

# Measure Theory

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## 1 $\sigma$ -Algebras

**Definition 1.1.** A  $\sigma$ -algebra  $\mathcal{A}$  on a set  $X$  is a family of subsets of  $X$  such that:

- $X \in \mathcal{A}$  ( $\Sigma_1$ )
- If  $A \in \mathcal{A}$ , then  $A^c \in \mathcal{A}$  ( $\Sigma_2$ )
- If  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ , then  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$  ( $\Sigma_3$ )

A set  $A \in \mathcal{A}$  is said to be *measurable* or  *$\mathcal{A}$ -measurable*.

**Example 1.1.**

1.  $\mathcal{P}(X)$  is a  $\sigma$ -algebra (the maximal  $\sigma$ -algebra on  $X$ ).
2.  $\{\emptyset, X\}$  is a  $\sigma$ -algebra (the minimal  $\sigma$ -algebra on  $X$ ).
3.  $\mathcal{A} := \{A \subseteq X : \#A < \infty \text{ or } \#A^c < \infty\}$  is a  $\sigma$ -algebra.
4. (Trace  $\sigma$ -algebra) Let  $E \subseteq X$  be any set and let  $\mathcal{A}$  be a  $\sigma$ -algebra on  $X$ . Then

$$\mathcal{A}_E := \{E \cap A : A \in \mathcal{A}\}$$

is a  $\sigma$ -algebra on  $E$ .

*Proof.* We verify the three defining properties of a  $\sigma$ -algebra on  $E$ :

- Since  $X \in \mathcal{A}$ , we have  $E = E \cap X \in \mathcal{A}_E$ .
- If  $E \cap A \in \mathcal{A}_E$ , then  $E \setminus (E \cap A) = E \cap A^c$ , and since  $A^c \in \mathcal{A}$ , it follows that  $E \cap A^c \in \mathcal{A}_E$ .
- If  $(E \cap A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}_E$ , then  $\bigcup_n (E \cap A_n) = E \cap \bigcup_n A_n$ , and since  $\bigcup_n A_n \in \mathcal{A}$ , we conclude that  $\bigcup_n (E \cap A_n) \in \mathcal{A}_E$ .

Hence,  $\mathcal{A}_E$  is a  $\sigma$ -algebra on  $E$ . □

5. (Pre-image  $\sigma$ -algebra) Let  $f : X \rightarrow X'$  be a function and let  $\mathcal{A}'$  be a  $\sigma$ -algebra on  $X'$ . Then

$$\mathcal{A} := \{f^{-1}(A') : A' \in \mathcal{A}'\}$$

is a  $\sigma$ -algebra on  $X$ .

**Theorem 1.1.** Let  $X$  be a set and let  $\{\mathcal{A}_i : i \in I\}$  be a family of  $\sigma$ -algebras on  $X$ . Define

$$\mathcal{A} := \bigcap_{i \in I} \mathcal{A}_i = \{A \subseteq X : A \in \mathcal{A}_i \text{ for all } i \in I\}.$$

Then,  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ .

*Proof.* We verify the  $\sigma$ -algebra properties for  $\mathcal{A}$ :

- Since  $X \in \mathcal{A}_i$  for all  $i \in I$ , we have  $X \in \mathcal{A}$ .
- If  $A \in \mathcal{A}$ , then  $A \in \mathcal{A}_i$  for all  $i \in I$ , so  $A^c = X \setminus A \in \mathcal{A}_i$  for all  $i \in I$ , hence  $A^c \in \mathcal{A}$ .
- If  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ , then  $A_n \in \mathcal{A}_i$  for all  $n$  and  $i$ , so  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_i$  for all  $i \in I$ , and thus  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ .

Therefore,  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ . □

**Definition 1.2.** Let  $X$  be a set and let  $\mathcal{E} \subseteq \mathcal{P}(X)$  be a collection of subsets of  $X$ . The  $\sigma$ -algebra generated by  $\mathcal{E}$ , denoted by  $\sigma(\mathcal{E})$ , is the smallest  $\sigma$ -algebra on  $X$  containing all sets in  $\mathcal{E}$ . That is,

$$\sigma(\mathcal{E}) := \bigcap \left\{ \mathcal{A} \subseteq \mathcal{P}(X) : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X, \mathcal{E} \subseteq \mathcal{A} \right\}.$$

**Remark 1.1** (Generated  $\sigma$ -algebras).

- If  $\mathcal{A}$  is a  $\sigma$ -algebra, then  $\sigma(\mathcal{A}) = \mathcal{A}$ .
- For  $A \subseteq X$ , we have  $\sigma(\{A\}) = \{\emptyset, A, A^c, X\}$ .
- If  $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{A}$ , then  $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G}) \subseteq \sigma(\mathcal{A})$ .

**Definition 1.3** (Topological Space). A *topological space* is a pair  $(X, \mathcal{T})$  where  $X$  is a set and  $\mathcal{T} \subseteq \mathcal{P}(X)$  is a collection of subsets of  $X$ , called *open sets*, satisfying the following properties:

- $\emptyset \in \mathcal{T}$  and  $X \in \mathcal{T}$ ,
- If  $\{U_\alpha \in \mathcal{T} : \alpha \in I\}$  is an arbitrary collection of open sets, then the union  $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$ ,
- If  $\{U_i \in \mathcal{T} : i = 1, \dots, n\}$  is a finite collection of open sets, then the intersection  $\bigcap_{i=1}^n U_i \in \mathcal{T}$ .

The collection  $\mathcal{T}$  is called a *topology* on  $X$ . The complement of an open set is called a *closed set*.

**Remark 1.2** (Standard Topology on  $\mathbb{R}^n$ ). A subset  $U \subseteq \mathbb{R}^n$  is called *open* if for every point  $x \in U$ , there exists an  $\varepsilon > 0$  such that the open ball

$$B_\varepsilon(x) := \{y \in \mathbb{R}^n : \|x - y\| < \varepsilon\},$$

where  $\|\cdot\|$  denotes the Euclidean norm, is contained in  $U$ ; that is,  $B_\varepsilon(x) \subseteq U$ .

The collection of all such open sets is denoted by  $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$  and forms the *standard topology* on  $\mathbb{R}^n$ .

**Definition 1.4** (Borel  $\sigma$ -algebra). The  $\sigma$ -algebra  $\sigma(\mathcal{O})$  generated by the collection of open sets  $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$  of  $\mathbb{R}^n$  is called the *Borel  $\sigma$ -algebra* on  $\mathbb{R}^n$ .

Its elements are called *Borel sets* or *Borel measurable sets*. We denote the Borel  $\sigma$ -algebra by  $\mathcal{B}(\mathbb{R}^n)$ .

**Definition 1.5.** Let  $X$  be a topological space and let  $A \subseteq X$ . A collection  $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$  of open sets is called an *open cover* of  $A$  if

$$A \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

A *subcover* is a subcollection that still covers  $A$ . The set  $A$  is called *compact* if every open cover of  $A$  admits a finite subcover.

**Remark 1.3.** In  $\mathbb{R}^n$ , a set is compact if and only if it is closed and bounded (Heine–Borel Theorem).

**Theorem 1.2** (Borel  $\sigma$ -algebra from Different Generators). Let  $\mathcal{O}, \mathcal{C}, \mathcal{K} \subseteq \mathcal{P}(\mathbb{R}^n)$  denote the collections of open, closed, and compact subsets of  $\mathbb{R}^n$ , respectively. Then,

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}) = \sigma(\mathcal{C}) = \sigma(\mathcal{K}).$$

*Proof.* Since compact sets are closed, we have  $\mathcal{K} \subseteq \mathcal{C}$ , and by Remark 1.1,  $\sigma(\mathcal{K}) \subseteq \sigma(\mathcal{C})$ . Conversely, for any  $C \in \mathcal{C}$ , define  $C_k := C \cap B_k(0)$ , where  $B_k(0)$  is the closed ball of radius  $k$  centered at the origin. Each  $C_k$  is closed and bounded, hence compact, so  $C_k \in \mathcal{K}$ . Since  $C = \bigcup_{k \in \mathbb{N}} C_k$ , it follows that  $C \in \sigma(\mathcal{K})$ , and thus  $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{K})$ .

Next, since  $\mathcal{C} = \mathcal{O}^c := \{U^c : U \in \mathcal{O}\}$ , and complements of sets in a  $\sigma$ -algebra are again in the  $\sigma$ -algebra, it follows that  $\mathcal{C} \subseteq \sigma(\mathcal{O})$ , hence  $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{O})$ . The reverse inclusion follows similarly from  $\mathcal{O} = \mathcal{C}^c$ . Therefore, we have:

$$\sigma(\mathcal{K}) = \sigma(\mathcal{C}) = \sigma(\mathcal{O}).$$

□

**Generating Sets of the Borel Algebra.** The Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}^n)$  can be generated by various systems of sets. Of particular importance are:

- The family of open rectangles:

$$\mathcal{J}_{o,n} := \{(a_1, b_1) \times \cdots \times (a_n, b_n) : a_i, b_i \in \mathbb{R}\},$$

- The family of half-open rectangles:

$$\mathcal{J}_n := \{[a_1, b_1) \times \cdots \times [a_n, b_n) : a_i, b_i \in \mathbb{R}\}.$$

We denote by  $\mathcal{J}_n^{\text{rat}}, \mathcal{J}_{o,n}^{\text{rat}}$  the subsets with rational endpoints. These sets represent intervals in  $\mathbb{R}$ , rectangles in  $\mathbb{R}^2$ , cuboids in  $\mathbb{R}^3$ , and hypercubes in higher dimensions.

**Theorem 1.3.** We have the following equality of Borel  $\sigma$ -algebras on  $\mathbb{R}^n$ :

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{J}_n^{\text{rat}}) = \sigma(\mathcal{J}_{o,n}^{\text{rat}}) = \sigma(\mathcal{J}_n) = \sigma(\mathcal{J}_{o,n}),$$

**Remark 1.4.** Let  $D \subseteq \mathbb{R}$  be a dense subset, for example  $D = \mathbb{Q}$  or  $D = \mathbb{R}$ . Then the Borel sets on  $\mathbb{R}$  can also be generated by any of the following families of intervals:

$$\{(-\infty, a) : a \in D\}, \quad \{(-\infty, a] : a \in D\}, \quad \{(a, \infty) : a \in D\}, \quad \{[a, \infty) : a \in D\}.$$

## 2 Measure Spaces

**Definition 2.1.** A (*positive*) *measure*  $\mu$  on  $X$  is a map  $\mu : \mathcal{A} \rightarrow [0, \infty]$ , where  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ , satisfying:

$$\mu(\emptyset) = 0, \tag{M1}$$

and for any pairwise disjoint sequence  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ ,

$$\mu\left(\bigsqcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n). \tag{M2}$$

Property (M2) is also called *countable additivity*.

