CS132 2020 Homework 10

Carlos Lopez ()

April 16, 2020

Question 1.

$$A^k = PD^kP^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 1^k \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 3(-2^k) + 4(1^k) & 12(2^k) - 12(1^k) \\ -2^k + 1^k & 4(2^k) - 3(1^k) \end{bmatrix} =$$

$$\boxed{\begin{bmatrix} -3(2^k) + 4 & 12 * 2^k - 12 \\ -2^k + 1 & 2^{k+2} - 3 \end{bmatrix}}$$

Question 2.

According to the theorem, all we need to do to find the eigenvectors that will be the basis of the eigenspace is look at the columns of P. In addition, to find the eigenvalues of A, we just have to look at the dieagonal entries of D. Hence:

$$\boxed{\begin{array}{l} \lambda_1 = 5 \ \lambda_2 = 5 \ \lambda_3 = 4 \\ x_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \end{array}}$$

Question 3.

$\boxed{\text{A is not diagonizable since it only has 1 eigenvalue}}$

Firstly, since A is a triangular matrix, we can see that $\lambda = 5$. This is because when we subtract 5 from A's diagonal values, we get the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ In other words, (A - 5I)x=0. The matrix then tells us that $x_2 = 0$ and that x_1 is a free variable. From this we can get an eigenvector like $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but we can't get another vector to create a basis. Hence, A is not diagonalizable.

Question 4.

First, let us find the eigenvalues of A:

$det(A - \lambda I) = det(\begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) - (3*4) = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$

So λ =5 and 2

Let's now find the basis vectors: (A - 5I) = $\begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix} \xrightarrow{R1 \div 3} ... \xrightarrow{R2 \div -4} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ This tells us that $x_1 =$ 1 and $x_2 = 1$. Hence, we get the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Similarly, let's use the other eigenvalue to find the other eigenvector:

$(\mathrm{A}+2\mathrm{I}) = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}$ We can then turn the matrix into equation form and solve: $\begin{bmatrix} 4x_1 + 3x_2 = 0 \\ 0 + 0 = 0 \end{bmatrix}$

This tells us that $x_1 = \frac{-3}{4}x_2$ and x_2 is a free variable. Hence, we get the vector $\begin{bmatrix} -3/4 \\ 1 \end{bmatrix}$

Let us now diagonalize:

$$\begin{bmatrix} 1 & -3/4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{4}{7} & \frac{3}{7} \\ \frac{-4}{7} & \frac{4}{7} \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}}$$

Question 5.

Let us find the eigenvectors:
First for when $\lambda = 2$:

$(\mathrm{A}-2\mathrm{I}) = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \xrightarrow{R2-R1} \ldots \xrightarrow{R3-R1} \ldots \xrightarrow{R1\div 2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ If we turn this into parametric form:

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ This gives us the eigenvectors:

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Now, let's repeat the process for the other eigenvalue, for when $\lambda = 8$:

$(\mathrm{A}-8\mathrm{I}) = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \xrightarrow{R1\div 4} \ldots \xrightarrow{R2-2R1} \ldots \xrightarrow{R3-2R1} \ldots \xrightarrow{R3+R2} \ldots \xrightarrow{R2\div 6} \ldots \xrightarrow{R1-R2} \ldots \xrightarrow{R2*-2}$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ If we turn this into parametric form:

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ This gives us the eigenvector of:

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now we just Diagonalize:

$$A = PDP^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} \frac{-1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \boxed{\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}}$$

Question 6.

$$\boxed{\begin{array}{c}\text{False}\\ \text{True}\\ \text{False}\\ \text{False}\end{array}}$$

a.)
A is diagonalizable if A has n linearly independent eigenvectors.

b.)
Since it could be diagonalizable without distinct eigenvalues due to multiplicity

c.)
According to the diagonalizable theorem: "$A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvalues in P."

d.)
Those two things aren't even related! A is diagonalizable if A has n linearly independent eigenvectors.

Question 7.

$\boxed{\text{No, A is not diagonalizable}}$ because each eigenspace is one-dimensional and there are only two eigevectors, but we would need at least three to diagonalize A.

