Measure Theory

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1 σ-Algebras

Definition 1.1. A σ-algebra $\mathcal{A}$ on a set X is a family of subsets of X such that:

- $X \in \mathcal{A}$ $\quad (\mathbf{\Sigma}_1)$
- If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ $\quad (\mathbf{\Sigma}_2)$
- If $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ $\quad (\mathbf{\Sigma}_3)$

A set $A \in \mathcal{A}$ is said to be *measurable* or $\mathcal{A}$-*measurable*.

Example 1.1.

1. $\mathcal{P}(X)$ is a σ-algebra (the maximal σ-algebra on X).
2. $\{\emptyset, X\}$ is a σ-algebra (the minimal σ-algebra on X).
3. $A := \{A \subseteq X : \#A < \infty \text{ or } \#A^c < \infty\}$ is a σ-algebra.
4. (Trace σ-algebra) Let $E \subseteq X$ be any set and let $\mathcal{A}$ be a σ-algebra on X. Then
$$\mathcal{A}_E := \{E \cap A : A \in \mathcal{A}\}$$
is a σ-algebra on E.

Proof. We verify the three defining properties of a σ-algebra on E:

- Since $X \in \mathcal{A}$, we have $E = E \cap X \in \mathcal{A}_E$.
- If $E \cap A \in \mathcal{A}_E$, then $E \setminus (E \cap A) = E \cap A^c$, and since $A^c \in \mathcal{A}$, it follows that $E \cap A^c \in \mathcal{A}_E$.
- If $(E \cap A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_E$, then $\bigcup_n (E \cap A_n) = E \cap \bigcup_n A_n$, and since $\bigcup_n A_n \in \mathcal{A}$, we conclude that $\bigcup_n (E \cap A_n) \in \mathcal{A}_E$.

Hence, $\mathcal{A}_E$ is a σ-algebra on E. □

5. (Pre-image σ-algebra) Let $f : X \to X'$ be a function and let $\mathcal{A}'$ be a σ-algebra on X'. Then
$$\mathcal{A} := \{f^{-1}(A') : A' \in \mathcal{A}'\}$$
is a σ-algebra on X.

Theorem 1.1. Let X be a set and let $\{\mathcal{A}_i : i \in I\}$ be a family of σ-algebras on X. Define

$$\mathcal{A} := \bigcap_{i \in I} \mathcal{A}_i = \{A \subseteq X : A \in \mathcal{A}_i \text{ for all } i \in I\}.$$

Then, $\mathcal{A}$ is a σ-algebra on X.

Proof. We verify the σ-algebra properties for $\mathcal{A}$:

- Since $X \in \mathcal{A}_i$ for all $i \in I$, we have $X \in \mathcal{A}$.
- If $A \in \mathcal{A}$, then $A \in \mathcal{A}_i$ for all $i \in I$, so $A^c = X \setminus A \in \mathcal{A}_i$ for all $i \in I$, hence $A^c \in \mathcal{A}$.
- If $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$, then $A_n \in \mathcal{A}_i$ for all n and i, so $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_i$ for all $i \in I$, and thus $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

Therefore, $\mathcal{A}$ is a σ-algebra on X. $\square$

Definition 1.2. Let X be a set and let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. The *σ-algebra generated by $\mathcal{E}$*, denoted by $\sigma(\mathcal{E})$, is the smallest σ-algebra on X containing all sets in $\mathcal{E}$. That is,

$$\sigma(\mathcal{E}) := \bigcap \Big\{ \mathcal{A} \subseteq \mathcal{P}(X) : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X,\ \mathcal{E} \subseteq \mathcal{A} \Big\}.$$

Remark 1.1 (Generated σ-algebras)**.**

- If $\mathcal{A}$ is a σ-algebra, then $\sigma(\mathcal{A}) = \mathcal{A}$.
- For $A \subseteq X$, we have $\sigma(\{A\}) = \{\emptyset, A, A^c, X\}$.
- If $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{A}$, then $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G}) \subseteq \sigma(\mathcal{A})$.

Definition 1.3 (Topological Space)**.** A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of subsets of X, called *open sets*, satisfying the following properties:

- $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
- If $\{U_\alpha \in \mathcal{T} : \alpha \in I\}$ is an arbitrary collection of open sets, then the union $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$,
- If $\{U_i \in \mathcal{T} : i = 1, \dots, n\}$ is a finite collection of open sets, then the intersection $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X. The complement of an open set is called a *closed set*.

Remark 1.2 (Standard Topology on $\mathbb{R}^n$)**.** A subset $U \subseteq \mathbb{R}^n$ is called *open* if for every point $x \in U$, there exists an $\varepsilon > 0$ such that the open ball

$$B_\varepsilon(x) := \{y \in \mathbb{R}^n : \|x - y\| < \varepsilon\},$$

where $\| \cdot \|$ denotes the Euclidean norm, is contained in U; that is, $B_\varepsilon(x) \subseteq U$.

The collection of all such open sets is denoted by $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ and forms the *standard topology* on $\mathbb{R}^n$.

Definition 1.4 (Borel σ-algebra)**.** The σ-algebra $\sigma(\mathcal{O})$ generated by the collection of open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called the *Borel σ-algebra* on $\mathbb{R}^n$.

Its elements are called *Borel sets* or *Borel measurable sets.* We denote the Borel σ-algebra by $\mathcal{B}(\mathbb{R}^n)$.

Definition 1.5. Let X be a topological space and let $A \subseteq X$. A collection $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ of open sets is called an *open cover* of A if

$$A \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

A *subcover* is a subcollection that still covers A. The set A is called *compact* if every open cover of A admits a finite subcover.

Remark 1.3. In $\mathbb{R}^n$, a set is compact if and only if it is closed and bounded (Heine–Borel Theorem).

Theorem 1.2 (Borel σ-algebra from Different Generators)**.** Let $\mathcal{O}, \mathcal{C}, \mathcal{K} \subseteq \mathcal{P}(\mathbb{R}^n)$ denote the collections of open, closed, and compact subsets of $\mathbb{R}^n$, respectively. Then,

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}) = \sigma(\mathcal{C}) = \sigma(\mathcal{K}).$$

Proof. Since compact sets are closed, we have $\mathcal{K} \subseteq \mathcal{C}$, and by Remark 1.1, $\sigma(\mathcal{K}) \subseteq \sigma(\mathcal{C})$. Conversely, for any $C \in \mathcal{C}$, define $C_k := C \cap B_k(0)$, where $B_k(0)$ is the closed ball of radius k centered at the origin. Each C_k is closed and bounded, hence compact, so $C_k \in \mathcal{K}$. Since $C = \bigcup_{k \in \mathbb{N}} C_k$, it follows that $C \in \sigma(\mathcal{K})$, and thus $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{K})$.

Next, since $\mathcal{C} = \mathcal{O}^c := \{U^c : U \in \mathcal{O}\}$, and complements of sets in a σ-algebra are again in the σ-algebra, it follows that $\mathcal{C} \subseteq \sigma(\mathcal{O})$, hence $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{O})$. The reverse inclusion follows similarly from $\mathcal{O} = \mathcal{C}^c$. Therefore, we have:

$$\sigma(\mathcal{K}) = \sigma(\mathcal{C}) = \sigma(\mathcal{O}).$$

□

Generating Sets of the Borel Algebra. The Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ can be generated by various systems of sets. Of particular importance are:

- The family of open rectangles:

$$\mathcal{J}_{o,n} := \{(a_1, b_1) \times \cdots \times (a_n, b_n) : a_i, b_i \in \mathbb{R}\},$$

- The family of half-open rectangles:

$$\mathcal{J}_n := \{[a_1, b_1) \times \cdots \times [a_n, b_n) : a_i, b_i \in \mathbb{R}\}.$$

We denote by $\mathcal{J}_n^{\mathrm{rat}}, \mathcal{J}_{o,n}^{\mathrm{rat}}$ the subsets with rational endpoints. These sets represent intervals in $\mathbb{R}$, rectangles in $\mathbb{R}^2$, cuboids in $\mathbb{R}^3$, and hypercubes in higher dimensions.

Theorem 1.3. We have the following equality of Borel σ-algebras on $\mathbb{R}^n$:

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{J}_n^{\mathrm{rat}}) = \sigma(\mathcal{J}_{o,n}^{\mathrm{rat}}) = \sigma(\mathcal{J}_n) = \sigma(\mathcal{J}_{o,n}),$$

Remark 1.4. Let $D \subseteq \mathbb{R}$ be a dense subset, for example $D = \mathbb{Q}$ or $D = \mathbb{R}$. Then the Borel sets on $\mathbb{R}$ can also be generated by any of the following families of intervals:

$$\{(-\infty, a) : a \in D\}, \quad \{(-\infty, a] : a \in D\}, \quad \{(a, \infty) : a \in D\}, \quad \{[a, \infty) : a \in D\}.$$

2 Measure Spaces

Definition 2.1. A *(positive) measure* μ on X is a map $\mu : \mathcal{A} \to [0, \infty]$, where $\mathcal{A}$ is a σ-algebra on X, satisfying:

$$\mu(\emptyset) = 0, \tag{M1}$$

and for any pairwise disjoint sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$,

$$\mu\left(\bigsqcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n). \tag{M2}$$

Property (M2) is also called *countable additivity.*

If μ satisfies (M1), (M2), but $\mathcal{A}$ is not a σ-algebra, then μ is called a *pre-measure.*

Remark 2.1. (M2) requires implicitly that $\bigsqcup_n A_n$ is again in $\mathcal{A}$ this is clearly the case for σ-algebras, but needs special attention when dealing with pre-measures.

