

# Measure Theory

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## 1 $\sigma$ -Algebras

**Definition 1.1.** A  $\sigma$ -algebra  $\mathcal{A}$  on a set  $X$  is a family of subsets of  $X$  such that:

- $X \in \mathcal{A}$  ( $\Sigma_1$ )

- If  $A \in \mathcal{A}$ , then  $A^c \in \mathcal{A}$  ( $\Sigma_2$ )

- If  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ , then  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$  ( $\Sigma_3$ )

A set  $A \in \mathcal{A}$  is said to be *measurable* or  $\mathcal{A}$ -*measurable*.

**Example 1.1.**

1.  $\mathcal{P}(X)$  is a  $\sigma$ -algebra (the maximal  $\sigma$ -algebra on  $X$ ).
2.  $\{\emptyset, X\}$  is a  $\sigma$ -algebra (the minimal  $\sigma$ -algebra on  $X$ ).
3.  $\mathcal{A} := \{A \subseteq X : \#A < \infty \text{ or } \#A^c < \infty\}$  is a  $\sigma$ -algebra.
4. (Trace  $\sigma$ -algebra) Let  $E \subseteq X$  be any set and let  $\mathcal{A}$  be a  $\sigma$ -algebra on  $X$ . Then

$$\mathcal{A}_E := \{E \cap A : A \in \mathcal{A}\}$$

is a  $\sigma$ -algebra on  $E$ .

*Proof.* We verify the three defining properties of a  $\sigma$ -algebra on  $E$ :

- Since  $X \in \mathcal{A}$ , we have  $E = E \cap X \in \mathcal{A}_E$ .
- If  $E \cap A \in \mathcal{A}_E$ , then  $E \setminus (E \cap A) = E \cap A^c$ , and since  $A^c \in \mathcal{A}$ , it follows that  $E \cap A^c \in \mathcal{A}_E$ .
- If  $(E \cap A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}_E$ , then  $\bigcup_n (E \cap A_n) = E \cap \bigcup_n A_n$ , and since  $\bigcup_n A_n \in \mathcal{A}$ , we conclude that  $\bigcup_n (E \cap A_n) \in \mathcal{A}_E$ .

Hence,  $\mathcal{A}_E$  is a  $\sigma$ -algebra on  $E$ .  $\square$

5. (Pre-image  $\sigma$ -algebra) Let  $f : X \rightarrow X'$  be a function and let  $\mathcal{A}'$  be a  $\sigma$ -algebra on  $X'$ . Then

$$\mathcal{A} := \{f^{-1}(A') : A' \in \mathcal{A}'\}$$

is a  $\sigma$ -algebra on  $X$ .

**Theorem 1.1.** Let  $X$  be a set and let  $\{\mathcal{A}_i : i \in I\}$  be a family of  $\sigma$ -algebras on  $X$ . Define

$$\mathcal{A} := \bigcap_{i \in I} \mathcal{A}_i = \{A \subseteq X : A \in \mathcal{A}_i \text{ for all } i \in I\}.$$

Then,  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ .

*Proof.* We verify the  $\sigma$ -algebra properties for  $\mathcal{A}$ :

- Since  $X \in \mathcal{A}_i$  for all  $i \in I$ , we have  $X \in \mathcal{A}$ .
- If  $A \in \mathcal{A}$ , then  $A \in \mathcal{A}_i$  for all  $i \in I$ , so  $A^c = X \setminus A \in \mathcal{A}_i$  for all  $i \in I$ , hence  $A^c \in \mathcal{A}$ .
- If  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ , then  $A_n \in \mathcal{A}_i$  for all  $n$  and  $i$ , so  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}_i$  for all  $i \in I$ , and thus  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ .

Therefore,  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ . □

**Definition 1.2.** Let  $X$  be a set and let  $\mathcal{E} \subseteq \mathcal{P}(X)$  be a collection of subsets of  $X$ . The  $\sigma$ -algebra generated by  $\mathcal{E}$ , denoted by  $\sigma(\mathcal{E})$ , is the smallest  $\sigma$ -algebra on  $X$  containing all sets in  $\mathcal{E}$ . That is,

$$\sigma(\mathcal{E}) := \bigcap \left\{ \mathcal{A} \subseteq \mathcal{P}(X) : \mathcal{A} \text{ is a } \sigma\text{-algebra on } X, \mathcal{E} \subseteq \mathcal{A} \right\}.$$

**Remark 1.1** (Generated  $\sigma$ -algebras).

- If  $\mathcal{A}$  is a  $\sigma$ -algebra, then  $\sigma(\mathcal{A}) = \mathcal{A}$ .
- For  $A \subseteq X$ , we have  $\sigma(\{A\}) = \{\emptyset, A, A^c, X\}$ .
- If  $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{A}$ , then  $\sigma(\mathcal{F}) \subseteq \sigma(\mathcal{G}) \subseteq \sigma(\mathcal{A})$ .

