Measure Theory

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1 σ-Algebras

Definition 1.1. A σ-algebra $\mathcal{A}$ on a set X is a family of subsets of X such that:

- $X \in \mathcal{A}$ $(\boldsymbol{\Sigma}_1)$
- If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ $(\boldsymbol{\Sigma}_2)$
- If $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ $(\boldsymbol{\Sigma}_3)$

A set $A \in \mathcal{A}$ is said to be *measurable* or $\mathcal{A}$-*measurable*.

Example 1.1.

1. $\mathcal{P}(X)$ is a σ-algebra (the maximal σ-algebra on X).
2. $\{\emptyset, X\}$ is a σ-algebra (the minimal σ-algebra on X).
3. $A := \{A \subseteq X : \#A < \infty \text{ or } \#A^c < \infty\}$ is a σ-algebra.
4. (Trace σ-algebra) Let $E \subseteq X$ be any set and let $\mathcal{A}$ be a σ-algebra on X. Then

$$\mathcal{A}_E := \{E \cap A : A \in \mathcal{A}\}$$

is a σ-algebra on E.

Proof. We verify the three defining properties of a σ-algebra on E:

- Since $X \in \mathcal{A}$, we have $E = E \cap X \in \mathcal{A}_E$.
- If $E \cap A \in \mathcal{A}_E$, then $E \setminus (E \cap A) = E \cap A^c$, and since $A^c \in \mathcal{A}$, it follows that $E \cap A^c \in \mathcal{A}_E$.
- If $(E \cap A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_E$, then $\bigcup_n (E \cap A_n) = E \cap \bigcup_n A_n$, and since $\bigcup_n A_n \in \mathcal{A}$, we conclude that $\bigcup_n (E \cap A_n) \in \mathcal{A}_E$.

Hence, $\mathcal{A}_E$ is a σ-algebra on E. □

5. (Pre-image σ-algebra) Let $f : X \to X'$ be a function and let $\mathcal{A}'$ be a σ-algebra on X'. Then

$$\mathcal{A} := \{f^{-1}(A') : A' \in \mathcal{A}'\}$$

is a σ-algebra on X.

Theorem 1.1. Let X be a set and let $\{\mathcal{A}_i : i \in I\}$ be a family of σ-algebras on X. Define

$$\mathcal{A} := \bigcap_{i \in I} \mathcal{A}_i = \{A \subseteq X : A \in \mathcal{A}_i \text{ for all } i \in I\}.$$

Then, $\mathcal{A}$ is a σ-algebra on X.

Proof. We verify the σ-algebra properties for $\mathcal{A}$:

- Since $X \in \mathcal{A}_i$ for all $i \in I$, we have $X \in \mathcal{A}$.
- If $A \in \mathcal{A}$, then $A \in \mathcal{A}_i$ for all $i \in I$, so $A^c = X \setminus A \in \mathcal{A}_i$ for all $i \in I$, hence $A^c \in \mathcal{A}$.
- If $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$, then $A_n \in \mathcal{A}_i$ for all n and i, so $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_i$ for all $i \in I$, and thus $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

Therefore, $\mathcal{A}$ is a σ-algebra on X. □

Definition 1.2. Let X be a set and let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. The *σ-algebra generated by $\mathcal{E}$*, denoted by $\sigma(\mathcal{E})$, is the smallest σ-algebra on X containing all sets in $\mathcal{E}$. That is,

$$\sigma(\mathcal{E}) := \bigcap \Big\{ \mathcal{A} \subseteq \mathcal{P}(X) : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X,\ \mathcal{E} \subseteq \mathcal{A} \Big\}.$$

Remark 1.1 (Generated σ-algebras)**.**

- If $\mathcal{A}$ is a σ-algebra, then $\sigma(\mathcal{A}) = \mathcal{A}$.
- For $A \subseteq X$, we have $\sigma(\{A\}) = \{\emptyset, A, A^c, X\}$.
- If $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{A}$, then $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G}) \subseteq \sigma(\mathcal{A})$.

Definition 1.3 (Topological Space)**.** A *topological space* is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ is a collection of subsets of X, called *open sets*, satisfying the following properties:

- $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
- If $\{U_\alpha \in \mathcal{T} : \alpha \in I\}$ is an arbitrary collection of open sets, then the union $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$,
- If $\{U_i \in \mathcal{T} : i = 1, \ldots, n\}$ is a finite collection of open sets, then the intersection $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X. The complement of an open set is called a *closed set*.

Remark 1.2 (Standard Topology on $\mathbb{R}^n$)**.** A subset $U \subseteq \mathbb{R}^n$ is called *open* if for every point $x \in U$, there exists an $\varepsilon > 0$ such that the open ball

$$B_\varepsilon(x) := \{y \in \mathbb{R}^n : \|x - y\| < \varepsilon\},$$

where $\|\cdot\|$ denotes the Euclidean norm, is contained in U; that is, $B_\varepsilon(x) \subseteq U$.

The collection of all such open sets is denoted by $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ and forms the *standard topology* on $\mathbb{R}^n$.

Definition 1.4 (Borel σ-algebra)**.** The σ-algebra $\sigma(\mathcal{O})$ generated by the collection of open sets $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ is called the *Borel σ-algebra* on $\mathbb{R}^n$.

Its elements are called *Borel sets* or *Borel measurable sets*. We denote the Borel σ-algebra by $\mathcal{B}(\mathbb{R}^n)$.

Definition 1.5. Let X be a topological space and let $A \subseteq X$. A collection $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ of open sets is called an *open cover* of A if

$$A \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

A *subcover* is a subcollection that still covers A. The set A is called *compact* if every open cover of A admits a finite subcover.

Remark 1.3. In $\mathbb{R}^n$, a set is compact if and only if it is closed and bounded (Heine–Borel Theorem).

