# A Best-of-Both-Worlds Algorithm for Bandits with Delayed Feedback

**Saeed Masoudian**
University of Copenhagen
saeed.masoudian@di.ku.dk

**Julian Zimmert**
Google Research
zimmert@google.com

**Yevgeny Seldin**
University of Copenhagen
seldin@di.ku.dk

## Abstract

We present a modified tuning of the algorithm of Zimmert and Seldin [2020] for adversarial multiarmed bandits with delayed feedback, which in addition to the minimax optimal adversarial regret guarantee shown by Zimmert and Seldin simultaneously achieves a near-optimal regret guarantee in the stochastic setting with fixed delays. Specifically, the adversarial regret guarantee is $\mathcal{O}(\sqrt{TK} + \sqrt{dT \log K})$, where $T$ is the time horizon, $K$ is the number of arms, and $d$ is the fixed delay, whereas the stochastic regret guarantee is $\mathcal{O}\left(\sum_{i \neq i^*}(\frac{1}{\Delta_i}\log(T) + \frac{d}{\Delta_i \log K}) + dK^{1/3}\log K\right)$, where $\Delta_i$ are the suboptimality gaps. We also present an extension of the algorithm to the case of arbitrary delays, which is based on an oracle knowledge of the maximal delay $d_{max}$ and achieves $\mathcal{O}(\sqrt{TK} + \sqrt{D \log K} + d_{max}K^{1/3}\log K)$ regret in the adversarial regime, where $D$ is the total delay, and $\mathcal{O}\left(\sum_{i \neq i^*}(\frac{1}{\Delta_i}\log(T) + \frac{\sigma_{max}}{\Delta_i \log K}) + d_{max}K^{1/3}\log K\right)$ regret in the stochastic regime, where $\sigma_{max}$ is the maximal number of outstanding observations. Finally, we present a lower bound that matches the refined adversarial regret upper bound achieved by the skipping technique of Zimmert and Seldin [2020] in the adversarial setting.

## 1 Introduction

Delayed feedback is a common challenge in many online learning problems, including multi-armed bandits. The literature studying multi-armed bandit games with delayed feedback builds on prior work on bandit problems with no delays. The researchers have traditionally separated the study of bandit games in stochastic environments [Thompson, 1933, Robbins, 1952, Lai and Robbins, 1985, Auer et al., 2002] and in adversarial environments[Auer et al., 2002b]. However, in practice the environments are rarely purely stochastic, whereas they may not be fully adversarial either. Furthermore, the exact nature of an environment is not always known in practice. Therefore, in recent years there has been an increasing interest in algorithms that perform well in both regimes with no prior knowledge of the regime [Bubeck and Slivkins, 2012, Seldin and Slivkins, 2014, Auer and Chiang, 2016, Seldin and Lugosi, 2017, Wei and Luo, 2018]. The quest for best-of-both-worlds algorithms for no-delay setting culminated with the Tsallis-INF algorithm proposed by Zimmert and Seldin [2019], which achieves the optimal regret bounds in both stochastic and adversarial environments. The algorithm and analysis were further improved by Zimmert and Seldin [2021] and Masoudian and Seldin [2021], who, in particular, derived improved regret bounds for intermediate

36th Conference on Neural Information Processing Systems (NeurIPS 2022).

regimes between stochastic and adversarial, while Ito [2021] removed an assumption on uniqueness of the best arm, which was used in the early works.

Our goal is to extend best-of-both-worlds results to multi-armed bandits with delayed feedback. So far the literature on multi-armed bandits with delayed feedback has followed the traditional separation into stochastic and adversarial. In the stochastic regime Joulani et al. [2013] showed that if the delays are random (generated i.i.d), then compared to the non-delayed stochastic multi-armed bandit setting, the regret only increases additively by a factor that is proportional to the expected delay. In the adversarial setting Cesa-Bianchi et al. [2019] have studied the case of uniform delays $d$. They derived a lower bound $\Omega(\max(\sqrt{KT}, \sqrt{dT\log K}))$ and an almost matching upper bound $\mathcal{O}(\sqrt{KT\log K} + \sqrt{dT\log K})$. Thune et al. [2019] and Bistritz et al. [2019] extended the results to arbitrary delays, achieving $\mathcal{O}(\sqrt{KT\log K} + \sqrt{D\log K})$ regret bounds based on oracle knowledge of the total delay $D$ and time horizon $T$. Thune et al. [2019] also proposed a skipping technique based on advance knowledge of the delays "at action time", which allowed to exclude excessively large delays from $D$. Finally, Zimmert and Seldin [2020] introduced an FTRL algorithm with a hybrid regularizer that achieved $\mathcal{O}(\sqrt{KT} + \sqrt{D\log K})$ regret bound, matching the lower bound in the case of uniform delays and requiring no prior knowledge of $D$ or $T$. The regularizer used by Zimmert and Seldin was a mix of the negative Tsallis entropy regularizer used in the Tsallis-INF algorithm for bandits and the negative entropy regularizer used in the Hedge algorithm for full information games, mixed with separate learning rates:

$$F_t(x) = -2\eta_t^{-1}\left(\sum_{i=1}^{K}\sqrt{x_i}\right) + \gamma_t^{-1}\left(\sum_{i=1}^{K}x_i(\log x_i - 1)\right). \tag{1}$$

Zimmert and Seldin [2020] also improved the skipping technique and achieved a refined regret bound $\mathcal{O}(\sqrt{KT} + \min_S(|S| + \sqrt{D_{\bar{S}}\log K}))$, where $S$ is a set of skipped rounds and $D_{\bar{S}}$ is the total delay in non-skipped rounds. The refined skipping technique requires no advance knowledge of the delays. Their key step toward elimination of the need of advance knowledge of delays was to base the analysis on the count of the number of outstanding observations rather than the delays. The great advantage of skipping is that a few rounds with excessively large or potentially even infinite delays have a very limited impact on the regret bound. One of our contributions in this paper is a lower bound for the case of non-uniform delays, which matches the refined regret upper bound achieved by skipping.

Even though the hybrid regularizer used by Zimmert and Seldin [2020] was sharing the Tsallis entropy part with their best-of-both-worlds Tsallis-INF algorithm from Zimmert and Seldin [2021], and even though the adversarial analysis was partly similar to the analysis of the Tsallis-INF algorithm, Zimmert and Seldin [2020] did not manage to derive a regret bound for their algorithm in the stochastic setting with delayed feedback and left it as an open problem. The stochastic analysis of the Tsallis-INF algorithm is based on the self-bounding technique [Zimmert and Seldin, 2021]. Application of this technique in the no delay setting is relatively straightforward, but in presence of delays it requires control of the drift of the playing distribution from the moment an action is played to the moment the feedback arrives. Cesa-Bianchi et al. [2019] have bounded the drift of the playing distribution of the EXP3 algorithm in the uniform delays setting with a fixed learning rate. But best-of-both-worlds algorithms require decreasing learning rates [Mourtada and Gaïffas, 2019], which makes the drift control much more challenging. The problem gets even more challenging in the case of arbitrary delays, because it requires drift control over arbitrary long periods of time.

We apply an FTRL algorithm with the same hybrid regularizer as the one used by Zimmert and Seldin [2020], but with a different tuning of the learning rates. The new tuning has a minor effect on the adversarial regret bound, but allows us to make progress with the stochastic analysis. For the stochastic analysis we use the self-bounding technique. One of our key contributions is a general lemma that bounds the drift of the playing distribution derived from the time-varying hybrid regularizer over arbitrary delays. Using this lemma we derive near-optimal best-of-both-worlds regret guarantees for the case of fixed delays. But even with the lemma at hand, application of the self-bounding technique in presence of arbitrary delays is still much more challenging than in the no delays or fixed delay setting. Therefore, we resort to introducing an assumption of oracle knowledge of the maximal delay, which limits the maximal period of time over which we need to keep control over the drift. Our contributions are summarized below. To keep the presentation simple we assume uniqueness of the best arm throughout the paper. Tools for eliminating the uniqueness of the best arm assumption were proposed by Ito [2021].

1. We show that in the arbitrary delays setting with an oracle knowledge of the maximal delay $d_{max}$, our algorithm achieves $\mathcal{O}(\sqrt{KT}+\sqrt{D\log K}+d_{max}K^{1/3}\log K)$ regret bound in the adversarial regime simultaneously with $\mathcal{O}\left(\sum_{i\neq i^*}(\frac{\log T}{\Delta_i}+\frac{\sigma_{max}}{\Delta_i\log K})+d_{max}K^{1/3}\log K\right)$ regret bound in the stochastic regime, where $\sigma_{max}$ is the maximal number of outstanding observations. We note that $\sigma_{max}\leq d_{max}$, but it may potentially be much smaller. For example, if the first observation has a delay of $T$ and all the remaining observations have zero delay, then $d_{max}=T$, but $\sigma_{max}=1$.
2. In the case of uniform delays the above bounds simplify to $\mathcal{O}(\sqrt{KT}+\sqrt{dT\log K}+dK^{1/3}\log K)$ in the adversarial case and $\mathcal{O}\left(\sum_{i\neq i^*}(\frac{\log T}{\Delta_i}+\frac{d}{\Delta_i\log K})+dK^{1/3}\log K\right)$ in the stochastic case. For $T\geq dK^{2/3}\log K$ the last term in the adversarial regret bound is dominated by the middle term, which leads to the minimax optimal $\mathcal{O}(\sqrt{KT}+\sqrt{dT\log K})$ adversarial regret. The stochastic regret lower bound is trivially $\Omega(\min\{d\frac{\sum_{i\neq i^*}\Delta_i}{K},\sum_{i\neq i^*}\frac{\log T}{\Delta_i}\})=\Omega(d\frac{\sum_{i\neq i^*}\Delta_i}{K}+\sum_{i\neq i^*}\frac{\log T}{\Delta_i})$ and, therefore, our stochastic regret upper bound is near-optimal.
3. We present an $\Omega\left(\sqrt{KT}+\min_S(|S|+\sqrt{D_{\bar{S}}\log K})\right)$ regret lower bound for adversarial multi-armed bandits with non-uniformly delayed feedback, which matches the refined regret upper bound achieved by the skipping technique of Zimmert and Seldin [2020].

## 2 Problem setting

We study the multi-armed bandit with delays problem, in which at time $t=1,2,\dots$ the learner chooses an arm $I_t$ among a set of $K$ arms and instantaneously suffers a loss $\ell_{t,I_t}$ from a loss vector $\ell_t\in[0,1]^K$ generated by the environment, but $\ell_{t,I_t}$ is not observed by the learner immediately. After a delay of $d_t$, at the end of round $t+d_t$, the learner observes the pair $(t,\ell_{t,I_t})$, namely, the loss and the index of the game round the loss is coming from. The sequence of delays $d_1,d_2,\dots$ is selected arbitrarily by the environment. Without loss of generality we can assume that all the outstanding observations are revealed at the end of the game, i.e., $t+d_t\leq T$ for all $t$, where $T$ is the time horizon, unknown to the learner. We consider two regimes, oblivious adversarial and stochastic. The performance of the learner is evaluated using pseudo-regret, which is defined as

$$\overline{Reg}_T=\mathbb{E}\left[\sum_{t=1}^{T}\ell_{t,I_t}\right]-\min_{i\in[K]}\mathbb{E}\left[\sum_{t=1}^{T}\ell_{t,i}\right]=\mathbb{E}\left[\sum_{t=1}^{T}\left(\ell_{t,I_t}-\ell_{t,i_T^*}\right)\right],$$

where $i_T^*\in\operatorname{argmin}_{i\in[K]}\mathbb{E}\left[\sum_{t=t}^{T}\ell_{t,i}\right]$ is a best arm in hindsight in expectation over the loss generation model and the randomness of the learner. In the oblivious adversarial setting the losses are independent of the actions taken by the algorithm and considered to be deterministic, and the pseudo-regret is equal to the expected regret.

**Additional Notation:** We use $\Delta^n$ to denote the probability simplex over $n+1$ points. The characteristic function of a closed convex set $\mathcal{A}$ is denoted by $\mathcal{I}_{\mathcal{A}}(x)$ and satisfies $\mathcal{I}_{\mathcal{A}}(x)=0$ for $x\in\mathcal{A}$ and $\mathcal{I}_{\mathcal{A}}(x)=\infty$ otherwise. The convex conjugate of a function $f:\mathbb{R}^n\to\mathbb{R}$ is defined by $f^*(y)=\sup_{x\in\mathbb{R}^n}\{\langle x,y\rangle-f(x)\}$. We also use bar to denote that the function domain is restricted to $\Delta^n$, e.g., $\bar{f}(x)=\begin{cases}f(x), & \text{if } x\in\Delta^n\\ \infty, & \text{otherwise}\end{cases}$. We denote the indicator function of an event $\mathcal{E}$ by $\mathbb{1}(\mathcal{E})$ and use $\mathbb{1}_t(i)$ as a shorthand for $\mathbb{1}(I_t=i)$. The probability distribution over arms that is played by the learner at round $t$ is denoted by $x_t\in\Delta^{K-1}$.

## 3 Algorithm

The algorithm is based on Follow The Regularized Leader (FTRL) algorithm with the hybrid regularizer used by Zimmert and Seldin [2020], stated in equation (1). At each time step $t$ let $\sigma_t=\sum_{s=1}^{t-1}\mathbb{1}(s+d_s\geq t)$ be the number of outstanding observations and $\mathcal{D}_t=\sum_{s=1}^{t}\sigma_t$ be the cumulative number of outstanding observations, then the learning rates are defined as

$$\eta_t^{-1} = \sqrt{t + \eta_0}, \qquad \gamma_t^{-1} = \sqrt{\frac{\sum_{s=1}^{t} \sigma_s + \gamma_0}{\log K}}, \tag{2}$$

where $\eta_0 = 10 d_{max} + d_{max}^2 / \left(K^{1/3} \log(K)\right)^2$ and $\gamma_0 = 24^2 d_{max}^2 K^{2/3} \log(K)$. The update rule for the distribution over actions played by the learner is

$$x_t = \nabla \bar{F}_t^*(-\hat{L}_t^{obs}) = \arg\min_{x \in \Delta^{K-1}} \langle \hat{L}_t^{obs}, x \rangle + F_t(x), \tag{3}$$

where $\hat{L}_t^{obs} = \sum_{s=1}^{t-1} \hat{\ell}_s \mathbb{1}(s + d_s < t)$ is the cumulative importance-weighted observed loss and $\hat{\ell}_s$ is an importance-weighted estimate of the loss vector $\ell_s$ defined by

$$\hat{\ell}_{t,i} = \frac{\ell_{t,i} \mathbb{1}(I_t = i)}{x_{t,i}}.$$

At the beginning of round $t$ the algorithm calculates the cumulative number of outstanding observations $\mathcal{D}_t$ and uses it to define the learning rate $\gamma_t$. Next, it uses the FTRL update rule defined in (3) to define a distribution over actions $x_t$ from which to draw action $I_t$. Finally, at the end of round $t$ it receives the delayed observations and updates the cumulative loss estimation vector accordingly, so that $\hat{L}_{t+1}^{obs} = \hat{L}_t^{obs} + \sum_{s=1}^{t} \hat{\ell}_s \mathbb{1}(s + d_s = t)$. The complete algorithm is provided in Algorithm 1.