**Definition 1.3** (Topological Space). A *topological space* is a pair  $(X, \mathcal{T})$  where  $X$  is a set and  $\mathcal{T} \subseteq \mathcal{P}(X)$  is a collection of subsets of  $X$ , called *open sets*, satisfying the following properties:

- $\emptyset \in \mathcal{T}$  and  $X \in \mathcal{T}$ ,
- If  $\{U_\alpha \in \mathcal{T} : \alpha \in I\}$  is an arbitrary collection of open sets, then the union  $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$ ,
- If  $\{U_i \in \mathcal{T} : i = 1, \dots, n\}$  is a finite collection of open sets, then the intersection  $\bigcap_{i=1}^n U_i \in \mathcal{T}$ .

The collection  $\mathcal{T}$  is called a *topology* on  $X$ . The complement of an open set is called a *closed set*.

**Remark 1.2** (Standard Topology on  $\mathbb{R}^n$ ). A subset  $U \subseteq \mathbb{R}^n$  is called *open* if for every point  $x \in U$ , there exists an  $\varepsilon > 0$  such that the open ball

$$B_\varepsilon(x) := \{y \in \mathbb{R}^n : \|x - y\| < \varepsilon\},$$

where  $\|\cdot\|$  denotes the Euclidean norm, is contained in  $U$ ; that is,  $B_\varepsilon(x) \subseteq U$ .

The collection of all such open sets is denoted by  $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$  and forms the *standard topology* on  $\mathbb{R}^n$ .

**Definition 1.4** (Borel  $\sigma$ -algebra). The  $\sigma$ -algebra  $\sigma(\mathcal{O})$  generated by the collection of open sets  $\mathcal{O} = \mathcal{O}_{\mathbb{R}^n}$  of  $\mathbb{R}^n$  is called the *Borel  $\sigma$ -algebra* on  $\mathbb{R}^n$ .

Its elements are called *Borel sets* or *Borel measurable sets*. We denote the Borel  $\sigma$ -algebra by  $\mathcal{B}(\mathbb{R}^n)$ .

**Definition 1.5.** Let  $X$  be a topological space and let  $A \subseteq X$ . A collection  $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$  of open sets is called an *open cover* of  $A$  if

$$A \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

A *subcover* is a subcollection that still covers  $A$ . The set  $A$  is called *compact* if every open cover of  $A$  admits a finite subcover.

**Remark 1.3.** In  $\mathbb{R}^n$ , a set is compact if and only if it is closed and bounded (Heine–Borel Theorem).

**Theorem 1.2** (Borel  $\sigma$ -algebra from Different Generators). Let  $\mathcal{O}, \mathcal{C}, \mathcal{K} \subseteq \mathcal{P}(\mathbb{R}^n)$  denote the collections of open, closed, and compact subsets of  $\mathbb{R}^n$ , respectively. Then,

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}) = \sigma(\mathcal{C}) = \sigma(\mathcal{K}).$$

*Proof.* Since compact sets are closed, we have  $\mathcal{K} \subseteq \mathcal{C}$ , and by Remark 1.1,  $\sigma(\mathcal{K}) \subseteq \sigma(\mathcal{C})$ . Conversely, for any  $C \in \mathcal{C}$ , define  $C_k := C \cap B_k(0)$ , where  $B_k(0)$  is the closed ball of radius  $k$  centered at the origin. Each  $C_k$  is closed and bounded, hence compact, so  $C_k \in \mathcal{K}$ . Since  $C = \bigcup_{k \in \mathbb{N}} C_k$ , it follows that  $C \in \sigma(\mathcal{K})$ , and thus  $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{K})$ .

Next, since  $\mathcal{C} = \mathcal{O}^c := \{U^c : U \in \mathcal{O}\}$ , and complements of sets in a  $\sigma$ -algebra are again in the  $\sigma$ -algebra, it follows that  $\mathcal{C} \subseteq \sigma(\mathcal{O})$ , hence  $\sigma(\mathcal{C}) \subseteq \sigma(\mathcal{O})$ . The reverse inclusion follows similarly from  $\mathcal{O} = \mathcal{C}^c$ . Therefore, we have:

$$\sigma(\mathcal{K}) = \sigma(\mathcal{C}) = \sigma(\mathcal{O}).$$

□

**Generating Sets of the Borel Algebra.** The Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}^n)$  can be generated by various systems of sets. Of particular importance are:

- The family of open rectangles:

$$\mathcal{J}_{o,n} := \{(a_1, b_1) \times \cdots \times (a_n, b_n) : a_i, b_i \in \mathbb{R}\},$$

- The family of half-open rectangles:

$$\mathcal{J}_n := \{[a_1, b_1) \times \cdots \times [a_n, b_n) : a_i, b_i \in \mathbb{R}\}.$$

We denote by  $\mathcal{J}_n^{\text{rat}}, \mathcal{J}_{o,n}^{\text{rat}}$  the subsets with rational endpoints. These sets represent intervals in  $\mathbb{R}$ , rectangles in  $\mathbb{R}^2$ , cuboids in  $\mathbb{R}^3$ , and hypercubes in higher dimensions.