Theorem 1.2 (Borel σ-algebra from Different Generators)**.** Let $\mathcal{O}, \mathcal{C}, \mathcal{K} \subseteq \mathcal{P}(\mathbb{R}^n)$ denote the collections of open, closed, and compact subsets of $\mathbb{R}^n$, respectively. Then,

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}) = \sigma(\mathcal{C}) = \sigma(\mathcal{K}).$$

Proof. Since compact sets are closed, we have $\mathcal{K} \subseteq \mathcal{C}$, and by Remark 1.1, $\sigma(\mathcal{K}) \subseteq \sigma(\mathcal{C})$. Conversely, for any $C \in \mathcal{C}$, define $C_k := C \cap B_k(0)$, where $B_k(0)$ is the closed ball of radius k centered at the origin. Each C_k is closed and bounded, hence compact, so $C_k \in \mathcal{K}$. Since $C = \bigcup_{k \in \mathbb{N}} C_k$, it follows that $C \in \sigma(\mathcal{K})$, and thus $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{K})$.

Next, since $\mathcal{C} = \mathcal{O}^c := \{U^c : U \in \mathcal{O}\}$, and complements of sets in a σ-algebra are again in the σ-algebra, it follows that $\mathcal{C} \subseteq \sigma(\mathcal{O})$, hence $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{O})$. The reverse inclusion follows similarly from $\mathcal{O} = \mathcal{C}^c$. Therefore, we have:

$$\sigma(\mathcal{K}) = \sigma(\mathcal{C}) = \sigma(\mathcal{O}).$$

$\square$

Generating Sets of the Borel Algebra. The Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ can be generated by various systems of sets. Of particular importance are:

- The family of open rectangles:

$$\mathcal{J}_{o,n} := \{(a_1, b_1) \times \cdots \times (a_n, b_n) : a_i, b_i \in \mathbb{R}\},$$

- The family of half-open rectangles:

$$\mathcal{J}_n := \{[a_1, b_1) \times \cdots \times [a_n, b_n) : a_i, b_i \in \mathbb{R}\}.$$

We denote by $\mathcal{J}_n^{\text{rat}}, \mathcal{J}_{o,n}^{\text{rat}}$ the subsets with rational endpoints. These sets represent intervals in $\mathbb{R}$, rectangles in $\mathbb{R}^2$, cuboids in $\mathbb{R}^3$, and hypercubes in higher dimensions.

Theorem 1.3. We have the following equality of Borel σ-algebras on $\mathbb{R}^n$:

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{J}_n^{\text{rat}}) = \sigma(\mathcal{J}_{o,n}^{\text{rat}}) = \sigma(\mathcal{J}_n) = \sigma(\mathcal{J}_{o,n}),$$

Remark 1.4. Let $D \subseteq \mathbb{R}$ be a dense subset, for example $D = \mathbb{Q}$ or $D = \mathbb{R}$. Then the Borel sets on $\mathbb{R}$ can also be generated by any of the following families of intervals:

$$\{(-\infty, a) : a \in D\}, \quad \{(-\infty, a] : a \in D\}, \quad \{(a, \infty) : a \in D\}, \quad \{[a, \infty) : a \in D\}.$$

2 Measure Spaces

Definition 2.1. A *(positive) measure* μ on X is a map $\mu : \mathcal{A} \to [0, \infty]$, where $\mathcal{A}$ is a σ-algebra on X, satisfying:

$$\mu(\emptyset) = 0, \tag{M1}$$

and for any pairwise disjoint sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$,

$$\mu\left(\bigsqcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n). \tag{M2}$$

Property (M2) is also called *countable additivity.*

If μ satisfies (M1), (M2), but $\mathcal{A}$ is not a σ-algebra, then μ is called a *pre-measure.*

Remark 2.1. (M2) requires implicitly that $\bigsqcup_n A_n$ is again in $\mathcal{A}$ this is clearly the case for σ-algebras, but needs special attention when dealing with pre-measures.

Definition 2.2 (Monotone sequences of sets)**.** Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be sequences of subsets of X.

We say (A_n) is *increasing* if

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

and write $A_n \uparrow A$ where

$$A := \bigcup_{n\in\mathbb{N}} A_n$$

Similarly, (B_n) is *decreasing* if

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots$$

and write $B_n \downarrow B$ where

$$B := \bigcap_{n\in\mathbb{N}} B_n$$

Definition 2.3. Let X be a set and $\mathcal{A}$ a σ-algebra on X. The pair $(X, \mathcal{A})$ is called a *measurable space.* If μ is a measure on $(X, \mathcal{A})$, then $(X, \mathcal{A}, \mu)$ is called a *measure space.*

A measure μ is called:

- *finite* if $\mu(X) < \infty$,
- a *probability measure* if $\mu(X) = 1$.

Accordingly, we speak of a *finite measure space* and a *probability space.*

Definition 2.4. A measure μ on a measurable space $(X, \mathcal{A})$ is called σ*-finite* if there exists a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that:

$$A_n \uparrow X \quad \text{and} \quad \mu(A_n) < \infty \quad \text{for all } n \in \mathbb{N}.$$

In this case, the measure space $(X, \mathcal{A}, \mu)$ is called σ*-finite.*

Lemma 2.1 (Basic properties of measures)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Then:

(i) If $A_0, \dots, A_k \in \mathcal{A}$ are pairwise disjoint, then $\mu\Big(\bigcup_{n=1}^{k} A_n\Big) = \sum_{n=1}^{k} \mu(A_n)$.

(ii) If $A, B \in \mathcal{A}$ with $A \subseteq B$, then $\mu(A) \le \mu(B)$.

(iii) If $A, B \in \mathcal{A}$, $A \subseteq B$, and $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

Proof. (i) Extend (A_n) by $A_n = \varnothing$ for $n > k$. Then by countable additivity,

$$\mu\Big(\bigsqcup_{n=1}^{k} A_n\Big) = \mu\Big(\bigsqcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{k} \mu(A_n).$$

(ii) Since $B = A \sqcup (B \setminus A)$ we have

$$\mu(B) = \mu(A) + \mu(B \setminus A) \ge \mu(A).$$

(iii) Rearranging gives

$$\mu(B \setminus A) = \mu(B) - \mu(A),$$

which is well-defined if $\mu(A) < \infty$. $\square$

Lemma 2.2 (Main properties of measures)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Then:

(i) **Countable subadditivity:** For any countable family $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$,

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leq \sum_{i\in\mathbb{N}} \mu(A_i).$$

