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Minimal vertex separators of chordal graphs

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Abstract

Chordal graphs form an important and widely studied subclass of perfect graphs. The set of minimal vertex separators constitute an unique class of separators of a chordal graph and capture the structure of the graph. In this paper, we explore the connection between perfect elimination orderings of a chordal graph and its minimal vertex separators. Specifically, we prove a characterization of these separators in terms of the monotone adjacency sets of the vertices of the graph, numbered by the maximum cardinality search (MCS) scheme. This leads to a simple linear-time algorithm to identify the minimal vertex separators of a chordal graph using the MCS scheme. We also introduce the notion of multiplicity of a minimal vertex separator which indicates the number of different pairs of vertices separated by it. We prove a useful property of the lexicographic breadth first scheme (LBFS) that enables us to determine the multiplicities of minimal vertex separators of a chordal graph.

Keywords: Chordal graphs; Minimal vertex separators; Maximum cardinality search; Lexicographic breadth first search

1. Introduction

Chordal graphs form an important subclass of perfect graphs. Chordal graphs and subclasses of chordal graphs arise in many practical situations such as scheduling [6], Gaussian elimination on sparse matrices [7, 8] relational database systems [2].

Minimal vertex separators of a chordal graph constitute a unique set of separators of the graph. Identification of the set of minimal vertex separators of a chordal graph enables us to decompose the graph into subgraphs that are again chordal. These subgraphs can in turn be decomposed further into chordal subgraphs and the process can be continued until the subgraphs are separator-free chordal graphs, namely cliques.

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In this paper, we carry out a thorough study of minimal separators and minimal vertex separators of chordal graphs. We first introduce the notions of a *base set* for chordal graphs and *multiplicity* of base sets. Base sets are monotone adjacency sets of certain vertices of the chordal graph numbered by a perfect elimination ordering scheme and thus have an algorithmic definition. We then characterize minimal separators of a chordal graph using base sets. Next, we prove that the notion of base sets is equivalent to that of minimal vertex separators thus providing an algorithmic definition for the latter. We next relate base sets to specific schemes for generating perfect elimination orderings (peos). We show that MCS scheme for generating peos can be used to characterize the minimal vertex separators of a chordal graph which leads us to an efficient algorithm to identify all the minimal vertex separators of a chordal graph. Multiplicity of a base set indicates the number of distinct pairs of vertices separated by it. We prove that the lexicographic BFS scheme can be used to efficiently determine the multiplicities of base sets.

The size and multiplicity of minimal vertex separators are two parameters on which if we impose conditions, we can obtain several different subclasses of chordal graphs. These classes would be useful in gaining insights into the nature of problems that are hard for the class of chordal graphs; we can restrict the problems to the new subclasses of chordal graphs and study their behaviour. We have obtained few results of this nature by introducing what we call *k-separator-chordal* (briefly, *k-sep-chordal*) graphs. This is the class of chordal graphs for which all the minimal vertex separators are of size exactly k. The class of 2-sep-chordal graphs properly contains 2-trees. We have shown that the Hamiltonian Circuit problem can be solved efficiently on 2-sep-chordal graphs [12], whereas it remains NP-complete for 3-sep-chordal graphs. Hamiltonian 2-sep-chordal graphs properly contain the class of maximal outer planar graphs (mops) and the linear-time isomorphism result on mops can be extended to Hamiltonian 2-sep-chordal graphs [14]. Further just as for mops, 2-sep-chordal graphs can be shown to be visibility graphs of monotone polygons. We believe that more results of this nature can be obtained.

The notion of a *clique tree* of a chordal graph is used to provide an intersection model for chordal graphs. It is a tree with nodes corresponding to the maximal cliques of the graph and where the nodes corresponding to the set of maximal cliques that contain any particular vertex of the graph induce a subtree of the tree. Associating these subtrees with the vertices of the graph gives an intersection model for the graph. There can be several clique trees for a given chordal graph. A structure that generalizes the clique tree notion, called the *reduced clique hypergraph* (*rch*), and from which all the clique trees can be obtained was proposed in [10]. Minimal vertex separators are useful in characterizing the edges of the *rch* and algorithmic characterization of these separators leads to an efficient algorithm for constructing the *rch* of chordal graphs [11].

The rest of the paper is organized as follows: Section 2 gives the relevant definitions and characterizations concerning chordal graphs. Base sets are introduced in Section 3. Section 4 contains the characterizations of minimal separators and minimal vertex separators in terms of base sets. In Section 5 MCS and LBFS schemes are related to base sets. Section 6 concludes the paper.

