CS132 Homework 5

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Question 1.

$$\begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{bmatrix} \xrightarrow{R2-R1} \dots \xrightarrow{R4+2R1} \dots \xrightarrow{R2\longleftrightarrow R3} \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix} \xrightarrow{R3+3R2} \dots \xrightarrow{R4-9R2}$$

$$\dots \xrightarrow{R3\div 3} \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -18 & 0 \end{bmatrix} \xrightarrow{R4+18R3} \dots \xrightarrow{R2-3R3} \dots \xrightarrow{R1-2R3} \dots \xrightarrow{R1-3R2} \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence:
$x_1 = -3x_3$
$x_2 = -2x_3$
x_3 is a free variable
$x_4 = 0$

Question 2.

Let us make another matrix where A and b are one:

$$\begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 1 & 0 & 3 & -4 & 3 \\ 0 & 1 & 2 & 3 & -1 \\ -2 & 3 & 0 & 5 & 4 \end{bmatrix} \xrightarrow{R2-R1} \dots \xrightarrow{R4+2R1} \dots \xrightarrow{R2\longleftrightarrow R3} \begin{bmatrix} 1 & 3 & 9 & 2 & -1 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & -3 & -6 & -6 & 4 \\ 0 & 9 & 18 & 9 & 2 \end{bmatrix} \xrightarrow{R1-3R2} \dots \xrightarrow{R3+3R2}$$

$$\dots \xrightarrow{R4-9R2} \begin{bmatrix} 1 & 0 & 3 & -7 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -18 & 11 \end{bmatrix} \xrightarrow{R4+6R3} \dots \xrightarrow{R2-3R3} \dots \xrightarrow{R1+7R3} \begin{bmatrix} 1 & 0 & 3 & 0 & \frac{13}{3} \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 17 \end{bmatrix}$$

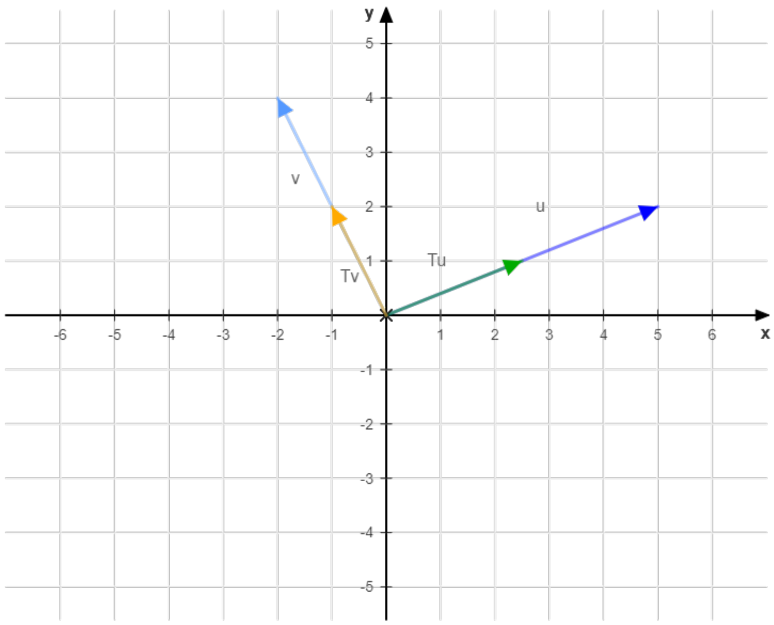
Since the system is inconsistent, b is not in the range of the linear transformation x $\rightarrow Ax$.

Question 3.

