Problem 7.3 R+L Solution

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Problem: R+L Chapter 7, P7.3

Problem Statement: Show that the photon energy in the electron rest frame is small compared to mc^2 for the following cases:

(a) Electrons with $\gamma \sim 10^4$ scattering synchrotron photons produced in a magnetic field $B \sim 0.1G$ (typical of compact radio sources).

(b) Electrons with $\gamma \sim 10^4$ scattering the 3 K photons of the cosmic microwave background.

Problem 1: My Comprehension

1 Synchrotron Photons

Using Equation 6.17C in R+L: The characteristic synchrotron frequency is given by:

$$\omega_c = \frac{3}{2}\gamma^2 \frac{qB}{mc} \sin\alpha$$

Here: where:

- ω_c angular frequency of the emitted synchrotron radiation
- γ is the Lorentz factor of the electron
- B is the magnetic field strength
- α is the pitch angle (defined as the angle between velocity and magnetic field direction)
- m is the mass of the e^-
- all calculation will be done in CGS but symbolic manipulation remains unchanged.

As the book did I take $\sin\alpha = 3^{-1/2}$ or $\sin\alpha = \frac{1}{\sqrt{3}}$ After a bit more reading this is a common convention to average out scattering angles over a range of frequencies?

Problem 2: Angular Frequency and Photon Energy

We know frequency and energy are related simply by Plank's constant:

$$E = h\nu = \hbar\omega$$

where $h = 2\pi\hbar$. Substituting for ω_c:

$$E_c = \hbar\omega_c = \frac{3}{2}\gamma^2 \frac{\hbar qB}{mc} \sin\alpha$$

Since $\hbar = h/(2\pi)$, we rewrite:

$$E_c = \frac{3}{2}\gamma^2 \frac{h}{2\pi} \frac{qB}{mc} \sin\alpha \tag{1}$$

This is where I believe I might have had some misunderstandings. I will state all my assumptions about frames below for the reader to check.

Problem 3: Boosting/Understanding Reference Frames

- The problem asks whether the *photon energy* in the e^- is small compared to mc^2.
- I believe this means we are looking for the photon's energy *in the electron's rest frame* rather than the 'lab frame'

I will transform the synchrotron photon energy to the *electron's rest frame.* Given Equation (1):

$$E_c = \hbar\omega_c = \frac{3}{2}\gamma^2\frac{\hbar qB}{mc}\sin\alpha$$

I understand this to be the typical energy of synchrotron photons as measured in the lab frame. When the electron moves with relativistic speed ($\gamma \gg 1$), the photon in the lab frame will appear more energetic in the e^- rest frame from a Doppler boost. In the electron's rest frame, that photon energy is increased by an additional factor of γ

$$E' = \gamma E_c$$

This boost will account for how the electron 'see's' the incoming photon at relativistic speeds. Such that:

- E_c is the characteristic energy of synchrotron photon in the lab frame.
- E' is that same photon's energy but transformed into the electron's rest frame / how the electron would percieve it.

Now that we have transformed the synchrotron photon energy, we can compare it to the electrons rest energy.

$$\frac{E'}{mc^2} = \frac{\gamma E_c}{mc^2}$$

This is where another factor γ factor comes from. We spoke about this on our 1/28/2025 group meeting, where I mistakenly had it 2 factors of gamma higher. We boost the synchrotron photon energy into the electrons rest frame.

Okay, that was the sticky business for me. If that is wrong I need to relook at it. However, I will continue from here.

Problem 4: Express in Terms of Electron Rest Energy

The rest energy of an electron, e^- is:

$$E_e = mc^2$$

$$\approx 0.511 \text{ MeV} = 5.11 \times 10^5 \text{ eV}$$

Problem 5: Computing the Ratio

We can now compute the ratio

$$\frac{E'}{mc^2} = \frac{\gamma E_c}{mc^2} = \gamma \frac{\frac{3}{2}\gamma^2 \frac{hqB}{2\pi mc} \sin\alpha}{mc^2}$$

where:

- $E_c = \frac{3}{2}\gamma^2 \frac{hqB}{2\pi mc} \sin\alpha$ (synchrotron photon energy in the lab frame),
- $E' = \gamma E_c$ (synchrotron photon energy in the electron's rest frame).

