CS132 2020 Homework 9

Carlos Lopez ()

April 9, 2020

Question 1.

$\boxed{Yes, -2}$

Since $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} =$ -2 $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$,

then yes, $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$ with an eigenvalue of (-2)

Question 2.

$\boxed{\begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}}$

Av $= \lambda v$

$Av = (\lambda I)v$

$Av - (\lambda I)v = 0$

$(A - (\lambda I))v = 0$

And since we're trying to get a vector that is not zero, we want to solve for:

$(A - (\lambda I)) = 0$

$(A - (3I)) = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

Then, we'll reduce this with a 0 row. $\begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix} \xrightarrow{R1 \div -2} \ldots \xrightarrow{R2-3R1} \ldots \xrightarrow{R2 \div -2} \ldots \xrightarrow{R1+R3}$

$\ldots \xrightarrow{R3-R2} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Hence, we get the eigenvector of $\begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$

Question 3.

Since A $= \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$, $\lambda = 4$, and Av $= \lambda v$,

we can see that (A - 4I)v = 0. Therefore, let's try and solve for v:

(A - 4I)= $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$- $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$= $\begin{bmatrix} -6 & 9 \\ -4 & 6 \end{bmatrix}$

We can then turn this into an augmented matrix with the extra zero column and row-reduce to find the entries in v:

$$\begin{bmatrix} -6 & 9 & 0 \\ -4 & 6 & 0 \end{bmatrix} \xrightarrow{R2-\frac{2}{3}R_1} \ldots \xrightarrow{R1\div-6} \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This tells us that $x_1 = \frac{3}{2}x_2$ and that x_2 is a free variable.

Therefore, if we make $x_2 = 1$, we can get a basis for the eigenspace that equals = $\boxed{\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}}$

Question 4.

False
False
True
False
True

a.)
The vector must also not be zero for this to be true

b.)
Only the converse statement is true.

c.)
This is true because a steady-state vector has the property that Axx = x. (λ is one in this case)

d.)
this is only true for triangular matrices

e.)
Yeah, that's true. It's the nullspace of $(A - \lambda I$

Question 5.

If we take a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and then look at the characteristic equation for it: $(a-\lambda)(d-\lambda)-bc = 0$. We can see that this equation will have a single root if $a = d$ and $bc = 0$. Hence, all we gotta do is set up a matrix like this: $\begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$ and plug in whatever values for a and c. For example, a = 1 and c = 0 which gives us the following matrix: $\boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

Question 6.

$$\boxed{\begin{array}{c}\lambda^2 - 10\lambda + 16 \\ 8 \text{ and } 2\end{array}}$$

$A - \lambda I = \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}$

$det(A - \lambda I) = det(\begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}) = (5-\lambda)(5-\lambda) - 3 * 3 = \lambda^2 - 10\lambda + 16$

We then solve for the roots of the equation:

$\lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2)$

which give us the eigenvalues of 8 and 2.

Question 7.

$$\boxed{\begin{array}{c}\lambda^2 - 8\lambda + 3 \\ 4 \pm\sqrt{13}\end{array}}$$

$A - \lambda I = \begin{bmatrix} 5-\lambda & -3 \\ -4 & 3-\lambda \end{bmatrix}$

$det(A - \lambda I) = det(\begin{bmatrix} 5-\lambda & -3 \\ -4 & 3-\lambda \end{bmatrix}) = (5-\lambda)(3-\lambda) - (-3) * (-4) = \lambda^2 - 8\lambda + 3$

We then solve for the roots of the equation with the quadratic formula:

$\lambda = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{8\pm\sqrt{64-12}}{2} = 4 \pm \sqrt{13}$

Question 8.

$$\boxed{\begin{array}{c}\text{False} \\ \text{True} \\ \text{False}\end{array}}$$

b.)
False since $detA^T = detA$

c.)
True by simple definition

d.)
False. Here's a counter-example:
$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has an eigenvalue of 1 and 0, but:
$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ only has an eigenvalue of 0.

Question 9.

a.)