T(u)= $\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$

T(v)= $\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

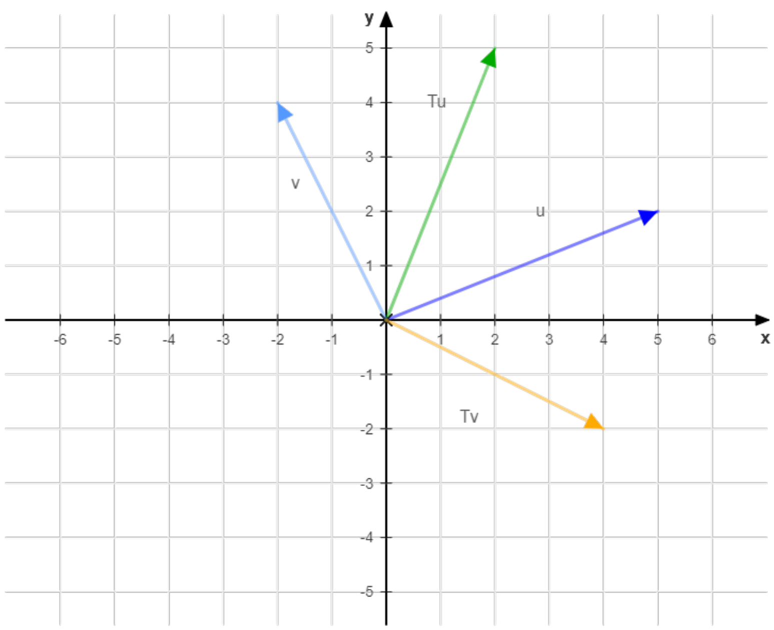
Hence, the length of the vectors is reduced by half, but the direction stays the same.



Question 4.

$$T(u)=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix}=\begin{bmatrix}2\\5\end{bmatrix}$$
$$T(v)=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}-2\\4\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix}$$

Hence, the transformation is a reflection through the line y=x



Question 5.

$T(x) = x_1v_1 + x_2v_2$
$T(x) = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$T(x) = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$ x
Hence:

$\mathbf{A} = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$

Question 6.

They gave us all three vectors e_1, e_2, and e_3, so we just need to put them in a matrix:

$\mathbf{T}= \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}$

Question 7.

According to example 3 in section 1.9, general rotation is given by:

$\begin{bmatrix} cos(\phi) & -sin(\phi) \\ sin(\phi) & cos(\phi) \end{bmatrix}$

Therefore, if we plug in our desired angle of $-\frac{\pi}{4}$ and solve, we can see that:

T$= \begin{bmatrix} cos(-\frac{\pi}{4}) & -sin(-\frac{\pi}{4}) \\ sin(-\frac{\pi}{4}) & cos(-\frac{\pi}{4}) \end{bmatrix}$

$\mathbf{T}= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Question 8.

Since e_1 must be left alone, we must leave the first column of the matrix as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ a.k.a e_1.

Afterwards, since they asked for $e_2 + 3e_1$ we just need to grab e_2 ($\begin{bmatrix} 0 \\ 1 \end{bmatrix}$) and add it to $3e_1$ ($\begin{bmatrix} 3 \\ 0 \end{bmatrix}$)

As a result, we get the matrix:

$\mathbf{T}= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

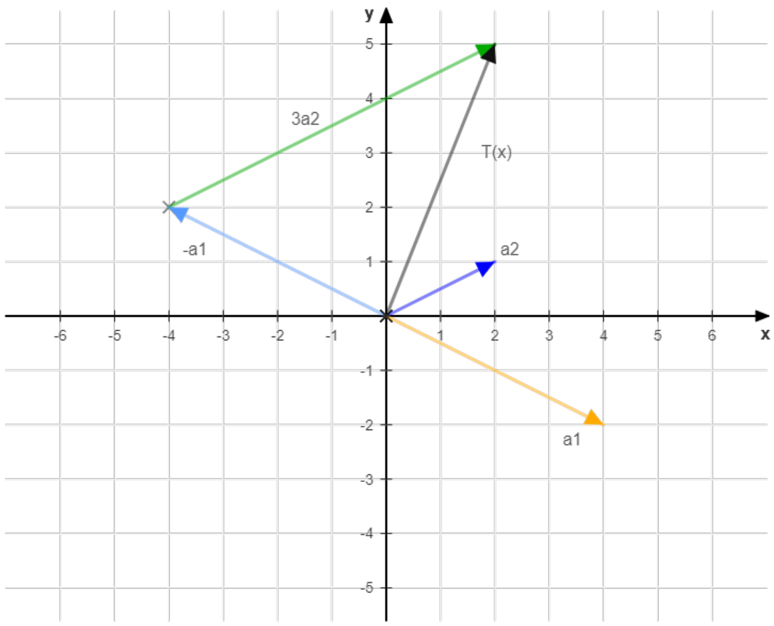
Question 9.

