CS132 2019 Homework 12

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Question 1.

Let us confirm orthogonality through the dot product:

$u_1 \cdot u_2 = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = (-12) + (12) = 0$ Since this equals 0, the vectors are orthogonal. Hence, let's now use the formula:

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{30}{25} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \frac{-15}{25} \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}$$

Question 2.

Let's use the same formula again:

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{30}{25} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{7}{14} \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{bmatrix}$$

Then, we can plug this into the following:

$$y - \hat{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-5}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}$$

Hence: $\boxed{\begin{bmatrix} \frac{3}{2} \\ \frac{7}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{-5}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}}$

Question 3.

Let's use the formula for orthogonal projection again:

$$\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{30}{10} \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \frac{26}{26} \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}}$$

Question 4.

By using the same formula again, we get:

$$\hat{z} = \frac{z \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{z \cdot v_2}{v_2 \cdot v_2} v_2 = \boxed{\frac{7}{14} \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} + \frac{0}{49} \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}}$$

Question 5.

Since every least squares solution for Ax=b agrees with $A^TAx = A^Tb$, let's find A^TAandA^Tb:
As you can see:

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

Then, we can solve for x since:

$$A^TAx \;=\; A^Tb \;\rightarrow\; \boxed{\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 3 & 11 & 14 \end{bmatrix} \xrightarrow{R2-R1} \ldots \xrightarrow{R1\div 3} \ldots \xrightarrow{R2\div 8}$$

$$\ldots \xrightarrow{R1-3R2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$ which means that $x_1 = 1$ and $x_2 = 1$.

Hence: $\boxed{\hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

Question 6.

Following the same process as in the last problem, let's find A^TAandA^Tb:
As you can see:

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}$$

Then, we can solve for x since:

$$A^TAx = A^Tb \rightarrow \boxed{\begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 & 27 \\ 3 & 3 & 0 & 12 \\ 3 & 0 & 3 & 15 \end{bmatrix} \xrightarrow{R1\div 6} \ldots \xrightarrow{R2-3R1} \ldots \xrightarrow{R3-3R1}$$

$$\ldots \xrightarrow{\frac{2}{3}R2} \ldots \xrightarrow{R3\frac{3}{2}R2} \rightarrow \xrightarrow{R1-\frac{1}{2}R2} \begin{bmatrix} 1 & 0 & 1 & 5 \\ & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ which means that

$x + z = 5$
$y - z = -1$
and z is a free variable.

Hence: $\boxed{\hat{x} = \begin{bmatrix} 5-z \\ z-1 \\ z \end{bmatrix} \text{ where z is any real number}}$

Question 7.

Using the same formula:

$\hat{b} = \frac{b \cdot a_1}{a_1 \cdot a_1} + \frac{b \cdot a_2}{a_2 \cdot a_2} a_2 = \boxed{\frac{9}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}}$

In addition, you can see that the boxed in equation above is just the parametric version of the equation $Ax = b$. Hence, solving for is not necessary since we can just rearrange the equation and get that $\boxed{\hat{x} = \begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix}}$

Question 8.

$Au = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}$

$b - Au = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}$ and $||b - Au|| = \sqrt{(2^2) + (-4^2) + (2^2)} = \sqrt{24}$

Similarly:

$Av = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 8 \end{bmatrix}$

$b - Av = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}$ and $||b - Av|| = \sqrt{(-2^2) + (2^2) + (-4^2)} = \sqrt{24}$

Due to the results above,

it is not possible for either Au or Av to be a least-squares solution for b
since by definition, the orthogonal projection is the UNIQUE closest point in A to b.
Hence, since Au and Av are at an equal distance from b, neither can be the least-squares solution to Ax=b.

Question 9.

We know that the columns of A are linearly independent whenever Ax=0 (where x is the column vector)
Therefore, that means that if and only if $A^T Ax = 0$ then $A^T A$ is invertible.
Therefore, since $A^T A$ is invertible, that implies that $A^T Ax = 0$ which means that $x = 0$ which then implies that $Ax = 0$ since $A(0) = 0$. Hence, the columns of A are linearly independent.

Question 10.

By theorem 6, whenever the columns of A are orthonormal, then $A^TA = I$, and we know that if we use the formula of $A^TA = A^TB$ we can find the solution of least squares. therefore, by combining those two facts, we get this formula: $\boxed{x = A^TB}$

Question 11.