If  $\mu$  satisfies (M1), (M2), but  $\mathcal{A}$  is not a  $\sigma$ -algebra, then  $\mu$  is called a *pre-measure*.

**Remark 2.1.** (M2) requires implicitly that  $\bigsqcup_n A_n$  is again in  $\mathcal{A}$  this is clearly the case for  $\sigma$ -algebras, but needs special attention when dealing with pre-measures.

**Definition 2.2** (Monotone sequences of sets). Let  $(A_n)_{n \in \mathbb{N}}$  and  $(B_n)_{n \in \mathbb{N}}$  be sequences of subsets of  $X$ .

We say  $(A_n)$  is *increasing* if

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$$

and write  $A_n \uparrow A$  where

$$A := \bigcup_{n \in \mathbb{N}} A_n$$

Similarly,  $(B_n)$  is *decreasing* if

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$$

and write  $B_n \downarrow B$  where

$$B := \bigcap_{n \in \mathbb{N}} B_n$$

**Definition 2.3.** Let  $X$  be a set and  $\mathcal{A}$  a  $\sigma$ -algebra on  $X$ . The pair  $(X, \mathcal{A})$  is called a *measurable space*. If  $\mu$  is a measure on  $(X, \mathcal{A})$ , then  $(X, \mathcal{A}, \mu)$  is called a *measure space*.

A measure  $\mu$  is called:

- *finite* if  $\mu(X) < \infty$ ,
- a *probability measure* if  $\mu(X) = 1$ .

Accordingly, we speak of a *finite measure space* and a *probability space*.

**Definition 2.4.** A measure  $\mu$  on a measurable space  $(X, \mathcal{A})$  is called  *$\sigma$ -finite* if there exists a sequence  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$  such that:

$$A_n \uparrow X \quad \text{and} \quad \mu(A_n) < \infty \quad \text{for all } n \in \mathbb{N}.$$

In this case, the measure space  $(X, \mathcal{A}, \mu)$  is called  *$\sigma$ -finite*.

**Lemma 2.1** (Basic properties of measures). Let  $(X, \mathcal{A}, \mu)$  be a measure space. Then:

- (i) If  $A_0, \dots, A_k \in \mathcal{A}$  are pairwise disjoint, then  $\mu\left(\bigsqcup_{n=1}^k A_n\right) = \sum_{n=1}^k \mu(A_n)$ .
- (ii) If  $A, B \in \mathcal{A}$  with  $A \subseteq B$ , then  $\mu(A) \leq \mu(B)$ .
- (iii) If  $A, B \in \mathcal{A}$ ,  $A \subseteq B$ , and  $\mu(A) < \infty$ , then  $\mu(B \setminus A) = \mu(B) - \mu(A)$ .

*Proof.* (i) Extend  $(A_n)$  by  $A_n = \emptyset$  for  $n > k$ . Then by countable additivity,

$$\mu\left(\bigsqcup_{n=1}^k A_n\right) = \mu\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^k \mu(A_n).$$

- (ii) Since  $B = A \sqcup (B \setminus A)$  we have

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

- (iii) Rearranging gives

$$\mu(B \setminus A) = \mu(B) - \mu(A),$$

which is well-defined if  $\mu(A) < \infty$ . □

**Lemma 2.2** (Main properties of measures). Let  $(X, \mathcal{A}, \mu)$  be a measure space. Then:

(i) **Countable subadditivity:** For any countable family  $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ ,

$$\mu \left( \bigcup_{i \in \mathbb{N}} A_i \right) \leq \sum_{i \in \mathbb{N}} \mu(A_i).$$

(ii) **Continuity from below (increasing sequence):** If  $A_1 \subseteq A_2 \subseteq \cdots$  (i.e.,  $A_n \uparrow A$ ), then

$$\mu \left( \bigcup_{n \in \mathbb{N}} A_n \right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

(iii) **Continuity from above (decreasing sequence):** If  $B_1 \supseteq B_2 \supseteq \cdots$  (i.e.,  $B_n \downarrow B$ ), then

$$\mu \left( \bigcap_{n \in \mathbb{N}} B_n \right) = \lim_{n \rightarrow \infty} \mu(B_n).$$

*Proof.* (i) For countable subadditivity, set  $B_k := A_k \setminus \bigcup_{i=1}^{k-1} A_i$ , so that  $(B_k)$  are disjoint with  $B_k \subseteq A_k$ . Then,

$$\mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \mu \left( \bigsqcup_{i=1}^{\infty} B_i \right) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

(ii) Let  $A_n \uparrow A$ , i.e.,  $A_n \subseteq A_{n+1}$  and  $A = \bigcup_{n \in \mathbb{N}} A_n$ . Define  $B_n := A_n \setminus A_{n-1}$  with  $A_0 := \emptyset$ . Then  $(B_n)$  is disjoint and  $\bigsqcup_n B_n = A$ . By countable additivity,

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(B_k) = \lim_{n \rightarrow \infty} \mu(A_n)$$

(iii) Assume  $B_n \downarrow B$ , i.e.,  $B_n \supseteq B_{n+1}$  and  $B = \bigcap_n B_n$ , with  $\mu(B_1) < \infty$ . Set  $A_n := B_1 \setminus B_n$ , so  $A_n \uparrow A := B_1 \setminus B$ . Then

$$\mu(B) = \mu(B_1) - \mu(A) = \mu(B_1) - \lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \mu(B_n)$$

□

**Remark 2.2.** With appropriate modifications, these properties also hold for pre-measures, i.e., when  $\mathcal{A}$  is not necessarily a  $\sigma$ -algebra.

**Example 2.1** (Dirac measure). Let  $(X, \mathcal{A})$  be a measurable space and let  $x \in X$ . Define  $\delta_x : \mathcal{A} \rightarrow \{0, 1\}$  by

$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then  $\delta_x$  is a measure on  $(X, \mathcal{A})$ , called the *Dirac measure* (or unit mass) at the point  $x$ .

**Example 2.2** (Counting measure). Let  $(X, \mathcal{A})$  be a measurable space. Define  $\#A : \mathcal{A} \rightarrow [0, \infty]$  by

$$\#A := \begin{cases} \text{number of elements in } A & \text{if } A \text{ is finite,} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$$

Then  $\#$  is a measure on  $(X, \mathcal{A})$ , called the *counting measure*.

**Example 2.3** (Discrete probability measure). Let  $\Omega = \{\omega_1, \omega_2, \dots\}$  be a countable set, and let  $(p_n)_{n \in \mathbb{N}} \subseteq [0, 1]$  be a sequence such that  $\sum_{n \in \mathbb{N}} p_n = 1$ . Define the set function  $P : \mathcal{P}(\Omega) \rightarrow [0, 1]$  by

$$P(A) := \sum_{\{n \in \mathbb{N} : \omega_n \in A\}} p_n = \sum_{n \in \mathbb{N}} p_n \delta_{\omega_n}(A), \quad A \subseteq \Omega,$$

where  $\delta_{\omega_n}$  denotes the Dirac measure at  $\omega_n$ . Then  $P$  is a probability measure on  $(\Omega, \mathcal{P}(\Omega))$ , and the triplet  $(\Omega, \mathcal{P}(\Omega), P)$  is called a *discrete probability space*.

**Example 2.4** (Linear combination of measures). Let  $(X, \mathcal{A})$  be a measurable space, and let  $(\mu_n)_{n \in \mathbb{N}}$  be a sequence of measures on  $(X, \mathcal{A})$ . Let  $(x_n)_{n \in \mathbb{N}} \subseteq [0, \infty]$ . Then the set function

$$\mu := \sum_{n \in \mathbb{N}} x_n \mu_n$$

defined by

$$\mu(A) := \sum_{n \in \mathbb{N}} x_n \mu_n(A), \quad \text{for all } A \in \mathcal{A},$$

is a measure on  $(X, \mathcal{A})$

*Proof.* We verify the axioms of a measure:

(M1) (*Null empty set*): For all  $n \in \mathbb{N}$ ,  $\mu_n(\emptyset) = 0$ , so

$$\mu(\emptyset) = \sum_{n \in \mathbb{N}} x_n \mu_n(\emptyset) = \sum_{n \in \mathbb{N}} x_n \cdot 0 = 0.$$

(M2) (*Countable additivity*): Let  $(A_k)_{k \in \mathbb{N}} \subseteq \mathcal{A}$  be pairwise disjoint. Since each  $\mu_n$  is a measure, we have

$$\mu_n \left( \bigsqcup_{k \in \mathbb{N}} A_k \right) = \sum_{k \in \mathbb{N}} \mu_n(A_k), \quad \text{for all } n \in \mathbb{N}.$$

Then,

$$\mu \left( \bigsqcup_{k \in \mathbb{N}} A_k \right) = \sum_{n \in \mathbb{N}} x_n \mu_n \left( \bigsqcup_{k \in \mathbb{N}} A_k \right) = \sum_{n \in \mathbb{N}} x_n \sum_{k \in \mathbb{N}} \mu_n(A_k).$$

Since all terms are non-negative, we may exchange the order of summation:

$$\sum_{n \in \mathbb{N}} x_n \sum_{k \in \mathbb{N}} \mu_n(A_k) = \sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} x_n \mu_n(A_k) = \sum_{k \in \mathbb{N}} \mu(A_k).$$

Therefore,  $\mu$  is countably additive. □

**Example 2.5** (Restriction of a measure). Let  $(X, \mathcal{A}, \mu)$  be a measure space and let  $A \in \mathcal{A}$ . Define the set function  $\mu_A : \mathcal{A} \rightarrow [0, \infty]$  by

$$\mu_A(B) := \mu(A \cap B), \quad \text{for all } B \in \mathcal{A}.$$

Then  $\mu_A$  is a measure on  $(X, \mathcal{A})$ , called the *restriction of  $\mu$  to  $A$* .

*Proof.* We verify the two defining properties of a measure:

(M1):  $\mu_A(\emptyset) = \mu(A \cap \emptyset) = \mu(\emptyset) = 0$ .