(ii) **Continuity from below (increasing sequence):** If $A_1 \subseteq A_2 \subseteq \cdots$ (i.e., $A_n \uparrow A$), then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(iii) **Continuity from above (decreasing sequence):** If $B_1 \supseteq B_2 \supseteq \cdots$ (i.e., $B_n \downarrow B$), then

$$\mu\left(\bigcap_{n\in\mathbb{N}} B_n\right) = \lim_{n\to\infty} \mu(B_n).$$

Proof. (i) For countable subadditivity, set $B_k := A_k \setminus \bigcup_{i=1}^{k-1} A_i$, so that (B_k) are disjoint with $B_k \subseteq A_k$. Then,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigsqcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

(ii) Let $A_n \uparrow A$, i.e., $A_n \subseteq A_{n+1}$ and $A = \bigcup_{n\in\mathbb{N}} A_n$. Define $B_n := A_n \setminus A_{n-1}$ with $A_0 := \emptyset$. Then (B_n) is disjoint and $\bigsqcup_n B_n = A$. By countable additivity,

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n\to\infty} \mu(A_n)$$

(iii) Assume $B_n \downarrow B$, i.e., $B_n \supseteq B_{n+1}$ and $B = \bigcap_n B_n$, with $\mu(B_1) < \infty$. Set $A_n := B_1 \setminus B_n$, so $A_n \uparrow A := B_1 \setminus B$. Then

$$\mu(B) = \mu(B_1) - \mu(A) = \mu(B_1) - \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu(B_n)$$

□

Remark 2.2. With appropriate modifications, these properties also hold for pre-measures, i.e., when $\mathcal{A}$ is not necessarily a σ-algebra.

Example 2.1 (Dirac measure)**.** Let $(X, \mathcal{A})$ be a measurable space and let $x \in X$. Define $\delta_x : \mathcal{A} \to \{0, 1\}$ by

$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then δ_x is a measure on $(X, \mathcal{A})$, called the *Dirac measure* (or unit mass) at the point x.

Example 2.2 (Counting measure)**.** Let $(X, \mathcal{A})$ be a measurable space. Define $\#A : \mathcal{A} \to [0, \infty]$ by

$$\#A := \begin{cases} \text{number of elements in } A & \text{if } A \text{ is finite,} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$$

Then $\#$ is a measure on $(X, \mathcal{A})$, called the *counting measure*.

Example 2.3 (Discrete probability measure)**.** Let $\Omega = \{\omega_1, \omega_2, \dots\}$ be a countable set, and let $(p_n)_{n\in\mathbb{N}} \subseteq [0, 1]$ be a sequence such that $\sum_{n\in\mathbb{N}} p_n = 1$. Define the set function $P : \mathcal{P}(\Omega) \to [0, 1]$ by

$$P(A) := \sum_{\{n\in\mathbb{N}:\omega_n\in A\}} p_n = \sum_{n\in\mathbb{N}} p_n\, \delta_{\omega_n}(A), \quad A \subseteq \Omega,$$

where δ_{ω_n} denotes the Dirac measure at ω_n. Then P is a probability measure on $(\Omega, \mathcal{P}(\Omega))$, and the triplet $(\Omega, \mathcal{P}(\Omega), P)$ is called a *discrete probability space*.

Example 2.4 (Linear combination of measures)**.** Let $(X, \mathcal{A})$ be a measurable space, and let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures on $(X, \mathcal{A})$. Let $(x_n)_{n\in\mathbb{N}} \subseteq [0, \infty]$. Then the set function

$$\mu := \sum_{n\in\mathbb{N}} x_n \mu_n$$

defined by

$$\mu(A) := \sum_{n\in\mathbb{N}} x_n \mu_n(A), \quad \text{for all } A \in \mathcal{A},$$

is a measure on $(X, \mathcal{A})$

Proof. We verify the axioms of a measure:

(M1) *(Null empty set):* For all $n \in \mathbb{N}$, $\mu_n(\emptyset) = 0$, so

$$\mu(\emptyset) = \sum_{n\in\mathbb{N}} x_n \mu_n(\emptyset) = \sum_{n\in\mathbb{N}} x_n \cdot 0 = 0.$$

(M2) *(Countable additivity):* Let $(A_k)_{k\in\mathbb{N}} \subseteq \mathcal{A}$ be pairwise disjoint. Since each μ_n is a measure, we have

$$\mu_n\left(\bigsqcup_{k\in\mathbb{N}} A_k\right) = \sum_{k\in\mathbb{N}} \mu_n(A_k), \quad \text{for all } n \in \mathbb{N}.$$

Then,

$$\mu\left(\bigsqcup_{k\in\mathbb{N}} A_k\right) = \sum_{n\in\mathbb{N}} x_n \mu_n \left(\bigsqcup_{k\in\mathbb{N}} A_k\right) = \sum_{n\in\mathbb{N}} x_n \sum_{k\in\mathbb{N}} \mu_n(A_k).$$

Since all terms are non-negative, we may exchange the order of summation:

$$\sum_{n\in\mathbb{N}} x_n \sum_{k\in\mathbb{N}} \mu_n(A_k) = \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} x_n \mu_n(A_k) = \sum_{k\in\mathbb{N}} \mu(A_k).$$

Therefore, μ is countably additive. $\square$

Example 2.5 (Restriction of a measure)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and let $A \in \mathcal{A}$. Define the set function $\mu_A : \mathcal{A} \to [0, \infty]$ by

$$\mu_A(B) := \mu(A \cap B), \quad \text{for all } B \in \mathcal{A}.$$

Then μ_A is a measure on $(X, \mathcal{A})$, called the *restriction of μ to A*.

Proof. We verify the two defining properties of a measure:
(M1): $\mu_A(\emptyset) = \mu(A \cap \emptyset) = \mu(\emptyset) = 0$.

(M2): Let $(B_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be pairwise disjoint. Then $(A \cap B_n)_{n\in\mathbb{N}}$ are also pairwise disjoint, and

$$\mu_A\left(\bigsqcup_{n\in\mathbb{N}} B_n\right) = \mu\left(A \cap \bigsqcup_{n\in\mathbb{N}} B_n\right) = \mu\left(\bigsqcup_{n\in\mathbb{N}} (A \cap B_n)\right) = \sum_{n\in\mathbb{N}} \mu(A \cap B_n) = \sum_{n\in\mathbb{N}} \mu_A(B_n).$$

Hence, μ_A is a measure. □

Definition 2.5 (Lebesgue measure on $\mathbb{R}^n$)**.** Define the set function λ_n on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ by

$$\lambda_n\left([\![a, b)\!)\right) := \prod_{i=1}^{n} (b_i - a_i),$$

for all $[\![a, b)\!) := [a_1, b_1) \times \cdots \times [a_n, b_n) \in \mathcal{J}_n$. This is called the n-dimensional Lebesgue measure.