**Algorithm 1:** FTRL with advance tuning for delayed bandit

```
Initialize D_0 = 0 and L̂_1^obs = 0_K (where 0_K is a zero vector in R^K)
for t = 1, ..., n do
    Set σ_t = Σ_{s=1}^{t-1} 1(s + d_s > t)
    Update D_t = D_{t-1} + σ_t
    Set x_t = arg min_{x∈Δ^{K-1}} ⟨L̂_t^obs, x⟩ + F_t(x)        // F_t is defined in (1) and η_t and γ_t in (2)
    Sample I_t ~ x_t
    Observe (s, ℓ_{s,I_s}) for all s that satisfy s + d_s = t
    L̂_{t+1}^obs = L̂_t^obs + Σ_{s=1}^{t} ℓ̂_s 1(s + d_s = t)
```

## 4 Best-of-both-worlds regret bounds for Algorithm 1

In this section we provide best-of-both-worlds regret bounds for Algorithm 1. First, in Theorem 1 we provide regret bounds for an arbitrary delay setting, where we assume an oracle access to $d_{max}$. Then, in Corollary 2 we specialize the result to a fixed delay setting.

**Theorem 1.** *Assume that Algorithm 1 is given an oracle knowledge of $d_{max}$. Then its pseudo-regret for any sequence of delays and losses satisfies*

$$\overline{Reg}_T = \mathcal{O}(\sqrt{TK} + \sqrt{D \log K} + d_{max} K^{1/3} \log K).$$

*Furthermore, in the stochastic regime the pseudo-regret additionally satisfies*

$$\overline{Reg}_T = \mathcal{O}\left(\sum_{i \neq i^*} \left(\frac{1}{\Delta_i} \log(T) + \frac{\sigma_{max}}{\Delta_i \log K}\right) + d_{max} K^{1/3} \log K\right).$$

A sketch of the proof is provided in Section 5 and detailed constants are worked out in Appendix C. For fixed delays Theorem 1 gives the following corollary.

**Corollary 2.** *If the delays are fixed and equal to $d$, and $T \geq dK^{2/3} \log K$, then the pseudo-regret of Algorithm 1 always satisfies*

$$\overline{Reg}_T = \mathcal{O}(\sqrt{TK} + \sqrt{dT \log K})$$

*and in the stochastic setting it additionally satisfies*

$$\overline{Reg}_T = \mathcal{O}\left(\sum_{i \neq i^*} \left(\frac{1}{\Delta_i} \log(T) + \frac{d}{\Delta_i \log K}\right) + dK^{1/3} \log K\right).$$

In the adversarial regime with fixed delays $d$, regret lower bound is $\Omega\left(\sqrt{KT}+\sqrt{dT\log K}\right)$, whereas in the stochastic regime with fixed delays the regret lower bound is trivially $\Omega(d\frac{\sum_{i\neq i*}\Delta_i}{K}+\sum_{i\neq i^*}\frac{\log T}{\Delta_i})$. Thus, in the adversarial regime the corollary yields the minimax optimal regret bound and in the stochastic regime it is near-optimal. More explicitly, it is optimal within a multiplicative factor of $\sum_{i\neq i^*}\frac{1}{\Delta_i\log K}+\frac{K^{4/3}\log K}{\sum_{i\neq i*}\Delta_i}$ in front of $d$.

If we fix a total delay budget $D$, then uniform delays $d=D/T$ is a special case, and in this sense Theorem 1 is also optimal in the adversarial regime and near-optimal in the stochastic regime, although for non-uniform delays improved regret bounds can potentially be achieved by skipping. We also note that having the dependence on $\sigma_{max}$ in the middle term of the stochastic regret bound in Theorem 1 is better than having a dependence on $d_{max}$, since $\sigma_{max}\leq d_{max}$, and in some cases it can be significantly smaller, as shown in the example in the Introduction and quantified by the following lemma.

**Lemma 3.** *Let $d_{max}(S)=\max_{s\in S}d_s$, where $S\subseteq\{1,\dots,T\}$ is a subset of rounds. Let $\bar{S}=\{1,\dots,T\}\setminus S$ be the remaining rounds. Then*

$$\sigma_{max}\leq\min_{S\subseteq\{1,\dots,T\}}\left\{|S|+d_{max}(\bar{S})\right\}.$$

A proof of Lemma 3 is provided in Appendix A.

Finally, we note that the result in Theorem 1 is easily extendable to the corrupted regime, because the proof relies on the same self-bounding technique as the one used by Zimmert and Seldin [2021]. If we denote by $B_T^{stoch}$ the regret upper bound in the stochastic regime in Theorem 1 and by $C$ the total corruption budget, then in the corrupted regime the regret would be $\mathcal{O}(B_T^{stoch}+\sqrt{B_T^{stoch}C})$. The proof is straightforward, following the lines of Zimmert and Seldin [2021], and, therefore, left out.

# 5 A proof sketch of Theorem 1

In this section we provide a sketch of a proof of Theorem 1. We provide a proof sketch for the stochastic bound in Section 5.1. Afterwards, in Section 5.2, we show how the analysis of Zimmert and Seldin [2020] gives the adversarial bound stated in Theorem 1.

## 5.1 Stochastic Bound

We start by providing a key lemma (Lemma 4) that controls the drift of the playing distribution derived from the time-varying hybrid regularizer over arbitrary delays. We then introduce a drifted version of the pseudo-regret defined in (4), for which we use the key lemma to show that the drifted version of the pseudo-regret is close to the actual one. As a result, it is sufficient to bound the drifted version. The analysis of the drifted pseudo-regret follows by the standard analysis of the FTRL algorithm [Lattimore and Szepesvári, 2020] that decomposes the pseudo-regret (drifted pseudo-regret in our case) into stability and penalty terms. Thereafter, we proceed by using Lemma 4 again, this time to bound the stability term in order to apply the self-bounding technique [Zimmert and Seldin, 2019], which yields logarithmic regret in the stochastic setting. Our key lemma is the following.

**Lemma 4** (The Key Lemma)**.** *For any $i\in[K]$ and $s,t\in[T]$, where $s\leq t$ and $t-s\leq d_{max}$, we have*

$$x_{t,i}\leq 2x_{s,i}.$$

A detailed proof of the lemma is provided in Appendix B. Below we explain the high level idea behind the proof.

*Proof sketch.* We know that $x_t=\nabla\bar{F}_t^*(-\hat{L}_t^{obs})$ and $x_s=\nabla\bar{F}_s^*(-\hat{L}_s^{obs})$, so we introduce $\tilde{x}=\nabla\bar{F}_s^*(-\hat{L}_t^{obs})$ as an auxiliary variable to bridge between $x_t$ and $x_s$. The analysis consists of two key steps and is based on induction on $(t,s)$.

**Deviation Induced by the Loss Shift:** This step controls the drift when we fix the learning rates and shift the cumulative loss. We prove the following inequality:

$$\tilde{x}_i\leq\frac{3}{2}x_{s,i}.$$

Note that this step uses the induction assumption for $(s, s-d_r)$ for all $r < s : r + d_r = s$.
**Deviation Induced by the Change of Regularizer:** In this step we bound the drift when the cumulative loss vector is fixed and we change the regularizer. We show that

$$x_{t,i} \leq \frac{4}{3}\tilde{x}_i.$$

Combining these two steps gives us the desired bound. A proof of these steps is provided in Appendix B. ■

We use Lemma 4 to relate the drifted pseudo-regret to the actual pseudo-regret. Let $A_t = \{s : s \leq t \text{ and } s + d_s = t\}$ be the set of rounds for which feedback arrives at round $t$. We define the observed loss vector at time $t$ as $\hat{\ell}_t^{obs} = \sum_{s \in A_t} \hat{\ell}_s$ and the drifted pseudo-regret as

$$\overline{Reg}_T^{drift} = \mathbb{E}\left[\sum_{t=1}^{T}\left(\langle x_t, \hat{\ell}_t^{obs}\rangle - \hat{\ell}_{t,i_T^*}^{obs}\right)\right]. \tag{4}$$

We rewrite the drifted regret as

$$\begin{aligned}
\overline{Reg}_T^{drift} &= \mathbb{E}\left[\sum_{t=1}^{T}\sum_{s\in A_t}\left(\langle x_t, \hat{\ell}_s\rangle - \hat{\ell}_{s,i_T^*}\right)\right]\\
&= \sum_{t=1}^{T}\sum_{s\in A_t}\sum_{i=1}^{K}\mathbb{E}[x_{t,i}(\hat{\ell}_{s,i} - \hat{\ell}_{s,i_T^*})]\\
&= \sum_{t=1}^{T}\sum_{s\in A_t}\sum_{i=1}^{K}\mathbb{E}[x_{t,i}]\Delta_i = \sum_{t=1}^{T}\sum_{i=1}^{K}\mathbb{E}[x_{t+d_t,i}]\Delta_i,
\end{aligned}$$

where when taking the expectation we use the facts that $\hat{\ell}_s$ has no impact on the determination of $x_t$ and that the loss estimators are unbiased. Using Lemma 4 we make a connection between pseudo-regret and the drifted version:

$$\begin{aligned}
\overline{Reg}_T^{drift} = \sum_{t=1}^{T}\sum_{i=1}^{K}\mathbb{E}[x_{t+d_t,i}]\Delta_i &\geq \sum_{t=1}^{T-d_{max}}\sum_{i=1}^{K}\frac{1}{2}\mathbb{E}[x_{t+d_{max},i}]\Delta_i\\
&= \frac{1}{2}\sum_{t=d_{max}+1}^{T}\sum_{i=1}^{K}\mathbb{E}[x_{t,i}]\Delta_i\\
&\geq \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{K}\mathbb{E}[x_{t,i}]\Delta_i - \frac{d_{max}}{2} = \frac{1}{2}\overline{Reg}_T - \frac{d_{max}}{2},
\end{aligned}$$

where the first inequality follows by Lemma 4, and the second inequality uses $\sum_{t=1}^{d_{max}}\mathbb{E}[x_{t,i}]\Delta_i \leq d_{max}$. As a result, we have $\overline{Reg}_T \leq 2\overline{Reg}_T^{drift} + d_{max}$ and it suffices to upper bound $\overline{Reg}_T^{drift}$. We follow the standard analysis of FTRL, which decomposes the drifted pseudo-regret into *stabiltiy* and *penalty* terms as

$$\overline{Reg}_T^{drift} = \mathbb{E}\left[\underbrace{\sum_{t=1}^{T}\langle x_t, \hat{\ell}_t^{obs}\rangle + \bar{F}_t^*(-\hat{L}_{t+1}^{obs}) - \bar{F}_t^*(-\hat{L}_t^{obs})}_{stability}\right] + \mathbb{E}\left[\underbrace{\sum_{t=1}^{T}\bar{F}_t^*(-\hat{L}_t^{obs}) - \bar{F}_t^*(-\hat{L}_{t+1}^{obs}) - \ell_{t,i_T^*}}_{penalty}\right].$$

For the penalty term we have the following bound by Abernethy et al. [2015]

$$penalty \leq \sum_{t=2}^{T}(F_{t-1}(x_t) - F_t(x_t)) + F_T(\mathrm{e}_{i_T^*}) - F_1(x_1),$$

where $e_{i_T^*}$ denotes a the unit vector in $\mathbb{R}^K$ with the $i_T^*$-th element being one and zero elsewhere. By replacing the closed form of the regularizer in this bound and using the facts that $\eta_t^{-1} - \eta_{t-1}^{-1} = \mathcal{O}(\eta_t)$, $\gamma_t^{-1} - \gamma_{t-1}^{-1} = \mathcal{O}(\sigma_t\gamma_t/\log K)$, and $x_{t,i_T^*}^{\frac{1}{2}} - 1 \leq 0$, we obtain

$$penalty \leq \mathcal{O}\left(\sum_{t=2}^{T}\sum_{i\neq i^*}\eta_t x_{t,i}^{\frac{1}{2}} + \sum_{t=2}^{T}\sum_{i=1}^{K}\frac{\sigma_t\gamma_t x_{t,i}\log(1/x_{t,i})}{\log K}\right) + 2\sqrt{\eta_0(K-1)} + \sqrt{\gamma_0 \log K}. \tag{5}$$

In order to control the stability term we derive Lemma 5.

**Lemma 5** (Stability)**.** *Let $\upsilon_t = |A_t|$. For any $\alpha_t \leq \gamma_t^{-1}$ we have*

$$stability \leq \sum_{t=1}^{T}\sum_{i=1}^{K} 2f_t^{''}(x_{t,i})^{-1}(\hat{\ell}_{t,i}^{obs} - \alpha_t)^2.$$

*Furthermore, $\alpha_t = \frac{\sum_{j=1}^{K} f''(x_{t,j})^{-1}\hat{\ell}_{t,j}^{obs}}{\sum_{j=1}^{K} f''(x_{t,j})^{-1}}$ satisfies $\alpha_t \leq \gamma_t^{-1}$ and yields*

$$\mathbb{E}[stability] \leq \sum_{t=1}^{T}\sum_{i\neq i^*} 2\gamma_t(\upsilon_t - 1)\upsilon_t\mathbb{E}[x_{t,i}]\Delta_i + \sum_{t=1}^{T}\sum_{s\in A_t}\sum_{i=1}^{K} 2\eta_t\mathbb{E}[x_{t,i}^{3/2}x_{s,i}^{-1}(1-x_{s,i})]. \tag{6}$$

A proof of the stability lemma is provided in Appendix A.3. We apply Lemma 4 to (6) to give bounds $\upsilon_t x_{t,i} = \sum_{s\in A_t} x_{t,i} \leq 2\sum_{s\in A_t} x_{s,i}$ and $x_{t,i}^{3/2}x_{s,i}^{-1}(1-x_{s,i}) \leq 2^{3/2}x_{s,i}^{1/2}(1-x_{s,i})$. Moreover, in order to remove the best arm $i^*$ from the summation in the later bound we use $x_{s,i^*}^{1/2}(1-x_{s,i^*}) \leq \sum_{i\neq i^*} x_{s,i} \leq \sum_{i\neq i^*} x_{s,i}^{1/2}$. These bounds together with the facts that we can change the order of the summations and that each $t$ belongs to exactly one $A_s$, gives us the following stability bound

$$\mathbb{E}[stability] = \mathcal{O}\left(\sum_{t=1}^{T}\sum_{i\neq i^*}\eta_t\mathbb{E}[x_{t,i}^{1/2}] + \sum_{t=1}^{T}\sum_{i\neq i^*}\gamma_{t+d_t}(\upsilon_{t+d_t} - 1)\mathbb{E}[x_{t,i}]\Delta_i\right). \tag{7}$$

By combining (7), (5), and the fact that $\overline{Reg}_T \leq 2\overline{Reg}_T^{drift} + d_{max}$, we show that there exist constants $a, b, c \geq 0$, such that

$$\overline{Reg}_T \leq \mathbb{E}\left[a\underbrace{\sum_{t=1}^{T}\sum_{i\neq i^*}\eta_t x_{t,i}^{1/2}}_{A} + b\underbrace{\sum_{t=1}^{T}\sum_{i\neq i^*}\gamma_{t+d_t}(\upsilon_{t+d_t} - 1)x_{t,i}\Delta_i}_{B} + c\underbrace{\sum_{t=2}^{T}\sum_{i=1}^{K}\frac{\sigma_t\gamma_t x_{t,i}\log(1/x_{t,i})}{\log K}}_{C}\right]$$
$$+ \underbrace{4\sqrt{\eta_0(K-1)} + 2\sqrt{\gamma_0\log K} + d_{max}}_{D}. \tag{8}$$

**Self bounding analysis:** We use the self-bounding technique to write $\overline{Reg}_T = 4\overline{Reg}_T - 3\overline{Reg}_T$, and then based on (8) we have

$$\overline{Reg}_T \leq \mathbb{E}\left[4aA - \overline{Reg}_T\right] + \mathbb{E}\left[4bB - \overline{Reg}_T\right] + \mathbb{E}\left[4cC - \overline{Reg}_T\right] + 4D. \tag{9}$$

For $D$ we can substitute the values of $\gamma_0$ and $\eta_0$ and get

$$D = \mathcal{O}(d_{max}(K-1)^{1/3}\log K). \tag{10}$$

Upper bounding $A, B$, and $C$ requires separate and elaborate analysis, which we do in Lemmas 6, 7 and 8, respectively. Proofs of these lemmas are provided in Appendix A.2.