Question 8.

a.)

$$\boxed{\begin{bmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix} \text{ and } \frac{1}{12}\begin{bmatrix} 3 \\ 3 \\ 2 \\ 2 \\ 2 \end{bmatrix}}$$

Each value in the matrix corresponds to 1 over the amount of connections that the point (represented by the number of the column from left to right) has to other points (which are represented by the number of the column. In addition, since there are no columns full of 0s, this is a regular matrix which means that the eigenvector corresponding with the eigenvalue of 1 is the steady-state vector of the markov chain. Therefore, to get the steady-state vector we count the total amount of non-zero values (12) and make a vector that accounts for the number of non-zero values.

b.)

$$\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \text{ and } \frac{1}{12}\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

The same logic and process as above was followed

Question 9.

Part I.

Preamble:
Here's the program I used to get all of the following answers:

```python
# -*- coding: utf-8 -*-
"""
Created on Wed Apr 15 22:11:35 2020

@author: Carlos
"""

import numpy as np

# Part A
alpha = .01
P1 = np.array([[0,0,1,0,0],[1/3,0,0,1/2,0],[1/3,0,0,1/2,0],[1/3,1/2,0,0,0],[0,1/2,0,0,0]])
P2 = np.array([[0,1/2,1/4,0,0,0],[0,0,1/4,0,0,0],[0,1/2,0,1/2,0,0],[0,0,1/4,0,1/2,0],[0,0,1/4,1/2,0,0],[0,0,0,0,1/2,0]])
print('\n' "Here's the transition matrix for I-1:""\n" , P1)
print('\n' "Here's the transition matrix for I-2:""\n", P2)



#Part B
P1p = np.array([[0,0,1,0,1/5],[1/3,0,0,1/2,1/5],[1/3,0,0,1/2,1/5],[1/3,1/2,0,0,1/5],[0,1/2,0,0,1/5]])
P2p = np.array([[0,1/2,1/4,0,0,1/6],[0,0,1/4,0,0,1/6],[0,1/2,0,1/2,0,1/6],[0,0,1/4,0,1/2,1/6],[0,0,1/4,1/2,0,1/6],[0,0,0,0,1/2,1/6]])
FP1 = ((1-alpha)*P1p + (alpha/5))
FP2 = ((1-alpha)*P2p + (alpha/6))
print('\n' "Here's the P'' for I-1:""\n", FP1)
print('\n' "Here's the P'' for I-2:""\n", FP2)


#Part C
def PartC(graph, x):
    eigenvalues, eigenvectors = np.linalg.eig(x)
    MaxVal = np.argmax(eigenvalues)
    print("\n" "Here's the steady-state vector for I-", graph, ": ""\n", eigenvectors[MaxVal])
    return eigenvectors[MaxVal]
Vector1 = PartC("1", FP1)
Vector2 = PartC("2", FP2)

#Part D
def PartD(graph, x):
    #Note that this is sorted from least to greatest (backwards)
    reverseOrder = np.argsort(x)
    ActualAnswer = 1 + reverseOrder[::-1]
    print("\n""The order in which the pages for I-",graph, " would be presented would be: ", ActualAnswer)
    print()

PartD("1", Vector1)
PartD("2", Vector2)

```

a.)

```
>>> runfile('C:/Users/Carlos/Desktop/Class Notes/CS132/Hws/HW10/Computational.py', wdir='C:/Users/Carlos/
Desktop/Class Notes/CS132/Hws/HW10')

Here's the transition matrix for I-1:
 [[0.         0.         1.         0.         0.        ]
 [0.33333333 0.         0.         0.5        0.        ]
 [0.33333333 0.         0.         0.5        0.        ]
 [0.33333333 0.5        0.         0.         0.        ]
 [0.         0.5        0.         0.         0.        ]]

Here's the transition matrix for I-2:
 [[0.   0.5  0.25 0.   0.   0.  ]
 [0.   0.   0.25 0.   0.   0.  ]
 [0.   0.5  0.   0.5  0.   0.  ]
 [0.   0.   0.25 0.   0.5  0.  ]
 [0.   0.   0.25 0.5  0.   0.  ]
 [0.   0.   0.   0.   0.5  0.  ]]
```