2. Preliminaries

An undirected graph $G=(V,E)$ is *chordal* if every cycle of length at least four has a *chord*, i.e., an edge between two nonconsecutive vertices of the cycle. A subset $S \subset V$ is called a *u–v separator* of G if in $G-S$, the vertices u and v are in two different connected components. A u–v separator is called a *minimal u–v separator* if no proper subset of it is a u–v separator. A *minimal vertex separator* is a minimal u–v separator for some u and v. One of the many characterizations of chordal graphs is in terms of minimal vertex separators.

Theorem 2.1 (Dirac [3]). *An undirected graph is chordal if and only if every minimal vertex separator of it induces a clique.*

A vertex v of a graph G is called *simplicial* if its adjacency set $adj(v)$ induces a clique. An *elimination ordering* σ of a graph G is a bijection $\sigma:\{1,2,\ldots,n\} \to V$, where $|V|=n$. Accordingly, $\sigma(i)$ is the ith vertex in the elimination ordering and $\sigma^{-1}(v)$, $v \in V$ gives the position of v in σ. A *perfect elimination ordering* (peo) is an elimination ordering $\sigma=(v_1,v_2,\ldots,v_n)$ where v_i, $(1 \leqslant i \leqslant n)$ is a simplicial vertex in the subgraph induced by $\{v_i,v_{i+1},\ldots,v_n\}$. The following characterization is well known.

Theorem 2.2 (Fulkerson and Gross [4]; Golumbic [5]). *An undirected graph is chordal if and only if it has a perfect elimination ordering.*

Given a peo σ of a chordal graph G, $N(v,\sigma)$ denotes the set of vertices adjacent to v that appear later than v in σ. That is,

$$\mathrm{N}(v,\sigma)=\{x \in adj(v):\ \sigma^{-1}(x) > \sigma^{-1}(v)\}.$$

The set $\mathrm{N}(v,\sigma)$ is called the *monotone adjacency set* of v with respect to σ. The graph G can be constructed by starting with an empty graph and adding vertices in the order $\sigma(n),\sigma(n-1),\ldots,\sigma(1)$, making each added vertex v adjacent to all the vertices in $\mathrm{N}(v,\sigma)$. We call this process the *reconstruction* of G with respect to σ. For a chordal graph G and peo σ, $G_i(\sigma)$ is the subgraph of G induced by the set $V_i(\sigma)=\{\sigma(j):\ j \geqslant i\}$.

The class of k-trees is a proper subclass of chordal graphs defined recursively as follows: A k-clique is a k-tree and if H is a k-tree with a k-clique Q then adding a vertex v to H and making it adjacent to all the vertices in Q gives another k-tree. k-trees generalize the class of trees which are nothing but 1-trees.

3. Base sets of chordal graphs

Consider the reconstruction of a chordal graph G with respect to a peo σ. During the reconstructing of a tree (in general a k-tree), addition of every vertex after the first two (respectively, $k+1$) vertices, brings in a new maximal clique. In the case of a chordal graph this is not the case. This observation motivates the following formulation of the notion of base sets of chordal graphs.

We first define the notion of a base set of a chordal graph with the help of a peo σ. This way of defining is simple and is useful for algorithmic purposes. We later show that any peo of the chordal graph leads to the same set of base sets.

We define a set $B \subset V(G)$ to be a *base set* of a chordal graph G with respect to a peo σ if there exists a vertex v with $\sigma^{-1}(v)=t$ such that (i) $B=\mathrm{N}(v,\sigma)$ and (ii) B is not a maximal clique in $G_{t+1}(\sigma)$. For a base set B, the vertices which satisfy (i) and (ii) above are called its *dependent vertices* with respect to σ. We denote this set of vertices by $D(B,\sigma)$. The size of the set $D(B,\sigma)$ is called the *multiplicity* of the base set B with respect to σ and is denoted by $\mu(B,\sigma)$. Fig. 1 shows a chordal graph along with one of its peos σ and the set of base sets with respect to σ. Note that in the case of a k-tree, for any vertex v with $\sigma^{-1}(v) < n-k$, $\mathrm{N}(v,\sigma)$ is a base set with respect to σ. The size of a base set of G is at most $\omega(G)-1$.

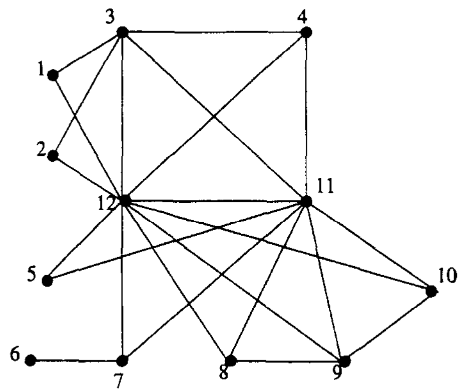
In the following lemmas we state certain fundamental properties of base sets.

