Measure Theory

1 σ-Algebras

Definition 1.1. A σ-algebra $\mathcal{A}$ on a set X is a family of subsets of X such that:

1. $X \in \mathcal{A}$ (M1)
2. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ (M2)
3. If $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ (M3)

A set $A \in \mathcal{A}$ is said to be *measurable* or $\mathcal{A}$-*measurable*.

Example 1.1.

1. $\mathcal{P}(X)$ is a σ-algebra (the maximal σ-algebra on X).
2. $\{\emptyset, X\}$ is a σ-algebra (the minimal σ-algebra on X).
3. $A := \{A \subseteq X : \#A < \infty \text{ or } \#A^c < \infty\}$ is a σ-algebra.
4. (Trace σ-algebra) Let $E \subseteq X$ be any set and let $\mathcal{A}$ be a σ-algebra on X. Then
$$\mathcal{A}_E := \{E \cap A : A \in \mathcal{A}\}$$
is a σ-algebra on E.

Proof. We verify the three defining properties of a σ-algebra on E:

(a) Since $X \in \mathcal{A}$, we have $E = E \cap X \in \mathcal{A}_E$.
(b) If $E \cap A \in \mathcal{A}_E$, then $E \setminus (E \cap A) = E \cap A^c$, and since $A^c \in \mathcal{A}$, it follows that $E \cap A^c \in \mathcal{A}_E$.
(c) If $(E \cap A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_E$, then $\bigcup_n (E \cap A_n) = E \cap \bigcup_n A_n$, and since $\bigcup_n A_n \in \mathcal{A}$, we conclude that $\bigcup_n (E \cap A_n) \in \mathcal{A}_E$.

Hence, $\mathcal{A}_E$ is a σ-algebra on E. □

5. (Pre-image σ-algebra) Let $f : X \to X'$ be a function and let $\mathcal{A}'$ be a σ-algebra on X'. Then

$$\mathcal{A} := \{f^{-1}(A') : A' \in \mathcal{A}'\}$$

is a σ-algebra on X.

Theorem 1.1. Let X be a set and let $\{\mathcal{A}_i : i \in I\}$ be a family of σ-algebras on X. Define

$$\mathcal{A} := \bigcap_{i \in I} \mathcal{A}_i = \{A \subseteq X : A \in \mathcal{A}_i \text{ for all } i \in I\}.$$

Then, $\mathcal{A}$ is a σ-algebra on X.

Proof. We verify the σ-algebra properties for $\mathcal{A}$:

1. Since $X \in \mathcal{A}_i$ for all $i \in I$, we have $X \in \mathcal{A}$.
2. If $A \in \mathcal{A}$, then $A \in \mathcal{A}_i$ for all $i \in I$, so $A^c = X \setminus A \in \mathcal{A}_i$ for all $i \in I$, hence $A^c \in \mathcal{A}$.
3. If $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$, then $A_n \in \mathcal{A}_i$ for all n and i, so $\bigcup_{n=0}^{\infty} A_n \in \mathcal{A}_i$ for all $i \in I$, and thus $\bigcup_{n=0}^{\infty} A_n \in \mathcal{A}$.

Therefore, $\mathcal{A}$ is a σ-algebra on X. $\square$

Definition 1.2. Let X be a set and let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. The *σ-algebra generated by* $\mathcal{E}$, denoted by $\sigma(\mathcal{E})$, is the smallest σ-algebra on X containing all sets in $\mathcal{E}$. That is,

$$\sigma(\mathcal{E}) := \bigcap \big\{\mathcal{A} \subseteq \mathcal{P}(X) : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X,\ \mathcal{E} \subseteq \mathcal{A}\big\}.$$

Remark 1.1 (Generated σ-algebras)**.**

1. If $\mathcal{A}$ is a σ-algebra, then $\sigma(\mathcal{A}) = \mathcal{A}$.
2. For $A \subseteq X$, we have $\sigma(\{A\}) = \{\emptyset, A, A^c, X\}$.
3. If $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{A}$, then $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G}) \subseteq \sigma(\mathcal{A})$.

Definition 1.3 (Topological Space)**.** A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of subsets of X, called *open sets*, satisfying the following properties:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,

2. If $\{U_\alpha \in \mathcal{T} : \alpha \in I\}$ is an arbitrary collection of open sets, then the union $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$,
3. If $\{U_i \in \mathcal{T} : i = 1, \ldots, N\}$ is a finite collection of open sets, then the intersection $\bigcap_{i=1}^{N} U_i \in \mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X. The complement of an open set is called a *closed set*.

Remark 1.2 (Standard Topology on $\mathbb{R}^n$)**.** A subset $U \subseteq \mathbb{R}^n$ is called *open* if for every point $x \in U$, there exists an $\varepsilon > 0$ such that the open ball

$$B_\varepsilon(x) := \{y \in \mathbb{R}^n : \|x - y\| < \varepsilon\},$$

where $\|\cdot\|$ denotes the Euclidean norm, is contained in U; that is, $B_\varepsilon(x) \subseteq U$.

The collection of all such open sets is denoted by $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ and forms the *standard topology* on $\mathbb{R}^n$.