**Theorem 1.3.** We have the following equality of Borel  $\sigma$ -algebras on  $\mathbb{R}^n$ :

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{J}_n^{\text{rat}}) = \sigma(\mathcal{J}_{o,n}^{\text{rat}}) = \sigma(\mathcal{J}_n) = \sigma(\mathcal{J}_{o,n}),$$

**Remark 1.4.** Let  $D \subseteq \mathbb{R}$  be a dense subset, for example  $D = \mathbb{Q}$  or  $D = \mathbb{R}$ . Then the Borel sets on  $\mathbb{R}$  can also be generated by any of the following families of intervals:

$$\{(-\infty, a) : a \in D\}, \quad \{(-\infty, a] : a \in D\}, \quad \{(a, \infty) : a \in D\}, \quad \{[a, \infty) : a \in D\}.$$

## 2 Measure Spaces

**Definition 2.1.** A (*positive*) *measure*  $\mu$  on  $X$  is a map  $\mu : \mathcal{A} \rightarrow [0, \infty]$ , where  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ , satisfying:

$$\mu(\emptyset) = 0, \tag{M1}$$

and for any pairwise disjoint sequence  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ ,

$$\mu\left(\bigsqcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n). \tag{M2}$$

Property (M2) is also called *countable additivity*.

If  $\mu$  satisfies (M1), (M2), but  $\mathcal{A}$  is not a  $\sigma$ -algebra, then  $\mu$  is called a *pre-measure*.

**Remark 2.1.** (M2) requires implicitly that  $\bigsqcup_n A_n$  is again in  $\mathcal{A}$  this is clearly the case for  $\sigma$ -algebras, but needs special attention when dealing with pre-measures.

**Definition 2.2** (Monotone sequences of sets). Let  $(A_n)_{n \in \mathbb{N}}$  and  $(B_n)_{n \in \mathbb{N}}$  be sequences of subsets of  $X$ .

We say  $(A_n)$  is *increasing* if

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$$

and write  $A_n \uparrow A$  where

$$A := \bigcup_{n \in \mathbb{N}} A_n$$

Similarly,  $(B_n)$  is *decreasing* if

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$$

and write  $B_n \downarrow B$  where

$$B := \bigcap_{n \in \mathbb{N}} B_n$$

**Definition 2.3.** Let  $X$  be a set and  $\mathcal{A}$  a  $\sigma$ -algebra on  $X$ . The pair  $(X, \mathcal{A})$  is called a *measurable space*. If  $\mu$  is a measure on  $(X, \mathcal{A})$ , then  $(X, \mathcal{A}, \mu)$  is called a *measure space*.

A measure  $\mu$  is called:

- *finite* if  $\mu(X) < \infty$ ,
- a *probability measure* if  $\mu(X) = 1$ .

Accordingly, we speak of a *finite measure space* and a *probability space*.

**Definition 2.4.** A measure  $\mu$  on a measurable space  $(X, \mathcal{A})$  is called  *$\sigma$ -finite* if there exists a sequence  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$  such that:

$$A_n \uparrow X \quad \text{and} \quad \mu(A_n) < \infty \quad \text{for all } n \in \mathbb{N}.$$

In this case, the measure space  $(X, \mathcal{A}, \mu)$  is called  *$\sigma$ -finite*.

**Lemma 2.1** (Basic properties of measures). Let  $(X, \mathcal{A}, \mu)$  be a measure space. Then:

- (i) If  $A_0, \dots, A_k \in \mathcal{A}$  are pairwise disjoint, then  $\mu\left(\bigcup_{n=1}^k A_n\right) = \sum_{n=1}^k \mu(A_n)$ .
- (ii) If  $A, B \in \mathcal{A}$  with  $A \subseteq B$ , then  $\mu(A) \leq \mu(B)$ .
- (iii) If  $A, B \in \mathcal{A}$ ,  $A \subseteq B$ , and  $\mu(A) < \infty$ , then  $\mu(B \setminus A) = \mu(B) - \mu(A)$ .