By the formula of Ax = b, we can see that:

$\begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ =T(x)

$\begin{bmatrix} (-1)a_1 & (3)a_2 \end{bmatrix}$ =T(x)

Therefore, we can apply this to the picture and just connect the vectors from point to tail like so:



Question 10.

Since $T(x) = Ax$, and they gave us T(x), then all we need to do is $\frac{T(x)}{x} = A$ which just implies that we should remove the x from the elements that they gave us and then we should arrange them in a matrix to get:

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Question 11.

By following the same procedure as in the last problem, we get that:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -3 & 4 \end{bmatrix}$$

Question 12.

We can turn A(x) and T(x) into an augmented matrix like so and solve for x:

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 4 \\ 3 & -2 & 9 \end{bmatrix} \xrightarrow{R2+R1} \dots \xrightarrow{R3-3R1} \dots \xrightarrow{R3-4R2} \dots \xrightarrow{R1+2R2} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $\mathbf{x}= \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Question 13.

In addition to x_k being the total population for a given year, let r_k and s_k represent the amount of residents in the city and suburbs (respectively) for any given year. We can see that $r_0 = 10,000,000$ and $s_0 = 800,000$

In addition, we can see that every year, 6% of the city population moves to the suburbs and hence 94% must stay in the city. Similarly, every year 4% of the suburban population moves to the city and hence 96% stay in the suburbs. We can use this information and the general Ax=b formula to form a matrix like so:

$b = Ax$

$$x_{k+1} = \begin{bmatrix} r_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{bmatrix} .06 & .04 \\ .94 & .96 \end{bmatrix} \begin{bmatrix} r_k \\ s_k \end{bmatrix}$$

We can then recursively use this formula for calculating the total population for two years later:

$$x_2 = \left(\begin{bmatrix} .06 & .04 \\ .94 & .96 \end{bmatrix} \left(\begin{bmatrix} .06 & .04 \\ .94 & .96 \end{bmatrix} \begin{bmatrix} r_0 \\ s_0 \end{bmatrix} \right) \right)$$

$$\text{x}_2 = \left(\begin{bmatrix} .06 & .04 \\ .94 & .96 \end{bmatrix} \left(\begin{bmatrix} .06 & .04 \\ .94 & .96 \end{bmatrix} \begin{bmatrix} 10000000 \\ 800000 \end{bmatrix} \right) \right)$$

$$\text{x}_2 = \begin{bmatrix} 8,920,800 \\ 1,879,200 \end{bmatrix}$$

Question 14.

Let's follow the same procedure as in the last problem:

Anaconda Prompt (Anaconda3) - python

```
(base) C:\Users\Carlos>python
Python 3.7.3 (default, Apr 24 2019, 15:29:51) [MSC v.1915 64 bit (AMD64)] :: Anaconda, Inc. on win32
Type "help", "copyright", "credits" or "license" for more information.
>>> import numpy as np
>>>
>>> A = np.array([[.97,.05,.1],[0,.9,.05],[.03,.05,.85]])
>>> x=np.array([295,55,150])
>>> print(A@(A@x))
[311.543  58.255 130.202]
>>>
```

Therefore, the approximate distribution of cars on Wednesday will be 311.543 cars in the airport, 58.225 cars in the East side office, and 130.202 cars in the West Side Office

Question 15.