$det(A - \lambda I) = 0$

$det(\begin{bmatrix} .6-\lambda & .3 \\ .4 & .7-\lambda \end{bmatrix} = (.6-\lambda)(.7-\lambda) - .3 * .4 = \lambda^2 - 1.3\lambda + .3 = (\lambda - 1)(\lambda - .3)$

which give us the eigen-values of 1 and .3

We'll then use the eigenvalue of .3 in the formula to solve for a v_2:

(A - .3I) = ($\begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$ - $\begin{bmatrix} .3 & 0 \\ 0 & .3 \end{bmatrix}$) = 0

$\begin{bmatrix} .3 & .3 & 0 \\ .4 & .4 & 0 \end{bmatrix} \xrightarrow{R1 \div .3} ... \xrightarrow{R2-4R1} ... \xrightarrow{R1*10} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, we end up with $x_1 = -x_2$ and x_2 being a free variable. Hence, if we let $x_2 = 1$, we'll get the eigen-vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Combining this vector with the one given to us in the problem we can make a basis for R^2 composed of the two vectors like so: $\boxed{\begin{bmatrix} 3/7 & -1 \\ 4/7 & 1 \end{bmatrix}}$

b.)

$x_0 = v - 1 + cv_2$

$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ = $\begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix}$ + c $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

1/2 = 3/7 - c

c = 3/7 - 1/2 = -1/14

Yep, x_0 can be written in the form $x_0 = v_1 + cv_2$ like this:

$$\boxed{\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} + (\frac{-1}{14}) \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

c.)

Since $x_1 = v_1 - (\frac{-1}{14})(.3)v_2$, then $x_2 = v_1 - (\frac{-1}{14})(.3)^2$.

By extension, $x_k = v_1 - (\frac{-1}{14})(.3)^k v_2$. Therefore,

as k approaches infinity, x_k will approach v_1

since the term on that is being subtracted from it $((\frac{-1}{14})(.3)^k v_2)$ will approach zero.

Question 10.

4x4 Matrix (on next page):

Anaconda Prompt (Anaconda3) - python

```
(base) C:\Users\Carlos>python
Python 3.7.3 (default, Apr 24 2019, 15:29:51) [MSC v.1915 64 bit (AMD64)] :: Anaconda, Inc. on win32
Type "help", "copyright", "credits" or "license" for more information.
>>> import numpy as np
>>> A = np.random.randint(5, size=(4,4))
>>> print(A)
[[1 3 3 2]
 [1 0 0 3]
 [0 3 2 3]
 [1 1 3 1]]
>>> np.linalg.eig(A)
(array([ 6.44221747+0.j        ,  0.54874851+0.j        ,
       -1.49548299+1.77147249j, -1.49548299-1.77147249j]), array([[ 0.63618112+0.j        ,  0.82118357+0.j
     ,
        -0.37448804-0.06940696j, -0.37448804+0.06940696j],
       [ 0.31569273+0.j        ,  0.38302556+0.j        ,
         0.65433474+0.j        ,  0.65433474-0.j        ],
       [ 0.52781258+0.j        , -0.37076814+0.j        ,
        -0.16760584-0.43640667j, -0.16760584+0.43640667j],
       [ 0.46586003+0.j        , -0.20366629+0.j        ,
        -0.20135281+0.40951432j, -0.20135281-0.40951432j]]))
>>> At = np.transpose(A)
>>> At
array([[1, 1, 0, 1],
       [3, 0, 3, 1],
       [3, 0, 2, 3],
       [2, 3, 3, 1]])
>>> np.linalg.eig(At)
(array([ 6.44221747+0.j        ,  0.54874851+0.j        ,
       -1.49548299+1.77147249j, -1.49548299-1.77147249j]), array([[-0.20347129+0.j        ,  0.70596344+0.j
     ,
        -0.05989388-0.26608053j, -0.05989388+0.26608053j],
       [-0.46190906+0.j        ,  0.12962798+0.j        ,
        -0.00160497+0.55789908j, -0.00160497-0.55789908j],
       [-0.57329292+0.j        , -0.53285404+0.j        ,
        -0.47621483-0.01297671j, -0.47621483+0.01297671j],
       [-0.64542597+0.j        , -0.44819503+0.j        ,
         0.62242346+0.j        ,  0.62242346-0.j        ]]))
>>>
```