**Lemma 6** (A bound for $4aA - \overline{Reg}_T$)**.** *We have the following bound for any $a \geq 0$:*

$$4aA - \overline{Reg}_T \leq \sum_{i\neq i^*}\frac{4a^2}{\Delta_i}\log(T/\eta_0 + 1) + 1. \tag{11}$$

Lemma 6 contributes the logarithmic (in $T$) term to the regret bound.

**Lemma 7** (A bound for $4bB - \overline{Reg}_T$)**.** *Let $\upsilon_{max} = \max_{t\in[T]} \upsilon_t$, then for any $b \geq 0$:*

$$4bB - \overline{Reg}_T \leq 64b^2\upsilon_{max}\log K. \tag{12}$$

It is evident that $\upsilon_{max} \leq \sigma_{max} \leq d_{max}$, so the bound in Lemma 7 contributes an $\mathcal{O}(d_{max}\log K)$ term to the regret bound.

**Lemma 8** (A bound for $4cC - \overline{Reg}_T$)**.** *For any $c \geq 0$:*

$$4cC - \overline{Reg}_T \leq \sum_{i\neq i^*} \frac{128c^2\sigma_{max}}{\Delta_i \log K}. \tag{13}$$

Part of the pseudo-regret bound that corresponds to Lemma 8 comes from the penalty term related to the negative entropy part of the regularizer. In this part, despite the fact that $\sigma_{max}$ can be much smaller than $d_{max}$ (Lemma 3), the $\sum_{i\neq i^*}\frac{\sigma_{max}}{\Delta_i \log K}$ term could be very large when the suboptimality gaps are small. In Appendix D we show how an asymmetric oracle learning rate $\gamma_{t,i} \simeq \gamma_t/\sqrt{\Delta_i}$ for the negative entropy regularizer can be used to remove the $\sum_{i\neq i^*} 1/\Delta_i$ factor in front of $\sigma_{max}$. The possibility of removing this factor without the oracle knowledge is left as an open question.

Finally, by plugging (10),(11),(12),(13) into (9) we obtain the desired regret bound.

### 5.2 Adversarial bound

For the adversarial regime we use the final bound of Zimmert and Seldin [2021], which holds for any non-increasing learning rates:

$$\overline{Reg}_T \leq \sum_{t=1}^{T}\eta_t\sqrt{K} + \sum_{t=1}^{T}\gamma_t\sigma_t + 2\eta_T^{-1}\sqrt{K} + \gamma_T^{-1}\log K.$$

It suffices to substitute the values of the learning rates and use Lemma 11 for function $\frac{1}{\sqrt{x}}$:

$$\begin{aligned}\overline{Reg}_T &\leq \sum_{t=1}^{T}\frac{\sqrt{K}}{\sqrt{t+\eta_0}} + \sum_{t=1}^{T}\frac{\sigma_t\sqrt{\log K}}{\sqrt{D_t+\gamma_0}} + 2\sqrt{KT+K\eta_0} + \sqrt{\log(K)D_T + \gamma_0\log(K)}\\ &= \mathcal{O}\left(\sqrt{KT} + \sqrt{\log(K)D_T} + d_{max}K^{1/3}\log K\right).\end{aligned}$$

## 6 Refined lower bound

In this section, we prove a tight lower bound for adversarial regret with arbitrary delays. Thune et al. [2019] have proposed a skipping technique to achieve refined regret upper bounds in the adversarial regime with non-uniform delays. The technique was improved by Zimmert and Seldin [2020], but it remained unknown whether the refined regret bounds for regimes with non-uniform delays are tight. We answer this question positively by showing that the regret bound of Zimmert and Seldin [2020] is not improvable without additional assumptions. We first derive a refined lower bound for full-information games with variable loss ranges, which might be of independent interest. A proof is provided in Appendix E.

**Theorem 9.** *Let $L_T \geq L_{T-1} \geq \cdots \geq L_1 \geq 0$ be a non-increasing sequence of positive reals and assume that there exists a permutation $\rho : [T] \to [T]$, such that the losses at time t are bounded in $[0, L_{\rho(t)}]^K$. The minimax regret $Reg^*$ in the corresponding adversarial full-information game satisfies*

$$Reg^* \geq \max\left\{\frac{1}{2}\sum_{t=1}^{\lfloor\log_2(K)\rfloor} L_t, \frac{1}{32}\sqrt{\sum_{t=\lfloor\log_2(K)\rfloor}^{T} L_t^2\log(K)}\right\}.$$

From here we can directly obtain a lower bound for the full-information game with variable delays. This implies the same lower bound for bandits, since we have strictly less information available.

**Corollary 10.** *Let $(d_t)_{t=1}^T$ be a sequence of non-increasing delays, such that $d_t \le T+1-t$ and let an oblivious adversary select all loss vectors $(\ell_t)_{t=1}^T$ in $[0,1]^K$ before the start of the game. The minimax regret of the full-information game is bounded from below by*

$$Reg^* = \Omega\left(\min_{S\subset[T]} |S| + \sqrt{D_{\bar S}\log(K)}\right)\text{, where } D_{\bar S} = \sum_{t\in[T]\setminus S} d_t\,.$$

*Proof.* We divide the time horizon greedily into $M$ buckets, such that the actions for all timesteps inside a bucket have to be chosen before the first feedback from any timestep inside the bucket is received. In other words, let bucket $B_m = \{b_m,\dots,b_{m+1}-1\}$, then $\forall t\in B_m : t+d_t > b_{m+1}-1$, while $\exists t\in B_m : t+d_t = b_{m+1}$. This division of buckets has the following properties:

*(i)* monotonically decreasing sizes: $|B_1| \ge |B_2| \ge \dots \ge |B_M|$.

*(ii)* upper bound on the sum of delays: $\forall m\in[M-1] : |B_m|^2 \ge \sum_{t\in B_{m+1}} d_t$.

Both properties follow directly from the non-decreasing nature of the delays.

$$\begin{aligned}
&|B_m| = b_{m+1} - b_m \le b_m + d_{b_m} - b_m = d_{b_m}\\
&|B_m| = \min_{t\in B_m}\{d_t + t - b_m\} \ge d_{b_{m+1}-1} + \min_{t\in B_m}\{t-b_m\} \ge d_{b_{m+1}-1}\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&|B_m| \ge d_{b_{m+1}-1} \ge d_{b_{m+1}} \ge |B_{m+1}|\,,\\
&\textstyle\sum_{t\in B_{m+1}} d_t \le |B_{m+1}|\cdot d_{b_{m+1}} \le |B_{m+1}|\cdot|B_m| \le |B_m|^2\,.
\end{aligned}$$

Set $S' = \bigcup_{m=1}^{\lfloor\log_2(K)\rfloor} B_m$ and let the adversary set all losses within a bucket to the same value, then the game reduces to a full information game over $M$ rounds with loss ranges $|B_1|, |B_2|, \dots, |B_M|$. Applying Theorem 9 yields

$$\begin{aligned}
Reg^* &\ge \max\left\{\frac{1}{2}\sum_{m=1}^{\lfloor\log_2(K)\rfloor}|B_m|, \frac{1}{32}\sqrt{\sum_{m=\lfloor\log_2(K)\rfloor}^{M}|B_m|^2\log(K)}\right\}\\
&\ge \max\left\{\frac{1}{2}|S'|, \frac{1}{32}\sqrt{\sum_{t\in\bar S'} d_t\log(K)}\right\} = \Omega\left(\min_{S\subset[T]}|S| + \sqrt{\sum_{t\in\bar S} d_t\log(K)}\right)\,.
\end{aligned}$$

■

## 7 Discussion

We have presented a best-of-both-worlds analysis of a slightly modified version of the algorithm of Zimmert and Seldin [2020] for bandits with delayed feedback. The key novelty of our analysis is the control of the drift of the playing distribution over arbitrary, but bounded, time intervals when the learning rate is changing over time. This control is necessary for best-of-both-worlds guarantees, but it is much more challenging than the drift control over fixed time intervals with fixed learning rate that appeared in prior work.

We also presented an adversarial regret lower bound matching the skipping-based refined regret upper bound of Zimmert and Seldin [2020] within constants.

Our work leads to several exciting open questions. The main one is whether skipping can be used to eliminate the need in oracle knowledge of $d_{max}$. If possible, this would remedy the deterioration of the adversarial bound by the additive factor of $d_{max}$, because the skipping threshold would be dominated by $\sqrt{D_{\bar S}\log K}$. Another open question is whether the $\frac{\sigma_{max}}{\Delta_i}$ term can be eliminated from the stochastic bound. Yet another open question is whether the $d_{max}$ factor in the stochastic bound can be reduced to $\sigma_{max}$ and whether the multiplicative terms dependent on $K$ can be eliminated. An extension of the results to first order bounds, that depend on the cumulative loss of the best action rather than $T$, and extension to arm dependent delays are also open questions. For now it was only done in the adversarial setting [Gyorgy and Joulani, 2021, Van Der Hoeven and Cesa-Bianchi, 2022].

## Acknowledgments and Disclosure of Funding

This project has received funding from European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 801199. YS acknowledges partial support by the Independent Research Fund Denmark, grant number 9040-00361B.

## References

Jacob D Abernethy, Chansoo Lee, and Ambuj Tewari. Fighting bandits with a new kind of smoothness. In *Advances in Neural Information Processing Systems (NeurIPS)*. 2015.

Peter Auer and Chao-Kai Chiang. An algorithm with nearly optimal pseudo-regret for both stochastic and adversarial bandits. In *Proceedings of the Conference on Learning Theory (COLT)*, 2016.

Peter Auer, Nicolò Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine Learning*, 47, 2002.

Peter Auer, Nicolò Cesa-Bianchi, Yoav Freund, and Robert E. Schapire. The nonstochastic multiarmed bandit problem. *SIAM Journal on Computing*, 32, 2002b.

Ilai Bistritz, Zhengyuan Zhou, Xi Chen, Nicholas Bambos, and Jose Blanchet. Online exp3 learning in adversarial bandits with delayed feedback. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2019.

Sébastien Bubeck and Aleksandrs Slivkins. The best of both worlds: Stochastic and adversarial bandits. In *Proceedings of the Conference on Learning Theory (COLT)*, 2012.

Nicol'o Cesa-Bianchi, Claudio Gentile, Yishay Mansour, and Alberto Minora. Delay and cooperation in nonstochastic bandits. In *Journal of Machine Learning Research*, 2019.

Andras Gyorgy and Pooria Joulani. Adapting to delays and data in adversarial multi-armed bandits. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2021.

Shinji Ito. Parameter-free multi-armed bandit algorithms with hybrid data-dependent regret bounds. In *Proceedings of the Conference on Learning Theory (COLT)*, 2021.

Pooria Joulani, Andras Gyorgy, and Csaba Szepesvari. Online learning under delayed feedback. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2013.

Tze Leung Lai and Herbert Robbins. Asymptotically efficient adaptive allocation rules. *Advances in Applied Mathematics*, 6, 1985.

Tor Lattimore and Csaba Szepesvári. *Bandit Algorithms*. Cambridge University Press, 2020.

Saeed Masoudian and Yevgeny Seldin. Improved analysis of the tsallis-inf algorithm in stochastically constrained adversarial bandits and stochastic bandits with adversarial corruptions. In *Proceedings of the Conference on Learning Theory (COLT)*, 2021.

Jaouad Mourtada and Stéphane Gaïffas. On the optimality of the hedge algorithm in the stochastic regime. *Journal of Machine Learning Research*, 20, 2019.

Francesco Orabona. A modern introduction to online learning. `https://arxiv.org/abs/1912.13213`, 2019.

Herbert Robbins. Some aspects of the sequential design of experiments. *Bulletin of the American Mathematical Society*, 58, 1952.

Yevgeny Seldin and Gábor Lugosi. An improved parametrization and analysis of the EXP3++ algorithm for stochastic and adversarial bandits. In *Proceedings of the Conference on Learning Theory (COLT)*, 2017.

Yevgeny Seldin and Aleksandrs Slivkins. One practical algorithm for both stochastic and adversarial bandits. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2014.

William R. Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. *Biometrika*, 25, 1933.

Tobias Sommer Thune, Nicolò Cesa-Bianchi, and Yevgeny Seldin. Nonstochastic multiarmed bandits with unrestricted delays. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2019.

Dirk Van Der Hoeven and Nicolò Cesa-Bianchi. Nonstochastic bandits and experts with arm-dependent delays. In *Proceedings on the International Conference on Artificial Intelligence and Statistics (AISTATS)*, 2022.

Chen-Yu Wei and Haipeng Luo. More adaptive algorithms for adversarial bandits. In *Proceedings of the Conference on Learning Theory (COLT)*, 2018.

Julian Zimmert and Yevgeny Seldin. An optimal algorithm for stochastic and adversarial bandits. In *Proceedings on the International Conference on Artificial Intelligence and Statistics (AISTATS)*, 2019.

Julian Zimmert and Yevgeny Seldin. An optimal algorithm for adversarial bandits with arbitrary delays. In *Proceedings on the International Conference on Artificial Intelligence and Statistics (AISTATS)*, 2020.

Julian Zimmert and Yevgeny Seldin. Tsallis-INF: An optimal algorithm for stochastic and adversarial bandits. *Journal of Machine Learning Research*, 2021.

# Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] All assumptions are stated in the statements of the theorems.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A] The main applications of our work are in theoretical and guarantees for Multi-armed bandit setting with delays. Multi-armed bandit is a very fundamental problem in sequential decision making which is base for many online learning problem. So this is not a relevant issue in our work.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes]
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [N/A]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [N/A]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A]
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [N/A]
   (b) Did you mention the license of the assets? [N/A]
   (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

# A Proofs of the lemmas for the analysis of Algorithm 1

## A.1 A proof of Lemma 3

*Proof.* Let $S \subseteq \{1, \ldots, T\}$ and $\bar{S} = \{1, \ldots, T\} \setminus S$ be an arbitrary split of the game rounds. Consider the number of outstanding observations $\sigma_t$ at an arbitrary round $t$. The number $\sigma_t$ is bounded by the sum of the number of outstanding observations from actions taken in the rounds in $S$ and the number of outstanding observations from actions taken in the rounds in $\bar{S}$. The former is bounded by $|S|$, and the latter is bounded by $d_{max}(\bar{S})$, since by definition of $d_{max}(\bar{S})$ any observation from an action taken in a round in $\bar{S}$ can be outstanding for at most $d_{max}(\bar{S})$ rounds. Since this holds for any split of the rounds $\{1, \ldots, T\}$ into $S$ and $\bar{S}$, we have $\sigma_{max} = \max_t \sigma_t \leq \min_{S \subseteq \{1,\ldots,T\}} \left(|S| + d_{max}(\bar{S})\right)$. ■

## A.2 Proofs of the lemmas supporting the proof of Theorem 1

We start with providing some auxiliary lemmas.