(M2): Let  $(B_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$  be pairwise disjoint. Then  $(A \cap B_n)_{n \in \mathbb{N}}$  are also pairwise disjoint, and

$$\mu_A\left(\bigsqcup_{n \in \mathbb{N}} B_n\right) = \mu\left(A \cap \bigsqcup_{n \in \mathbb{N}} B_n\right) = \mu\left(\bigsqcup_{n \in \mathbb{N}} (A \cap B_n)\right) = \sum_{n \in \mathbb{N}} \mu(A \cap B_n) = \sum_{n \in \mathbb{N}} \mu_A(B_n).$$

Hence,  $\mu_A$  is a measure. □

**Definition 2.5** (Lebesgue measure on  $\mathbb{R}^n$ ). Define the set function  $\lambda_n$  on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$  by

$$\lambda_n([a, b]) := \prod_{i=1}^n (b_i - a_i),$$

for all  $[a, b] := [a_1, b_1] \times \cdots \times [a_n, b_n] \in \mathcal{J}_n$ . This is called the  $n$ -dimensional Lebesgue measure.

**Remark 2.3.** The set function  $\lambda_n$  is defined only on the family  $\mathcal{J}_n$  of half-open rectangles and hence is not yet a measure. Extending  $\lambda_n$  to a measure on  $\mathcal{B}(\mathbb{R}^n)$  requires the Carathéodory extension theorem, which will be developed later.

**Lemma 2.3.** Let  $(X, \mathcal{A})$  be a measure space, and let  $\mu : \mathcal{A} \rightarrow [0, \infty]$  be an additive set function with  $\mu(\emptyset) = 0$ . Then  $\mu$  is a measure if and only if it is **continuous from below**, i.e., for every increasing sequence  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$  with  $A_n \uparrow A$ , we have

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n) = \sup_{n \in \mathbb{N}} \mu(A_n).$$

*Proof.* Any measure  $\mu$  is continuous from below.

Conversely, suppose  $\mu$  is finitely additive,  $\mu(\emptyset) = 0$ , and  $\mu$  is continuous from below. Let  $(B_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$  be disjoint, and define  $A_n := \bigcup_{i=1}^n B_i$ . Then  $(A_n)$  is increasing with  $\bigcup_{n=1}^{\infty} A_n = \bigsqcup_{n=1}^{\infty} B_n$ . By finite additivity,

$$\mu(A_n) = \sum_{i=1}^n \mu(B_i),$$

and by continuity from below,

$$\mu\left(\bigsqcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu(B_n).$$

Hence  $\mu$  is countably additive, i.e., a measure. □



**Lemma 2.4.** Let  $(X, \mathcal{A})$  be a measurable space and  $\mu : \mathcal{A} \rightarrow [0, \infty)$  an additive set function with  $\mu(\emptyset) = 0$  and  $\mu(A) < \infty$  for all  $A \in \mathcal{A}$ . Then  $\mu$  is a measure if and only if it satisfies one of the following continuity properties:

- (i)  $\mu$  is continuous from below;
- (ii)  $\mu$  is continuous from above;
- (iii)  $\mu$  is continuous at  $\emptyset$ , i.e., for every decreasing sequence  $(B_n)_{n \in \mathbb{N}}$  in  $\mathcal{A}$  with  $\bigcap_{n=1}^{\infty} B_n = \emptyset$ , we have

$$\lim_{n \rightarrow \infty} \mu(B_n) = 0.$$

*Proof.* Clearly, every measure satisfies properties (i)–(iii), so we only need to show that (iii) implies countable additivity.

Assume  $\mu$  is additive,  $\mu(\emptyset) = 0$ , and satisfies continuity at  $\emptyset$ . Let  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$  be pairwise disjoint and define  $A := \bigsqcup_{n \in \mathbb{N}} A_n$ . For each  $n$ , let

$$B_n := A \setminus \bigcup_{i=1}^n A_i.$$

Then  $(B_n)$  is a decreasing sequence in  $\mathcal{A}$  with  $\bigcap_{n \in \mathbb{N}} B_n = \emptyset$ , so by continuity at  $\emptyset$ , we have  $\mu(B_n) \rightarrow 0$ .

Using additivity, we compute

$$\mu(A) = \mu\left(B_n \sqcup \bigcup_{i=1}^n A_i\right) = \mu(B_n) + \sum_{i=1}^n \mu(A_i).$$

Taking the limit as  $n \rightarrow \infty$ , we get

$$\mu(A) = \lim_{n \rightarrow \infty} \left( \mu(B_n) + \sum_{i=1}^n \mu(A_i) \right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Thus,  $\mu$  is countably additive, hence a measure. □

### 3 Uniqueness of Measures

**Definition 3.1.** A *Dynkin system* (or  $\lambda$ -system)  $\mathcal{D} \subseteq \mathcal{P}(X)$  is a collection of subsets of  $X$  such that:

- $X \in \mathcal{D}$  (D1)

- If  $D \in \mathcal{D}$ , then  $D^c \in \mathcal{D}$  (D2)

- If  $(D_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}$  are pairwise disjoint, then  $\bigsqcup_{n \in \mathbb{N}} D_n \in \mathcal{D}$  (D3)

**Remark 3.1.** As with  $\sigma$ -algebras one easily checks that  $\emptyset \in \mathcal{D}$  and that finite disjoint unions are in  $\mathcal{D}$ : if  $D, E \in \mathcal{D}$  with  $D \cap E = \emptyset$ , then  $D \sqcup E \in \mathcal{D}$ . Every  $\sigma$ -algebra is a Dynkin system, but the converse is not true in general.

**Lemma 3.1.** Let  $\mathcal{E} \subseteq \mathcal{P}(X)$ . Then there exists a smallest Dynkin system  $\mathcal{D}(\mathcal{E})$  containing  $\mathcal{E}$ , called the *Dynkin system generated by  $\mathcal{E}$* . Moreover,

$$\mathcal{E} \subseteq \mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E}),$$

where  $\sigma(\mathcal{E})$  denotes the  $\sigma$ -algebra generated by  $\mathcal{E}$ .

*Proof.* The proof is analogous to that of Theorem 1.1 for  $\sigma$ -algebras. Let  $\mathcal{F}$  be the family of all Dynkin systems on  $X$  that contain  $\mathcal{E}$ . Then  $\mathcal{F}$  is nonempty, since  $\mathcal{P}(X)$  is a Dynkin system containing  $\mathcal{E}$ . Define

$$\mathcal{D}(\mathcal{E}) := \bigcap_{\mathcal{D} \in \mathcal{F}} \mathcal{D}.$$

Then  $\mathcal{D}(\mathcal{E})$  is a Dynkin system, being the intersection of Dynkin systems (which are closed under complements, disjoint unions, and contain  $X$ ). Moreover, it is the smallest such system containing  $\mathcal{E}$  by construction. Since every  $\sigma$ -algebra is in particular a Dynkin system, we also have

$$\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E}).$$

□

**Lemma 3.2.** A Dynkin system  $\mathcal{D}$  is a  $\sigma$ -algebra if and only if it is closed under finite intersections; that is,

$$D, E \in \mathcal{D} \quad \Rightarrow \quad D \cap E \in \mathcal{D}.$$

*Proof.* The “only if” direction follows immediately from Remark 3.1 and the fact that every  $\sigma$ -algebra is closed under finite intersections.

For the converse, assume  $\mathcal{D}$  is a Dynkin system closed under finite intersections. Let  $(D_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}$ , and define

$$E_1 := D_1 \in \mathcal{D}, \quad E_{n+1} := D_{n+1} \setminus \bigcup_{k=1}^n D_k = D_{n+1} \cap \bigcap_{k=1}^n D_k^c.$$

Each  $E_n \in \mathcal{D}$  by the Dynkin properties and the assumed stability under finite intersections. The sets  $(E_n)$  are disjoint and satisfy

$$\bigcup_{n=1}^{\infty} D_n = \bigsqcup_{n=1}^{\infty} E_n \in \mathcal{D},$$

so  $\mathcal{D}$  is closed under countable unions. Hence,  $\mathcal{D}$  is a  $\sigma$ -algebra. □

While Lemma 3.2 characterizes when a Dynkin system is a  $\sigma$ -algebra, it is not directly applicable when the Dynkin system  $\mathcal{D}$  is defined via a generator  $\mathcal{E} \subseteq \mathcal{P}(X)$ , as is often the case in practice. The following theorem overcomes this limitation and plays a central role in many applications.

**Theorem 3.3** (Dynkin’s  $\pi$ - $\lambda$  Theorem). Let  $\mathcal{E} \subseteq \mathcal{P}(X)$  be a collection of sets that is closed under finite intersections. Then,

$$\mathcal{D}(\mathcal{E}) = \sigma(\mathcal{E}).$$

*Proof.* By Lemma 3.1, we have  $\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E})$ . To show equality, it suffices to prove that  $\mathcal{D}(\mathcal{E})$  is a  $\sigma$ -algebra. Since it contains  $\mathcal{E}$ , it would then contain  $\sigma(\mathcal{E})$  by minimality.

By Lemma 3.2, it is enough to show that  $\mathcal{D}(\mathcal{E})$  is closed under finite intersections.

Fix  $D \in \mathcal{D}(\mathcal{E})$ , and define

$$\mathcal{D}_D := \{A \subseteq X : A \cap D \in \mathcal{D}(\mathcal{E})\}.$$

We claim that  $\mathcal{D}_D$  is a Dynkin system:

(D1): Since  $D = X \cap D \in \mathcal{D}(\mathcal{E})$ , we have  $X \in \mathcal{D}_D$ .

(D2): If  $A \in \mathcal{D}_D$ , then

$$A^c \cap D = ((A \cap D) \sqcup D^c)^c \cap D \in \mathcal{D}(\mathcal{E}),$$

using that  $A \cap D \in \mathcal{D}(\mathcal{E})$ ,  $D^c \in \mathcal{D}(\mathcal{E})$ , and that Dynkin systems are closed under disjoint unions and complements.