Lemma 3.1. *Let B be a base set of a chordal graph G with respect to a peo σ and let $t=\max\{\sigma^{-1}(x)\colon x \in D(B,\sigma)\}$. Then B is a clique separator of $G_j(\sigma)$, $1 \leqslant j \leqslant t$.*

Proof. We show that B is a separator in G_{t-i} by induction on i. Consider the basis case where $i=0$. Since $\sigma(t) \in D(B,\sigma)$, $\sigma(t)$ is adjacent to all vertices of B. Further since B is not a maximal clique in $G_{t+1}(\sigma)$, there is a vertex $x \in V_{t+1}$ which is adjacent to all vertices of B. Therefore, in the subgraph $G_t(\sigma)$, the removal of B separates $\sigma(t)$ and x. Now we show that if B is a separator in G_{t-j} then it is also a separator in G_{t-j-1}. Suppose B is a separator in G_{t-j} but not a separator $G_{t-(j+1)}$. Let $\{u\}=V_{t-j-1}-V_{t-j}$. Then $\mathrm{N}(u,\sigma)$ must contain at least one vertex from each of the connected components of $G_{t-j}-B$. Since $\mathrm{N}(u,\sigma)$ is a clique, this contradicts the fact that B is a separator in G_{t-j}. Thus B must be a separator in $G_{t-(j+1)}$. Hence, by induction, B is a separator in G_j, $1 \leqslant j \leqslant t$. Since any $\mathrm{N}(v,\sigma)$ is a clique, B is a clique separator. □

The following Lemma can be proved by induction and we skip the proof.

Lemma 3.2. *The number of maximal cliques in a chordal graph equals* $1+\sum_{B\in\mathscr{B}} \mu(B,\sigma)$ *where $\mathscr{B}$ is the set of base sets of G with respect to σ.*



peo : 1, 2, ..., 12

Base set B	$D(B,\sigma)$	$\mu(B,\sigma)$
$\{9,11,12\}$	$\{8\}$	1
$\{11,12\}$	$\{4,5,7\}$	3
$\{7\}$	$\{6\}$	1
$\{3,12\}$	$\{1,2\}$	2

Fig. 1. Base sets of a chordal graph.

Corollary 3.3. *The number of base sets of a chordal graph G with respect to peo σ is at most one less than the number of maximal cliques in G.*

We now show that any two different peos α and β of a chordal graph lead to the same set of base sets. Thus the set of base sets is a unique or canonical set of separators of a chordal graph. Further, we will also show that a base set has the same multiplicity with respect to *any* peo. To this end we introduce the following definitions and notation.

Let B be a base set of a chordal graph G with respect to a peo σ. We call a connected component S of $G(V_i - B)$ a *dependent component* of B in the subgraph $G_i(\sigma)$ if there exists a vertex $x \in V(S)$ such that for all $y \in B$, $(x,y) \in E(G_i(\sigma))$. We denote the number of dependent components of B in the subgraph $G_i(\sigma)$ by $d_i(B,\sigma)$. Note that $d_1(B,\alpha) = d_1(B,\beta)$ for any peo's α and β as $G_1(\alpha) = G_1(\beta) = G$. So, we denote $d_1(B,\alpha)$ by just $d(B)$. In the following lemma we prove the relation between $d(B)$ and the multiplicity of B with respect to any peo σ.

Lemma 3.4. *Let B be a base set of a chordal graph G with respect to peo σ. Then $d(B) = \mu(B,\sigma) + 1$.*

Proof. Let $t = \max\{\sigma^{-1}(x)\colon x \in D(B,\sigma)\}$. By the definition of t, B is not a base set in $G_{t+1}(\sigma)$. We first show that $d_{t+1}(B,\sigma) = 1$. Since $\sigma(t) \in D(B,\sigma)$, B is not a maximal clique in $G_{t+1}(\sigma)$. Let B be contained in a maximal clique Q of $G_{t+1}(\sigma)$. Any vertex in $Q - B$ is adjacent to all vertices in B. Thus $d_{t+1}(B,\sigma) \geqslant 1$. Suppose $d_{t+1}(B,\sigma) \geqslant 2$. Let s be an integer, where $t+1 \leqslant s \leqslant n$, such that $d_s(B,\sigma) = 2$ but $d_{s+1}(B,\sigma) = 1$. Let $\{v\} = V_s - V_{s+1}$. Since $d_s(B,\sigma) = 2$, in the process of obtaining $G_s(\sigma)$ from $G_{s+1}(\sigma)$, v must have been made adjacent to all vertices of B i.e., $\mathrm{N}(v,\sigma) = B$. Further, since $d_{s+1}(B,\sigma) = 1$, there exists a vertex $x \in V_{s+1}$ which is adjacent to all vertices of B. Therefore, B is not a maximal clique of $G_{s+1}(\sigma)$. Since $\mathrm{N}(v,\sigma) = B$ and B is not a maximal clique of $G_{s+1}(\sigma)$, by definition, B is a base set of $G_s(\sigma)$. As $G_s(\sigma)$ is a subgraph of $G_{t+1}(\sigma)$, B is also a base set of $G_{t+1}(\sigma)$, a contradiction. Hence, $d_{t+1}(B,\sigma) = 1$.

