of Points.

Assumptions
For each of the above, assume σ_s (standard deviation of distance measurements) are all equal to 0.02. Coordinates of the known point A are errorless. Coordinates of the known points B and C are of equal precision ($\sigma_{x_B} = \sigma_{y_B} = \sigma_{x_c} = \sigma_{y_c} = 0.01$ and uncorrelated.)
Ask:

1) Compute the estimated coordinates of points D and E and their corresponding cofactor matrix for cases I and II.

2) Compare the results against those you have obtained in HomeworkNo.7.

Notice that now points B and C are with uncertainty. So does the length of line section, but with different precision.
distance: $\sigma_s = 0.02$, coordinate:$\sigma_c = 0.01$

3 Case I

3.1 Unified Case

There are 4 unknown parameters corresponded to the point D and point E's coordinates, $D(x_D, y_D), E(x_E, y_E)$
We can write 4 condition equation while we know the length of each bold line segment $\overline{AE}, \overline{AD}, \overline{CD}$, and $\overline{CE}$. As following:

$$\begin{cases} \overline{AE} = s_1 & F_1 : (x_E - x_A)^2 + (y_E - y_A)^2 - s_1^2 = 0 \\ \overline{AD} = s_2 & F_2 : (x_D - x_A)^2 + (y_D - y_A)^2 - s_2^2 = 0 \\ \overline{CD} = s_3 & F_3 : (x_D - x_C)^2 + (y_D - y_C)^2 - s_3^2 = 0 \\ \overline{CE} = s_4 & F_4 : (x_E - x_C)^2 + (y_E - y_C)^2 - s_4^2 = 0 \end{cases} \tag{1}$$

$(x_A, y_A) = (0, 4); (x_C, y_C) = (8, 0); (s_1, s_2, s_3, s_4) = (12.41, 8.06, 3.87, 8.83)$

$$\begin{cases} \mathbf{A}\mathbf{v} + \mathbf{B}\Delta = \mathbf{f} \\ \mathbf{A}_c\mathbf{v}_c + \mathbf{C}\Delta = \mathbf{f}_c \\ \mathbf{v}_x + \Delta = \mathbf{f}_x \end{cases} \tag{2}$$

$$\underbrace{\begin{bmatrix} \mathbf{A} & \phi & \phi \\ \phi & \mathbf{A}_c & \phi \\ \phi & \phi & \mathbf{I} \end{bmatrix}}_{\mathbf{A}_T} \underbrace{\begin{bmatrix} \mathbf{v} \\ \mathbf{v}_c \\ \mathbf{v}_\lambda \end{bmatrix}}_{\mathbf{v}_T} + \underbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{C} \\ -\mathbf{I} \end{bmatrix}}_{\mathbf{B}_T} = \underbrace{\begin{bmatrix} \mathbf{f} \\ \mathbf{f}_c \\ \mathbf{f}_x \end{bmatrix}}_{\mathbf{f}_T} \tag{3}$$

No constraint equations;

$$\begin{aligned} \Delta &= \mathbf{N}^{-1}\mathbf{t}_T \\ &= (\mathbf{B}_T^T\mathbf{w}_{eT}\mathbf{B}_T)^{-1}((\mathbf{B}_T^T\mathbf{w}_{eT}\mathbf{B}_T)), \text{where } \mathbf{w}_{et} = (\mathbf{A_T}\mathbf{Q_T}\mathbf{A_T^T})^{-1} \end{aligned} \tag{4}$$

Solve the variables $\mathbf{x}$ by following procedures:
The parameters can be solve by iteration steps as following:

(i) Pick initial $\mathbf{x}_0 = [8, 12, 4, 8]^T$ by observation of Fig1

(ii) Compute Δ

(iii) Update $\mathbf{x}_{0,new} = \mathbf{x}_0 + \Delta\mathbf{x}$ (My code using x-dx, because the side of $\mathbf{f}$)

(iv) Repeat (i) to (iii) until $\Delta\mathbf{x} \to 0$ (I set the threshold 10^{-10} here)