Definition 2.2 (Monotone sequences of sets)**.** Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be sequences of subsets of X.

We say (A_n) is *increasing* if

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

and write $A_n \uparrow A$ where

$$A := \bigcup_{n\in\mathbb{N}} A_n$$

Similarly, (B_n) is *decreasing* if

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$$

and write $B_n \downarrow B$ where

$$B := \bigcap_{n\in\mathbb{N}} B_n$$

Definition 2.3. Let X be a set and $\mathcal{A}$ a σ-algebra on X. The pair $(X, \mathcal{A})$ is called a *measurable space*. If μ is a measure on $(X, \mathcal{A})$, then $(X, \mathcal{A}, \mu)$ is called a *measure space*.

A measure μ is called:

- *finite* if $\mu(X) < \infty$,
- a *probability measure* if $\mu(X) = 1$.

Accordingly, we speak of a *finite measure space* and a *probability space*.

Definition 2.4. A measure μ on a measurable space $(X, \mathcal{A})$ is called σ-*finite* if there exists a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that:

$$A_n \uparrow X \quad \text{and} \quad \mu(A_n) < \infty \quad \text{for all } n \in \mathbb{N}.$$

In this case, the measure space $(X, \mathcal{A}, \mu)$ is called σ-*finite*.

Lemma 2.1 (Basic properties of measures)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Then:

(i) If $A_0, \ldots, A_k \in \mathcal{A}$ are pairwise disjoint, then $\mu\Big(\bigsqcup_{n=1}^k A_n\Big) = \sum_{n=1}^k \mu(A_n)$.

(ii) If $A, B \in \mathcal{A}$ with $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

(iii) If $A, B \in \mathcal{A}$, $A \subseteq B$, and $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

Proof. (i) Extend (A_n) by $A_n = \varnothing$ for $n > k$. Then by countable additivity,

$$\mu\Big(\bigsqcup_{n=1}^{k} A_n\Big) = \mu\Big(\bigsqcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{k} \mu(A_n).$$

(ii) Since $B = A \sqcup (B \setminus A)$ we have

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

(iii) Rearranging gives

$$\mu(B \setminus A) = \mu(B) - \mu(A),$$

which is well-defined if $\mu(A) < \infty$. $\square$

Lemma 2.2 (Main properties of measures)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Then:

(i) **Countable subadditivity:** For any countable family $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$,

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leq \sum_{i\in\mathbb{N}} \mu(A_i).$$

(ii) **Continuity from below (increasing sequence):** If $A_1 \subseteq A_2 \subseteq \cdots$ (i.e., $A_n \uparrow A$), then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(iii) **Continuity from above (decreasing sequence):** If $B_1 \supseteq B_2 \supseteq \cdots$ (i.e., $B_n \downarrow B$), then

$$\mu\left(\bigcap_{n\in\mathbb{N}} B_n\right) = \lim_{n\to\infty} \mu(B_n).$$

Proof. (i) For countable subadditivity, set $B_k := A_k \setminus \bigcup_{i=1}^{k-1} A_i$, so that (B_k) are disjoint with $B_k \subseteq A_k$. Then,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigsqcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

(ii) Let $A_n \uparrow A$, i.e., $A_n \subseteq A_{n+1}$ and $A = \bigcup_{n\in\mathbb{N}} A_n$. Define $B_n := A_n \setminus A_{n-1}$ with $A_0 := \emptyset$. Then (B_n) is disjoint and $\bigsqcup_n B_n = A$. By countable additivity,

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n\to\infty} \mu(A_n)$$

(iii) Assume $B_n \downarrow B$, i.e., $B_n \supseteq B_{n+1}$ and $B = \bigcap_n B_n$, with $\mu(B_1) < \infty$. Set $A_n := B_1 \setminus B_n$, so $A_n \uparrow A := B_1 \setminus B$. Then

$$\mu(B) = \mu(B_1) - \mu(A) = \mu(B_1) - \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(B_n)$$

□

Remark 2.2. With appropriate modifications, these properties also hold for pre-measures, i.e., when $\mathcal{A}$ is not necessarily a σ-algebra.

Example 2.1 (Dirac measure)**.** Let $(X, \mathcal{A})$ be a measurable space and let $x \in X$. Define $\delta_x : \mathcal{A} \to \{0, 1\}$ by

$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then δ_x is a measure on $(X, \mathcal{A})$, called the *Dirac measure* (or unit mass) at the point x.

Example 2.2 (Counting measure)**.** Let $(X, \mathcal{A})$ be a measurable space. Define $\#A : \mathcal{A} \to [0, \infty]$ by

$$\#A := \begin{cases} \text{number of elements in } A & \text{if } A \text{ is finite,} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$$

Then $\#$ is a measure on $(X, \mathcal{A})$, called the *counting measure*.

Example 2.3 (Discrete probability measure)**.** Let $\Omega = \{\omega_1, \omega_2, \dots\}$ be a countable set, and let $(p_n)_{n\in\mathbb{N}} \subseteq [0, 1]$ be a sequence such that $\sum_{n\in\mathbb{N}} p_n = 1$. Define the set function $P : \mathcal{P}(\Omega) \to [0, 1]$ by

$$P(A) := \sum_{\{n\in\mathbb{N}:\omega_n\in A\}} p_n = \sum_{n\in\mathbb{N}} p_n\, \delta_{\omega_n}(A), \quad A \subseteq \Omega,$$

where δ_{ω_n} denotes the Dirac measure at ω_n. Then P is a probability measure on $(\Omega, \mathcal{P}(\Omega))$, and the triplet $(\Omega, \mathcal{P}(\Omega), P)$ is called a *discrete probability space*.

Example 2.4 (Linear combination of measures)**.** Let $(X, \mathcal{A})$ be a measurable space, and let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures on $(X, \mathcal{A})$. Let $(x_n)_{n\in\mathbb{N}} \subseteq [0, \infty]$. Then the set function

$$\mu := \sum_{n\in\mathbb{N}} x_n\mu_n$$

defined by

$$\mu(A) := \sum_{n\in\mathbb{N}} x_n\mu_n(A), \quad \text{for all } A \in \mathcal{A},$$

is a measure on $(X, \mathcal{A})$

Proof. We verify the axioms of a measure:

(M1) *(Null empty set):* For all $n \in \mathbb{N}$, $\mu_n(\emptyset) = 0$, so

$$\mu(\emptyset) = \sum_{n\in\mathbb{N}} x_n\mu_n(\emptyset) = \sum_{n\in\mathbb{N}} x_n \cdot 0 = 0.$$

(M2) *(Countable additivity):* Let $(A_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ be pairwise disjoint. Since each μ_n is a measure, we have

$$\mu_n\left(\bigsqcup_{k\in\mathbb{N}} A_k\right) = \sum_{k\in\mathbb{N}} \mu_n(A_k), \quad \text{for all } n \in \mathbb{N}.$$

Then,

$$\mu\left(\bigsqcup_{k\in\mathbb{N}} A_k\right) = \sum_{n\in\mathbb{N}} x_n\mu_n\left(\bigsqcup_{k\in\mathbb{N}} A_k\right) = \sum_{n\in\mathbb{N}} x_n \sum_{k\in\mathbb{N}} \mu_n(A_k).$$

Since all terms are non-negative, we may exchange the order of summation:

$$\sum_{n\in\mathbb{N}} x_n \sum_{k\in\mathbb{N}} \mu_n(A_k) = \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} x_n\mu_n(A_k) = \sum_{k\in\mathbb{N}} \mu(A_k).$$

Therefore, μ is countably additive. $\square$

Example 2.5 (Restriction of a measure)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $A \in \mathcal{A}$. Define the set function $\mu_A : \mathcal{A} \to [0, \infty]$ by

$$\mu_A(B) := \mu(A \cap B), \quad \text{for all } B \in \mathcal{A}.$$

Then μ_A is a measure on $(X, \mathcal{A})$, called the *restriction of μ to A*.

Proof. We verify the two defining properties of a measure:

(M1): $\mu_A(\emptyset) = \mu(A \cap \emptyset) = \mu(\emptyset) = 0$.

(M2): Let $(B_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be pairwise disjoint. Then $(A \cap B_n)_{n\in\mathbb{N}}$ are also pairwise disjoint, and

$$\mu_A\left(\bigsqcup_{n\in\mathbb{N}} B_n\right) = \mu\left(A \cap \bigsqcup_{n\in\mathbb{N}} B_n\right) = \mu\left(\bigsqcup_{n\in\mathbb{N}} (A \cap B_n)\right) = \sum_{n\in\mathbb{N}} \mu(A \cap B_n) = \sum_{n\in\mathbb{N}} \mu_A(B_n).$$

Hence, μ_A is a measure. □

Definition 2.5 (Lebesgue measure on $\mathbb{R}^n$)**.** Define the set function λ_n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ by

$$\lambda_n\left([\![a, b)\!)\right) := \prod_{i=1}^{n} (b_i - a_i),$$

for all $[\![a, b)\!) := [a_1, b_1) \times \cdots \times [a_n, b_n) \in \mathcal{J}_n$. This is called the n-dimensional Lebesgue measure.

