Differential Geometry

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1 Smooth Manifolds

Definition 1.1 (Chart)**.** An n-dimensional chart on a set M is a map $\varphi : U \to \tilde{U}$, where:

- $U \subseteq M$ is a subset,
- $\tilde{U} \subseteq \mathbb{R}^n$ is a non-empty open subset,
- φ is a bijection.

A chart is often denoted by (U, φ), and $\tilde{U}$ always refers to the range of φ.

Then U is called a *coordinate domain* and φ the *coordinate map.* The coordinate map is a vector-valued function with components usually denoted as $x^1, \dots, x^n$ (or $y^1, \dots, y^n$, etc.), and called *coordinates on* U determined by the chart (U, φ). Accordingly, we often write $(U, \varphi = (x^1, \dots, x^n))$.

Definition 1.2 (Chart Around a Point)**.** A chart on M *around a point* $p \in M$ is a chart (U, φ) such that $p \in U$.

Definition 1.3 (Centered Chart)**.** A chart (U, φ) around a point $p \in M$ is said to be *centered at* p if $\tilde{U} = \varphi(U) \ni 0$ and $\varphi(p) = 0$.

Example 1.1. The identity map $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is an n-dimensional chart on $\mathbb{R}^n$ whose associated coordinates are the standard coordinates $t^1, \dots, t^n$.

The chart $(\mathbb{R}^n, \mathrm{id})$ is also called the *standard chart* on $\mathbb{R}^n$.

The standard chart on $\mathbb{R}^n$ is centered at 0.

Definition 1.4 (Coordinate Map Associated with a Frame)**.** Let V be an n-dimensional real vector space, and let $R = (e_1, \dots, e_n)$ be a basis (frame) of V. The *coordinate map associated with* R is the vector space isomorphism

$$\varphi_R : V \to \mathbb{R}^n, \quad v \mapsto \varphi_R(v) := \Big(x^1(v), \dots, x^n(v)\Big),$$

where the components $x^i(v)$ are determined by the unique expression

$$v = \sum_{i=1}^{n} x^i(v) e_i.$$

Example 1.2. Example 1.1 can be generalized by considering any n-dimensional real vector space V and any frame $R = (e_1, \ldots, e_n)$ of V.

The coordinate map $\varphi_R : V \to \mathbb{R}^n$ defined as in Definition 1.4 maps each vector $v \in V$ to its coordinate tuple with respect to the frame R.

The pair (V, φ_R) forms an n-dimensional chart on V, and it is centered at 0, since $\varphi_R(0) = 0$ and $\varphi_R(V) = \mathbb{R}^n$.

Example 1.3 (Stereographic Charts on the Sphere)**.** Consider the Euclidean space $\mathbb{R}^{n+1}$ with standard coordinates $t^1, \ldots, t^{n+1}$. For a point $P = (P^1, \ldots, P^{n+1}) \in \mathbb{R}^{n+1}$, define the Euclidean norm by

$$\|P\| := \sqrt{(P^1)^2 + \cdots + (P^{n+1})^2}.$$

The n-dimensional unit sphere is the subset

$$S^n := \left\{ P \in \mathbb{R}^{n+1} \mid \|P\| = 1 \right\}.$$

Define the north and south poles as

$$P_+ := (0, \ldots, 0, 1), \quad P_- := (0, \ldots, 0, -1),$$

and the corresponding open subsets

$$U_+ := S^n \setminus \{P_+\}, \quad U_- := S^n \setminus \{P_-\}.$$

For $P \in U_+$, the line through P and P_+ intersects the hyperplane $t^{n+1} = 0$ at a unique point with coordinates $(X_+^1(P), \ldots, X_+^n(P))$. This defines the *stereographic projection from the north*:

$$\varphi_+ : U_+ \to \mathbb{R}^n, \quad \varphi_+(P) := (X_+^1(P), \ldots, X_+^n(P)).$$

Similarly, projection from P_- defines the *stereographic projection from the south*:

$$\varphi_- : U_- \to \mathbb{R}^n, \quad \varphi_-(P) := (X_-^1(P), \ldots, X_-^n(P)).$$

Both (U_+, φ_+) and (U_-, φ_-) are n-dimensional charts on S^n. The chart from the north is centered at the south pole P_-, and the chart from the south is centered at the north pole P_+.

Example 1.4 (Orthogonal Projection Charts on the Sphere)**.** Let $S^n \subset \mathbb{R}^{n+1}$ be the n-dimensional sphere defined as before.

We first define the open unit n-disk:

$$D^n := \{P \in \mathbb{R}^n \mid \|P\| < 1\},$$

where $\|x\|$ denotes the standard Euclidean norm in $\mathbb{R}^n$.

For each index $i = 1, \ldots, n+1$, we consider the open subsets of S^n:

$$U_{i,\pm} := \left\{ P \in S^n \mid \pm P^i > 0 \right\},$$

which are the portions of the sphere where the i-th coordinate is strictly positive (for $U_{i,+}$) or strictly negative (for $U_{i,-}$).

Now define the orthogonal projection map:

$$\pi_i : \mathbb{R}^{n+1} \to \mathbb{R}^n, \quad \pi_i(P^1, \dots, P^{n+1}) := (P^1, \dots, \widehat{P^i}, \dots, P^{n+1}),$$

where the hat $\widehat{P^i}$ means that the i-th component is omitted. This is simply the projection of a point in $\mathbb{R}^{n+1}$ onto the hyperplane $t^i = 0$.

