Measure Theory

1 σ-Algebras

Definition 1.1. A σ-algebra $\mathcal{A}$ on a set X is a family of subsets of X such that:

1. $X \in \mathcal{A}$,
2. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$,
3. If $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

A set $A \in \mathcal{A}$ is said to be *measurable* or $\mathcal{A}$-*measurable*.

Example 1.1.

1. $\mathcal{P}(X)$ is a σ-algebra (the maximal σ-algebra on X).
2. $\{\emptyset, X\}$ is a σ-algebra (the minimal σ-algebra on X).
3. $A := \{A \subseteq X : \#A < \infty \text{ or } \#A^c < \infty\}$ is a σ-algebra.
4. (Trace σ-algebra) Let $E \subseteq X$ be any set and let $\mathcal{A}$ be a σ-algebra on X. Then
$$\mathcal{A}_E := \{E \cap A : A \in \mathcal{A}\}$$
is a σ-algebra on E.

Proof. We verify the three defining properties of a σ-algebra on E:

(a) Since $X \in \mathcal{A}$, we have
$$E = E \cap X \in \mathcal{A}_E.$$

(b) If $E \cap A \in \mathcal{A}_E$, then
$$E \setminus (E \cap A) = E \cap A^c,$$
and since $A^c \in \mathcal{A}$, it follows that
$$E \cap A^c \in \mathcal{A}_E.$$

(c) If $(E \cap A_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}_E$, then

$$\bigcup_n (E \cap A_n) = E \cap \bigcup_n A_n,$$

and since $\bigcup_n A_n \in \mathcal{A}$, we conclude that

$$\bigcup_n (E \cap A_n) \in \mathcal{A}_E.$$

Hence, $\mathcal{A}_E$ is a σ-algebra on E. □

5. (Pre-image σ-algebra) Let $f : X \to X'$ be a function and let $\mathcal{A}'$ be a σ-algebra on X'. Then

$$\mathcal{A} := \{f^{-1}(A') : A' \in \mathcal{A}'\}$$

is a σ-algebra on X.

Theorem 1.1. Let X be a set and let $\{\mathcal{A}_i : i \in I\}$ be a family of σ-algebras on X. Define

$$\mathcal{A} := \bigcap_{i\in I} \mathcal{A}_i = \{A \subseteq X : A \in \mathcal{A}_i \text{ for all } i \in I\}.$$

Then, $\mathcal{A}$ is a σ-algebra on X.

Proof. We verify the σ-algebra properties for $\mathcal{A}$:

1. Since $X \in \mathcal{A}_i$ for all $i \in I$, we have $X \in \mathcal{A}$.
2. If $A \in \mathcal{A}$, then $A \in \mathcal{A}_i$ for all $i \in I$, so

$$A^c = X \setminus A \in \mathcal{A}_i \quad \forall i \in I,$$

hence $A^c \in \mathcal{A}$.

3. If $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then for all n and i,

$$A_n \in \mathcal{A}_i,$$

so by closure of each $\mathcal{A}_i$ under countable unions,

$$\bigcup_{n=0}^{\infty} A_n \in \mathcal{A}_i \quad \forall i \in I,$$

and thus

$$\bigcup_{n=0}^{\infty} A_n \in \mathcal{A}.$$

Therefore, $\mathcal{A}$ is a σ-algebra on X. $\square$

Definition 1.2. Let X be a set and let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. The *σ-algebra generated by* $\mathcal{E}$, denoted by $\sigma(\mathcal{E})$, is the smallest σ-algebra on X containing all sets in $\mathcal{E}$. That is,

$$\sigma(\mathcal{E}) := \bigcap \big\{ \mathcal{A} \subseteq \mathcal{P}(X) : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X,\ \mathcal{E} \subseteq \mathcal{A} \big\}.$$

Remark 1.1 (Generated σ-algebras)**.**

1. If $\mathcal{A}$ is a σ-algebra, then $\sigma(\mathcal{A}) = \mathcal{A}$.
2. For $A \subseteq X$, we have $\sigma(\{A\}) = \{\emptyset, A, A^c, X\}$.
3. If $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{A}$, then $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G}) \subseteq \sigma(\mathcal{A})$.

Definition 1.3 (Topological Space)**.** A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of subsets of X, called *open sets*, satisfying the following properties:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. If $\{U_\alpha \in \mathcal{T} : \alpha \in I\}$ is an arbitrary collection of open sets, then the union $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$,
3. If $\{U_i \in \mathcal{T} : i = 1, \ldots, N\}$ is a finite collection of open sets, then the intersection $\bigcap_{i=1}^{N} U_i \in \mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X. The complement of an open set is called a *closed set*.

Remark 1.2 (Standard Topology on $\mathbb{R}^n$)**.** A subset $U \subseteq \mathbb{R}^n$ is called *open* if for every point $x \in U$, there exists an $\varepsilon > 0$ such that the open ball

$$B_\varepsilon(x) := \{y \in \mathbb{R}^n : \|x - y\| < \varepsilon\},$$

where $\|\cdot\|$ denotes the Euclidean norm, is contained in U; that is, $B_\varepsilon(x) \subseteq U$.

The collection of all such open sets is denoted by $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ and forms the *standard topology* on $\mathbb{R}^n$.

Definition 1.4 (Borel σ-algebra)**.** The σ-algebra $\sigma(\mathcal{O})$ generated by the collection of open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called the *Borel σ-algebra* on $\mathbb{R}^n$.

Its elements are called *Borel sets* or *Borel measurable sets*. We denote the Borel σ-algebra by $\mathcal{B}(\mathbb{R}^n)$.