Now, we show that $d_i(B,\sigma) = d_{i+1}(B,\sigma)+1$, where $i \leqslant t$, if and only if $\sigma(i) \in D(B,\sigma)$. Suppose $\sigma(i) \in D(B,\sigma)$. In $G(V_i - B)$, $\sigma(i)$ is an isolated vertex and forms a dependent component of B. Further, all the dependent components of B in G_{i+1} are also the dependent components of B in G_i as G_{i+1} is a subgraph of G_i. Therefore, $d_i(B,\sigma) = d_{i+1}(B,\sigma) + 1$. Suppose $d_i(B,\sigma) = d_{i+1}(B,\sigma) + 1$. It is straightforward to see that $\sigma(i) \in D(B,\sigma)$.

Since there are $\mu(B,\sigma)$ dependent vertices of B in G, $d_1(B,\sigma) = d(B) = \mu(B,\sigma) + 1$. □

Corollary 3.5. *If B is a base set of a chordal graph G with respect to peo σ, then B is a proper subset of at least $\mu(B,\sigma)+1$ maximal cliques of G.*

Corollary 3.6. *Let B be a base set of a chordal graph with respect to σ. No two vertices of $D(B,\sigma)$ belong to the same dependent component of B.*

Theorem 3.7. *Let α and β be two peo's of a chordal graph G.* (i) *If B is a base set of G with respect to α then it is also a base set of G with respect to β, and* (ii) $\mu(B,\alpha) = \mu(B,\beta)$.

Proof. (i) Let t be such that B is properly contained in one maximal clique in $G_{t+1}(\beta)$, but is contained in two maximal cliques in $G_t(\beta)$. Such a t must exist because, being a base set of G with respect to α, by Corollary 3.5, B is contained in at least two maximal cliques of G. If $v = \beta^{-1}(t)$ then $\mathrm{N}(v,\beta) = B$ and B is not a maximal clique of $G_{t+1}(\beta)$. Thus B is a base set of G with respect to β also. (ii) By Lemma 3.4, $\mu(B,\alpha) = \mu(B,\beta) = d(B) - 1$. □

We will henceforth drop the phrase 'with respect to a peo' while referring to base sets.

In the following theorem we prove a characterization of base sets.

Theorem 3.8. *A clique S of a chordal graph G is a base set of G if and only if there exist two maximal cliques Q_0, Q_1 of G such that $Q_0 \cap Q_1 = S$ and $Q_0 - S$ and $Q_1 - S$ are in different components of $G - S$.*

Proof. (If) Let σ be an arbitrary peo of G. Let t be the largest integer such that in $G_t(\sigma)$, S is properly contained in a maximal clique Q where either $Q \subseteq Q_0$ or $Q \subseteq Q_1$. Without loss of generality, assume that $Q \subseteq Q_0$. From the assumptions in the statement of the theorem, it follows that Q_1 contains a vertex v such that $\mathrm{N}(v, \sigma) = S$. Thus, S is a base set of G. (Only if) Follows from Corollary 3.5. □

In fact, the above theorem can be stated in a general form as follows and its proof is similar to that of the above theorem.

Theorem 3.9. *A clique S of a chordal graph G is a base set with multiplicity r if and only if there exists $(r+1)$ maximal cliques $Q_0, Q_1, \ldots, Q_r$ such that $Q_i \cap Q_j = S$ and $Q_i - S$, $Q_j - S$ are in different components of $G - S$ for all $0 \leqslant i < j \leqslant r$.*

4. Characterizations of separators

4.1. Minimal separators

A vertex separator is *minimal* if it contains no other separator. The minimal separators of a chordal graph have the following interesting property.