5x5 Matrix (on next page):

```
(base) C:\Users\Carlos>python
Python 3.7.3 (default, Apr 24 2019, 15:29:51) [MSC v.1915 64 bit (AMD64)] :: Anaconda, Inc. on win32
Type "help", "copyright", "credits" or "license" for more information.
>>> import numpy as np
>>> A = np.random.randint(5, size=(5,5))
>>> print(A)
[[1 2 1 0 0]
 [1 4 4 3 1]
 [0 1 0 1 1]
 [0 0 2 2 4]
 [3 1 4 0 2]]
>>> np.linalg.eig(A)
(array([ 7.54241338+0.j        ,  1.62956963+2.59090877j,
        1.62956963-2.59090877j, -1.45024269+0.j        ,
       -0.35130995+0.j        ]), array([[-0.26085471+0.j        , -0.36416808+0.09679291j,
        -0.36416808-0.09679291j,  0.21477768+0.j        ,
        -0.62168313+0.j        ],
       [-0.75007821+0.j        , -0.26973237-0.40642431j,
        -0.26973237+0.40642431j, -0.49379332+0.j        ,
         0.29777681+0.j        ],
       [-0.20646294+0.j        ,  0.05941397-0.06973976j,
         0.05941397+0.06973976j,  0.46132919+0.j        ,
         0.24453299+0.j        ],
       [-0.38161502+0.j        ,  0.63583068+0.j        ,
         0.63583068-0.j        ,  0.40322246+0.j        ,
        -0.63424216+0.j        ],
       [-0.42553558+0.j        , -0.08858974+0.4467147j ,
        -0.08858974-0.4467147j , -0.57846844+0.j        ,
         0.25055848+0.j        ]]))
>>> At = np.transpose(A)
>>> At
array([[1, 1, 0, 0, 3],
       [2, 4, 1, 0, 1],
       [1, 4, 0, 2, 4],
       [0, 3, 1, 2, 0],
       [0, 1, 1, 4, 2]])
>>> np.linalg.eig(At)
(array([ 7.54241338+0.j        ,  1.62956963+2.59090877j,
        1.62956963-2.59090877j, -0.35130995+0.j        ,
       -1.45024269+0.j        ]), array([[-2.77744226e-01+0.j        ,  1.14712511e-01-0.47880112j,
         1.14712511e-01+0.47880112j,  4.86919389e-01+0.j        ,
        -2.56974880e-01+0.j        ],
       [-4.58372472e-01+0.j        , -4.28344482e-01-0.00422899j,
        -4.28344482e-01+0.00422899j,  6.44659842e-03+0.j        ,
         2.40225332e-01+0.j        ],
       [-6.15341301e-01+0.j        ,  2.16528005e-01-0.14217471j,
         2.16528005e-01+0.14217471j, -7.80414721e-01+0.j        ,
        -9.25145093e-01+0.j        ],
       [-3.59132129e-01+0.j        , -7.92056547e-04+0.4125189j ,
        -7.92056547e-04-0.4125189j ,  3.23681243e-01+0.j        ,
         5.92622365e-02+0.j        ],
       [-4.52915024e-01+0.j        ,  5.80364671e-01+0.j        ,
         5.80364671e-01-0.j        , -2.21475204e-01+0.j        ,
         1.29808496e-01+0.j        ]]))
>>>
```

As you can see above, the eigenvalues for A and its transpose A^T were the same. However, their eigenvectors were not the same. This was true for both the 4x4 and 5x5 matrices.

Question 11.

a.)

Since $x_{k+1} = x_k + x_{k-1}$, then we can see that when we multiply A$\begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix}$ it will equal $\begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix}$.

Therefore, $\boxed{\text{A must equal } \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}$ since as you can see:

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} = \begin{bmatrix} x_k + x_{k-1} \\ x_k \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix}$

b.)

det(A - $\lambda I) = 0$ and $A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix}$

$det(\begin{bmatrix} 1-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix}) = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1 = 0$

We then solve for the roots of the equation with the quadratic formula:

$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

Now, let's find the eigen-vectors:

$0 = $(A - λI)$= \begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 & 0 \\ 1 & 0 - \frac{1+\sqrt{5}}{2} & 0 \end{bmatrix} \xrightarrow{RowReduction} \begin{bmatrix} 1 & -\frac{1+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

This gives us a solution that $x_1 = \frac{1+\sqrt{5}}{2}x_2$ and x_2 is a free variable. This gives us an eigen-vector of $\begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$

Then, we do the same for the other lambda:

$0 = (A - \lambda I) = \begin{bmatrix} 1 - \frac{1-\sqrt{5}}{2} & 1 & 0 \\ 1 & 0 - \frac{1-\sqrt{5}}{2} & 0 \end{bmatrix} \xrightarrow{RowReduction} \begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

This gives us a solution that $x_1 = \frac{1-\sqrt{5}}{2}x_2$ and x_2 is a free variable. This gives us an eigen-vector of $\begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$

$\boxed{\text{Hence, our eigen-vectors are } \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \text{ and our eigenvalues are } \frac{1-\sqrt{5}}{2} \text{ and } \frac{1+\sqrt{5}}{2}}$

c.)