Remark 2.3. The set function λ_n is defined only on the family $\mathcal{J}_n$ of half-open rectangles and hence is not yet a measure. Extending λ_n to a measure on $\mathcal{B}(\mathbb{R}^n)$ requires the Carathéodory extension theorem, which will be developed later.

Lemma 2.3. Let $(X, \mathcal{A})$ be a measure space, and let $\mu : \mathcal{A} \to [0, \infty]$ be an additive set function with $\mu(\emptyset) = 0$. Then μ is a measure if and only if it is **continuous from below**, i.e., for every increasing sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ with $A_n \uparrow A$, we have

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) = \sup_{n\in\mathbb{N}} \mu(A_n).$$

Proof. Any measure μ is continuous from below.

Conversely, suppose μ is finitely additive, $\mu(\emptyset) = 0$, and μ is continuous from below. Let $(B_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be disjoint, and define $A_n := \bigcup_{i=1}^{n} B_i$. Then (A_n) is increasing with $\bigcup_{n=1}^{\infty} A_n = \bigsqcup_{n=1}^{\infty} B_n$. By finite additivity,

$$\mu(A_n) = \sum_{i=1}^{n} \mu(B_i),$$

and by continuity from below,

$$\mu\left(\bigsqcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu(B_n).$$

Hence μ is countably additive, i.e., a measure. □

Lemma 2.4. Let $(X, \mathcal{A})$ be a measurable space and $\mu : \mathcal{A} \to [0, \infty)$ an additive set function with $\mu(\emptyset) = 0$ and $\mu(A) < \infty$ for all $A \in \mathcal{A}$. Then μ is a measure if and only if it satisfies one of the following continuity properties:

(i) μ is continuous from below;

(ii) μ is continuous from above;

(iii) μ is continuous at $\emptyset$, i.e., for every decreasing sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ with $\bigcap_{n=1}^{\infty} B_n = \emptyset$, we have
$$\lim_{n\to\infty} \mu(B_n) = 0.$$

Proof. Clearly, every measure satisfies properties (i)–(iii), so we only need to show that (iii) implies countable additivity.

Assume μ is additive, $\mu(\emptyset) = 0$, and satisfies continuity at $\emptyset$. Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be pairwise disjoint and define $A := \bigsqcup_{n\in\mathbb{N}} A_n$. For each n, let

$$B_n := A \setminus \bigcup_{i=1}^{n} A_i.$$

Then (B_n) is a decreasing sequence in $\mathcal{A}$ with $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$, so by continuity at $\emptyset$, we have $\mu(B_n) \to 0$.

Using additivity, we compute

$$\mu(A) = \mu\left(B_n \sqcup \bigcup_{i=1}^{n} A_i\right) = \mu(B_n) + \sum_{i=1}^{n} \mu(A_i).$$

Taking the limit as $n \to \infty$, we get

$$\mu(A) = \lim_{n\to\infty} \left(\mu(B_n) + \sum_{i=1}^{n} \mu(A_i)\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Thus, μ is countably additive, hence a measure. □

3 Uniqueness of Measures

Definition 3.1. A *Dynkin system* (or λ-system) $\mathcal{D} \subseteq \mathcal{P}(X)$ is a collection of subsets of X such that:

- $X \in \mathcal{D}$ **(D1)**
- If $D \in \mathcal{D}$, then $D^c \in \mathcal{D}$ **(D2)**
- If $(D_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ are pairwise disjoint, then $\bigsqcup_{n\in\mathbb{N}} D_n \in \mathcal{D}$ **(D3)**

Remark 3.1. As with σ-algebras one easily checks that $\emptyset \in \mathcal{D}$ and that finite disjoint unions are in $\mathcal{D}$: if $D, E \in \mathcal{D}$ with $D \cap E = \emptyset$, then $D \sqcup E \in \mathcal{D}$. Every σ-algebra is a Dynkin system, but the converse is not true in general.

Lemma 3.1. Let $\mathcal{E} \subseteq \mathcal{P}(X)$. Then there exists a smallest Dynkin system $\mathcal{D}(\mathcal{E})$ containing $\mathcal{E}$, called the *Dynkin system generated by* $\mathcal{E}$. Moreover,

$$\mathcal{E} \subseteq \mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E}),$$

where $\sigma(\mathcal{E})$ denotes the σ-algebra generated by $\mathcal{E}$.

Proof. The proof is analogous to that of Theorem 1.1 for σ-algebras. Let $\mathcal{F}$ be the family of all Dynkin systems on X that contain $\mathcal{E}$. Then $\mathcal{F}$ is nonempty, since $\mathcal{P}(X)$ is a Dynkin system containing $\mathcal{E}$. Define

$$\mathcal{D}(\mathcal{E}) := \bigcap_{\mathcal{D} \in \mathcal{F}} \mathcal{D}.$$

Then $\mathcal{D}(\mathcal{E})$ is a Dynkin system, being the intersection of Dynkin systems (which are closed under complements, disjoint unions, and contain X). Moreover, it is the smallest such system containing $\mathcal{E}$ by construction. Since every σ-algebra is in particular a Dynkin system, we also have

$$\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E}).$$

□

Lemma 3.2. A Dynkin system $\mathcal{D}$ is a σ-algebra if and only if it is closed under finite intersections; that is,

$$D, E \in \mathcal{D} \quad \Rightarrow \quad D \cap E \in \mathcal{D}.$$

Proof. The "only if" direction follows immediately from Remark 3.1 and the fact that every σ-algebra is closed under finite intersections.

For the converse, assume $\mathcal{D}$ is a Dynkin system closed under finite intersections. Let $(D_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}$, and define

$$E_1 := D_1 \in \mathcal{D}, \quad E_{n+1} := D_{n+1} \setminus \bigcup_{k=1}^{n} D_k = D_{n+1} \cap \bigcap_{k=1}^{n} D_k^c.$$

Each $E_n \in \mathcal{D}$ by the Dynkin properties and the assumed stability under finite intersections. The sets (E_n) are disjoint and satisfy

$$\bigcup_{n=1}^{\infty} D_n = \bigsqcup_{n=1}^{\infty} E_n \in \mathcal{D},$$

so $\mathcal{D}$ is closed under countable unions. Hence, $\mathcal{D}$ is a σ-algebra. □

While Lemma 3.2 characterizes when a Dynkin system is a σ-algebra, it is not directly applicable when the Dynkin system $\mathcal{D}$ is defined via a generator $\mathcal{E} \subseteq \mathcal{P}(X)$, as is often the case in practice. The following theorem overcomes this limitation and plays a central role in many applications.

Theorem 3.3 (Dynkin's π-λ Theorem)**.** Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of sets that is closed under finite intersections. Then,

$$\mathcal{D}(\mathcal{E}) = \sigma(\mathcal{E}).$$

Proof. By Lemma 3.1, we have $\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E})$. To show equality, it suffices to prove that $\mathcal{D}(\mathcal{E})$ is a σ-algebra. Since it contains $\mathcal{E}$, it would then contain $\sigma(\mathcal{E})$ by minimality.

By Lemma 3.2, it is enough to show that $\mathcal{D}(\mathcal{E})$ is closed under finite intersections.

Fix $D \in \mathcal{D}(\mathcal{E})$, and define

$$\mathcal{D}_D := \{A \subseteq X : A \cap D \in \mathcal{D}(\mathcal{E})\}.$$

We claim that $\mathcal{D}_D$ is a Dynkin system:

(D1): Since $D = X \cap D \in \mathcal{D}(\mathcal{E})$, we have $X \in \mathcal{D}_D$.

(D2): If $A \in \mathcal{D}_D$, then

$$A^c \cap D = ((A \cap D) \sqcup D^c)^c \cap D \in \mathcal{D}(\mathcal{E}),$$

using that $A \cap D \in \mathcal{D}(\mathcal{E})$, $D^c \in \mathcal{D}(\mathcal{E})$, and that Dynkin systems are closed under disjoint unions and complements.

(D3): Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_D$ be disjoint. Then the sets $A_n \cap D \in \mathcal{D}(\mathcal{E})$ are disjoint, and

$$\left(\bigsqcup_{n=1}^{\infty} A_n\right) \cap D = \bigsqcup_{n=1}^{\infty} (A_n \cap D) \in \mathcal{D}(\mathcal{E}).$$

Thus, $\mathcal{D}_D$ is a Dynkin system. Since $\mathcal{E} \subseteq \mathcal{D}_G$ for all $G \in \mathcal{E}$ by the assumed $\cap$-stability of $\mathcal{E}$, and each $\mathcal{D}_G$ is a Dynkin system, it follows that

$$\mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}_G \quad \text{for all } G \in \mathcal{E}.$$

Hence, for all $D \in \mathcal{D}(\mathcal{E})$ and $G \in \mathcal{E}$, we have $D \cap G \in \mathcal{D}(\mathcal{E})$, i.e., $\mathcal{D}(\mathcal{E})$ is closed under finite intersections.