Lemma 4.1. *If S is a minimal separator of a chordal graph G, then each of the connected components of $G - S$ has a vertex that is adjacent to all the vertices of S.*

Proof. As G is chordal, S is a clique. Let M be a connected component of $G - S$ that has no vertex which is adjacent to all the vertices of S. If $|M| = 1$, then it follows that S is not minimal. So, let $|M| > 1$. For a vertex $v \in M$, let $S_v = S \cap adj(v)$. By assumption, for each vertex v of S, S_v is a proper subset of S. Consider a vertex $x \in M$. If for every vertex $y \in adj(x)$, $S_y \subset S_x$, then removal of S_x in G separates x, contradicting the minimality of separator S. Thus there is a vertex $y \in adj(x)$ such that S_x and S_y are not inclusion comparable. Now, let $w \in S_x - S_y$ and $z \in S_y - s_x$. w and z are adjacent as they belong to S. Clearly, w, x, y, z, w is a cycle of length four without a chord. A contradiction. Thus, each of the connected components of $G - S$ has a vertex that is adjacent to all the vertices of S. □

Corollary 4.2. *If S is a minimal u–v separator of a chordal graph G, then the connected components of $G - S$ that contain either u or v have a vertex that is adjacent to all the vertices of S.*

In the following lemma, we characterize the minimal separators of a chordal graph. Note that the minimal separators of a chordal graph are different from its minimal vertex separators.

Lemma 4.3. *A clique S is a minimal separator of a chordal graph G if and only if S is a minimal base set of G.*

Proof. (If) Suppose $S_1 \subset S$ is also a separator of G and further let S_1 be a minimal separator contained in S. Consider the connected components of $G - S_1$. By Lemma 4.1, each of them must contain at least one vertex that is adjacent to all the vertices of S_1. Now, it is easy to find two maximal cliques Q_1 and Q_2 such that $Q_1 \cap Q_2 = S_1$ and $Q_1 - S_1$ and $Q_2 - S_1$ are in different components of $G - S_1$. By Theorem 3.8, S_1 is also a base set of G, contradicting the minimality of base set S. Thus S is a minimal separator of G.

(Only if) Let S be a minimal separator of G. As G is chordal, S is a clique. By an argument similar to the one used in the "if" part of the proof it follows that S is base set of G. Since no vertex separator of G is properly contained in S, there can be no base set properly contained in S. Thus S is a minimal base set of G. □

Therefore, to determine the vertex connectivity number of a chordal graph, we can scan through its base sets and determine the size of the minimum base set which by the above theorem is the connectivity. We will see later that this can be done in linear time.

4.2. Minimal vertex separators

Recall that a subset S of the vertices $V(G)$ of a chordal graph G is a *minimal u–v separator* of G if

- $G(V - S)$ has at least two connected components and the vertices u and v are in different components, and
- No proper subset of S has this property with respect to u and v.

A *minimal vertex separator* is a minimal u–v separator for some vertices u and v of G. Note that it is possible for a minimal vertex separator to contain another minimal vertex separator for a different pair of vertices. Thus minimal vertex separators of a chordal graph are different from minimal separators. Every minimal separator is also a minimal vertex separator, but not every minimal vertex separator is a minimal separator. Fig. 2 illustrates this. The minimal 1–6 separator $\{2,4\}$ is a subset of the minimal 1–5 separator $\{2,3,4\}$.

Theorem 4.4. *A subset $S \subset V(G)$ is a minimal vertex separator of a chordal graph G if and only if S is a base set of G.*

Proof. (If) Let σ be any perfect elimination ordering of G. By the definition of base set, there exists a vertex v of G such that $\mathrm{N}(v,\sigma) = S$ and S is not a maximal clique in $G_{t+1}(\sigma)$ where $t = \sigma^{-1}(v)$. Let Q be a maximal clique of $G_{t+1}(\sigma)$ that contains S and let $u \in Q - S$. In $G_t(\sigma)$, S is a u–v separator and since u and v are adjacent to all the vertices of S, S is a minimal u–v separator. Now we show that if S is a u–v

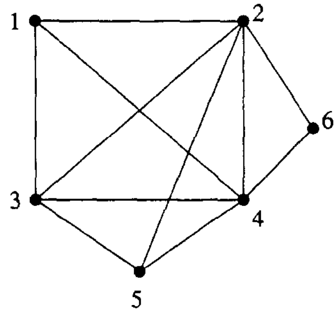


Fig. 2. Minimal vertex separators, minimal separators.

separator of $G_i(\sigma)$ then it is also a u–v separator of $G_{i-1}(\sigma)$, $i \leqslant t$. Let $\{x\} = V_{i-1} - V_i$. Let M_j be the subgraph induced by $C_j \cup S$, where C_j, are the connected components of $G(V_i - S)$. Either $\mathrm{N}(x,\sigma) \subseteq S$ or $\mathrm{N}(x,\sigma)$ is contained in exactly one of M_j, for otherwise $\mathrm{N}(x,\sigma)$ cannot be a clique in $G_i(\sigma)$. Thus S is a u–v separator in $G_{i-1}(\sigma)$ also. Since the addition of vertices does not affect the minimality of a separator, S is a minimal u–v separator in $G_{i-1}(\sigma)$ also. It follows that S is a minimal u–v separator of $G_1(\sigma) = G$.