The basis for the eigen-space is v_1, v_2 where v_1 and v_2 are the eigen-vectors that were gotten in the last problem. I'll use their approximations (1.61 and -.62) Hence, let us re-write this in the form $x_0 = c_1v_1 + c_2v_2$.

First, let us rearrange this formula into $\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = x_0$. Furthermore, since $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$ is invertible, we can turn this into $\begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} x_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} 1.61 & -.62 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(\frac{1}{1.61+.62}\right) \begin{bmatrix} 1 & .62 \\ -1 & 1.61 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \approx \begin{bmatrix} .28 \\ .72 \end{bmatrix}$$

We can verify this answer by seeing that: $X_0 = (.28) \begin{bmatrix} 1.61 \\ 1 \end{bmatrix} + (.72) \begin{bmatrix} -.62 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

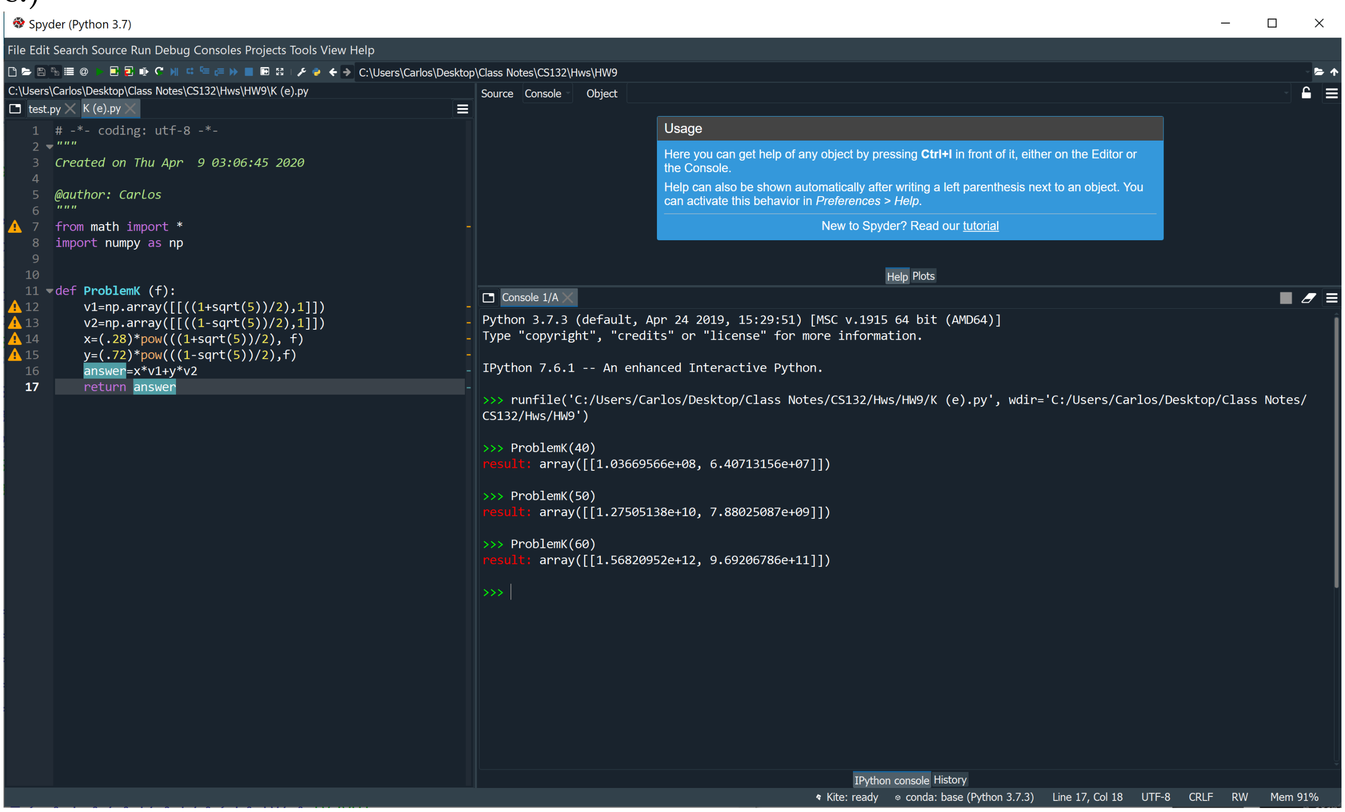
Therefore, we can see that the starting condition in the basis given by the eigen spaces is roughly:

$$\boxed{\begin{bmatrix} .28 \\ .72 \end{bmatrix}}$$

d.)