**Lemma 11** (Integral inequality: Lemma 4.13 of Orabona [2019]). *Let $g(x)$ be a positive nonincreasing function, then for any non-negative sequence $\{z_n\}_{n\in\{0,\ldots,N\}}$ we have*

$$\sum_{n=1}^{N} z_n g\left(\sum_{i=0}^{n} z_i\right) \leq \int_{z_0}^{\sum_{i=0}^{N} z_i} g(x)dx.$$

**Lemma 12.** *Let $\sigma_t$ and $\upsilon_t$ be the number of outstanding observations and arriving observations at time $t$, respectively, then the following inequality holds for all $t$*

$$\sum_{s=1}^{t} \sigma_s \geq \sum_{s=1}^{t} \frac{\upsilon_s^2 - \upsilon_s}{2}.$$

*Proof.* Note that $A_s = \{r : r + d_r = t\}$. We define $D_s = \{d_r : r \in A_s\}$ be the set of delays corresponding to observations that arrive at round $s$, then $D_s$ must have $\upsilon_s = |A_s|$ different number of elements, because $\forall r \in A_s : r + d_r = s$. As a result, we have

$$\sum_{r\in A_s} d_r \geq 0 + 1 + \ldots + (\upsilon_s - 1) = \frac{\upsilon_s(\upsilon_s - 1)}{2}.$$

This gives us the following inequality

$$\begin{aligned}\sum_{s=1}^{t} \frac{\upsilon_s^2 - \upsilon_s}{2} &\leq \sum_{s=1}^{t} \sum_{r\in A_s} d_r \\ &= \sum_{r: r+d_r \leq t} d_r.\end{aligned}$$

On the other hand, $\sum_{s=1}^{t} \sigma_s \geq \sum_{r:r+d_r\leq t} d_r$, since every observation from an action taken at round $r$ with delay $d_r$ counts as outstanding over $d_r$ rounds, i.e., contributes 1 to $\sigma_{r+1}, \ldots, \sigma_{r+d_r}$, and observations that have not arrived by round $t$ contribute only to the left hand side of the inequality. Together with the preceding inequality this completes the proof. ■

### A.2.1 A proof of Lemma 6

*Proof.* We bound $4aA - \overline{Reg}_T$.

$$\begin{aligned}4aA - \overline{Reg}_T &= \sum_{t=1}^{T} \sum_{i\neq i^*} \left(\frac{4ax_{t,i}^{\frac{1}{2}}}{\sqrt{t+\eta_0}} - x_{t,i}\Delta_i\right) \\ &\leq \sum_{t=1}^{T} \sum_{i\neq i^*} \frac{4a^2}{(t+\eta_0)\Delta_i} \leq \sum_{i\neq i^*} \frac{4a^2}{\Delta_i} \log(T/\eta_0 + 1) + 1, \qquad (14)\end{aligned}$$

where the first inequality uses the AM-GM inequality, by which for any $z$ and $y$ we have $z + y \geq 2\sqrt{zy} \Rightarrow 2\sqrt{zy} - y \leq z$. The second inequality follows by the integral bound on the harmonic series, by which $\sum_{t=1}^{T} 1/(t + \eta_0) \leq \log(T + \eta_0) - \log(\eta_0) + 1$. ■

### A.2.2 Proof of Lemma 7

*Proof.* We have

$$4bB - \overline{Reg}_T = \sum_{t=1}^{T} \sum_{i \neq i^*} x_{t,i} \Delta_i \left(4b(\upsilon_{t+d_t} - 1)\gamma_{t+d_t} - 1\right).$$

We define $T_0$ to be the first round $t$ with $\gamma_t^{-1} \geq 4b(\upsilon_{max} - 1)$, where $\upsilon_{max} = \max_{s \in [T]}\{\upsilon_s\}$. Then in the summation over time, the rounds with $t + d_t \geq T_0$ provide a negative contribution, since $4b(\upsilon_{t+d_t} - 1)\gamma_{t+d_t} - 1 \leq \frac{4b(\upsilon_{t+d_t}-1)}{4b(\upsilon_{max}-1)} - 1 \leq 0$. Therefore,

$$\begin{aligned} 4bB - \overline{Reg}_T &\leq \sum_{t+d_t < T_0} \sum_{i \neq i^*} x_{t,i} \Delta_i \left(4b(\upsilon_{t+d_t} - 1)\gamma_{t+d_t} - 1\right) \\ &\leq \sum_{t+d_t < T_0} 4b(\upsilon_{t+d_t} - 1)\gamma_{t+d_t} = \sum_{t=1}^{T_0 - 1} \sum_{s+d_s = t} 4b(\upsilon_t - 1)\gamma_t = \sum_{t=1}^{T_0-1} 4b\upsilon_t(\upsilon_t - 1)\gamma_t, \end{aligned} \tag{15}$$

where the second inequality holds because $\sum_{i \neq i^*} x_{t,i}\Delta_i \leq 1$ and $\upsilon_{t+d_t} \geq 1$. For simplicity of notation, we denote $\tilde{\upsilon}_t = \upsilon_t(\upsilon_t - 1)/2$, for which Lemma 12 gives us $\sum_{s=1}^{t} \tilde{\upsilon}_t \leq \sum_{s=1}^{t} \sigma_s$. Therefore, we have

$$\begin{aligned} \sum_{t=1}^{T_0-1} 4b\upsilon_t(\upsilon_t - 1)\gamma_t &\leq \sum_{t=1}^{T_0-1} \frac{8b\sqrt{\log K}\tilde{\upsilon}_t}{\sqrt{\sum_{s=1}^{t} \tilde{\upsilon}_t}} \\ &\leq 16b\sqrt{(\log K)\sum_{t=1}^{T_0-1} \tilde{\upsilon}_t} \leq 16b\sqrt{(\log K)\sum_{t=1}^{T_0-1} \sigma_t} \leq 16b(\log K)\gamma_{T_0-1}^{-1}, \end{aligned} \tag{16}$$

where the second inequality uses integral inequality Lemma 11 for $g(x) = \frac{1}{\sqrt{x}}$. Moreover, by the choice of $T_0$ we have $\gamma_{T_0-1}^{-1} \leq 4b(\upsilon_{max} - 1)$. Combining this with (15) and (16) gives us $4bB - \overline{Reg}_T \leq 64b^2\upsilon_{max}\log K$. ■

### A.2.3 Proof of Lemma 8

*Proof.* First, we remove $i^*$ from the summation in $C$ by using the following inequality

$$-x_{t,i^*}\log(x_{t,i^*}) \leq (1 - x_{t,i^*}) = \sum_{i \neq i^*} x_{t,i},$$

which follows by the fact that $z\log(z) + 1 - z$ is a decreasing function for $z \in [0,1]$, and the minimum value is zero, therefore, it is non-negative for $z \in [0,1]$. By using this inequality we have

$$\sum_{t=2}^{T}\sum_{i=1}^{K} \frac{-4c\sigma_t x_{t,i}\log(x_{t,i})}{\sqrt{(S_t + \gamma_0)\log K}} \leq 4c\underbrace{\sum_{t=1}^{T}\sum_{i \neq i^*} \frac{-\sigma_t x_{t,i}\log(x_{t,i})}{\sqrt{(S_t+\gamma_0)\log K}}}_{C_1} + 4c\underbrace{\sum_{t=1}^{T}\sum_{i \neq i^*} \frac{\sigma_t x_{t,i}}{\sqrt{(S_t+\gamma_0)\log K}}}_{C_2},$$

where $S_t = \sum_{s=1}^{t}\sigma_s$. We break the expression $4cC - \overline{Reg}_T$, into $4\left(cC_1 - \alpha\overline{Reg}_T\right) + 4\left(cC_2 - \beta\overline{Reg}_T\right)$, where $\alpha + \beta = 1/4$.

**Controlling** $cC_2 - \beta\overline{Reg}_T$

Let $\sigma_{max} = \max_{t \in [T]}\{\sigma_t\}$ and let $T_i$ be the first round $t$ when $S_t + \gamma_0 \geq \frac{c^2\sigma_{max}^2}{\beta^2\Delta_i^2\log K}$. Then for all $t \geq T_i$ we have

$$\frac{c\sigma_t x_{t,i}}{\sqrt{(S_t+\gamma_0)\log K}} - \beta x_{t,i}\Delta_i \leq 0.$$

Therefore, rounds after $T_i$ provide negative contribution to the summation, and we have

$$\begin{aligned}
cC_2 - \beta\overline{Reg}_T &\le \beta \sum_{i\ne i^*}\sum_{t=1}^{T_i-1} x_{t,i}\left(\frac{c\sigma_t}{\beta\sqrt{(S_t+\gamma_0)\log K}} - \Delta_i\right) \\
&\le \sum_{i\ne i^*}\sum_{t=1}^{T_i-1} \frac{c\sigma_t}{\sqrt{(S_t+\gamma_0)\log K}} \\
&\le \sum_{i\ne i^*} \frac{2c(\sqrt{S_{T_i-1}+\gamma_0}-\sqrt{\gamma_0})}{\sqrt{\log K}} \\
&\le \sum_{i\ne i^*} \frac{2c^2\sigma_{max}}{\beta\Delta_i\log K}, \qquad (17)
\end{aligned}$$

where the third inequality uses Lemma 11 for $g(x)=\frac{1}{\sqrt{x}}$ and the last inequality follows by the choice of $T_i$, which gives $S_{T_i-1}+\gamma_0 \le \frac{c^2\sigma_{max}^2}{\beta^2\Delta_i^2\log K}$.

**Controlling** $cC_1 - \alpha\overline{Reg}_T$

For $cC_1 - \alpha\overline{Reg}_T$, let $b_t = \frac{c\sigma_t}{\alpha\sqrt{(S_t+\gamma_0)\log K}}$, then

$$\begin{aligned}
cC_1 - \alpha\overline{Reg}_T &= \alpha\sum_{t=1}^{T}\sum_{i\ne i^*}\left(-b_t x_{t,i}\log(x_{t,i}) - \Delta_i x_{t,i}\right) \\
&\le \alpha\sum_{t=1}^{T}\sum_{i\ne i^*}\max_{z\in[0,1]}\left\{-b_t z\log(z) - \Delta_i z\right\}.
\end{aligned}$$

The function $g(z) = -b_t z\log(z) - \Delta_i z$ is a concave function for $z\in[0,1]$ and the maximum occurs when the derivative is zero. So we must have $-b_t\log(z) - b_t - \Delta_i = 0 \Rightarrow z = e^{-\frac{\Delta_i}{b_t}-1}$, and by substitution $\max_{z\in[0,1]} g(z) = b_t e^{-\frac{\Delta_i}{b_t}-1}$. Therefore,

$$\begin{aligned}
cC_1 - \alpha\overline{Reg}_T &\le \alpha\sum_{t=1}^{T}\sum_{i\ne i^*} b_t e^{-\frac{\Delta_i}{b_t}-1} \\
&= \sum_{i\ne i^*}\sum_{t=1}^{T}\frac{c\sigma_t}{\sqrt{(S_t+\gamma_0)\log K}}\exp\left(-\frac{\alpha\Delta_i\sqrt{(S_t+\gamma_0)\log K}}{c\sigma_t} - 1\right) \\
&\le \sum_{i\ne i^*}\sum_{t=1}^{T}\sigma_t \times \frac{c}{\sqrt{(S_t+\gamma_0)\log K}}\exp\left(-\frac{\alpha\Delta_i\sqrt{(S_t+\gamma_0)\log K}}{c\sigma_{max}} - 1\right),
\end{aligned}$$

where $\sigma_{max} = \max_{t\in[T]}\{\sigma_t\}$. Let $g_i(x) = \frac{c}{\sqrt{x\log K}}\exp\left(-\frac{\alpha\Delta_i\sqrt{x\log K}}{c\sigma_{max}} - 1\right)$, then for each $i$ we need to upper bound $\sum_{t=1}^{T}\sigma_t g_i(S_t+\gamma_0)$, which by Lemma 11 can be upper bounded by $\int_{\gamma_0}^{S_T+\gamma_0} g_i(x)dx$, because $g$ is nonincreasing. On the other hand, for any $\delta, a\ge 0$, we have $\int \frac{a}{\sqrt{x}}\exp(-\frac{\delta\sqrt{x}}{a}-1)dx = -\frac{2a^2}{\delta}\exp(-\frac{\delta\sqrt{x}}{a}-1)$. So, using the closed form of $\int g_i(x)dx$ with $\delta = \frac{\alpha\Delta_i}{\sigma_{max}}$, $a = \frac{c}{\sqrt{\log K}}$, we have

$$\begin{aligned}
cC_1 - \alpha\overline{Reg}_T &\le \sum_{i\ne i^*}\int_{\gamma_0}^{S_T+\gamma_0} g_i(x)dx \\
&= \sum_{i\ne i^*}\frac{-2c^2\sigma_{max}}{\alpha\Delta_i\log K}\exp\left(-\frac{\alpha\Delta_i\sqrt{x\log K}}{c\sigma_{max}} - 1\right)\Bigg|_{x=\gamma_0}^{x=S_T+\gamma_0} \\
&= \frac{2c^2\sigma_{max}\left(\exp\left(-\frac{\alpha\Delta_i\sqrt{\gamma_0\log K}}{c\sigma_{max}} - 1\right) - \exp\left(-\frac{\alpha\Delta_i\sqrt{(S_T+\gamma_0)\log K}}{c\sigma_{max}} - 1\right)\right)}{\alpha\Delta_i\log K} \\
&\le \sum_{i\ne i^*}\frac{2c^2\sigma_{max}}{\alpha\Delta_i\log K}. \qquad (18)
\end{aligned}$$

Taking together (17) and (18) gives us

$$4cC - \overline{Reg}_T \leq \sum_{i \neq i^*} \frac{8c^2\sigma_{max}}{\Delta_i \log K}\left(\frac{1}{\beta} + \frac{1}{\alpha}\right) = \sum_{i \neq i^*} \frac{8c^2\sigma_{max}}{\Delta_i \log K}\left(\frac{1}{1/4 - \alpha} + \frac{1}{\alpha}\right)$$
$$\leq \sum_{i \neq i^*} \frac{128c^2\sigma_{max}}{\Delta_i \log K}, \tag{19}$$

where the second inequality uses $\alpha = \frac{1}{8}$.

■

## A.3 Proof of the stability lemma

The lemma has two parts, the first part is the general bound for the stability term and the second is a special case of that bound where we set $\alpha$ to a specific value to get the desirable bound.

Before starting the proof we provide one fact and one lemma that help us in the proof of the stability lemma. We recall that our regularization function is $F_t(x) = \sum_{i=1}^K f_t(x)$, where $f_t(x) = -2\eta_t^{-1}\sqrt{x} + \gamma_t^{-1}x(\log x - 1)$.