(D3): Let  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}_D$  be disjoint. Then the sets  $A_n \cap D \in \mathcal{D}(\mathcal{E})$  are disjoint, and

$$\left( \bigsqcup_{n=1}^{\infty} A_n \right) \cap D = \bigsqcup_{n=1}^{\infty} (A_n \cap D) \in \mathcal{D}(\mathcal{E}).$$

Thus,  $\mathcal{D}_D$  is a Dynkin system. Since  $\mathcal{E} \subseteq \mathcal{D}_G$  for all  $G \in \mathcal{E}$  by the assumed  $\cap$ -stability of  $\mathcal{E}$ , and each  $\mathcal{D}_G$  is a Dynkin system, it follows that

$$\mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}_G \quad \text{for all } G \in \mathcal{E}.$$

Hence, for all  $D \in \mathcal{D}(\mathcal{E})$  and  $G \in \mathcal{E}$ , we have  $D \cap G \in \mathcal{D}(\mathcal{E})$ , i.e.,  $\mathcal{D}(\mathcal{E})$  is closed under finite intersections.

By Lemma 3.2, we conclude that  $\mathcal{D}(\mathcal{E})$  is a  $\sigma$ -algebra. Since  $\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E})$  and both are  $\sigma$ -algebras containing  $\mathcal{E}$ , we have

$$\mathcal{D}(\mathcal{E}) = \sigma(\mathcal{E}).$$

□

**Theorem 3.4** (Uniqueness of Measures). Let  $(X, \mathcal{A})$  be a measurable space with  $\mathcal{A} = \sigma(\mathcal{E})$ , where  $\mathcal{E} \subseteq \mathcal{P}(X)$  satisfies:

- $\mathcal{E}$  is closed under finite intersections;
- there exists an increasing sequence  $(E_n)_{n \in \mathbb{N}} \subseteq \mathcal{E}$  with  $E_n \uparrow X$ .

Suppose  $\mu$  and  $\nu$  are measures on  $\mathcal{A}$  such that  $\mu(E) = \nu(E)$  for all  $E \in \mathcal{E}$ , and  $\mu(E_n) = \nu(E_n) < \infty$  for all  $n \in \mathbb{N}$ . Then  $\mu = \nu$  on  $\mathcal{A}$ ; that is,

$$\mu(A) = \nu(A) \quad \text{for all } A \in \mathcal{A}.$$

*Proof.* Fix  $n \in \mathbb{N}$ , and define

$$\mathcal{D}_n := \{A \in \mathcal{A} : \mu(E_n \cap A) = \nu(E_n \cap A)\}.$$

We claim that  $\mathcal{D}_n$  is a Dynkin system:

(D1): Since  $E_n \in \mathcal{E} \subseteq \mathcal{A}$ , and  $\mu(E_n) = \nu(E_n)$ , it follows that  $X \in \mathcal{D}_n$ .

(D2): If  $A \in \mathcal{D}_n$ , then

$$\mu(E_n \cap A^c) = \mu(E_n) - \mu(E_n \cap A) = \nu(E_n) - \nu(E_n \cap A) = \nu(E_n \cap A^c),$$

so  $A^c \in \mathcal{D}_n$ .

(D3): Let  $(A_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}_n$  be disjoint. Then:

$$\mu\left(E_n \cap \bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(E_n \cap A_k) = \sum_{k=1}^{\infty} \nu(E_n \cap A_k) = \nu\left(E_n \cap \bigsqcup_{k=1}^{\infty} A_k\right),$$

so  $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{D}_n$ .

Thus,  $\mathcal{D}_n$  is a Dynkin system. Since  $\mathcal{E} \subseteq \mathcal{D}_n$  (as  $\mu(E_n \cap E) = \nu(E_n \cap E)$  for all  $E \in \mathcal{E}$ , by the  $\cap$ -stability of  $\mathcal{E}$ ), and since  $\sigma(\mathcal{E}) = \mathcal{A}$ , Theorem 3.3 yields

$$\mathcal{A} = \sigma(\mathcal{E}) = \mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}_n.$$

Hence,

$$\mu(E_n \cap A) = \nu(E_n \cap A) \quad \text{for all } A \in \mathcal{A}, n \in \mathbb{N}.$$

Now fix  $A \in \mathcal{A}$ . Since  $E_n \uparrow X$ , we have  $E_n \cap A \uparrow A$ , and by continuity from below,

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(E_n \cap A) = \lim_{n \rightarrow \infty} \nu(E_n \cap A) = \nu(A).$$

Therefore,  $\mu = \nu$  on  $\mathcal{A}$ . □

**Theorem 3.5** (Translation Invariance and Uniqueness of Lebesgue Measure). Let  $\lambda^n$  denote the  $n$ -dimensional Lebesgue measure on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ . Then:

(i) **(Translation invariance)** For all  $x \in \mathbb{R}^n$  and all  $B \in \mathcal{B}(\mathbb{R}^n)$ , we have

$$\lambda^n(x + B) = \lambda^n(B),$$

where  $x + B := \{x + y : y \in B\}$  is the translation of  $B$  by  $x$ .

(ii) **(Uniqueness up to scalar)** Let  $\mu$  be a measure on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$  that is translation invariant and finite on the unit cube:

$$\mu(x + B) = \mu(B) \quad \text{for all } x \in \mathbb{R}^n, B \in \mathcal{B}(\mathbb{R}^n), \quad \text{and} \quad \mu([0, 1]^n) < \infty.$$

Then  $\mu$  is a scalar multiple of Lebesgue measure:

$$\mu = \mu([0, 1]^n) \cdot \lambda^n.$$

## 4 Existence of Measures

**Definition 4.1** (Semi-ring). Let  $X$  be a set. A family  $\mathcal{S} \subseteq \mathcal{P}(X)$  is called a *semi-ring* if:

- $\emptyset \in \mathcal{S}$  (S1)

- For all  $S, T \in \mathcal{S}$ , we have  $S \cap T \in \mathcal{S}$  (S2)

- For all  $S, T \in \mathcal{S}$ , there exist disjoint sets  $S_1, \dots, S_M \in \mathcal{S}$  such that

$$S \setminus T = \bigsqcup_{i=1}^M S_i \quad (\text{S3})$$

**Theorem 4.1** (Carathéodory Extension Theorem). Let  $\mathcal{S} \subseteq \mathcal{P}(X)$  be a semi-ring and let  $\mu : \mathcal{S} \rightarrow [0, \infty]$  be a pre-measure, i.e.,

- $\mu(\emptyset) = 0$ ,
- For every sequence  $(S_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}$  of disjoint sets with  $\bigsqcup_{n \in \mathbb{N}} S_n \in \mathcal{S}$ , we have

$$\mu \left( \bigsqcup_{n \in \mathbb{N}} S_n \right) = \sum_{n \in \mathbb{N}} \mu(S_n).$$

Then  $\mu$  has an extension to a measure  $\bar{\mu}$  on  $\sigma(\mathcal{S})$ .

Moreover, if  $\mathcal{S}$  contains an increasing sequence  $(S_n)_{n \in \mathbb{N}}$  with  $S_n \uparrow X$  and  $\mu(S_n) < \infty$  for all  $n$ , then the extension is unique.

*Idea of the proof.* The fundamental problem is how to extend the pre-measure  $\mu$ . The following auxiliary set function  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  will play a central role. For any  $A \subseteq X$ , define the family of countable  $\mathcal{S}$ -coverings

$$\mathcal{C}(A) := \left\{ (S_n)_{n \in \mathbb{N}} \subseteq \mathcal{S} : A \subseteq \bigcup_{n \in \mathbb{N}} S_n \right\},$$

and the set function

$$\mu^*(A) := \inf \left\{ \sum_{n \in \mathbb{N}} \mu(S_n) : (S_n)_{n \in \mathbb{N}} \in \mathcal{C}(A) \right\}.$$

If  $A$  cannot be covered by sets from  $\mathcal{S}$ , we define  $\mathcal{C}(A) = \emptyset$  and hence  $\mu^*(A) := \inf \emptyset = \infty$ .

The proof proceeds in four main steps:

1. **(Outer measure)** Show that  $\mu^*$  is an outer measure, i.e., it satisfies:

$$(\text{OM1}) \quad \mu^*(\emptyset) = 0,$$

$$(\text{OM2}) \quad A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B),$$

$$(\text{OM3}) \quad \mu^* \left( \bigcup_{n \in \mathbb{N}} A_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(A_n).$$

2. **(Extension)** Show that  $\mu^*$  extends  $\mu$ , i.e.,  $\mu^*(S) = \mu(S)$  for all  $S \in \mathcal{S}$ .

3. **( $\mu^*$ -measurable sets)** Define the collection of  $\mu^*$ -measurable sets by

$$\mathcal{A}_{\mu^*} := \{A \subseteq X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \setminus A) \text{ for all } Q \subseteq X\}.$$

Then  $\mathcal{A}_{\mu^*}$  is a  $\sigma$ -algebra with  $\mathcal{S} \subseteq \mathcal{A}_{\mu^*}$  and  $\sigma(\mathcal{S}) \subseteq \mathcal{A}_{\mu^*}$ .

4. **(Measure on  $\sigma$ -algebra)** The restriction of  $\mu^*$  to  $\mathcal{A}_{\mu^*}$  is a measure. In particular,  $\mu^*|_{\sigma(\mathcal{S})}$  is a measure extending  $\mu$ .

If  $\mathcal{S}$  contains an increasing sequence  $(S_n)_{n \in \mathbb{N}}$  with  $S_n \uparrow X$  and  $\mu(S_n) < \infty$  for all  $n$ , then the extension is unique.  $\square$

### Existence of Lebesgue Measure on $\mathbb{R}$

**Lemma 4.2.** Let  $\mathcal{J}_1 := \{[a, b) \subseteq \mathbb{R} : a < b\}$  be the family of half-open intervals. Define the set function

$$\lambda_1([a, b)) := b - a \text{ for all } [a, b) \in \mathcal{J}_1.$$

Then  $\lambda_1 : \mathcal{J}_1 \rightarrow [0, \infty)$  is a pre-measure.

*Proof.* Let  $[a, b) \in \mathcal{J}_1$ , and suppose it can be written as a disjoint union of intervals:

$$[a, b) = \bigsqcup_{n \in \mathbb{N}} I_n, \text{ with } I_n \in \mathcal{J}_1 \text{ for all } n.$$

Our goal is to show that

$$\lambda_1([a, b)) = \sum_{n=1}^{\infty} \lambda_1(I_n).$$

Fix  $\varepsilon > 0$ . For each  $n \in \mathbb{N}$ , choose a closed interval  $I_n^{(\varepsilon)}$  such that

$$I_n \subseteq I_n^{(\varepsilon)} \text{ and } \lambda_1(I_n^{(\varepsilon)}) \leq \lambda_1(I_n) + \frac{\varepsilon}{2^n}.$$

These intervals slightly extend each  $I_n$ , allowing us to approximate the union  $\bigsqcup I_n$  from above.