By Lemma 3.2, we conclude that $\mathcal{D}(\mathcal{E})$ is a σ-algebra. Since $\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E})$ and both are σ-algebras containing $\mathcal{E}$, we have

$$\mathcal{D}(\mathcal{E}) = \sigma(\mathcal{E}).$$

□

Theorem 3.4 (Uniqueness of Measures)**.** Let $(X, \mathcal{A})$ be a measurable space with $\mathcal{A} = \sigma(\mathcal{E})$, where $\mathcal{E} \subseteq \mathcal{P}(X)$ satisfies:

- $\mathcal{E}$ is closed under finite intersections;
- there exists an increasing sequence $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}$ with $E_n \uparrow X$.

Suppose μ and ν are measures on $\mathcal{A}$ such that $\mu(E) = \nu(E)$ for all $E \in \mathcal{E}$, and $\mu(E_n) = \nu(E_n) < \infty$ for all $n \in \mathbb{N}$. Then $\mu = \nu$ on $\mathcal{A}$; that is,

$$\mu(A) = \nu(A) \quad \text{for all } A \in \mathcal{A}.$$

Proof. Fix $n \in \mathbb{N}$, and define

$$\mathcal{D}_n := \{A \in \mathcal{A} : \mu(E_n \cap A) = \nu(E_n \cap A)\}.$$

We claim that $\mathcal{D}_n$ is a Dynkin system:

(D1): Since $E_n \in \mathcal{E} \subseteq \mathcal{A}$, and $\mu(E_n) = \nu(E_n)$, it follows that $X \in \mathcal{D}_n$.

(D2): If $A \in \mathcal{D}_n$, then

$$\mu(E_n \cap A^c) = \mu(E_n) - \mu(E_n \cap A) = \nu(E_n) - \nu(E_n \cap A) = \nu(E_n \cap A^c),$$

so $A^c \in \mathcal{D}_n$.

(D3): Let $(A_k)_{k\in\mathbb{N}} \subseteq \mathcal{D}_n$ be disjoint. Then:

$$\mu\left(E_n \cap \bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(E_n \cap A_k) = \sum_{k=1}^{\infty} \nu(E_n \cap A_k) = \nu\left(E_n \cap \bigsqcup_{k=1}^{\infty} A_k\right),$$

so $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{D}_n$.

Thus, $\mathcal{D}_n$ is a Dynkin system. Since $\mathcal{E} \subseteq \mathcal{D}_n$ (as $\mu(E_n \cap E) = \nu(E_n \cap E)$ for all $E \in \mathcal{E}$, by the $\cap$-stability of $\mathcal{E}$), and since $\sigma(\mathcal{E}) = \mathcal{A}$, Theorem 3.3 yields

$$\mathcal{A} = \sigma(\mathcal{E}) = \mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}_n.$$

Hence,

$$\mu(E_n \cap A) = \nu(E_n \cap A) \quad \text{for all } A \in \mathcal{A},\ n \in \mathbb{N}.$$

Now fix $A \in \mathcal{A}$. Since $E_n \uparrow X$, we have $E_n \cap A \uparrow A$, and by continuity from below,

$$\mu(A) = \lim_{n\to\infty} \mu(E_n \cap A) = \lim_{n\to\infty} \nu(E_n \cap A) = \nu(A).$$

Therefore, $\mu = \nu$ on $\mathcal{A}$. □

Theorem 3.5 (Translation Invariance and Uniqueness of Lebesgue Measure)**.** Let λ^n denote the n-dimensional Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then:

(i) **(Translation invariance)** For all $x \in \mathbb{R}^n$ and all $B \in \mathcal{B}(\mathbb{R}^n)$, we have

$$\lambda^n(x + B) = \lambda^n(B),$$

where $x + B := \{x + y : y \in B\}$ is the translation of B by x.

(ii) **(Uniqueness up to scalar)** Let μ be a measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ that is translation invariant and finite on the unit cube:

$$\mu(x + B) = \mu(B) \quad \text{for all } x \in \mathbb{R}^n,\ B \in \mathcal{B}(\mathbb{R}^n), \quad \text{and} \quad \mu([0,1)^n) < \infty.$$

Then μ is a scalar multiple of Lebesgue measure:

$$\mu = \mu([0,1)^n) \cdot \lambda^n.$$

4 Existence of Measures

Definition 4.1 (Semi-ring)**.** Let X be a set. A family $\mathcal{S} \subseteq \mathcal{P}(X)$ is called a *semi-ring* if:

- $\emptyset \in \mathcal{S}$ (S1)
- For all $S, T \in \mathcal{S}$, we have $S \cap T \in \mathcal{S}$ (S2)
- For all $S, T \in \mathcal{S}$, there exist disjoint sets $S_1, \dots, S_M \in \mathcal{S}$ such that

$$S \setminus T = \bigsqcup_{i=1}^{M} S_i \tag{S3}$$

Theorem 4.1 (Carathéodory Extension Theorem)**.** Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be a semi-ring and let $\mu : \mathcal{S} \to [0, \infty]$ be a pre-measure, i.e.,

- $\mu(\emptyset) = 0$,
- For every sequence $(S_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}$ of disjoint sets with $\bigsqcup_{n \in \mathbb{N}} S_n \in \mathcal{S}$, we have

$$\mu\left(\bigsqcup_{n \in \mathbb{N}} S_n\right) = \sum_{n \in \mathbb{N}} \mu(S_n).$$

Then μ has an extension to a measure $\bar{\mu}$ on $\sigma(\mathcal{S})$.

Moreover, if $\mathcal{S}$ contains an increasing sequence $(S_n)_{n \in \mathbb{N}}$ with $S_n \uparrow X$ and $\mu(S_n) < \infty$ for all n, then the extension is unique.

Idea of the proof. The fundamental problem is how to extend the pre-measure μ. The following auxiliary set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ will play a central role. For any $A \subseteq X$, define the family of countable $\mathcal{S}$-coverings

$$\mathcal{C}(A) := \left\{ (S_n)_{n \in \mathbb{N}} \subseteq \mathcal{S} : A \subseteq \bigcup_{n \in \mathbb{N}} S_n \right\},$$

and the set function

$$\mu^*(A) := \inf \left\{ \sum_{n \in \mathbb{N}} \mu(S_n) : (S_n)_{n \in \mathbb{N}} \in \mathcal{C}(A) \right\}.$$

If A cannot be covered by sets from $\mathcal{S}$, we define $\mathcal{C}(A) = \emptyset$ and hence $\mu^*(A) := \inf \emptyset = \infty$.

The proof proceeds in four main steps:

1. **(Outer measure)** Show that μ^* is an outer measure, i.e., it satisfies:

$$\begin{aligned} &\text{(OM1)} \quad \mu^*(\emptyset) = 0, \\ &\text{(OM2)} \quad A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B), \\ &\text{(OM3)} \quad \mu^*\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n). \end{aligned}$$

2. **(Extension)** Show that μ^* extends μ, i.e., $\mu^*(S) = \mu(S)$ for all $S \in \mathcal{S}$.
3. **(μ^*-measurable sets)** Define the collection of μ^*-measurable sets by

$$\mathcal{A}_{\mu^*} := \{A \subseteq X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \setminus A) \quad \text{for all } Q \subseteq X\}.$$

   Then $\mathcal{A}_{\mu^*}$ is a σ-algebra with $\mathcal{S} \subseteq \mathcal{A}_{\mu^*}$ and $\sigma(\mathcal{S}) \subseteq \mathcal{A}_{\mu^*}$.
4. **(Measure on σ-algebra)** The restriction of μ^* to $\mathcal{A}_{\mu^*}$ is a measure. In particular, $\mu^*|_{\sigma(\mathcal{S})}$ is a measure extending μ.

If $\mathcal{S}$ contains an increasing sequence $(S_n)_{n\in\mathbb{N}}$ with $S_n \uparrow X$ and $\mu(S_n) < \infty$ for all n, then the extension is unique. □

Existence of Lebesgue Measure on $\mathbb{R}$

Lemma 4.2. Let $\mathcal{J}_1 := \{[a,b) \subseteq \mathbb{R} : a < b\}$ be the family of half-open intervals. Define the set function

$$\lambda_1([a,b)) := b - a \quad \text{for all } [a,b) \in \mathcal{J}_1.$$

Then $\lambda_1 : \mathcal{J}_1 \to [0,\infty)$ is a pre-measure.

Proof. Let $[a,b) \in \mathcal{J}_1$, and suppose it can be written as a disjoint union of intervals:

$$[a,b) = \bigsqcup_{n\in\mathbb{N}} I_n, \quad \text{with } I_n \in \mathcal{J}_1 \text{ for all } n.$$

Our goal is to show that

$$\lambda_1([a,b)) = \sum_{n=1}^{\infty} \lambda_1(I_n).$$

Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, choose a closed interval $I_n^{(\varepsilon)}$ such that

$$I_n \subseteq I_n^{(\varepsilon)} \quad \text{and} \quad \lambda_1(I_n^{(\varepsilon)}) \leq \lambda_1(I_n) + \frac{\varepsilon}{2^n}.$$

These intervals slightly extend each I_n, allowing us to approximate the union $\bigsqcup I_n$ from above.