Here are the expressions of some parameters be used during computation:

$$\underbrace{\begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{I} \end{bmatrix}}_{\mathbf{A}_T} \underbrace{\begin{bmatrix} \mathbf{v} \\ \mathbf{v}_\lambda \end{bmatrix}}_{\mathbf{v}_T} + \underbrace{\begin{bmatrix} \mathbf{B} \\ -\mathbf{I} \end{bmatrix}}_{\mathbf{B}_T} = \underbrace{\begin{bmatrix} \mathbf{f} \\ \mathbf{f}_x \end{bmatrix}}_{\mathbf{f}_T} \quad \text{(No constraint)} \tag{5}$$

$$\mathbf{x} = \begin{bmatrix} x_D \\ x_E \\ y_D \\ y_E \end{bmatrix}; \mathbf{l} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}; \mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{l}} \begin{bmatrix} -2s_1 & 0 & 0 & 0 \\ 0 & -2s_2 & 0 & 0 \\ 0 & 0 & -2s_3 & 0 \\ 0 & 0 & 0 & -2s_4 \end{bmatrix} \tag{6}$$

$$\mathbf{B} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 2x_E & 0 & 2y_E - 8 \\ 2x_D & 0 & 2y_D - 8 & 0 \\ 2x_D - 16 & 0 & 2y_D & 0 \\ 0 & 2x_E - 16 & 0 & 2y_E \end{bmatrix} \tag{7}$$

$\mathbf{f_T} = [\mathbf{f} \quad \mathbf{f}_x]^T$ is composed by the $\mathbf{F}$ and the precision of point B, C's coordinate. $\mathbf{f_x} = [\phi_{4\times1}]$
The Total Cofactor Matrix

$$\mathbf{Q_T} = \begin{bmatrix} \mathbf{Q_{ll}} & \phi \\ \phi & \mathbf{Q_{xx}} \end{bmatrix} \tag{8}$$

$$\mathbf{Q_{ll}} = \mathbf{Q_{ll,s}} + \mathbf{J_C} \mathbf{Q_{ll,c}} \mathbf{J_C^T}, \quad \text{that } \mathbf{J_C} = \frac{\partial \mathbf{F}}{\partial [x_c, y_c]}$$

$$\mathbf{J_C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -2(x_D - x_C) & -2(y_D - y_C) \\ -2(x_E - x_C) & -2(y_E - y_C) \end{bmatrix} \qquad = \sigma_s^2 \mathbf{I_4} + \mathbf{J_C} \sigma_c^2 \mathbf{I_2} \mathbf{J_C^T} \tag{9}$$

And $\mathbf{Q_{xx}}$ is set to be $10^8 \times \mathbf{I_4}$, an appropriate size so that the iteration steps are not too large, which would lead to calculation errors. And the setting also can let the result converge.

After computation, got the parameter $\hat{\mathbf{x}}$ and the cofactor matrix $\mathbf{Q}_{\hat{x}\hat{x}}$

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_D \\ \hat{x}_E \\ \hat{y}_D \\ \hat{y}_E \end{bmatrix} = \begin{bmatrix} 8.06 \\ 11.75 \\ 3.87 \\ 7.99 \end{bmatrix};$$

$$\mathbf{Q}_{\hat{x}\hat{x}} = (\mathbf{B_T}^T \mathbf{w_{ET}} \mathbf{B_T})^{-1} = 10^{-3} \times \begin{bmatrix} 0.40 & -0.09 & 0.10 & 0.27 \\ -0.09 & 6.94 & -5.57 & -18.84 \\ 0.10 & -5.57 & 6.39 & 16.37 \\ 0.27 & -18.84 & 16.37 & 54.68 \end{bmatrix} \tag{10}$$

3.2 The result of last report

For report 7, my result of case I is

$$\begin{aligned}\hat{\mathbf{x}} &= \begin{bmatrix}\hat{x}_D\\ \hat{x}_E\\ \hat{y}_D\\ \hat{y}_E\end{bmatrix} = \begin{bmatrix}8.06\\ 11.75\\ 3.87\\ 8.00\end{bmatrix};\\ \mathbf{Q}_{\hat{x}\hat{x}} &= (\mathbf{B}^T\mathbf{w}\mathbf{B})^{-1} = 10^{-3}\times\begin{bmatrix}3.85 & 0 & 0.21 & 0\\ 0 & 3.20 & 0 & -3.08\\ 0.21 & 0 & 16.69 & 0\\ 0 & -3.08 & 0 & 6.10\end{bmatrix}\end{aligned} \tag{11}$$

The difference of cofactor matrix arises is from the uncertainty of the paramters has been changed. The shift in paramerter uncertainty from zero to non-zero causes the weight decreases, further cause the the $\mathbf{Q_{xx}}$ increases.