Since the  $I_n$  cover  $[a, b)$  disjointly, the union of the extended intervals will eventually cover most of  $[a, b)$ . More precisely, for sufficiently large  $N$ , we have

$$[a, b - \varepsilon) \subseteq \bigcup_{n=1}^N I_n^{(\varepsilon)}.$$

Now we estimate the difference:

$$\begin{aligned} \lambda_1([a, b)) - \sum_{n=1}^N \lambda_1(I_n) &= (\lambda_1([a, b)) - \lambda_1([a, b - \varepsilon))) \\ &\quad + \left( \lambda_1([a, b - \varepsilon)) - \sum_{n=1}^N \lambda_1(I_n^{(\varepsilon)}) \right) \\ &\quad + \sum_{n=1}^N (\lambda_1(I_n^{(\varepsilon)}) - \lambda_1(I_n)) \\ &\leq \varepsilon + 0 + \sum_{n=1}^N \frac{\varepsilon}{2^n} \leq 2\varepsilon. \end{aligned}$$

On the other hand, since  $\bigsqcup_{n=1}^N I_n \subseteq [a, b)$  and the intervals  $I_n$  are disjoint, finite additivity and monotonicity of  $\lambda_1$  imply:

$$\sum_{n=1}^N \lambda_1(I_n) = \lambda_1\left(\bigsqcup_{n=1}^N I_n\right) \leq \lambda_1([a, b)).$$

Therefore,

$$0 \leq \lambda_1([a, b)) - \sum_{n=1}^N \lambda_1(I_n),$$

which justifies the lower bound in the previous inequality.

Combining both sides, we have

$$0 \leq \lambda_1([a, b)) - \sum_{n=1}^N \lambda_1(I_n) \leq 2\varepsilon.$$

Letting  $N \rightarrow \infty$  and then  $\varepsilon \rightarrow 0$ , we conclude:

$$\lambda_1([a, b)) = \sum_{n=1}^{\infty} \lambda_1(I_n).$$

Thus,  $\lambda_1$  is countably additive on  $\mathcal{J}_1$ , and hence a pre-measure.  $\square$

**Lemma 4.3** (Lebesgue measure on  $\mathbb{R}$ ). The set function  $\lambda_1$ , defined on  $\mathcal{J}_1$  by  $\lambda_1([a, b)) = b - a$  for  $a < b$ , extends to a measure on  $\mathcal{B}(\mathbb{R})$ . This extension is the unique measure  $\mu$  on  $\mathcal{B}(\mathbb{R})$  such that

$$\mu([a, b)) = b - a \quad \text{for all } a < b.$$

*Proof.* We have already shown that  $\lambda_1$  is a pre-measure on  $\mathcal{J}_1$ . By Theorem 1.3,  $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{J}_1)$ , i.e., the Borel  $\sigma$ -algebra is generated by  $\mathcal{J}_1$ .

Consider the sequence of half-open intervals  $[-k, k) \subseteq \mathbb{R}$  for  $k \in \mathbb{N}$ . This forms an increasing sequence for  $\mathbb{R}$ , and we have

$$\lambda_1([-k, k)) = 2k < \infty \quad \text{for all } k \in \mathbb{N}.$$

Thus, all the conditions of Theorem 4.1 (Carathéodory's extension theorem) are satisfied. It follows that  $\lambda_1$  extends uniquely to a measure on  $\mathcal{B}(\mathbb{R})$ , yielding the one-dimensional Lebesgue measure on  $\mathbb{R}$ .  $\square$

### Existence of Lebesgue Measure on $\mathbb{R}^n$

**Lemma 4.4.** Let  $\mathcal{J}_n$  denote the collection of half-open rectangles in  $\mathbb{R}^n$  of the form

$$\llbracket a, b \rrbracket = \prod_{i=1}^n [a_i, b_i), \quad \text{where } a = (a_1, \dots, a_n), \quad b = (b_1, \dots, b_n), \quad a_i < b_i.$$

Then  $\mathcal{J}_n$  is a semi-ring.

*Proof.* We prove the statement by induction on  $n$ . Assume  $\mathcal{J}_n \subset \mathbb{R}^n$  is a semi-ring. Define

$$\mathcal{J}_{n+1} := \mathcal{J}_n \times \mathcal{J}_1,$$

i.e., the collection of rectangles of the form  $R = R_n \times R_1$ , where  $R_n \in \mathcal{J}_n$  and  $R_1 \in \mathcal{J}_1$ .

We verify the properties of a semi-ring:

(S1) *Closure under the empty set:* Since  $\emptyset \in \mathcal{J}_n$  and  $\mathcal{J}_1$ , we have

$$\emptyset = \emptyset \times [a, b] \in \mathcal{J}_{n+1}.$$

(S2) *Closure under intersection:* Let  $R = R_n \times R_1$  and  $S = S_n \times S_1$  be in  $\mathcal{J}_{n+1}$ . Then

$$R \cap S = (R_n \cap S_n) \times (R_1 \cap S_1),$$

which belongs to  $\mathcal{J}_{n+1}$ , since both  $R_n \cap S_n \in \mathcal{J}_n$  and  $R_1 \cap S_1 \in \mathcal{J}_1$ , by the inductive hypothesis.

(S3) *Closure under set difference (finite disjoint union):* Consider

$$R \setminus S = (R_n \times R_1) \setminus (S_n \times S_1).$$

This set can be decomposed as

$$(R_n \setminus S_n) \times (R_1 \setminus S_1) \sqcup (R_n \cap S_n) \times (R_1 \setminus S_1) \sqcup (R_n \setminus S_n) \times (R_1 \cap S_1).$$

Each of the components  $R_n \setminus S_n$ ,  $R_n \cap S_n$ ,  $R_1 \setminus S_1$ , and  $R_1 \cap S_1$  can be written as finite disjoint unions of sets in  $\mathcal{J}_n$  and  $\mathcal{J}_1$ , respectively. Therefore, their Cartesian products yield finite disjoint unions of elements in  $\mathcal{J}_{n+1}$ .

Hence,  $\mathcal{J}_{n+1}$  is a semi-ring. By induction, it follows that  $\mathcal{J}_n$  is a semi-ring for all  $n \in \mathbb{N}$ .  $\square$

**Lemma 4.5.** The function  $\lambda_n: \mathcal{J}_n \rightarrow [0, \infty)$ , defined by

$$\lambda_n([a_1, b_1] \times \cdots \times [a_n, b_n]) = \prod_{i=1}^n (b_i - a_i),$$

is a pre-measure on the semi-ring  $\mathcal{J}_n$ .

**Corollary 4.5.1** (Lebesgue measure on  $\mathbb{R}^n$ ). The set function  $\lambda_n$  extends to a measure on the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}^n)$ , called the *Lebesgue measure*. It is the unique measure satisfying

$$\lambda_n([a_1, b_1] \times \cdots \times [a_n, b_n]) = \prod_{i=1}^n (b_i - a_i), \quad \text{for all } a_i < b_i.$$

**Remark 4.1** (Relation to Elementary Volume). The uniqueness of Lebesgue measure and its properties imply that Lebesgue measure coincides with the familiar volume function  $\text{vol}^{(n)}(\cdot)$  from elementary geometry. More precisely,  $\text{vol}^{(n)}$  can be extended to a measure on the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}^n)$  in only one way namely, as the Lebesgue measure  $\lambda^n$ .



## 5 Measurable Mappings

**Definition 5.1** (Measurable Map). Let  $(X, \mathcal{A})$ ,  $(X', \mathcal{A}')$  be measurable spaces. A map  $T : X \rightarrow X'$  is called  $\mathcal{A}/\mathcal{A}'$ -measurable (or simply measurable) if the pre-image of every measurable set is measurable:

$$T^{-1}(A') \in \mathcal{A} \quad \text{for all } A' \in \mathcal{A}'.$$

**Remark 5.1.**

- Probabilists often refer to a measurable map defined on a probability space as a *random variable*.
- The symbolic notation  $T^{-1}(\mathcal{A}') := \{T^{-1}(A') : A' \in \mathcal{A}'\}$  is often used. We also write  $T^{-1}(\mathcal{A}') \subset \mathcal{A}$  as shorthand for measurability.
- It is common to write  $T : (X, \mathcal{A}) \rightarrow (X', \mathcal{A}')$  to indicate that  $T$  is measurable.
- A measurable map between  $\mathcal{B}(\mathbb{R}^n)$  and  $\mathcal{B}(\mathbb{R}^m)$  is often called a *Borel measurable map*.

**Example 5.1.** Let  $(X, \mathcal{A})$  be a measurable space and let  $A \in \mathcal{A}$ . We show that the indicator function

$$\mathbf{1}_A : X \rightarrow \{0, 1\}, \quad \mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$$

is  $\mathcal{A}/\mathcal{P}(\{0, 1\})$ -measurable.

We check that the preimage of each subset of  $\{0, 1\}$  lies in  $\mathcal{A}$ :

- $\mathbf{1}_A^{-1}(\emptyset) = \emptyset \in \mathcal{A}$ ,
- $\mathbf{1}_A^{-1}(\{0\}) = A^c \in \mathcal{A}$ ,
- $\mathbf{1}_A^{-1}(\{1\}) = A \in \mathcal{A}$ ,
- $\mathbf{1}_A^{-1}(\{0, 1\}) = X \in \mathcal{A}$ .

Therefore,  $\mathbf{1}_A$  is measurable.

**Lemma 5.1.** Let  $(X, \mathcal{A})$ ,  $(X', \mathcal{A}')$  be measurable spaces, and suppose  $\mathcal{A}' = \sigma(\mathcal{E}')$ . Then a map  $T : X \rightarrow X'$  is  $\mathcal{A}/\mathcal{A}'$ -measurable if and only if  $T^{-1}(\mathcal{E}') \subseteq \mathcal{A}$ , i.e. if

$$T^{-1}(E') \in \mathcal{A} \quad \text{for all } E' \in \mathcal{E}'.$$

*Proof.* If  $T$  is measurable, then by definition  $T^{-1}(A') \in \mathcal{A}$  for all  $A' \in \mathcal{A}'$ . Since  $\mathcal{E}' \subset \mathcal{A}'$ , it follows immediately that  $T^{-1}(E') \in \mathcal{A}$  for all  $E' \in \mathcal{E}'$ .

