IV054 2019 Homework 1

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April 23, 2020

Question 1.

$$(\frac{x \cdot w}{x \cdot x})x = (\frac{\begin{bmatrix}6 & -2 & 3\end{bmatrix} \cdot \begin{bmatrix}3\\-1\\-5\end{bmatrix}}{\begin{bmatrix}6 & -2 & 3\end{bmatrix} \cdot \begin{bmatrix}6\\-2\\3\end{bmatrix}}) \begin{bmatrix}6\\-2\\3\end{bmatrix} = (\frac{(6*3)+(-2*-1)+(3*-5)}{(6*6)+(-2*-2)+(3*3)}) \begin{bmatrix}6\\-2\\3\end{bmatrix} = \frac{5}{49} \begin{bmatrix}6\\-2\\3\end{bmatrix} = \boxed{\begin{bmatrix}\frac{30}{49}\\ \frac{-10}{49}\\ \frac{15}{49}\end{bmatrix}}$$

Question 2.

$$||x|| = \sqrt{(6)^2 + (-2)^2 + (3)^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = \boxed{7}$$

Question 3.

Let v be the vector given $||v|| = \sqrt{(-6)^2 + (4)^2 + (-3)^2} = \sqrt{36 + 16 + 9} = \sqrt{61}$

Hence, we can make the following vector: $\boxed{\begin{bmatrix}\frac{-6}{\sqrt{61}}\\ \frac{4}{\sqrt{61}}\\ \frac{-3}{\sqrt{61}}\end{bmatrix}}$

Question 4.

$$||u - v|| = || \begin{bmatrix}0\\-5\\2\end{bmatrix} - \begin{bmatrix}-4\\-1\\8\end{bmatrix} || = || \begin{bmatrix}4\\-4\\-6\end{bmatrix} || = \sqrt{(4)^2 + (-4)^2 + (-6)^2} = \sqrt{16 + 16 + 36} =$$
$$\sqrt{68} = \boxed{2\sqrt{17}}$$

Question 5.

Two vectors are orthogonal if their dot product equals 0, and as you can see:

$$u \cdot v = \begin{bmatrix}12 & 3 & -5\end{bmatrix} \cdot \begin{bmatrix}2\\-3\\3\end{bmatrix} = (12 * 2) + (3 * -3) + (-5 * 3) = 24 - 9 - 15 = 0.$$

Hence, $\boxed{\text{yes, the vectors are orthogonal}}$

Question 6.

Two vectors are orthogonal if their dot product equals 0, and as you can see:

$$u \cdot v = \begin{bmatrix} -3 & 7 & 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix} = (-3 * 1) + (7 * -8) + (4 * 15) + (0 * -7) = -3 - 56 + 60 + 0 = 1.$$

Hence, $\boxed{\text{no, the vectors are } \textbf{NOT} \text{ orthogonal}}$

Question 7.

$\boxed{\begin{array}{c} \text{True} \\ \text{False} \\ \text{True} \end{array}}$

a.)
Since the dot product is commutative, we can see that $u \cdot v = v \cdot u$. If we just move the right side to the left, we get: $u \cdot v - v \cdot u = 0$.

b.)
Actually,$||cv|| = ||c||||v||$,but here's an example just to prove it:
If we let c=-3 and v $= \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, then we can see that $||cv|| = || \begin{bmatrix} -6 \\ -6 \end{bmatrix} || = \sqrt{(-6)^2 + (-6)^2} = \sqrt{72}$
which is not the same as $c||v|| = -3|| \begin{bmatrix} 2 \\ 2 \end{bmatrix} || = -3\sqrt{(2)^2 + (2)^2} = -3\sqrt{8}$

d.)
Here's the algebra:
$||u||^2 + ||v||^2 = ||u + v||^2$
u $\cdot u + v \cdot v = (u + v) \cdot (u + v)$
$u \cdot u + v \cdot v = u \cdot u + 2(u \cdot v) + v \cdot v$
$0 = 2(u \cdot v)$
$0 = u \cdot v$

Question 8.

Since $u \cdot u$ yields the vector $\begin{bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix}$, and we know that any number (besides 0) times itself is bigger than 0, then we know that $u_1, u_2, and u_3$ are all bigger than zero. The only time where the vector will be equal to 0 will be when u_1, u_2, and u_3 all equal 0.

Question 9.