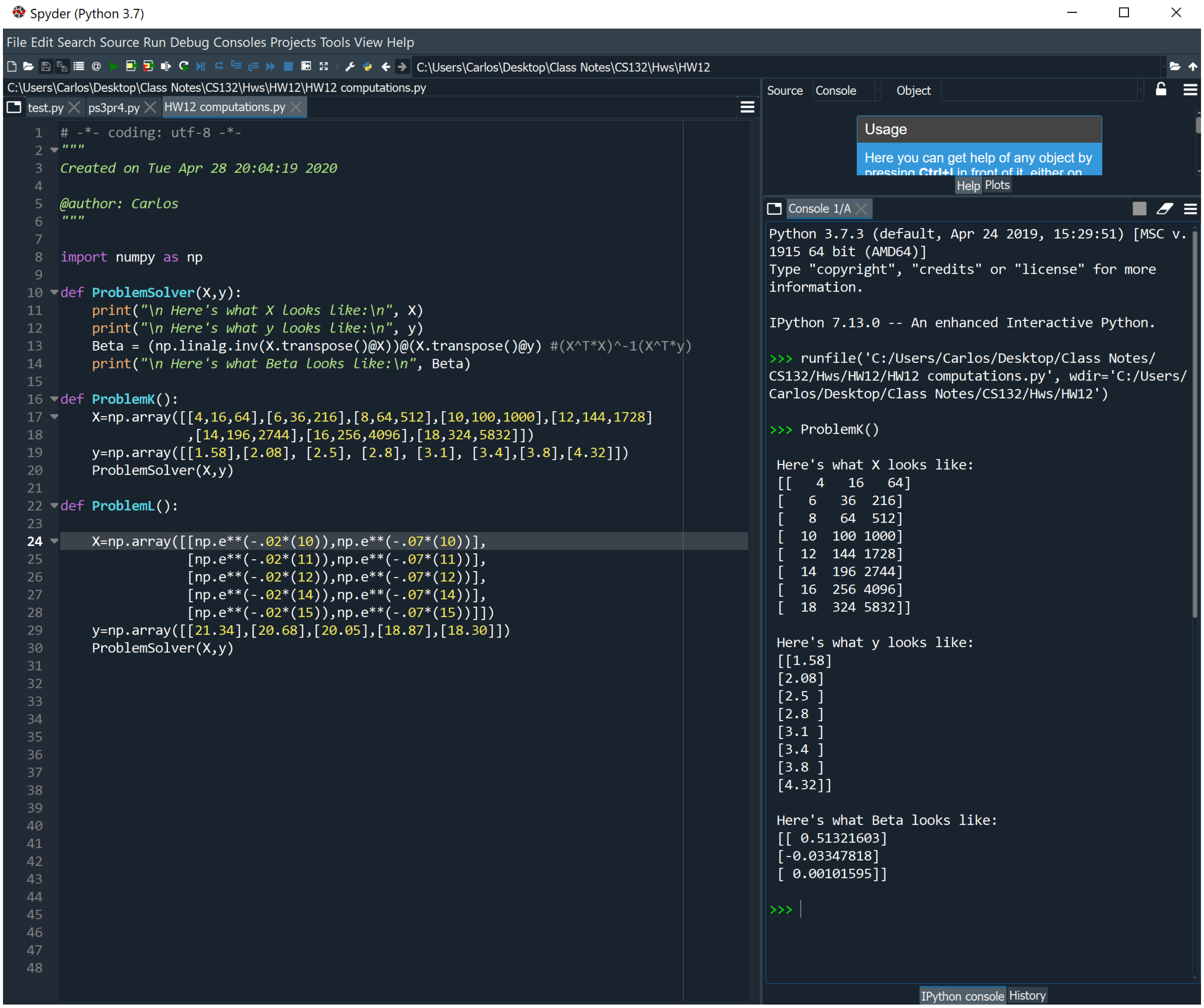
a.)

We can use the linear model $y = X\beta + \epsilon$ where:

y= $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$,$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 & x_1^2 & x_1^3 \\ \vdots & \vdots & \vdots \\ x_n & x_n^2 & x_n^3 \end{bmatrix}$, $\epsilon = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}$

(β represents the parameter vector and X represents the design matrix)

b.)



