CS132 Homework 6

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Question 1.

Since D is invertible, we can multiply both sides by D^{-1}

$$\begin{aligned}(B-C)D &= 0\\(B-C)DD^{-1} &= 0D^{-1}\\(B-C)I &= 0\\B-C &= 0\\&\boxed{\text{B=C}}\end{aligned} \tag{1}$$

Q.E.D.

Question 2.

Via Theorem 6 (page 107):

$$\begin{aligned}C &= AB\\CB^{-1} &= ABB^{-1}\\CB^{-1} &= AI\\&\boxed{\text{CB}^{-1} = A}\end{aligned} \tag{2}$$

Question 3.

$$\begin{aligned}A &= PBP^{-1}\\P^{-1}A &= IBP\\P^{-1}AP &= BI\\&\boxed{\text{P}^{-1}AP = B}\end{aligned} \tag{3}$$

Question 4.

If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

Then, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Hence, as long as ad - bc $\neq 0$, then we can see that:

AA^{-1} = $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

In addition, we can see that:

A^{-1}A = $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\boxed{\text{Since both of those equal each other that means that the formula for } A^{-1} \text{ works}}$

Question 5.

Ax = I_2

$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ x = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+c+e & b+d+f \\ c+e+g & d+f+h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, we can derive the following system of linear equations:

$\begin{bmatrix} a & c & e & 1 \\ b & d & f & 0 \\ c & e & g & 0 \\ d & f & h & 1 \end{bmatrix}$ Therefore, we can construct the answer by making everything in the middle rows 0 by the exception of the two values that aren't in the middle two rows (a h). Hence, we'll set these two values to 1 like so:
b=c=d=e=f=g=h=0
a=h=1

This gives us the answer of $\boxed{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \text{D}}$

And to answer part 2:

$\boxed{\begin{array}{c}\text{No, because the columns of A are linearly dependent. Hence, for some x, Ax = 0} \\ \text{which means that for that x, CAx = 0}\end{array}}$

Question 6.

The matrix A was made by combining the matrix given in the problem and the second and third columns of the identity matrix($\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$)

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Python 3.7.3 (default, Apr 24 2019, 15:29:51) [MSC v.1915 64 bit (AMD64)]
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IPython 7.6.1 -- An enhanced Interactive Python.

>>> runfile('C:/Users/Carlos/Desktop/Class Notes/hw2GESolutions.py', wdir='C:/Users/Carlos/Desktop/Class Notes')

>>> A=np.array([[-25,-9,-27,0,0],[546,180,537,1,0],[154,50,149,0,1]])

>>> backsubstitution(forwardElimination(A))
result:
array([[  1.        ,   0.        ,   0.        ,   1.5       ,
         -4.5       ],
       [  0.        ,   1.        ,   0.        , -72.16666667,
        219.5       ],
       [ -0.        ,  -0.        ,   1.        ,  22.66666667,
        -69.        ]])

>>> 
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This tells us the last 2 columns of A^{-1} are: $\boxed{\begin{bmatrix} 1.5 & -4.5 \\ -72.17 & 219.5 \\ 22.6 & -69 \end{bmatrix}}$

Question 7.

$\boxed{\text{No}}$, according to Invertible Matrix theorem (Theorem 8), this is not possible.

Question 8.

$\boxed{\text{We can say that Hx = 0 will have a non-trivial solution}}$ because according to theorem 8, if one of the conditions is false (g in this case), then so will all the other conditions in the theorem.

Question 9.

$\boxed{\text{Yes}}$ again, by theorem 8. Since condition d. is met on the Theorem (Lx = 0 has only the trivial solution), then all the conditions in the theorem are true, including the columns of n spanning R^n

Question 10.

Since at the beginning of the problems it is stated that "unless otherwise specified, assume that all matrices in these exercises are n x n" (a.k.a. they're all square), then we know that the matrix A must be a square matrix, and by extension A^2 is also a square matrix.

In addition, we know that A is invertible due to its columns being independent (Theorem 8 tells us this), so A^2 must also be invertible.

Therefore, since A^2 is invertible and a square matrix, then A^2 must also span R^n.

Question 11.

a.):

$$A= \boxed{\begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}}$$

b.):

$$x_1 = Ax_0 = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix}$$

$$x_2 = Ax_1 = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix} \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} = \begin{bmatrix} .375 \\ .3125 \\ .3125 \end{bmatrix}$$

Hence, the probability that the lab animal will eat food 2 on the second trial will be .3125 (31.25%)

Question 12.

a.):

$$A= \boxed{\begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}}$$

b.):

$$x_1 = Ax_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix} \begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

Hence, the probability of bad weather tomorrow will be .2 (20%)

c.):

$$x_1 = Ax_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix} \begin{bmatrix} 0 \\ .4 \\ .6 \end{bmatrix} = \begin{bmatrix} .4 \\ .42 \\ .18 \end{bmatrix}$$

$$x_2 = Ax_1 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix} \begin{bmatrix} .4 \\ .42 \\ .18 \end{bmatrix} = \begin{bmatrix} .480 \\ .336 \\ .184 \end{bmatrix}$$

Hence, the probability of good weather on Wednesday will be .480 (48%)

Question 13.

$\boxed{\text{No}}$, P is not a regular stochastic matrix since regardless of how many times you multiply P times itself, it will always have a zero on the bottom left of the matrix.

Question 14.

To find the steady-state vector, we need to find q such that Pq = q, as according to the definition before Example 4 in the same lesson.

To do this we can use the formula of (P-I)x = 0 which is given to us in example 5 on this same lesson.

(P-I)x = 0

$$\left(\begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} -.5 & .25 & .25 \\ .25 & -.5 & .25 \\ .25 & .25 & -.5 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can then row-reduce the matrix on the left an apply it to the x vector like so:
(Note the during the reduction, the matrix was multiplied by 4 to end up with whole numbers.)

$$\begin{bmatrix} -.5 & .25 & .25 \\ .25 & -.5 & .25 \\ .25 & .25 & -.5 \end{bmatrix} \xrightarrow{Reduction} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 & -x_3 = 0 \\ x_2 & -x_3 = 0 \end{bmatrix}$$

Then, let us choose the solution space of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for the sake of simplicity.

Afterwards, we'll divide them by their sum (3) to get:
$X_1 = \frac{1}{3}$ $X_2 = \frac{1}{3}$ $X_3 = \frac{1}{3}$

$\boxed{\text{This means that the animal will prefer all three foods equally after many trials}}$

Question 15.

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IPython 7.6.1 -- An enhanced Interactive Python.

>>> runfile('C:/Users/Carlos/Desktop/Class Notes/hw2GESolutions.py', wdir='C:/Users/Carlos/
Desktop/Class Notes')

>>> P = np.array([[.9,.01,.09],[.01,.90,.01],[.09,.09,.90]])

>>> I=np.array([[1,0,0],[0,1,0],[0,0,1]])

>>> A=P-I

>>> backsubstitution(forwardElimination(A))
result:
array([[ 1.        ,  0.        , -0.91919192],
       [-0.        ,  1.        , -0.19191919],
       [ 0.        ,  0.        ,  0.        ]])

>>> A=backsubstitution(forwardElimination(A))

>>> summation = .91919192+.19191919+1

>>> q=np.array([[.91919192/summation],[.19191919/summation],[1/summation]])

>>> print(q)
[[0.4354067 ]
 [0.09090909]
 [0.47368421]]

>>>
```

Hence, the amount of cars rented or ready from the downtown location on a typical day will be $(.090909*2000) = \boxed{\text{182 cars}}$