Remark 2.3. The set function λ_n is defined only on the family $\mathcal{J}_n$ of half-open rectangles and hence is not yet a measure. Extending λ_n to a measure on $\mathcal{B}(\mathbb{R}^n)$ requires the Carathéodory extension theorem, which will be developed later.

Lemma 2.3. Let $(X, \mathcal{A})$ be a measure space, and let $\mu : \mathcal{A} \to [0, \infty]$ be an additive set function with $\mu(\emptyset) = 0$. Then μ is a measure if and only if it is **continuous from below**, i.e., for every increasing sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ with $A_n \uparrow A$, we have

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) = \sup_{n\in\mathbb{N}} \mu(A_n).$$

Proof. Any measure μ is continuous from below.

Conversely, suppose μ is finitely additive, $\mu(\emptyset) = 0$, and μ is continuous from below. Let $(B_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be disjoint, and define $A_n := \bigcup_{i=1}^n B_i$. Then (A_n) is increasing with $\bigcup_{n=1}^\infty A_n = \bigsqcup_{n=1}^\infty B_n$. By finite additivity,

$$\mu(A_n) = \sum_{i=1}^{n} \mu(B_i),$$

and by continuity from below,

$$\mu\left(\bigsqcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu(B_n).$$

Hence μ is countably additive, i.e., a measure. □

Lemma 2.4. Let $(X, \mathcal{A})$ be a measurable space and $\mu : \mathcal{A} \to [0, \infty)$ an additive set function with $\mu(\emptyset) = 0$ and $\mu(A) < \infty$ for all $A \in \mathcal{A}$. Then μ is a measure if and only if it satisfies one of the following continuity properties:

(i) μ is continuous from below;

(ii) μ is continuous from above;

(iii) μ is continuous at $\emptyset$, i.e., for every decreasing sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ with $\bigcap_{n=1}^{\infty} B_n = \emptyset$, we have

$$\lim_{n\to\infty} \mu(B_n) = 0.$$

Proof. Clearly, every measure satisfies properties (i)–(iii), so we only need to show that (iii) implies countable additivity.

Assume μ is additive, $\mu(\emptyset) = 0$, and satisfies continuity at $\emptyset$. Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be pairwise disjoint and define $A := \bigsqcup_{n\in\mathbb{N}} A_n$. For each n, let

$$B_n := A \setminus \bigcup_{i=1}^{n} A_i.$$

Then (B_n) is a decreasing sequence in $\mathcal{A}$ with $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$, so by continuity at $\emptyset$, we have $\mu(B_n) \to 0$.

Using additivity, we compute

$$\mu(A) = \mu\left(B_n \sqcup \bigcup_{i=1}^{n} A_i\right) = \mu(B_n) + \sum_{i=1}^{n} \mu(A_i).$$

Taking the limit as $n \to \infty$, we get

$$\mu(A) = \lim_{n\to\infty}\left(\mu(B_n) + \sum_{i=1}^{n} \mu(A_i)\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Thus, μ is countably additive, hence a measure. □

3 Uniqueness of Measures

Definition 3.1. A *Dynkin system* (or λ-system) $\mathcal{D} \subseteq \mathcal{P}(X)$ is a collection of subsets of X such that:

- $X \in \mathcal{D}$ **(D1)**
- If $D \in \mathcal{D}$, then $D^c \in \mathcal{D}$ **(D2)**
- If $(D_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ are pairwise disjoint, then $\bigsqcup_{n\in\mathbb{N}} D_n \in \mathcal{D}$ **(D3)**

Remark 3.1. As with σ-algebras one easily checks that $\emptyset \in \mathcal{D}$ and that finite disjoint unions are in $\mathcal{D}$: if $D, E \in \mathcal{D}$ with $D \cap E = \emptyset$, then $D \sqcup E \in \mathcal{D}$. Every σ-algebra is a Dynkin system, but the converse is not true in general.

Lemma 3.1. Let $\mathcal{E} \subseteq \mathcal{P}(X)$. Then there exists a smallest Dynkin system $\mathcal{D}(\mathcal{E})$ containing $\mathcal{E}$, called the *Dynkin system generated by* $\mathcal{E}$. Moreover,

$$\mathcal{E} \subseteq \mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E}),$$

where $\sigma(\mathcal{E})$ denotes the σ-algebra generated by $\mathcal{E}$.

Proof. The proof is analogous to that of Theorem 1.1 for σ-algebras. Let $\mathcal{F}$ be the family of all Dynkin systems on X that contain $\mathcal{E}$. Then $\mathcal{F}$ is nonempty, since $\mathcal{P}(X)$ is a Dynkin system containing $\mathcal{E}$. Define

$$\mathcal{D}(\mathcal{E}) := \bigcap_{\mathcal{D} \in \mathcal{F}} \mathcal{D}.$$

Then $\mathcal{D}(\mathcal{E})$ is a Dynkin system, being the intersection of Dynkin systems (which are closed under complements, disjoint unions, and contain X). Moreover, it is the smallest such system containing $\mathcal{E}$ by construction. Since every σ-algebra is in particular a Dynkin system, we also have

$$\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E}).$$

□

Lemma 3.2. A Dynkin system $\mathcal{D}$ is a σ-algebra if and only if it is closed under finite intersections; that is,

$$D, E \in \mathcal{D} \quad \Rightarrow \quad D \cap E \in \mathcal{D}.$$

Proof. The "only if" direction follows immediately from Remark 3.1 and the fact that every σ-algebra is closed under finite intersections.