4 Case II

B, D, and E on a straight line, assume the line equation:

$$\frac{y_E - y_D}{x_E - x_D} = \frac{y_E - y_B}{x_E - x_B} \tag{12}$$

for case x_E, x_D, and x_B are all different (if all same, it is truely a vertical line)

$$\begin{aligned} & (y_E - y_D)(x_E - x_B) - (y_E - y_B)(x_E - x_D) = 0\\ \mathbf{h}_1 \qquad & -y_E x_B - y_D x_E + y_D x_B + y_E x_D + y_B x_E - y_B x_D = 0\end{aligned} \tag{13}$$

let $\mathbf{H} = [\mathbf{h}_1]$ represent constraint equation. Thus, we can write

$$\mathbf{C} = \frac{\partial \mathbf{H}}{\partial \mathbf{x}} = \begin{bmatrix} y_E & -y_D & 4 - x_E & x_D - 4\end{bmatrix}; \quad \mathbf{A_c} = [1]; \tag{14}$$

And consider the error of variable (x_B, y_B). $\mathbf{Q_{CC}} \neq 0$

$$\mathbf{Q_{CC}} = \mathbf{J_B Q_{BB} J_B^T} \tag{15}$$

$\mathbf{Q_{BB}}$ is the cofactor matrix of coordinate point B, $\mathbf{B} = [x_B, y_B]$, and $\mathbf{J_B} = \frac{\partial \mathbf{H}}{\partial \mathbf{B}}$

$$\begin{aligned}\mathbf{Q_{CC}} &= \mathbf{J_B Q_{BB} J_B^T}\\ &= \begin{bmatrix} y_D - y_E & x_E - x_D\end{bmatrix}\begin{bmatrix}\sigma_c^2 & 0\\ 0 & \sigma_c^2\end{bmatrix}\begin{bmatrix} y_D - y_E\\ x_E - x_D\end{bmatrix}\\ &= \sigma_c^2[(y_D - y_E)^2 + (x_E - x_D)^2]\end{aligned} \tag{16}$$

And with similar procedure with Case I

$$\underbrace{\begin{bmatrix}\mathbf{A} & \phi & \phi\\ \phi & \mathbf{A}_c & \phi\\ \phi & \phi & \mathbf{I}\end{bmatrix}}_{\mathbf{A}_T}\underbrace{\begin{bmatrix}\mathbf{v}\\ \mathbf{v}_c\\ \mathbf{v}_\lambda\end{bmatrix}}_{\mathbf{v}_T} + \underbrace{\begin{bmatrix}\mathbf{B}\\ \mathbf{C}\\ -\mathbf{I}\end{bmatrix}}_{\mathbf{B}_T} = \underbrace{\begin{bmatrix}\mathbf{f}\\ \mathbf{f}_c\\ \mathbf{f}_x\end{bmatrix}}_{\mathbf{f}_T} \tag{17}$$

The symbols are with same meanings as Case I. But now is with constraint. For the threshold $||\Delta x|| \leq 10^{-10}$. We have parameter $\hat{\mathbf{x}}$ and the cofactor matrix $\mathbf{Q}_{\hat{x}\hat{x}}$

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_D \\ \hat{x}_E \\ \hat{y}_D \\ \hat{y}_E \end{bmatrix} = \begin{bmatrix} 8.04 \\ 11.96 \\ 3.74 \\ 7.36 \end{bmatrix};$$
$$\mathbf{Q}_{\hat{x}\hat{x}} = 10^{-3} \times \begin{bmatrix} 0.38 & 0.18 & -0.01 & -0.55 \\ 0.18 & 1.14 & -1.94 & -3.13 \\ -0.01 & -1.94 & 4.67 & 7.45 \\ -0.55 & -3.13 & 7.45 & 12.97 \end{bmatrix} \tag{18}$$

4.1 The result of last report

For report 7, my result of case II is

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x}_D \\ \hat{x}_E \\ \hat{y}_D \\ \hat{y}_E \end{bmatrix} = \begin{bmatrix} 8.01 \\ 11.79 \\ 4.08 \\ 7.93 \end{bmatrix};$$
$$\mathbf{Q}_{\hat{x}\hat{x}} = \mathbf{N}^{-1} - \mathbf{N}^{-1}\mathbf{C}^T\mathbf{M}^{-1}\mathbf{C}\mathbf{N}^{-1} \tag{19}$$
$$= 10^{-3} \times \begin{bmatrix} 3.18 & 0.57 & 2.51 & -0.84 \\ 0.57 & 2.72 & -2.09 & -2.42 \\ 2.51 & -2.09 & 5.24 & 3.10 \\ -0.84 & -2.43 & 3.10 & 5.23 \end{bmatrix}$$

Although the norm of the cofactor matrix ($\mathbf{Q}_{\hat{\mathbf{x}}\hat{\mathbf{x}}}$) in the corrected Unified LSQ (ULS) method is not obvious to be larger than that of Report 7, it is theoretically true ($0.0182 > 0.0099$) that accounting for the uncertainty in both coordinates and distance observations leads to lower precision in the final estimated parameters $\hat{x}$.

5 Appendix

The programs can view and download code by visit this Github website: https://github.com/Awecean/Adjustment-and-Analysis-of-Spatial-Information/tree/main/HW7.

5.1 Code of Case I