CS132 2019 Homework 4

Carlos Lopez ()

February 20, 2020

Question 1.

(a) Due to the fact that Flow in = Flow out, we can turn the following data into a matrix which we can then solve:

Intersection	Flow in	Flow out
A	x_1	$x_3 + x_4 + 40$
B	200	$x_1 + x_2$
C	$x_2 + x_3$	$x_5 + 100$
D	$x_4 + x_5$	60
$Total Flow:$	200	200

$$\begin{bmatrix} x_1 & 0 & -x_3 & -x_4 & 0 & 40 \\ x_1 & x_2 & 0 & 0 & 0 & 200 \\ 0 & x_2 & x_3 & 0 & -x_5 & 100 \\ 0 & 0 & 0 & x_4 & x_5 & 60 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{bmatrix} \xrightarrow{R3-R2} \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{bmatrix} \xrightarrow{R4+R}$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & -1 & -1 & -60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R3\div -1} \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1+R3} \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 1 & 0 & 160 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2-R3}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the solutions are:

$x_1 = 100 + x_3 - x_5$
$x_2 = 100 - x_3 + x_5$
x_3 is a free variable
$x_4 = 60 - x_5$
x_5 is a free variable

(b) If we look at the last matrix for the last part and the solution set for the last part and we make $x_4 = 0$, then we get this:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & \mathbf{0} & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This means that $x_5 = 60$. Therefore, overall we get that:

$x_1 = 100 + x_3 - x_5$
$x_2 = 100 - x_3 + x_5$
x_3 is a free variable
$x_4 = \mathbf{0}$
$x_5 = \mathbf{60}$

(c) If we use the results from the previous problem, and plug them into the equation for x_1, then we can see that:

$$\begin{aligned} x_1 &= 100 + x_3 - x_5 \\ x_1 &= 100 + x_3 - 60 \\ x_1 &= 40 + x_3 \end{aligned} \tag{1}$$

And since x_3 has to be positive (since you can't have negative traffic flowing through somewhere), then the minimum x_1 can equal is 40.

Question 2.

We can repeat the process we used in the previous problem:

Intersection	Flow in	Flow out
A	x_1	$x_2 + 100$
B	$x_2 + 50$	x_3
C	x_3	$x_4 + 120$
D	$x_4 + 150$	x_5
E	x_5	$x_6 + 80$
F	$x_6 + 100$	x_1
$TotalFlow:$	300	300

$$\begin{bmatrix} x_1 & -x_2 & 0 & 0 & 0 & 0 & 100 \\ 0 & x_2 & -x_3 & 0 & 0 & 0 & -50 \\ 0 & 0 & x_3 & -x_4 & 0 & 0 & 120 \\ 0 & 0 & 0 & x_4 & -x_5 & 0 & -150 \\ 0 & 0 & 0 & 0 & x_5 & -x_6 & 80 \\ -x_1 & 0 & 0 & 0 & 0 & x_6 & -100 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix} \xrightarrow{R7+(R1+R2+R3+R4+R5+R6)} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R4+R5}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R3+R4} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2+R3} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1+R2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 50 \\ 0 & 0 & 0 & 1 & 0 & -1 & -70 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the solutions are:

$x_1 = 100 + x_6$
$x_2 = x_6$
$x_3 = 50 + x_6$
$x_4 = -70 + x_6$
$x_5 = 80 + x_6$
x_6 is a free variable
(Note that x_6 needs to be 70 or more though since x_4 can't be negative)

Question 3.

$$\begin{bmatrix} 0 & 0 & -3 & 0 \\ 0 & 5 & 4 & 0 \\ 2 & -8 & 1 & 0 \end{bmatrix} \xrightarrow{R1 \longleftrightarrow R3} \begin{bmatrix} 2 & -8 & 1 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \xrightarrow{R3 \div -3} \begin{bmatrix} 2 & -8 & 1 & 0 \\ 0 & 5 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R2-4R3} \begin{bmatrix} 2 & -8 & 1 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R1-R3}$$

$$\begin{bmatrix} 2 & -8 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R2 \div 5} \begin{bmatrix} 2 & -8 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R1 \div 2} \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R1+4R2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Since the matrix is solvable, then the vectors are **linearly independent.**

Question 4.

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The vectors are **linearly independent** because when they are arranged in a reduced row-echelon matrix there are no free variables(no zeros in the last row). You also can't multiply one by anything to get the other.

Question 5.