Conversely, suppose  $T^{-1}(E') \in \mathcal{A}$  for every  $E' \in \mathcal{E}'$ . Define

$$\mathcal{D}' := \{A' \subseteq X' : T^{-1}(A') \in \mathcal{A}\}.$$

By assumption,  $\mathcal{E}' \subseteq \mathcal{D}'$ . We now show that  $\mathcal{D}'$  is a  $\sigma$ -algebra:

- Since  $T^{-1}(X') = X \in \mathcal{A}$ , we have  $X' \in \mathcal{D}'$ .
- If  $A' \in \mathcal{D}'$ , then  $T^{-1}(A'^c) = T^{-1}(A')^c \in \mathcal{A}$ , so  $A'^c \in \mathcal{D}'$ .
- If  $A'_1, A'_2, \dots \in \mathcal{D}'$ , then

$$T^{-1}\left(\bigcup_{i=1}^{\infty} A'_i\right) = \bigcup_{i=1}^{\infty} T^{-1}(A'_i) \in \mathcal{A},$$

hence  $\bigcup_{i=1}^{\infty} A'_i \in \mathcal{D}'$ .

Thus,  $\mathcal{D}'$  is a  $\sigma$ -algebra containing  $\mathcal{E}'$ , so it contains  $\sigma(\mathcal{E}') = \mathcal{A}'$ . Therefore,  $T^{-1}(A') \in \mathcal{A}$  for all  $A' \in \mathcal{A}'$ , i.e.,  $T$  is measurable.  $\square$

**Example 5.2.** Let  $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a continuous function. Then  $T$  is  $\mathcal{B}(\mathbb{R}^m)/\mathcal{B}(\mathbb{R}^n)$ -measurable.

Indeed, from elementary analysis, we know that  $T$  is continuous if and only if

$$T^{-1}(A') \subset \mathbb{R}^m \text{ is open for every open set } A' \subset \mathbb{R}^n.$$

Since the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}^n)$  is generated by the open sets  $\mathcal{O}_{\mathbb{R}^n}$ , it follows that

$$T^{-1}(\mathcal{O}_{\mathbb{R}^n}) \subset \mathcal{O}_{\mathbb{R}^m} \subset \sigma(\mathcal{O}_{\mathbb{R}^m}) = \mathcal{B}(\mathbb{R}^m).$$

Hence, by Lemma 5.1,  $T$  is  $\mathcal{B}(\mathbb{R}^m)/\mathcal{B}(\mathbb{R}^n)$ -measurable.

**Theorem 5.2.** Let  $(X_i, \mathcal{A}_i)$ ,  $i = 1, 2, 3$ , be measurable spaces, and let

$$T : X_1 \rightarrow X_2, \quad S : X_2 \rightarrow X_3$$

be  $\mathcal{A}_1/\mathcal{A}_2$ - and  $\mathcal{A}_2/\mathcal{A}_3$ -measurable maps, respectively. Then the composition

$$S \circ T : X_1 \rightarrow X_3$$

is  $\mathcal{A}_1/\mathcal{A}_3$ -measurable.

*Proof.* Let  $A_3 \in \mathcal{A}_3$ . Then

$$(S \circ T)^{-1}(A_3) = T^{-1}\left(S^{-1}(A_3)\right).$$

Since  $S$  is  $\mathcal{A}_2/\mathcal{A}_3$ -measurable, we have  $S^{-1}(A_3) \in \mathcal{A}_2$ . Since  $T$  is  $\mathcal{A}_1/\mathcal{A}_2$ -measurable, it follows that  $T^{-1}(S^{-1}(A_3)) \in \mathcal{A}_1$ . Therefore,  $S \circ T$  is  $\mathcal{A}_1/\mathcal{A}_3$ -measurable.  $\square$

**Remark 5.2.** Given a map  $T : X \rightarrow X'$ , where  $X'$  carries a natural  $\sigma$ -algebra  $\mathcal{A}'$  (e.g.,  $\mathcal{B}(\mathbb{R})$ ), but no  $\sigma$ -algebra is specified on  $X$ , one may ask: is there a smallest  $\sigma$ -algebra on  $X$  that makes  $T$  measurable?

While  $\mathcal{P}(X)$  trivially works, it is too large to be useful. On the other hand,  $T^{-1}(\mathcal{A}')$  is a  $\sigma$ -algebra, and removing any set from it may break measurability. This leads to the following definition.

**Definition 5.2.** Let  $(T_i)_{i \in I}$ , with  $T_i : X \rightarrow X_i$ , be an arbitrary family of mappings from the same space  $X$  into measurable spaces  $(X_i, \mathcal{A}_i)$ . The smallest  $\sigma$ -algebra on  $X$  that makes all  $T_i$  simultaneously measurable is given by

$$\sigma(T_i : i \in I) := \sigma \left( \bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i) \right).$$

We say that  $\sigma(T_i : i \in I)$  is the  $\sigma$ -algebra generated by the family  $(T_i)_{i \in I}$ .

Although each  $T_i^{-1}(\mathcal{A}_i)$  is a  $\sigma$ -algebra, the union  $\bigcup_{i \in I} T_i^{-1}(\mathcal{A}_i)$  is, in general, not a  $\sigma$ -algebra if  $\#I > 1$ ; this is why we must take the  $\sigma$ -hull in the definition above.

**Theorem 5.3.** Let  $(X, \mathcal{A})$ ,  $(X', \mathcal{A}')$  be measurable spaces and let  $T : X \rightarrow X'$  be an  $\mathcal{A}/\mathcal{A}'$ -measurable map. For every measure  $\mu$  on  $(X, \mathcal{A})$ ,

$$\mu'(A') := \mu(T^{-1}(A')), \quad A' \in \mathcal{A}'$$

defines a measure  $\mu'$  on  $(X', \mathcal{A}')$ .

*Proof.* If  $A' = \emptyset$ , then  $T^{-1}(\emptyset) = \emptyset$  and  $\mu'(\emptyset) = \mu(\emptyset) = 0$ .

Let  $(A'_n)_{n \in \mathbb{N}} \subset \mathcal{A}'$  be a sequence of pairwise disjoint sets. Then

$$\mu' \left( \bigsqcup_{n \in \mathbb{N}} A'_n \right) = \mu \left( T^{-1} \left( \bigsqcup_{n \in \mathbb{N}} A'_n \right) \right) = \mu \left( \bigsqcup_{n \in \mathbb{N}} T^{-1}(A'_n) \right) = \sum_{n \in \mathbb{N}} \mu(T^{-1}(A'_n)) = \sum_{n \in \mathbb{N}} \mu'(A'_n).$$

Hence,  $\mu'$  is a measure on  $(X', \mathcal{A}')$ . □

**Definition 5.3.** The measure  $\mu'(\cdot)$  from Theorem 7.6 is called the *image measure* or *pushforward* of  $\mu$  under  $T$ . It is commonly denoted by one of the following:

- $T(\mu)(\cdot)$ ,
- $T_*\mu(\cdot)$ ,
- $\mu \circ T^{-1}(\cdot)$ .

**Example 5.3.** Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space and let  $\xi : \Omega \rightarrow \mathbb{R}$  be a random variable, i.e., an  $\mathcal{A}/\mathcal{B}(\mathbb{R})$ -measurable map. Then the pushforward measure

$$\xi(\mathbb{P})(A') = \mathbb{P}(\xi^{-1}(A')) = \mathbb{P}(\{\omega \in \Omega : \xi(\omega) \in A'\}) = \mathbb{P}(\xi \in A')$$

defines a probability measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , which is called the *law* or *distribution* of the random variable  $\xi$ .

**Example 5.4.** Suppose we model the experiment of rolling two fair six-sided dice. The underlying probability space is given by

$$\Omega := \{(i, k) : 1 \leq i, k \leq 6\}, \quad \mathcal{A} := \mathcal{P}(\Omega), \quad \mathbb{P}(\{(i, k)\}) := \frac{1}{36}.$$

Each outcome  $(i, k)$  represents the result of the first and second die, respectively. Define the map

$$\xi : \Omega \rightarrow \{2, 3, \dots, 12\}, \quad \xi(i, k) := i + k,$$

which assigns to each outcome the total number of points rolled. This function  $\xi$  is measurable and thus a random variable.

The pushforward measure  $\xi(\mathbb{P})$ , also called the *law* or *distribution* of  $\xi$ , gives the probabilities of the possible total number of points. For instance,

$$\xi(\mathbb{P})(7) = \mathbb{P}(\xi^{-1}(\{7\})) = \mathbb{P}(\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}) = \frac{6}{36} = \frac{1}{6}.$$

**Remark 5.3.** A matrix  $T \in \mathbb{R}^{n \times n}$  is called orthogonal if and only if

$$T^\top T = I,$$

i.e., the transpose of  $T$  is equal to its inverse.

Orthogonal matrices preserve lengths and angles. That is, for all  $x, y \in \mathbb{R}^n$ , we have

$$\begin{aligned} \langle x, y \rangle &= \langle Tx, Ty \rangle, \\ \|x\| &= \|Tx\|, \end{aligned}$$

where the standard Euclidean inner product and norm are defined by

$$\langle x, y \rangle := \sum_{i=1}^n x_i y_i, \quad \|x\|^2 := \langle x, x \rangle.$$

**Theorem 5.4.** Let  $T \in \mathbb{R}^{n \times n}$  be an orthogonal matrix. Then the Lebesgue measure  $\lambda^n$  is invariant under  $T$ , i.e.,

$$T(\lambda^n) = \lambda^n.$$

*Proof.* The matrix  $T \in \mathbb{R}^{n \times n}$  defines a linear map  $x \mapsto Tx$ , i.e.,

$$T(ax + by) = aTx + bTy \quad \text{for all } a, b \in \mathbb{R}, x, y \in \mathbb{R}^n.$$

From the orthogonality condition, it follows that  $T$  is an isometry:

$$\|Tx - Ty\| = \|T(x - y)\| = \|x - y\|.$$

Thus,  $T$  is continuous and therefore  $\mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$ -measurable by Example 5.2. Furthermore by Theorem 5.3, the pushforward measure

$$\nu(B) := \lambda^n(T^{-1}(B))$$

is well-defined on  $\mathcal{B}(\mathbb{R}^n)$ .