(Only if) Suppose S is a minimal u–v separator of G. Let M_u and M_v be the connected components of $G(V - S)$ that contain u and v, respectively. By Corollary 4.2, M_u (respectively, M_v) contains a vertex x (respectively, y) such that $S \subseteq adj(x)$ (respectively, $adj(y)$). Let Q_1 and Q_2 be two maximal cliques of G that contain $S \cup \{x\}$ and $S \cup \{y\}$, respectively. Thus S is a clique separator satisfying the conditions of the characterization of base sets. By Theorem 3.8, S is a base set of G. □

5. Base sets, maximum cardinality search and lexicographic BFS

5.1. Maximum cardinality search (MCS)

We first review the *maximum cardinality search* (*MCS*) scheme for obtaining an elimination ordering σ of a graph. This scheme has the property that the elimination ordering obtained is a perfect elimination ordering if and only if the graph is chordal. Thus it can be used to recognize chordal graphs. The MCS method was proposed by Tarjan and Yannakakis [13] and runs in $O(|V| + |E|)$ time. An MCS ordering σ of a graph with n vertices is obtained as follows: Number the vertices from n to 1 in decreasing order. $\sigma(i)$ is the vertex that receives the number i. $\sigma(n)$ is chosen arbitrarily. As the next vertex to number, select the vertex that is adjacent to the largest number of currently numbered vertices, breaking ties arbitrarily (Fig. 3). After computing an MCS ordering σ, we compute the fill-in $F(\sigma)$ generated by it. The graph under consideration is chordal if and only if $F(\sigma) = \emptyset$. The fill-in can be computed in linear time [5]. Thus the MCS method gives a linear-time recognition algorithm for chordal graphs. It turns

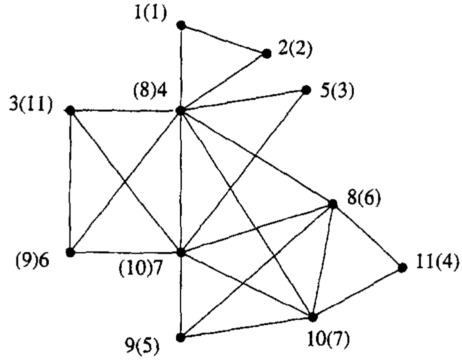


Fig. 3. Illustration of the MCS scheme. Numbers in brackets indicate the position of the vertex in the peo σ.

out that the notion of base sets can be used to obtain an alternative proof of correctness of the MCS method.

We now propose an algorithmic characterization of base sets of a chordal graph.

Lemma 5.1. *Let σ be an MCS peo of a chordal graph G and $\sigma(t)=v$, $\sigma(t+1)=u$. If $|\mathrm{N}(v,\sigma)|>|\mathrm{N}(u,\sigma)|$ then $|\mathrm{N}(v,\sigma)|=|\mathrm{N}(u,\sigma)|+1$ and $\mathrm{N}(v,\sigma)=\mathrm{N}(u,\sigma)\cup\{u\}$. Further, $\mathrm{N}(v,\sigma)$ is not a base set of $G_t(\sigma)$.*

Proof. Let $|\mathrm{N}(u,\sigma)|=i$. Since σ is an MCS peo and $\sigma(t+1)=u$ every vertex of G is adjacent to at most i vertices of the subgraph $G_{t+2}(\sigma)$. In particular, v is adjacent to at most i vertices of the subgraph $G_{t+2}(\sigma)$. Since $|\mathrm{N}(v,\sigma)|\geqslant i+1$, v is adjacent to at least $i+1$ vertices of $G_{t+1}(\sigma)$. As $V_{t+1}(\sigma)=V_{t+2}(\sigma)\cup\{u\}$, v is adjacent to i vertices of $G_{t+2}(\sigma)$ and $i+1$ vertices of $G_{t+1}(\sigma)$ and further u must be adjacent to v. Thus $u\in\mathrm{N}(v,\sigma)$. Since $\mathrm{N}(v,\sigma)$ is a clique, u must also be adjacent to all the i vertices of $G_{t+2}(\sigma)$ that are adjacent to v. Hence $\mathrm{N}(v,\sigma)=\mathrm{N}(u,\sigma)\cup\{u\}$. Finally, since $\mathrm{N}(u,\sigma)\cup\{u\}$ is a maximal clique of $G_{t+1}(\sigma)$, $\mathrm{N}(v,\sigma)$ is not a base set of $G_t(\sigma)$. □

Lemma 5.2. *Let σ be an MCS peo of a chordal graph G and $\sigma(t)=v$, $\sigma(t+1)=u$. If $|\mathrm{N}(v,\sigma)|\leqslant|\mathrm{N}(u,\sigma)|$ then $\mathrm{N}(v,\sigma)$ is a base set of G.*

Proof. Two cases arise.