Since y is orthogonal to u and v, then $y \cdot u = 0$ and $y \cdot v = 0$. Also, since w spans u,v, then $w = c_1 u + c_2 v$ Therefore:

$w \cdot y = (c_1 u + c_2 v) \cdot y$
$= c_1 u \cdot y + c_2 v \cdot y$
$= c_1(0) + c_2(0)$
$= 0 + 0$
$= 0$
Hence, $\boxed{w \cdot y = 0}$

Question 10.

Let us check for orthogonlity:

$$u_1 \cdot u_2 = \begin{bmatrix} 3 & -3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 6 - 6 + 0 = 0$$

$$u_1 \cdot u_3 = \begin{bmatrix} 3 & -3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 3 - 3 + 0 = 0$$

$$u_2 \cdot u_3 = \begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 2 + 2 - 4 = 0$$

Since everything equals 0, they are all orthogonal and hence by Theorem 4 form a basis for R^3. Now let's find x in terms of the u's by making an augmented matrix out of the vectors and x and then solving the augmented matrix:

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ -3 & 2 & 1 & -3 \\ 0 & -1 & 4 & 1 \end{bmatrix} \xrightarrow{R2+R1} \ldots \xrightarrow{R2+4R3} \ldots \xrightarrow{R3 \longleftrightarrow R2} \ldots \xrightarrow{R3 \div 18} \ldots \xrightarrow{R2 \div -1} \begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix} \xrightarrow{R2+4R3}$$

$$\ldots \xrightarrow{R1-2R2} \ldots \xrightarrow{R1-R3} \ldots \xrightarrow{R1 \div 3} \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

Hence,

$$\boxed{x = \tfrac{4}{3} \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} + \tfrac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \tfrac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}}$$

and we know that u_1, u_2, u_3 form a basis for R^3
since they are orthogonal due to the multiplications above all equaling 0 and what Theorem 4 states

Question 11.

Using the formula: $\hat{y} = \frac{y \cdot u}{u \cdot u} u$ we get:

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u = \frac{\begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}}{\begin{bmatrix} -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{-1-3}{1+9} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{-4}{10} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{2}{5} \\ \frac{-6}{5} \end{bmatrix}}$$

Question 12.

Again, using the formula:$\hat{y} = \frac{y \cdot u}{u \cdot u} u$ we get:

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u = \frac{\begin{bmatrix} 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix}}{\begin{bmatrix} 7 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix}} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{14+6}{49+1} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{20}{50} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{14}{5} \\ \\ \frac{2}{5} \end{bmatrix}$$

Now that we have the part that spans u, we need to get the other part like so:

$$y - \hat{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{14}{5} \\ \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{-4}{5} \\ \\ \frac{28}{5} \end{bmatrix}$$

This leaves us with the final answer of $\boxed{y = \begin{bmatrix} \frac{14}{5} \\ \\ \frac{2}{5} \end{bmatrix} + \begin{bmatrix} \frac{-4}{5} \\ \\ \frac{28}{5} \end{bmatrix}}$

Question 13.

False
False
True
True
True

a.)
Orthogonality implies linear independence

b.)
S is an orthogonal set, but S is not an orthonormal set since this requires the magnitudes of all the vectors in the set to be 1 which is not specified to be the case.

c.)
This is clearly stated below Theorem 7 of this lesson

d.)
Here's the projection of y onto v: $\hat{y} = \frac{y \cdot v}{v \cdot v} v$
Now, here's the projection of y onto cv:$\hat{y} = \frac{y \cdot (cv)}{(cv) \cdot (cv0} (cv) = \frac{c(y \cdot v)}{c^2 (v \cdot v)} (cv) = \frac{y \cdot v}{v \cdot v} v$
As you can see, they are the same

e.)
The columns are linearly independent due to the vectors being orthogonal. Hence, the invertible matrix theorem tells us that an orthogonal matrix is invertible.

Question 14.

We know that the determinant of any orthogonal matrix is either 1 or -1. Hence, $det(UV) = det(U) * det(V) = \pm 1 * \pm 1$ which can't equal 0.
Therefore, due to the determinant not being 0, UV is an invertible matrix.

Now we can see that UV's inverse is $(UV)^T$ since:
$(UV)(UV)^T = (UV)V^TU^T = U(I)U^T = UU^T = I$ and
$(UV)^T(UV) = (V^TU^T)(UV) = (V^T)(I)(V) = (V^T)(V) = I$

We just used the definition of an orthogonal matrix which is that $UU^T = U^TU = I$, and since UV has an inverse and $UV(UV)^T = (UV)^TUV = I$, then $\boxed{\text{UV must be an orthogonal matrix}}$

Question 15.