$$f_k = A^k x_0$$
$$= A^k(.28v_1 + (.72)v_2)$$
$$= (.28)A^k v_1 + (.72)A^k v_2$$
$$= \boxed{(.28)(1.62)^k \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} + (.72)(-.62)^k \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}}$$

e.)



f.)
Since

$$\lim_{k\to\infty} (-.62)^k = 0$$

, then

$$\lim_{k\to\infty} \frac{f_{k+1}}{f_k} = (1.62)$$

$\boxed{\text{Therefore, as k approaches infinity, the ratio of } f_{k+1} \text{ to } f_k \text{ approaches a constant of 1.62.}}$

Question 12.

a.)

Anaconda Prompt (Anaconda3) - python

```
>>> import numpy as np
>>> A = np.array([[0,0,.1,.1,.1,.1,],[1/7,1,0,0,0,0],[.1,.1,0,0,.1,.1],[0,0,1/7,1,0,0],[.1,.1,.1,.1,0,0],[0,0,0,0,1/7,1]])
>>> A
array([[0.        , 0.        , 0.1       , 0.1       , 0.1       ,
        0.1       ],
       [0.14285714, 1.        , 0.        , 0.        , 0.        ,
        0.        ],
       [0.1       , 0.1       , 0.        , 0.        , 0.1       ,
        0.1       ],
       [0.        , 0.        , 0.14285714, 1.        , 0.        ,
        0.        ],
       [0.1       , 0.1       , 0.1       , 0.1       , 0.        ,
        0.        ],
       [0.        , 0.        , 0.        , 0.        , 0.14285714,
        1.        ]])
>>> np.linalg.eig(A)
(array([ 0.16575188,  1.03424812, -0.08685593, -0.08685593,  0.98685593,
        0.98685593]), array([[-0.56906708, -0.13459823,  0.80953351, -0.31174598,  0.07480866,
         0.02626707],
       [ 0.09744738, -0.56144158, -0.10640568,  0.04097612, -0.81306232,
        -0.28548524],
       [-0.56906708, -0.13459823, -0.40476675, -0.49113489, -0.03740433,
         0.04752767],
       [ 0.09744738, -0.56144158,  0.05320284,  0.06455513,  0.40653116,
        -0.5165573 ],
       [-0.56906708, -0.13459823, -0.40476675,  0.80288088, -0.03740433,
        -0.07379474],
       [ 0.09744738, -0.56144158,  0.05320284, -0.10553125,  0.40653116,
         0.80204254]]))
>>>
```

$\boxed{\text{Eigen-values are the first array in the last command executed above while the eigen-vectors are the columns in the second array in the last command executed above}}$

b.)

$\boxed{\text{The largest eigenvalue is 1.034... and since it is bigger than 1, it tells us that the growth rate is exponential}}$

c.)

Anaconda Prompt (Anaconda3) - python

```
(base) C:\Users\Carlos>python
Python 3.7.3 (default, Apr 24 2019, 15:29:51) [MSC v.1915 64 bit (AMD64)] :: Anaconda, Inc. on win32
Type "help", "copyright", "credits" or "license" for more information.
>>> import numpy as np
>>> A = np.array([[0,0,.1,.1,.1,.1,],[1/7,1,0,0,0,0],[.1,.1,0,0,.1,.1],[0,0,1/7,1,0,0],[.1,.1,.1,.1,0,0],[0,0,0,0,1/7,1]
])
>>> a,b = np.linalg.eig(A)
>>> OneMortal = np.array([0,0,.1,.1,.1,.1])
>>> answer = np.linalg.inv(b)@OneMortal
>>> print(answer)
[-8.55532526e-02 -1.33591116e-01 -9.10299214e-02 -8.60422844e-16
  9.39076449e-02 -4.16333634e-17]
>>>
```

d.)