We now show that  $\nu$  is translation invariant. For any  $x \in \mathbb{R}^n$  and  $B \in \mathcal{B}(\mathbb{R}^n)$ , we compute

$$\begin{aligned} \nu(x + B) &= \lambda^n(T^{-1}(x + B)) \\ &= \lambda^n(T^{-1}x + T^{-1}B) \\ &= \lambda^n(T^{-1}B) = \nu(B), \end{aligned}$$

where we used linearity of  $T$  and translation invariance of Lebesgue measure (Theorem 3.5(i)).

Hence  $\nu$  is a translation-invariant measure on  $\mathbb{R}^n$ . By Theorem 3.5(ii), since  $\nu$  is also finite on bounded sets (e.g., the unit ball), it must be a scalar multiple of Lebesgue measure:

$$\nu = c\lambda^n \quad \text{for some } c > 0.$$

To determine the constant  $c$ , consider the unit ball  $B_1(0) := \{x \in \mathbb{R}^n : \|x\| < 1\}$ . Since  $T$  is orthogonal, we have

$$x \in B_1(0) \iff \|x\| < 1 \iff \|Tx\| < 1 \iff x \in T^{-1}(B_1(0)),$$

so  $T^{-1}(B_1(0)) = B_1(0)$ . Therefore,

$$\lambda^n(B_1(0)) = \lambda^n(T^{-1}(B_1(0))) = \nu(B_1(0)) = c\lambda^n(B_1(0)),$$

which implies  $c = 1$ , since  $0 < \lambda^n(B_1(0)) < \infty$ . Thus,  $\nu = \lambda^n$ , and the theorem follows.  $\square$

**Remark 5.4.** Theorem 5.4 is a special case of the following general change-of-variable formula for Lebesgue measure. Recall that a matrix  $S \in \mathbb{R}^{n \times n}$  is invertible if and only if  $\det S \neq 0$ .

**Theorem 5.5** (Change of Variables). Let  $S \in \mathbb{R}^{n \times n}$  be an invertible matrix. Then

$$S(\lambda^n) = |\det S^{-1}| \lambda^n = |\det S|^{-1} \lambda^n. \quad (7.7)$$

*Proof.* Since  $S$  is invertible, both  $S$  and  $S^{-1}$  are linear maps on  $\mathbb{R}^n$ , and hence continuous and measurable (by Example 5.2). Define a measure  $\nu$  on  $\mathbb{R}^n$  by

$$\nu(B) := \lambda^n(S^{-1}(B)), \quad B \in \mathcal{B}(\mathbb{R}^n).$$

For any  $x \in \mathbb{R}^n$ , we have

$$\nu(x + B) = \lambda^n(S^{-1}(x + B)) = \lambda^n(S^{-1}x + S^{-1}B) = \lambda^n(S^{-1}B) = \nu(B),$$

so  $\nu$  is translation invariant.

By Theorem 3.5(ii), any translation-invariant measure finite on the unit cube must be a scalar multiple of Lebesgue measure, so there exists a constant  $c > 0$  such that

$$\nu = c\lambda^n.$$

To determine  $c$ , we evaluate both sides on the unit cube  $[0, 1]^n$ , which satisfies  $\lambda^n([0, 1]^n) = 1$ :

$$\nu([0, 1]^n) = \lambda^n(S^{-1}([0, 1]^n)).$$

The set  $S^{-1}([0, 1]^n)$  is a parallelepiped spanned by the vectors  $S^{-1}e_i$ , where  $(e_i)_{i=1}^n$  is the standard basis of  $\mathbb{R}^n$ . Its volume is given by the absolute value of the determinant:

$$\text{vol}^{(n)}(S^{-1}([0, 1]^n)) = |\det S^{-1}| = |\det S|^{-1}.$$

By Remark 4.1, Lebesgue measure coincides with this volume on Borel sets, so

$$\nu([0, 1]^n) = \lambda^n(S^{-1}([0, 1]^n)) = |\det S|^{-1}.$$

Hence,

$$\nu = |\det S|^{-1} \lambda^n,$$

which completes the proof.  $\square$

**Definition 5.4.** A *motion* in  $\mathbb{R}^n$  is a linear transformation of the form

$$Mx = \tau_x \circ T(x),$$

where  $\tau_x(y) = y + x$  denotes translation by  $x \in \mathbb{R}^n$ , and  $T \in \mathbb{R}^{n \times n}$  is an orthogonal matrix (i.e.,  $T^\top T = \text{id}_n$ ). In particular, two sets are said to be *congruent* if one can be obtained from the other by a motion.

**Theorem 5.6** (Invariance under Motions). Lebesgue measure is invariant under motions: for any motion  $M$  in  $\mathbb{R}^n$ , we have

$$\lambda^n = M(\lambda^n).$$

In particular, congruent sets have the same Lebesgue measure.

*Proof.* By definition, any motion  $M$  can be written as  $M = \tau_x \circ T$ , where  $T$  is orthogonal (so  $|\det T| = 1$ ). By Theorem 5.4,

$$T(\lambda^n) = \lambda^n.$$

Translation invariance of Lebesgue measure (Theorem 3.5(i)) gives

$$\tau_x(\lambda^n) = \lambda^n.$$

Hence,

$$M(\lambda^n) = \tau_x(T(\lambda^n)) = \tau_x(\lambda^n) = \lambda^n. \quad \square$$

## 6 Measurable Functions

**Definition 6.1** (Measurable Function). Let  $(X, \mathcal{A})$  be a measurable space. A function  $u : X \rightarrow \mathbb{R}$  is called *measurable* if it is  $\mathcal{A}$ - $\mathcal{B}(\mathbb{R})$ -measurable; that is,

$$u^{-1}(B) \in \mathcal{A} \quad \text{for all } B \in \mathcal{B}(\mathbb{R}).$$

**Remark 6.1.** By Lemma 5.1, a function  $u : X \rightarrow \mathbb{R}$  is  $\mathcal{A}/\mathcal{B}(\mathbb{R})$ -measurable if and only if

$$u^{-1}(G) \in \mathcal{A} \quad \text{for all } G \in \mathcal{E},$$

where  $\mathcal{E}$  is a generator of  $\mathcal{B}(\mathbb{R})$ .

**Definition 6.2** (Level Set). Let  $(X, \mathcal{A})$  be a measurable space, and let  $u : X \rightarrow \mathbb{R}$  be a function. For any  $y \in \mathbb{R}$ , the *level set* of  $u$  at the value  $y$  is defined as

$$\{u = y\} := \{x \in X : u(x) = y\}.$$

More generally, we define:

- Strict upper level set:  $\{u > y\} := \{x \in X : u(x) > y\}$
- Strict lower level set:  $\{u < y\} := \{x \in X : u(x) < y\}$
- Upper level set:  $\{u \geq y\} := \{x \in X : u(x) \geq y\}$
- Lower level set:  $\{u \leq y\} := \{x \in X : u(x) \leq y\}$

**Remark 6.2.** As noted in Remark 1.4, the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  is generated by intervals of the form  $[a, \infty)$ ,  $(a, \infty)$ ,  $(-\infty, a)$ , or  $(-\infty, a]$ , with  $a \in \mathbb{R}$  (or  $\mathbb{Q}$ ). To verify that a function  $u : X \rightarrow \mathbb{R}$  is measurable, it suffices to check that

$$u^{-1}([a, \infty)) = \{x \in X : u(x) \in [a, \infty)\} = \{x \in X : u(x) \geq a\} \in \mathcal{A}$$

for all such  $a$ , and likewise for the other interval types.

We write

$$\{u > v\} := \{x \in X : u(x) > v(x)\},$$

and similarly  $\{u < v\}$ ,  $\{u \leq v\}$ ,  $\{u = v\}$ ,  $\{u \neq v\}$ ,  $\{u \in B\}$ , etc., for measurable  $u, v : X \rightarrow \mathbb{R}$  and Borel sets  $B \subseteq \mathbb{R}$ .

**Lemma 6.1.** Let  $(X, \mathcal{A})$  be a measurable space. A function  $u : X \rightarrow \mathbb{R}$  is  $\mathcal{A}/\mathcal{B}(\mathbb{R})$ -measurable if and only if any one (and hence all) of the following equivalent conditions hold:

- (i)  $\{u > a\} \in \mathcal{A} \quad \forall a \in \mathbb{R} \text{ or } a \in \mathbb{Q}$
- (ii)  $\{u \geq a\} \in \mathcal{A} \quad \forall a \in \mathbb{R} \text{ or } a \in \mathbb{Q}$
- (iii)  $\{u < a\} \in \mathcal{A} \quad \forall a \in \mathbb{R} \text{ or } a \in \mathbb{Q}$
- (iv)  $\{u \leq a\} \in \mathcal{A} \quad \forall a \in \mathbb{R} \text{ or } a \in \mathbb{Q}$

**Remark 6.3.** It is often helpful to use the values  $+\infty$  and  $-\infty$  in calculations. To do this properly, we consider the extended real line  $\overline{\mathbb{R}} := [-\infty, +\infty]$ . If we agree that  $-\infty < x$  and  $y < +\infty$  for all  $x, y \in \mathbb{R}$ , then  $\overline{\mathbb{R}}$  inherits the usual ordering from  $\mathbb{R}$ , as well as the standard rules of addition and multiplication for real numbers. The latter, however, must be augmented as shown below.

Addition in $\overline{\mathbb{R}}$					Multiplication in $\overline{\mathbb{R}}$				
$+$	$0$	$x$	$+\infty$	$-\infty$	$\cdot$	$0$	$a$	$+\infty$	$-\infty$
$0$	$0$	$x$	$+\infty$	$-\infty$	$0$	$0$	$0$	$0$	$0$
$y$	$y$	$x + y$	$+\infty$	$-\infty$	$b$	$0$	$ab$	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	$+\infty$	undef.	$+\infty$	$0$	$+\infty$	$+\infty$	$-\infty$
$-\infty$	$-\infty$	$-\infty$	undef.	$-\infty$	$-\infty$	$0$	$-\infty$	$-\infty$	$+\infty$

**Remark 6.4. Caution:** The extended real line  $\overline{\mathbb{R}} = [-\infty, +\infty]$  is *not* a field. Expressions such as  $\infty - \infty$  or  $\frac{\infty}{\infty}$  are undefined and must be avoided.

The Borel  $\sigma$ -algebra on  $\overline{\mathbb{R}}$ , denoted  $\mathcal{B}(\overline{\mathbb{R}})$ , is defined by

$$B^* \in \mathcal{B}(\overline{\mathbb{R}}) \iff B^* = B \cup S,$$

for some  $B \in \mathcal{B}(\mathbb{R})$  and  $S \in \{\emptyset, \{-\infty\}, \{+\infty\}, \{-\infty, +\infty\}\}$ .