*Proof.* (i) Extend  $(A_n)$  by  $A_n = \emptyset$  for  $n > k$ . Then by countable additivity,

$$\mu\left(\bigcup_{n=1}^k A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^k \mu(A_n).$$

- (ii) Since  $B = A \sqcup (B \setminus A)$  we have

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

- (iii) Rearranging gives

$$\mu(B \setminus A) = \mu(B) - \mu(A),$$

which is well-defined if  $\mu(A) < \infty$ . □

**Lemma 2.2** (Main properties of measures). Let  $(X, \mathcal{A}, \mu)$  be a measure space. Then:

(i) **Countable subadditivity:** For any countable family  $\{A_i\}_{i \in \mathbb{N}} \subseteq \mathcal{A}$ ,

$$\mu \left( \bigcup_{i \in \mathbb{N}} A_i \right) \leq \sum_{i \in \mathbb{N}} \mu(A_i).$$

(ii) **Continuity from below (increasing sequence):** If  $A_1 \subseteq A_2 \subseteq \cdots$  (i.e.,  $A_n \uparrow A$ ), then

$$\mu \left( \bigcup_{n \in \mathbb{N}} A_n \right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

(iii) **Continuity from above (decreasing sequence):** If  $B_1 \supseteq B_2 \supseteq \cdots$  (i.e.,  $B_n \downarrow B$ ), then

$$\mu \left( \bigcap_{n \in \mathbb{N}} B_n \right) = \lim_{n \rightarrow \infty} \mu(B_n).$$

*Proof.* (i) For countable subadditivity, set  $B_k := A_k \setminus \bigcup_{i=1}^{k-1} A_i$ , so that  $(B_k)$  are disjoint with  $B_k \subseteq A_k$ . Then,

$$\mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \mu \left( \bigsqcup_{i=1}^{\infty} B_i \right) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

(ii) Let  $A_n \uparrow A$ , i.e.,  $A_n \subseteq A_{n+1}$  and  $A = \bigcup_{n \in \mathbb{N}} A_n$ . Define  $B_n := A_n \setminus A_{n-1}$  with  $A_0 := \emptyset$ . Then  $(B_n)$  is disjoint and  $\bigsqcup_n B_n = A$ . By countable additivity,

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(B_k) = \lim_{n \rightarrow \infty} \mu(A_n)$$

(iii) Assume  $B_n \downarrow B$ , i.e.,  $B_n \supseteq B_{n+1}$  and  $B = \bigcap_n B_n$ , with  $\mu(B_1) < \infty$ . Set  $A_n := B_1 \setminus B_n$ , so  $A_n \uparrow A := B_1 \setminus B$ . Then

$$\mu(B) = \mu(B_1) - \mu(A) = \mu(B_1) - \lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \mu(B_n)$$

□

**Remark 2.2.** With appropriate modifications, these properties also hold for pre-measures, i.e., when  $\mathcal{A}$  is not necessarily a  $\sigma$ -algebra.

**Example 2.1** (Dirac measure). Let  $(X, \mathcal{A})$  be a measurable space and let  $x \in X$ . Define  $\delta_x : \mathcal{A} \rightarrow \{0, 1\}$  by

$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Then  $\delta_x$  is a measure on  $(X, \mathcal{A})$ , called the *Dirac measure* (or unit mass) at the point  $x$ .

**Example 2.2** (Counting measure). Let  $(X, \mathcal{A})$  be a measurable space. Define  $\#A : \mathcal{A} \rightarrow [0, \infty]$  by

$$\#A := \begin{cases} \text{number of elements in } A & \text{if } A \text{ is finite,} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$$

Then  $\#$  is a measure on  $(X, \mathcal{A})$ , called the *counting measure*.

**Example 2.3** (Discrete probability measure). Let  $\Omega = \{\omega_1, \omega_2, \dots\}$  be a countable set, and let  $(p_n)_{n \in \mathbb{N}} \subseteq [0, 1]$  be a sequence such that  $\sum_{n \in \mathbb{N}} p_n = 1$ . Define the set function  $P : \mathcal{P}(\Omega) \rightarrow [0, 1]$  by

$$P(A) := \sum_{\{n \in \mathbb{N} : \omega_n \in A\}} p_n = \sum_{n \in \mathbb{N}} p_n \delta_{\omega_n}(A), \quad A \subseteq \Omega,$$

where  $\delta_{\omega_n}$  denotes the Dirac measure at  $\omega_n$ . Then  $P$  is a probability measure on  $(\Omega, \mathcal{P}(\Omega))$ , and the triplet  $(\Omega, \mathcal{P}(\Omega), P)$  is called a *discrete probability space*.