**Fact 13** ([Zimmert and Seldin, 2020]). *$f_t^{*'}(x)$ is a convex monotonically increasing function.*

*Proof.* The proof is available in Section 7.3 of the supplementary material of Zimmert and Seldin [2020]. ■

**Lemma 14.** *Let $D_F(x,y) = F(x) - F(y) - \langle x - y, \nabla F(y)\rangle$ be the Bergman divergence of a function $F$. Then for any $x \in$ **dom**$(f_t)$, and any $\ell$ such that $\ell \geq -\gamma_t^{-1}$:*

$$D_{f_t^*}(f_t^{'}(x) - \ell, f_t^{'}(x)) \leq \frac{\ell^2}{2f_t^{''}(ex)}.$$

*Moreover, it is easy to see $(f_t^{''}(ex))^{-1} \leq 4(f_t^{''}(x))^{-1}$, which implies $D_{f_t^*}(f_t^{'}(x) - \ell, f_t^{'}(x)) \leq \frac{2\ell^2}{f_t^{''}(x)}$.*

*Proof.* By Taylor's theorem there exists $\tilde{x} \in \left[f_t^{*'}(f_t^{'}(x) - \ell), f_t^{*'}(f_t^{'}(x))\right]$, such that

$$D_{f_t^*}(f_t^{'}(x) - \ell, f_t^{'}(x)) = \frac{1}{2}\ell^2 f_t^{*''}(f_t^{'}(\tilde{x})) = \frac{1}{2}\ell^2 f_t^{''}(\tilde{x})^{-1},$$

where the second equality is a property of the convex conjugate operation. We have two cases for $\ell$:

1. If $\ell \geq 0$, then based on Fact 13 we know that $f_t^{*'}$ is increasing, so $\tilde{x} \leq x$. On the other hand, $f^{''}(x)^{-1}$ is increasing, so $f_t^{''}(\tilde{x})^{-1} \leq f_t^{''}(x)^{-1} \leq f_t^{''}(ex)^{-1}$.

2. If $\ell < 0$, then $\tilde{x} \in \left[f_t^{*'}(f_t^{'}(x)), f_t^{*'}(f_t^{'}(x) - \ell)\right]$. We show that $f_t^{*'}(f_t^{'}(x) - \ell) \leq ex$, which by the choice of $\tilde{x}$ implies $\tilde{x} \leq ex$, and consequently, like in the other case, we end up having $f_t^{''}(\tilde{x})^{-1} \leq f_t^{''}(ex)^{-1}$.

   Since $f^{*'}$ is increasing and $ex = f^{*'}(f^{'}(ex))$, it suffices to prove that $f^{'}(ex) \geq f^{'}(x) - \ell$, or, equivalently, $f^{'}(ex) - f^{'}(x) \geq -\ell$. So

$$f^{'}(ex) - f^{'}(x) = \left(-\eta_t^{-1}(ex)^{-1/2} + \gamma_t^{-1}\log(ex)\right) - \left(-\eta_t^{-1}x^{-1/2} + \gamma_t^{-1}\log(x)\right)$$
$$= \eta_t^{-1}x^{-1/2}\left(1 - \frac{1}{\sqrt{2}}\right) + \gamma_t^{-1} \geq \gamma_t^{-1} \geq -\ell$$

■

***Proof of the First Part of the Stability Lemma.*** We have $x_t = \arg\min_{x \in \Delta^{K-1}} \langle \hat{L}_t^{obs}, x\rangle + F_t(x)$, so by the KKT conditions there exists $c_0 \in \mathbb{R}$, such that $-\hat{L}_t^{obs} = \nabla F_t(x_t) - c_0\mathbf{1}_K$. On the other hand, $\bar{F}_t(-L + c\mathbf{1}_K) = \bar{F}_t(-L) + c$ for any $c \in \mathbb{R}$ and $L \in \mathbb{R}^K$ and the equality holds iff $c = 0$. Therefore, using these two facts we can rewrite the stability term as

$$
\begin{aligned}
\sum_{t=1}^{T}\langle x_t,\hat{\ell}_t^{obs}\rangle+\bar{F}_t^*(-\hat{L}_{t+1}^{obs})-\bar{F}_t^*(-\hat{L}_t^{obs}) &= \sum_{t=1}^{T}\langle x_t,\hat{\ell}_t^{obs}-\alpha_t\mathbf{1}_K\rangle+\bar{F}_t^*(-\hat{L}_{t+1}^{obs}+(\alpha_t+c_0)\mathbf{1}_K)-\bar{F}_t^*(-\hat{L}_t^{obs}+c_0\mathbf{1}_K)\\
&= \sum_{t=1}^{T}\langle x_t,\hat{\ell}_t^{obs}-\alpha_t\mathbf{1}_K\rangle+\bar{F}_t^*(\nabla F_t(x_t)-(\hat{\ell}_t^{obs}-\alpha_t\mathbf{1}_K))-\bar{F}_t^*(\nabla F_t(x_t))\\
&\le \sum_{t=1}^{T}\langle x_t,\hat{\ell}_t^{obs}-\alpha_t\mathbf{1}_K\rangle+F_t^*(\nabla F_t(x_t)-(\hat{\ell}_t^{obs}-\alpha_t\mathbf{1}_K))-F_t^*(\nabla F_t(x_t))\\
&= \sum_{i=1}^{K}D_{f_t^*}\left(f_t^{'}(x_{t,i})-(\hat{\ell}_{t,i}^{obs}-\alpha_t),f_t^{'}(x_{t,i})\right), \qquad (20)
\end{aligned}
$$

where the inequality holds because $\bar{F}_t^*(L)\le F_t^*(L)$ for all $L\in\mathbb{R}^K$ and $\bar{F}_t^*(\nabla F_t(x))=F_t^*(\nabla F_t(x))$ for all $x\in\mathbb{R}^K$. Hence, since $\alpha_t\le\gamma_t^{-1}$, we have $\hat{\ell}_{t,i}^{obs}-\alpha_t\ge-\alpha_t\ge-\gamma_t^{-1}$. This implies that we can apply Lemma 14 to get the following bound for (20)

$$
stability\le\sum_{i=1}^{K}2f_t^{''}(x_{t,i})^{-1}(\hat{\ell}_{t,i}^{obs}-\alpha_t)^2.
$$

■

***Proof of the Second Part of the Stability Lemma***. First, we must check whether $\alpha_t=\frac{\sum_{j=1}^K f^{''}(x_{t,j})^{-1}\tilde{\ell}_{t,j}}{\sum_{j=1}^K f^{''}(x_{t,j})^{-1}}$ satisfies $\alpha_t\le\gamma_t^{-1}$ or not:

$$
\begin{aligned}
\alpha_t &= \frac{\sum_{j=1}^K f^{''}(x_{t,j})^{-1}\tilde{\ell}_{t,j}}{\sum_{j=1}^K f^{''}(x_{t,j})^{-1}}\\
&= \frac{\sum_{j=1}^K f^{''}(x_{t,j})^{-1}\sum_{s\in A_t}\hat{\ell}_{s,j}}{\sum_{j=1}^K f^{''}(x_{t,j})^{-1}}\\
&\le 8|A_t|(K-1)^{\frac{1}{3}}\le 8d_{max}(K-1)^{\frac{1}{3}}\le\gamma_t^{-1},
\end{aligned}
$$

where the first inequality uses Lemma 17. To simplify the analysis, for all $i$ let $z_i=f_t^{''}(x_{t,i})^{-1}$, then by substitution of the value of $\alpha_t$ in the stability expression we have

$$
\begin{aligned}
\sum_{i=1}^{K}z_i(\tilde{\ell}_{t,i}-\alpha_t)^2 &= \sum_{i=1}^{K}z_i\tilde{\ell}_{t,i}^2-2\sum_{i=1}^{K}z_i\tilde{\ell}_{t,i}\alpha_t+\sum_{i=1}^{K}z_i\alpha_t^2\\
&= \sum_{i=1}^{K}z_i\tilde{\ell}_{t,i}^2-\frac{(\sum_{i=1}^K z_i\tilde{\ell}_{t,i})^2}{\sum_{i=1}^K z_i}\\
&= \sum_{i=1}^{K}z_i\tilde{\ell}_{t,i}^2-\frac{\sum_{i=1}^K z_i^2\tilde{\ell}_{t,i}^2}{\sum_{i=1}^K z_i}-\frac{\sum_{i,j,i\neq j}z_iz_j\tilde{\ell}_{t,i}\tilde{\ell}_{t,j}}{\sum_{i=1}^K z_i}\\
&= \sum_{i=1}^{K}\left(z_i-\frac{z_i^2}{\sum_{j=1}^K z_j}\right)\left(\sum_{s\in A_t}\hat{\ell}_{s,i}\right)^2-\frac{\sum_{i,j,i\neq j}z_iz_j\left(\sum_{r,s\in A_t}\hat{\ell}_{r,i}\hat{\ell}_{s,j}\right)}{\sum_{i=1}^K z_i}\\
&= \sum_{i=1}^{K}\left(z_i-\frac{z_i^2}{\sum_{j=1}^K z_j}\right)\left(\sum_{s\in A_t}\hat{\ell}_{s,i}^2\right) \qquad (21)\\
&+\sum_{i=1}^{K}\left(z_i-\frac{z_i^2}{\sum_{j=1}^K z_j}\right)\left(\sum_{r,s\in A_t,r\neq s}\hat{\ell}_{r,i}\hat{\ell}_{s,i}\right)-\frac{\sum_{i,j,i\neq j}z_iz_j\left(\sum_{r,s\in A_t}\hat{\ell}_{s,i}\hat{\ell}_{r,j}\right)}{\sum_{i=1}^K z_i}. \qquad (22)
\end{aligned}
$$

We call the term in line (21) Stab1 and the two terms in line (22) Stab2. We first bound the expectation of Stab1.

$$
\begin{aligned}
\mathbb{E}[\text{Stab1}] &= \mathbb{E}\left[\sum_{i=1}^{K}\left(z_i - \frac{z_i^2}{\sum_{i=1}^{K} z_i}\right)\left(\sum_{s\in A_t} \hat{\ell}_{s,i}^2\right)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{K}\left(z_i - \frac{z_i^2}{\sum_{i=1}^{K} z_i}\right)\left(\sum_{s\in A_t} \mathbb{E}_s[\hat{\ell}_{s,i}^2]\right)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{K}\left(z_i - \frac{z_i^2}{\sum_{i=1}^{K} z_i}\right)\left(\sum_{s\in A_t} \ell_{s,i}^2 x_{s,i}^{-1}\right)\right] \\
&\le \sum_{s\in A_t} \mathbb{E}\left[\sum_{i=1}^{K} z_i x_{s,i}^{-1} - \frac{\sum_{i=1}^{K} z_i^2 x_{s,i}^{-1}}{\sum_{i=1}^{K} z_i}\right] \\
&\le \sum_{s\in A_t} \mathbb{E}\left[\sum_{i=1}^{K} z_i x_{s,i}^{-1}(1 - x_{s,i})\right] \\
&\le \sum_{s\in A_t} \mathbb{E}\left[\sum_{i=1}^{K} 2\eta_t x_{t,i}^{3/2} x_{s,i}^{-1}(1 - x_{s,i})\right],
\end{aligned} \tag{23}
$$

where the first inequality bounds losses by one and changes the order of summations, the second inequality uses Cauchy-Schwarz inequality $\sum_{i=1}^{K} z_i^2 x_{s,i}^{-1} = \left(\sum_{i=1}^{K} z_i^2 x_{s,i}^{-1}\right)\underbrace{\left(\sum_{i=1}^{K} x_{s,i}\right)}_{=1} \ge \left(\sum_{i=1}^{K} z_i\right)^2$, and the last inequality uses the fact that $z_i = f_t^{''}(x_{t,i})^{-1} \le 2\eta_t x_{t,i}^{3/2}$.

For Stab2 we have

$$
\begin{aligned}
\mathbb{E}[\text{Stab2}] &= \mathbb{E}\left[\frac{1}{\sum_{i=1}^{K} z_i}\left(\sum_{i=1}^{K}\sum_{r,s\in A_t, r\neq s}\sum_{j\neq i} z_i z_j \hat{\ell}_{r,i}\hat{\ell}_{s,i} - \sum_{i,j,i\neq j}\sum_{r,s\in A_t} z_i z_j \hat{\ell}_{s,i}\hat{\ell}_{r,j}\right)\right] \\
&= \mathbb{E}\left[\frac{1}{\sum_{i=1}^{K} z_i}\left(\sum_{i=1}^{K}\sum_{r,s\in A_t, r\neq s}\sum_{j\neq i} z_i z_j \mu_i^2 - \sum_{i,j,i\neq j}\sum_{r,s\in A_t} z_i z_j \mu_i\mu_j\right)\right] \\
&= \mathbb{E}\left[\frac{1}{\sum_{i=1}^{K} z_i}\left(\upsilon_t(\upsilon_t - 1)\sum_{i=1}^{K}\sum_{j\neq i} z_i z_j \mu_i^2 - \upsilon_t^2 \sum_{i,j,i\neq j} z_i z_j \mu_i\mu_j\right)\right] \qquad (24) \\
&\le \mathbb{E}\left[\frac{\upsilon_t(\upsilon_t - 1)}{\sum_{i=1}^{K} z_i}\left(\sum_{i=1}^{K} z_i(\sum_{j=1}^{K} z_j)\mu_i^2 - \sum_{i=1}^{K} z_i^2\mu_i^2 - \sum_{i,j,i\neq j} z_i z_j \mu_i\mu_j\right)\right] \\
&= \mathbb{E}\left[\frac{\upsilon_t(\upsilon_t - 1)}{\sum_{i=1}^{K} z_i}\left((\sum_{i=1}^{K} z_i\mu_i^2)(\sum_{i=1}^{K} z_i) - (\sum_{i=1}^{K} z_i\mu_i)^2\right)\right] \\
&\le \mathbb{E}\left[\frac{\upsilon_t(\upsilon_t - 1)}{\sum_{i=1}^{K} z_i}\left((\sum_{i=1}^{K} z_i\mu_i^2)(\sum_{i=1}^{K} z_i) - (\sum_{i=1}^{K} z_i)^2\mu_{i^*}^2\right)\right] \\
&= \mathbb{E}\left[\upsilon_t(\upsilon_t - 1)(\sum_{i=1}^{K} z_i\mu_i^2 - \sum_{i=1}^{K} z_i\mu_{i^*}^2)\right] \\
&\le \mathbb{E}\left[\upsilon_t(\upsilon_t - 1)(\sum_{i\neq i^*} 2z_i\Delta_i)\right] \\
&\le \mathbb{E}\left[\sum_{i\neq i^*} 2\upsilon_t(\upsilon_t - 1)\gamma_t x_{t,i}\Delta_i\right],
\end{aligned} \tag{25}
$$

where the second equality follows by the fact that for all $s \in A_t$, $x_s$ has no impact on $x_t$, and for all different elements of $A_t$, such as $r, s \in A_t$ and $r < s$, $x_r$ has no impact on $x_s$. Regarding the inequalities, the first one follows by $\upsilon_t^2 \ge \upsilon_t(\upsilon_t - 1)$, the second one holds because for all $i$ we have $\mu_i^* \le \mu_i$, the third inequality follows by $\mu_i + \mu_{i^*} \le 2$ and $\mu_i - \mu_{i^*} = \Delta_i$, and the last one substitutes $z_i = f^{''}(x_{t,i})^{-1} \le \gamma_t x_{t,i}$.

Combining (23) and (25) completes the proof. ■

# B Proof of the Key Lemma

## B.1 Auxiliary results for the proof of the key lemma

First, we provide two facts and a lemma, which are needed for the proof of the key lemma. We recall that $f_t(x) = -2\eta_t^{-1}\sqrt{x} + \gamma_t^{-1}x(\log x - 1)$.

