Derivation of Fractional Differencing

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Lemma 1: Binomial Series is the Maclaurin Series for $f(x) = (1+x)^d$

Explicitly:

$$(1+x)^d = \sum_{k=0}^{\infty} \binom{d}{k} x^k = 1 + dx + \frac{d(d-1)}{2!}x^2 + \frac{d(d-1)(d-2)}{3!}x^3 + ... = \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1}(d-i)}{k!} x^k \tag{1}$$

Lemma 2: Backshift Operator B has the following Properties:

Given a matrix of time series values $X = X_1, X_2, ...$

$$\begin{aligned} B * B &= B^2, B + B = 2B \\ B^k X_t &= X_{t-k} \,\text{for}\, t > k \,\text{, and for all integers}\,, k > 0 \\ B^{-1} X_t &= X_{t+k} \end{aligned} \tag{2}$$

For example: $(1-B)^2 = 1 - 2B + B^2$ and $(1-B)^2 X_t = X_t - 2X_{t-1} + X_{t-2}$

Derivation: Show that $(1-B)^d$, the formula used to derive weights for fractional differencing, converges to an infinite series of weights $w = \{w_0, w_1, w_2, w_3, ...\}$

$$\begin{aligned} (1-B)^d = \sum_{k=0}^{\infty} \binom{d}{k} (-B)^k &= \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1}(d-i)}{k!} (-B)^k \\ &= \sum_{k=0}^{\infty} (-B)^k \frac{\prod_{i=0}^{k-1}(d-i)}{k!} \\ &= \sum_{k=0}^{\infty} (-B)^k \prod_{i=0}^{k-1} \frac{d-i}{k-i} \\ &= 1 - dB + \frac{d(d-1)}{2!}B^2 - \frac{d(d-1)(d-2)}{3!}B^3 + ... \end{aligned} \tag{3}$$

Also, note an interesting property below with the weighting scheme derived above:

$$w_k = -w_{k-1} \frac{d-k+1}{k}$$

Consider the following: $w_3 = -w_2 \frac{d-3+1}{3} = -(\frac{d(d-1)}{2!})(\frac{d-2}{3}) = -\frac{d(d-1)(d-2)}{3!}$