**Example 2.4** (Linear combination of measures). Let  $(X, \mathcal{A})$  be a measurable space, and let  $(\mu_n)_{n \in \mathbb{N}}$  be a sequence of measures on  $(X, \mathcal{A})$ . Let  $(x_n)_{n \in \mathbb{N}} \subseteq [0, \infty]$ . Then the set function

$$\mu := \sum_{n \in \mathbb{N}} x_n \mu_n$$

defined by

$$\mu(A) := \sum_{n \in \mathbb{N}} x_n \mu_n(A), \quad \text{for all } A \in \mathcal{A},$$

is a measure on  $(X, \mathcal{A})$

*Proof.* We verify the axioms of a measure:

(M1) (*Null empty set*): For all  $n \in \mathbb{N}$ ,  $\mu_n(\emptyset) = 0$ , so

$$\mu(\emptyset) = \sum_{n \in \mathbb{N}} x_n \mu_n(\emptyset) = \sum_{n \in \mathbb{N}} x_n \cdot 0 = 0.$$

(M2) (*Countable additivity*): Let  $(A_k)_{k \in \mathbb{N}} \subseteq \mathcal{A}$  be pairwise disjoint. Since each  $\mu_n$  is a measure, we have

$$\mu_n \left( \bigsqcup_{k \in \mathbb{N}} A_k \right) = \sum_{k \in \mathbb{N}} \mu_n(A_k), \quad \text{for all } n \in \mathbb{N}.$$

Then,

$$\mu \left( \bigsqcup_{k \in \mathbb{N}} A_k \right) = \sum_{n \in \mathbb{N}} x_n \mu_n \left( \bigsqcup_{k \in \mathbb{N}} A_k \right) = \sum_{n \in \mathbb{N}} x_n \sum_{k \in \mathbb{N}} \mu_n(A_k).$$

Since all terms are non-negative, we may exchange the order of summation:

$$\sum_{n \in \mathbb{N}} x_n \sum_{k \in \mathbb{N}} \mu_n(A_k) = \sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} x_n \mu_n(A_k) = \sum_{k \in \mathbb{N}} \mu(A_k).$$

Therefore,  $\mu$  is countably additive. □

**Example 2.5** (Restriction of a measure). Let  $(X, \mathcal{A}, \mu)$  be a measure space and let  $A \in \mathcal{A}$ . Define the set function  $\mu_A : \mathcal{A} \rightarrow [0, \infty]$  by

$$\mu_A(B) := \mu(A \cap B), \quad \text{for all } B \in \mathcal{A}.$$

Then  $\mu_A$  is a measure on  $(X, \mathcal{A})$ , called the *restriction of  $\mu$  to  $A$* .

*Proof.* We verify the two defining properties of a measure:

(M1):  $\mu_A(\emptyset) = \mu(A \cap \emptyset) = \mu(\emptyset) = 0$ .

(M2): Let  $(B_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$  be pairwise disjoint. Then  $(A \cap B_n)_{n \in \mathbb{N}}$  are also pairwise disjoint, and

$$\mu_A\left(\bigsqcup_{n \in \mathbb{N}} B_n\right) = \mu\left(A \cap \bigsqcup_{n \in \mathbb{N}} B_n\right) = \mu\left(\bigsqcup_{n \in \mathbb{N}} (A \cap B_n)\right) = \sum_{n \in \mathbb{N}} \mu(A \cap B_n) = \sum_{n \in \mathbb{N}} \mu_A(B_n).$$

Hence,  $\mu_A$  is a measure. □

**Definition 2.5** (Lebesgue measure on  $\mathbb{R}^n$ ). Define the set function  $\lambda_n$  on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$  by

$$\lambda_n([a, b]) := \prod_{i=1}^n (b_i - a_i),$$

for all  $[a, b] := [a_1, b_1) \times \cdots \times [a_n, b_n) \in \mathcal{J}_n$ . This is called the  $n$ -dimensional Lebesgue measure.

**Remark 2.3.** The set function  $\lambda_n$  is defined only on the family  $\mathcal{J}_n$  of half-open rectangles and hence is not yet a measure. Extending  $\lambda_n$  to a measure on  $\mathcal{B}(\mathbb{R}^n)$  requires the Carathéodory extension theorem, which will be developed later.

**Lemma 2.3.** Let  $(X, \mathcal{A})$  be a measure space, and let  $\mu : \mathcal{A} \rightarrow [0, \infty]$  be an additive set function with  $\mu(\emptyset) = 0$ . Then  $\mu$  is a measure if and only if it is **continuous from below**, i.e., for every increasing sequence  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$  with  $A_n \uparrow A$ , we have

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n) = \sup_{n \in \mathbb{N}} \mu(A_n).$$

*Proof.* Any measure  $\mu$  is continuous from below.