For the converse, assume $\mathcal{D}$ is a Dynkin system closed under finite intersections. Let $(D_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}$, and define

$$E_1 := D_1 \in \mathcal{D}, \quad E_{n+1} := D_{n+1} \setminus \bigcup_{k=1}^{n} D_k = D_{n+1} \cap \bigcap_{k=1}^{n} D_k^c.$$

Each $E_n \in \mathcal{D}$ by the Dynkin properties and the assumed stability under finite intersections. The sets (E_n) are disjoint and satisfy

$$\bigcup_{n=1}^{\infty} D_n = \bigsqcup_{n=1}^{\infty} E_n \in \mathcal{D},$$

so $\mathcal{D}$ is closed under countable unions. Hence, $\mathcal{D}$ is a σ-algebra. □

While Lemma 3.2 characterizes when a Dynkin system is a σ-algebra, it is not directly applicable when the Dynkin system $\mathcal{D}$ is defined via a generator $\mathcal{E} \subseteq \mathcal{P}(X)$, as is often the case in practice. The following theorem overcomes this limitation and plays a central role in many applications.

Theorem 3.3 (Dynkin's π-λ Theorem)**.** Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a collection of sets that is closed under finite intersections. Then,

$$\mathcal{D}(\mathcal{E}) = \sigma(\mathcal{E}).$$

Proof. By Lemma 3.1, we have $\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E})$. To show equality, it suffices to prove that $\mathcal{D}(\mathcal{E})$ is a σ-algebra. Since it contains $\mathcal{E}$, it would then contain $\sigma(\mathcal{E})$ by minimality.

By Lemma 3.2, it is enough to show that $\mathcal{D}(\mathcal{E})$ is closed under finite intersections.

Fix $D \in \mathcal{D}(\mathcal{E})$, and define

$$\mathcal{D}_D := \{A \subseteq X : A \cap D \in \mathcal{D}(\mathcal{E})\}.$$

We claim that $\mathcal{D}_D$ is a Dynkin system:

(D1): Since $D = X \cap D \in \mathcal{D}(\mathcal{E})$, we have $X \in \mathcal{D}_D$.

(D2): If $A \in \mathcal{D}_D$, then

$$A^c \cap D = ((A \cap D) \sqcup D^c)^c \cap D \in \mathcal{D}(\mathcal{E}),$$

using that $A \cap D \in \mathcal{D}(\mathcal{E})$, $D^c \in \mathcal{D}(\mathcal{E})$, and that Dynkin systems are closed under disjoint unions and complements.

(D3): Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_D$ be disjoint. Then the sets $A_n \cap D \in \mathcal{D}(\mathcal{E})$ are disjoint, and

$$\left(\bigsqcup_{n=1}^{\infty} A_n\right) \cap D = \bigsqcup_{n=1}^{\infty} (A_n \cap D) \in \mathcal{D}(\mathcal{E}).$$

Thus, $\mathcal{D}_D$ is a Dynkin system. Since $\mathcal{E} \subseteq \mathcal{D}_G$ for all $G \in \mathcal{E}$ by the assumed $\cap$-stability of $\mathcal{E}$, and each $\mathcal{D}_G$ is a Dynkin system, it follows that

$$\mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}_G \quad \text{for all } G \in \mathcal{E}.$$

Hence, for all $D \in \mathcal{D}(\mathcal{E})$ and $G \in \mathcal{E}$, we have $D \cap G \in \mathcal{D}(\mathcal{E})$, i.e., $\mathcal{D}(\mathcal{E})$ is closed under finite intersections.

By Lemma 3.2, we conclude that $\mathcal{D}(\mathcal{E})$ is a σ-algebra. Since $\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E})$ and both are σ-algebras containing $\mathcal{E}$, we have

$$\mathcal{D}(\mathcal{E}) = \sigma(\mathcal{E}).$$

$\square$

Theorem 3.4 (Uniqueness of Measures)**.** Let $(X, \mathcal{A})$ be a measurable space with $\mathcal{A} = \sigma(\mathcal{E})$, where $\mathcal{E} \subseteq \mathcal{P}(X)$ satisfies:

- $\mathcal{E}$ is closed under finite intersections;
- there exists an increasing sequence $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}$ with $E_n \uparrow X$.

Suppose μ and ν are measures on $\mathcal{A}$ such that $\mu(E) = \nu(E)$ for all $E \in \mathcal{E}$, and $\mu(E_n) = \nu(E_n) < \infty$ for all $n \in \mathbb{N}$. Then $\mu = \nu$ on $\mathcal{A}$; that is,

$$\mu(A) = \nu(A) \quad \text{for all } A \in \mathcal{A}.$$

Proof. Fix $n \in \mathbb{N}$, and define

$$\mathcal{D}_n := \{A \in \mathcal{A} : \mu(E_n \cap A) = \nu(E_n \cap A)\}.$$

We claim that $\mathcal{D}_n$ is a Dynkin system:

(D1): Since $E_n \in \mathcal{E} \subseteq \mathcal{A}$, and $\mu(E_n) = \nu(E_n)$, it follows that $X \in \mathcal{D}_n$.

(D2): If $A \in \mathcal{D}_n$, then

$$\mu(E_n \cap A^c) = \mu(E_n) - \mu(E_n \cap A) = \nu(E_n) - \nu(E_n \cap A) = \nu(E_n \cap A^c),$$

so $A^c \in \mathcal{D}_n$.

(D3): Let $(A_k)_{k\in\mathbb{N}} \subseteq \mathcal{D}_n$ be disjoint. Then:

$$\mu\left(E_n \cap \bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(E_n \cap A_k) = \sum_{k=1}^{\infty} \nu(E_n \cap A_k) = \nu\left(E_n \cap \bigsqcup_{k=1}^{\infty} A_k\right),$$

so $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{D}_n$.

Thus, $\mathcal{D}_n$ is a Dynkin system. Since $\mathcal{E} \subseteq \mathcal{D}_n$ (as $\mu(E_n \cap E) = \nu(E_n \cap E)$ for all $E \in \mathcal{E}$, by the $\cap$-stability of $\mathcal{E}$), and since $\sigma(\mathcal{E}) = \mathcal{A}$, Theorem 3.3 yields

$$\mathcal{A} = \sigma(\mathcal{E}) = \mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}_n.$$

Hence,

$$\mu(E_n \cap A) = \nu(E_n \cap A) \quad \text{for all } A \in \mathcal{A},\ n \in \mathbb{N}.$$

Now fix $A \in \mathcal{A}$. Since $E_n \uparrow X$, we have $E_n \cap A \uparrow A$, and by continuity from below,

$$\mu(A) = \lim_{n\to\infty} \mu(E_n \cap A) = \lim_{n\to\infty} \nu(E_n \cap A) = \nu(A).$$

Therefore, $\mu = \nu$ on $\mathcal{A}$. □

Theorem 3.5 (Translation Invariance and Uniqueness of Lebesgue Measure)**.** Let λ^n denote the n-dimensional Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Then:

(i) **(Translation invariance)** For all $x \in \mathbb{R}^n$ and all $B \in \mathcal{B}(\mathbb{R}^n)$, we have

$$\lambda^n(x + B) = \lambda^n(B),$$

where $x + B := \{x + y : y \in B\}$ is the translation of B by x.

