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① a HO has energy eigenstates $|\psi_n\rangle$, $n = 0, 1, 2 \ldots$

↳ states are ortho. $\langle \psi_m | \psi_n \rangle = \delta_{mn}$

↳ ladder satisfy: $a|\psi_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle$, $a^\dagger|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle$

↳ comutation: $[a, a^\dagger] = 1$

lets express x and p in terms of $a, a^\dagger$

(a.)

i) $\langle \psi_m | (a^\dagger)^s | \psi_n \rangle$: let $s > 0 \in \mathbb{R}$, integer.

$$a^\dagger|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle$$

$$(a^\dagger)^2|\psi_n\rangle = a^\dagger(\sqrt{n+1}\,|\psi_{n+1}\rangle) = \sqrt{n+1}\sqrt{n+2}\,|\psi_{n+2}\rangle$$

$$= \sqrt{(n+1)(n+2)}\,|\psi_{n+2}\rangle \quad \ldots (1)$$

$$(a^\dagger)^3|\psi_n\rangle = a^\dagger(\sqrt{(n+1)(n+2)}\,|\psi_{n+2}\rangle) = \sqrt{(n+1)(n+2)}\cdot\sqrt{n+3}\,|\psi_{n+3}\rangle \quad \ldots (2)$$

Thus we can see from (1) and (2) the general result

$$(a^\dagger)^s|\psi_n\rangle = \sqrt{(n+1)(n+2)\ldots+(n+s)}\,|\psi_{n+s}\rangle = \sqrt{\frac{(n+s)!}{n!}}\,|\psi_{n+s}\rangle, \quad \ldots (3)$$

Where s is consecutive integers, Note if $s = 0 \rightarrow$ Identity operator $\rightarrow$ unchan

$$\therefore \langle \psi_m | (a^\dagger)^s | \psi_n \rangle = \sqrt{\frac{(n+s)!}{n!}}\langle \psi_m | \psi_{n+s} \rangle = \boxed{\sqrt{\frac{(n+s)!}{n!}}\,\delta_{m,n+s} = \langle \psi_m | (a^\dagger)^s | \psi_n \rangle}$$

★ Note, only non-zero if $m = n+s$ (bra state is at $n+s$)

Pr 1 a. cont

1.a

ii Similarly for annihilation/lowering, lowers by s.

using: $a|\varphi_n\rangle = \sqrt{n}|\varphi_{n-1}\rangle$

$$\hookrightarrow a^s|\varphi_n\rangle = \sqrt{n(n-1)\ldots(n-s+1)}\,|\varphi_{n-s}\rangle$$

$$= \sqrt{\frac{n!}{(n-s)!}}\,|\varphi_{n-s}\rangle \quad \ldots (4)$$

if $n \geq s$, if $n < s \rightarrow a^s|\varphi_n\rangle = 0$ ($\rightarrow$ grond/vacuum state)

thus: $\langle\varphi_m|a^s|\varphi_n\rangle = \sqrt{\frac{n!}{(n-s)!}}\,\langle\varphi_m|\varphi_{n-s}\rangle$

$$\boxed{\langle\varphi_m|a^s|\varphi_n\rangle = \sqrt{\frac{n!}{(n-s)!}}\,\delta_{m,n-s}}$$

non zero when $m = n-s$

$\therefore \langle\varphi_m|(a^+)^s|\varphi_n\rangle = \sqrt{\frac{(n+s)!}{n!}}\,\delta_{m,n+s}$; $\langle\varphi_m|a^s|\varphi_n\rangle = \sqrt{\frac{n!}{(n-s)!}}\,\delta_{m,n-s}$

$(a^+)^s$ connects $|n\rangle$ to $|n+s\rangle$ and a^s connect $|n\rangle$ to $|n-s\rangle$

1b $x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^+)$: becuse $a = \sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{i}{\sqrt{2m\hbar\omega}}\,p$ (hamilton)

- $\langle\varphi_m|x|\varphi_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle\varphi_m|a|\varphi_n\rangle + \langle\varphi_m|a^+|\varphi_n\rangle\right)$

and single step $\langle\varphi_m|a|\varphi_n\rangle = \sqrt{n}\,\delta_{m,n-1}$; $\langle\varphi_m|a^+|\varphi_n\rangle = \sqrt{n+1}\,\delta_{m,n+1}$

$$\boxed{\langle\varphi_m|x|\varphi_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right)}$$

- non zero if $m = n \pm 1$

1b cont

1b ii $\langle \varphi_m | x^2 | \varphi_n \rangle$:

$$x^2 = \frac{\hbar}{2m\omega}(a+a^\dagger)^2 = \frac{\hbar}{2m\omega}(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2)$$