Definition 1.4 (Borel σ-algebra)**.** The σ-algebra $\sigma(\mathcal{O})$ generated by the collection of open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called the *Borel σ-algebra* on $\mathbb{R}^n$.

Its elements are called *Borel sets* or *Borel measurable sets*. We denote the Borel σ-algebra by $\mathcal{B}(\mathbb{R}^n)$.

Definition 1.5. Let X be a topological space and let $A \subseteq X$. A collection $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ of open sets is called an *open cover* of A if

$$A \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

A *subcover* is a subcollection that still covers A. The set A is called *compact* if every open cover of A admits a finite subcover.

Remark 1.3. In $\mathbb{R}^n$, a set is compact if and only if it is closed and bounded (Heine–Borel Theorem).

Theorem 1.2 (Borel σ-algebra from Different Generators)**.** Let $\mathcal{O}, \mathcal{C}, \mathcal{K} \subseteq \mathcal{P}(\mathbb{R}^n)$ denote the collections of open, closed, and compact subsets of $\mathbb{R}^n$, respectively. Then,

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}) = \sigma(\mathcal{C}) = \sigma(\mathcal{K}).$$

Proof. Since compact sets are closed, we have $\mathcal{K} \subseteq \mathcal{C}$, and by Remark 1.1(3), $\sigma(\mathcal{K}) \subseteq \sigma(\mathcal{C})$. Conversely, for any $C \in \mathcal{C}$, define $C_k := C \cap B_k(0)$, where $B_k(0)$ is the closed ball of radius k centered at the origin. Each C_k is closed and bounded, hence compact, so $C_k \in \mathcal{K}$. Since $C = \bigcup_{k\in\mathbb{N}} C_k$, it follows that $C \in \sigma(\mathcal{K})$, and thus $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{K})$.

Next, since $\mathcal{C} = \mathcal{O}^c := \{U^c : U \in \mathcal{O}\}$, and complements of sets in a σ-algebra are again in the σ-algebra, it follows that $\mathcal{C} \subseteq \sigma(\mathcal{O})$, hence $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{O})$. The reverse inclusion follows similarly from $\mathcal{O} = \mathcal{C}^c$. Therefore, we have:

$$\sigma(\mathcal{K}) = \sigma(\mathcal{C}) = \sigma(\mathcal{O}).$$

□

Generating Sets of the Borel Algebra. The Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ can be generated by various systems of sets. Of particular importance are:

- The family of open rectangles:

$$\mathcal{J}_{o,n} := \{(a_1, b_1) \times \cdots \times (a_n, b_n) : a_i, b_i \in \mathbb{R}\},$$

- The family of half-open rectangles:

$$\mathcal{J}_n := \{[a_1, b_1) \times \cdots \times [a_n, b_n) : a_i, b_i \in \mathbb{R}\}.$$

We denote by $\mathcal{J}_n^{\mathrm{rat}}, \mathcal{J}_{o,n}^{\mathrm{rat}}$ the subsets with rational endpoints. These sets represent intervals in $\mathbb{R}$, rectangles in $\mathbb{R}^2$, cuboids in $\mathbb{R}^3$, and hypercubes in higher dimensions.

Theorem 1.3. We have the following equality of Borel σ-algebras on $\mathbb{R}^n$:

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{J}_n^{\mathrm{rat}}) = \sigma(\mathcal{J}_{o,n}^{\mathrm{rat}}) = \sigma(\mathcal{J}_n) = \sigma(\mathcal{J}_{o,n}),$$

Remark 1.4. Let $D \subseteq \mathbb{R}$ be a dense subset, for example $D = \mathbb{Q}$ or $D = \mathbb{R}$. Then the Borel sets on $\mathbb{R}$ can also be generated by any of the following families of intervals:

$$\{(-\infty, a) : a \in D\}, \quad \{(-\infty, a] : a \in D\}, \quad \{(a, \infty) : a \in D\}, \quad \{[a, \infty) : a \in D\}.$$

2 Measure Spaces

Definition 2.1. A *(positive) measure* μ on X is a map $\mu : \mathcal{A} \to [0, \infty]$ where $\mathcal{A}$ is a σ-algebra on X, satisfying:

$$\mu(\emptyset) = 0, \tag{M1}$$

and for any pairwise disjoint sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$,

$$\mu\left(\bigsqcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n). \tag{M2}$$

If μ satisfies (M1), (M2) but $\mathcal{A}$ is not a σ-algebra, then μ is called a *premeasure*.

Remark 2.1. (M2) requires implicitly that $\bigsqcup_n A_n$ is again in $\mathcal{A}$ this is clearly the case for σ-algebras, but needs special attention when dealing with premeasures.

Definition 2.2 (Monotone sequences of sets)**.** Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be sequences of subsets of X.

We say (A_n) is *increasing* if

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots,$$

and write $A_n \nearrow A$ where

$$A := \bigcup_{n\in\mathbb{N}} A_n.$$

Similarly, (B_n) is *decreasing* if

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots,$$

and write $B_n \searrow B$ where

$$B := \bigcap_{n\in\mathbb{N}} B_n.$$