Therefore, the least-squares curve is:

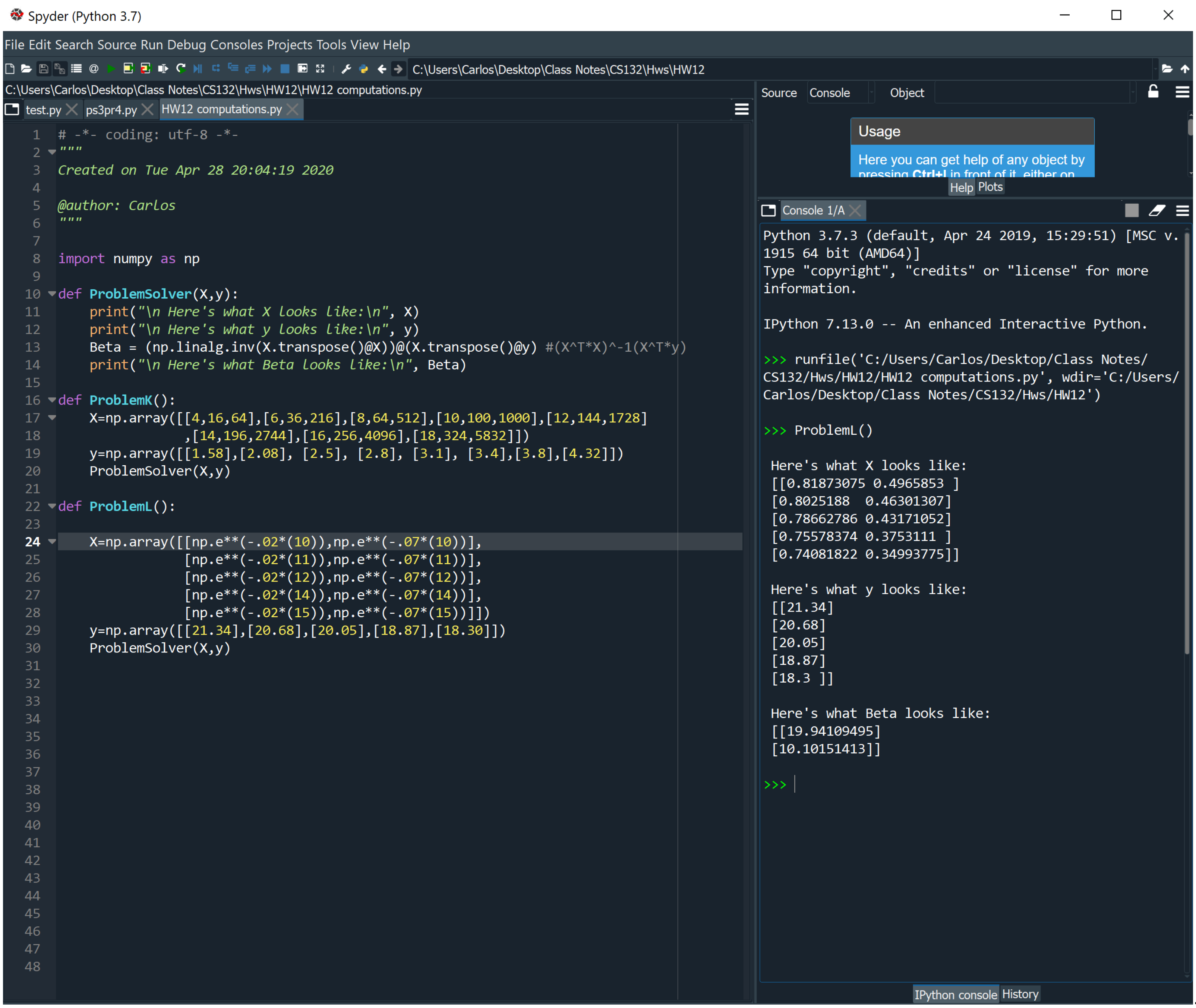
$$\boxed{y = .51321603x - .03347818x^2 + .00101595x^3}$$

Question 12.

a.) We can use the linear model $y = X\beta + \epsilon$ where:

$$\text{y}=\begin{bmatrix}21.34\\20.68\\20.05\\18.87\\18.30\end{bmatrix}, \beta = \begin{bmatrix}M_A\\M_B\end{bmatrix}, X = \begin{bmatrix}e^{-.02(10)} & e^{-.07(10)}\\e^{-.02(11)} & e^{-.07(11)}\\e^{-.02(12)} & e^{-.07(12)}\\e^{-.02(13)} & e^{-.07(13)}\\e^{-.02(14)} & e^{-.07(14)}\\e^{-.02(15)} & e^{-.07(15)}\end{bmatrix}, \epsilon = \begin{bmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\\\epsilon_4\\\epsilon_5\end{bmatrix}$$

b.)