Just breaking into numerator and denominator

$$\text{Numerator: } \gamma E_c = \gamma \times (\frac{3}{2}\gamma^2 \frac{hqB}{2\pi mc} \sin\alpha) = \frac{3}{2}\gamma^3 \frac{hqB}{2\pi mc} \sin\alpha$$

$$\text{Denominator: } mc^2$$

Therefore:

$$\frac{\gamma E_c}{mc^2} = \frac{\frac{3}{2}\gamma^3 \frac{hqB}{2\pi mc} \sin\alpha}{mc^2}$$

Simplifying the expression:

$$\frac{\gamma E_c}{mc^2} = \frac{3}{2} \frac{hqB}{2\pi} \frac{\gamma^3}{m^2 c^3} \sin\alpha$$

Which is the final expression for ratio of synchrotron photon energy in the electrons rest frame vs. the electrons rest energy.

Problem 6: Solving Numerically

Using the following numerical values (will always stay in CGS for R+L at Brandon's advice)

- $h = 6.63 \times 10^{-27}$ erg·s
- $q = 4.8 \times 10^{-10}$ statC
- $m = 9.11 \times 10^{-28}$ g
- $c = 3 \times 10^{10}$ cm/s
- $B = 0.1$ G (Gauss)
- $\sin\alpha = 3^{-1/2}$

Evaluating:

$$\frac{\gamma E_c}{mc^2} \approx 1.96 \times 10^{-3} \ll 1$$

Thus, we can conclude that the photon energy in the electron's rest frame is **very small** compared to mc^2.

Numpy Implementation - 7.3a

```
import numpy as np

# double check before presenting, using CGS units
h = 6.63e-27  # erg·s (planck's constant)
q = 4.8e-10  # statC (electron charge)
m = 9.11e-28  # g (electron mass)
c = 3e10  # cm/s (speed of light)
B = 0.1  # G (magnetic field strength in Gauss)

sin_alpha = 1 / np.sqrt(3)  # average over pitch angle

mc2 = 0.511e6  # eV (electron rest energy)
gamma = 10**4  # given Lorentz factor

# calculate ratio in CGS
numerical_ratio = (3/2) * (h * q * B / (2 * np.pi)) * #new line for formatting
(gamma**3 / (m**2 * c**3)) * sin_alpha

# display in scientific notation
print(f"Numerical ratio: {numerical_ratio:.2e}")
```

Output:

$$\text{Numerical ratio: } 1.96 \times 10^{-3} \approx 2 \times 10^{-3} \text{ (R+L solution)}$$

Problem 7: CMB Background

For part B, we are looking at the Cosmic Microwave Background (CMB) photons. The energy of a blackbody photon at a temperature T is:

$$E_{\text{CMB}} = 2.8 k_B T$$

Given our CMB with $T = 3K$ and the Boltzmann constant:

$$k_B = 1.38 \times 10^{-16} \text{ erg/K}$$

we compute:

$$E_{\text{CMB}} = 2.8 \times (1.38 \times 10^{-16} \text{ erg/K}) \times 3K$$

which is:

$$E_{\text{CMB}} \approx 1.416 \times 10^{-5} \text{ erg}$$

Problem 8: Transforming to e^- Rest Frame

Again we expect the CMB photon in the electron's rest frame to be boosted by a factor of γ, that is:

$$E' = \gamma E_{\text{CMB}}$$

Computing the ratio to the electron rest mass:

$$\frac{\gamma E_{\text{CMB}}}{mc^2} = \frac{\gamma(7.22 \times 10^{-4}\ \text{eV})}{0.511 \times 10^6\ \text{eV}}$$

Simplifying:

$$\frac{\gamma E_{\text{CMB}}}{mc^2} \approx 1.41 \times 10^{-5}$$

Since $1.41 \times 10^{-5} \ll 1$, we confirm that the CMB photon energy in the electron rest frame is also small compared to mc^2.