Case (i) $u\in\mathrm{N}(v,\sigma)$. Since $\mathrm{N}(v,\sigma)$ is a clique, u is adjacent to all the other vertices in $\mathrm{N}(v,\sigma)$. Therefore, $\mathrm{N}(v,\sigma)$ is a subset of $\mathrm{N}(u,\sigma)\cup\{u\}$ which is a maximal clique in $G_{t+1}(\sigma)$. Hence $\mathrm{N}(v,\sigma)$ is a base set of G.

Case (ii) $u\notin\mathrm{N}(v,\sigma)$. Let $w=\sigma(s)$ where $s=\min\{\sigma^{-1}(x)\colon x\in\mathrm{N}(v,\sigma)\}$. Any vertex $\in\mathrm{N}(v,\sigma)-\{w\}$ is adjacent to w as $\mathrm{N}(v,\sigma)$ is a clique and $w\in\mathrm{N}(v,\sigma)$. Therefore, $\mathrm{N}(v,\sigma)$ is a subset of $\mathrm{N}(w,\sigma)\cup\{w\}$. If there exists a vertex in $\mathrm{N}(w,\sigma)$ that is not in

i	$\sigma(i)$	$\mathrm{N}(\sigma(i),\sigma)$	$\lvert\mathrm{N}(\sigma(i),\sigma)\rvert$	Base set
11	3	$\emptyset$	0	–
10	7	$\{3\}$	1	–
9	6	$\{3,7\}$	2	–
8	4	$\{3,7,6\}$	3	–
7	10	$\{4,7\}$	2	$\{4,7\}$
6	8	$\{4,7,10\}$	3	–
5	9	$\{7,8,10\}$	3	$\{7,8,10\}$
4	11	$\{8,10\}$	2	$\{8,10\}$
3	5	$\{4,7\}$	2	$\{4,7\}$
2	2	$\{4\}$	1	$\{4\}$
1	1	$\{2,4\}$	2	–

Fig. 4. Determining the base sets of a chordal graph.

$\mathrm{N}(v,\sigma)$ then $\mathrm{N}(v,\sigma)$ is a base set (because $\mathrm{N}(v,\sigma)$ is not a maximal clique in $G_{t+1}(\sigma)$) and we are through. So, assume that $\mathrm{N}(v,\sigma)=\mathrm{N}(w,\sigma)\cup\{w\}$. Let z be the vertex chosen immediately after w i.e. $\sigma(s-1)=z$. By the definition of w, v is adjacent to $|\mathrm{N}(v,\sigma)|$ numbered vertices after w is numbered. Since z is chosen immediately after w in preference to v, $|\mathrm{N}(z,\sigma)| \geqslant |\mathrm{N}(v,\sigma)|$. Thus $|\mathrm{N}(z,\sigma)|>|\mathrm{N}(w,\sigma)|$. By Lemma 5.1, $\mathrm{N}(z,\sigma)=\mathrm{N}(w,\sigma)\cup\{w\}=\mathrm{N}(v,\sigma)$. Thus $\mathrm{N}(v,\sigma)$ is a proper subset of $\mathrm{N}(z,\sigma)\cup\{z\}$. Hence $\mathrm{N}(v,\sigma)$ is a base set of G. □

As an MCS peo of a chordal graph can be obtained in $O(|V|+|E|)$ time [13], the above characterization of base sets can be used to detect all the base sets of a chordal graph in linear time. Fig. 4 shows the process of finding the base sets for the graph shown in Fig. 3.

5.2. Lexicographic breadth first search (LBFS)

Another scheme of obtaining an elimination ordering of a graph is the *lexicographic breadth first search* (*LBFS*) scheme, proposed by Rose et al. [9]. This scheme is a predecessor of the MCS scheme. The LBFS scheme also has the property that the elimination ordering generated by it is perfect if and only if the graph is chordal. An LBFS ordering σ of a graph with n vertices is obtained by the following labeling algorithm. As before, we number the vertices from n to 1 in decreasing order and $\sigma(i)$ is the vertex that receives the number i. In addition, we also assign labels to vertices which are used in selecting the next vertex to be numbered.