It can be shown that the restriction of π_i to $U_{i,\pm}$ maps onto the open unit disk D^n:

$$\pi_i : U_{i,\pm} \to D^n.$$

Thus, each pair $(U_{i,\pm}, \pi_i)$ defines an n-dimensional chart on S^n. These are called the *orthogonal projection charts* onto the $t^i = 0$ hyperplane.

Each such chart is centered at the point

$$(0, \dots, \pm 1, \dots, 0) \in S^n,$$

where the value ± 1 occurs in the i-th coordinate.

Example 1.5 (Affine Charts on the Projective Space)**.** Consider $\mathbb{R}^{n+1} \setminus \{0\}$ with standard coordinates $(t^0, \dots, t^n)$. Define an equivalence relation $\sim$ by declaring

$$P \sim Q \quad \text{if and only if} \quad P = \lambda Q \quad \text{for some } \lambda \in \mathbb{R} \setminus \{0\}.$$

That is, two points are equivalent if they lie on the same line through the origin.

The n-dimensional real projective space is defined as the quotient

$$\mathbb{RP}^n := \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \Big/ \sim .$$

Equivalently, $\mathbb{RP}^n$ is the set of lines through the origin in $\mathbb{R}^{n+1}$. An element of $\mathbb{RP}^n$ is denoted by an equivalence class

$$[P] = [P^0 : \cdots : P^n],$$

where $(P^0, \dots, P^n) \in \mathbb{R}^{n+1} \setminus \{0\}$ and the P^i are called *homogeneous coordinates* of $[P]$.

To define charts on $\mathbb{RP}^n$, fix $i \in \{0, \dots, n\}$ and consider the open subset

$$U_i := \left\{[P^0 : \cdots : P^n] \in \mathbb{RP}^n \mid P^i \neq 0\right\}.$$

On U_i, we define the coordinate map $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i([P^0 : \cdots : P^n]) := \left(\frac{P^0}{P^i}, \dots, \frac{\widehat{P^i}}{P^i}, \dots, \frac{P^n}{P^i}\right),$$

where the hat $\widehat{\cdot}$ indicates omission of the i-th coordinate. This map is well-defined and bijective.

The inverse is given by

$$\varphi_i^{-1}(Q^1, \dots, Q^n) = [Q^1 : \cdots : 1 : \cdots : Q^n],$$

where the number 1 is placed in the i-th position (i.e., the coordinate that was omitted).

Each pair (U_i, φ_i) defines an n-dimensional chart on $\mathbb{RP}^n$, called an *affine chart*. The chart (U_i, φ_i) is centered at the point

$$[0 : \cdots : 1 : \cdots : 0],$$

where the 1 appears in the i-th position.

Remark 1.1. Given an n-dimensional chart (U, φ) on a set M, we can use the coordinate map φ to identify the coordinate domain U with its image $\tilde{U} := \varphi(U) \subset \mathbb{R}^n$. This allows us to transfer notions from calculus such as continuity, differentiability, and smoothness from $\tilde{U}$ to U.

For example, a function $f : U \to \mathbb{R}$ is said to be *smooth at a point* $p \in U$ if the composition $f \circ \varphi^{-1} : \tilde{U} \to \mathbb{R}$ is smooth at $\varphi(p)$.

To extend this idea to the entire set M, we require a collection of charts $\mathcal{A} = \{(U, \varphi)\}$ that covers M i.e., $M = \bigcup_{(U,\varphi)\in\mathcal{A}} U$. Then, for a function $f : M \to \mathbb{R}$ and a point $p \in M$, we define f to be *smooth at* p if there exists a chart $(U, \varphi) \in \mathcal{A}$ around p such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth at $\varphi(p)$.

However, this definition may depend on the choice of chart unless the family $\mathcal{A}$ satisfies certain compatibility conditions. This leads us to the notion of a *smooth atlas*.

Definition 1.5 (Compatible Charts)**.** Let $(U, \varphi = (x^1, \ldots, x^n))$ and $(V, \psi = (y^1, \ldots, y^n))$ be two n-dimensional charts on a set M. These charts are said to be *compatible* if either:

- The domains do not overlap: $U \cap V = \emptyset$, or
- The domains overlap ($U \cap V \neq \emptyset$), and the following two conditions are satisfied:
    1. The images $\varphi(U \cap V) \subset \mathbb{R}^n$ and $\psi(U \cap V) \subset \mathbb{R}^n$ are open subsets.
    2. The map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a *diffeomorphism*, that is, a smooth bijection with a smooth inverse.

The map $\psi \circ \varphi^{-1}$ is called the *transition map* between the charts (U, φ) and (V, ψ), or between the coordinate systems $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$.

Definition 1.6 (Atlas)**.** An *n-dimensional atlas* (or *smooth atlas*) on a set M is a collection $\mathcal{A} = \{(U, \varphi)\}$ of n-dimensional charts satisfying the following conditions:

- **Covering:** The charts cover M, that is,

$$M = \bigcup_{(U,\varphi)\in\mathcal{A}} U.$$

- **Compatibility:** Any two charts in $\mathcal{A}$ are pairwise compatible.

An atlas allows us to transfer the tools of calculus from $\mathbb{R}^n$ to the set M by using coordinate charts.

Before introducing examples, we note that there are several ways to generate new charts that are compatible with those already in the atlas.