Since

$$aa^\dagger = a^\dagger a + 1$$

$$x^2 = \frac{\hbar}{2m\omega}(a^2 + (a^\dagger)^2 + 2a^\dagger a + 1)$$

from part a: $\langle \varphi_m | a^2 | \varphi_n \rangle = \sqrt{n(n-1)}\,\delta_{m,n-2}$

$$\langle \varphi_m | (a^\dagger)^2 | \varphi_n \rangle = \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}$$

$$\langle \varphi_m | a^\dagger a | \varphi_n \rangle = n\,\delta_{m,n}$$

$$\langle \varphi_m | 1 | \varphi_n \rangle = \delta_{m,n}$$

$$\boxed{\therefore \langle \varphi_m | x^2 | \varphi_n \rangle = \frac{\hbar}{2m\omega}\left(\sqrt{n(n-1)}\,\delta_{m,n-2} + (2n+1)\,\delta_{m,n} + \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right)}$$

here if $m=n \rightarrow$ no change: $m=n+2 \rightarrow \frac{\hbar}{2m\omega}\sqrt{(n+1)(n+2)}$ or $m=n-2 \rightarrow \frac{\hbar}{2m\omega}\sqrt{n(n-1)}$

1c p (conjugate to x): $p = i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a)$

$$\therefore \langle \varphi_m | p | \varphi_n \rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\langle \varphi_m | a^\dagger | \varphi_n \rangle - \langle \varphi_m | a | \varphi_n \rangle\right)$$

Since for single step: $\langle \varphi_m | a^\dagger | \varphi_n \rangle = \sqrt{n+1}\,\delta_{m,n+1}$; $\langle \varphi_m | a | \varphi_n \rangle = \sqrt{n}\,\delta_{m,n-1}$

Similar to x: p only connects adjacent states ($m = n \pm 1$) below:

$$\boxed{\langle \varphi_m | p | \varphi_n \rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n+1}\,\delta_{m,n+1} - \sqrt{n}\,\delta_{m,n-1}\right)}$$

for ± 1: $\langle \varphi_{n+1} | p | \varphi_n \rangle = i\sqrt{\frac{\hbar m\omega}{2}}\sqrt{n+1}$; $\langle \varphi_{n-1} | p | \varphi_n \rangle = -i\sqrt{\frac{\hbar m\omega}{2}}\sqrt{n}$

raising gets $+i$, and lowering $-i$ (expected as $a^\dagger - a$ is anti hermitian)

(1c)

ii $\langle \psi_m | p^2 | \psi_n \rangle$: we know p from previous step.

$$p^2 = \left(i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a) \right)^2 = -\frac{\hbar m\omega}{2}(a^\dagger - a)^2 = -\frac{\hbar m\omega}{2}\left((a^\dagger)^2 - a^\dagger a - aa^\dagger + a^2\right)$$

since $aa^\dagger = a^\dagger a + 1$: $p^2 = -\frac{\hbar m\omega}{2}\left((a^\dagger)^2 + a^2 - 2a^\dagger a - 1\right)$

$$p^2 = \frac{\hbar m\omega}{2}\left(2a^\dagger a + 1 - a^2 - (a^\dagger)^2\right)$$

bc. $\langle \psi_m | a^2 | \psi_n \rangle = \sqrt{n(n-1)}\,\delta_{m,n-2}$ $\qquad$ $\langle \psi_m | a^\dagger a | \psi_n \rangle = n\,\delta_{m,n}$

$\langle \psi_m | (a^\dagger)^2 | \psi_n \rangle = \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}$ $\qquad$ $\langle \psi_m | 1 | \psi_n \rangle = \delta_{m,n}$

$$\boxed{\text{then: } \langle \psi_m | p^2 | \psi_n \rangle = \frac{\hbar m\omega}{2}\left((2n+1)\,\delta_{m,n} - \sqrt{n(n-1)}\,\delta_{m,n-2} - \sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right)}$$

could also write off diagonal [matrix is hermitian → $\langle m | p^2 | n \rangle = \langle n | p^2 | m \rangle$]

such that $\Delta n = \pm 2$.

diagonal: $\langle \psi_n | p^2 | \psi_n \rangle = \frac{\hbar m\omega}{2}(2n+1)$; off-diag: $\langle \psi_{n-2} | p^2 \psi_n \rangle = -\frac{\hbar m\omega}{2}\sqrt{(n\pm1)(n\pm0)}$

bc negative → hermitian matrix p^2 has symmetric/real off diagonal.