**Fact 15.** *$f_t^{'}(x)$ is a concave function.*

*Proof.* $f_t^{'}(x) = -\eta^{-1}x^{-1/2} + \gamma_t^{-1}\log x$, so the second derivative is $-\frac{3}{4}\eta^{-1}x^{-5/2} - \gamma_t^{-1}x^{-2} \le 0$. ■

**Fact 16.** *$f_t^{''}(x)^{-1}$ is a convex function.*

*Proof.* Let $g(x) = f_t^{''}(x)^{-1} = (\frac{\eta_t^{-1}x^{-3/2}}{2} + \gamma_t^{-1}x^{-1})^{-1}$, then the second derivative of $g(x)$ is

$$g^{''}(x) = \frac{\eta_t\gamma_t^2 \cdot \left(2\eta_t x^{\frac{7}{2}} + 3\gamma_t x^3\right)}{2\sqrt{x}\left(2\eta_t x^{\frac{3}{2}} + \gamma_t x\right)^3},$$

which is positive. ■

**Lemma 17.** *Fix $t$ and $s$ where $t \ge s$, and assume that there exists $\alpha$, such that $x_{t,i} \le \alpha x_{s,i}$ for all $i \in [K]$, and let $f(x) = \left(-2\eta_t^{-1}\sqrt{x} + \gamma_t^{-1}x(\log x - 1)\right)$, then we have the following inequality*

$$\frac{\sum_{j=1}^K f^{''}(x_{t,j})^{-1}\hat{\ell}_{s,j}}{\sum_{j=1}^K f^{''}(x_{t,j})^{-1}} \le 2\alpha(K-1)^{\frac{1}{3}}.$$

*Proof for Lemma 17.* We begin the proof as the following

$$\begin{aligned}
\frac{\sum_{i=1}^K f^{''}(x_{t,i})^{-1}\hat{\ell}_{s,i}}{\sum_{i=1}^K f^{''}(x_{t,i})^{-1}} &= \frac{f^{''}(x_{t,i_s})^{-1}x_{s,i_s}^{-1}\ell_{s,i_s}}{\sum_{i=1}^K f^{''}(x_{t,i})^{-1}} \\
&\le \frac{f^{''}(x_{t,i_s})^{-1}x_{t,i_s}^{-1}\,(x_{t,i_s}/x_{s,i_s})}{\sum_{i=1}^K f^{''}(x_{t,i})^{-1}} \\
&\le \frac{f^{''}(x_{t,i_s})^{-1}\alpha x_{t,i_s}^{-1}}{\sum_{i=1}^K f^{''}(x_{t,i})^{-1}} \\
&\le \frac{\alpha f^{''}(x_{t,i_s})^{-1}x_{t,i_s}^{-1}}{(K-1)f^{''}\left(\frac{1-x_{t,i_s}}{K-1}\right)^{-1} + f^{''}(x_{t,i_s})^{-1}} \quad \text{Define } z := x_{t,i_s} \\
&= \frac{\alpha\left(\eta_t^{-1}z^{-3/2} + 2\gamma_t^{-1}z^{-1}\right)^{-1}z^{-1}}{(K-1)\left(\eta_t^{-1}(\frac{1-z}{K-1})^{-3/2} + 2\gamma_t^{-1}(\frac{1-z}{K-1})^{-1}\right)^{-1} + \left(\eta_t^{-1}z^{-3/2} + 2\gamma_t^{-1}z^{-1}\right)^{-1}} \\
&= \alpha\left((1-z)\frac{\eta_t^{-1}z^{-1/2} + 2\gamma_t^{-1}}{\eta_t^{-1}\sqrt{K-1}(1-z)^{-1/2} + 2\gamma_t^{-1}} + z\right)^{-1},
\end{aligned} \tag{26}$$

where the first inequality follows by $\ell_{s,i_s} \le 1$, the second one uses the assumption of the lemma that $x_{t,i} \le \alpha x_{s,i}$, and the third inequality is due to convexity of $f^{''}(x)^{-1}$ from Fact 16. We consider two cases for $z$: $z < \frac{1}{K}$ and $z \ge \frac{1}{K}$.

a) $z \le \frac{1}{K}$: This case implies

$$\begin{aligned}
\frac{1-z}{z} = \frac{1}{z} - 1 \ge K - 1 &\Rightarrow (1-z)^{-1/2}\sqrt{K-1} \le z^{-1/2} \\
&\Rightarrow 1 \le \frac{\eta_t^{-1}z^{-1/2} + 2\gamma_t^{-1}}{\eta_t^{-1}\sqrt{K-1}(1-z)^{-1/2} + 2\gamma_t^{-1}}.
\end{aligned} \tag{27}$$

Plugging (27) into (26) gives

$$\frac{\sum_{i=1}^K f''(x_{t,i})^{-1}\hat{\ell}_{s,i}}{\sum_{i=1}^K f''(x_{t,i})^{-1}} \leq \alpha\left(1-z+z\right)^{-1} = \alpha.$$

b) $z \geq \frac{1}{K}$: Similar to the previous case, $z \geq \frac{1}{K}$ implies $\eta_t^{-1}z^{-1/2} \leq \eta_t^{-1}\sqrt{K-1}(1-z)^{-1/2}$, so the minimum of $\frac{\eta_t^{-1}z^{-1/2}+2\gamma_t^{-1}}{\eta_t^{-1}\sqrt{K-1}(1-z)^{-1/2}+2\gamma_t^{-1}}$ occurs when $2\gamma_t^{-1} = 0$. Substitution of $2\gamma_t^{-1} = 0$ in (26) gives

$$\frac{\sum_{i=1}^K f''(x_{t,i})^{-1}\hat{\ell}_{s,i}}{\sum_{i=1}^K f''(x_{t,i})^{-1}} \leq \alpha\left((1-z)^{3/2}z^{-1/2}(K-1)^{-1/2}+z\right)^{-1}. \tag{28}$$

Here we have the following two subcases

b1) $z \geq \frac{1}{(K-1)^{1/3}+1}$: This gives

$$\alpha\left((1-z)^{3/2}z^{-1/2}(K-1)^{-1/2}+z\right)^{-1} \leq \alpha z^{-1} \leq \alpha\left((K-1)^{1/3}+1\right) \leq 2\alpha(K-1)^{1/3}.$$

b2) $z \leq \frac{1}{(K-1)^{1/3}+1}$: This implies $(1-z) \geq \frac{(K-1)^{1/3}}{(K-1)^{1/3}+1} \geq \frac{1}{2}$ and we can use it in (28) in the following way

$$\begin{aligned}\alpha\left((1-z)^{3/2}z^{-1/2}(K-1)^{-1/2}+z\right)^{-1} &\leq \alpha\left(\frac{z^{-1/2}(K-1)^{-1/2}}{\sqrt{8}}+z\right)^{-1}\\ &= \alpha\left(\frac{z^{-1/2}(K-1)^{-1/2}}{2\sqrt{8}}+\frac{z^{-1/2}(K-1)^{-1/2}}{2\sqrt{8}}+z\right)^{-1}\\ &\leq \frac{\alpha}{3}\left(\frac{(K-1)^{-1}}{32}\right)^{-1/3} \leq 2\alpha(K-1)^{1/3},\end{aligned}$$

where the second inequality is by the AM-GM inequality.

Combining the results for all cases and setting $\alpha = 4$ we obtain the upper bound $8(K-1)^{1/3}$. ■

## B.2 Proof of the key lemma

*Proof of Lemma 4.* To show $x_{t,i} \leq 2x_{s,i}$ for all $i$ we do induction on *valid* pairs $(t,s)$, where we call a pair $(t,s)$ valid if $s \leq t$ and $t-s \leq d_{max}$. The induction step for $(t,s)$ uses the induction assumption for all valid pairs $(t',s')$, such that $s',t' < t$, and all valid pairs $(t',s')$, such that $t'=t$ and $s < s' \leq t$. Thus, the induction base would be all the pairs of $(t',t')$ for all $t' \in [T]$, for which the statement $x_{t',i} \leq 2x_{t',i}$ trivially holds. Hence, it suffices to prove the induction step for the valid pair $(t,s)$.

As we mentioned in the proof sketch, we have $x_t = \bar{F}_t^*(-\hat{L}_t^{obs})$ and $x_s = \bar{F}_s^*(-\hat{L}_s^{obs})$, and we introduce $\tilde{x} = \bar{F}_s^*(-\hat{L}_t^{obs})$ as an auxiliary variable to bridge from $x_t$ and $x_s$. We bridge from $x_t$ to $x_s$ via $\tilde{x}$ in the following way.

**Deviation Induced by the Loss Shift:** This step controls the drift when we fix the regularization (more precisely, the learning rates) and shift the cumulative loss. We prove the following inequality:

$$\tilde{x}_i \leq \frac{3}{2}x_{s,i}.$$

Note that this step uses the induction assumption for $(s, s-d_r)$ for all $r < s : r+d_r = s$.
**Deviation Induced by the Change of Regularizer:** In this step we bound the drift when the cumulative loss vector is fixed and we change the regularizer. We show that

$$x_{t,i} \leq \frac{4}{3}\tilde{x}_i.$$

### Deviation induced by the change of regularizer

The regularizer at any round $r$ is $F_r(x) = \sum_{i=1}^K f_r(x_i) = \sum_{i=1}^K\left(-2\eta_r^{-1}\sqrt{x_i}+\gamma_r^{-1}x_i(\log x_i - 1)\right)$. Since $x_t = \nabla\bar{F}_t^*(-\hat{L}_t^{obs})$ and $\tilde{x} = \nabla\bar{F}_s^*(-\hat{L}_t^{obs})$, by the KKT conditions $\exists\mu,\tilde{\mu}$ s.t. $\forall i$:

$$f_s'(\tilde{x}_i) = -L_{s,i}^{obs}+\mu,$$
$$f_t'(x_{t,i}) = -L_{t,i}^{obs}+\tilde{\mu}.$$

We also know that $\exists j : \tilde{x}_j \geq x_{t,j}$ which leads to

$$-L^{obs}_{t,j} + \tilde{\mu} = f_t^{'}(x_{t,j}) \leq f_s^{'}(x_{t,j}) \leq f_s^{'}(\tilde{x}_j) = -L^{obs}_{s,j} + \mu,$$

where the first inequality holds because the learning rates are decreasing, and the second inequality is due to the fact that $f_s^{'}(x)$ is increasing. This implies that $\tilde{\mu} \leq \mu$, which gives us the following inequality for all $i$:

$$f_t^{'}(x_{t,i}) = -\frac{1}{\eta_t\sqrt{x_{t,i}}} + \frac{\log(x_{t,i})}{\gamma_t} \leq -\frac{1}{\eta_s\sqrt{\tilde{x}_i}} + \frac{\log(\tilde{x}_i)}{\gamma_s} = f_s^{'}(\tilde{x}_i).$$

Define $\alpha = x_{t,i}/\tilde{x}_i$. Using the above inequality we have

$$\begin{aligned}
\frac{1}{\eta_s\sqrt{\tilde{x}_i}} - \frac{\log(\tilde{x}_i)}{\gamma_s} &\leq \frac{1}{\eta_t\sqrt{\alpha\tilde{x}_i}} - \frac{\log(\tilde{x}_i)}{\gamma_t} - \frac{\log(\alpha)}{\gamma_t} \qquad \text{(multiply both sides by } \eta_t\sqrt{\tilde{x}_i} \text{ and rearrange)}\\
\Rightarrow \frac{1}{\sqrt{\alpha}} &\geq \frac{\eta_t}{\eta_s} + 2\sqrt{\tilde{x}_i}\log(\sqrt{\tilde{x}_i})\left(\frac{\eta_t}{\gamma_t} - \frac{\eta_t}{\gamma_s}\right) + \log(\alpha)\frac{\eta_t}{\gamma_t}\sqrt{\tilde{x}_i}\\
&\geq \frac{\eta_t}{\eta_s} + \min_{0\leq z\leq 1}\left\{2z\log(z)\left(\frac{\eta_t}{\gamma_t} - \frac{\eta_t}{\gamma_s}\right) + \log(\alpha)\frac{\eta_t}{\gamma_t}z\right\}\\
&\overset{(a)}{=} \frac{\eta_t}{\eta_s} - \frac{2}{e}\left(\frac{\eta_t}{\gamma_t} - \frac{\eta_t}{\gamma_s}\right)\left(\frac{1}{\sqrt{\alpha}}\right)^{\frac{\gamma_t^{-1}}{\gamma_t^{-1}-\gamma_s^{-1}}}\\
&\overset{(b)}{\geq} \frac{\eta_t}{\eta_s} - \left(\frac{\eta_t}{\gamma_t} - \frac{\eta_t}{\gamma_s}\right)\frac{1}{\sqrt{\alpha}},
\end{aligned}$$

where (a) holds because the minimized function is convex and equating the first derivative to zero gives $z = \left(\frac{1}{\sqrt{\alpha}}\right)^{\frac{\gamma_t^{-1}}{\gamma_t^{-1}-\gamma_s^{-1}}}$, and (b) follows by $\frac{\gamma_t^{-1}}{\gamma_t^{-1}-\gamma_s^{-1}} \geq 1$ and $e \geq 2$. Rearranging the above result gives

$$\alpha \leq \left(\frac{\eta_s}{\gamma_t} - \frac{\eta_s}{\gamma_s} + \frac{\eta_s}{\eta_t}\right)^2 = \left(\eta_s(\gamma_t^{-1} - \gamma_s^{-1}) + \frac{\eta_s}{\eta_t}\right)^2. \tag{29}$$

Now we need to substitute the closed form of learning rates to obtain an upper bound for $\alpha$. As a reminder, the learning rates are

$$\gamma_s^{-1} = \frac{1}{\sqrt{\log K}}\sqrt{\sum_{r=1}^{s}\sigma_r + \gamma_0},\ \eta_s^{-1} = \sqrt{s+\eta_0},$$

$$\gamma_t^{-1} = \frac{1}{\sqrt{\log K}}\sqrt{\sum_{r=1}^{s+d}\sigma_r + \gamma_0},\ \eta_t^{-1} = \sqrt{s+d+\eta_0},$$

where $d = t - s$, $\eta_0 = 10d_{max} + d_{max}^2/\left(K^{1/3}\log(K)\right)^2$, and $\gamma_0 = 24^2 d_{max}^2 K^{2/3}\log(K)$. Therefore, in (29) we have

$$\begin{aligned}
\eta_s\left(\gamma_t^{-1} - \gamma_s^{-1}\right) &\leq \eta_s\frac{\sum_{r=s+1}^{s+d}\sigma_r}{\sqrt{\log(K)\left(\sum_{r=1}^{s+d}\sigma_r + \gamma_0\right)}}\\
&\leq \eta_s\frac{\sum_{r=s+1}^{s+d}\sigma_r}{\sqrt{\log(K)\gamma_0}}\\
&\leq \frac{d_{max}^2}{\sqrt{\log(K)\gamma_0\eta_0}} \leq \frac{d_{max}^2}{\sqrt{24^2 d_{max}^4}} = \frac{1}{24},
\end{aligned} \tag{30}$$

where the third inequality follows by $d, \sigma_r \leq d_{max}$ for all $r$ and $\eta_s \leq \frac{1}{\sqrt{\eta_0}}$, and the last inequality holds because $\eta_0 \geq 16d_{max}^2/K^{2/3}$. On the other hand, for $\frac{\eta_s}{\eta_t}$ in (29) we have

$$\begin{aligned}
\frac{\eta_s}{\eta_t} = \sqrt{\frac{s+d+\eta_0}{s+\eta_0}} &= \sqrt{1 + \frac{d}{s+\eta_0}}\\
&\leq \sqrt{1 + \frac{d}{10d_{max}}}\\
&\leq \sqrt{1 + \frac{d_{max}}{10d_{max}}} = \sqrt{\frac{11}{10}},
\end{aligned} \tag{31}$$

where the first and the second inequalities hold because $\eta_0 \geq 10d_{max}$ and $d \leq d_{max}$, respectively. Plugging (30) and (31) into (29) gives us the following bound for $\alpha$:

$$\alpha \leq \left(\sqrt{\frac{11}{10}} + \frac{1}{24}\right)^2 \leq \frac{4}{3}. \tag{32}$$