(ii) **(Uniqueness up to scalar)** Let μ be a measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ that is translation invariant and finite on the unit cube:

$$\mu(x + B) = \mu(B) \quad \text{for all } x \in \mathbb{R}^n,\ B \in \mathcal{B}(\mathbb{R}^n), \quad \text{and} \quad \mu([0,1)^n) < \infty.$$

Then μ is a scalar multiple of Lebesgue measure:

$$\mu = \mu([0,1)^n) \cdot \lambda^n.$$

4 Existence of Measures

Definition 4.1 (Semi-ring)**.** Let X be a set. A family $\mathcal{S} \subseteq \mathcal{P}(X)$ is called a *semi-ring* if:

- $\emptyset \in \mathcal{S}$ (S1)
- For all $S, T \in \mathcal{S}$, we have $S \cap T \in \mathcal{S}$ (S2)
- For all $S, T \in \mathcal{S}$, there exist disjoint sets $S_1, \dots, S_M \in \mathcal{S}$ such that

$$S \setminus T = \bigsqcup_{i=1}^{M} S_i \tag{S3}$$

Theorem 4.1 (Carathéodory Extension Theorem)**.** Let $\mathcal{S} \subseteq \mathcal{P}(X)$ be a semi-ring and let $\mu : \mathcal{S} \to [0, \infty]$ be a pre-measure, i.e.,

- $\mu(\emptyset) = 0$,
- For every sequence $(S_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}$ of disjoint sets with $\bigsqcup_{n\in\mathbb{N}} S_n \in \mathcal{S}$, we have

$$\mu\left(\bigsqcup_{n\in\mathbb{N}} S_n\right) = \sum_{n\in\mathbb{N}} \mu(S_n).$$

Then μ has an extension to a measure $\bar{\mu}$ on $\sigma(\mathcal{S})$.

Moreover, if $\mathcal{S}$ contains an increasing sequence $(S_n)_{n\in\mathbb{N}}$ with $S_n \uparrow X$ and $\mu(S_n) < \infty$ for all n, then the extension is unique.

Idea of the proof. The fundamental problem is how to extend the pre-measure μ. The following auxiliary set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ will play a central role. For any $A \subseteq X$, define the family of countable $\mathcal{S}$-coverings

$$\mathcal{C}(A) := \left\{ (S_n)_{n\in\mathbb{N}} \subseteq \mathcal{S} : A \subseteq \bigcup_{n\in\mathbb{N}} S_n \right\},$$

and the set function

$$\mu^*(A) := \inf \left\{ \sum_{n\in\mathbb{N}} \mu(S_n) : (S_n)_{n\in\mathbb{N}} \in \mathcal{C}(A) \right\}.$$

If A cannot be covered by sets from $\mathcal{S}$, we define $\mathcal{C}(A) = \emptyset$ and hence $\mu^*(A) := \inf \emptyset = \infty$.

The proof proceeds in four main steps:

1. **(Outer measure)** Show that μ^* is an outer measure, i.e., it satisfies:

$$\begin{aligned} &\text{(OM1)} \quad \mu^*(\emptyset) = 0, \\ &\text{(OM2)} \quad A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B), \\ &\text{(OM3)} \quad \mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n). \end{aligned}$$

2. **(Extension)** Show that μ^* extends μ, i.e., $\mu^*(S) = \mu(S)$ for all $S \in \mathcal{S}$.
3. **(μ^*-measurable sets)** Define the collection of μ^*-measurable sets by

$$\mathcal{A}_{\mu^*} := \{A \subseteq X : \mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \setminus A) \quad \text{for all } Q \subseteq X\}.$$

   Then $\mathcal{A}_{\mu^*}$ is a σ-algebra with $\mathcal{S} \subseteq \mathcal{A}_{\mu^*}$ and $\sigma(\mathcal{S}) \subseteq \mathcal{A}_{\mu^*}$.
4. **(Measure on σ-algebra)** The restriction of μ^* to $\mathcal{A}_{\mu^*}$ is a measure. In particular, $\mu^*|_{\sigma(\mathcal{S})}$ is a measure extending μ.

If $\mathcal{S}$ contains an increasing sequence $(S_n)_{n\in\mathbb{N}}$ with $S_n \uparrow X$ and $\mu(S_n) < \infty$ for all n, then the extension is unique. □

Existence of Lebesgue Measure on $\mathbb{R}$

Lemma 4.2. Let $\mathcal{J}_1 := \{[a, b) \subseteq \mathbb{R} : a < b\}$ be the family of half-open intervals. Define the set function

$$\lambda_1([a, b)) := b - a \quad \text{for all } [a, b) \in \mathcal{J}_1.$$

Then $\lambda_1 : \mathcal{J}_1 \to [0, \infty)$ is a pre-measure.

Proof. Let $[a, b) \in \mathcal{J}_1$, and suppose it can be written as a disjoint union of intervals:

$$[a, b) = \bigsqcup_{n\in\mathbb{N}} I_n, \quad \text{with } I_n \in \mathcal{J}_1 \text{ for all } n.$$

Our goal is to show that

$$\lambda_1([a, b)) = \sum_{n=1}^{\infty} \lambda_1(I_n).$$

Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, choose a closed interval $I_n^{(\varepsilon)}$ such that

$$I_n \subseteq I_n^{(\varepsilon)} \quad \text{and} \quad \lambda_1(I_n^{(\varepsilon)}) \le \lambda_1(I_n) + \frac{\varepsilon}{2^n}.$$

These intervals slightly extend each I_n, allowing us to approximate the union $\bigsqcup I_n$ from above.