Conversely, suppose  $\mu$  is finitely additive,  $\mu(\emptyset) = 0$ , and  $\mu$  is continuous from below. Let  $(B_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$  be disjoint, and define  $A_n := \bigcup_{i=1}^n B_i$ . Then  $(A_n)$  is increasing with  $\bigcup_{n=1}^{\infty} A_n = \bigsqcup_{n=1}^{\infty} B_n$ . By finite additivity,

$$\mu(A_n) = \sum_{i=1}^n \mu(B_i),$$

and by continuity from below,

$$\mu\left(\bigsqcup_{n=1}^{\infty} B_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu(B_n).$$

Hence  $\mu$  is countably additive, i.e., a measure. □



**Lemma 2.4.** Let  $(X, \mathcal{A})$  be a measurable space and  $\mu : \mathcal{A} \rightarrow [0, \infty)$  an additive set function with  $\mu(\emptyset) = 0$  and  $\mu(A) < \infty$  for all  $A \in \mathcal{A}$ . Then  $\mu$  is a measure if and only if it satisfies one of the following continuity properties:

- (i)  $\mu$  is continuous from below;
- (ii)  $\mu$  is continuous from above;
- (iii)  $\mu$  is continuous at  $\emptyset$ , i.e., for every decreasing sequence  $(B_n)_{n \in \mathbb{N}}$  in  $\mathcal{A}$  with  $\bigcap_{n=1}^{\infty} B_n = \emptyset$ , we have

$$\lim_{n \rightarrow \infty} \mu(B_n) = 0.$$

*Proof.* Clearly, every measure satisfies properties (i)–(iii), so we only need to show that (iii) implies countable additivity.

Assume  $\mu$  is additive,  $\mu(\emptyset) = 0$ , and satisfies continuity at  $\emptyset$ . Let  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$  be pairwise disjoint and define  $A := \bigsqcup_{n \in \mathbb{N}} A_n$ . For each  $n$ , let

$$B_n := A \setminus \bigcup_{i=1}^n A_i.$$

Then  $(B_n)$  is a decreasing sequence in  $\mathcal{A}$  with  $\bigcap_{n \in \mathbb{N}} B_n = \emptyset$ , so by continuity at  $\emptyset$ , we have  $\mu(B_n) \rightarrow 0$ .

Using additivity, we compute

$$\mu(A) = \mu\left(B_n \sqcup \bigcup_{i=1}^n A_i\right) = \mu(B_n) + \sum_{i=1}^n \mu(A_i).$$

Taking the limit as  $n \rightarrow \infty$ , we get

$$\mu(A) = \lim_{n \rightarrow \infty} \left( \mu(B_n) + \sum_{i=1}^n \mu(A_i) \right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Thus,  $\mu$  is countably additive, hence a measure. □

### 3 Uniqueness of Measures

**Definition 3.1.** A *Dynkin system* (or  $\lambda$ -system)  $\mathcal{D} \subseteq \mathcal{P}(X)$  is a collection of subsets of  $X$  such that:

- $X \in \mathcal{D}$  (D1)

- If  $D \in \mathcal{D}$ , then  $D^c \in \mathcal{D}$  (D2)

- If  $(D_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}$  are pairwise disjoint, then  $\bigsqcup_{n \in \mathbb{N}} D_n \in \mathcal{D}$  (D3)

**Remark 3.1.** As with  $\sigma$ -algebras one easily checks that  $\emptyset \in \mathcal{D}$  and that finite disjoint unions are in  $\mathcal{D}$ : if  $D, E \in \mathcal{D}$  with  $D \cap E = \emptyset$ , then  $D \sqcup E \in \mathcal{D}$ . Every  $\sigma$ -algebra is a Dynkin system, but the converse is not true in general.

**Lemma 3.1.** Let  $\mathcal{E} \subseteq \mathcal{P}(X)$ . Then there exists a smallest Dynkin system  $\mathcal{D}(\mathcal{E})$  containing  $\mathcal{E}$ , called the *Dynkin system generated by  $\mathcal{E}$* . Moreover,

$$\mathcal{E} \subseteq \mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E}),$$

where  $\sigma(\mathcal{E})$  denotes the  $\sigma$ -algebra generated by  $\mathcal{E}$ .

*Proof.* The proof is analogous to that of Theorem 1.1 for  $\sigma$ -algebras. Let  $\mathcal{F}$  be the family of all Dynkin systems on  $X$  that contain  $\mathcal{E}$ . Then  $\mathcal{F}$  is nonempty, since  $\mathcal{P}(X)$  is a Dynkin system containing  $\mathcal{E}$ . Define

$$\mathcal{D}(\mathcal{E}) := \bigcap_{\mathcal{D} \in \mathcal{F}} \mathcal{D}.$$

Then  $\mathcal{D}(\mathcal{E})$  is a Dynkin system, being the intersection of Dynkin systems (which are closed under complements, disjoint unions, and contain  $X$ ). Moreover, it is the smallest such system containing  $\mathcal{E}$  by construction. Since every  $\sigma$ -algebra is in particular a Dynkin system, we also have

$$\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E}).$$

□

**Lemma 3.2.** A Dynkin system  $\mathcal{D}$  is a  $\sigma$ -algebra if and only if it is closed under finite intersections; that is,

$$D, E \in \mathcal{D} \quad \Rightarrow \quad D \cap E \in \mathcal{D}.$$

*Proof.* The “only if” direction follows immediately from Remark 3.1 and the fact that every  $\sigma$ -algebra is closed under finite intersections.