(2.) Similar to in class: $\hat{H} = L_z^2/2mR^2, \; L_z = (\hbar/i)\partial_\phi$
HW pr 1

$$\frac{-\hbar^2}{2mR^2}\frac{\partial^2\psi}{\partial\phi^2} = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t}; \; \psi = \psi(\phi, t) \rightarrow \text{Solve to find } E_m \text{ and } \psi_m, \text{ degens?}$$

No perterbation thry bc. particle is free w/ no potential.
exact/ soluable w/ sep of vars.

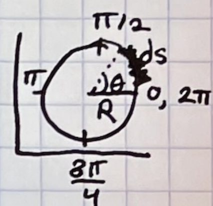
free particle schrö. in arclength $S = R\theta$

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi(s)}{\partial s^2} = E\psi(s).$$

in terms of θ. $\frac{\partial^2}{\partial s^2} = \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} \rightarrow \frac{-\hbar^2}{2mR^2}\frac{\partial^2\psi(\theta)}{\partial\theta^2} = E\psi(\theta)$

moment of Inertia: $I = MR^2$

$$\therefore \frac{-\hbar^2}{2I}\cdot\frac{\partial^2\psi(\theta)}{\partial\theta^2} = E\psi(\theta) \quad \ldots (1)$$



Periodic: thus $\psi(\theta + 2\pi) = \psi(\theta)$

We have ODE w/ general sol.: $\psi(\theta) = Ae^{ik\theta} + Be^{-ik\theta}$

let $\psi(\theta) = e^{ik\theta}$ in eq (1): $-\frac{\hbar^2}{2I}(-k^2)e^{ik\theta} = Ee^{ik\theta} \rightarrow E = \frac{\hbar^2 k^2}{2I}$ where k relates to Energy.

↳ Quantize + bandry: $[\psi(\theta + 2\pi) = \psi(\theta)]$

$$Ae^{ik(\theta+2\pi)} + Be^{-ik(\theta+2\pi)} = Ae^{ik\theta} + Be^{-ik\theta}$$

for this to hold for all non-trivial solutions, for all θ, A, B:

$$e^{ik2\pi} = 1; \; e^{-ik2\pi} = 1$$

$$k \cdot 2\pi = 2\pi m \text{ for all } m \Rightarrow k = m \in \mathbb{Z}$$

pr 2. cont.

for each int. m: $\boxed{\psi_m(\theta) = \frac{1}{\sqrt{2\pi}} e^{im\theta}, \quad m = 0, \pm 1, \pm 2}$

$\psi_m(\theta + 2\pi) = \frac{1}{\sqrt{2\pi}} e^{im(\theta+2\pi)} = e^{i2\pi m}\psi_m(\theta) = \psi_m(\theta)$ ✓ periodic

Since Eigenfunctions are orthonormal:

$$\int_0^{2\pi} \psi_m^*(\theta)\psi_{m'}(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m'-m)\theta}\,d\theta = \delta_{m,m'} \quad ✓$$

for allowed E: $k = m \rightarrow E = \frac{\hbar^2 k^2}{2I}$

$$\boxed{E_m = \frac{\hbar^2 m^2}{2I} = \frac{\hbar^2 m^2}{2MR^2}}$$

- $E \propto m^2$ $[E_m = E_{-m}]$
- $m = 0 \rightarrow E = 0$; stationary
- $|m| \propto E^2$

Degen:
- $m \neq 0$, $+m \neq -m$ [same energy, opposite angular momentum]
- $m = 0$; one state $\left(\psi_0(\theta) = \frac{1}{\sqrt{2\pi}}\right)$

$E_0 \rightarrow$ Non-degen $\rightarrow$ one state

$E_{m\neq 0} \rightarrow$ 2-fold degen $\rightarrow +m, -m$

Will note $\hat{P}_\theta = -i\hbar\frac{\partial}{\partial\theta}$

So: $\hat{P}_\theta \psi_m(\theta) = -i\hbar \frac{\partial}{\partial\theta}\left(\frac{1}{\sqrt{2\pi}} e^{im\theta}\right)$

$= \hbar m\, \psi_m(\theta)$ which are eigenstates of $\hat{P}$ w/ e-vals $\hbar m$.

$|\psi_m(\theta)|^2 = \frac{1}{2\pi} \rightarrow$ free motion

No preferred location on ring.

Summary pr 2:

- Eigenfunction: $\psi_m(\theta) = \frac{1}{\sqrt{2\pi}} e^{im\theta}$, $m \in \mathbb{Z}$
- $E_m = \frac{\hbar^2 m^2}{2MR^2}$
- $E_0 \rightarrow$ non-degen ☆
- $E_{m\neq 0} \rightarrow$ 2-fold degen
- $\hat{P}\psi_m(\theta) = \hbar m\, \psi_m(\theta)$
- $|\psi_m(\theta)|^2 = 1/2\pi$