Since the I_n cover $[a, b)$ disjointly, the union of the extended intervals will eventually cover most of $[a, b)$. More precisely, for sufficiently large N, we have

$$[a, b - \varepsilon) \subseteq \bigcup_{n=1}^{N} I_n^{(\varepsilon)}.$$

Now we estimate the difference:

$$\begin{aligned}
\lambda_1([a, b)) - \sum_{n=1}^{N} \lambda_1(I_n) &= (\lambda_1([a, b)) - \lambda_1([a, b - \varepsilon))) \\
&\quad + \left(\lambda_1([a, b - \varepsilon)) - \sum_{n=1}^{N} \lambda_1(I_n^{(\varepsilon)})\right) \\
&\quad + \sum_{n=1}^{N} \left(\lambda_1(I_n^{(\varepsilon)}) - \lambda_1(I_n)\right) \\
&\le \varepsilon + 0 + \sum_{n=1}^{N} \frac{\varepsilon}{2^n} \le 2\varepsilon.
\end{aligned}$$

On the other hand, since $\bigsqcup_{n=1}^N I_n \subseteq [a,b)$ and the intervals I_n are disjoint, finite additivity and monotonicity of λ_1 imply:

$$\sum_{n=1}^N \lambda_1(I_n) = \lambda_1\left(\bigsqcup_{n=1}^N I_n\right) \le \lambda_1([a,b)).$$

Therefore,

$$0 \le \lambda_1([a,b)) - \sum_{n=1}^N \lambda_1(I_n),$$

which justifies the lower bound in the previous inequality.

Combining both sides, we have

$$0 \le \lambda_1([a,b)) - \sum_{n=1}^N \lambda_1(I_n) \le 2\varepsilon.$$

Letting $N \to \infty$ and then $\varepsilon \to 0$, we conclude:

$$\lambda_1([a,b)) = \sum_{n=1}^\infty \lambda_1(I_n).$$

Thus, λ_1 is countably additive on $\mathcal{J}_1$, and hence a pre-measure. □

Lemma 4.3 (Lebesgue measure on $\mathbb{R}$)**.** The set function λ_1, defined on $\mathcal{J}_1$ by $\lambda_1([a,b)) = b-a$ for $a<b$, extends to a measure on $\mathcal{B}(\mathbb{R})$. This extension is the unique measure μ on $\mathcal{B}(\mathbb{R})$ such that

$$\mu([a,b)) = b-a \quad \text{for all } a<b.$$

Proof. We have already shown that λ_1 is a pre-measure on $\mathcal{J}_1$. By Theorem 1.3, $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{J}_1)$, i.e., the Borel σ-algebra is generated by $\mathcal{J}_1$.

Consider the sequence of half-open intervals $[-k,k) \subseteq \mathbb{R}$ for $k \in \mathbb{N}$. This forms an increasing sequence for $\mathbb{R}$, and we have

$$\lambda_1([-k,k)) = 2k < \infty \quad \text{for all } k \in \mathbb{N}.$$

Thus, all the conditions of Theorem 4.1 (Carathéodory's extension theorem) are satisfied. It follows that λ_1 extends uniquely to a measure on $\mathcal{B}(\mathbb{R})$, yielding the one-dimensional Lebesgue measure on $\mathbb{R}$. □

Existence of Lebesgue Measure on $\mathbb{R}^n$

Lemma 4.4. Let $\mathcal{J}_n$ denote the collection of half-open rectangles in $\mathbb{R}^n$ of the form

$$[\![a,b)\!) = \prod_{i=1}^n [a_i,b_i), \quad \text{where } a=(a_1,\dots,a_n),\ b=(b_1,\dots,b_n),\ a_i<b_i.$$

Then $\mathcal{J}_n$ is a semi-ring.

Proof. We prove the statement by induction on n. Assume $\mathcal{J}_n \subset \mathbb{R}^n$ is a semi-ring. Define

$$\mathcal{J}_{n+1} := \mathcal{J}_n \times \mathcal{J}_1,$$

i.e., the collection of rectangles of the form $R = R_n \times R_1$, where $R_n \in \mathcal{J}_n$ and $R_1 \in \mathcal{J}_1$.

We verify the properties of a semi-ring:

(S1) *Closure under the empty set:* Since $\emptyset \in \mathcal{J}_n$ and $\mathcal{J}_1$, we have

$$\emptyset = \emptyset \times [a, b) \in \mathcal{J}_{n+1}.$$

(S2) *Closure under intersection:* Let $R = R_n \times R_1$ and $S = S_n \times S_1$ be in $\mathcal{J}_{n+1}$. Then

$$R \cap S = (R_n \cap S_n) \times (R_1 \cap S_1),$$

which belongs to $\mathcal{J}_{n+1}$, since both $R_n \cap S_n \in \mathcal{J}_n$ and $R_1 \cap S_1 \in \mathcal{J}_1$, by the inductive hypothesis.

(S3) *Closure under set difference (finite disjoint union):* Consider

$$R \setminus S = (R_n \times R_1) \setminus (S_n \times S_1).$$

This set can be decomposed as

$$(R_n \setminus S_n) \times (R_1 \setminus S_1) \sqcup (R_n \cap S_n) \times (R_1 \setminus S_1) \sqcup (R_n \setminus S_n) \times (R_1 \cap S_1).$$

Each of the components $R_n \setminus S_n$, $R_n \cap S_n$, $R_1 \setminus S_1$, and $R_1 \cap S_1$ can be written as finite disjoint unions of sets in $\mathcal{J}_n$ and $\mathcal{J}_1$, respectively. Therefore, their Cartesian products yield finite disjoint unions of elements in $\mathcal{J}_{n+1}$.

Hence, $\mathcal{J}_{n+1}$ is a semi-ring. By induction, it follows that $\mathcal{J}_n$ is a semi-ring for all $n \in \mathbb{N}$. □

Lemma 4.5. The function $\lambda_n \colon \mathcal{J}_n \to [0, \infty)$, defined by

$$\lambda_n\Big([a_1, b_1) \times \cdots \times [a_n, b_n)\Big) = \prod_{i=1}^{n} (b_i - a_i),$$

is a pre-measure on the semi-ring $\mathcal{J}_n$.