It is straightforward to verify that  $\mathcal{B}(\overline{\mathbb{R}})$  is a  $\sigma$ -algebra, and its trace on  $\mathbb{R}$  coincides with the usual Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$ .

**Lemma 6.2.** The Borel  $\sigma$ -algebra on the extended real line  $\overline{\mathbb{R}} = [-\infty, +\infty]$  satisfies

$$\mathcal{B}(\mathbb{R}) = \mathbb{R} \cap \mathcal{B}(\overline{\mathbb{R}}) \quad \text{or equivalently,} \quad \mathcal{B}(\mathbb{R}) = \{A \cap \mathbb{R} : A \in \mathcal{B}(\overline{\mathbb{R}})\}.$$

**Lemma 6.3.** The Borel  $\sigma$ -algebra  $\mathcal{B}(\overline{\mathbb{R}})$  is generated by any one of the following families of sets:

$$[a, +\infty], \quad (a, +\infty], \quad [-\infty, a), \quad \text{or} \quad [-\infty, a],$$

with  $a \in \mathbb{R}$  or  $\mathbb{Q}$ .

*Proof.* Let  $\mathcal{E} := \sigma(\{[a, +\infty] : a \in \mathbb{R}\})$ . Since

$$[a, +\infty] = [a, +\infty) \cup \{+\infty\} \quad \text{with} \quad [a, +\infty) \in \mathcal{B}(\mathbb{R}),$$

we see that  $[a, +\infty] \in \mathcal{B}(\overline{\mathbb{R}})$ , hence  $\mathcal{E} \subseteq \mathcal{B}(\overline{\mathbb{R}})$ .

Conversely, for  $-\infty < a \leq b < +\infty$ ,

$$[a, b) = [a, +\infty] \setminus [b, +\infty] \in \mathcal{E},$$

so  $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{E}$ . Moreover,

$$\{+\infty\} = \bigcap_{j \in \mathbb{N}} [j, +\infty], \quad \{-\infty\} = \bigcap_{j \in \mathbb{N}} [-\infty, -j) = \bigcap_{j \in \mathbb{N}} [-j, +\infty]^c,$$

so  $\{-\infty\}, \{+\infty\} \in \mathcal{E}$ . Hence for any  $B \in \mathcal{B}(\mathbb{R})$ ,

$$B, \quad B \cup \{+\infty\}, \quad B \cup \{-\infty\}, \quad B \cup \{-\infty, +\infty\} \in \mathcal{E},$$

implying  $\mathcal{B}(\overline{\mathbb{R}}) \subseteq \mathcal{E}$ . Therefore,  $\mathcal{B}(\overline{\mathbb{R}}) = \mathcal{E}$ .

The same argument applies if the generating system uses  $a \in \mathbb{Q}$ , or other families like  $(a, +\infty]$ ,  $[-\infty, a)$ , or  $[-\infty, a]$ .  $\square$

**Definition 6.3.** Let  $(X, \mathcal{A})$  be a measurable space. We define

$$\mathcal{M} := \mathcal{M}(\mathcal{A}) \quad \text{and} \quad \mathcal{M}_{\overline{\mathbb{R}}} := \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$$

as the collections of real-valued and extended real-valued measurable functions, respectively:

$$\begin{aligned} \mathcal{M} &= \{u : X \rightarrow \mathbb{R} \mid u \text{ is } \mathcal{A}/\mathcal{B}(\mathbb{R})\text{-measurable}\}, \\ \mathcal{M}_{\overline{\mathbb{R}}} &= \{u : X \rightarrow \overline{\mathbb{R}} \mid u \text{ is } \mathcal{A}/\mathcal{B}(\overline{\mathbb{R}})\text{-measurable}\}. \end{aligned}$$



**Example 6.1.** Let  $(X, \mathcal{A})$  be a measurable space. The indicator function  $f(x) := \mathbf{1}_A(x)$  is measurable if and only if  $A \in \mathcal{A}$ .

*Proof.* Recall that a function  $f : X \rightarrow \mathbb{R}$  is measurable if for all  $\alpha \in \mathbb{R}$ , the set  $\{f > \alpha\} \in \mathcal{A}$ . Now observe:

$$\{\mathbf{1}_A > \alpha\} = \begin{cases} \emptyset & \text{if } \alpha \geq 1, \\ A & \text{if } 0 < \alpha < 1, \\ X & \text{if } \alpha \leq 0. \end{cases}$$

Thus,  $\{\mathbf{1}_A > \alpha\} \in \mathcal{A}$  for all  $\alpha \in \mathbb{R}$  if and only if  $A \in \mathcal{A}$ , proving the claim.  $\square$

**Example 6.2.** Let  $(X, \mathcal{A})$  be a measurable space. Suppose  $A_1, \dots, A_M \in \mathcal{A}$  are pairwise disjoint and  $y_1, \dots, y_M \in \mathbb{R}$ . Define the function

$$f(x) := \sum_{m=1}^M y_m \mathbf{1}_{A_m}(x).$$

Then  $f \in \mathcal{E}(\mathcal{A})$ , and in particular,  $f$  is measurable.

**Definition 6.4.** Let  $(X, \mathcal{A})$  be a measurable space.

A *simple function* is a function  $f : X \rightarrow \mathbb{R}$  of the form

$$f(x) = \sum_{m=1}^M y_m \mathbf{1}_{A_m}(x),$$

where  $M \in \mathbb{N}$ ,  $y_m \in \mathbb{R}$ , and  $A_1, \dots, A_M \in \mathcal{A}$  are pairwise disjoint.

A representation of the form

$$f(x) = \sum_{n=1}^N z_n \mathbf{1}_{B_n}(x),$$

with  $N \in \mathbb{N}$ ,  $z_n \in \mathbb{R}$ ,  $B_n \in \mathcal{A}$ , and  $\bigsqcup_{n=1}^N B_n = X$ , is called a *standard representation* of  $f$ .

The set of all simple functions on  $(X, \mathcal{A})$  is denoted by  $\mathcal{E}$  or  $\mathcal{E}(\mathcal{A})$ .

**Remark 6.5.** Simple functions may have multiple representations; in particular, standard representations are not unique.

**Example 6.3.** A measurable function  $h : X \rightarrow \mathbb{R}$  that attains only finitely many values is a simple function.

Indeed, let  $h(X) = \{y_0, \dots, y_M\}$ . The sets  $\{h = \beta\}$  with  $\beta \in h(X)$  are mutually disjoint and satisfy

$$\{h = \beta\} = \{h \leq \beta\} \setminus \{h < \beta\} \in \mathcal{A},$$

and

$$\bigcup_{\beta \in h(X)} \{h = \beta\} = X.$$

Thus,  $h$  admits the standard representation

$$h(x) = \sum_{\beta \in h(X)} \beta \mathbf{1}_{\{h=\beta\}}(x).$$

This shows that every measurable function with finitely many values is a simple function. Conversely, since every simple function only takes finitely many values, it always admits at least one standard representation.

In particular,  $\mathcal{E}(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$ , where  $\mathcal{M}(\mathcal{A})$  is the space of measurable functions.

**Example 6.4.** If  $f, g \in \mathcal{E}(\mathcal{A})$ , then

$$f \pm g \in \mathcal{E}(\mathcal{A}) \quad \text{and} \quad f \cdot g \in \mathcal{E}(\mathcal{A}).$$

*Proof.* Let

$$f = \sum_{m=1}^M y_m \mathbf{1}_{A_m}, \quad g = \sum_{n=1}^N z_n \mathbf{1}_{B_n}$$

be standard representations, where  $A_m, B_n \in \mathcal{A}$  are disjoint families and  $y_m, z_n \in \mathbb{R}$ .

Consider the intersections  $A_m \cap B_n$ . Then:

- Each  $A_m \cap B_n \in \mathcal{A}$ , since  $\mathcal{A}$  is closed under intersections.
- The sets  $A_m \cap B_n$  are pairwise disjoint for distinct pairs  $(m, n) \neq (m', n')$ .
- Their union covers  $X$ , because

$$X = \bigcup_{m=1}^M A_m = \bigcup_{n=1}^N B_n \quad \Rightarrow \quad X = \bigcup_{m=1}^M \bigcup_{n=1}^N (A_m \cap B_n).$$

On each set  $A_m \cap B_n$ , the functions  $f$  and  $g$  are constant with values  $y_m$  and  $z_n$ , respectively. Thus,

$$f(x) \pm g(x) = y_m \pm z_n, \quad f(x) \cdot g(x) = y_m z_n \quad \text{for } x \in A_m \cap B_n.$$

Therefore, we obtain:

$$f \pm g = \sum_{m=1}^M \sum_{n=1}^N (y_m \pm z_n) \mathbf{1}_{A_m \cap B_n}, \quad f \cdot g = \sum_{m=1}^M \sum_{n=1}^N (y_m z_n) \mathbf{1}_{A_m \cap B_n}.$$

These are finite sums over pairwise disjoint measurable sets, so  $f \pm g$  and  $f \cdot g$  are both simple functions.

Hence,  $f \pm g, f \cdot g \in \mathcal{E}(\mathcal{A})$ . □

**Definition 6.5.** Let  $f \in \mathcal{E}(\mathcal{A})$ . Define its *positive part*, *negative part*, and *absolute value* by

$$f^+(x) := \max(f(x), 0), \quad f^-(x) := \max(-f(x), 0), \quad |f(x)| := f^+(x) + f^-(x).$$

Then  $f^+, f^-, |f| \in \mathcal{E}(\mathcal{A})$ , and

$$f = f^+ - f^-, \quad |f| = f^+ + f^-.$$

**Theorem 6.4** (Sombrero Lemma). Let  $(X, \mathcal{A})$  be a measurable space. Every positive  $\mathcal{A}/\mathcal{B}([0, \infty])$ -measurable function  $u : X \rightarrow [0, \infty]$  is the pointwise limit of an increasing sequence of simple functions  $f_n \in \mathcal{E}(\mathcal{A})$ , with  $f_n \geq 0$ , that is,

$$u(x) = \sup_{n \in \mathbb{N}} f_n(x) = \lim_{n \rightarrow \infty} f_n(x), \quad \text{with } f_1 \leq f_2 \leq f_3 \leq \dots$$