Therefore, the least-squares curve is:

$$\boxed{y = 19.9410495e^{-.002t} + 10.10151413e^{-.002t}}$$

Question 13.

First, let's use the characteristic equation to find the matrix's eigenvalues:
$det(A-\lambda I) = det(\begin{bmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{bmatrix} = (6-\lambda)(9-\lambda)-4 = \lambda^2-15\lambda+50 = (\lambda-5)(\lambda-10$
Therefore, $\lambda = 5$ and $\lambda = 10$
Then, we can calculate the eigenvectors:
Let's solve for when $\lambda = 5 \begin{bmatrix} 6-5 & -2 \\ -2 & 9-5 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$
Turn it into an augmented matrix and solve for Bx=0:
$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \end{bmatrix} \xrightarrow{R2+2R1} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$and we get that$ $x_1 = -2x_2$ or $\frac{-1}{2}x_1 = x_2$, so let us make $x_2 = 1$ to obtain the eigenvector of $\begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix}$

Now, let's solve for when $\lambda = 10 \begin{bmatrix} 6-10 & -2 \\ -2 & 9-10 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}$
Turn it into an augmented matrix and solve for Bx=0:
$\begin{bmatrix} -4 & -2 & 0 \\ -2 & -1 & 0 \end{bmatrix} \xrightarrow{R1-2R2} ... \xrightarrow{R1 \leftrightarrow R2} ... \xrightarrow{R1 \div -1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
and we get that $2x_1 = -x_2$, so let us make $x_2 = 1$ to obtain the eigenvector of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Now, let's normalize the eigenvectors via the formula $\hat{v} = \frac{v}{||v||}$
$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow ||v|| = \sqrt{(2^2)+(1^2)} = \sqrt{5}$
$\frac{v}{||v||} = \begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix} \rightarrow ||v|| = \sqrt{(\frac{-1}{2}^2)+(1^2)} = \sqrt{1.25}$
Therefore, now we can now form the P matrix out of the eigenvectors and the D matrix out of the eigenvalues like so:
P$= \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-.5}{\sqrt{1.25}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{1.25}} \end{bmatrix}$, D$= \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$

And to verify, we can just see that indeed: $\boxed{A = PDP^{-1} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-.5}{\sqrt{1.25}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{1.25}} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1.25}} & \frac{.5}{\sqrt{1.25}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}}$

Question 14.

Let us first turn the quadratic into a matrix and find its eigenvalues:
$det(A-\lambda I) = det(\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}) = (-6-\lambda)(2-\lambda)-9 = \lambda^2+4\lambda-21 = (\lambda+7)(\lambda-3)$Therefore, $\lambda = -7$ and $\lambda = 3$
Now, let's find the eigenvectors:
When $\lambda = 3$: $\begin{bmatrix} -1 & 3 & 0 \\ 3 & -9 & 0 \end{bmatrix} \xrightarrow{R2+3R1} ... \xrightarrow{R1 \div -1} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = -3x_2$. Hence, if we make $X_2 = 1$ we get the vector $\begin{bmatrix} \frac{-1}{3} \\ 1 \end{bmatrix}$

or for avoiding work later: $\begin{bmatrix}-1\\3\end{bmatrix}$ which we can normalize to $\begin{bmatrix}\frac{-1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{bmatrix}$ since $\hat{v} = \frac{v}{||v||}$ and $||v|| = \sqrt{(-1^2)+(3^2)} = \sqrt{10}$

When $\lambda = -7$: $\begin{bmatrix}9&3&0\\3&1&0\end{bmatrix} \xrightarrow{R2-\frac{1}{3}R1} ... \xrightarrow{R1\div 3} \begin{bmatrix}3&1&0\\0&0&0\end{bmatrix}$ so $3\text{x}_1 = x_2$. Hence, if we make $x_2 = 1$ we get the vector $\begin{bmatrix}3\\1\end{bmatrix}$ which we can then normalize to $\begin{bmatrix}\frac{3}{\sqrt{10}}\\\frac{1}{\sqrt{10}}\end{bmatrix}$ since $\hat{v} = \frac{v}{||v||}$ and $||v|| = \sqrt{(3^2)+(1^2)} = \sqrt{10}$

Now we can make $P = \frac{1}{\sqrt{10}}\begin{bmatrix}-1&3\\3&1\end{bmatrix}$ and $D = \begin{bmatrix}-7&0\\0&3\end{bmatrix}$ However, we really don't need P since $x^TAx = (Py)^TA(Py) = y^TP^TAPy = y^TDy$ Hence, we arrive at our answer of $3y_2^2 - 7y_1^2$.
So to reitrate,

$P = \frac{1}{\sqrt{10}}\begin{bmatrix}-1&3\\3&1\end{bmatrix}$, the change of variable turns the equation into $3y_2^2 - 7y_1^2$, and by theorem 5, since the matrix has both positive and negative eigenvalues, then the function is indefinite.