Corollary 4.5.1 (Lebesgue measure on $\mathbb{R}^n$)**.** The set function λ_n extends to a measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$, called the *Lebesgue measure.* It is the unique measure satisfying

$$\lambda_n \left([a_1, b_1) \times \cdots \times [a_n, b_n)\right) = \prod_{i=1}^{n} (b_i - a_i), \quad \text{for all } a_i < b_i.$$

5 Measurable Mappings

Definition 5.1 (Measurable Map)**.** Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces. A map $T : X \to X'$ is called $\mathcal{A}/\mathcal{A}'$-measurable (or simply measurable) if the pre-image of every measurable set is measurable:

$$T^{-1}(A') \in \mathcal{A} \quad \text{for all } A' \in \mathcal{A}'.$$

Remark 5.1.

- Probabilists often refer to a measurable map defined on a probability space as a *random variable.*
- The symbolic notation $T^{-1}(\mathcal{A}') := \{T^{-1}(A') : A' \in \mathcal{A}'\}$ is often used. We also write $T^{-1}(\mathcal{A}') \subset \mathcal{A}$ as shorthand for measurability.
- It is common to write $T : (X, \mathcal{A}) \to (X', \mathcal{A}')$ to indicate that T is measurable.
- A measurable map between $\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^m)$ is often called a *Borel measurable map.*

Example 5.1. Let $(X, \mathcal{A})$ be a measurable space and let $A \in \mathcal{A}$. We show that the indicator function

$$\mathbf{1}_A : X \to \{0, 1\}, \quad \mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$$

is $\mathcal{A}/\mathcal{P}(\{0, 1\})$-measurable.

We check that the preimage of each subset of $\{0, 1\}$ lies in $\mathcal{A}$:

- $\mathbf{1}_A^{-1}(\emptyset) = \emptyset \in \mathcal{A}$,
- $\mathbf{1}_A^{-1}(\{0\}) = A^c \in \mathcal{A}$,
- $\mathbf{1}_A^{-1}(\{1\}) = A \in \mathcal{A}$,
- $\mathbf{1}_A^{-1}(\{0, 1\}) = X \in \mathcal{A}$.

Therefore, $\mathbf{1}_A$ is measurable.

Lemma 5.1. Let $(X, \mathcal{A})$, $(X', \mathcal{A}')$ be measurable spaces, and suppose $\mathcal{A}' = \sigma(\mathcal{E}')$. Then a map $T : X \to X'$ is $\mathcal{A}/\mathcal{A}'$-measurable if and only if $T^{-1}(\mathcal{E}') \subseteq \mathcal{A}$, i.e. if

$$T^{-1}(E') \in \mathcal{A} \quad \text{for all } E' \in \mathcal{E}'.$$

Proof. If T is measurable, then by definition $T^{-1}(A') \in \mathcal{A}$ for all $A' \in \mathcal{A}'$. Since $\mathcal{E}' \subset \mathcal{A}'$, it follows immediately that $T^{-1}(E') \in \mathcal{A}$ for all $E' \in \mathcal{E}'$.

Conversely, suppose $T^{-1}(E') \in \mathcal{A}$ for every $E' \in \mathcal{E}'$. Define

$$\mathcal{D}' := \{A' \subseteq X' : T^{-1}(A') \in \mathcal{A}\}.$$

By assumption, $\mathcal{E}' \subseteq \mathcal{D}'$. We now show that $\mathcal{D}'$ is a σ-algebra:

- Since $T^{-1}(X') = X \in \mathcal{A}$, we have $X' \in \mathcal{D}'$.
- If $A' \in \mathcal{D}'$, then $T^{-1}(A'^c) = T^{-1}(A')^c \in \mathcal{A}$, so $A'^c \in \mathcal{D}'$.
- If $A'_1, A'_2, \cdots \in \mathcal{D}'$, then

$$T^{-1}\left(\bigcup_{i=1}^{\infty} A'_i\right) = \bigcup_{i=1}^{\infty} T^{-1}(A'_i) \in \mathcal{A},$$

  hence $\bigcup_{i=1}^{\infty} A'_i \in \mathcal{D}'$.

Thus, $\mathcal{D}'$ is a σ-algebra containing $\mathcal{E}'$, so it contains $\sigma(\mathcal{E}') = \mathcal{A}'$. Therefore, $T^{-1}(A') \in \mathcal{A}$ for all $A' \in \mathcal{A}'$, i.e., T is measurable. □

Example 5.2. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function. Then T is $\mathcal{B}(\mathbb{R}^m)/\mathcal{B}(\mathbb{R}^n)$-measurable.

Indeed, from elementary analysis, we know that T is continuous if and only if

$$T^{-1}(A') \subset \mathbb{R}^m \text{ is open for every open set } A' \subset \mathbb{R}^n.$$

Since the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ is generated by the open sets $\mathcal{O}_{\mathbb{R}^n}$, it follows that

$$T^{-1}(\mathcal{O}_{\mathbb{R}^n}) \subset \mathcal{O}_{\mathbb{R}^m} \subset \sigma(\mathcal{O}_{\mathbb{R}^m}) = \mathcal{B}(\mathbb{R}^m).$$

Hence, by Lemma 5.1, T is $\mathcal{B}(\mathbb{R}^m)/\mathcal{B}(\mathbb{R}^n)$-measurable.

Theorem 5.2. Let $(X_i, \mathcal{A}_i)$, $i = 1, 2, 3$, be measurable spaces, and let

$$T : X_1 \to X_2, \quad S : X_2 \to X_3$$

be $\mathcal{A}_1/\mathcal{A}_2$- and $\mathcal{A}_2/\mathcal{A}_3$-measurable maps, respectively. Then the composition

$$S \circ T : X_1 \to X_3$$

is $\mathcal{A}_1/\mathcal{A}_3$-measurable.

Proof. Let $A_3 \in \mathcal{A}_3$. Then

$$(S \circ T)^{-1}(A_3) = T^{-1}\left(S^{-1}(A_3)\right).$$

Since S is $\mathcal{A}_2/\mathcal{A}_3$-measurable, we have $S^{-1}(A_3) \in \mathcal{A}_2$. Since T is $\mathcal{A}_1/\mathcal{A}_2$-measurable, it follows that $T^{-1}(S^{-1}(A_3)) \in \mathcal{A}_1$. Therefore, $S \circ T$ is $\mathcal{A}_1/\mathcal{A}_3$-measurable. □