Algorithm LBFS
begin
  label all vertices of G with $\emptyset$;
  for $i=n$ downto 1 **do begin**
    among the vertices that have not been given a number,
    choose a vertex v with lexicographically largest label,

(breaking ties arbitrarily) and set $\sigma(i) \leftarrow v$;
for each $w \in adj(v)$ **do** append i to the label of w;
end
end

We prove below a useful property of the LBFS scheme that enables us to determine the multiplicities of base sets of a chordal graph.

Lemma 5.3. *Let α be an LBFS peo of a chordal graph G and B be a base set of G. Let β be the subsequence of α obtained by deleting all vertices x such that $|\mathrm{N}(x,\alpha)| > |B|$. Then the dependent vertices of B w.r.t. α occur consecutively in the sequence β.*

Proof. Let $D(B,\alpha) = \{y_1, y_2, \ldots, y_r\}$ where the y_i's are ordered according to α, i.e. $\alpha^{-1}(y_1) < \alpha^{-1}(y_2) < \cdots < \alpha^{-1}(y_r)$. Recall that $\mathrm{N}(y_i,\alpha) = B$, $1 \leqslant i \leqslant r$ and when y_i are selected they all have the same label. Consider y_{j-1} and y_j for some j, $2 \leqslant j \leqslant r$. It is clear from the operation of LBFS, that the labels of vertices selected for numbering subsequent to the selection of y_j and before the selection of y_{j-1} must have labels larger than $label(y_j) = label(y_{j-1})$. No label of length $|label(y_j)| = |B|$ or less can be larger than $label(y_j)$ as the numbers assigned to vertices in LBFS keep decreasing. Thus there is no vertex x such that $\alpha^{-1}(y_{j-1}) < \alpha^{-1}(x) < \alpha^{-1}(y_j)$ and $|\mathrm{N}(x,\alpha)| = |B|$. This shows that y_{j-1} and y_j appear consecutively in the subsequence β. $\square$

5.3. Algorithmic implications

We now make use of the characterization of base sets presented above to determine the base sets of a chordal graph along with their multiplicities. We assume that the chordal graph G is available in the adjacency list form. We also assume that the adjacency lists are doubly linked and the entries in them are sorted in increasing order. Such a representation can be obtained in linear time [1].

Algorithm Base-Sets
Input: A chordal graph G.
Output: The set of base sets $\mathscr{B}$ along with the multiplicities.
begin
compute an MCS peo σ of G;
for all $v \in V(G)$ determine $\mathrm{N}(v,\sigma)$; $\mathscr{B} = \emptyset$;
for $i = n-1$ downto 1 **do**
begin
if $|\mathrm{N}(\sigma(i),\sigma)| \leqslant |\mathrm{N}(\sigma(i+1),\sigma)|$ then
if $\mathrm{N}(\sigma(i),\sigma) \in \mathscr{B}$ then $\mu(\mathrm{N}(\sigma(i),\sigma) \leftarrow \mu(\mathrm{N}(\sigma(i),\sigma)) + 1$
else $\mathscr{B} \leftarrow \mathscr{B} \cup \{\mathrm{N}(\sigma(i),\sigma)\}$
end
end

We first compute an MCS peo σ of G. This can be done in $O(|V| + |E|)$ time using the algorithm in [13]. We next compute the monotone adjacency sets of all vertices as follows: Traverse the adjacency list $adj(v)$ and delete the vertices x with $\sigma^{-1}(x) > \sigma^{-1}(v)$. Thus the $N(v, \sigma)$ sets for all vertices v can be computed in linear time. The set $\mathscr{B}$ can be implemented as a balanced binary tree [1] such that the base sets are maintained in a lexicographic order. As the sets that are entered into $\mathscr{B}$ are all sorted lists of length $\leqslant \omega(G)$, comparision between two sets takes $O(\omega(G))$ time. Since there are at most $|\mathscr{B}|$ base sets, each search and insert operation on $\mathscr{B}$ takes $O(\omega(G)\log|\mathscr{B}|)$ time. Thus we have

Theorem 5.4. *The base sets of a chordal graph G and their multiplicities can be computed in* $O(\omega(G)|V|\log|\mathscr{B}| + |E|)$ *time.*

6. Conclusions

In this paper, we have proposed an algorithmic definition for the important notion of minimal vertex separators of chordal graphs and provided a characterization for them in terms the monotone adjacency sets of the vertices numbered by the MCS scheme. This has lead to an efficient algorithm to list all the minimal vertex separators. We also proposed a framework for introducing new subclasses of chordal graphs namely that of imposing restrictions on the size and multiplicity of the minimal vertex separators. It would be interesting to investigate if new results on hard problems for the class of chordal graphs can be obtained using the new subclasses.

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