Since the I_n cover $[a,b)$ disjointly, the union of the extended intervals will eventually cover most of $[a,b)$. More precisely, for sufficiently large N, we have

$$[a, b-\varepsilon) \subseteq \bigcup_{n=1}^{N} I_n^{(\varepsilon)}.$$

Now we estimate the difference:

$$\begin{aligned}\lambda_1([a,b)) - \sum_{n=1}^{N} \lambda_1(I_n) &= (\lambda_1([a,b)) - \lambda_1([a,b-\varepsilon))) \\ &\quad + \left(\lambda_1([a,b-\varepsilon)) - \sum_{n=1}^{N} \lambda_1(I_n^{(\varepsilon)})\right) \\ &\quad + \sum_{n=1}^{N} \left(\lambda_1(I_n^{(\varepsilon)}) - \lambda_1(I_n)\right) \\ &\leq \varepsilon + 0 + \sum_{n=1}^{N} \frac{\varepsilon}{2^n} \leq 2\varepsilon.\end{aligned}$$

On the other hand, since $\bigsqcup_{n=1}^{N} I_n \subseteq [a,b)$ and the intervals I_n are disjoint, finite additivity and monotonicity of λ_1 imply:

$$\sum_{n=1}^{N} \lambda_1(I_n) = \lambda_1\left(\bigsqcup_{n=1}^{N} I_n\right) \leq \lambda_1([a,b)).$$

Therefore,

$$0 \leq \lambda_1([a,b)) - \sum_{n=1}^{N} \lambda_1(I_n),$$

which justifies the lower bound in the previous inequality.

Combining both sides, we have

$$0 \leq \lambda_1([a,b)) - \sum_{n=1}^{N} \lambda_1(I_n) \leq 2\varepsilon.$$

Letting $N \to \infty$ and then $\varepsilon \to 0$, we conclude:

$$\lambda_1([a,b)) = \sum_{n=1}^{\infty} \lambda_1(I_n).$$

Thus, λ_1 is countably additive on $\mathcal{J}_1$, and hence a pre-measure. $\square$

Lemma 4.3 (Lebesgue measure on $\mathbb{R}$)**.** The set function λ_1, defined on $\mathcal{J}_1$ by $\lambda_1([a,b)) = b - a$ for $a < b$, extends to a measure on $\mathcal{B}(\mathbb{R})$. This extension is the unique measure μ on $\mathcal{B}(\mathbb{R})$ such that

$$\mu([a,b)) = b - a \quad \text{for all } a < b.$$

Proof. We have already shown that λ_1 is a pre-measure on $\mathcal{J}_1$. By Theorem 1.3, $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{J}_1)$, i.e., the Borel σ-algebra is generated by $\mathcal{J}_1$.

Consider the sequence of half-open intervals $[-k,k) \subseteq \mathbb{R}$ for $k \in \mathbb{N}$. This forms an increasing sequence for $\mathbb{R}$, and we have

$$\lambda_1([-k,k)) = 2k < \infty \quad \text{for all } k \in \mathbb{N}.$$

Thus, all the conditions of Theorem 4.1 (Carathéodory's extension theorem) are satisfied. It follows that λ_1 extends uniquely to a measure on $\mathcal{B}(\mathbb{R})$, yielding the one-dimensional Lebesgue measure on $\mathbb{R}$. $\square$

Existence of Lebesgue Measure on $\mathbb{R}^n$

Lemma 4.4. Let $\mathcal{J}_n$ denote the collection of half-open rectangles in $\mathbb{R}^n$ of the form

$$[\![a,b)\!) = \prod_{i=1}^{n} [a_i, b_i), \quad \text{where } a = (a_1, \ldots, a_n),\ b = (b_1, \ldots, b_n),\ a_i < b_i.$$

Then $\mathcal{J}_n$ is a semi-ring.

Proof. We prove the statement by induction on n. Assume $\mathcal{J}_n \subset \mathbb{R}^n$ is a semi-ring. Define

$$\mathcal{J}_{n+1} := \mathcal{J}_n \times \mathcal{J}_1,$$

i.e., the collection of rectangles of the form $R = R_n \times R_1$, where $R_n \in \mathcal{J}_n$ and $R_1 \in \mathcal{J}_1$.

We verify the properties of a semi-ring:

(S1) *Closure under the empty set:* Since $\emptyset \in \mathcal{J}_n$ and $\mathcal{J}_1$, we have

$$\emptyset = \emptyset \times [a, b) \in \mathcal{J}_{n+1}.$$

(S2) *Closure under intersection:* Let $R = R_n \times R_1$ and $S = S_n \times S_1$ be in $\mathcal{J}_{n+1}$. Then

$$R \cap S = (R_n \cap S_n) \times (R_1 \cap S_1),$$

which belongs to $\mathcal{J}_{n+1}$, since both $R_n \cap S_n \in \mathcal{J}_n$ and $R_1 \cap S_1 \in \mathcal{J}_1$, by the inductive hypothesis.

(S3) *Closure under set difference (finite disjoint union):* Consider

$$R \setminus S = (R_n \times R_1) \setminus (S_n \times S_1).$$

This set can be decomposed as

$$(R_n \setminus S_n) \times (R_1 \setminus S_1) \sqcup (R_n \cap S_n) \times (R_1 \setminus S_1) \sqcup (R_n \setminus S_n) \times (R_1 \cap S_1).$$

Each of the components $R_n \setminus S_n$, $R_n \cap S_n$, $R_1 \setminus S_1$, and $R_1 \cap S_1$ can be written as finite disjoint unions of sets in $\mathcal{J}_n$ and $\mathcal{J}_1$, respectively. Therefore, their Cartesian products yield finite disjoint unions of elements in $\mathcal{J}_{n+1}$.

Hence, $\mathcal{J}_{n+1}$ is a semi-ring. By induction, it follows that $\mathcal{J}_n$ is a semi-ring for all $n \in \mathbb{N}$. □

Lemma 4.5. The function $\lambda_n \colon \mathcal{J}_n \to [0, \infty)$, defined by

$$\lambda_n\Big([a_1, b_1) \times \cdots \times [a_n, b_n)\Big) = \prod_{i=1}^{n} (b_i - a_i),$$

is a pre-measure on the semi-ring $\mathcal{J}_n$.

Corollary 4.5.1 (Lebesgue measure on $\mathbb{R}^n$)**.** The set function λ_n extends to a measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$, called the *Lebesgue measure*. It is the unique measure satisfying

$$\lambda_n \left([a_1, b_1) \times \cdots \times [a_n, b_n)\right) = \prod_{i=1}^{n} (b_i - a_i), \quad \text{for all } a_i < b_i.$$

Remark 4.1 (Relation to Elementary Volume)**.** The uniqueness of Lebesgue measure and its properties imply that Lebesgue measure coincides with the familiar volume function $\mathrm{vol}^{(n)}(\,\cdot\,)$ from elementary geometry. More precisely, $\mathrm{vol}^{(n)}$ can be extended to a measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ in only one way namely, as the Lebesgue measure λ^n.

5 Measurable Mappings

Definition 5.1 (Measurable Map)**.** Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$-measurable (or simply measurable) if the pre-image of every measurable set is measurable:

$$T^{-1}(A') \in \mathcal{A} \quad \text{for all } A' \in \mathcal{A}'.$$

Remark 5.1.

- Probabilists often refer to a measurable map defined on a probability space as a *random variable.*
- The symbolic notation $T^{-1}(\mathcal{A}') := \{T^{-1}(A') : A' \in \mathcal{A}'\}$ is often used. We also write $T^{-1}(\mathcal{A}') \subset \mathcal{A}$ as shorthand for measurability.
- It is common to write $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ to indicate that T is measurable.
- A measurable map between $\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^m)$ is often called a *Borel measurable map.*

Example 5.1. Let $(X, \mathcal{A})$ be a measurable space and let $A \in \mathcal{A}$. We show that the indicator function

$$\mathbf{1}_A : X \to \{0, 1\}, \quad \mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$$

is $\mathcal{A}/\mathcal{P}(\{0, 1\})$-measurable.

We check that the preimage of each subset of $\{0, 1\}$ lies in $\mathcal{A}$:

- $\mathbf{1}_A^{-1}(\emptyset) = \emptyset \in \mathcal{A}$,
- $\mathbf{1}_A^{-1}(\{0\}) = A^c \in \mathcal{A}$,
- $\mathbf{1}_A^{-1}(\{1\}) = A \in \mathcal{A}$,
- $\mathbf{1}_A^{-1}(\{0, 1\}) = X \in \mathcal{A}$.

Therefore, $\mathbf{1}_A$ is measurable.

Lemma 5.1. Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces, and suppose $\mathcal{A}' = \sigma(\mathcal{E}')$. Then a map $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable if and only if $T^{-1}(\mathcal{E}') \subseteq \mathcal{A}$, i.e. if

$$T^{-1}(E') \in \mathcal{A} \quad \text{for all } E' \in \mathcal{E}'.$$

Proof. If T is measurable, then by definition $T^{-1}(A') \in \mathcal{A}$ for all $A' \in \mathcal{A}'$. Since $\mathcal{E}' \subset \mathcal{A}'$, it follows immediately that $T^{-1}(E') \in \mathcal{A}$ for all $E' \in \mathcal{E}'$.