For the converse, assume  $\mathcal{D}$  is a Dynkin system closed under finite intersections. Let  $(D_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}$ , and define

$$E_1 := D_1 \in \mathcal{D}, \quad E_{n+1} := D_{n+1} \setminus \bigcup_{k=1}^n D_k = D_{n+1} \cap \bigcap_{k=1}^n D_k^c.$$

Each  $E_n \in \mathcal{D}$  by the Dynkin properties and the assumed stability under finite intersections. The sets  $(E_n)$  are disjoint and satisfy

$$\bigcup_{n=1}^{\infty} D_n = \bigsqcup_{n=1}^{\infty} E_n \in \mathcal{D},$$

so  $\mathcal{D}$  is closed under countable unions. Hence,  $\mathcal{D}$  is a  $\sigma$ -algebra. □

While Lemma 3.2 characterizes when a Dynkin system is a  $\sigma$ -algebra, it is not directly applicable when the Dynkin system  $\mathcal{D}$  is defined via a generator  $\mathcal{E} \subseteq \mathcal{P}(X)$ , as is often the case in practice. The following theorem overcomes this limitation and plays a central role in many applications.

**Theorem 3.3** (Dynkin’s  $\pi$ - $\lambda$  Theorem). Let  $\mathcal{E} \subseteq \mathcal{P}(X)$  be a collection of sets that is closed under finite intersections. Then,

$$\mathcal{D}(\mathcal{E}) = \sigma(\mathcal{E}).$$

*Proof.* By Lemma 3.1, we have  $\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E})$ . To show equality, it suffices to prove that  $\mathcal{D}(\mathcal{E})$  is a  $\sigma$ -algebra. Since it contains  $\mathcal{E}$ , it would then contain  $\sigma(\mathcal{E})$  by minimality.

By Lemma 3.2, it is enough to show that  $\mathcal{D}(\mathcal{E})$  is closed under finite intersections.

Fix  $D \in \mathcal{D}(\mathcal{E})$ , and define

$$\mathcal{D}_D := \{A \subseteq X : A \cap D \in \mathcal{D}(\mathcal{E})\}.$$

We claim that  $\mathcal{D}_D$  is a Dynkin system:

(D1): Since  $D = X \cap D \in \mathcal{D}(\mathcal{E})$ , we have  $X \in \mathcal{D}_D$ .

(D2): If  $A \in \mathcal{D}_D$ , then

$$A^c \cap D = ((A \cap D) \sqcup D^c)^c \cap D \in \mathcal{D}(\mathcal{E}),$$

using that  $A \cap D \in \mathcal{D}(\mathcal{E})$ ,  $D^c \in \mathcal{D}(\mathcal{E})$ , and that Dynkin systems are closed under disjoint unions and complements.

(D3): Let  $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}_D$  be disjoint. Then the sets  $A_n \cap D \in \mathcal{D}(\mathcal{E})$  are disjoint, and

$$\left( \bigsqcup_{n=1}^{\infty} A_n \right) \cap D = \bigsqcup_{n=1}^{\infty} (A_n \cap D) \in \mathcal{D}(\mathcal{E}).$$

Thus,  $\mathcal{D}_D$  is a Dynkin system. Since  $\mathcal{E} \subseteq \mathcal{D}_G$  for all  $G \in \mathcal{E}$  by the assumed  $\cap$ -stability of  $\mathcal{E}$ , and each  $\mathcal{D}_G$  is a Dynkin system, it follows that

$$\mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}_G \quad \text{for all } G \in \mathcal{E}.$$

Hence, for all  $D \in \mathcal{D}(\mathcal{E})$  and  $G \in \mathcal{E}$ , we have  $D \cap G \in \mathcal{D}(\mathcal{E})$ , i.e.,  $\mathcal{D}(\mathcal{E})$  is closed under finite intersections.

By Lemma 3.2, we conclude that  $\mathcal{D}(\mathcal{E})$  is a  $\sigma$ -algebra. Since  $\mathcal{D}(\mathcal{E}) \subseteq \sigma(\mathcal{E})$  and both are  $\sigma$ -algebras containing  $\mathcal{E}$ , we have

$$\mathcal{D}(\mathcal{E}) = \sigma(\mathcal{E}).$$

□

**Theorem 3.4** (Uniqueness of Measures). Let  $(X, \mathcal{A})$  be a measurable space with  $\mathcal{A} = \sigma(\mathcal{E})$ , where  $\mathcal{E} \subseteq \mathcal{P}(X)$  satisfies:

- $\mathcal{E}$  is closed under finite intersections;
- there exists an increasing sequence  $(E_n)_{n \in \mathbb{N}} \subseteq \mathcal{E}$  with  $E_n \uparrow X$ .