(The last column of this matrix is omitted since it's all 0's)

$$\begin{bmatrix}-4 & -3 & 0\\ 0 & -1 & 4\\ 1 & 0 & 3\\ 5 & 4 & 6\end{bmatrix} \xrightarrow{R1 \leftrightarrow R3} \begin{bmatrix}1 & 0 & 3\\ 0 & -1 & 4\\ -4 & -3 & 0\\ 5 & 4 & 6\end{bmatrix} \xrightarrow{R3+4R1} \begin{bmatrix}1 & 0 & 3\\ 0 & -1 & 4\\ 0 & -3 & 12\\ 5 & 4 & 6\end{bmatrix} \xrightarrow{R4-5R1} \begin{bmatrix}1 & 0 & 3\\ 0 & -1 & 4\\ 0 & -3 & 12\\ 0 & 4 & -9\end{bmatrix} \xrightarrow{R3-3R2}$$

$$\begin{bmatrix}1 & 0 & 3\\ 0 & -1 & 4\\ 0 & 0 & 0\\ 0 & 4 & -9\end{bmatrix} \xrightarrow{R4 \leftrightarrow R3} \begin{bmatrix}1 & 0 & 3\\ 0 & -1 & 4\\ 0 & 4 & -9\\ 0 & 0 & 0\end{bmatrix} \xrightarrow{R3+4R2} \begin{bmatrix}1 & 0 & 3\\ 0 & -1 & 4\\ 0 & 0 & 7\\ 0 & 0 & 0\end{bmatrix} \xrightarrow{R3 \div 7} \begin{bmatrix}1 & 0 & 3\\ 0 & -1 & 4\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} \xrightarrow{R2 \div -1} \begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -4\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} \xrightarrow{R2+4R3}$$

$$\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} \xrightarrow{R1-3R3} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Since x_1, x_2, and x_3 would all have to be 0 for this to work, then the columns form a linearly independent set.

Question 6.

According to Theorem 8 in page 60, since the set has more vectors than the vectors have entries, then the set is linearly dependent. ($p > n$ when p and n are defined by: $(v_1 ... v_p)$ and R^n)

Question 7.

(a) part a

$$\begin{bmatrix}1 & -2 & 2\\ -5 & 10 & -9\\ -3 & 6 & h\end{bmatrix} \xrightarrow{R2+5R1} \dots \xrightarrow{R3+3R1} \begin{bmatrix}1 & -2 & 2\\ 0 & 0 & 1\\ 0 & 0 & h+6\end{bmatrix}$$

Since the second row is inconsistent, then v_3 is not within the span of v_2 and v_1.

(b) part b

Let us make a matrix with a row of zeros at the end:

$$\begin{bmatrix}1 & -2 & 2 & 0\\ -5 & 10 & -9 & 0\\ -3 & 6 & h & 0\end{bmatrix} \xrightarrow{R2+5R1} \dots \xrightarrow{R3+3R1} \begin{bmatrix}1 & -2 & 2 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & h+6 & 0\end{bmatrix} \xrightarrow{(R3-(6+h)R2} \begin{bmatrix}1 & -2 & 2 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

This is a linearly dependent set for all h since for every value of h, x_2 is a free variable.

Question 8.

$$\begin{bmatrix} 2 & -6 & 8 & 0 \\ -4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{bmatrix} \xrightarrow{R2+2R1} \begin{bmatrix} 2 & -6 & 8 & 0 \\ 0 & -5 & h+16 & 0 \\ 1 & -3 & 4 & 0 \end{bmatrix} \xrightarrow{R3-\frac{1}{2}R1} \begin{bmatrix} 2 & -6 & 8 & 0 \\ 0 & -5 & h+16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the matrix has a free variable, represented by the last row of zeroes, then that means that the vectors will be linearly dependent regardless of whatever value h is.

Question 9.

$$\begin{bmatrix} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ 3 & 8 & h & 0 \end{bmatrix} \xrightarrow{R2+R1} \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 3 & 8 & h & 0 \end{bmatrix} \xrightarrow{R3-3R1} \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 23 & h-3 & 0 \end{bmatrix} \xrightarrow{R2\div\frac{1}{2}} \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 23 & h-3 & 0 \end{bmatrix}$$
$$\xrightarrow{R3-23R2} \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & h-26 & 0 \end{bmatrix}$$

If h=26, then x_3 will be a free variable and hence the vectors will be linearly dependent.

Question 10.

Let * represent any value and let let **X** represent a pivot point.

There truly are only a few combinations of a 2 x 2 matrix that have the above elements. The combinations are the following:

$$\begin{bmatrix} \mathbf{X} & * \\ 0 & \mathbf{X} \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{X} \\ 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} \mathbf{X} & * \\ 0 & 0 \end{bmatrix}$$

Since there are no free variables on the first case, then we only have two cases where there can be a 2x2 matrix with linearly dependent columns.

(Side Note: Technically: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

is another possible case, but this doesn't really talk about vectors as much as the origin. Hence, it is disregarded.)

Question 11.

Let * represent any value and let let **X** represent a pivot point.

there seems to only be: $\begin{bmatrix} \mathbf{X} & * & * \\ 0 & \mathbf{X} & * \\ 0 & 0 & \mathbf{X} \\ 0 & 0 & 0 \end{bmatrix}$

This is because A $= (a_1, a_2, a_3)$, (a_1, a_2) are linearly dependent, and a_3 is not within the span (a_1, a_2). Hence, a_3 can be accommodated in the row below which yields the above matrix.

Question 12.

The matrix must have 5 pivot columns. This is dictated by Theorem 4 and the fact that for a matrix to span R^5, the matrix must have a pivot in five different rows.

Question 13.