Conversely, suppose $T^{-1}(E') \in \mathcal{A}$ for every $E' \in \mathcal{E}'$. Define

$$\mathcal{D}' := \{A' \subseteq X' : T^{-1}(A') \in \mathcal{A}\}.$$

By assumption, $\mathcal{E}' \subseteq \mathcal{D}'$. We now show that $\mathcal{D}'$ is a σ-algebra:

- Since $T^{-1}(X') = X \in \mathcal{A}$, we have $X' \in \mathcal{D}'$.
- If $A' \in \mathcal{D}'$, then $T^{-1}(A'^c) = T^{-1}(A')^c \in \mathcal{A}$, so $A'^c \in \mathcal{D}'$.
- If $A'_1, A'_2, \cdots \in \mathcal{D}'$, then

$$T^{-1}\left(\bigcup_{i=1}^{\infty} A'_i\right) = \bigcup_{i=1}^{\infty} T^{-1}(A'_i) \in \mathcal{A},$$

  hence $\bigcup_{i=1}^{\infty} A'_i \in \mathcal{D}'$.

Thus, $\mathcal{D}'$ is a σ-algebra containing $\mathcal{E}'$, so it contains $\sigma(\mathcal{E}') = \mathcal{A}'$. Therefore, $T^{-1}(A') \in \mathcal{A}$ for all $A' \in \mathcal{A}'$, i.e., T is measurable. □

Example 5.2. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function. Then T is $\mathcal{B}(\mathbb{R}^m)/\mathcal{B}(\mathbb{R}^n)$-measurable.

Indeed, from elementary analysis, we know that T is continuous if and only if

$$T^{-1}(A') \subset \mathbb{R}^m \text{ is open for every open set } A' \subset \mathbb{R}^n.$$

Since the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ is generated by the open sets $\mathcal{O}_{\mathbb{R}^n}$, it follows that

$$T^{-1}(\mathcal{O}_{\mathbb{R}^n}) \subset \mathcal{O}_{\mathbb{R}^m} \subset \sigma(\mathcal{O}_{\mathbb{R}^m}) = \mathcal{B}(\mathbb{R}^m).$$

Hence, by Lemma 5.1, T is $\mathcal{B}(\mathbb{R}^m)/\mathcal{B}(\mathbb{R}^n)$-measurable.

Theorem 5.2. Let $(X_i, \mathcal{A}_i)$, $i = 1, 2, 3$, be measurable spaces, and let

$$T : X_1 \to X_2, \quad S : X_2 \to X_3$$

be $\mathcal{A}_1/\mathcal{A}_2$- and $\mathcal{A}_2/\mathcal{A}_3$-measurable maps, respectively. Then the composition

$$S \circ T : X_1 \to X_3$$

is $\mathcal{A}_1/\mathcal{A}_3$-measurable.

Proof. Let $A_3 \in \mathcal{A}_3$. Then

$$(S \circ T)^{-1}(A_3) = T^{-1}\left(S^{-1}(A_3)\right).$$

Since S is $\mathcal{A}_2/\mathcal{A}_3$-measurable, we have $S^{-1}(A_3) \in \mathcal{A}_2$. Since T is $\mathcal{A}_1/\mathcal{A}_2$-measurable, it follows that $T^{-1}(S^{-1}(A_3)) \in \mathcal{A}_1$. Therefore, $S \circ T$ is $\mathcal{A}_1/\mathcal{A}_3$-measurable. □

Remark 5.2. Given a map $T : X \to X'$, where X' carries a natural σ-algebra $\mathcal{A}'$ (e.g., $\mathcal{B}(\mathbb{R})$), but no σ-algebra is specified on X, one may ask: is there a smallest σ-algebra on X that makes T measurable?

While $\mathcal{P}(X)$ trivially works, it is too large to be useful. On the other hand, $T^{-1}(\mathcal{A}')$ is a σ-algebra, and removing any set from it may break measurability. This leads to the following definition.

Definition 5.2. Let $(T_i)_{i\in I}$, with $T_i : X \to X_i$, be an arbitrary family of mappings from the same space X into measurable spaces $(X_i, \mathcal{A}_i)$. The smallest σ-algebra on X that makes all T_i simultaneously measurable is given by

$$\sigma(T_i : i \in I) := \sigma\left(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\right).$$

We say that $\sigma(T_i : i \in I)$ is the σ-algebra generated by the family $(T_i)_{i\in I}$.

Although each $T_i^{-1}(\mathcal{A}_i)$ is a σ-algebra, the union $\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)$ is, in general, not a σ-algebra if $\#I > 1$; this is why we must take the σ-hull in the definition above.

Theorem 5.3. Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces and let $T : X \to X'$ be an $\mathcal{A}/\mathcal{A}'$-measurable map. For every measure μ on $(X, \mathcal{A})$,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}'$$

defines a measure μ' on $(X', \mathcal{A}')$.

Proof. If $A' = \emptyset$, then $T^{-1}(\emptyset) = \emptyset$ and $\mu'(\emptyset) = \mu(\emptyset) = 0$.

Let $(A'_n)_{n\in\mathbb{N}} \subset \mathcal{A}'$ be a sequence of pairwise disjoint sets. Then

$$\mu'\left(\bigsqcup_{n\in\mathbb{N}} A'_n\right) = \mu\left(T^{-1}\left(\bigsqcup_{n\in\mathbb{N}} A'_n\right)\right) = \mu\left(\bigsqcup_{n\in\mathbb{N}} T^{-1}(A'_n)\right) = \sum_{n\in\mathbb{N}} \mu\left(T^{-1}(A'_n)\right) = \sum_{n\in\mathbb{N}} \mu'(A'_n).$$

Hence, μ' is a measure on $(X', \mathcal{A}')$. □

Definition 5.3. The measure $\mu'(\cdot)$ from Theorem 7.6 is called the *image measure* or *pushforward* of μ under T. It is commonly denoted by one of the following:

- $T(\mu)(\cdot)$,
- $T_*\mu(\cdot)$,
- $\mu \circ T^{-1}(\cdot)$.

Example 5.3. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $\xi : \Omega \to \mathbb{R}$ be a random variable, i.e., an $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable map. Then the pushforward measure

$$\xi(\mathbb{P})(A') = \mathbb{P}(\xi^{-1}(A')) = \mathbb{P}(\{\omega \in \Omega : \xi(\omega) \in A'\}) = \mathbb{P}(\xi \in A')$$

defines a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, which is called the *law* or *distribution* of the random variable ξ.

Example 5.4. Suppose we model the experiment of rolling two fair six-sided dice. The underlying probability space is given by

$$\Omega := \{(i, k) : 1 \le i, k \le 6\}, \quad \mathcal{A} := \mathcal{P}(\Omega), \quad \mathbb{P}(\{(i, k)\}) := \frac{1}{36}.$$

Each outcome (i,k) represents the result of the first and second die, respectively. Define the map

$$\xi : \Omega \to \{2,3,\dots,12\}, \quad \xi(i,k) := i+k,$$

which assigns to each outcome the total number of points rolled. This function ξ is measurable and thus a random variable.

The pushforward measure $\xi(\mathbb{P})$, also called the *law* or *distribution* of ξ, gives the probabilities of the possible total number of points. For instance,

$$\xi(\mathbb{P})(7) = \mathbb{P}(\xi^{-1}(\{7\})) = \mathbb{P}(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) = \frac{6}{36} = \frac{1}{6}.$$

Remark 5.3. A matrix $T \in \mathbb{R}^{n\times n}$ is called orthogonal if and only if

$$T^\top T = I,$$

i.e., the transpose of T is equal to its inverse.

Orthogonal matrices preserve lengths and angles. That is, for all $x,y \in \mathbb{R}^n$, we have

$$\begin{aligned}\langle x,y\rangle &= \langle Tx,Ty\rangle,\\ \|x\| &= \|Tx\|,\end{aligned}$$

where the standard Euclidean inner product and norm are defined by

$$\langle x,y\rangle := \sum_{i=1}^{n} x_i y_i, \qquad \|x\|^2 := \langle x,x\rangle.$$

Theorem 5.4. Let $T \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Then the Lebesgue measure λ^n is invariant under T, i.e.,

$$T(\lambda^n) = \lambda^n.$$

Proof. The matrix $T \in \mathbb{R}^{n\times n}$ defines a linear map $x \mapsto Tx$, i.e.,

$$T(ax+by) = aTx + bTy \quad \text{for all } a,b \in \mathbb{R},\ x,y \in \mathbb{R}^n.$$

From the orthogonality condition, it follows that T is an isometry:

$$\|Tx - Ty\| = \|T(x-y)\| = \|x-y\|.$$

Thus, T is continuous and therefore $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$-measurable by Example 5.2. Furthermore by Theorem 5.3, the pushforward measure

$$\nu(B) := \lambda^n(T^{-1}(B))$$

is well-defined on $\mathcal{B}(\mathbb{R}^n)$.

We now show that ν is translation invariant. For any $x \in \mathbb{R}^n$ and $B \in \mathcal{B}(\mathbb{R}^n)$, we compute

$$\begin{aligned}\nu(x+B) &= \lambda^n\left(T^{-1}(x+B)\right)\\ &= \lambda^n\left(T^{-1}x + T^{-1}B\right)\\ &= \lambda^n\left(T^{-1}B\right) = \nu(B),\end{aligned}$$

where we used linearity of T and translation invariance of Lebesgue measure (Theorem 3.5(i)).