Numpy Implementation 7.3b

```
import numpy as np
#redo cgs
k_B = 1.38e-16  # erg/K (google says this is kb in cgs)
T_CMB = 3  #K (temp of CMB)
mc2 = 0.511e6 * 1.602e-12  #erg electron rm in cgs

#calculate CMB photon energy in CGS
E_CMB = 2.8 * k_B * T_CMB  #erg

#transform to electron rest frame by 1 factor of gamma
E_prime = gamma * E_CMB  #transform

#compute ratio
ratio_cmb = E_prime / mc2

#display in scientific notation
print(f"Ratio: {ratio_cmb:.3e}", 'unitless')
```

Output:

$$\text{Numerical ratio: } 1.41 \times 10^{-5}$$

In part A and B, the ratios show that the photon energy in the electron's rest frame is way wayyyy smaller than mc^2 thus the electrons interact with low-energy photons in their own rest frame??

Problem 9: Preemptively thinking about other questions from this problem

- **Choice of reference frame influence on observed photon energy rest and lab frame**

  I know reference frame is important, it affects photons energy bc relativistic transforms. In lab, photon energy is measured directly, but in e^- rest frame the photon energy is boosted by a factor of γ from the relative motion between the e^- and the photon. Thus higher observed photon energy in electron's rest frame. This results in higher observed photon energy in the electron's rest frame compared to the lab frame. I explore this a bit later w/ plot

- **How scattering of low energy photons (like CMB) differ from higher-energy synchrtron scattering**

  I believe CMB/low energy photons are boosted by a smaller factor in the electrons rest frame and thus the energy shift is smaller for low-energy photons like the CMB. They are boosted by a smaller factor in the electron rest frame compared to synchrotron photons. Since the energy shift is smaller for CMB photons and I imagine the subsequent effects when interacting with higher energy photons (like in synchrotron) would be more pronounced? compton/energy transfer