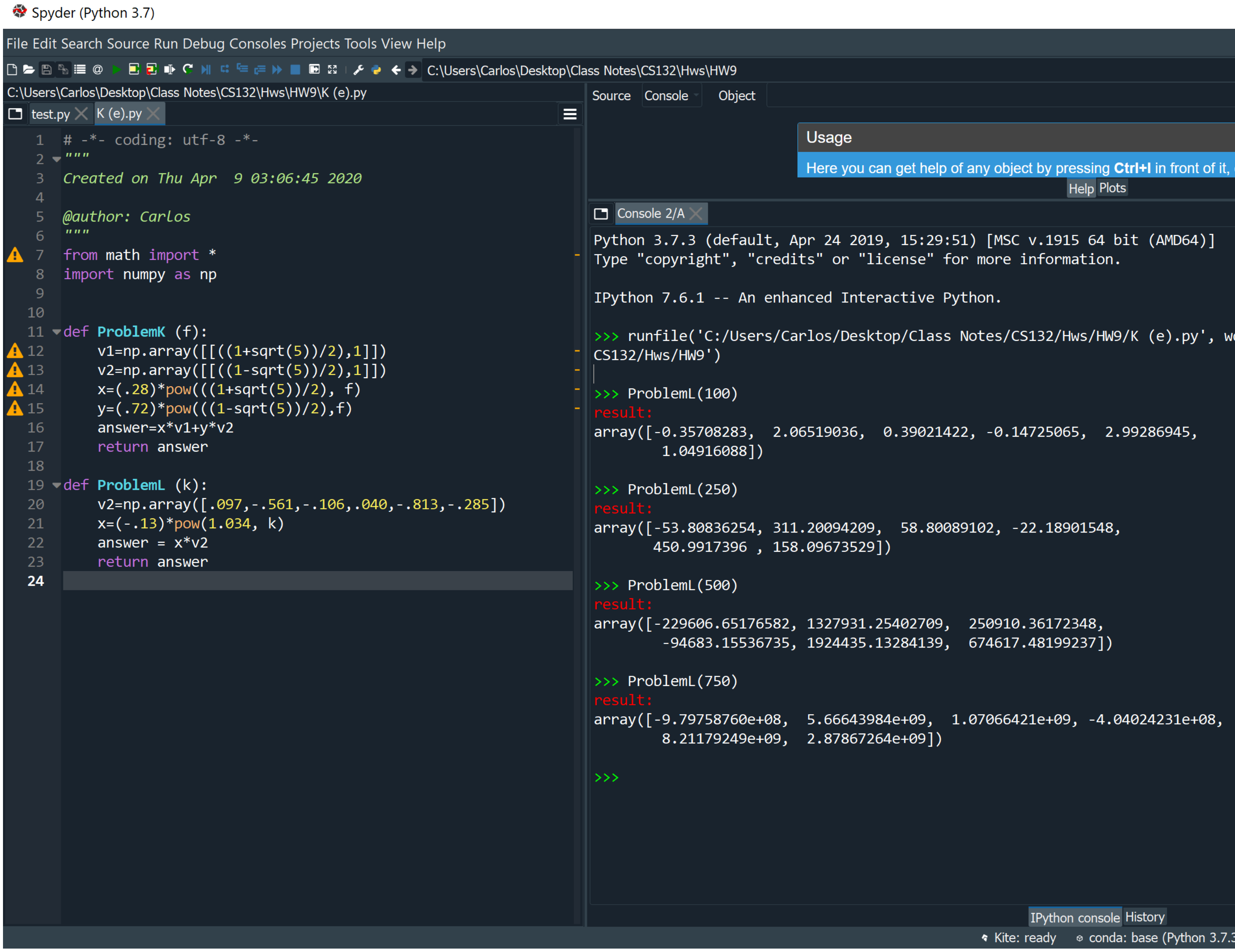
Since, $\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = x_0$, then we can see that:

$A^k x_0 = A^k(c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + c_5v_5 + c_6v_6)$ but according to the problem we can ignore all other vectors besides the dominant one which is v_2, so this turns into:

$A^k(c_2v_2) =$
$c_2A^kv_2 =$

$$\boxed{(-.13)(1.034)^k(\begin{bmatrix} 0.097 \\ -0.561 \\ -0.106 \\ 0.040 \\ -0.813 \\ -0.285 \end{bmatrix})}$$

e.)



Answers can be seen in console above