b.)

```
Here's the P'' for I-1:
 [[0.002 0.002 0.992 0.002 0.2  ]
 [0.332 0.002 0.002 0.497 0.2  ]
 [0.332 0.002 0.002 0.497 0.2  ]
 [0.332 0.497 0.002 0.002 0.2  ]
 [0.002 0.497 0.002 0.002 0.2  ]]

Here's the P'' for I-2:
 [[0.00166667 0.49666667 0.24916667 0.00166667 0.00166667 0.16666667]
 [0.00166667 0.00166667 0.24916667 0.00166667 0.00166667 0.16666667]
 [0.00166667 0.49666667 0.00166667 0.49666667 0.00166667 0.16666667]
 [0.00166667 0.00166667 0.24916667 0.00166667 0.49666667 0.16666667]
 [0.00166667 0.00166667 0.24916667 0.49666667 0.00166667 0.16666667]
 [0.00166667 0.00166667 0.00166667 0.00166667 0.49666667 0.16666667]]
```

c.)

```
Here's the steady-state vector for I- 1 :
 [ 5.20753953e-01  7.43611087e-01  2.83506411e-01 -7.69431315e-16
  7.22315119e-01]

Here's the steady-state vector for I- 2 :
 [-3.31361564e-01+0.j -6.53575051e-02+0.j -9.99950004e-01+0.j
 -8.35833700e-01+0.j -8.35833700e-01-0.j -3.78182662e-15+0.j]
```

d.)

```
The order in which the pages for I- 1  would be presented would be:  [2 5 1 3 4]


The order in which the pages for I- 2  would be presented would be:  [6 2 1 5 4 3]
```

Question 10.

Part II.

a.) A= $\begin{bmatrix} p00 & p01 \\ p10 & p11 \end{bmatrix} \xrightarrow{p_1 p_0 terms} \begin{bmatrix} p0 & 1-p1 \\ 1-p0 & p1 \end{bmatrix}$
Then, we'll calculate the eigenvalues through $det(A - \lambda I) = 0$:

$$det(\begin{bmatrix} p0 & 1-p1 \\ 1-p0 & p1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0$$

$$det(\begin{bmatrix} p0-\lambda & 1-p1 \\ 1-p0 & p1 \end{bmatrix}) = (P_0 - \lambda)(P_1 - \lambda) - (1-P_0)(1-P_1) = \lambda^2 - (P_0+P_1)\lambda + P_0 + P_1 - 1 = 0$$

Hence, when we solve for lambda we get:

$$\lambda = \frac{(P_0+P_1)+\sqrt{(P_0+P_1)^2-4(P_0+P_1-1)}}{2} = \frac{(P_0+P_1)+\sqrt{(P_0+P_1-2)^2}}{2} = P_0 + P_1 - 1$$

In addition, due to the quadratic formula λ could also equal:

$$\lambda = \frac{(P_0+P_1)+\sqrt{(P_0+P_1)^2-4(P_0+P_1-1)}}{2} = \frac{(P_0+P_1)+\sqrt{(P_0+P_1-2)^2}}{2} = \frac{2}{2} = 1$$

1 is the largest eigenvalue since we established that P_0 and P_1 are less than 1.
Hence, "$P_0 + P_1 - 1$" has to also be less than 1

b.)
$(A - \lambda I)x = 0 \begin{bmatrix} p0 & 1-p1 \\ 1-p0 & p1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\begin{bmatrix} p0-1 & 1-p1 \\ 1-p0 & p1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So we get that:$(P_0 - 1)x_1 + (1 - P_1)x_2 = 0$
Therefore, we'll set $x_2 = 1$ as stated by the problem and we get that:
$x_1 = \frac{1-P_1}{1-P_0}$ and $x_2 = 1$
which gives us the vector:

$\boxed{\begin{bmatrix} \frac{1-P_1}{1-P_0} \\ 1 \end{bmatrix}}$

c.)
As stated by the problem: $P_0 = .95$, $P_1 = .97$, and $N = 1000$. Hence,

$1000 * \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \begin{bmatrix} .6 \\ 1 \end{bmatrix} = 1000 * \begin{bmatrix} .6 \\ 1 \end{bmatrix} = \begin{bmatrix} 600 \\ 1 \end{bmatrix}$
Therefore,

600 packets on average will be on the network each time step at steady-state