### Deviation Induced by the Loss Shift

We have $x_s = \nabla \bar{F}_s^*(-L_s^{obs})$ and $\tilde{x} = \nabla \bar{F}_s^*(-L_t^{obs})$. Since they both share the same regularizer $F_s(x) = \sum_{i=1}^K f_s(x_i)$, to simplify the notation we drop $s$ and use $f(x)$ to refer to $f_s(x)$. By the KKT conditions $\exists \mu, \tilde{\mu}$ s.t. $\forall i$:

$$\begin{aligned} f^{'}(x_{s,i}) &= -L_{s,i}^{obs} + \mu, \\ f^{'}(\tilde{x}_i) &= -L_{t,i}^{obs} + \tilde{\mu}. \end{aligned}$$

Let $\tilde{\ell} = L_t^{obs} - L_s^{obs}$, then by the concavity of $f^{'}(x)$ from Fact 15, we have

$$(x_{s,i} - \tilde{x}_i) f^{''}(x_{s,i}) \leq \underbrace{f^{'}(x_{s,i}) - f^{'}(\tilde{x}_i)}_{\mu - \tilde{\mu} + \tilde{\ell}_i} \leq (x_{s,i} - \tilde{x}_i) f^{''}(\tilde{x}_i). \tag{33}$$

Since $f^{''}(x_{s,i}) \geq 0$, from the left side of (33) we get $x_{s,i} - \tilde{x}_i \leq f^{''}(x_{s,i})^{-1}\left(\mu - \tilde{\mu} + \tilde{\ell}_i\right)$. Taking summation over all $i$ and using the fact that both vectors $x_s$ and $\tilde{x}$ are probability vectors, we have

$$\begin{aligned} 0 = \sum_{i=1}^K x_{s,i} - \tilde{x}_i &\leq \sum_{i=1}^K f^{''}(x_{s,i})^{-1}\left(\mu - \tilde{\mu} + \tilde{\ell}_i\right) \\ &\Rightarrow \tilde{\mu} - \mu \leq \frac{\sum_{i=1}^K f^{''}(x_{s,i})^{-1}\tilde{\ell}_i}{\sum_{i=1}^K f^{''}(x_{s,i})^{-1}}. \end{aligned} \tag{34}$$

Combining the right hand sides of (33) and (34) gives

$$(\tilde{x}_i - x_{s,i}) f^{''}(\tilde{x}_i) \leq \tilde{\mu} - \mu - \tilde{\ell}_i \leq \frac{\sum_{j=1}^K f^{''}(x_{s,j})^{-1}\tilde{\ell}_j}{\sum_{j=1}^K, f^{''}(x_{s,j})^{-1}}$$

and by rearrangement

$$\begin{aligned} \tilde{x}_i &\leq x_{s,i} + f^{''}(\tilde{x}_i)^{-1} \times \frac{\sum_{j=1}^K f^{''}(x_{s,j})^{-1}\tilde{\ell}_j}{\sum_{j=1}^K f^{''}(x_{s,j})^{-1}} \\ &\leq x_{s,i} + \gamma_s \tilde{x}_i \times \frac{\sum_{j=1}^K f^{''}(x_{s,j})^{-1}\tilde{\ell}_j}{\sum_{j=1}^K f^{''}(x_{s,j})^{-1}}, \end{aligned} \tag{35}$$

where the last inequality holds because $f^{''}(\tilde{x}_i)^{-1} = \left(\eta_s^{-1}\frac{1}{2}\tilde{x}_i^{-3/2} + \gamma_s^{-1}\tilde{x}_i^{-1}\right)^{-1}$. The next step for bounding $\tilde{x}_i$ is to bound $\frac{\sum_{j=1}^K f^{''}(x_{s,j})^{-1}\tilde{\ell}_j}{\sum_{j=1}^K f^{''}(x_{s,j})^{-1}}$ in (35), where $\tilde{\ell}_j = \sum_{r \in A} \hat{\ell}_{r,j}$ and $A = \{r : s \leq r + d_r < t\}$.

If there exists $r \in A$, such that $r > s$ and $2x_{r,i} \leq x_{s,i}$, then combining it with the induction assumption for $(t, r)$, i.e., $x_{t,i} \leq 2x_{r,i}$, leads to $x_{t,i} \leq 2x_{r,i} \leq x_{s,i}$, which completes the proof. Otherwise, that for all $r \in A$ we have either $r \leq s$ or $x_{s,i} \leq 2x_{r,i}$. If $r \leq s$, we can use the induction assumption for $(s, r)$, which gives $x_{s,i} \leq 2x_{r,i}$. Consequently, in either case, the inequality $x_{s,i} \leq 2x_{r,i}$ holds for all $r \in A$, and we can plug it into Lemma 17 to get the following bound for all $r \in A$:

$$\frac{\sum_{j=1}^K f^{''}(x_{s,j})^{-1}\hat{\ell}_{r,j}}{\sum_{j=1}^K f^{''}(x_{s,j})^{-1}} \leq 4(K-1)^{\frac{1}{3}}. \tag{36}$$

We then proceed by doing a summation over all $r \in A$ on both sides of the above inequality and get $\frac{\sum_{j=1}^{K} f''(x_{s,j})^{-1}\tilde{\ell}_j}{\sum_{j=1}^{K} f''(x_{s,j})^{-1}} \leq 4|A|(K-1)^{\frac{1}{3}}$. Now it suffices to plug this result into (35):

$$\begin{aligned}
&\tilde{x}_i \leq x_{s,i} + 4|A|\gamma_s \tilde{x}_i (K-1)^{\frac{1}{3}} \Rightarrow \\
&\tilde{x}_i \leq x_{s,i} \times \left( \frac{1}{1 - 4|A|\gamma_s (K-1)^{1/3}} \right) \qquad (37) \\
&\quad \leq x_{s,i} \times \left( \frac{1}{1 - 8\gamma_s d_{max} (K-1)^{1/3}} \right) \\
&\quad \leq x_{s,i} \times \left( \frac{1}{1 - 8\sqrt{\log K/\gamma_0}\, d_{max} (K-1)^{1/3}} \right) = \frac{x_{s,i}}{1 - 1/3} = \frac{3}{2} x_{s,i}, \qquad (38)
\end{aligned}$$

where the third inequality uses $|A| \leq d_{max} + t - s \leq 2d_{max}$, and the last one uses the facts that $\gamma_s \leq \sqrt{\log(K)/\gamma_0}$ and $\gamma_0 = 24^2 d_{max}^2 (K-1)^{2/3} \log(K)$.

Combining (38) and (32) completes the proof. ■

## C Detailed constant factors in the regret bound for Algorithm 1

In this section we provide a detailed regret bound for Algorithm 1.
As we proved in Section 5 we have the following inequality for the drifted regret:

$$\overline{Reg}_T \leq 2\overline{Reg}_T^{drift} + d_{max} \qquad (39)$$

We first derive a bound for the drifted regret by splitting the drifted regret into stability and penalty terms, as mentioned in Section 5. Following the general analysis of the penalty term for FTRL [Abernethy et al., 2015], we have

$$penalty \leq \sum_{t=2}^{T} (F_{t-1}(x_t) - F_t(x_t)) + F_T(x^*) - F_1(x_1),$$

which gives us

$$\begin{aligned}
penalty &= \sum_{t=2}^{T} \left( 2(\sum_{i=1}^{K} x_{t,i}^{\frac{1}{2}} - 1)(\eta_t^{-1} - \eta_{t-1}^{-1}) - \sum_{i=1}^{K} x_{t,i} \log(x_{t,i})(\gamma_t^{-1} - \gamma_{t-1}^{-1}) \right) - 2\eta_1^{-1} + 2\sqrt{K}\eta_1^{-1} + \gamma_1^{-1}\log K \\
&\leq \sum_{t=2}^{T} \left( 2\sum_{i \neq i^*} x_{t,i}^{\frac{1}{2}}(\eta_t^{-1} - \eta_{t-1}^{-1}) - \sum_{i=1}^{K} x_{t,i} \log(x_{t,i})(\gamma_t^{-1} - \gamma_{t-1}^{-1}) \right) + 2\sqrt{\eta_0(K-1)} + \sqrt{\gamma_0 \log K} \\
&\leq \sum_{t=2}^{T} \left( 2\sum_{i \neq i^*} \eta_t x_{t,i}^{\frac{1}{2}} - \sum_{i=1}^{K} \frac{\sigma_t \gamma_t x_{t,i} \log(x_{t,i})}{\sqrt{\log K}} \right) + 2\sqrt{\eta_0(K-1)} + \sqrt{\gamma_0 \log K}, \qquad (40)
\end{aligned}$$

where the first inequality holds because $x_{t,i^*}^{\frac{1}{2}} \leq 1$ and the second inequality follows by $\eta_t^{-1} - \eta_{t-1}^{-1} = \sqrt{t+\eta_0} - \sqrt{t-1+\eta_0} \leq \frac{1}{\sqrt{t+\eta_0}} = \eta_t$ and $\gamma_t^{-1} - \gamma_{t-1}^{-1} = \frac{\gamma_t^{-2} - \gamma_{t-1}^{-2}}{\gamma_t^{-1} + \gamma_{t-1}^{-1}} \leq \frac{\gamma_t^{-2} - \gamma_{t-1}^{-2}}{\gamma_t^{-1}}$.

For the stability term, we start from the bound given by Lemma 5:

$$\mathbb{E}[stability] \leq \sum_{t=1}^{T} \sum_{i \neq i^*} 2\gamma_t(\upsilon_t - 1)\upsilon_t \mathbb{E}[x_{t,i}]\Delta_i + \sum_{t=1}^{T} \sum_{s \in A_t} \sum_{i=1}^{K} \eta_t \mathbb{E}[x_{t,i}^{3/2} x_{s,i}^{-1}(1 - x_{s,i})]. \qquad (41)$$

In above inequality, we know that $\upsilon_t x_{t,i} = \sum_{s \in A_t} x_{t,i}$, and by Lemma 4 we have $x_{t,i} \leq 2x_{s,i}$ for $s \in A_t$. Then for the first term in (41):

$$\sum_{t=1}^{T} \sum_{i \neq i^*} 2\gamma_t(\upsilon_t - 1)\upsilon_t x_{t,i}\Delta_i \leq \sum_{t=1}^{T} \sum_{i \neq i^*} \sum_{s \in A_t} 4\gamma_t(\upsilon_t - 1)\upsilon_t x_{s,i}\Delta_i = \sum_{t=1}^{T} \sum_{i \neq i^*} 4\gamma_{t+d_t}(\upsilon_{t+d_t} - 1)x_{t,i}\Delta_i. \qquad (42)$$

Furthermore, we can bound $x_{t,i}^{3/2} x_{s,i}^{-1}(1 - x_{s,i}) \leq 2^{3/2} x_{s,i}^{1/2}(1 - x_{s,i})$. Moreover, in order to remove the best arm $i^*$ from the summation in the later bound we use $x_{t,i^*}^{3/2} x_{s,i^*}^{-1}(1 - x_{s,i^*}) \leq 2\sum_{i \neq i^*} x_{s,i} \leq \sum_{i \neq i^*} 2x_{s,i}^{1/2}$.

For the second term in (41) we have

$$\begin{aligned}\sum_{t=1}^{T}\sum_{s\in A_t}\sum_{i=1}^{K}\eta_t x_{t,i}^{3/2}x_{s,i}^{-1}(1-x_{s,i}) &\leq \sum_{t=1}^{T}\sum_{s\in A_t}\sum_{i=1}^{K}\eta_t 2^{3/2}x_{s,i}^{1/2}(1-x_{s,i})\\ &\leq \sum_{t=1}^{T}\sum_{s\in A_t}\sum_{i\neq i^*}\sqrt{8}\eta_t x_{s,i}^{1/2} + \sum_{t=1}^{T}\sum_{s\in A_t}\sum_{i\neq i^*}2\eta_t x_{s,i}^{1/2}\\ &\leq \sum_{t=1}^{T}\sum_{i\neq i^*}5\eta_t x_{t,i}^{1/2}, \qquad (43)\end{aligned}$$

where the last inequality follows by the facts that we can change the order of the summations and that each $t$ belongs to exactly one $A_s$. Plugging (42) and (43) into (41) we have

$$\mathbb{E}[stability] \leq \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i\neq i^*}4\gamma_{t+d_t}(\upsilon_{t+d_t}-1)x_{t,i}\Delta_i + \sum_{t=1}^{T}\sum_{i\neq i^*}5\eta_t x_{t,i}^{1/2}\right]. \qquad (44)$$

Now it suffices to combine (44), (40), and (39) to get

$$\begin{aligned}\overline{Reg}_T \leq \mathbb{E}&\left[14\underbrace{\sum_{t=1}^{T}\sum_{i\neq i^*}\eta_t x_{t,i}^{1/2}}_{A} + 8\underbrace{\sum_{t=1}^{T}\sum_{i\neq i^*}\gamma_{t+d_t}(\upsilon_{t+d_t}-1)x_{t,i}\Delta_i}_{B} + 2\underbrace{\sum_{t=2}^{T}\sum_{i=1}^{K}\frac{\sigma_t\gamma_t x_{t,i}\log(1/x_{t,i})}{\log K}}_{C}\right]\\ &+ \underbrace{4\sqrt{\eta_0(K-1)} + 2\sqrt{\gamma_0\log K} + d_{max}}_{D}. \qquad (45)\end{aligned}$$

We rewrite the regret as

$$\overline{Reg}_T = 4\overline{Reg}_T - 3\overline{Reg}_T \leq 4\times 14A - \overline{Reg}_T + 4\times 8B - \overline{Reg}_T + 4\times 2C - \overline{Reg}_T + 4D,$$

where by applying Lemmas 6, 7, and 8 we achieve

$$\begin{aligned}4\times 14A - \overline{Reg}_T &\leq \sum_{i\neq i^*}\frac{28^2}{\Delta_i}\log(T/\eta_0+1)\\ 4\times 8B - \overline{Reg}_T &\leq 64^2\upsilon_{max}\log K\\ 4\times 2C - \overline{Reg}_T &\leq \sum_{i\neq i^*}\frac{512\sigma_{max}}{\Delta_i\log K}.\end{aligned}$$

Therefore, the final regret bound is

$$\begin{aligned}\overline{Reg}_T \leq &\sum_{i\neq i^*}\frac{28^2}{\Delta_i}\log(T/\eta_0+1) + 64^2\upsilon_{max}\log K + \sum_{i\neq i^*}\frac{512\sigma_{max}}{\Delta_i\log K}\\ &+ 16\sqrt{\eta_0(K-1)} + 8\sqrt{\gamma_0\log K} + 4d_{max}.\end{aligned}$$

# D Removing the multiplicative factor $1/\Delta_i$ from $\sigma_{max}/\Delta_i$ in the regret bound

In this section we discuss how an asymmetric *oracle* learning rate $\gamma_{t,i} \simeq \gamma_t/\sqrt{\Delta_i}$ for negative entropy regularizer can be used to remove the factor $\sum_{i\neq i^*} 1/\Delta_i$ in front of $\sigma_{max}$ in the regret bound.