Hence ν is a translation-invariant measure on $\mathbb{R}^n$. By Theorem 3.5(ii), since ν is also finite on bounded sets (e.g., the unit ball), it must be a scalar multiple of Lebesgue measure:

$$\nu = c\lambda^n \quad \text{for some } c > 0.$$

To determine the constant c, consider the unit ball $B_1(0) := \{x \in \mathbb{R}^n : \|x\| < 1\}$. Since T is orthogonal, we have

$$x \in B_1(0) \iff \|x\| < 1 \iff \|Tx\| < 1 \iff x \in T^{-1}(B_1(0)),$$

so $T^{-1}(B_1(0)) = B_1(0)$. Therefore,

$$\lambda^n(B_1(0)) = \lambda^n(T^{-1}(B_1(0))) = \nu(B_1(0)) = c\lambda^n(B_1(0)),$$

which implies $c = 1$, since $0 < \lambda^n(B_1(0)) < \infty$. Thus, $\nu = \lambda^n$, and the theorem follows. □

Remark 5.4. Theorem 5.4 is a special case of the following general change-of-variable formula for Lebesgue measure. Recall that a matrix $S \in \mathbb{R}^{n\times n}$ is invertible if and only if $\det S \neq 0$.

Theorem 5.5 (Change of Variables)**.** Let $S \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then

$$S(\lambda^n) = |\det S^{-1}|\,\lambda^n = |\det S|^{-1}\,\lambda^n. \tag{7.7}$$

Proof. Since S is invertible, both S and S^{-1} are linear maps on $\mathbb{R}^n$, and hence continuous and measurable (by Example 5.2). Define a measure ν on $\mathbb{R}^n$ by

$$\nu(B) := \lambda^n(S^{-1}(B)), \quad B \in \mathcal{B}(\mathbb{R}^n).$$

For any $x \in \mathbb{R}^n$, we have

$$\nu(x + B) = \lambda^n(S^{-1}(x + B)) = \lambda^n(S^{-1}x + S^{-1}B) = \lambda^n(S^{-1}B) = \nu(B),$$

so ν is translation invariant.

By Theorem 3.5(ii), any translation-invariant measure finite on the unit cube must be a scalar multiple of Lebesgue measure, so there exists a constant $c > 0$ such that

$$\nu = c\lambda^n.$$

To determine c, we evaluate both sides on the unit cube $[0,1)^n$, which satisfies $\lambda^n([0,1)^n) = 1$:

$$\nu([0,1)^n) = \lambda^n\Big(S^{-1}([0,1)^n)\Big).$$

The set $S^{-1}([0,1)^n)$ is a parallelepiped spanned by the vectors $S^{-1}e_i$, where $(e_i)_{i=1}^n$ is the standard basis of $\mathbb{R}^n$. Its volume is given by the absolute value of the determinant:

$$\mathrm{vol}^{(n)}\Big(S^{-1}([0,1)^n)\Big) = |\det S^{-1}| = |\det S|^{-1}.$$

By Remark 4.1, Lebesgue measure coincides with this volume on Borel sets, so

$$\nu([0,1)^n) = \lambda^n\Big(S^{-1}([0,1)^n)\Big) = |\det S|^{-1}.$$

Hence,

$$\nu = |\det S|^{-1}\lambda^n,$$

which completes the proof. □

Definition 5.4. A *motion* in $\mathbb{R}^n$ is a linear transformation of the form

$$Mx = \tau_x \circ T(x),$$

where $\tau_x(y) = y + x$ denotes translation by $x \in \mathbb{R}^n$, and $T \in \mathbb{R}^{n \times n}$ is an orthogonal matrix (i.e., $T^\top T = \mathrm{id}_n$). In particular, two sets are said to be *congruent* if one can be obtained from the other by a motion.

Theorem 5.6 (Invariance under Motions)**.** Lebesgue measure is invariant under motions: for any motion M in $\mathbb{R}^n$, we have

$$\lambda^n = M(\lambda^n).$$

In particular, congruent sets have the same Lebesgue measure.

Proof. By definition, any motion M can be written as $M = \tau_x \circ T$, where T is orthogonal (so $|\det T| = 1$). By Theorem 5.4,

$$T(\lambda^n) = \lambda^n.$$

Translation invariance of Lebesgue measure (Theorem 3.5(i)) gives

$$\tau_x(\lambda^n) = \lambda^n.$$

Hence,

$$M(\lambda^n) = \tau_x(T(\lambda^n)) = \tau_x(\lambda^n) = \lambda^n. \qquad \square$$

6 Measurable Functions

Definition 6.1 (Measurable Function)**.** Let $(X, \mathcal{A})$ be a measurable space. A function $u : X \to \mathbb{R}$ is called *measurable* if it is $\mathcal{A}$-$\mathcal{B}(\mathbb{R})$-measurable; that is,

$$u^{-1}(B) \in \mathcal{A} \quad \text{for all } B \in \mathcal{B}(\mathbb{R}).$$

Remark 6.1. By Lemma 5.1, a function $u : X \to \mathbb{R}$ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable if and only if

$$u^{-1}(G) \in \mathcal{A} \quad \text{for all } G \in \mathcal{E},$$

where $\mathcal{E}$ is a generator of $\mathcal{B}(\mathbb{R})$.

Definition 6.2 (Level Set)**.** Let $(X, \mathcal{A})$ be a measurable space, and let $u : X \to \mathbb{R}$ be a function. For any $y \in \mathbb{R}$, the *level set* of u at the value y is defined as

$$\{u = y\} := \{x \in X : u(x) = y\}.$$

More generally, we define:

- Strict upper level set: $\{u > y\} := \{x \in X : u(x) > y\}$
- Strict lower level set: $\{u < y\} := \{x \in X : u(x) < y\}$
- Upper level set: $\{u \geq y\} := \{x \in X : u(x) \geq y\}$
- Lower level set: $\{u \leq y\} := \{x \in X : u(x) \leq y\}$

Remark 6.2. As noted in Remark 1.4, the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is generated by intervals of the form $[a, \infty)$, (a, ∞), $(-\infty, a)$, or $(-\infty, a]$, with $a \in \mathbb{R}$ (or $\mathbb{Q}$). To verify that a function $u : X \to \mathbb{R}$ is measurable, it suffices to check that

$$u^{-1}([a, \infty)) = \{x \in X : u(x) \in [a, \infty)\} = \{x \in X : u(x) \geq a\} \in \mathcal{A}$$

for all such a, and likewise for the other interval types.

We write

$$\{u > v\} := \{x \in X : u(x) > v(x)\},$$

and similarly $\{u < v\}$, $\{u \leq v\}$, $\{u = v\}$, $\{u \neq v\}$, $\{u \in B\}$, etc., for measurable $u, v : X \to \mathbb{R}$ and Borel sets $B \subseteq \mathbb{R}$.

Lemma 6.1. Let $(X, \mathcal{A})$ be a measurable space. A function $u : X \to \mathbb{R}$ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable if and only if any one (and hence all) of the following equivalent conditions hold:

(i) $\{u > a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$ or $a \in \mathbb{Q}$

(ii) $\{u \geq a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$ or $a \in \mathbb{Q}$

(iii) $\{u < a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$ or $a \in \mathbb{Q}$

(iv) $\{u \leq a\} \in \mathcal{A} \quad \forall a \in \mathbb{R}$ or $a \in \mathbb{Q}$

Remark 6.3. It is often helpful to use the values $+\infty$ and $-\infty$ in calculations. To do this properly, we consider the extended real line $\overline{\mathbb{R}} := [-\infty, +\infty]$. If we agree that $-\infty < x$ and $y < +\infty$ for all $x, y \in \mathbb{R}$, then $\overline{\mathbb{R}}$ inherits the usual ordering from $\mathbb{R}$, as well as the standard rules of addition and multiplication for real numbers. The latter, however, must be augmented as shown below.

Addition in $\overline{\mathbb{R}}$

$+$	0	x	$+\infty$	$-\infty$
0	0	x	$+\infty$	$-\infty$
y	y	$x+y$	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	$+\infty$	undef.
$-\infty$	$-\infty$	$-\infty$	undef.	$-\infty$

Multiplication in $\overline{\mathbb{R}}$

$\cdot$	0	a	$+\infty$	$-\infty$
0	0	0	0	0
b	0	ab	$+\infty$	$-\infty$
$+\infty$	0	$+\infty$	$+\infty$	$-\infty$
$-\infty$	0	$-\infty$	$-\infty$	$+\infty$

Remark 6.4. Caution: The extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$ is *not* a field. Expressions such as $\infty - \infty$ or $\frac{\infty}{\infty}$ are undefined and must be avoided.

The Borel σ-algebra on $\overline{\mathbb{R}}$, denoted $\mathcal{B}(\overline{\mathbb{R}})$, is defined by

$$B^* \in \mathcal{B}(\overline{\mathbb{R}}) \quad \Longleftrightarrow \quad B^* = B \cup S,$$

for some $B \in \mathcal{B}(\mathbb{R})$ and $S \in \{\emptyset, \{-\infty\}, \{+\infty\}, \{-\infty, +\infty\}\}$.

