Problem 8.2 R+L Solution

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Problem: R+L Chapter 8, Problem 8.2

Problem 1: We are given:

We are given wavefunction:

$$\psi(r,t) = \int_{-\infty}^{\infty} A(k) e^{i(kr-\omega(k)t)}\, dk$$

- $A(k)$ this is the spectral amplitude
- $\omega(k)$ dispersion relation (real)
- looks like superposition of monochome plane wave

Problem 2: Define the 'centroid'

we let the avg position be defined:

$$\langle r(t) \rangle = \frac{\int r|\psi(r,t)|^2\, dr}{\int |\psi(r,t)|^2\, dr}$$

- this is exactly like COM which here is just a weighted average of position over its probability density

Problem 3: time derivative of the numerator

lets take time derivative of the numerator:

$$\frac{d}{dt}\int r|\psi(r,t)|^2\, dr = \int r\frac{\partial}{\partial t}|\psi(r,t)|^2\, dr$$

then we can compute:

$$\frac{\partial}{\partial t}|\psi(r,t)|^2 = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}$$

Problem 4: Compute $\partial\psi/\partial t$

pick up at:

$$\psi(r,t) = \int A(k) e^{i(kr-\omega(k)t)}\, dk$$

then:

$$\frac{\partial \psi}{\partial t} = \int A(k)\left(-i\omega(k)\right) e^{i(kr-\omega(k)t)}\, dk = -i\int \omega(k) A(k) e^{i(kr-\omega(k)t)}\, dk$$

therefore:

$$\frac{\partial \psi^*}{\partial t} = i\int \omega(k) A^*(k) e^{-i(kr-\omega(k)t)}\, dk$$

Problem 5: put it back into time dervative of density

$$\frac{\partial}{\partial t}|\psi|^2 = -i\psi^* \int \omega(k)A(k)e^{i(kr-\omega(k)t)}dk + i\psi \int \omega(k)A^*(k)e^{-i(kr-\omega(k)t)}dk$$

I calculate this as:

$$\frac{\partial}{\partial t}|\psi|^2 = 2\,\mathrm{Im}\left[\psi^* \cdot \left(\int \omega(k)A(k)e^{i(kr-\omega(k)t)}\,dk\right)\right]$$

Problem 6: put that into derivative of $\langle r \rangle$

$$\frac{d}{dt}\langle r(t)\rangle = \frac{\int r \cdot \frac{\partial}{\partial t}|\psi|^2\,dr}{\int |\psi|^2\,dr}$$

after this the numerator becomes:

$$2\int r \cdot \mathrm{Im}\left[\psi^*(r,t) \cdot \left(\int \omega(k)A(k)e^{i(kr-\omega(k)t)}\,dk\right)\right]dr$$

Problem 7: do a transform to to k-space

this is where i got hung up. I was spinning my wheels in position space, super gross. I took this into k-space. I did this by using:

$$\psi(r,t) = \int A(k)e^{i(kr-\omega(k)t)}dk \Rightarrow \tilde{\psi}(k,t) = A(k)e^{-i\omega(k)t}$$

then:

$$|\psi(r,t)|^2 = \int\int A(k)A^*(k')e^{i[(k-k')r-(\omega(k)-\omega(k'))t]}dk\,dk'$$

when i plug this into numerator $\int r|\psi|^2 dr$, and can clearly see:

$$\int re^{i(k-k')r}dr = i\delta'(k-k')$$

this was the eurika for me, this identity simplifies the position integral:

$$\int r|\psi(r,t)|^2\,dr = \int A(k)A^*(k') \cdot i\delta'(k-k')e^{-i(\omega(k)-\omega(k'))t}\,dk\,dk'$$

from propretys of the delta function + IBP (sorry a lot to latex type) i see it reduces to:

$$\int |A(k)|^2\frac{d\omega}{dk}\,dk$$

Problem 8: Final Result

so, if we take the time dervative of the centroid:

$$\frac{d}{dt}\langle r(t)\rangle = \frac{\int \frac{d\omega}{dk}|A(k)|^2 dk}{\int |A(k)|^2 dk} = \left\langle \frac{d\omega}{dk} \right\rangle$$

$$\boxed{\frac{d}{dt}\langle r(t)\rangle = \left\langle \frac{d\omega}{dk} \right\rangle}$$