- **i'm sure there will be more**

- Graphs etc

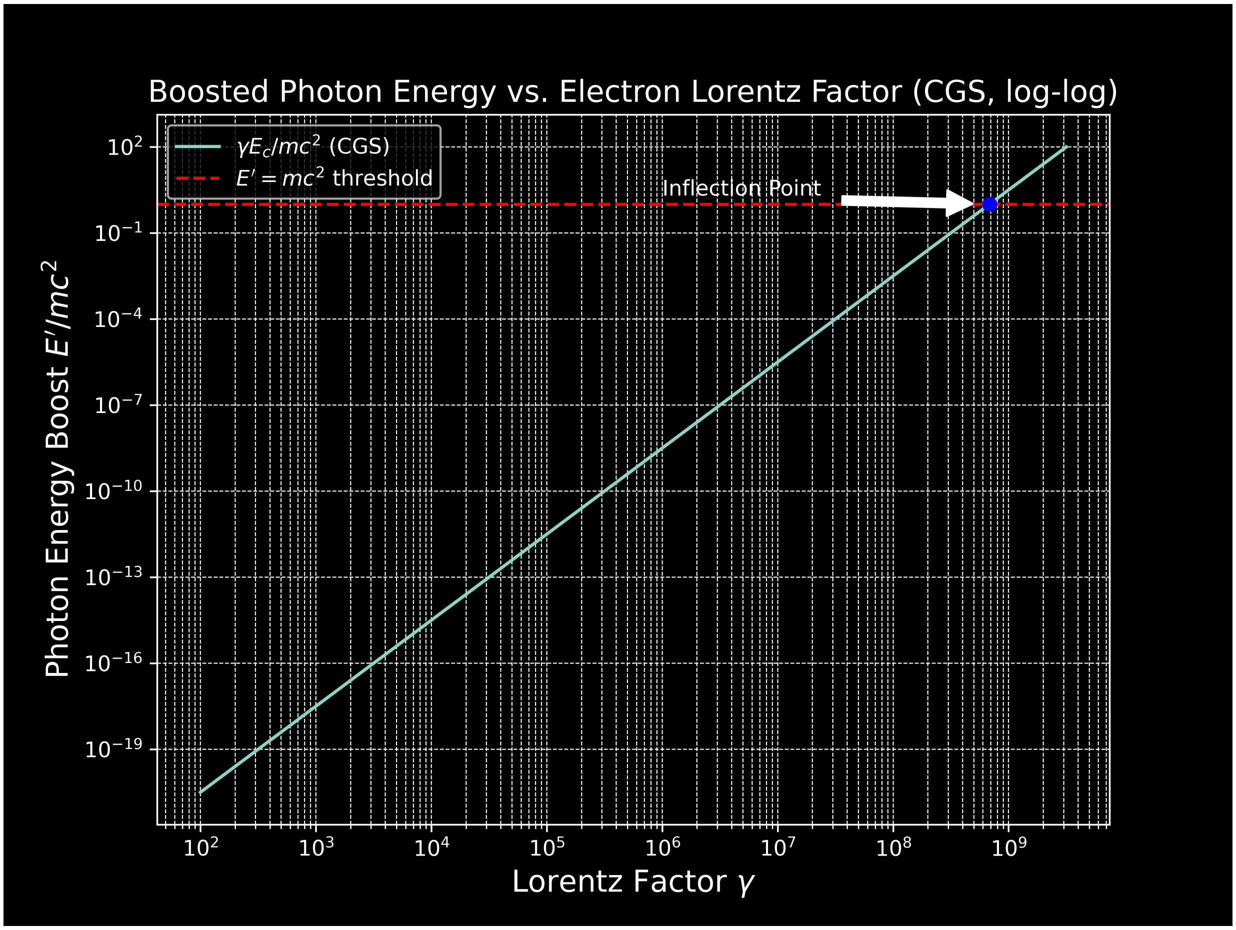

Figure 1: Lorentz Factor as a Function of Photon Energy Boost

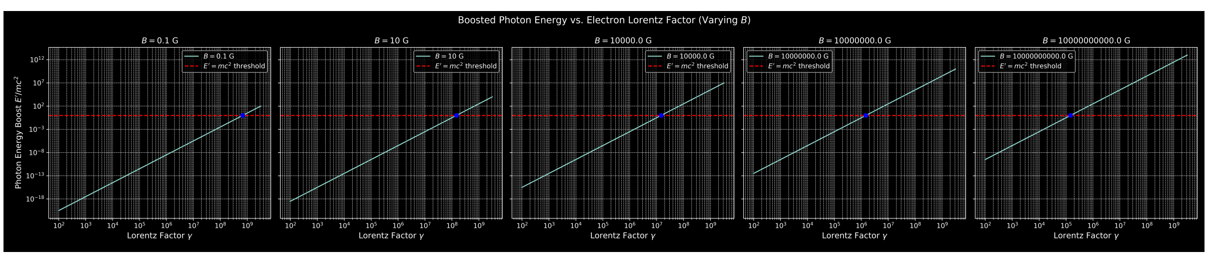

Figure 2: Boosted Photons with Different B Strength

Click here for full size:
This graph is to show the overall shape on the scenario.

- As γ increases, the photon energy in the electrons rest frame scales as γ^3 not γ^4 as I previously had.
- log, log scale, follows a power-law increase
- below $\gamma \leq 10^5$ is the threshold. our scenario is well below that

- not sure what happens when we cross the threshold, does this make sense in a physics context?
- if i exteded this to higher B fields or different photon sources (CMB v Synchrton) i think the scaling would hold but values would shift
- will likely try to play this out a bit more by next week
- been reading a bit on the distinction between compton/thompson and KN effects?

Github Links, Solutions, Graphs

- R+L Aaron's Solution Repo, always up-to-date
- R+L P7.3 Solution Folder, this specfic problem
- R+L 7.3 Solution Notebook, python implementation of 7.3

Full-Size Graphs

- Figure 1: Photon Energy vs Gamma
- Figure 2: Boosted Photons with Different B Strength
- Animation 3: Boosted Energy Animation