It is straightforward to verify that $\mathcal{B}(\overline{\mathbb{R}})$ is a σ-algebra, and its trace on $\mathbb{R}$ coincides with the usual Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

Lemma 6.2. The Borel σ-algebra on the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$ satisfies

$$\mathcal{B}(\mathbb{R}) = \mathbb{R} \cap \mathcal{B}(\overline{\mathbb{R}}) \quad \text{or equivalently,} \quad \mathcal{B}(\mathbb{R}) = \{A \cap \mathbb{R} : A \in \mathcal{B}(\overline{\mathbb{R}})\}.$$

Lemma 6.3. The Borel σ-algebra $\mathcal{B}(\overline{\mathbb{R}})$ is generated by any one of the following families of sets:

$$[a, +\infty], \quad (a, +\infty], \quad [-\infty, a), \quad \text{or} \quad [-\infty, a],$$

with $a \in \mathbb{R}$ or $\mathbb{Q}$.

Proof. Let $\mathcal{E} := \sigma\Big(\{[a, +\infty] : a \in \mathbb{R}\}\Big)$. Since

$$[a, +\infty] = [a, +\infty) \cup \{+\infty\} \quad \text{with} \quad [a, +\infty) \in \mathcal{B}(\mathbb{R}),$$

we see that $[a, +\infty] \in \mathcal{B}(\overline{\mathbb{R}})$, hence $\mathcal{E} \subseteq \mathcal{B}(\overline{\mathbb{R}})$.

Conversely, for $-\infty < a \leq b < +\infty$,

$$[a, b) = [a, +\infty] \setminus [b, +\infty] \in \mathcal{E},$$

so $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{E}$. Moreover,

$$\{+\infty\} = \bigcap_{j \in \mathbb{N}} [j, +\infty], \qquad \{-\infty\} = \bigcap_{j \in \mathbb{N}} [-\infty, -j) = \bigcap_{j \in \mathbb{N}} [-j, +\infty]^c,$$

so $\{-\infty\}, \{+\infty\} \in \mathcal{E}$. Hence for any $B \in \mathcal{B}(\mathbb{R})$,

$$B, \quad B \cup \{+\infty\}, \quad B \cup \{-\infty\}, \quad B \cup \{-\infty, +\infty\} \in \mathcal{E},$$

implying $\mathcal{B}(\overline{\mathbb{R}}) \subseteq \mathcal{E}$. Therefore, $\mathcal{B}(\overline{\mathbb{R}}) = \mathcal{E}$.

The same argument applies if the generating system uses $a \in \mathbb{Q}$, or other families like $(a, +\infty]$, $[-\infty, a)$, or $[-\infty, a]$. $\square$

Definition 6.3. Let $(X, \mathcal{A})$ be a measurable space. We define

$$\mathcal{M} := \mathcal{M}(\mathcal{A}) \quad \text{and} \quad \mathcal{M}_{\overline{\mathbb{R}}} := \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$$

as the collections of real-valued and extended real-valued measurable functions, respectively:

$$\mathcal{M} = \{u : X \to \mathbb{R} \mid u \text{ is } \mathcal{A}/\mathcal{B}(\mathbb{R})\text{-measurable}\},$$
$$\mathcal{M}_{\overline{\mathbb{R}}} = \{u : X \to \overline{\mathbb{R}} \mid u \text{ is } \mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})\text{-measurable}\}.$$

Example 6.1. Let $(X, \mathcal{A})$ be a measurable space. The indicator function $f(x) := \mathbf{1}_A(x)$ is measurable if and only if $A \in \mathcal{A}$.

Proof. Recall that a function $f : X \to \mathbb{R}$ is measurable if for all $\alpha \in \mathbb{R}$, the set $\{f > \alpha\} \in \mathcal{A}$. Now observe:

$$\{\mathbf{1}_A > \alpha\} = \begin{cases} \varnothing & \text{if } \alpha \geq 1, \\ A & \text{if } 0 < \alpha < 1, \\ X & \text{if } \alpha \leq 0. \end{cases}$$

Thus, $\{\mathbf{1}_A > \alpha\} \in \mathcal{A}$ for all $\alpha \in \mathbb{R}$ if and only if $A \in \mathcal{A}$, proving the claim. $\square$

Definition 6.4. Let $(X, \mathcal{A})$ be a measurable space.

A *simple function* is a function $f : X \to \mathbb{R}$ of the form

$$f(x) = \sum_{m=1}^{M} y_m \mathbf{1}_{A_m}(x),$$

where $M \in \mathbb{N}$, $y_m \in \mathbb{R}$, and $A_1, \ldots, A_M \in \mathcal{A}$ are pairwise disjoint.

A representation of the form

$$f(x) = \sum_{n=1}^{N} z_n \mathbf{1}_{B_n}(x),$$

with $N \in \mathbb{N}$, $z_n \in \mathbb{R}$, $B_n \in \mathcal{A}$, and $\bigsqcup_{n=1}^{N} B_n = X$, is called a *standard representation* of f.

The set of all simple functions on $(X, \mathcal{A})$ is denoted by $\mathcal{E}$ or $\mathcal{E}(\mathcal{A})$.

Remark 6.5. Simple functions may have multiple representations; in particular, standard representations are not unique.

Example 6.2. A measurable function $h : X \to \mathbb{R}$ that attains only finitely many values is a simple function.

Indeed, let $h(X) = \{y_0, \ldots, y_M\}$. The sets $\{h = \beta\}$ with $\beta \in h(X)$ are mutually disjoint and satisfy

$$\{h = \beta\} = \{h \leq \beta\} \setminus \{h < \beta\} \in \mathcal{A},$$

and

$$\bigcup_{\beta \in h(X)} \{h = \beta\} = X.$$

Thus, h admits the standard representation

$$h(x) = \sum_{\beta \in h(X)} \beta \, \mathbf{1}_{\{h = \beta\}}(x).$$

This shows that every measurable function with finitely many values is a simple function. Conversely, since every simple function only takes finitely many values, it always admits at least one standard representation.

In particular, $\mathcal{E}(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$, where $\mathcal{M}(\mathcal{A})$ is the space of measurable functions.

Example 6.3. If $f, g \in \mathcal{E}(\mathcal{A})$, then

$$f \pm g \in \mathcal{E}(\mathcal{A}) \quad \text{and} \quad f \cdot g \in \mathcal{E}(\mathcal{A}).$$

Proof. Let

$$f = \sum_{m=1}^{M} y_m \, \mathbf{1}_{A_m}, \qquad g = \sum_{n=1}^{N} z_n \, \mathbf{1}_{B_n}$$

be standard representations, where $A_m, B_n \in \mathcal{A}$ are disjoint families and $y_m, z_n \in \mathbb{R}$.

Consider the intersections $A_m \cap B_n$. Then:

- Each $A_m \cap B_n \in \mathcal{A}$, since $\mathcal{A}$ is closed under intersections.
- The sets $A_m \cap B_n$ are pairwise disjoint for distinct pairs $(m, n) \neq (m', n')$.
- Their union covers X, because

$$X = \bigcup_{m=1}^{M} A_m = \bigcup_{n=1}^{N} B_n \quad \Rightarrow \quad X = \bigcup_{m=1}^{M} \bigcup_{n=1}^{N} (A_m \cap B_n).$$

On each set $A_m \cap B_n$, the functions f and g are constant with values y_m and z_n, respectively. Thus,

$$f(x) \pm g(x) = y_m \pm z_n, \qquad f(x) \cdot g(x) = y_m z_n \quad \text{for } x \in A_m \cap B_n.$$

Therefore, we obtain:

$$f \pm g = \sum_{m=1}^{M} \sum_{n=1}^{N} (y_m \pm z_n) \, \mathbf{1}_{A_m \cap B_n}, \qquad f \cdot g = \sum_{m=1}^{M} \sum_{n=1}^{N} (y_m z_n) \, \mathbf{1}_{A_m \cap B_n}.$$

These are finite sums over pairwise disjoint measurable sets, so $f \pm g$ and $f \cdot g$ are both simple functions.

Hence, $f \pm g, \; f \cdot g \in \mathcal{E}(\mathcal{A})$. □

Definition 6.5. Let $f \in \mathcal{E}(\mathcal{A})$. Define its *positive part*, *negative part*, and *absolute value* by

$$f^+(x) := \max(f(x), 0), \qquad f^-(x) := \max(-f(x), 0), \qquad |f(x)| := f^+(x) + f^-(x).$$

Then $f^+, f^-, |f| \in \mathcal{E}(\mathcal{A})$, and

$$f = f^+ - f^-, \qquad |f| = f^+ + f^-.$$

Theorem 6.4 (Sombrero Lemma)**.** Let $(X, \mathcal{A})$ be a measurable space. Every positive $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable function $u : X \to [0, \infty]$ is the pointwise limit of an increasing sequence of simple functions $f_n \in \mathcal{E}(\mathcal{A})$, with $f_n \geq 0$, that is,

$$u(x) = \sup_{n \in \mathbb{N}} f_n(x) = \lim_{n \to \infty} f_n(x), \qquad \text{with} \quad f_1 \leq f_2 \leq f_3 \leq \dots$$

Corollary 6.4.1. Let $(X, \mathcal{A})$ be a measurable space. Every $\mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})$-measurable function $u : X \to \overline{\mathbb{R}}$ is the pointwise limit of simple functions $f_n \in \mathcal{E}(\mathcal{A})$ such that

$$|f_n(x)| \leq |u(x)| \quad \text{for all } x \in X.$$

If u is bounded, the convergence is uniform.