Since AB=0, then we can get:

AB= $\begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

AB= $\begin{bmatrix} 3w-6y & 3x-6z \\ -w+2y & -x+2z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

After simplifying, we get that:

$w = 2y$ and $x = 2z$

Hence, there are an infinite amount of answers.

Here's an example of one answer you get when you make $y = 1$ and $z = 1$:

$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

Question 16.

(a) False

(b) True

(c) False

(d) False

(e) True

Question 17.

According to the problem, $b_1 = b_2$ and $B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$

Hence, we can substitute b_2 with b_1 to see that $B = \begin{bmatrix} b_1 & b_1 & b_3 \end{bmatrix}$

Afterwards, we can multiply B by A to see -that:

$AB = \begin{bmatrix} Ab_1 & Ab_1 & Ab_3 \end{bmatrix}$

Therefore, we can see that the first two columns of the AB matrix are equal.

Question 18.

The second column of AB will also be all zeros

Question 19.

Since the columns of B are linearly dependent, then there exists a non-zero vector, x, such that $Bx = 0$. If we then multiply the left side by A, we can see that:

A(Bx)=0
(AB)x=0

And since we established that x is a non-zero vector, then that means that AB must be linearly dependent.

Question 20.

Since both u and v exist in R^n, then that means that:

v $= \begin{bmatrix} v_1 \\ v_2 \\ ... \\ v_n \end{bmatrix}$ and u $= \begin{bmatrix} u_1 \\ u_2 \\ ... \\ v_n \end{bmatrix}$ Hence: $v^T = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix}$ and $u^T = \begin{bmatrix} u_1 & u_2 & ... & u_n \end{bmatrix}$

If we multiply these we can see that:

$v^T u = \begin{bmatrix} v_1 & v_2 & ... & v_n \end{bmatrix} = u^T v$ so:

$v^T u = u^T v$

For the second part of the question, we can see that:

$$vu^T = \begin{bmatrix} v_1(u_1) & v_1(u_2) & ... & v_1(u_n) \\ v_2(u_1) & v_2(u_2) & ... & v_2(u_n) \\ ... & ... & ... & ... \\ v_n(u_1) & v_n(u_2) & ... & v_n(u_n) \end{bmatrix} \text{ and } uv^T = \begin{bmatrix} u_1(v_1) & u_1(v_2) & ... & u_1(v_n) \\ u_2(v_1) & u_2(v_2) & ... & u_2(v_n) \\ ... & ... & ... & ... \\ u_n(v_1) & u_n(v_2) & ... & u_n(v_n) \end{bmatrix}$$

As you can see one is the inverse of the other. Hence:

$vu^T = (uv^T)^T$

Question 21.

(a) np.zeros((5,6))

(b) np.ones((3,5))

(c) np.identity(6)

(d) a = np.zeros((5,5))
np.fill_diagonal(a, [3,5,7,2,4])

Question 22.

np.random.rand(6,4)

The entries in the matrix will be floats ranging from 0 to 1.

However, you can make numpy generate a 3x3 matrix with random elements from -9 to 9 by running:
np.random.randint(-9,10, size=(3, 3))

Question 23.

Here's the code used for the A and I matrices (note that I made the matrix of type int since it's easier to read than a matrix of floats):
A = np.random.randint(6, size=(4,4))
I = np.identity(4)
difference = (A+I)@(A-I)-((A@A)-I)
print(difference)

Here's the code used for the A and B matrices:
A = np.random.randint(6, size=(4,4))
B = np.random.randint(6, size=(4,4))
difference2 = (A+B)@(A-B)-((A@A)-(B@B))
print(difference2)

Conclusion: When running the code with the identity matrix(I), an array of zeros was always gotten which means that **the first equation given in the problem is true.**
However, this is not the case for when we use two random arrays (A B) since the results were always a new array with random numbers which means that **the second equation given in the problem is false.**