In the analysis of Algorithm 1 we divided the regret into stability and penalty expressions. Moreover, in each of the bounds for stability and penalty we have two terms which correspond to negative entropy and Tsallis parts of the hybrid regularizer. The terms related to negative entropy part in both stability and penalty bounds are

$$\underbrace{\sum_{t=1}^{T}\sum_{i\neq i^*}\gamma_{t+d_t}(\upsilon_{t+d_t}-1)\mathbb{E}[x_{t,i}]\Delta_i}_{B} + \underbrace{\sum_{i=1}^{K}\mathbb{E}[x_{t,i}\log(1/x_{t,i})](\gamma_t^{-1}-\gamma_{t-1}^{-1})}_{C},$$

where $B$ and $C$, as we have seen in Section 5, are due to stability and penalty terms,respectively. The idea here is to scale-up $\gamma_t$ to decrease $C$, however increasing $\gamma_t$ increases $B$. Hence, we are facing a trade off here. To deal with this trade-off we change the learning rates for negative entropy from symmetric $\gamma_t$ to asymmetric

$\gamma_{t,i}$, and we expect this change only affect the parts of regret bound come from the negative entropy part of the regularizer, which are $B$ and $C$. This change results in to having two following terms instead,

$$\underbrace{\sum_{t=1}^{T}\sum_{i\neq i^*}\gamma_{t+d_t,i}(\upsilon_{t+d_t}-1)\mathbb{E}[x_{t,i}]\Delta_i}_{B_{new}}+\underbrace{\sum_{i=1}^{K}\mathbb{E}[x_{t,i}\log(1/x_{t,i})](\gamma_{t,i}^{-1}-\gamma_{t-1,i}^{-1})}_{C_{new}}.$$

Here if we could choose $\gamma_{t,i}=\gamma_t/\sqrt{\Delta_i}$, then using the definition of $\gamma_t$ we would be able to rewrite $B_{new}$ and $C_{new}$ as

$$B_{new}=\mathcal{O}\left(\sum_{t=1}^{T}\sum_{i\neq i^*}\gamma_{t+d_t}(\upsilon_{t+d_t}-1)\mathbb{E}[x_{t,i}]\sqrt{\Delta_i}\right)$$
$$C_{new}=\mathcal{O}\left(\sum_{i=1}^{K}\frac{\sigma_t\gamma_t\mathbb{E}[x_{t,i}\log(1/x_{t,i})]\sqrt{\Delta_i}}{\sqrt{\log K}}\right).$$

Now we must see what is the result of applying the self-bounding technique on these new terms. For $B_{new}$ and $C_{new}$, following the similar analysis as Lemma 7 and Lemma 8 we can get

$$4B_{new}-\overline{Reg}_T=\mathcal{O}(\upsilon_{max}\log K)=\mathcal{O}(d_{max}\log K)$$
$$4C_{new}-\overline{Reg}_T=\mathcal{O}(\frac{\sigma_{max}}{\log K}).$$

This implies that injecting $\sqrt{1/\Delta_i}$ in the negative entropy learning rates removes the factor $\sum_{i\neq i^*}\frac{1}{\Delta_i}$ in front of the $\sigma_{max}$. More interestingly this comes without having any significant changes in the other terms of regret bound.
As a result, we conjecture that replacing a good estimation of the suboptimal gaps namely $\hat{\Delta}_i$ in $\gamma_{t,i}$ as $\gamma_{t,i}=\gamma_t/\sqrt{\hat{\Delta}_i}$ might be also helpful to remove the multiplicative factors related to suboptimal gaps in front of the $\sigma_{max}$. We leave this problem to future work.

## E Lower bounds

**Algorithm 2:** Adversarial choice of $\ell$

**Input:** $x$

**Initialize** $\mathcal{I}=\{\operatorname{argmax}_i x_i\}$ **while** $\sum_{i\in\mathcal{I}}x_i+\min_{i\in\bar{\mathcal{I}}}x_i\leq\frac{2}{3}$ **do**

  Update $\mathcal{I}\leftarrow\mathcal{I}\cup\{\operatorname{argmin}_{i\in\bar{\mathcal{I}}}x_i\}$

**return** $\ell_i=\begin{cases}\min\{1,\frac{\sum_{i\in\bar{\mathcal{I}}}x_i}{\sum_{i\in\mathcal{I}}x_i}\} & \textit{for } i\in\mathcal{I}\\ \max\{-1,-\frac{\sum_{i\in\mathcal{I}}x_i}{\sum_{i\in\bar{\mathcal{I}}}x_i}\} & \textit{for } i\in\bar{\mathcal{I}}\end{cases}$

**Lemma 18.** *For any $x\in\Delta([K])$, such that $\max_i x_i\leq\frac{2}{3}$, the vector $\ell$ returned by Algorithm 2 satisfies $\ell\in[-1,1]$, $\langle x,\ell\rangle=0$, and $\sum_{i=1}^{K}x_i\ell_i^2\geq\frac{1}{2}$.*

*Proof.* The first two properties follow directly by construction. For the third property we bound the ratio of the two sets. Assume that $\sum_{i\in\mathcal{I}}x_i<\frac{1}{3}$, then $\operatorname{argmin}_{i\in\bar{\mathcal{I}}}x_i<\frac{1}{3}$ and the algorithm does not return yet. The quantity in question is therefore bounded by

$$\sum_{i=1}^{K}x_i\ell_i^2=\sum_{i\in\mathcal{I}}x_i\ell_i^2+\sum_{i\in\bar{\mathcal{I}}}x_i\ell_i^2=p+(1-p)\left(\frac{p}{1-p}\right)^2=\frac{p}{1-p}\geq\frac{1}{2}.$$

■

**Claim 19.** *For the negentropy potential $F(x)=\eta^{-1}\sum_{i=1}^{K}\log(x_i)x_i$, it holds that*

$$-\overline{F}^*(-L)-\min_i L_i=\eta^{-1}\log(\max_i\nabla\overline{F}^*(-L)_i).$$

*Proof.* Denote $i^*=\operatorname{argmin}_{i\in[K]}L_i$. It is well known that the exponential weights distribution is $(\nabla\overline{F}^*(-L))_i=\exp(-\eta L_i)/(\sum_{j\in[K]})\exp(-\eta L_j)$. Therefore, the negentropy has an explicit form of the constrained convex conjugate:

$$\overline{F}^*(-L) = \left\langle \nabla\overline{F}^*(-L), -L \right\rangle - F(\nabla\overline{F}^*(-L)) = \eta^{-1}\log(\sum_{i=1}^{K}\exp(-\eta L_i))\,.$$

Hence

$$\begin{aligned}-\overline{F}^*(-L) - L_{i^*} &= -\eta^{-1}\log\left(\sum_{i=1}^{K}\exp(-\eta L_i)\right) + \eta^{-1}\log(\exp(-\eta L_{i^*}))\\ &= -\eta^{-1}\log\left(\frac{\sum_{i=1}^{K}\exp(-\eta L_i)}{\exp(-\eta L_{i^*})}\right) = \eta^{-1}\log\left(\nabla\overline{F}^*(-L)_{i^*}\right).\end{aligned}$$

■

*Proof of Theorem 9.* For ease of presentation, we will work with loss ranges $[-L_t/2, L_t/2]$, which is equivalent to loss ranges of $[0, L_t]$ in full-information games. Assume that

$$\frac{1}{2}\sum_{t=1}^{\lfloor\log_2(K)\rfloor} L_t \geq \frac{1}{32}\sqrt{\sum_{t=\lfloor\log_2(K)\rfloor}^{T} L_t^2\log(K)}\,.$$

Define the active set $\mathcal{A}_1 = [K]$. At any time $t$, if $L_t$ is not among the largest loss ranges, we set $\ell_t$ to 0 and proceed with $\mathcal{A}_{t+1} = \mathcal{A}_t$. Otherwise, if $t \in \rho([\lfloor\log_2(K)\rfloor])$, we randomly select half of the arms in $\mathcal{A}_t$ to assign $\ell_{t,i} = -L_t/2$, and the other half $\ell_{t,i} = L_t/2$. (In case of an uneven number $|\mathcal{A}_t|$ we leave one arm at 0.) All other losses are 0. We reduce $\mathcal{A}_{t+1} = \{i \in \mathcal{A}_t \,|\, \ell_{t,i} < 0\}$ to the set of arms that were negative. The set $\mathcal{A}_n$ will not be empty since we can repeat halving the action set exactly $\lfloor\log_2(K)\rfloor$ many times. The expected loss of any player is always 0, while the loss of the best arm is $\min_a \sum_{t=1}^{T}\ell_{t,a} = -\sum_{t=1}^{\lfloor\log_2(K)\rfloor} L_t/2$, hence

$$\mathbb{R}^* \geq \sum_{t=1}^{\lfloor\log_2(K)\rfloor} L_t/2\,.$$

It remains to analyse the case

$$\frac{1}{2}\sum_{t=1}^{\lfloor\log_2(K)\rfloor} L_t < \frac{1}{32}\sqrt{\sum_{t=\lfloor\log_2(K)\rfloor}^{T} L_t^2\log(K)}\,.$$

In this case, note that we have

$$\sqrt{\sum_{t=\lfloor\log_2(K)\rfloor}^{T} L_t^2/\log(K)} > \frac{16}{\log(K)}\sum_{t=1}^{\lfloor\log_2(K)\rfloor} L_t > 16\frac{\lfloor\log_2(K)\rfloor}{\log(K)}L_{\lfloor\log_2(K)\rfloor} > 8L_{\lfloor\log_2(K)\rfloor}\,. \tag{46}$$

The high level idea is now to create a sequence of losses adapted to the choices of the algorithm. Let $x_{ti} = \mathbb{E}[I_t = i|\ell_{t-1},\dots,\ell_1]$ be the expected trajectory of the algorithm and let $z_{ti} = \exp(-\eta L_{ti})/\sum_{j=1}^{K}\exp(-\eta L_{tj})$ for $L_t = \sum_{s=1}^{t-1}\ell_t$ be the trajectory of EXP3. We show that it is possible to choose $\ell_t$, such that $0 = \langle z_t, \ell_t\rangle \leq \langle x_t, \ell_t\rangle$, i.e., the regret of the algorithm cannot be smaller than that of EXP3. Finally, we show that the construction of the losses ensures that the regret of EXP3 is lower bounded by the right quantity. Set $\eta = \sqrt{\log(K)/(\sum_{t=\lfloor\log_2(K)\rfloor}^{T} L_t^2)}$, and let the adversary follow Algorithm 3 for the selection of the losses. Denote $\tau := \operatorname{argmax}\{t \in [T+1] \,|\, \ell_{t-1} \neq 0\}$, then the regret of algorithm $\mathcal{A}$ can be bounded as

$$Reg_T(\mathcal{A}) = \sum_{t=1}^{T}\langle x_t, \ell_t\rangle - \min_{a^*\in\Delta([K])}\langle a^*, L_{T+1}\rangle \geq \sum_{t=1}^{T}\langle z_t, \ell_t\rangle - \min_{a^*\in\Delta([K])}\langle a^*, L_{T+1}\rangle = -\min_{a^*\in\Delta([K])}\langle a^*, L_{T+1}\rangle\,,$$

where we use $\langle x_t, \ell_t\rangle \geq \langle z_t, \ell_t\rangle = 0$. By expansion and Claim 19, we have

$$\begin{aligned}-\min_{a^*\in\Delta([K])}\langle a^*, L_{T+1}\rangle &= \eta^{-1}\log(K) - \eta^{-1}\log\left(\sum_{i=1}^{K}\exp(-\eta L_{T+1,i})\right) - \min_{a^*\in\Delta([K])}\langle a^*, L_{T+1}\rangle\\ &\quad + \sum_{t=1}^{T}\eta^{-1}\log\left(\sum_{i=1}^{K}\exp(-\eta L_{t+1,i})\right) - \eta^{-1}\log\left(\sum_{i=1}^{K}\exp(-\eta L_{t,i})\right)\\ &= \eta^{-1}\log(K) + \eta^{-1}\log(\max_{i\in[K]} z_{T+1,i}) + \sum_{t=1}^{T}\eta^{-1}\log(\sum_{i=1}^{K} z_{ti}\exp(-\eta\ell_{ti})).\end{aligned}$$

The learning rate and setting the largest $\log_2(K)$ loss ranges to zero ensures that $|\eta\ell_{ti}| \leq \frac{1}{2}\eta L_{\lfloor \log_2(K) \rfloor} \leq \frac{1}{2}$. Using that, by Taylor's theorem and the monotonicity of the second derivative of exp, we have for all $x \geq -\frac{1}{2}$: $\exp(x) \geq 1 + x + \frac{1}{2}\exp''(-\frac{1}{2})x^2 \geq 1 + x + \frac{3}{10}x^2$, as well as by concavity of log for all $0 \leq x \leq \frac{1}{4}$ we have $\log(1+x) \geq 4\log(5/4)x \geq \frac{5}{6}x$, we get for any $t \in [T]$ by Lemma 18

$$\eta^{-1}\log(\sum_{i=1}^{K} z_{ti}\exp(-\eta\ell_{ti}) \geq \eta^{-1}\log(1+\eta^2\frac{3}{10}\sum_{i=1}^{K} z_{ti}\ell_{ti}^2) \geq \frac{\eta}{4}\sum_{i=1}^{K} z_{ti}\ell_{ti}^2 \leq \mathbb{I}\{\max_i z_{ti} \leq \frac{2}{3}\}\frac{\eta}{32}L_{\rho^{-1}(t)}^2 \,.$$

Now we have two possible events, either $\forall t \in [T] : \max_i z_{ti} \leq \frac{2}{3}$ and

$$Reg_T(\mathcal{A}) \geq \frac{\eta}{32}\sum_{t=\lfloor \log_2(K) \rfloor} L_t^2 = \frac{1}{32}\sqrt{\sum_{t=\lfloor \log_2(K) \rfloor} L_t^2 \log(K)}\,,$$

or $\max_i z_{T+1,i} > \frac{2}{3}$ and

$$Reg_T(\mathcal{A}) \geq \eta^{-1}(\log(K) + \log(2/3)) \geq \frac{1}{32}\eta^{-1}\log(K) = \frac{1}{32}\sqrt{\sum_{t=\lfloor \log_2(K) \rfloor} L_t^2 \log(K)}\,.$$

■

**Algorithm 3:** Adversary

**Input:** Actor $\mathcal{A}$, learning rate $\eta$

```
for t = 1, ..., n do
    Set ∀i : z_ti = exp(−η L_ti) / Σ_{j=1}^{K} exp(−η L_tj)
    if max_{i∈[K]} z_ti > 2/3 or ρ(t) ≤ ⌊log_2(K)⌋ then
        ℓ_t = 0
    else
        Get ℓ from Algorithm 2 with x = z_t.
        Determine x_t = E[A((ℓ_s)_{s=1}^{t−1})]
        Set ℓ_t = sign(⟨x_t, ℓ⟩) L_{ρ^{−1}(t)} ℓ/2
```