Question 15.

Let us first turn the quadratic into a matrix and find its eigenvalues:
$det(A-\lambda I) = det(\begin{bmatrix}-1-\lambda & -1\\-1&-1-\lambda\end{bmatrix}) = (-1-\lambda)(-1-\lambda) - 1 = \lambda^2 + 2\lambda = (\lambda)(\lambda+2)$Therefore, $\lambda = -2$ and $\lambda = 0$
Now, let's find the eigenvectors:
When $\lambda = -2$: $\begin{bmatrix}1&-1&0\\-1&1&0\end{bmatrix} \xrightarrow{R2+R1} \begin{bmatrix}1&-1&0\\0&0&0\end{bmatrix}$ so $\text{x}_1 = -1x_2$. Hence, if we make $X_2 = 1$ we get the vector $\begin{bmatrix}-1\\1\end{bmatrix}$

which we can normalize to $\begin{bmatrix}\frac{-1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$ since $\hat{v} = \frac{v}{||v||}$ and $||v|| = \sqrt{(-1^2)+(1^2)} = \sqrt{2}$

When $\lambda = 0$: $\begin{bmatrix}-1&-1&0\\-1&-1&0\end{bmatrix} \xrightarrow{R2-R1} ... \xrightarrow{R1\div -1} \begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}$ so $\text{x}_1 = x_2$. Hence, if we make $x_2 = 1$ we get the vector $\begin{bmatrix}1\\1\end{bmatrix}$ which we can then normalize to $\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$ since $\hat{v} = \frac{v}{||v||}$ and $||v|| = \sqrt{(1^2)+(1^2)} = \sqrt{2}$

Now we can make $P = \frac{1}{\sqrt{2}}\begin{bmatrix}-1&1\\1&1\end{bmatrix}$ and $D = \begin{bmatrix}0&0\\0&-2\end{bmatrix}$ However, we really don't need P since $x^TAx = (Py)^TA(Py) = y^TP^TAPy = y^TDy$ Hence, we arrive at our answer of $-2y_1^2$.
So to reitrate,

$P = \frac{1}{\sqrt{2}}\begin{bmatrix}-1&1\\1&1\end{bmatrix}$, the change of variable turns the equation into $-2y_1^2$, and the polynomial, Q(x), is negative semi-definite since Q(x) is less than 0 for all x.

Question 16.

a.)
First, let's make a matrix for Q(x) and let's find its eigenvalues with the characteristic equation: $det(A - \lambda I) = det(\begin{bmatrix} 3-\lambda & 4 \\ 4 & 9-\lambda \end{bmatrix}) = (3-\lambda)(9-\lambda) - 16 = \lambda^2 - 12\lambda + 1 = (\lambda - 1)(\lambda - 11)$ Hence, the eigenvalues are 1 and 11.
In addition, theorem 6 states that if A is a symmetric matrix, then the greatest value of $x^T Ax$ when under the constraint that $x^T AX = 1$ is equal to A's greatest eigenvalue. Therefore, our answer here is $\boxed{11}$

b.)
To find the unit vector, u, where the maximum is attained, all we have to do is find the normalized eigenvector for the biggest eigenvalue, 11, so let's do that.
$\begin{bmatrix} -8 & 4 & 0 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow{R2+\frac{1}{2}R1} \dots \xrightarrow{R1\div 8} \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $x_1 = \frac{1}{2}x_2$. Hence, if we set $x_2 = 1$, we get the eigenvector of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ which can then be normalized to $\boxed{\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}}$ since $\hat{u} = \frac{u}{||u||}$ and $||u|| = \sqrt{1^2) + (2^2)} = \sqrt{5}$

c.)
Since the second largest eigenvalue is 1, then
$\boxed{\text{1 will be the highest value under the constraints } x^T Ax = 1 \text{ and } x^T u = 0.}$