Suppose  $\mu$  and  $\nu$  are measures on  $\mathcal{A}$  such that  $\mu(E) = \nu(E)$  for all  $E \in \mathcal{E}$ , and  $\mu(E_n) = \nu(E_n) < \infty$  for all  $n \in \mathbb{N}$ . Then  $\mu = \nu$  on  $\mathcal{A}$ ; that is,

$$\mu(A) = \nu(A) \quad \text{for all } A \in \mathcal{A}.$$

*Proof.* Fix  $n \in \mathbb{N}$ , and define

$$\mathcal{D}_n := \{A \in \mathcal{A} : \mu(E_n \cap A) = \nu(E_n \cap A)\}.$$

We claim that  $\mathcal{D}_n$  is a Dynkin system:

(D1): Since  $E_n \in \mathcal{E} \subseteq \mathcal{A}$ , and  $\mu(E_n) = \nu(E_n)$ , it follows that  $X \in \mathcal{D}_n$ .

(D2): If  $A \in \mathcal{D}_n$ , then

$$\mu(E_n \cap A^c) = \mu(E_n) - \mu(E_n \cap A) = \nu(E_n) - \nu(E_n \cap A) = \nu(E_n \cap A^c),$$

so  $A^c \in \mathcal{D}_n$ .

(D3): Let  $(A_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}_n$  be disjoint. Then:

$$\mu\left(E_n \cap \bigsqcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(E_n \cap A_k) = \sum_{k=1}^{\infty} \nu(E_n \cap A_k) = \nu\left(E_n \cap \bigsqcup_{k=1}^{\infty} A_k\right),$$

so  $\bigsqcup_{k=1}^{\infty} A_k \in \mathcal{D}_n$ .

Thus,  $\mathcal{D}_n$  is a Dynkin system. Since  $\mathcal{E} \subseteq \mathcal{D}_n$  (as  $\mu(E_n \cap E) = \nu(E_n \cap E)$  for all  $E \in \mathcal{E}$ , by the  $\cap$ -stability of  $\mathcal{E}$ ), and since  $\sigma(\mathcal{E}) = \mathcal{A}$ , Theorem 3.3 yields

$$\mathcal{A} = \sigma(\mathcal{E}) = \mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}_n.$$

Hence,

$$\mu(E_n \cap A) = \nu(E_n \cap A) \quad \text{for all } A \in \mathcal{A}, n \in \mathbb{N}.$$

Now fix  $A \in \mathcal{A}$ . Since  $E_n \uparrow X$ , we have  $E_n \cap A \uparrow A$ , and by continuity from below,

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(E_n \cap A) = \lim_{n \rightarrow \infty} \nu(E_n \cap A) = \nu(A).$$

Therefore,  $\mu = \nu$  on  $\mathcal{A}$ . □

**Theorem 3.5** (Translation Invariance and Uniqueness of Lebesgue Measure). Let  $\lambda^n$  denote the  $n$ -dimensional Lebesgue measure on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ . Then:

(i) **(Translation invariance)** For all  $x \in \mathbb{R}^n$  and all  $B \in \mathcal{B}(\mathbb{R}^n)$ , we have

$$\lambda^n(x + B) = \lambda^n(B),$$

where  $x + B := \{x + y : y \in B\}$  is the translation of  $B$  by  $x$ .

(ii) **(Uniqueness up to scalar)** Let  $\mu$  be a measure on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$  that is translation invariant and finite on the unit cube:

$$\mu(x + B) = \mu(B) \quad \text{for all } x \in \mathbb{R}^n, B \in \mathcal{B}(\mathbb{R}^n), \quad \text{and} \quad \mu([0, 1]^n) < \infty.$$

Then  $\mu$  is a scalar multiple of Lebesgue measure:

$$\mu = \mu([0, 1]^n) \cdot \lambda^n.$$

## 4 Existence of Measures

**Definition 4.1** (Semi-ring). Let  $X$  be a set. A family  $\mathcal{S} \subseteq \mathcal{P}(X)$  is called a *semi-ring* if:

- $\emptyset \in \mathcal{S}$  (S1)

- For all  $S, T \in \mathcal{S}$ , we have  $S \cap T \in \mathcal{S}$  (S2)

- For all  $S, T \in \mathcal{S}$ , there exist disjoint sets  $S_1, \dots, S_M \in \mathcal{S}$  such that

$$S \setminus T = \bigsqcup_{i=1}^M S_i \tag{S3}$$